Minimal free resolution of close to free arrangements

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§ Introduction

1. $\mathbb{K}$: field. , $V := \mathbb{K}^{\ell}$: vector space.

2. $S := \mathbb{K}[x_1, \cdots, x_\ell]$: the polynomial ring.

3. $\mathcal{A} := \{H_1, \cdots, H_p\}$: hyperplane arrangement

4. $\mathcal{A}_i =: \mathcal{A} \setminus \{H_i\}$, $\mathcal{A}_{i,j} =: \mathcal{A} \setminus \{H_i, H_j\}$.

5. Restriction $\mathcal{A}^{H_i} := \{ H_j \cap H_i \mid H_j \in \mathcal{A}_i \}$.

6. Localization $\mathcal{A}_{H_i \cap H_j} := \{ H \in \mathcal{A} \mid H_i \cap H_j \subset H \}$

7. The logarithmic derivation module $D(\mathcal{A})$:

$$D(\mathcal{A}) := \{ \theta \in \mathrm{Der}\, S \mid \theta(\alpha_j) \in S\alpha_j ,\ j = 1, \cdots, p \}.$$

(tangent condition)

- If $D(\mathcal{A})$ is free, then there exist θ_i with $\deg \theta_i = d_i$ such that $D(\mathcal{A}) = \bigoplus_{i=1}^{\ell} S\theta_i$.

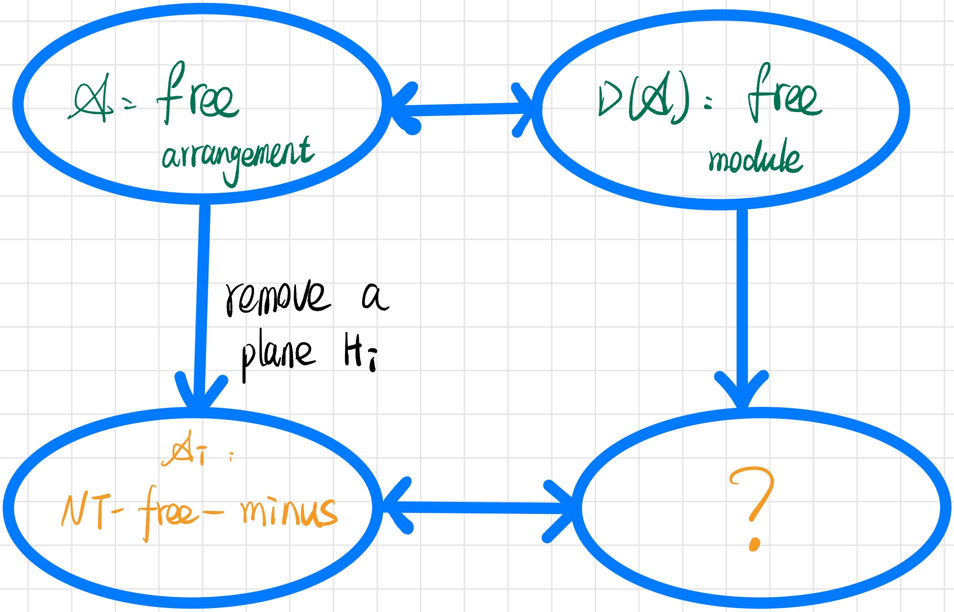
  In this case, we say $\mathcal{A}$ is free

  with exponents $\exp(\mathcal{A}) = (d_1, \cdots, d_\ell)$

- To study $D(\mathcal{A})$:

  $\rightarrow$ natural approach:

  - the min. generators.
  - the degrees of the gens
  - the min. free resolution.
  - ⋮



Def 1. [Abe, '21]

We say that $\mathcal{B}$ is next to free minus one (NT-Free-minus) if $\exists$ free arrangement $\mathcal{A}$ and $H_i \in \mathcal{A}$ such that $\mathcal{B} = \mathcal{A}_i$

Def 2. [A-, '21]

