Polynomial Interpolation of a Vector Field on a Convex Polygonal Domain

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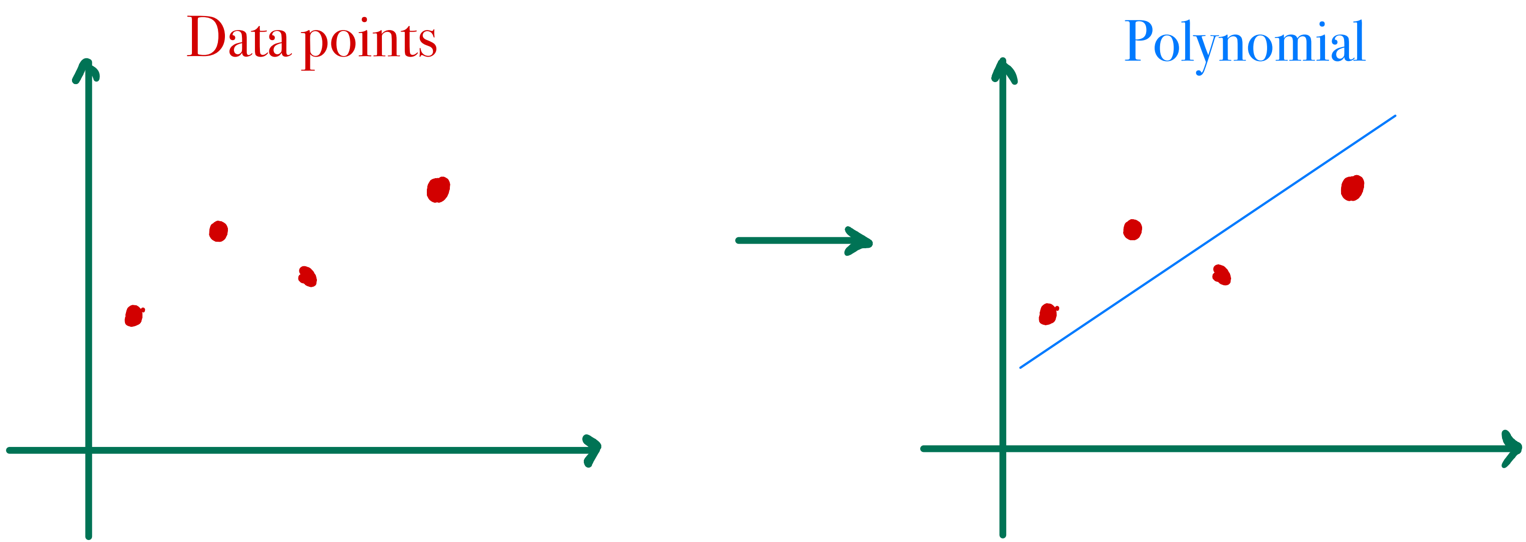
Joint work with S. Kaji

Curve fitting:

