Degrees of the logarithmic vector fields for close-to-free hyperplane arrangements

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§ Introduction

§ Line arrangement with cubic minimal log derivations.

§ Derivation Degree Sequences of Logarithmic Modules for close to Free Hyperplane Arrangements

https://github.com/jcwjmz/LogarithmicDerivationModule

or: https://github.com/jcwjmz

§ Introduction

1. $\mathbb{K}$: field. , $V := \mathbb{K}^\ell$: vector space.

2. $S := \mathbb{K}[x_1, \cdots, x_\ell]$: the polynomial ring.

3. $\mathcal{A} := \{H_1, \cdots, H_p\}$: hyperplane arrangement

   a finite set of linear hyperplanes $H_i = \ker \alpha_i \subset V$

4. $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \ (\in S)$: defining polynomial of $\mathcal{A}$

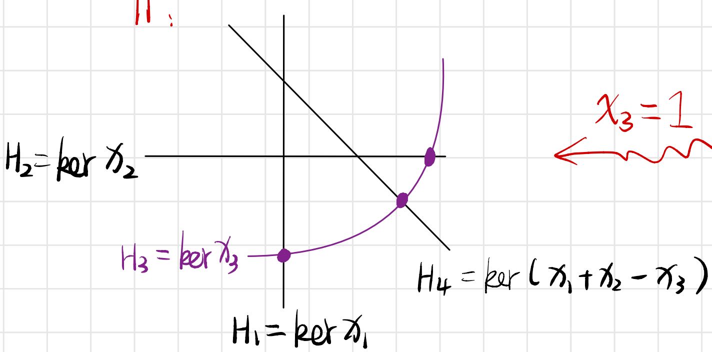
• Example :

Let $\underline{V = \mathbb{R}^3}$ and $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$.

projectivise

$$Q(\mathcal{A}) = x_1 x_2 x_3 (x_1 + x_2 - x_3)$$

$\mathbb{P}^2$:



$\xleftarrow{x_3 = 1}$

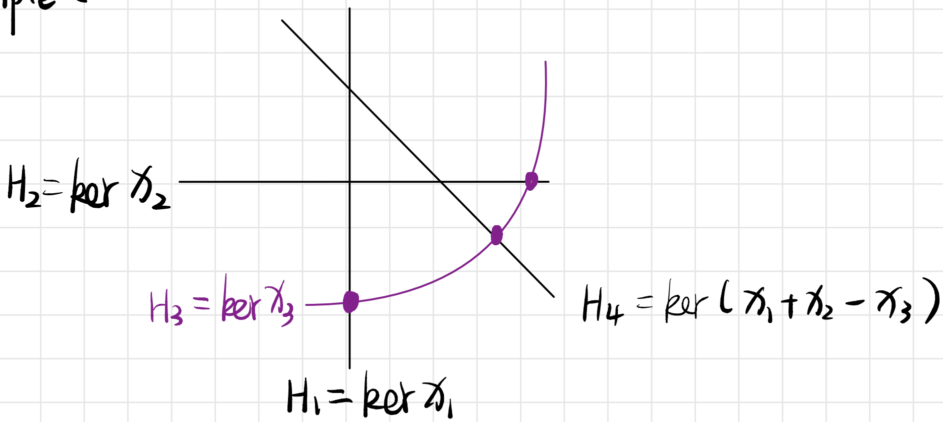
$$\begin{cases} H_1 = \ker x_1 \\ H_2 = \ker x_2 \\ H_3 = \ker x_3 \\ H_4 = \ker(x_1 + x_2 - x_3) \end{cases}$$

Combinatorics of hyperplane arrangements.

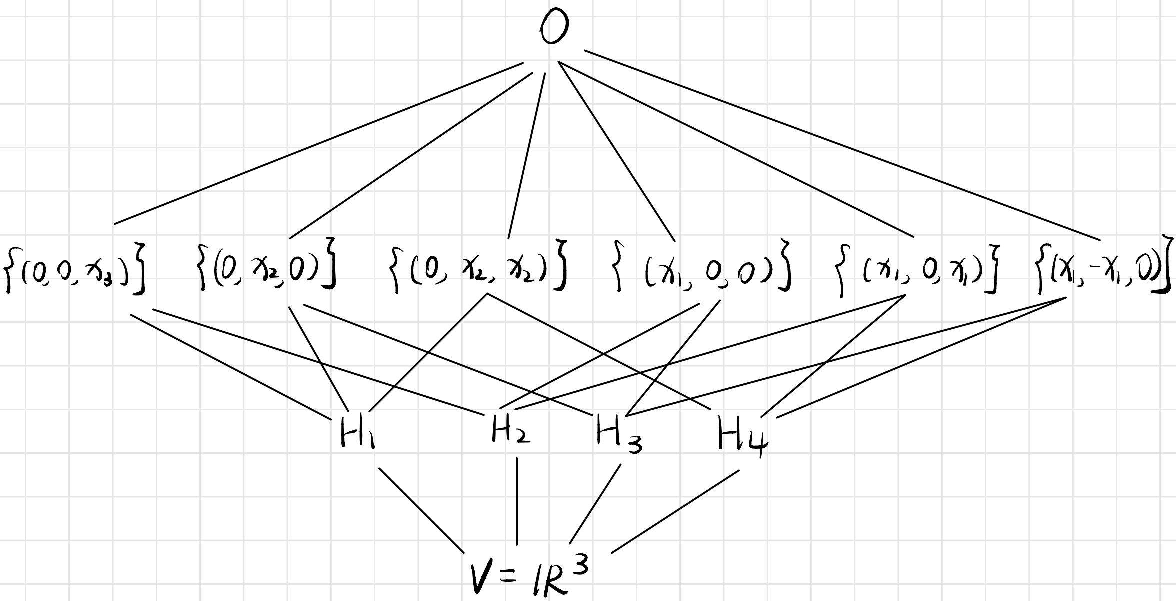
- $L(\mathcal{A}) := \{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A} \}$: intersection poset

partial order is reverse inclusion

Example:



$L(\mathcal{A})$:



- Some operations :