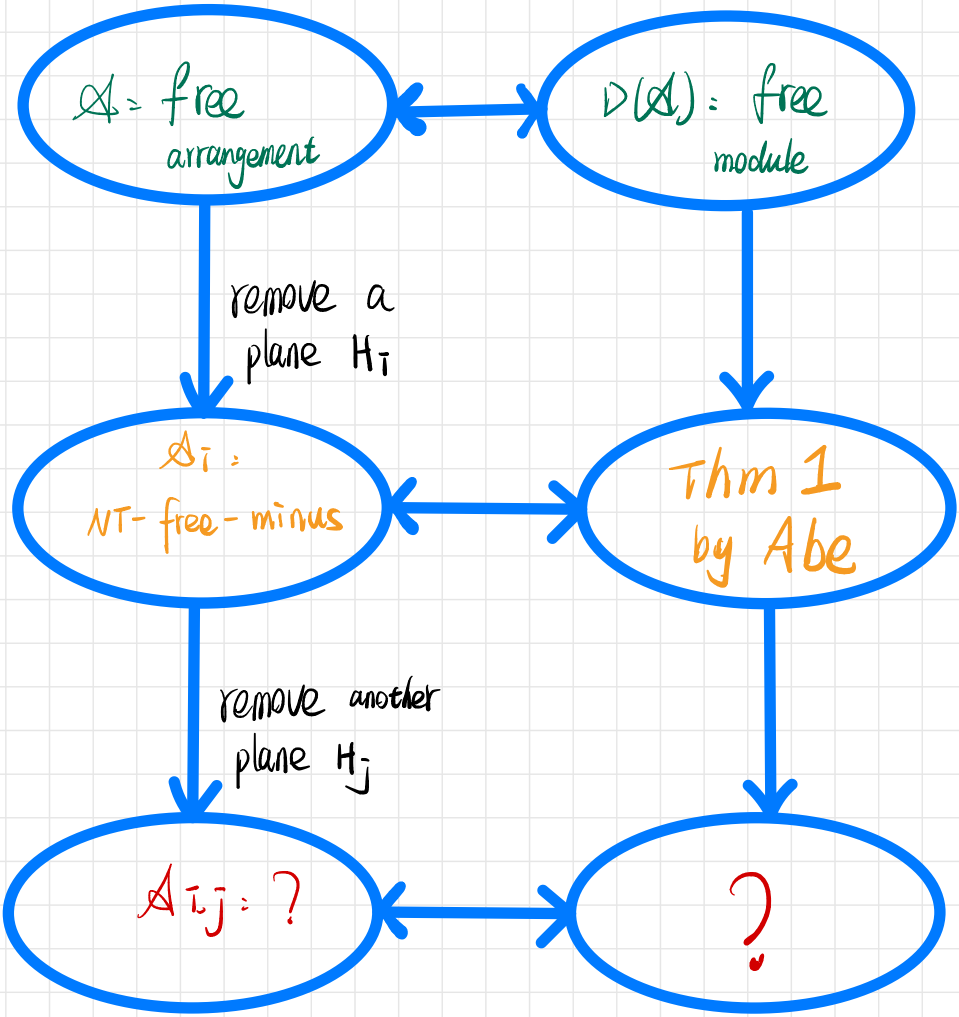
We say that $\mathcal{A}$ is strictly plus-one generated (SPOG) with exponents $\mathrm{POexp}(\mathcal{A}) = (d_1, \cdots, d_\ell)$ and level d iff $D(\mathcal{A})$ has a min. free resolution.

$$0 \to S[-d-1] \xrightarrow{(\alpha, f_1, \cdots, f_\ell)} S[-d] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i]\right) \longrightarrow D(\mathcal{A}) \longrightarrow 0$$

where: $f_i \in S$, $\alpha \in S_1 \setminus 0$.