Input $\xrightarrow{\text{fitting a polynomial}}$ Output



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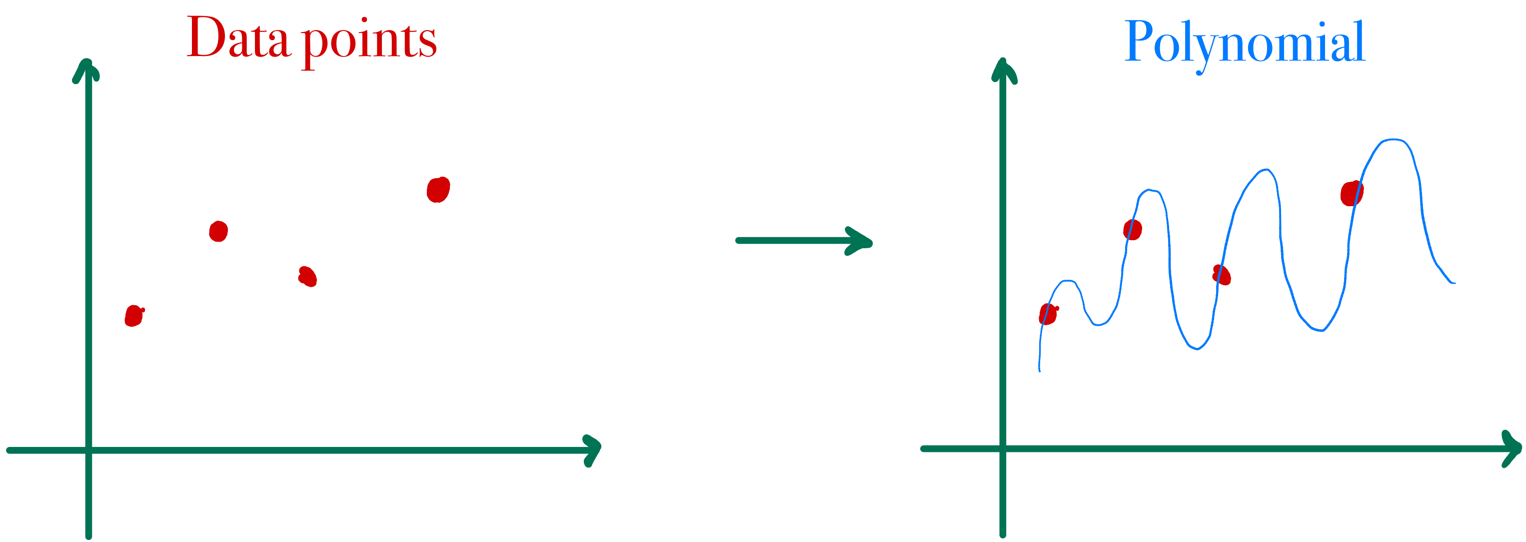
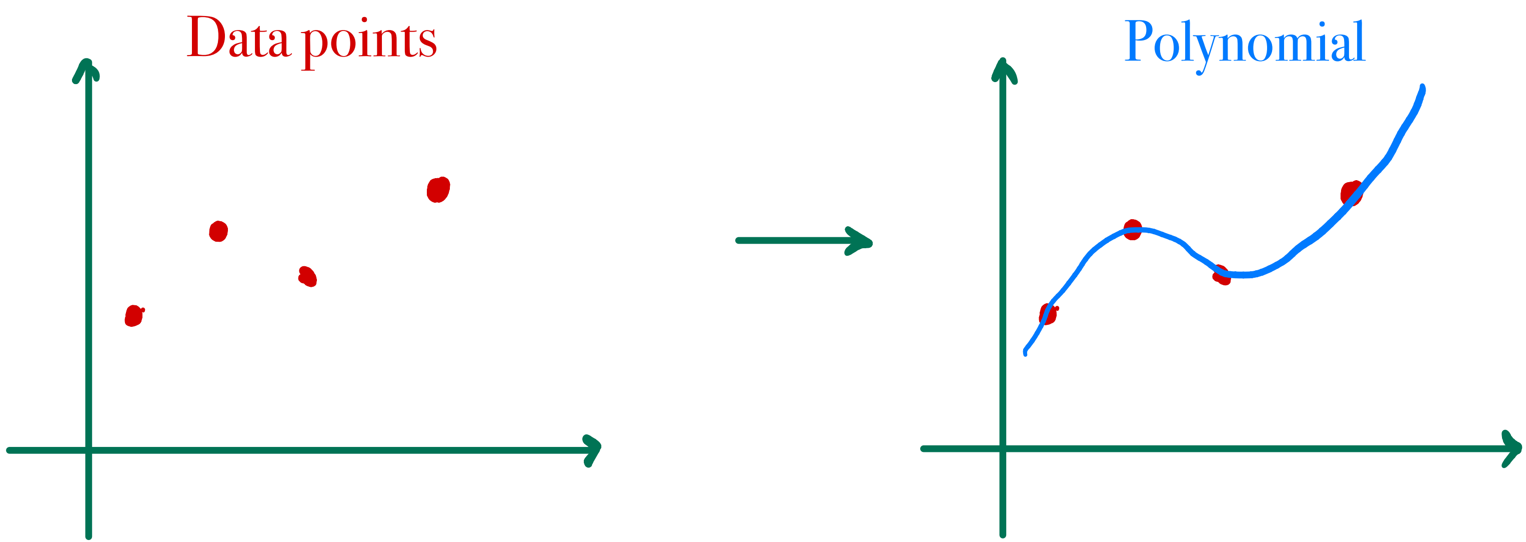


Curve fitting:

Input $\xmapsto{\text{fitting a polynomial}}$ Output

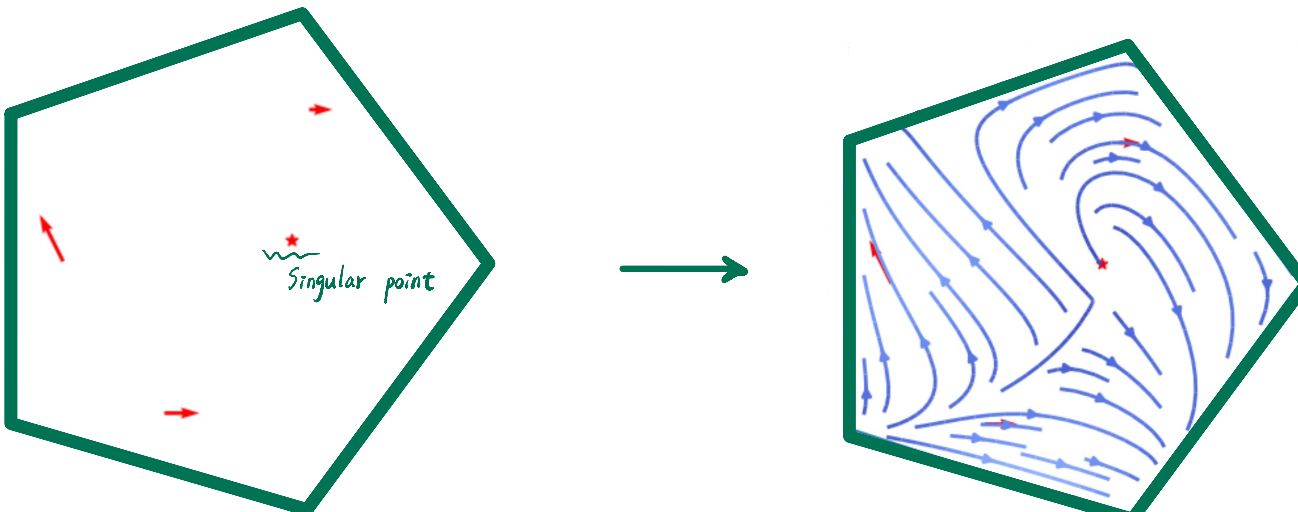


Goal:

Input $\xrightarrow{\text{fitting a polynomial vector field}}$ Output

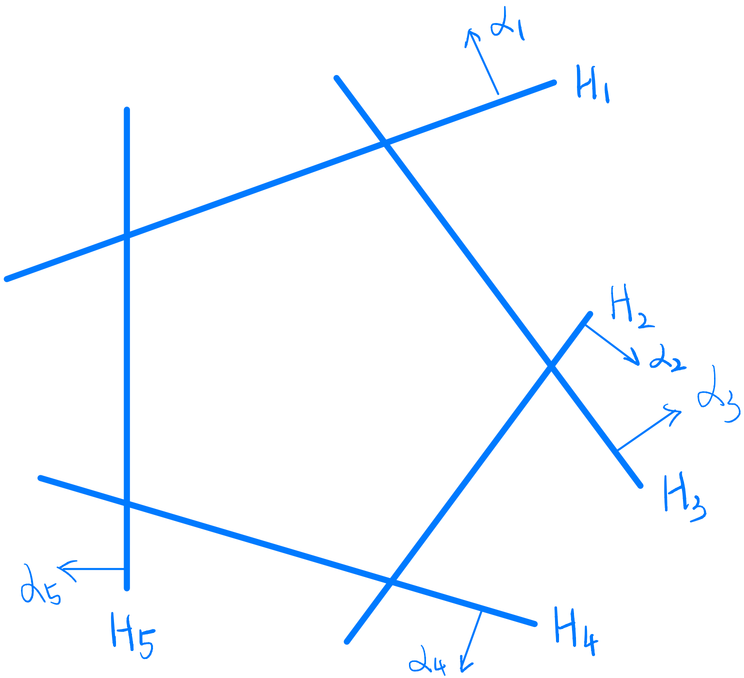
Key words:
Interpolation, Approximation, Data Fitting, Computing invariants (e.g., vorticity), Design

Data points | Polynomial vector field



P: (convex) polytope in $\mathbb{R}^d$

<u>Example:</u> $d = 2.$



hyperplane:

$H_i : \langle \alpha_i , x \rangle + l_i = 0$

P: (convex) polytope in $\mathbb{R}^d$

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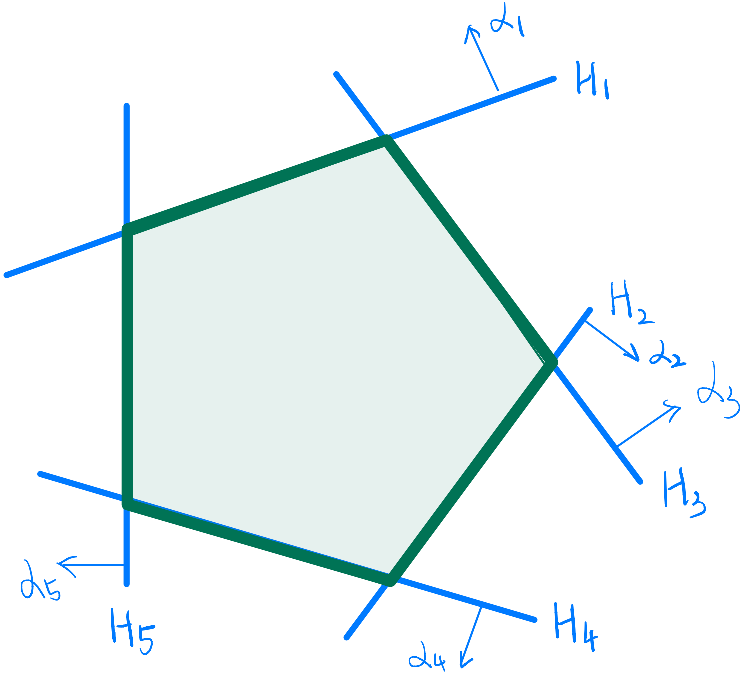

hyperplane :

$H_i : \langle \alpha_i , x \rangle + l_i = 0$

polytope :

