Coursework

Imperial College London

Department of Mathematics

Computational Stochastic Process

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This is all my own unaided work unless stated otherwise.

Q 1.1

Itô formula concludes that for $dX_t = b(t,\omega)dt + \rho(t,\omega)dW_t$, $f(t, X_t) \in C^{1,2}$:

$$df = (\frac{\partial f}{\partial t} + b_t + \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2} + \sigma_t \frac{\partial f}{\partial x} dW_t)$$

With the SDE:

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{1}$$

Setting $f(x) = In(x)$, we get that:

$$dln(X_t) = (\mu + \frac{\sigma^2}{2})dt + \sigma dW_t$$

Therefore using $X_0 = 1$:

$$X_t = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

We obtain a formula for $\mathbb{E}[|x|^2] = \mathbb{E}[X_t \bar{X}_t]$ such that:

$$\mathbb{E}[|x|^2] = \mathbb{E}[X_t \bar{X}_t] = \mathbb{E}[e^{(\mu + \bar{\mu} - \frac{\sigma^2}{2} - \frac{\bar{\sigma}^2}{2})t + (\sigma + \bar{\sigma})W_t}]$$

By using the geometric log normal distribution $\mathbb{E}[X] = e^{u + \frac{\sigma^2}{2}}, X \sim \mathcal{N}(\mu, \sigma)$ and $(\sigma + \bar{\sigma})W_t \sim \mathcal{N}(0, (\sigma + \bar{\sigma})^2 t)$, we find that:

$$\begin{aligned}\mathbb{E}[|x|^2] &= e^{(\mu + \bar{\mu} - \frac{\sigma^2}{2} - \frac{\bar{\sigma}^2}{2})t + \frac{\sigma^2}{2}t + \sigma\bar{\sigma}t + \frac{\bar{\sigma}^2}{2}t} \\ &= e^{(\mu + \bar{\mu} + \sigma\bar{\sigma})t} = e^{(2\mathbb{R}(\mu) + |\sigma|^2)t}\end{aligned}$$

therefore we can conclude: $\mathbb{E}[|x|^2] \to 0$ as $t \to \infty$ when $2\mathbb{R}(\mu) + |\sigma|^2 < 0$

Q 1.2

The θ Milstein scheme for test equation (1) is:

$$\begin{aligned} X_{n+1} &= X_n + ((1-\theta)\mu X_n + \theta\mu X_{n+1})\Delta t + \sigma \Delta W_n + \frac{\sigma X_n}{2}\sigma((\Delta W_n)^2 - \Delta t) \\ x_{n+1} &= \frac{1 + (1-\theta)\mu\Delta t + \sigma\Delta W_n + \frac{\sigma X_n}{2}\sigma((\Delta W_n)^2 - \Delta t)}{1 - \theta\mu\Delta t} X_n \end{aligned}$$

Since W_t is a wiener process and $W_t \sim \mathcal{N}(0, s)$,$\mathbb{E}(W_t^2) = s$, by Itô's formula:

$$W_t^4 = 4\int_0^t W_s^3 dW_s + 6\int_0^t W_s^2 ds$$

$$\mathbb{E}(\Delta W_t^4) = 6\int_0^t \mathbb{E}(W_s^2) ds$$

we have that $\mathbb{E}(\Delta W_t^4) = 3\Delta t^2$, setting $Z_{n+1} = \mathbb{E}[|X_{n+1}|^2]$ we have:

$$Z_{n+1} = \frac{1 - \theta\mu\Delta t + \mu\Delta t - \theta\bar{\mu}\Delta t + \bar{\mu}\Delta t - 2\theta\mu\bar{\mu}\Delta t^2 + \mu\bar{\mu}\Delta t^2 + \theta^2\mu\bar{\mu}\Delta t^2 + \sigma\bar{\sigma}\Delta t + \frac{\sigma^2\bar{\sigma}^2}{2}\Delta t^2}{(1-\theta\mu\Delta t)(1-\theta\bar{\mu}\Delta t)} Z_n$$