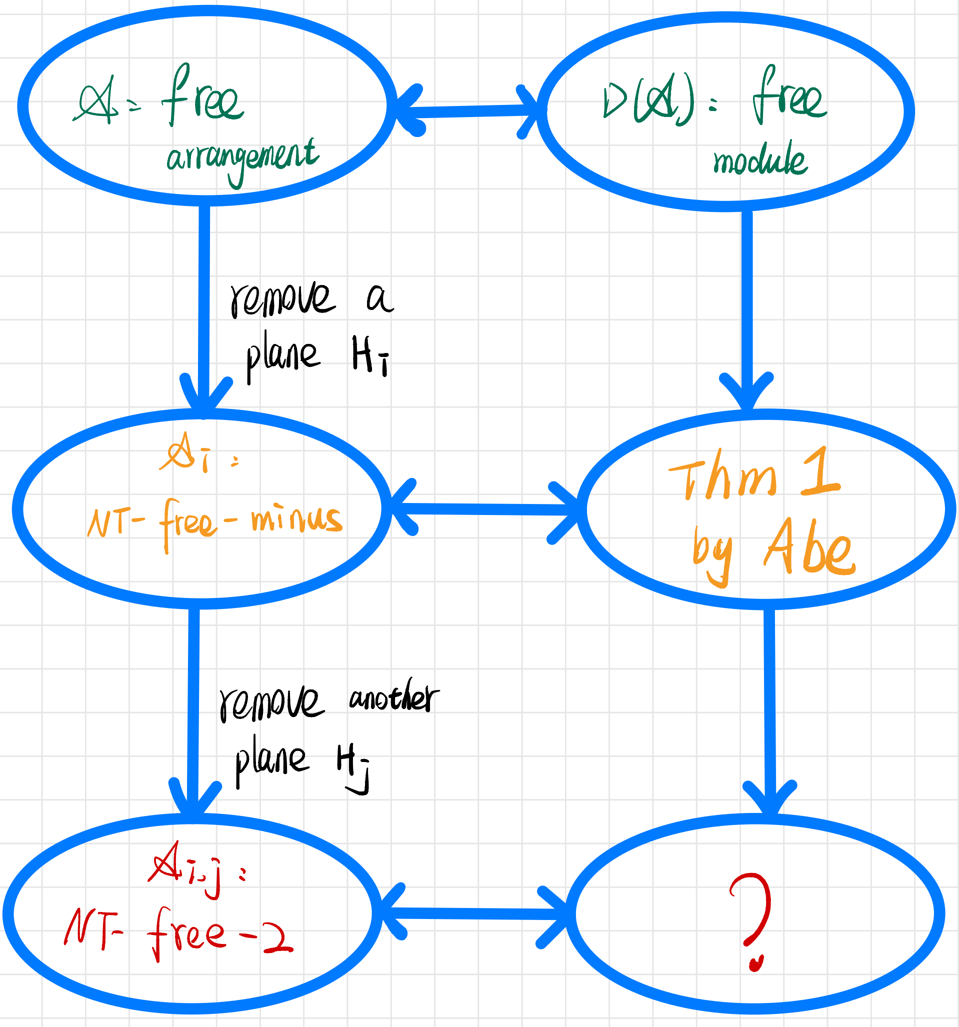
Theorem 1 [A-, '21]

Let $\mathcal{A}$ be free with $\exp(\mathcal{A}) = (d_1, \cdots, d_\ell)$ and $H \in \mathcal{A}$. Then $\mathcal{A}_i$ is free, or SPOG with $\mathrm{POexp}(\mathcal{A}_i) = (d_1, \cdots, d_\ell)$ and level $d = |\mathcal{A}_i| - |\mathcal{A}^{H_i}|$



$\mathcal{A}$: free arrangement
$D(\mathcal{A})$: free module
remove a plane H_i
$\mathcal{A}_i$: NT-free-minus
Thm 1 by Abe
remove another plane H_j
$\mathcal{A}_{ij}$: ?
?