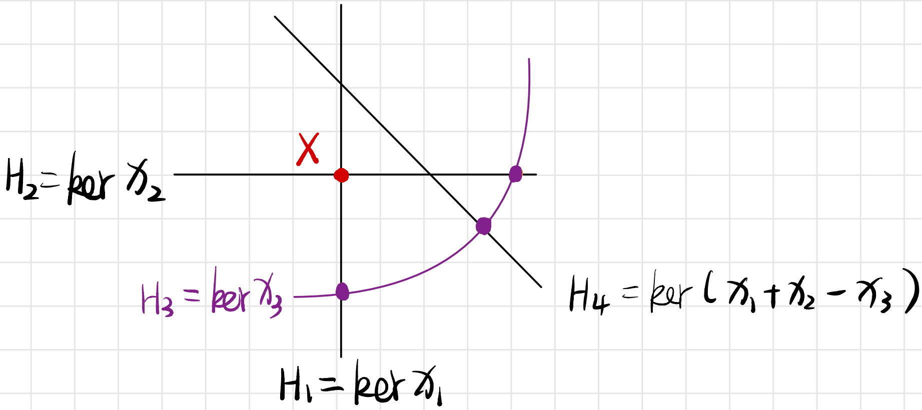
1. For $X \in L(\mathcal{A})$

(1) $\mathcal{A}_X := \{ H \in \mathcal{A} \mid X \subset H \}$: Localization of $\mathcal{A}$ to X. [$\mathcal{A}_X$ is "in" $V/X = \mathbb{K}^{\operatorname{codim} X}$]

(2) $\mathcal{A}^X := \{ H \cap X \mid H \in \mathcal{A} \setminus \mathcal{A}_X \}$: restriction of $\mathcal{A}$ to X [$\mathcal{A}^X$ is in $X = \mathbb{K}^{\dim X}$]

- Example:

Let $\ell = 3$ and $Q(\mathcal{A}) = x_1 x_2 x_3 (x_1 + x_2 - x_3)$



Then : $X = H_1 \cap H_2 \Rightarrow \mathcal{A}_X = \{ H_1, H_2 \}$

$X = H_3 \Rightarrow \mathcal{A}^X = \{ x_1 = 0, x_1 + x_2 = 0, x_2 = 0 \}$

Algebraic invariant of hyperplane arrangements.

Def 1. (1) The free S-module of derivations:

$$\mathrm{Der}\, S := \left\{ \sum_{i=1}^{\ell} f_i \partial_i \;\middle|\; \partial_i := \frac{\partial}{\partial x_i},\ f_i \in S,\ i = 1, \cdots, \ell \right\}$$

(2) The logarithmic derivation module $D(\mathcal{A})$:

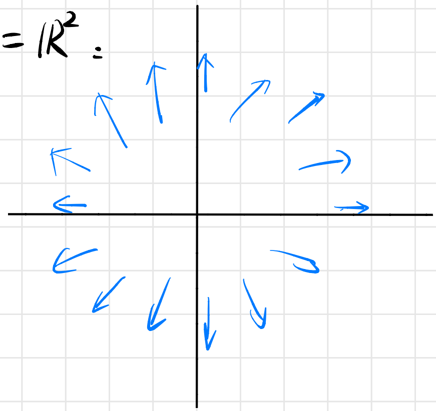
$$D(\mathcal{A}) := \{ \theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in S\alpha_H \ \forall H \in \mathcal{A} \},$$

(tangent condition)

Def 2: The Euler derivation $\theta_E \in \mathrm{Der}\, S$ is:

$$\theta_E = \sum_{i=1}^{\ell} x_i \partial_i$$

eg: $V = \mathbb{R}^2$:



- For any hyperplane $H := \ker \alpha$ with $\alpha = \sum a_i x_i$

$$\theta_E(\alpha) = \sum a_i x_i = \alpha$$

$\Rightarrow \theta_E \in D(\mathcal{A})$ for any arrangement $\mathcal{A}$.

• Lemma: $\boxed{D(\mathcal{A}) = S\theta_E \oplus D_H(\mathcal{A})}$.

$$D_H(\mathcal{A}) = \{ \theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0 \}$$

parallel to H

Def 3: We say $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2, \cdots, d_\ell)$ if

$$D(\mathcal{A}) = S\theta_1 \oplus S\theta_2 \oplus \cdots \oplus S\theta_\ell$$

with $\deg \theta_i = d_i$ $(i = 1, \cdots, \ell)$.

Examples.

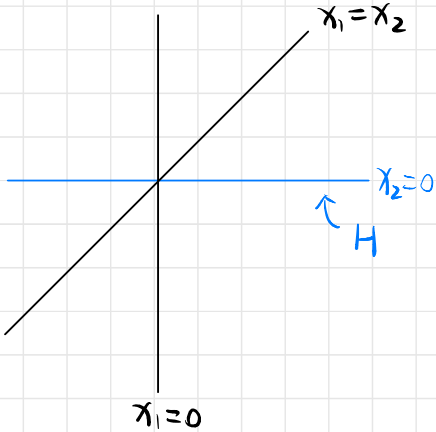
(1) Let $\mathcal{A} = \{ H_i = \ker x_i \mid i = 1, \cdots, \ell \}$. coordinate planes

Then $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, \cdots, 1)$.

In fact, $D(\mathcal{A}) = \bigoplus_{i=1}^{\ell} S\theta_i$, where $\theta_i = x_i \partial_i$.

(2) Any $\mathcal{A}$ in $\mathbb{k}^2$ is free with $\exp(\mathcal{A}) = (1, |\mathcal{A}|-1)$

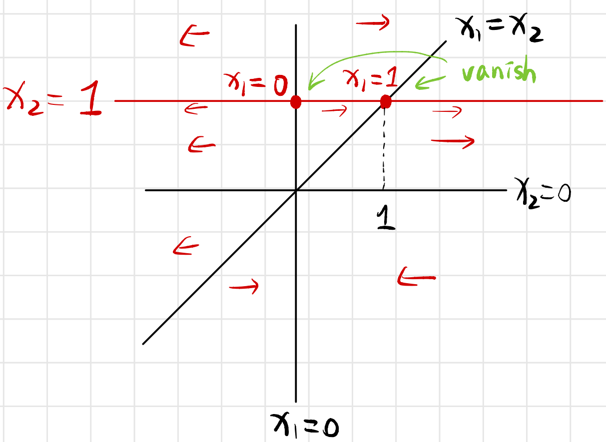
e.g.: $Q(\mathcal{A}) = x_1 x_2 (x_1 - x_2)$



$$\begin{cases} D(\mathcal{A}) = S\theta_E \oplus D_H(\mathcal{A}) \\ \theta_E = x_1\partial_1 + x_2\partial_2 \end{cases}$$