Since $1 - \theta\mu\Delta t - \theta\bar{\mu}\Delta tt + \mu\bar{\mu}\Delta t^2 = (1-\theta\mu\Delta t)(1-\theta\bar{\mu}\Delta t)$, we can simplify that:

$$\begin{aligned} Z_{n+1} &= (1 + \frac{(\mu+\bar{\mu})\Delta t + (1-2\theta)\mu\bar{\mu}\Delta t^2 + \sigma\bar{\sigma}\Delta t^2 + \frac{\sigma^2\bar{\sigma}^2}{2}\Delta t^2}{(1-\theta\mu\Delta t)(1-\theta\bar{\mu}\Delta t)}) Z_n \\ &= (1 + \frac{2\mathbb{R}(\mu)\Delta t + (1-2\theta)|\mu|^2\Delta t^2 + |\sigma|^2\Delta t^2 + \frac{|\sigma|^4}{2}\Delta t^2}{(1-\theta\mu\Delta t)(1-\theta\bar{\mu}\Delta t)}) Z_n \end{aligned}$$

Q 1.3

With previous result, we can see that:

$$Z_{n+1} = Z_0 \prod_0^{n+1} (1 + \frac{2\mathbb{R}(\mu)\Delta t + (1-2\theta)|\mu|^2\Delta t^2 + |\sigma|^2\Delta t^2 + \frac{|\sigma|^4}{2}\Delta t^2}{(1-\theta\mu\Delta t)(1-\theta\bar{\mu}\Delta t)})$$

For Z_{n+1} to converge, we need:

$$(1 + \frac{2\mathbb{R}(\mu)\Delta t + (1-2\theta)|\mu|^2\Delta t^2 + |\sigma|^2\Delta t^2 + \frac{|\sigma|^4}{2}\Delta t^2}{(1-\theta\mu\Delta t)(1-\theta\bar{\mu}\Delta t)}) < 1$$

By calculation, we get:

$$0 < \Delta t < \frac{-\sigma^2 - 2\mu}{\frac{1}{2}\sigma^4 + (1-2\theta)\mu^2}$$

Set $x = \mu\Delta t, y = \sigma\Delta t$, the mean square stability condition is:

$$-(1-2\theta) - 2x > y$$

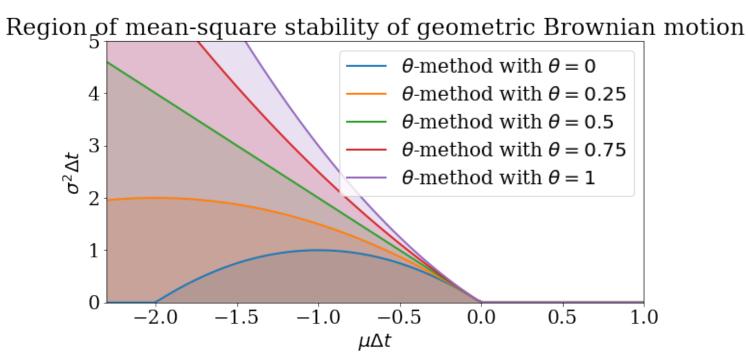

Figure 1: Q1.3

Q 1.4

With condition in $Q1.3$ we have:

$$2\mu\Delta t + (1-2\theta)\mu^2\Delta t^2 + \sigma^2\Delta t + \frac{1}{2}\sigma^4\Delta t^2 < 0$$
$$\rightarrow 2\mu + (1-2\theta)\mu^2\Delta t + \sigma^2 + \frac{1}{2}\sigma^4\Delta t < 0$$