Def 2: We say that $\mathcal{B}$ is next to free minus two (NT-free-2) if $\exists$ free arrangement $\mathcal{A}$ and $H_i \neq H_j \in \mathcal{A}$ s.t. $\mathcal{B} = \mathcal{A}_{i,j}$.



Theorem 2 [Chu]

$\mathcal{A}$: free with $\exp(\mathcal{A}) = (d_1, \cdots, d_\ell)$

$\mathcal{A}_1$: SPOG with level c_1

$\rotatebox{90}{$\leq$}$ (i.e. $c_1 \geq c_2$)

$\mathcal{A}_2$: SPOG with level c_2

If $\#\{\text{minimal generators of } D(\mathcal{A}_{1,2})\} \leq \ell+2$,

then $D(\mathcal{A}_{1,2})$ has a minimal free resolution of one of the following form:

(I) : If $D(\mathcal{A}_1) = D(\mathcal{A}) + \ker \rho_1^2$, then

$$0 \to S[-c_1-1] \xrightarrow{(\alpha_1, \alpha_2, f_1, \cdots, f_\ell)} S[-c_1+1] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i]\right) \longrightarrow D(\mathcal{A}_{1,2}) \longrightarrow 0$$

(II): If $\begin{cases} D(\mathcal{A}_1) \subsetneq D(\mathcal{A}) + \ker \rho_1^2 \\ D(\mathcal{A}_2) = D(\mathcal{A}) + \ker \rho_2^1 \end{cases}$, then

$$0 \to S[-c_1-1] \xrightarrow{(\alpha_1, 0, f_1, \cdots, f_{\ell-1})} S[-c_1] \oplus S[-c_2+1] \oplus \left(\bigoplus_{i=1}^{\ell-1} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0$$

$$0 \to S[-c_1-1] \oplus S[-c_2] \xrightarrow{\substack{(\alpha_1, 0, f_1, \cdots, f_\ell) \\ (g, \alpha_2, g_1, \cdots, g_\ell)}} S[-c_1] \oplus S[-c_2+1] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0$$

(III) else:

$$0 \to S[-c_1-1] \oplus S[-c_2-1] \xrightarrow{\substack{(\alpha_1, 0, f_1, \cdots, f_\ell) \\ (g, \alpha_2, g_1, \cdots, g_\ell)}} S[-c_1] \oplus S[-c_2] \oplus \left(\bigoplus_{i=1}^{\ell} S[-d_i] \right) \to D(\mathcal{A}_{1,2}) \to 0$$

Theorem 3 [Buzunáriz - Morales, '10] (Simplified version)

If $D(\mathcal{B})$ has a free resolution given by:

$$0 \to \bigoplus_{i=1}^{r_k} S[-d_i^k] \to \cdots \to \bigoplus_{i=1}^{r_1} S[-d_i^1] \to \bigoplus_{i=0}^{r_0} S[-d_i^0] \to D(\mathcal{B}) \to 0$$

then $\sum_{j=0}^{k} (-1)^j \sum_{i=1}^{r_j} d_i^j = |\mathcal{B}|$.

Corollary. 4. [G]

In Theorem 2, if $\#\{\text{minimal generators of } D(\mathcal{A}_{1,2})\} > \ell+2$, then $\mathrm{pd}_S(D(\mathcal{A}_{1,2})) \geq 2$

Example 1: This is a counter-example to Orlik's conjecture, which can be found in [DiPasquale,'23] or [Nakashima-Tsujie,'23] by performing a coordinate change.

$$Q(\mathcal{A}) = x_1 x_2 x_3 x_4 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$
$$(x_3 - x_4)(x_2 - x_3 + x_4)(x_1 - x_2 + x_3 - x_4)$$

$\mathcal{A}$: free with $\exp(\mathcal{A}) = (1, 3, 3, 3)$

The order of $H_i \in \mathcal{A}$ is consistent with its order of appearance in $Q(\mathcal{A})$.