Consider $\theta \in D_H(\mathcal{A})$

$\theta = f\partial_1, \quad f \in S.$



$(x_1-0)(x_1-1) \mid f$

$$\Downarrow$$

$\theta \in S(x_1(x_1-x_2)\partial_1)$

$\Downarrow \quad \theta_2 := x_1(x_1-x_2)\partial_1$

$D(\mathcal{A}) = S\theta_E \oplus S\theta_2$

One of the most important conjecture in this field is Terao's Conjecture (1981):

- For arrangents $\mathcal{A}$ and $\mathcal{B}$, if $L(\mathcal{A}) \cong L(\mathcal{B})$, then $\mathcal{A}$ is free iff $\mathcal{B}$ is free.

$\longrightarrow$ Example on SageMath

A deeper understanding of the generators of $D(\mathcal{A})$ is of utmost importance to tackle this conjecture

Machinery to check freeness.

Theorem (Saito's criterion)

Let $\theta_1 \cdots \theta_\ell \in D(\mathcal{A})$ be homogeneous and linearly independent over S. Then $\mathcal{A}$ is free with basis $\theta_1, \cdots, \theta_\ell$ if and only if

$$\sum_{i=1}^{\ell} \deg \theta_i = |\mathcal{A}|.$$

Theorem (Addition-deletion Theorem, Terao, 1980)

Let $H \in \mathcal{A}$, $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' := \mathcal{A}^H$. Then two of the following imply the third:

(1) $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \cdots, d_{\ell-1}, d_\ell)$

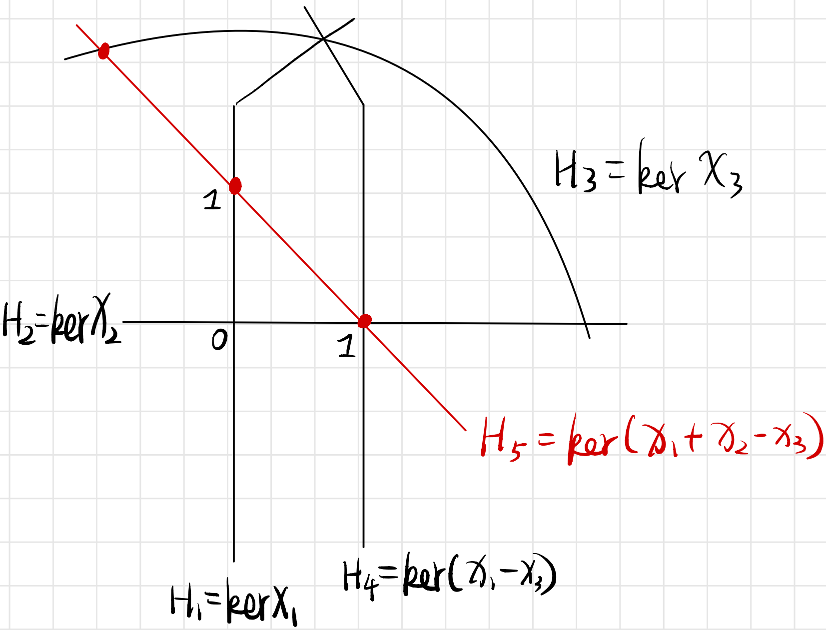
(2) $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (d_1, \cdots, d_{\ell-1}, d_\ell - 1)$

(3) $\mathcal{A}''$ is free with $\exp(\mathcal{A}'') = (d_1, \cdots, d_{\ell-1})$

Moreover, all the three above hold true if $\mathcal{A}$ and $\mathcal{A}'$ are free.

- Example:

Let $\ell = 3$ and $Q(\mathcal{A}) = x_1 x_2 x_3 (x_1 - x_3)(x_1 + x_2 - x_3)$



$\mathcal{A}' = \mathcal{A} \setminus \{H_5\}$ is free with $\exp(\mathcal{A}') = (1, 1, 2)$

$\mathcal{A}'' = \mathcal{A}^{H_5}$ is free with $\exp(\mathcal{A}'') = (1, 2)$

$\Rightarrow$ $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 2, 2)$

§ Line arrangement with cubic minimal log derivations.

The study of $D(\mathcal{A})$:

$\rightsquigarrow$ natural approach:

- the degrees of the gens.
- the min. generators.
- the min. free resolution
- ⋮

Recall that,

$$D(\mathcal{A}) = S\theta_E \oplus D_H(\mathcal{A}), \quad \forall H \in \mathcal{A}.$$

Let $\{\theta_E, \theta_2, \cdots, \theta_p \mid \theta_i \in D_H(\mathcal{A})\ \forall\, i = 2, \cdots, p\}$

is a minimal set of generators for $D(\mathcal{A})$.

The minimal degree of logarithmic derivations of arrangement $\mathcal{A}$ is defined as

$$\boxed{\min\{d_i \mid d_i = \deg \theta_i,\ i = 2, \cdots, p\}.}$$

From this view,

Burity and Tohăneanu give full classification of rank 3 line arrangents in $\mathbb{P}^2$ that have a minimal log. derivation of degree 1, 2, 3.

Theorem [Burity - Tohăneanu, 2021]:

Let $\mathcal{A} \subset \mathbb{P}^2$ be a line arrangement of rank 3, with $r(\mathcal{A}) = 3$. Then, up to a change of coordinates, $\mathcal{A}$ has one of the following defining polynomials (and corresponding (affine) pictures):

(Ia) $F = xyz(x+y)(bx+y)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 7.$

(Ib) $F = xyz(x+y)(bx+z)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 6.$

(Ic) $F = xyz(x+y)(y+z)(-x+z)\prod_{j=7}^{s}(t_jy+z), t_j \neq 0, 1, s \geq 7.$

(Id) $F = xyz(x+y)(ax+by+z)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 5.$

(IIa) $F = xyz(x+y+z)(ax+by+z)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, \ s \geq 5.$

(IIb) $F = xyz(x+y+z)(bx+y)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 6.$