$P = \bigcap \{ \langle \alpha_i , x \rangle + l_i \leq 0 \}$

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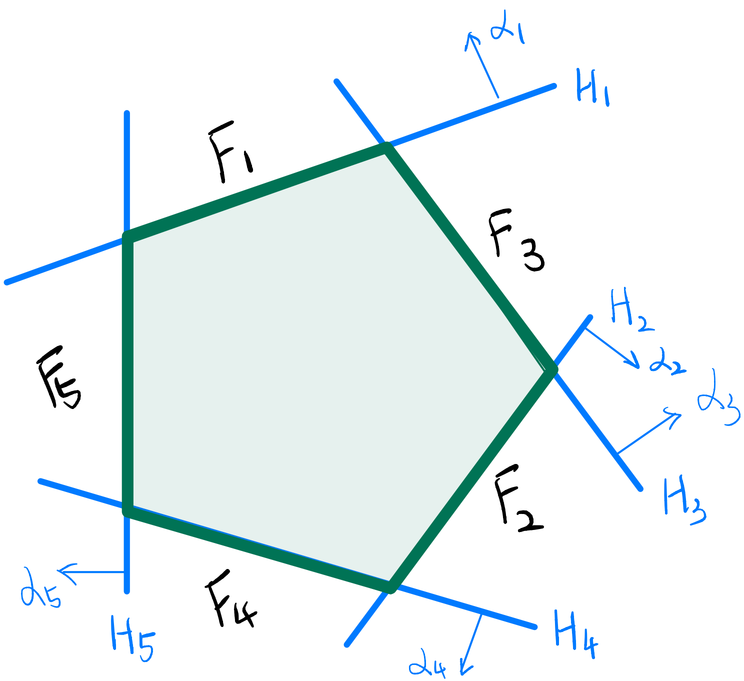


hyperplane :

$H_i := \langle \alpha_i, x \rangle + l_i = 0$

polytope :

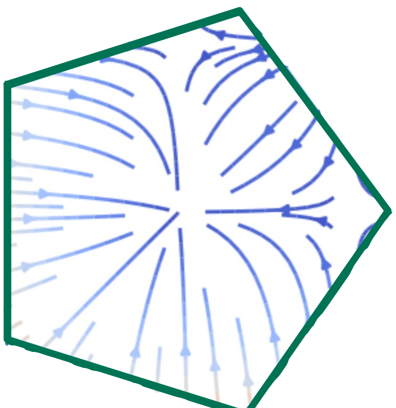
$P = \bigcap \{ \langle \alpha_i, x \rangle + l_i \leq 0 \}$

face :

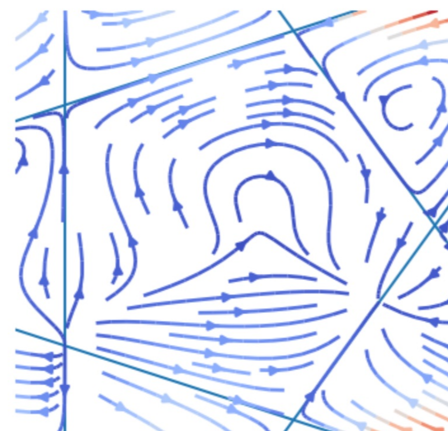
$F_i := H_i \cap \partial P$

$$\mathrm{Vect}(P) = \{\xi : P \longrightarrow \mathbb{R}^n\}$$: Space of vector fields on P

Example:



$\in \mathrm{Vect}(P) \ni$



$$\mathrm{Vect}(P) = \{ \xi : P \longrightarrow \mathbb{R}^n \}$$: Space of vector fields on P

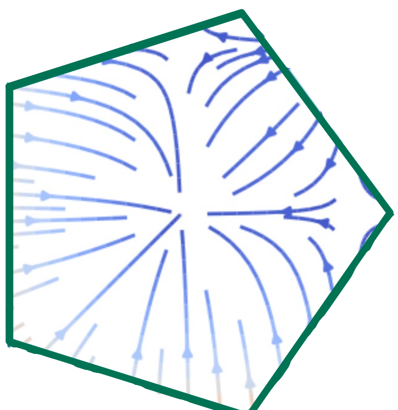
$$\rotatebox[origin=c]{90}{$\subseteq$}$$

$$\mathrm{Vect}_\partial(P) = \left\{ \xi \in \mathrm{Vect}(P) \;\middle|\; \langle \xi(x), \alpha_i \rangle = 0 \quad \forall\, x \in F_i \right\}$$

slip boundary condition

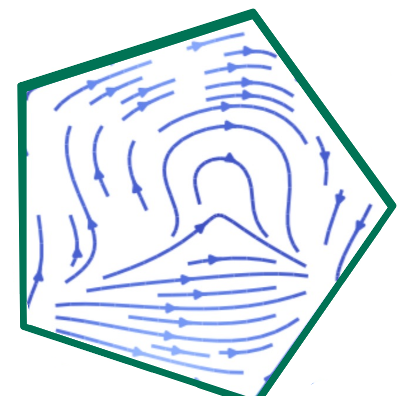
normal component is 0 along the boundary

Example:



$\in \mathrm{Vect}(P) \ni$

$\notin \mathrm{Vect}_\partial(P) \ni$



Space of polynomial vector fields:

$$\mathrm{Vect}(P)$$
$$\cup\!|$$

polynomial

- $\mathrm{Poly}(P) = \left\{ \xi \in \mathrm{Vect}(P) \;\middle|\; \xi = (f_1, \cdots, f_d),\ f_i \in \underbrace{\mathbb{R}[x]}_{\mathbb{R}[x_1, \cdots, x_d]} \right\}$

$$\cup\!|$$

- $\mathrm{Poly}(P)_k = \left\{ \xi \in \mathrm{Poly}(P) \;\middle|\; \max\{\deg f_i \mid \xi = (f_1, \cdots, f_d)\} \le k \right\}$

degree $\le k$

Space of polynomial vector fields:

$$\mathrm{Vect}(P) \supseteq \mathrm{Poly}(P) \supseteq \mathrm{Poly}(P)_k$$

- $\mathrm{Poly}(P) = \left\{ \xi \in \mathrm{Vect}(P) \;\middle|\; \xi = (f_1, \cdots, f_d),\ f_i \in \underbrace{\mathbb{R}[x]}_{\mathbb{R}[x_1, \cdots, x_d]} \right\}$

polynomial

- $\mathrm{Poly}(P)_k = \left\{ \xi \in \mathrm{Poly}(P) \;\middle|\; \max\{ \deg f_i \mid \xi = (f_1, \cdots, f_d) \} \le k \right\}$