Since $2\mu + \sigma^2 < 0$, to satisfy the above condition we need:

$$\mu^2 > \frac{\sigma^4}{2(2\theta - 1)}$$

From $Q1.1$ we know that $2\mathbb{R}(\mu) + |\sigma|^2 < 0$ then

$$2\mu < -\sigma^2$$
$$\rightarrow \mu^2 > \frac{\sigma^4}{4}$$

Thus for $\frac{\sigma^4}{4} \geq \frac{\sigma^4}{2(2\theta-1)}$ we need:

$$\theta \geq \frac{3}{2}$$

Q 1.5

By simulating the θ milstein scheme with $\Delta t_1 = 2\Delta t, \Delta t_2 = 0.5\Delta_t$ where $\Delta t = 1$. To find out the stable condition we need to substitute $\Delta t = 1$ into

$$\Delta t = \frac{-\sigma^2 - 2\mu}{\frac{1}{2}\sigma^4 + (1-2\theta)\mu^2}$$

to get $\theta = 0.25$, therefore our result is:

$$E_{100}^{2\Delta t} =$$

2. 1285603150251526e+57

$$E_{100}^{0.5\Delta t} =$$

6. 365828753983212e−59

Figure 2: Q1.5

We can find that the scheme is unstable when $\Delta t < 1$ as the expected variance approach to 0.

Q 2.1

Letting $Y_t = e^{\theta t}x_t$ and applying It's formula:

$$dY_t = \mu\theta e^{\theta t}dt + e^{\theta t}\sigma dW_t$$

therefore:

$$e^{\theta t}X_t = X_0 + \mu\theta\int_0^T e^{\theta s}ds + \sigma\int_0^T e^{\theta s}dW_s$$
$$X_t = X_0 e^{-\theta t} + \mu(1-e^{-\theta t}) + \sigma\int_0^T e^{\theta(s-t)}dW_s)$$

Therefore we have:

$$\mathbb{E}[X_t] = X_0 e^{-\theta t} + \mu(1-e^{-\theta t})$$
$$\mathbb{E}[X_t^2] = \mathbb{E}[(X_0 e^{-\theta t} + \mu(1-e^{-\theta t})^2] + 2\sigma\mathbb{E}[X_0 e^{-\theta t} + \mu(1-e^{-\theta t}]\mathbb{E}[\int_0^T e^{\theta s}dW_s] + \sigma^2\mathbb{E}[(\int_0^T e^{\theta s}dW_s)^2]$$

With $X_0 = 1 + x, x \sim \mathcal{U}(0,1), \mu = -1$, we can solve that:

$$\mathbb{E}[X_0] = 1 + \mathbb{E}[x] = \frac{3}{2}$$
$$\mathbb{E}[X_0^2] = \mathbb{E}[(1+x)^2] = \mathbb{E}[1 + 2x + x^2] = 1 + 1 + \mathbb{E}[x^2] = \frac{7}{3}$$

By using them, the definition that expected value of an stochastic integral is zero and It$\hat{o}$ isometry, we have that:

$$\mathbb{E}[(X_0 e^{-\theta t} + \mu(1-e^{-\theta t})^2] = \frac{7}{3}e^{-2\theta t} + 3\mu e^{-\theta t}(1-e^{-\theta t}) + \mu^2(1-e^{-\theta t})^2$$
$$\sigma^2\mathbb{E}[(\int_0^T e^{\theta s}dW_s)^2] = \sigma^2\mathbb{E}[(\int_0^T e^{\theta s}ds)^2] = \sigma^2\frac{1-e^{-2\theta t}}{2\theta}$$

therefore:

$$\mathbb{E}[X_t^2] = \frac{7}{3}e^{-2\theta t} + 3\mu e^{-\theta t}(1-e^{-\theta t}) + \mu^2(1-e^{-\theta t})^2 + \sigma^2\frac{1-e^{-2\theta t}}{2\theta}$$

substitute $T = 1, \theta = 1, \mu = -1, \sigma = \sqrt{2}$, we solve:

$$\mathbb{E}[X_t^2] = \frac{-15e + 16}{3e^2} + 2$$

Q 2.2

Set a class of stochastic process function f_N to be:

$$f_N(s) = f([\frac{s}{\Delta N}\Delta N]) = \sum_{j=0}^{N} c_j^N I_{j,j+1}$$

Taking for grant the sum of normally distributed random variables is normally distributed. By using it$\hat{o}$ intergral,we can rewrite as:

$$I_N = \int_0^T f_N(s)dW_s = \sum_{j=0}^{N} c_j^N (W_{t_{j+1}} - W_{tj})$$

$$\mathbb{E}[I_N] = \sum_{j=0}^{N} c_j^N \mathbb{E}[(W_{t_{j+1}} - W_{tj})] = 0$$

$$\mathbb{E}[(I_N)^2] = \mathbb{E}[(\int_0^T f_N dW_s)^2] = \int_0^T f_N^2 dt$$

Taking for granted that As N $\to \infty$ it holds that $f_N \to f$ in $L^2([0,T])$ and Convergence of a sequence of random variables in $L^2(\omega)$ implies convergence in distribution to the same limit. We have:

$$I := \int_0^T f(s)dW_s \sim \mathcal{N}(0, \int_0^T |f(s)|^2 ds)$$

Q 2.3

With the expression from $Q2.1$, we have:

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{\theta(s-t)} dW_s$$

$$\to e^{\theta t}(X_t - \mu) = X_0 - \mu + \sigma \int_0^t e^{\theta(s-t)} dW_s$$

TO find the numerical scheme, we need to link X_{t+1}, X_t, define:

$$X_{t+1} = e^{-\theta t+1}(X_0 - \mu + \sigma \int_0^t e^{\theta s} dW_s) + \mu + \sigma e^{-\theta t+1} \int_t^{t+1} e^{\theta s} dW_s$$

$$\to x_{t+1} = e^{-\theta \Delta t}(X_t - \mu) + \mu + \sigma e^{-\theta t+1} \int_t^{t+1} e^{\theta s} dW_s$$

With the answer from previous question, we can conclude that:

$$\int_t^{t+1} e^{\theta s} dW_s \sim \mathcal{N}(0, \int_t^{t+1} e^{2\theta s} dW_s)$$

Therefore:

$$\begin{aligned}\mathcal{S}(\sigma e^{-\theta t+1}\int_t^{t+1} e^{\theta s}dW_s) &= \sqrt{(\sigma e^{-\theta t+1}\int_t^{t+1} e^{\theta s}dW_s)^2} \\ &= \sqrt{\sigma^2 e^{-2\theta t+1}(\frac{e^{2\theta t+1}-e^{2\theta t}}{2\theta})} \\ &= \sigma\sqrt{\frac{1-e^{-2\theta\Delta t}}{2\theta}}\end{aligned}$$

this shows our iterative numerical scheme to be:

$$X_{n+1}^{\Delta t} = \mu + e^{-\theta\Delta t}(X_n^{\Delta t} - \mu) + \sigma\sqrt{\frac{1-e^{-2\theta\Delta t}}{2\theta}}\xi$$

Q 2.4

To calculate the 99% confidence interval, we need to calculate the mean I_m and standard deviation σ of $X_{100}^{\Delta t}$. Using the normally distributed confidence interval formula that:

$$(I_m - c_{0.005}\frac{\sigma}{\sqrt{m}}, I_m + c_{0.05}\frac{\sigma}{\sqrt{m}})$$

99% Confidence Interval:(0.8693690654654829, 0.8988461795919207)

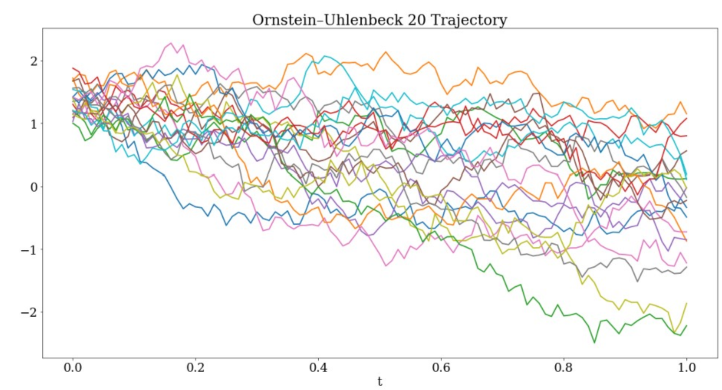

Figure 3: Q2.4

Q 2.5

For our equation. it is obvious that:

$$X_{t_{n+1}}|X_{t_n} \sim \mathcal{N}(X_{t_n} - \theta(X_{t_n} - \mu)\Delta t, \sigma^2\Delta t)$$

therefore our probability density function f_X^N where $X := (X_1, ...X_N)$is:

$$\mathbf{PDF} = f_X^N(X_1, ..., X_N; \theta) = |\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}|^N e^{-\frac{1}{2\sigma^2\Delta t}\sum_{k=1}^{N-1}|X_{k+1}-X_k+\theta(X_n-\mu)\Delta t|^2}$$