The minimal free resolution of $D(\mathcal{A}_{1,3})$:

$$0 \to S[-5] \to S[-4]^4 \to$$

$$\underbrace{S[-3]^6 \oplus S[-1]}_{7 > \ell + 2 = 6} \to D(\mathcal{A}_{1,3}) \to 0$$

Theorem 5 [C-]

Let $\mathcal{B}$ be a NT-free-2 arrangement.
then

$$\mathrm{pd}_S(D(\mathcal{B})) \leq 1 \iff$$
$$\#\{\text{minimal generators of } D(\mathcal{B})\} \leq \ell+2$$

Corollary

If $\ell = 3$, then

$$\#\{\text{minimal generators of } D(\mathcal{B})\} \leq \ell+2$$

Theorem 6. [C-]

$\mathcal{A}$: free with $\exp(\mathcal{A}) = (d_1, d_2, d_3)$

$\mathcal{A}_1$: SPOG with level c_1

$\mathcal{A}_2$: SPOG with level c_2

Then $D(\mathcal{A}_{1,2})$ has a minimal free resolution of the following form:

(I) $|\mathcal{A}_{H_1 \cap H_2}| = 2 \iff$

$$0 \longrightarrow S[-c_1-1] \oplus S[-c_2-1] \xrightarrow{\substack{(\alpha_1, 0, f_1, f_2, f_3) \\ (0, \alpha_2, g_1, g_2, g_3)}}$$

$$S[-c_1] \oplus S[-c_2] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i] \right) \longrightarrow D(\mathcal{A}_{1,2}) \longrightarrow 0$$

(II) $|\mathcal{A}_{H_1\cap H_2}| > 2$

(1) $c_1 = c_2 \iff$

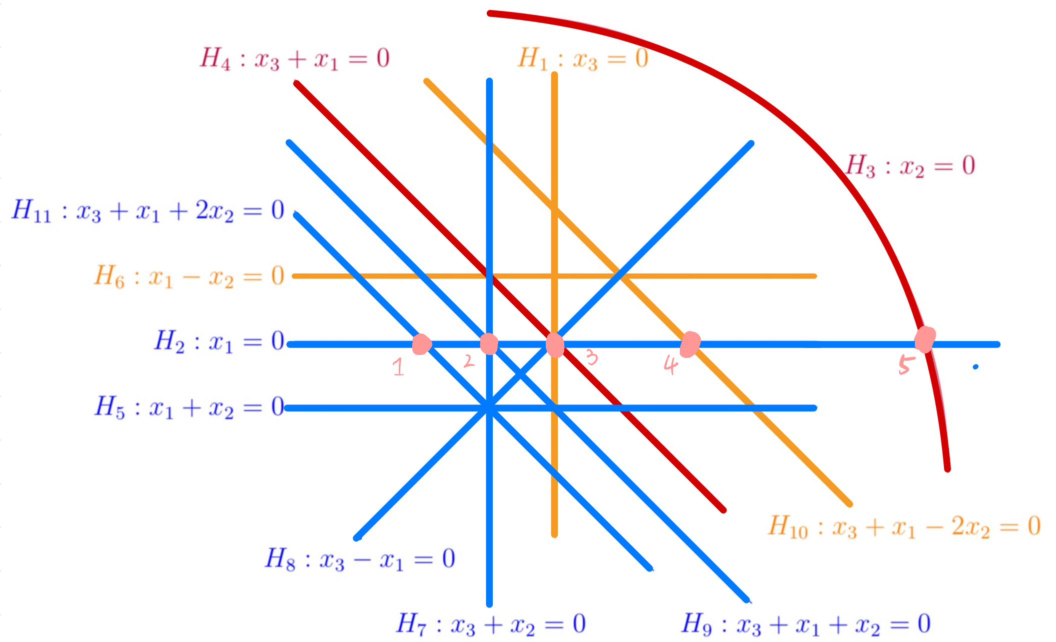
$$0 \longrightarrow S[-c_1-1] \xrightarrow{(\alpha_1\alpha_2,\ f_1,\ f_2,\ f_3)} S[-c_1+1] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i]\right) \longrightarrow D(\mathcal{A}_{1,2}) \longrightarrow 0$$