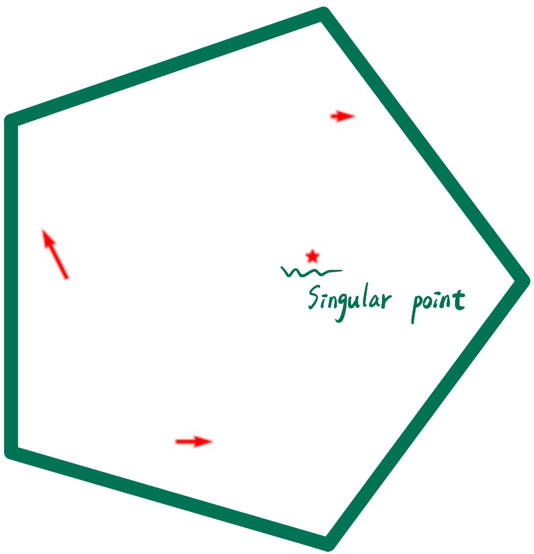
degree $\le k$

tangential poly. field

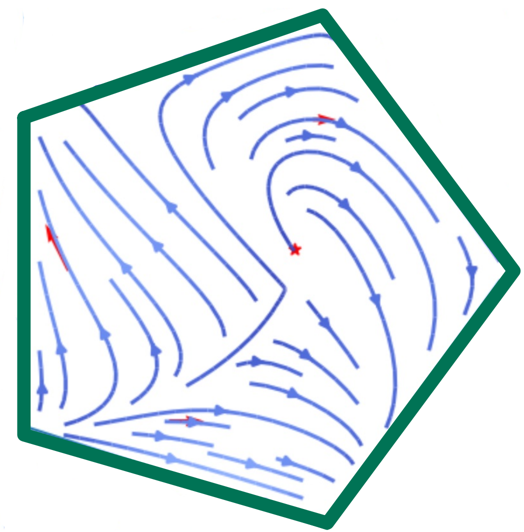
$$\boxed{\begin{aligned} \mathrm{Poly}_\partial(P) &= \mathrm{Poly}(P) \cap \mathrm{Vect}_\partial(P) \\ \rotatebox{90}{$\supseteq$}\ & \\ \mathrm{Poly}_\partial(P)_k &= \mathrm{Poly}(P)_k \cap \mathrm{Vect}_\partial(P) \end{aligned}}$$

Go back to our goal:

Data points



Polynomial vector field.



Given $\boxed{\bar{P} = \{p_i\}_{i=1}^{s} \subseteq P,}$

$$\boxed{\begin{array}{cccc} \bar{\xi}: & \bar{P} & \longrightarrow & \mathbb{R}^d \\ & \Downarrow & & \Downarrow \\ & p_i & \longmapsto & u_i \end{array}}$$

fitting a polynomial vector field

Tangential polynomial field:

$$\boxed{\xi: \quad P \longrightarrow \mathbb{R}^d}$$

Question:

Given $k \in \mathbb{Z}^+$, find $\xi \in Poly(P)_k$

- Minimises squared error $\sum_{i=1}^{S} \left| \xi(p_i) - \bar{\xi}(p_i) \right|^2$
- Satisfy boundary condition (as much as possible)

If we can identify the basis of $Poly_\partial(P)_k$ ....

Given $\boxed{\bar{P} = \{p_i\}_{i=1}^{s} \subseteq P,}$

$$\boxed{\begin{aligned}\bar{\xi}: \bar{P} &\longrightarrow \mathbb{R}^d \\ p_i &\longmapsto u_i\end{aligned}}$$

fitting

Tangential polynomial field:

$$\boxed{\xi: P \longrightarrow \mathbb{R}^d}$$

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Given $\boxed{\bar{P} = \{p_i\}_{i=1}^{s} \subseteq P,}$

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Tangential polynomial field:

$$\boxed{\xi: \quad P \longrightarrow \mathbb{R}^d}$$

$$Poly_\partial(P)_k = \langle \xi_1, \cdots, \xi_r \rangle_{\mathbb{R}}$$

$$\begin{cases} \xi = \sum_{j=1}^{r} c_j \xi_j \\ \xi(p_i) = u_i \end{cases}$$

least squares method

If we can identify the basis of $Poly_\partial(P)_k$ ....

__Problem 1__: Given $k \in \mathbb{Z}^+$, find $\xi \in Poly_\partial(P)_k$ minimises squared error $\sum_{i=1}^{s} \left| \xi(p_i) - \bar{\xi}(p_i) \right|^2$

$k \uparrow$ - error $\downarrow$

If we can identify the basis of $Poly_\partial(P)_k$ ....