Q 2.6

By using the previous probability distribution we can find it is maximized when:

$$0 = \frac{\partial}{\partial\theta}\sum_{k=0}^{N-1}|X_{k+1} - X_k + \theta(X_n - \mu)\Delta t|^2 = \sum_{k=0}^{N-1}(X_{k+1} - X_k)(X_k - \mu) + \theta|X_k - \mu|^2\Delta t$$

This indicates the MLE estimator:

$$\hat{\theta}_N = -\frac{\sum_{k=0}^{N-1}(X_{k+1} - X_k)(X_k - \mu)}{\Delta t\sum_{k=0}^{N-1}|X_k - \mu|^2}$$

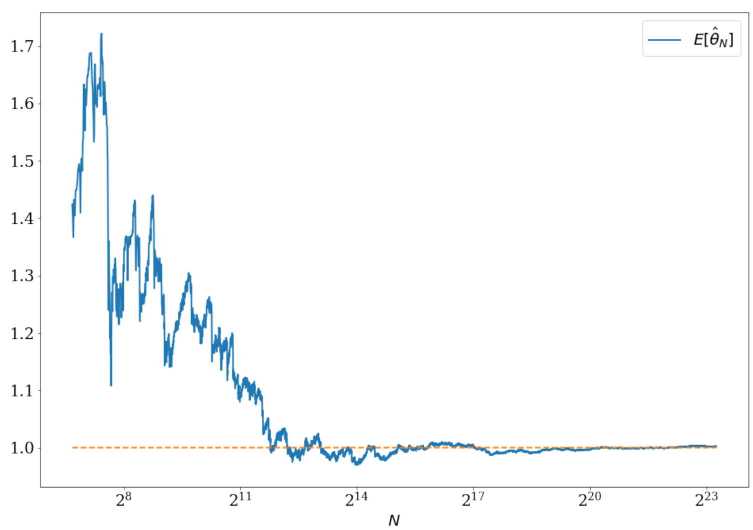

Figure 4: Q2.5

Q 3.1

Note that:

$$\begin{aligned}\int_0^T W_t \circ dW_t &= \lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{W_{t_j^N}+W_{t_{j+1}^N}}{2}(W_{t_{j+1}^N}-W_{t_j^N})\\ &= \frac{1}{2}\lim_{N\to\infty}\sum_{0}^{N-1}-(W_{t_j^N})^2+(W_{t_{j+1}^N})^2\\ &= \frac{1}{2}\lim_{N\to\infty}-(W_{t_0^N})^2+(W_{t_{j+1}^N})^2\end{aligned}$$

Since $\lim_{N\to\infty}(W_{t_{j+1}^N})^2 = W_t^2$, the sum can be viewed as the riemann sum of approximation therefore the value can be calculated using midpoint rule:

$$\int_0^T W_t \circ dW_t = \frac{W_t^2}{2}$$

Set $Y_t := W_t^{m+1}$, with the condition given:

$$W_t^{m+1} - W_0^{m+1} = (m+1)\int_0^t W_s^m \circ dW_s$$

therefore:

$$\int_0^t W_s^m \circ dW_s = \frac{W_t^{m+1}}{m+1}$$

Q 3.2

With the conversion formula from Stratonovich integral to it's formula we have:

$$\int_0^T b(X_t,t)\circ dW_t = \int_0^T b(X_t,t)dW_t + \frac{1}{2}\int_0^T \sigma(X_t,t)b'(X_t,t)dt$$

which implies:

$$b(X_t,t)\circ dW_t = b(X_t,t)dW_t + \frac{1}{2}\sigma(X_t,t)b'(X_t,t)dt$$

Now we can transfer the Stratonovich SDE to it$\hat{o}$ form such that:

$$dX_t = (\mu + \frac{\sigma^2}{2})X_t dt + \sigma X_t dW_t$$

we can set $b(X_t, t) = ln(X_t)$ and using it$\hat{o}$'s lemma to cancel out the $\frac{1}{2}\sigma^2$ term

$$dln(X_t) = \mu dt + \sigma dW_t \tag{2}$$

therfore:

$$\begin{aligned} ln(X_t) &= ln(X_0)(\int_0^t \mu dt + \int_0^t \sigma dW_t) \\ X_t &= X_0 e^{\int_0^t \mu dt + \int_0^t \sigma dW_t} \end{aligned}$$

Q 3.3

Q 3.4

Using $\mathcal{L}_0 f(x) = b(x)f'(x), \mathcal{L}_1 f'(x) = \sigma f'(x)$ and it$\hat{o}$'s formula, we can perform the following transformation:

$$X_t = X_s + \int_s^t b(x)du_1 + \int_s^t \sigma(x) \circ dW_{u_1} = X_0 + \int_s^t \mathcal{L}_0\iota(X_{u_1})du_1 + \int_s^t \mathcal{L}_1\iota(X_{u_1}) \circ dW_{u_1} \tag{3}$$

with similar recursion, take $\mathcal{L}_0\iota(X_{u_1})$and $\mathcal{L}_1\iota(X_{u_1})$ as the next approximation point:

$$\begin{aligned} \mathcal{L}_0\iota(X_{u_1}) &= \mathcal{L}_0\iota(X_s) + \int_s^{u_1} \mathcal{L}_0(\mathcal{L}_0\iota(X_{u_2}))du_2 + \int_s^{u_1} \mathcal{L}_1(\mathcal{L}_0\iota(X_{u_2})) \circ dW_{u_2} \\ \mathcal{L}_1\iota(X_{u_1}) &= \mathcal{L}_1\iota(X_s) + \int_s^{u_1} \mathcal{L}_0(\mathcal{L}_1\iota(X_{u_2}))du_2 + \int_s^{u_1} \mathcal{L}_1(\mathcal{L}_1\iota(X_{u_2})) \circ dW_{u_2} \end{aligned}$$

Once again we take $\mathcal{L}_0(\mathcal{L}_0\iota(X_{u_2})), \mathcal{L}_1(\mathcal{L}_0\iota(X_{u_2})), \mathcal{L}_0(\mathcal{L}_1\iota(X_{u_2})), \mathcal{L}_1(\mathcal{L}_1\iota(X_{u_2}))$ as the next approximation point:

$$\begin{aligned} \mathcal{L}_0(\mathcal{L}_0\iota(X_{u_2})) &= \mathcal{L}_0(\mathcal{L}_0\iota(X_s)) + \int_s^{u_2} \mathcal{L}_0(\mathcal{L}_0(\mathcal{L}_0\iota(X_{u_2})))du_3 + \int_s^{u_2} \mathcal{L}_1(\mathcal{L}_0(\mathcal{L}_0\iota(X_{u_2}))) \circ dW_{u_3} \\ \mathcal{L}_1(\mathcal{L}_0\iota(X_{u_2})) &= \mathcal{L}_1(\mathcal{L}_0\iota(X_s)) + \int_s^{u_2} \mathcal{L}_0(\mathcal{L}_1(\mathcal{L}_0\iota(X_{u_2})))du_3 + \int_s^{u_2} \mathcal{L}_1(\mathcal{L}_1(\mathcal{L}_0\iota(X_{u_2}))) \circ dW_{u_3} \\ \mathcal{L}_0(\mathcal{L}_1\iota(X_{u_2})) &= \mathcal{L}_0(\mathcal{L}_1\iota(X_s)) + \int_s^{u_2} \mathcal{L}_0(\mathcal{L}_0(\mathcal{L}_1\iota(X_{u_2})))du_3 + \int_s^{u_2} \mathcal{L}_1(\mathcal{L}_0(\mathcal{L}_1\iota(X_{u_2}))) \circ dW_{u_3} \\ \mathcal{L}_1(\mathcal{L}_1\iota(X_{u_2})) &= \mathcal{L}_1(\mathcal{L}_1\iota(X_s)) + \int_s^{u_2} \mathcal{L}_0(\mathcal{L}_1(\mathcal{L}_1\iota(X_{u_2})))du_3 + \int_s^{u_2} \mathcal{L}_1(\mathcal{L}_1(\mathcal{L}_1\iota(X_{u_2}))) \circ dW_{u_3} \end{aligned}$$

