Statistical Behavior and Consistency of Classification Methods Based on Convex Risk Minimization by Tong Zhang

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MAT 593 Practical Methods in Machine Learning

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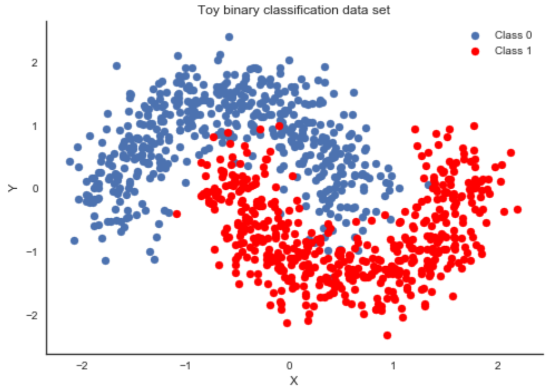
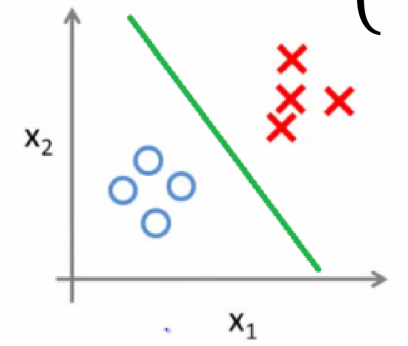
What is a Binary Classifier?

- Definition:
  I. Given data $\{x_i, y_i\}_{i=1}^{n}$ where:
  $x_i \in R^d$ are the features
  $y_i \in \{1, -1\}$ are the targets or class of the input x_i.
  II. Define a function $\mathcal{F}: R^d \to R$
  III. Now we consider the prediction rule $\hat{y}_i = \begin{cases} 1, & if\ f(x_i) \geq 0 \\ -1, & otherwise \end{cases}$



https://docs.pymc.io/notebooks/bayesian_neural_network_advi.html



https://utkuufuk.github.io/2018/06/03/one-vs-all-classification/

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What is Classification Error (Zero-One Loss)?

- Let $\widehat{y_i}$ = the predicted class y_i predicted by the classifier $f(x_i)$
- Classification Error (zero-one loss):

$$I(f(x_i), y_i) = \begin{cases} 1, & if\ \hat{y}_i \neq y_i \\ 0, & if\ \hat{y}_i = y_i \end{cases}$$

- The classifier seeks to minimize zero-one loss given by:

$$L(f(.)) = E_{X,Y} I(f(x), y)$$

- **PROBLEM:** $I(f(x), y)$ is not convex! Thus classifier algorithms seek a convex approximation of this problem.

Convex Approximation of Error Minimization

- Minimization of zero-one loss is very difficult!
- ML algorithms rely on a convex loss function ϕ which approximates zero-one loss:

$$\phi(f(x), y)$$

- Common classifier algorithms minimize the zero-one loss approximation:

$$Q\big(f(.)\big) = E_{X,Y}\phi(f(x), y)$$

- **How good is this approximation?**
- **How close is the resulting error rate to the lowest possible error rate we could expect?**
- This paper explores these questions and suggests that many common classifiers approach this lower bound.

Setting Goals for Classification Error

- **Bayes Error Rate:** The lowest error rate possible for classification of a given random outcome. Based on the Bayes decision rule for classification.
- Given the conditional probability of class 1:

$$\eta(x) = p(Y = 1|X = x)$$

- The Bayes decision rule is a classifier $f^*(x) = \begin{cases} 1, & if\ \eta(x) \geq .5 \\ -1, & otherwise \end{cases}$
- For any function f, $L(f(.)) - L(f^*(.)) \geq 0$
- Thus $L(f^*(.)) = inf_f L(f(.))$

Bayes Error Rate and Consistency

- Classifiers that achieve the Bayes Error rate:
  - Logistic regression: We know the MLE, thus we know it can achieve Bayes Error Rate under certain conditions.
- Other common classifiers meet or achieve similar classification error rates to logistic regression. *Is it possible that they also achieve Bayes Error Rate?*
- A classifier is said to be **consistent** if it's classification error rate approaches the Bayes Error as the number of empirical observations approaches infinity.
- This paper explores 5 common binary classifiers, showing that they are consistent, and that their error converges to the Bayes Error rate.