Problem 1: Given $k \in \mathbb{Z}^+$, find $\xi \in Poly_\partial(P)_k$

minimises squared error $\sum_{i=1}^{s} \left| \xi(p_i) - \bar{\xi}(p_i) \right|^2$

Problem 2: Given $\varepsilon \geq 0$, find a minimum degree

$\xi \in Poly_\partial(P)_k$ with $\sum_{i=1}^{s} \left| \xi(p_i) - \bar{\xi}(p_i) \right|^2 \leq \varepsilon$

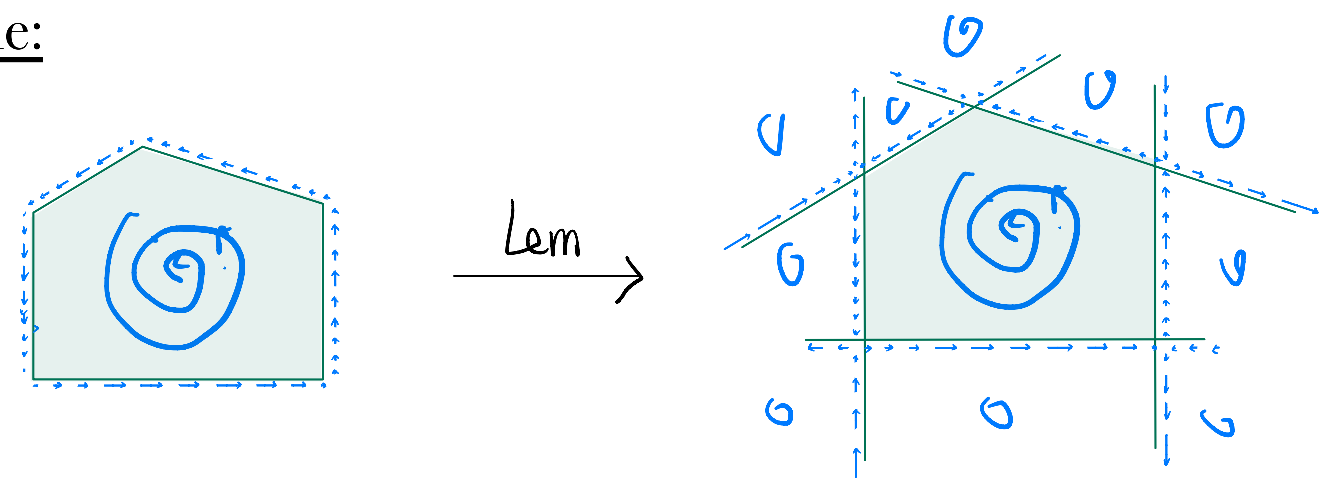
Question:

How can we identify the basis of $Poly_{\partial}(P)_k$?

Lemma: ξ tangent to ∂P $\xrightarrow{\exists!\ \text{extension}}$ $\tilde{\xi}$ tangent to hyperplanes

$\xi \in \mathrm{Poly}(P)$, $\quad \tilde{\xi} \in \mathrm{Poly}(\mathbb{R}^d)$

Example:



§ Theory of hyperplane arrangements

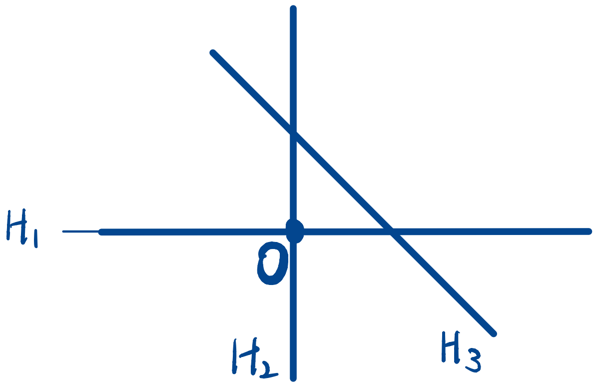
[c.f. P. Orlik and H. Terao, ***Arrangements of hyperplanes***. Springer-Verlag, Berlin, 1992.]

<u>Def</u>: affine (hyperplane) arrangement in $\mathbb{R}^d$:

$$\mathcal{A} = \left\{ H_i : \langle \alpha_i, x \rangle + l_i = 0 \;\middle|\; l_i \in \mathbb{R},\ i = 1, \cdots, t \right\}.$$

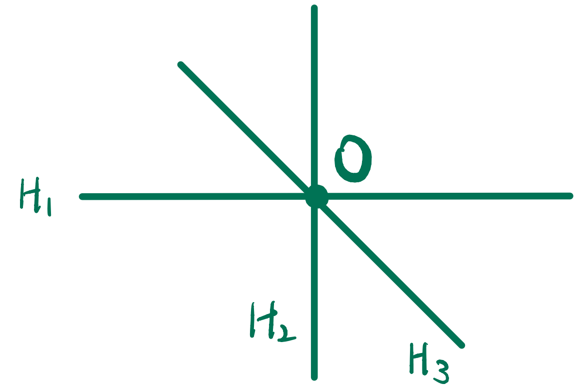
We say $\mathcal{A}$ is central if $l_i = 0$.