(IIIa) $F = xyz(x+z)(y+z)(x+y+z)(ax+by+z).$

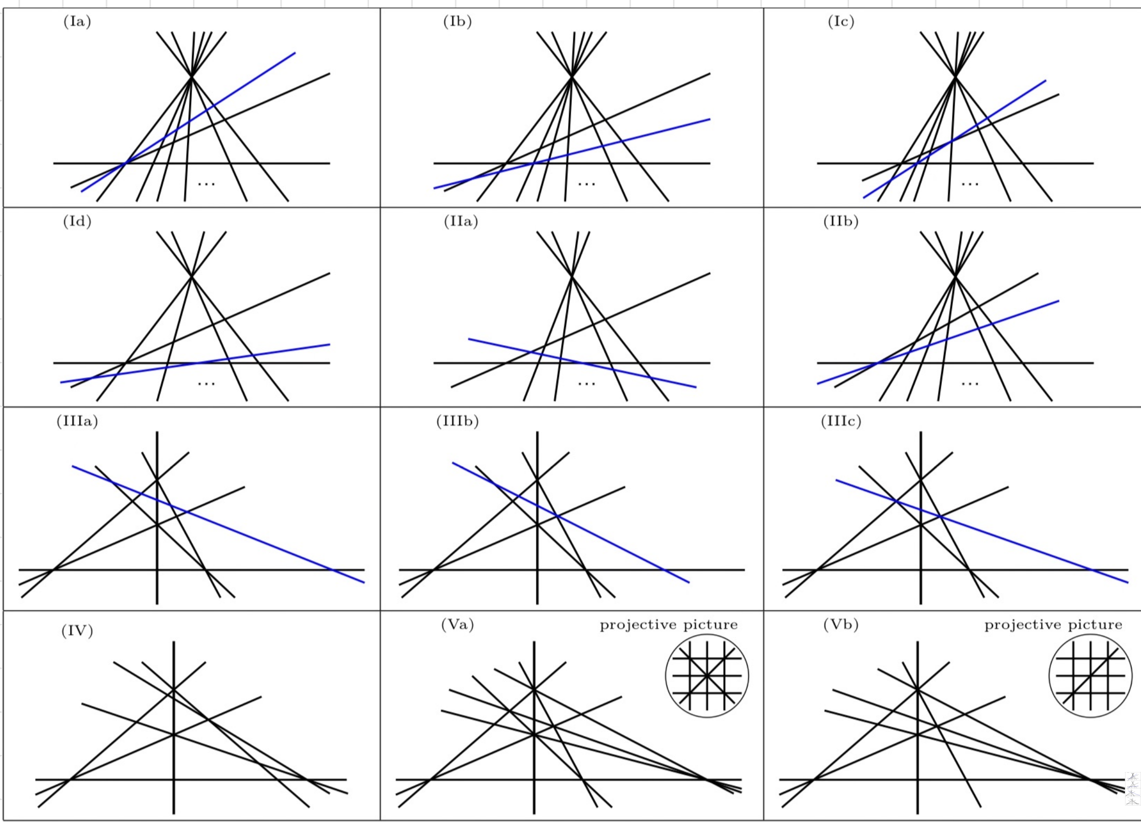
(IIIb) $F = xyz(x+z)(y+z)(x+y+z)(ax+ay+z).$

(IIIc) $F = xyz(x+z)(y+z)(x+y+z)(x+y+2z).$

(IV) $F = xy(x-z)(y-z)(x-2z)(y-2z)(x-y).$

(Va) $F = xyz(x^2-y^2)(x^2-z^2)(y^2-z^2).$

(Vb) $F = xyz(x^2-z^2)(y^2-z^2)(x-y).$



Theorem [Chu]

Let $\mathcal{A} \subset \mathbb{P}^2$ be a line arrangement of rank 3, with $r(\mathcal{A}) = 3$. Then, up to a change of coordinates, $\mathcal{A}$ has one of the following defining polynomials (and corresponding (affine) pictures):

(Ia) $F = xyz(x+y)(bx+y)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 7.$
(Ib) $F = xyz(x+y)(bx+z)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 6.$
(Ic) $F = xyz(x+y)(y+z)(-x+z)\prod_{j=7}^{s}(t_jy+z), t_j \neq 0, 1, s \geq 7.$
(Id) $F = xyz(x+y)(ax+by+z)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 5.$
(IIa) $F = xyz(x+y+z)(ax+by+z)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, \; s \geq 5.$
(IIb) $F = xyz(x+y+z)(bx+y)\prod_{j=6}^{s}(t_jy+z), t_j \neq 0, s \geq 6.$
(IIIa) $F = xyz(x+z)(y+z)(x+y+z)(ax+by+z).$
(IIIb) $F = xyz(x+z)(y+z)(x+y+z)(ax+ay+z).$
(IIIc) $F = xyz(x+z)(y+z)(x+y+z)(x+y+2z).$
(IV) $F = xy(x-z)(y-z)(x-2z)(y-2z)(x-y).$
(Va) $F = xyz(x^2-y^2)(x^2-z^2)(y^2-z^2).$
(Vb) $F = xyz(x^2-z^2)(y^2-z^2)(x-y).$

Table 1: Freeness of Theorem 2.1.

Cases	Freeness	Exponents
(Ia)	Free	$\{1,3,s-4\}$
(Ib)	$t_i = -b$	$\{1,3,s-4\}$
(Ic)	Free	$\{1,3,s-4\}$
(Id)	$t_i = -b$ and $t_j = b-a$	$\{1,3,s-4\}$
(IIa)	$t_i = 1$ and either $b=1$ or $t_j = b$, $a=1$ or $t_j = b$, $t_k = \frac{a-b}{a-1}$	$\{1,3,s-4\}$
(IIb)	$t_i = 1$ and either $b=1$ or $b(1-t_j)=1$	$\{1,3,s-4\}$
(IIIa)	either $a=1$ or $b=1$	$\{1,3,3\}$
(IIIb)	Not free	Non
(IIIc)	Free	$\{1,3,3\}$
(IV)	Not free	Non
(Va)	Free	$\{1,3,5\}$
(Vb)	Free	$\{1,3,4\}$

Theorem [C-]

Let $\ell=3$, $s\in\mathbb{N}$ and the defining polynomial of $\mathcal{A}$ be:

$$Q(\mathcal{A}) = x_1 x_2 x_3 (a x_1 + b x_2 + x_3) \prod_{j=1}^{s} (t_j x_2 + x_3)$$

with $a, b \in \mathbb{K}\setminus 0$.