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Preface to Main Result

- Zhang proves the classifiers are consistent intuitively:
  - Introduce the concept of approximation error.
  - Recall the convex approximation of zero-one loss:

$$Q\big(f(.)\big) = E_{X,Y}\phi(f(x), y)$$

  - Establish the convex minimizer of the function, which may not be uniquely determined:

$$f_\phi^*(.) = arg\min_{f \in R} Q\big(f(.)\big)$$

  - Determine the minimal error rate possible through convex approximation:

$$Q^* = Q(f_\phi^*(.))$$

  - Define the approximation error of a given estimator as the difference between its error rate and the quantity Q*:

$$\Delta Q\big(f(.)\big) = Q\big(f(.)\big) - Q^*$$

Main Result

- Zhang proves the consistency of binary estimators based on minimization of the convex approximation for zero-one loss.
- Demonstrate that the difference between the error rate of a given estimator and the Bayes error rate (L*) is bounded by approximation error:

$$If\ |.5 - \eta|^s \leq c^s \Delta Q(\eta, 0)$$
$$then\ L(f(x)) - L^* \leq 2c\Delta Q\big(f(.)\big)^{1/s}$$

- Calculate the quantity $\Delta Q\big(f(.)\big)$ for 5 common estimators and show that for each estimator $\hat{f}(.)$:

$$\lim_{n\to\infty} \Delta Q\big(\hat{f}(.)\big) = 0$$

- **RESULT:** The error rates of these 5 common loss functions converge to the Bayes Error Rate.

Demonstration

- To show the following condition for each estimator:

  1) If $|.5-\eta| \leq c^s \Delta Q(\eta, 0)$
  2) then $L(f(x)) - L^* \leq 2c\Delta Q\left(f(.)\right)^{1/s}$

  we calculate $\Delta Q(\eta, f)$ and show that 1) holds.

- $\Delta Q(\eta, f)$ can be calculated with the formula:

$$\Delta Q(\eta, p) = \eta d_\phi\left(f_\phi^*(\eta), p\right) + (1-\eta) d_\phi\left(-f_\phi^*(\eta), -p\right)$$

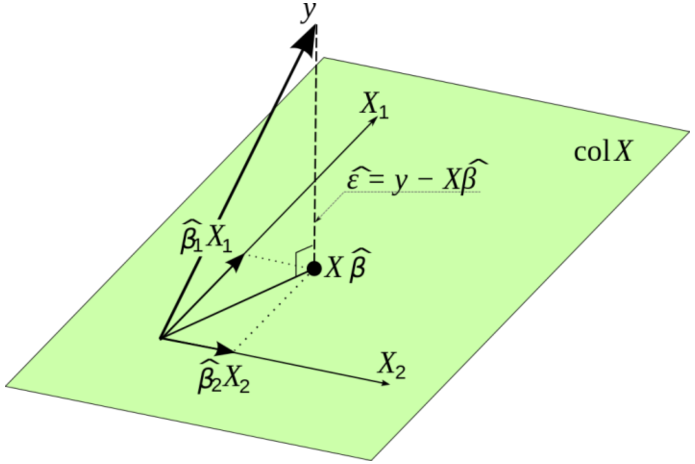
  where d_ϕ is Bregman divergence, calculated:

$$d_\phi(f_1, f_2) = (f_2) - \phi(f_1) - \phi'(f_2)(f_2 - f_1)$$

- To demonstrate this property, we need to calculate f_ϕ^*, $d_\phi(f_1, f_2)$, and $\Delta Q(\eta, p)$.

Example: OLS regression

- To find f_ϕ^*, we minimize the $Q(f(.))$ for the OLS loss function $\phi(v) = (1-v)^2$.
- $Q(f(.))$ can be re-formulated $Q(\eta, f) = \eta\phi(f) + (1-\eta)\phi(-f)$
- We solve the problem:
  $f_\phi^*(\eta) = \min_f Q(\eta, f) = \min_f \eta(1-f)^2 + (1-\eta)(1+f)^2 = 2\eta - 1$



https://en.wikipedia.org/wiki/Ordinary_least_squares

OLS Bregman Divergence and $\Delta Q(\eta, p)$

- $d_\phi(f_1, f_2) = \phi(f_2) - \phi(f_1) - \phi'(f_2)(f_2 - f_1)$
  $= (1 - f_2)^2 - (1 - f_1)^2 - 2(f_2 - 1)(f_2 - f_1)$
  $= (f_2 - f_1)^2$
- $\Delta Q(\eta, p) = \eta d_\phi\left(f_\phi^*(\eta), p\right) + (1 - \eta) d_\phi\left(-f_\phi^*(\eta), -p\right)$
  $= \eta(p - (2\eta - 1))^2 + (1 - \eta)\left((2\eta - 1) + p\right)^2$
  $= (2\eta - 1 - p)^2$