Moreover, $\mathcal{A}_{1,2}$ is SPOG $\iff c_1 = c_2 = \max\{d_i\}$

(2) $c_1 \leqslant c_2$. $\iff$

$$0 \longrightarrow S[-c_1-1] \oplus S[-c_2] \xrightarrow[(g,\ \alpha_2,\ g_1,\ g_2\ g_3)]{(\alpha_1,\ 0,\ f_1,\ f_2,\ f_3)} S[-c_1] \oplus S[-c_2+1] \oplus \left(\bigoplus_{i=1}^{3} S[-d_i]\right) \longrightarrow D(\mathcal{A}_{1,2}) \longrightarrow 0$$

Example 2 :

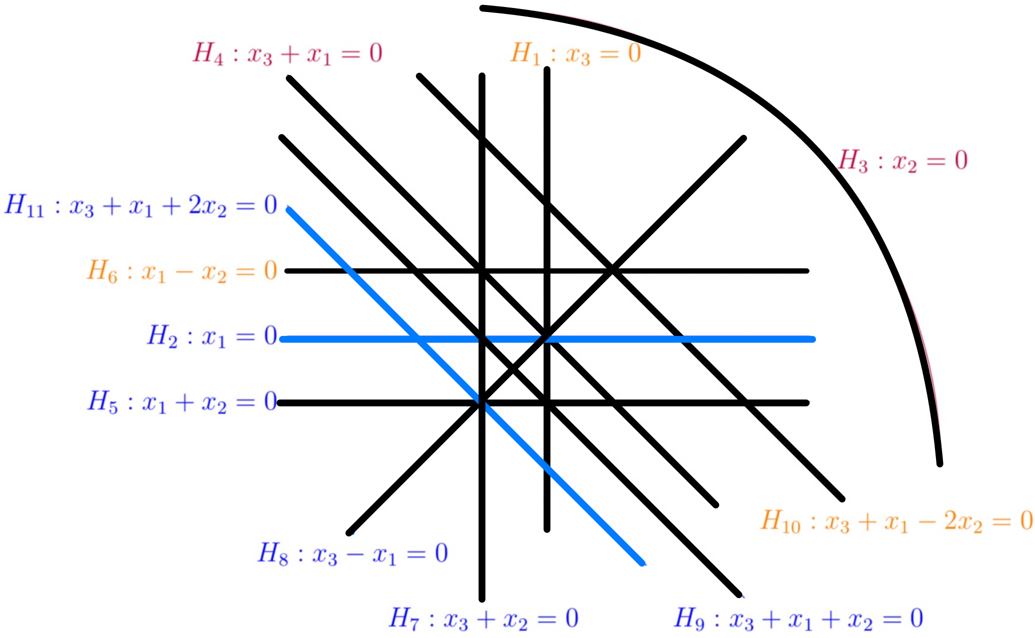


This is a free arrangent with $\exp(\mathcal{A}) = (1, 5, 5)$

$\mathcal{A}_i$: SPOG with level $c_i = 5 = |\mathcal{A}_2| - |\mathcal{A}^{H_2}|$

$\mathcal{A}_j$: SPOG with level $c_j = 6$

$\mathcal{A}_k$: free with $(1, 4, 5)$.