Then $\left[\mathcal{A} \text{ is free} \iff \exists!\, i \in \{1,\cdots,s\} \text{ s.t. } b = t_i.\right]$

Moreover, if $\mathcal{A}$ is free, we have $\exp(\mathcal{A}) = (1, 2, s+1)$ and a basis is as follows:

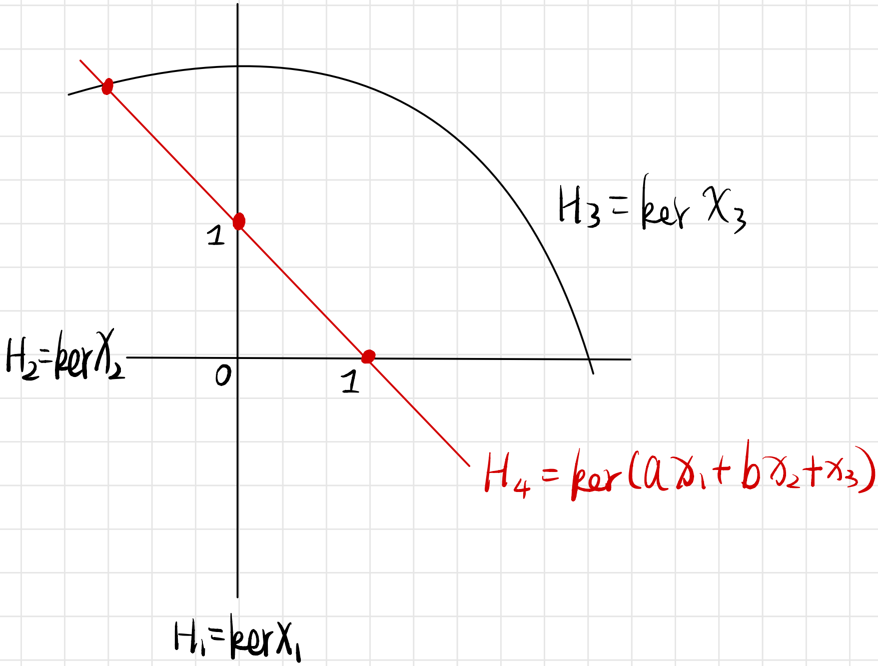
$$\xi_1 = \theta_E$$

$$\xi_2 = x_1 (a x_1 + b x_2 + x_3)\, \partial_{x_1}$$

$$\xi_3 = x_2 x_3 \prod_{\substack{j=1,\\ j\neq i}}^{s} (t_j x_2 + x_3) \left(\partial_{x_2} - t_i \partial_{x_3}\right)$$

Example:

Let $\ell = 3$ and $Q(\mathcal{A}) = x_1 x_2 x_3 (a x_1 + b x_2 + x_3)$



- $\mathcal{A}' = \mathcal{A} \setminus \{H_4\}$ is free with $\exp(\mathcal{A}') = (1,1,1)$
- $\mathcal{A}^{H_4}$ is free with $\exp(\mathcal{A}^{H}) = \begin{cases} (1,2), & a, b \in \mathbb{k} \setminus 0 \\ (1,1), & \text{"}a=0\text{" or "}b=0\text{"} \end{cases}$

By A-D Theorem, we may get that

$$\boxed{\mathcal{A} \text{ is free} \iff \text{either "}a=0\text{" or "}b=0\text{"}}$$

§ Derivation Degree Sequences of Logarithmic Modules for close to Free Hyperplane Arrangements

The study of $D(\mathcal{A})$:

→ natural approach:

- the min. generators.
- the degrees of the gens.
- the min. free resolution

⋮

- free arrangents
- non-free: much unexplored

Problem:

If $\mathcal{A}$ is free, then its deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is far from free, or not so far?

SPOG arrangements

Def [Abe, 2021]

We say that $\mathcal{A}$ is strictly plus-one generated (SPOG) with exponents $\mathrm{POexp}(\mathcal{A}) = (d_1, \cdots, d_\ell)$ and level d if $D(\mathcal{A})$ has a min. free resolution:

$$0 \to S[-d-1] \xrightarrow{(\alpha, f_1, \cdots, f_\ell)} S[-d] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \longrightarrow D(\mathcal{A}) \longrightarrow 0$$

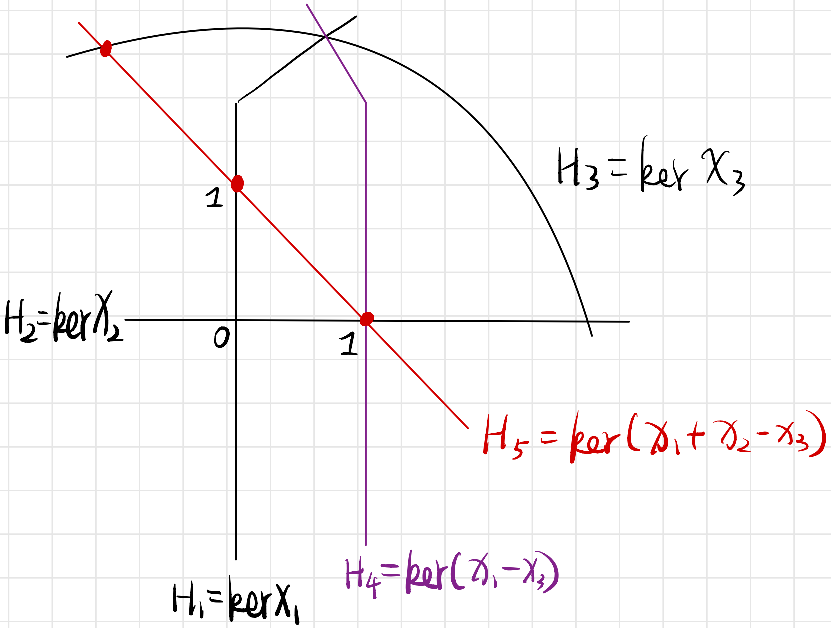
where: $f_i \in S$, $\alpha \in S_1 \setminus 0$.