Boundedness of OLS Result

- If $\Delta Q(\eta, p) = (2\eta - 1 - p)^2$, clearly $\Delta Q(\eta, 0) = (2\eta - 1)^2$
- Rearranging, we get:

$$|.5 - \eta|^2 \leq .5^2\, \Delta Q(\eta, 0)$$

which fits the condition for our key result

$$|.5 - \eta|^s \leq c^s \Delta Q(\eta, 0)$$

where c = .5 and s = 2.

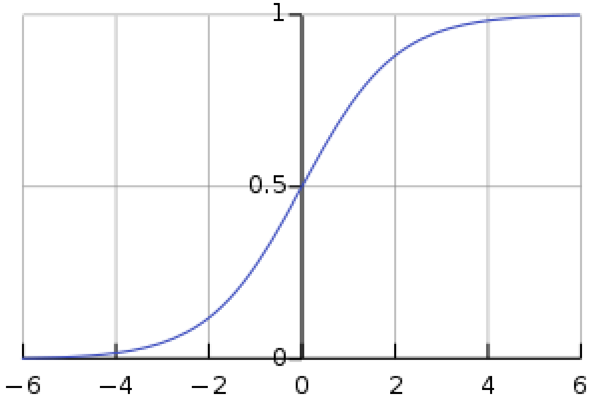
- Therefore, $L(f(x)) - L^* \leq 2c\Delta Q\left(f(.)\right)^{1/s}$

and since $\lim_{n\to\infty} \Delta Q\left(\hat{f}(.)\right) = 0$ for OLS, OLS is consistent.

Properties of OLS

- The function $p = 0$ minimizes $\Delta Q(\eta, p)$ when $\eta = .5$, since $f_{\phi}^{*} = 2\eta - 1$
- Using our classification decision rule, the function $f = 0$ estimates a probability of .5 that each observation is class 1.
- If $|.5 - \eta|^2 \leq .5^2\, \Delta Q(\eta, 0)$, we see that minimizing $\Delta Q(\eta, p)$ also minimizes the squared absolute difference of the predictor and in class probability η.
- Also, we see $(f(x) + 1)/2$ approximates the true in class probability.
- Therefore OLS provides a reliable estimate of the true in class probability.
- This provides us with a reliable estimate of how confident we can be in our prediction.

Example: Logistic Regression



https://en.wikipedia.org/wiki/Logistic_regression

- $f_\phi^*(\eta) = \ln \frac{\eta}{1-\eta}$
- $d_\phi(f_1, f_2) = -\eta \ln \eta - (1-\eta) \ln(1-\eta)$
- $\Delta Q(\eta, p) = \frac{1}{2\eta'(1-\eta')}(\eta - \bar{\eta}) \geq 2(\eta - \bar{\eta})^2$ where

  $\bar{\eta} = {f_\phi^*}^{-1}(p)$ and $\exists\, \eta'$, the Taylor expansion between $\bar{\eta}$ and η.

- Therefore:

$$\Delta Q(\eta, 0) \geq 2(\eta - .5)^2 \; or$$

$$|.5 - \eta|^2 \leq .5\Delta Q(\eta, 0) = 2^{-\frac{1}{2}^2} \Delta Q(\eta, 0)$$

with s = 2, c = $2^{-1/2}$

Properties of Logistic Regression

- The logistic transform $1/(1+e^{-f(x)})$ of f(x) approximates the true conditional in-class probability.
- Therefore, logistic regression is a maximum likelihood estimator.
- However, when the conditional in class probability is very close to 0 or 1, |f(x)| must be very large to approximate such a value given the logistic transform.
- Accordingly, logistic regression is poorly suited to predicting rare events.