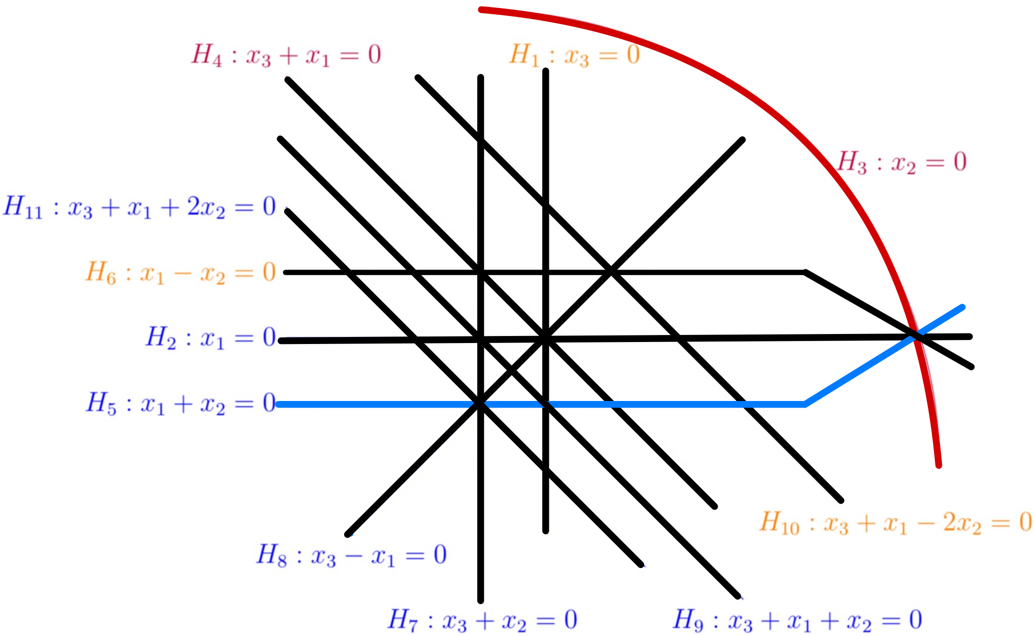


$\mathcal{A}$: free with $\exp(\mathcal{A}) = (1,5,5)$

$\mathcal{A}_i$: SPOG with level $c_i = 5$

$$\Downarrow \quad |\mathcal{A}_{H_2 \cap H_{11}}| = 2$$

$$0 \to S[-6]^2 \to S[-5]^4 \oplus S[-1] \to D(\mathcal{A}_{2,11}) \to 0$$



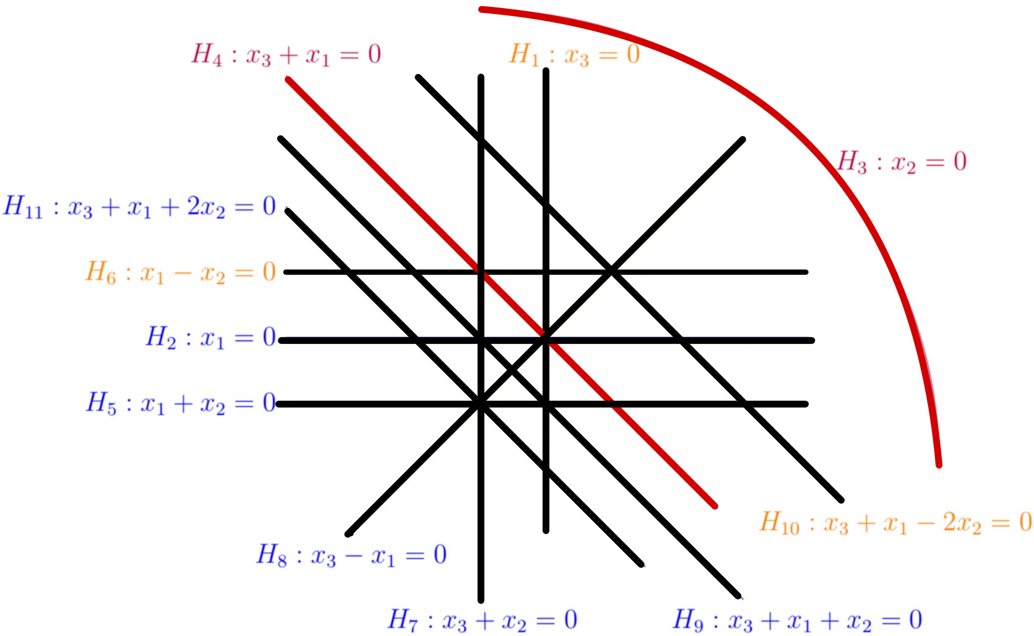
$\mathcal{A}$: free with $\exp(\mathcal{A}) = (1,5,5)$