Now we have the material to prove our scheme and by plugging the value back to equation (3), we get:

$$\begin{aligned}
X_t = X_s &+ \int_s^t \mathcal{L}_0\iota(X_s)du_1 + \int_s^t \mathcal{L}_1\iota(X_s)\circ dW_{u_1} \\
&+ \int_s^t\int_s^{u_1} \mathcal{L}_0(\mathcal{L}_0\iota(X_s))du_1du_2 + \int_s^t\int_s^{u_1} \mathcal{L}_1(\mathcal{L}_0\iota(X_s))du_1\circ dW_{u_2} \\
&+ \int_s^t\int_s^{u_1} \mathcal{L}_0(\mathcal{L}_1\iota(X_s))_{u_1}du_2 + \int_s^t\int_s^{u_1} \mathcal{L}_1(\mathcal{L}_1\iota(X_s))\circ dW_{u_1}\circ dW_{u_2} + ...
\end{aligned}$$

which shows

$$\begin{aligned}
X_t - X_0 =& J^{s,t}_{(0)}f_{(0)}(X_s) + J^{s,t}_{(1)}f_{(1)}(X_s) + J^{s,t}_{(0,0)}f_{(0,0)}(X_s) \\
&+ J^{s,t}_{(0,1)}f_{(0,1)}(X_s) + J^{s,t}_{(1,0)}f_{(1,0)}(X_s) + J^{s,t}_{(1,1)}f_{(1,1)}(X_s) + ...
\end{aligned}$$

Q 3.5

With the index set $\mathcal{A}_\gamma$ and $\gamma = 2$, we can find that the strong order 2 scheme to be:

$$\begin{aligned}
X^{\Delta t}_{n+1} =& X^{\Delta t}_n + J^{t_n,t_{n+1}}_{(0)}f_{(0)}(X^{\Delta t}_n) + J^{t_n,t_{n+1}}_{(1)}f_{(1)}(X^{\Delta t}_n) + J^{t_n,t_{n+1}}_{(0,0)}f_{(0,0)}(X^{\Delta t}_n) \\
&+ J^{t_n,t_{n+1}}_{(0,1)}f_{(0,1)}(X^{\Delta t}_n) + J^{t_n,t_{n+1}}_{(1,0)}f_{(1,0)}(X^{\Delta t}_n) + J^{t_n,t_{n+1}}_{(1,1)}f_{(1,1)}(X^{\Delta t}_n) \\
&+ J^{t_n,t_{n+1}}_{(1,1,0)}f_{(1,1,0)}(X^{\Delta t}_n) + J^{t_n,t_{n+1}}_{(1,0,1)}f_{(1,0,1)}(X^{\Delta t}_n) + J^{t_n,t_{n+1}}_{(0,1,1)}f_{(0,1,1)}(X^{\Delta t}_n) \\
&+ J^{t_n,t_{n+1}}_{(1,1,1)}f_{(1,1,1)}(X^{\Delta t}_n) + J^{t_n,t_{n+1}}_{(1,1,1,1)}f_{(1,1,1,1)}(X^{\Delta t}_n)
\end{aligned}$$

With our SDE $dx_t = \mu X_t dt + \sigma X_t \circ dW_t$ and by definite of $\mathcal{L}, f$, we have:

$$\begin{aligned}
f_{(0)} = \mu, f_{(1)} = \sigma, f_{(0,0)} = \mu^2 \\
f_{(0,1)} = \mu\sigma, f_{(1,1)} = \sigma^2, f_{(1,1,0)} = \mu\sigma^2 \\
f_{(1,0,1)} = \mu\sigma^2, f_{(0,1,1)} = \mu\sigma^2, f_{(1,1,1)} = \sigma^3, f_{(1,1,1,1)} = \sigma^4
\end{aligned}$$