Theorem [A-, 2021]

Let $\mathcal{A}$ be free with $\exp(\mathcal{A}) = (d_1, \cdots, d_\ell)$ and $H \in \mathcal{A}$. Then $\mathcal{A}' := \mathcal{A} \setminus \{H\}$ is free, or SPOG with $\mathrm{POexp}(\mathcal{A}') = (d_1, \cdots, d_\ell)$ and level $d = |\mathcal{A}'| - |\mathcal{A}^H|$

Example:

Let $V=\mathbb{R}^3$ and $Q(\mathcal{A}) = x_1 x_2 x_3 (x_1 - x_3)(x_1 + x_2 - x_3)$



- $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 2, 2)$

- $\mathcal{A}' = \mathcal{A} \setminus \{H_4\}$ is not free.

By the above theorem, we know that

$\mathcal{A}'$ is SPOG with $\mathrm{POexp}(\mathcal{A}') = (1, 2, 2)$ and level $|\mathcal{A}'| - |\mathcal{A}^{H_4}| = 4 - 2 = 2$.

Problem:

If $\mathcal{A}$ is free, then $\mathcal{A} \setminus \{H_1, H_2\}$ is far from free, or not so far?

Def: Assume that $\{\theta_1, \cdots, \theta_p\}$ is a min. set of gen. for $D(\mathcal{A})$ with $\deg \theta_i = d_i$.
The derivation degree sequence of $D(\mathcal{A})$ defined as $DS(\mathcal{A}) = (d_1, \cdots, d_p)$.
Moreover, $|DS(\mathcal{A})| := p$.

- Free minus two arrangement.

- $\mathcal{A}$: free with $\exp(\mathcal{A}) = (d_1, d_2, \cdots, d_\ell)_\leq$

- $\mathcal{A}_i := \mathcal{A} \setminus \{H_i\}$, $\mathcal{A}_{i,j} := \mathcal{A} \setminus \{H_i, H_j\}$, $H_i, H_j \in \mathcal{A}$

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are SPOG with level c_1 and c_2

$$\begin{tikzcd}
\mathcal{A} : \text{free} \arrow[r, "\setminus H_1"] \arrow[d, "\setminus H_2"'] & \mathcal{A}_1 = \mathcal{A} \setminus H_1 : \text{SPOG} \arrow[d, "\setminus H_2"] \\
\mathcal{A}_2 = \mathcal{A} \setminus H_2 : \text{SPOG} \arrow[r, "\setminus H_1"] & \mathcal{A}_{12} = \mathcal{A} \setminus \{H_1, H_2\}
\end{tikzcd}$$

Theorem [C-]: Let $\mathcal{A}_i$ be SPOG with $\mathrm{POexp}(\mathcal{A}_i) = (d_1, d_2, \cdots, d_\ell)$ and level c_i, where $i = 1, 2$. Then $DS(\mathcal{A}_{1,2})$ is as one of following:

(I) $|\mathcal{A}_{H_1 \cap H_2}| = 2$.

In this case, $DS(\mathcal{A}_{1,2}) = (d_1, d_2, \cdots, d_\ell, c_1, c_2)$

(II) $|\mathcal{A}_{H_1 \cap H_2}| > 2$.

(IIa) $DS(\mathcal{A}_{1,2}) = (d_1, \cdots, d_\ell, c_1 - 1)$

(IIb) $DS(\mathcal{A}_{1,2}) = (d_1, \cdots, d_\ell, c_1 - 1, c_2)$

(IIc) $DS(\mathcal{A}_{1,2}) \supseteq (d_1, \cdots, d_\ell, c_1, c_2)$

* If the relation in (IIc) is a proper subset, this shows that $|DS(\mathcal{A}_{1,2})| > \ell + 2$.

$H_i \in \mathcal{A}$ is in the same order as its in $Q(\mathcal{A})$.

- Example. Let $V = \mathbb{K}^4$ and

$$Q(\mathcal{A}) = x_1 x_2 x_3 x_4 (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)$$
$$(x_2 - x_4)(x_3 - x_4)(x_2 - x_3 + x_4)(x_1 - x_2 + x_3 - x_4)$$

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 3, 4, 4)$.

(II) $\mathcal{A}_1 . \mathcal{A}_2$: SPOG with level 4.
$\mathcal{A}_8 . \mathcal{A}_{10}$: SPOG with level 5

(IIa) $DS(\mathcal{A}_{1,2}) = (1, 3, 3, 4, 4)$

(IIb) $DS(\mathcal{A}_{1,8}) = (1, 3, 4, 4, 4, 5)$

(IIc) $DS(\mathcal{A}_{2,10}) = (1, 3, 4, 4, 4, 4)$

- Example. let $V = \mathbb{K}^4$ and

$$Q(\mathcal{A}) = x_1 (x_1 - x_2)(x_1 + x_2)(x_1 - x_3)(x_1 + x_3)(x_1 - x_4)$$
$$(x_1 + x_4)(x_2 - x_3)(x_2 + x_4)(x_3 - x_4)$$

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 3, 3, 3)$ By computer,

we have $DS(\mathcal{A}_{1,10}) = (1, 3, 3, 3, 3, 3, 3) \Rightarrow |DS(\mathcal{A}_{1,10})| = 7 > 4+2$

* This is a counter-example to Orlik's Conjecture.
[DiPasquale, 2023]

• Example. Let $V = \mathbb{K}^4$ and

$$Q(\mathcal{A}) = x_1 x_2 x_3 x_4 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_3 - x_4)$$
$$(x_2 - x_3 + x_4)(x_1 - x_2 + x_3 - x_4)$$

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 3, 3, 3)$

$\mathcal{A}_1, \mathcal{A}_3$ = SPOG with level 3.

By computer, we have $DS'(\mathcal{A}_{1,3}) = (1, 3, 3, 3, 3, 3, 3)$

*This is a counter-example to Orlik's Conjecture.

[Nakashima - Tsujie, 2023]

However, by computer, the first counter example to the Orlik's conjecture by Edelman and Reiner has at most $l+2$ min. generators.

Therefore, perhaps we cannot assert that there is a certain connection between Orlik's conjecture and the cardinality of minimal set of homogeneous generators.

- 3-dimsional case.