Example: $d = 2$.



Affine arrangement

$\mathcal{A} \subseteq \mathbb{R}^2$



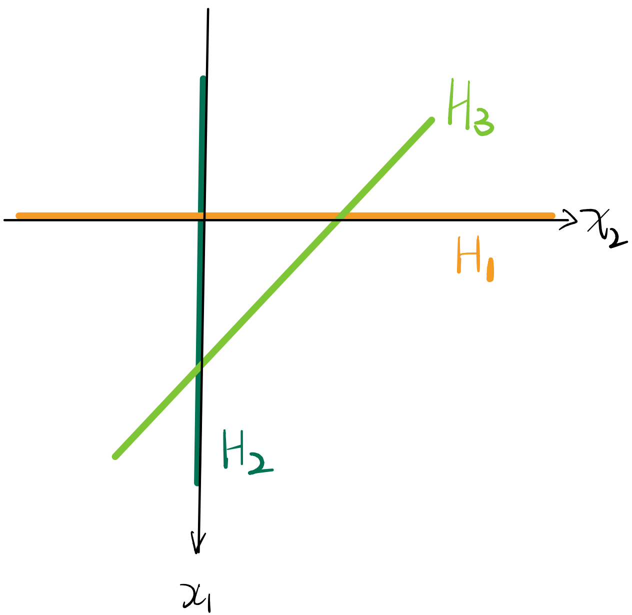
Central arrangement

$\mathcal{A} \subseteq \mathbb{R}^2$

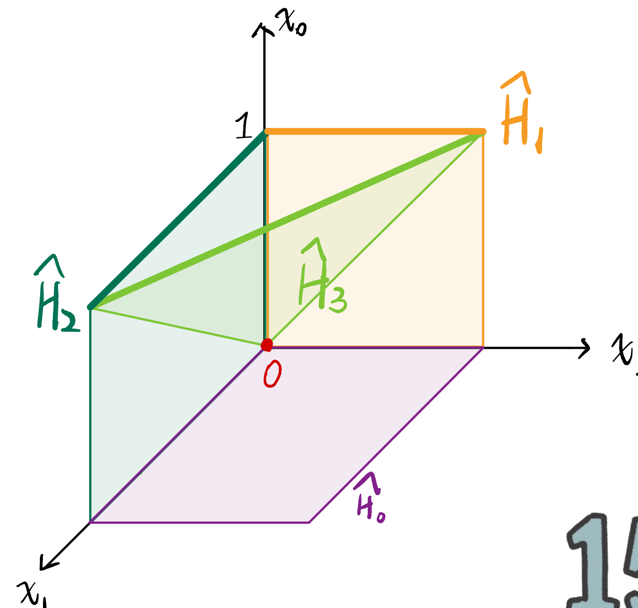
Example:

affine arrangement $\xrightarrow{\text{coning}}$ central arrangement

$\mathbb{R}^2 \supseteq \mathcal{A}:$ $\longleftrightarrow$ $\mathbb{R}^3 \supseteq \hat{\mathcal{A}}:$



$\longleftrightarrow$



affine arrangement in $\mathbb{R}^d$ $\xrightarrow{\text{coning}}$ central arrangement in $\mathbb{R}^{d+1}$

$$\mathcal{A} = \{ H_1, \cdots, H_t \} \longrightarrow \hat{\mathcal{A}} = \{ \hat{H}_0 = \{ x_0 = 0 \}, \hat{H}_1, \cdots, \hat{H}_t \}$$

H_i :

$\langle \alpha_i, x \rangle + l_i = 0$

$$\hat{\alpha}_i = (l_i, \alpha_i)$$

$$\hat{x} = (x_0, x)$$

$$\longmapsto$$

$$\hat{h}_i(x_0, x_1, \cdots, x_d) = x_0 h_i(x_1/x_0, \cdots, x_d/x_0)$$

$\hat{H}_i$:

$\langle \hat{\alpha}_i, \hat{x} \rangle = 0$

$$\hat{h}_i = \sum_{j=1}^{d} a_j x_j + l_i x_0$$

$$h_i = \sum_{j=1}^{d} a_j x_j + l_i$$

$$\alpha_i = (a_1, \cdots, a_d)$$

Homogenization

- How can we connect polytope and hyperplane arrangement?

polytope P $\longleftrightarrow$? central arr. $\hat{\mathcal{A}}$

- How can we connect polytope and hyperplane arrangement?