Example: Support Vector Machines

- $f_\phi^*(\eta) = sign(2\eta - 1)$
- Cannot calculate Bregman divergence since ϕ' is not uniquely defined.
- Calculated directly,
  $\Delta Q(\eta, p) = \eta \max(0, 1 - p) + (1 - \eta) \max(0, 1 + p) - 1 + |2\eta - 1|$
- Therefore:

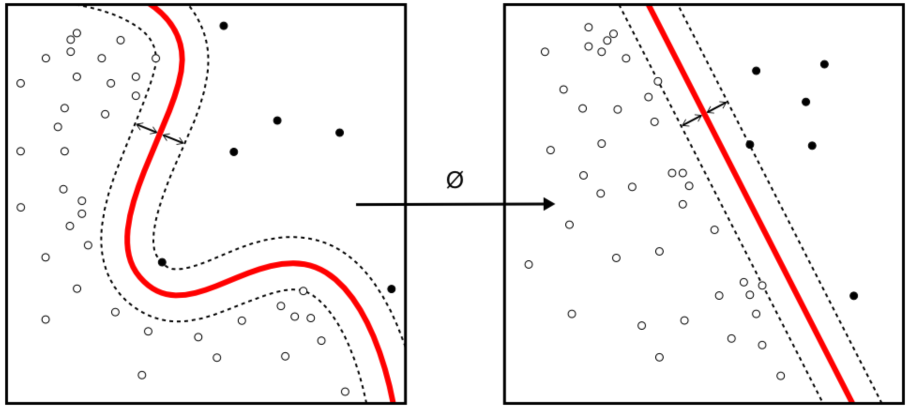
$$\Delta Q(\eta, 0) = \eta + (1 - \eta)1 + |2\eta - 1| = |2\eta - 1|$$

so

$$|\eta - .5| = .5\Delta Q(\eta, 0) \text{ with s= 1 and c= .5}$$

Properties of Support Vector Machines

- SVM delivers reliable predictions of class, and allows for conditions when $\eta \approx 1$ or $\eta \approx 0$
- However, even when $\eta(1-\eta)$ is not close to zero, f(x) clusters at 1 and -1.
- Therefore, SVM does not give reliable information about the confidence level of a prediction.



https://en.wikipedia.org/wiki/Support-vector_machine

Criticism

- Convexity, Classification and Risk Bounds by Bartlett Jordan, McAuliffe
  - “Need to find general quantitative relationships between approximation and estimation errors associated with Φ and those associated with 0-1 loss.”
  - Zhang presents several examples of these relationships.
  - This paper aims to simplify and extend Zhang’s results.
  - Similar comparison theorems to parts 1 and 3b of Theorem 1
  - However, authors’ conclusions hold under weaker conditions than those assumed by Zhang.
  - Pgs. 142-143

Criticism

- Convexity, Classification, and Risk Bounds by Bartlett, Jordan, and McAuliffe
  - Difficult of pattern classification is related to behavior of conditional in-class probability $\eta(X)$.
  - In practical problems, it is reasonable to assume for most X that $\eta(X)$ is not too close to ½
  - Tsybakov (2001) introduced formulation of this assumption
    - Under assumption of low noise, risk converges quickly to the minimum over the class
    - If minimum is nonzero, we expect a convergence rate as fast as $1/n$
  - Authors show that minimizing empirical Φ-risk leads to fast convergence rates under Tsybakov's assumption
  - If Φ is uniformly convex, empirical Φ-risk converges quickly to Φ-risk
  - Noise assumption allows improvement in relationship between excess Φ-risk and excess risk

Criticism

- Theory of Classification: A Survey of Some Recent Advances by Boucheron, Bousquet, and Lugosi
  - Excess misclassification error L(f)-L* is related to excess loss A(f)-A*
  - According to [27] (Bartlett, Jordan, McAuliffe), the above lemma may be improved under Mammen-Tsybakov noise conditions to yield

$$L(f) - L(f^*) \leq \left(\frac{2^s c}{\beta^{1-s}}(A(f) - A^*)\right)^{1/(s-s\alpha+\alpha)}.$$

  - The refined bounds may be carried over to analysis of classification rules based on the empirical minimization of a convex cost functional
  - The bounds are tighter and thus more precise, resulting in faster rates of convergence