Theorem [C-]:

Let $l=3$ and $\mathcal{A}_i$ be SPOG with $\mathrm{POexp}(\mathcal{A}_i)=(d_1,d_2,d_3)_{\leq}$ and level c_i, where $i=1,2$.
Then $DS(\mathcal{A}_{1,2})$ is as one of following:

(I) $|\mathcal{A}_{H_1\cap H_2}|=2 \iff DS(\mathcal{A}_{1,2})=(d_1,d_2,d_3,c_1,c_2)$

(II) $|\mathcal{A}_{H_1\cap H_2}|>2$. Assume: $c_1\leq c_2$.

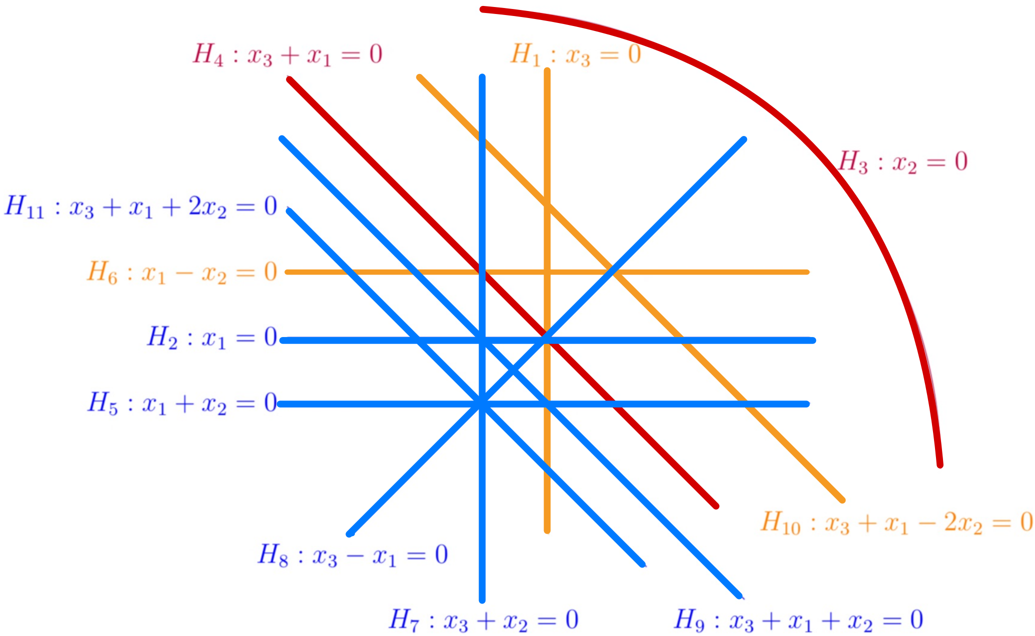
(IIa) $c_1=c_2 \iff DS(\mathcal{A}_{1,2})=(d_1,d_2,d_3,c_1-1)$

Moreover, $c_1=c_2=d_3$

$\iff \mathcal{A}_{1,2}$ is SPOG with level d_3

(IIb) $c_2>c_1 \iff DS(\mathcal{A}_{1,2})=(d_1,d_2,d_3,c_1,c_2-1)$

Example :

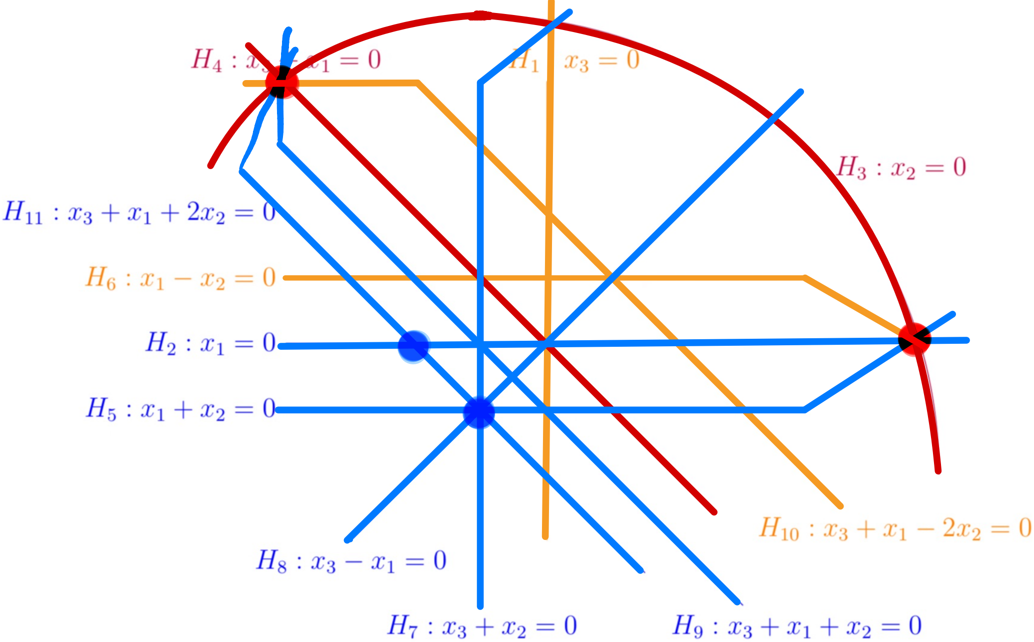


This is a free arrangent with $\exp(\mathcal{A}) = (1, 5, 5)$

$\mathcal{A}_i := \mathcal{A} \setminus H_i$: SPOG with level 5

$\mathcal{A}_j := \mathcal{A} \setminus H_j$: SPOG with level 6

$\mathcal{A}_k := \mathcal{A} \setminus H_k$: free with $(1, 4, 5)$.



By the above theorem, we may get that:

(1). $DS(\mathcal{A}_{2,11}) = (1, 5, 5, 5, 5)$

(2). $DS(\mathcal{A}_{3,4}) = (1, 5, 5, 5)$,

but $\mathcal{A}_{3,4}$ is not SPOG.