Using the result we have from $Q3.3$:

$$\begin{aligned}
\Delta t J^{t_n,t_{n+1}}_{(1)} &= J^{t_n,t_{n+1}}_{(0,0)} + J^{t_n,t_{n+1}}_{(0,1)} \\
\Delta t J^{t_n,t_{n+1}}_{(1,1)} &= J^{t_n,t_{n+1}}_{(1,1,0)} + J^{t_n,t_{n+1}}_{(1,0,1)} + J^{t_n,t_{n+1}}_{(0,1,1)}
\end{aligned}$$

With the result obtained from $Q3.1$, we have:

$$\begin{aligned}
J_{(1)}^{t_n,t_{n+1}} &= \int_{t_n}^{t_{n+1}} \circ dW_{u_1} = (W_{tn+1} - W_{tn}) \\
J_{(1,1)}^{t_n,t_{n+1}} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{u_1} \circ dW_{u_2} \circ dW_{u_1} = \frac{(W_{tn+1} - W_{tn})^2}{2} \\
J_{(1,1,1)}^{t_n,t_{n+1}} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{u_1} \int_{t_n}^{u_3} \circ dW_{u_3} \circ dW_{u_2} \circ dW_{u_1} = \frac{(W_{tn+1} - W_{tn})^3}{6} \\
J_{(1,1,1,1)}^{t_n,t_{n+1}} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{u_1} \int_{t_n}^{u_3} \int_{t_n}^{u_4} \circ dW_{u_4} \circ dW_{u_3} \circ dW_{u_2} \circ dW_{u_1} = \frac{(W_{tn+1} - W_{tn})^4}{24}
\end{aligned}$$

Substitute them back to our scheme, we end up with:

$$\begin{aligned}
X_{n+1}^{\Delta t} =& X_n^{\Delta t} + \Delta t \mu X_n^{\Delta t} + \Delta W \sigma X_n^{\Delta t} + \frac{\Delta t^2}{2}\mu^2 X_n^{\Delta t} + \Delta t \Delta W \mu \sigma X_n^{\Delta t} \\
&+ \frac{\Delta W^2}{2}\sigma^2 X_n^{\Delta t} + \frac{\Delta t \Delta W^2}{2}\mu\sigma^2 X_n^{\Delta t} + \frac{\Delta W^3}{6}\sigma^3 X_n^{\Delta t} + \frac{\Delta W^4}{24}\sigma^4 X_n^{\Delta t}
\end{aligned}$$

3.6

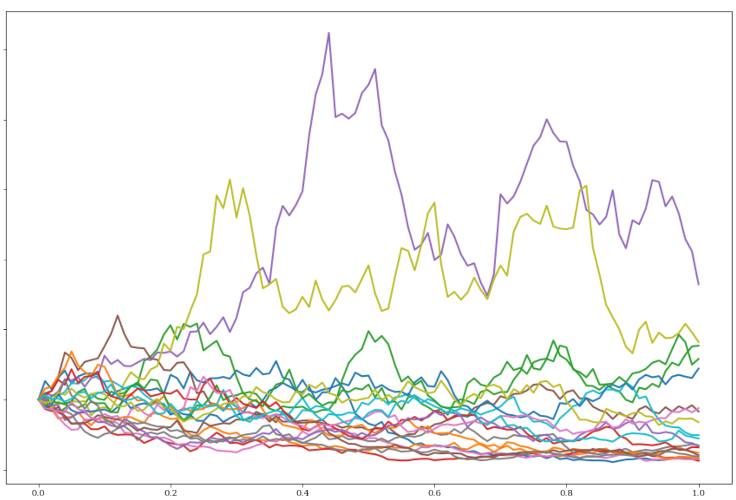

Figure 5: Q3.6