$\mathcal{A}_5$: SPOG with level $5 = c_5$

$\mathcal{A}_3$: SPOG with level $6 = c_3$

$\Downarrow$ $c_3 > c_5$, $|\mathcal{A}_{H_3 \cap H_5}| > 2$

$$0 \to S[-6]^2 \to S[-5]^4 \oplus S[-1] \to D(\mathcal{A}_{3,5}) \to 0$$

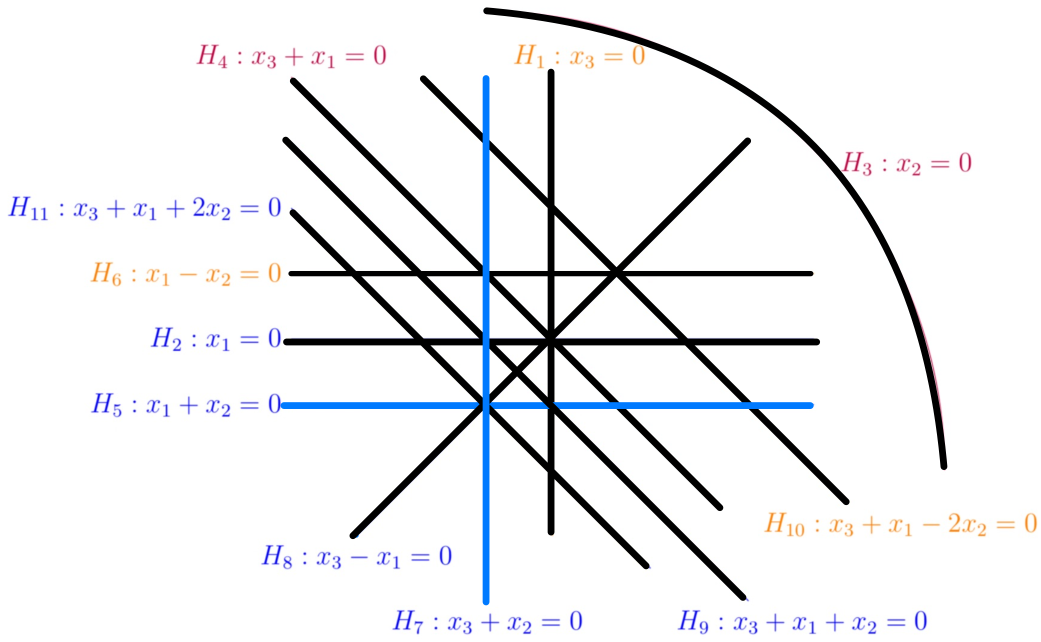


$\mathcal{A}$: free with $\exp(\mathcal{A}) = (1, 5, 5)$

$\mathcal{A}_j$: SPOG with level $c_j = 6$

$$\Downarrow \quad \begin{array}{l} |\mathcal{A}_{H_1 \cap H_2}| > 2 \\ c_1 = c_2 > 5 \end{array}$$

$$0 \to S[-7] \to S[-5]^3 \oplus S[-1] \to D(\mathcal{A}_3, 4) \to 0$$



$\mathcal{A}$: free with $\exp(\mathcal{A}) = (1,5,5)$

$\mathcal{A}_7$: SPOG with level $c_7 = 5$

$$\Downarrow \begin{array}{l} |\mathcal{A}_{H_5 \cap H_7}| > 2 \\ C_5 = C_7 = 5 \end{array}$$

$D(\mathcal{A}_{5,7})$ is SPOG

Thank you

for

your attention

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