(3). $\mathcal{A}_{7,8}$ is SPOG with level 5

and $DS(\mathcal{A}_{7,8}) = (1, 4, 5, 5)$

(4) $DS(\mathcal{A}_{2,3}) = (1, 5, 5, 5, 5)$

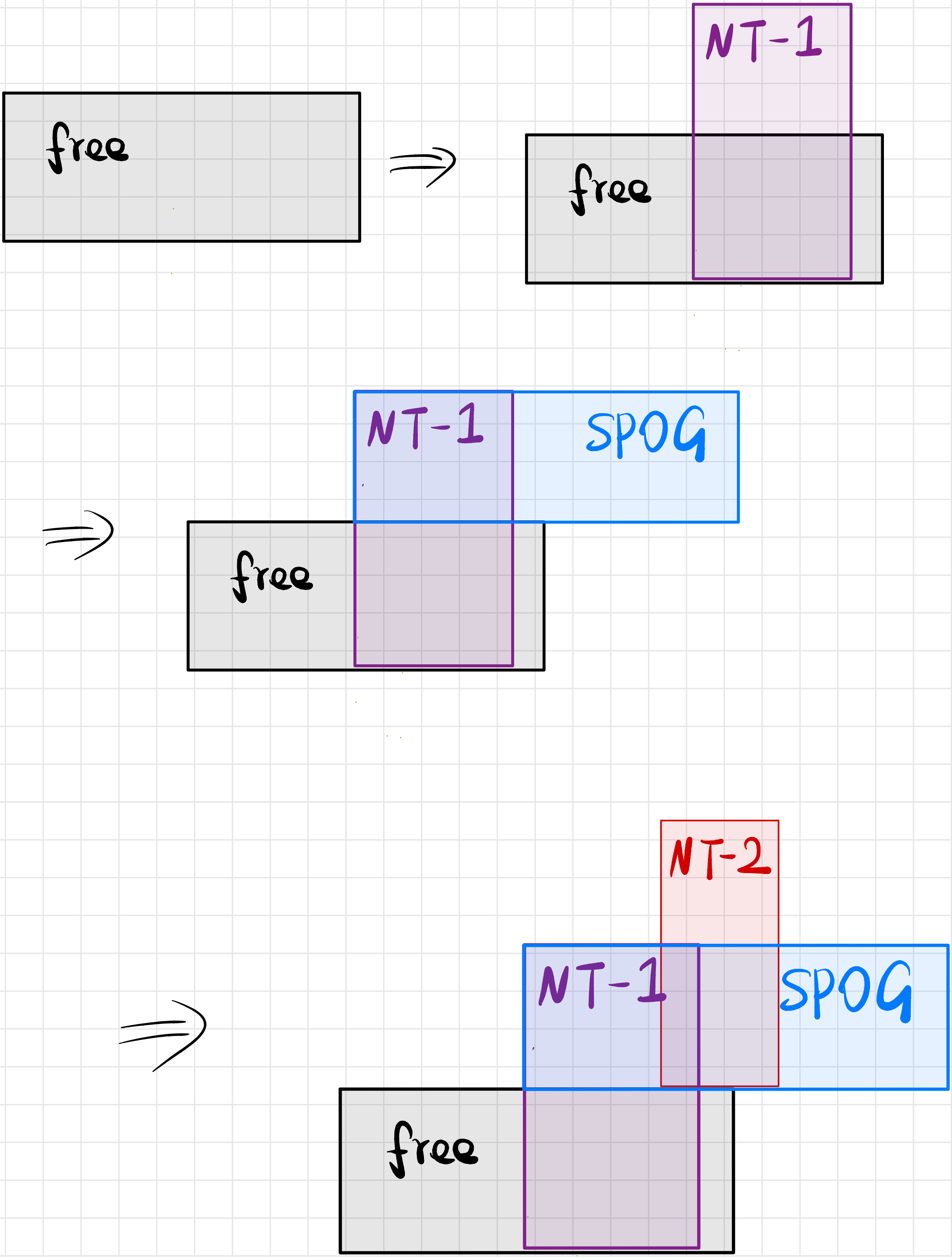
§ Conclusion & Conjecture.

Def 1 : We say that $\mathcal{A}$ is next to free minus one (NT-1) if $\exists$ free arrangement $\mathcal{B}$ and $H \in \mathcal{B}$ s.t. $\mathcal{A} = \mathcal{B} \setminus H$

- Consider $\mathcal{B} \setminus \{H_1, H_2\}$

$$\begin{tikzcd}
 & \mathcal{B} \setminus \{H_1\} \arrow[dr] & \\
\underset{\text{(free)}}{\mathcal{B}} \arrow[ur] \arrow[dr] & & \mathcal{B} \setminus \{H_1, H_2\} \\
 & \mathcal{B} \setminus \{H_2\} \arrow[ur] &
\end{tikzcd}$$

Def 2 : We say that $\mathcal{A}$ is strictly next to free minus two (NT-2) if $\exists$ free arrangement $\mathcal{B}$ and $H_1, H_2 \in \mathcal{B}$ s.t. $\mathcal{B} \setminus H_i$ is not free and $\mathcal{A} = \mathcal{B} \setminus \{H_1, H_2\}$, where $i = 1, 2$


free
NT-1
free
NT-1
SPOG
free
NT-2
NT-1
SPOG
free

Claim:

- $\{NT\text{-}2\} \cap \{\text{free}\} = \phi$
- $\{NT\text{-}2\} \cap \{NT\text{-}1\} \subseteq SPOG$.

Example:

Let $Q(\mathcal{A}) = x_1x_2x_3(x_1-x_2)(x_1+x_2)(x_1+2x_2)$
$(2x_1+x_2)(3x_1+x_2)(x_2+x_3)(3x_1+x_2+x_3)$

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 3, 6)$

$H_1 = \ker x_1$, $H_2 = \ker x_2$.

$\Rightarrow$ $\mathcal{A}_{1,2}$ is SPOG with

$POexp(\mathcal{A}_{1,2}) = (1, 3, 5)$ and level 6.

but $\mathcal{A}_{1,2}$ is not a NT-1. arrangent.

i.e., $\forall H \subset V$, $\mathcal{A}_{1,2} \cup H$ is not free.

Conjecture:

1. $\{NF-1\} \cap \{NF-2\} = \emptyset$.

2. If $\mathcal{A}$ is $NF-2$, then $pd_S D(\mathcal{A}) \leq 2$

Moreover, if # min. generators for $D(\mathcal{A})$ is at most $\ell+2$, then $pd_S D(\mathcal{A}) \leq 1$.

3. Free resolusion of $NF-2$.

4. $L(\mathcal{A}) \cong L(\mathcal{B}) \Rightarrow pd_S D(\mathcal{A}) = pd_S D(\mathcal{B})$

Thank you

for

your attention

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