Recall our Lemma:

Lemma: $\underset{\mathrm{Poly}(P)}{\underset{\cup}{\xi}}$ tangent to ∂P $\xrightarrow{\exists!\ \text{extension}}$ $\underset{\mathrm{Poly}(\mathbb{R}^d)}{\underset{\cup}{\tilde{\xi}}}$ tangent to hyperplanes

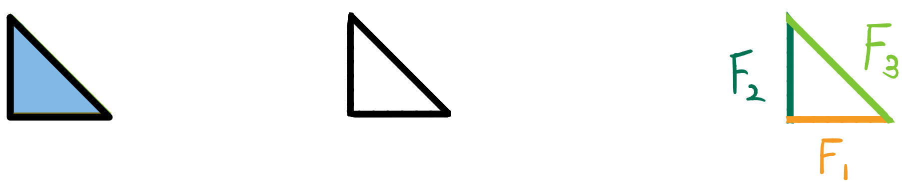
- How can we connect polytope and hyperplane arrangement?

polytope		boundary
P	$\longleftrightarrow$	∂P

- How can we connect polytope and hyperplane arrangement?

polytope — boundary — faces

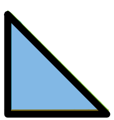
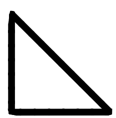
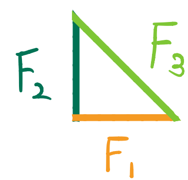
$$P \longleftrightarrow \partial P \longleftrightarrow \{F_i = H_i \cap \partial P\}$$

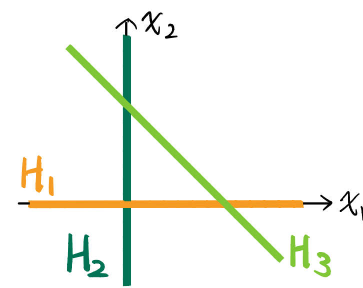


- How can we connect polytope and hyperplane arrangement?

polytope — boundary — faces — affine arr.

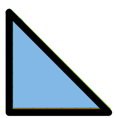
$$P \longleftrightarrow \partial P \longleftrightarrow \{F_i = H_i \cap \partial P\} \longleftrightarrow \{H_i\} = \mathcal{A}$$

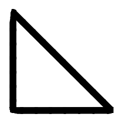




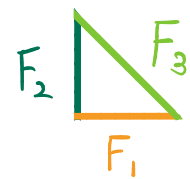


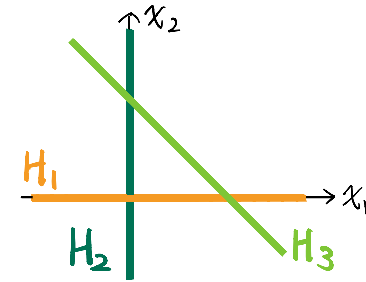
polytope	boundary	faces	affine arr.
P	∂P	$\{F_i = H_i \cap \partial P\}$	$\{H_i\} = \mathcal{A}$

$$P \longleftrightarrow \partial P \longleftrightarrow \{F_i = H_i \cap \partial P\} \longleftrightarrow \{H_i\} = \mathcal{A}$$

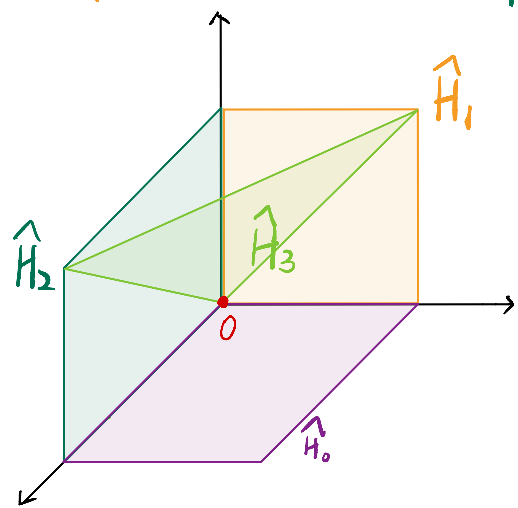






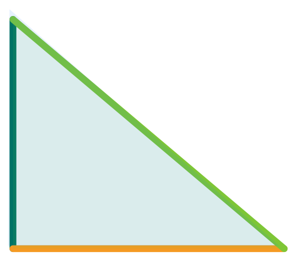


$\xrightarrow{\text{coning}}$ central arr. $\hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1}$

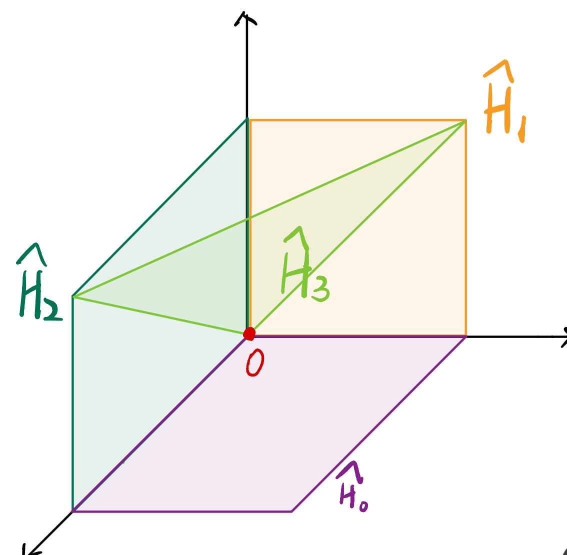


polytope $P \subseteq \mathbb{R}^d$ $\longleftrightarrow$ central arrangement $\hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1}$

<u>Example:</u> $d=2$.



$\longleftrightarrow$



Why do we want to connect with

the central arrangement?

Def: logarithmic derivation module of $\hat{A}$:

$$D(\hat{A}) = \left\{ \theta = \sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i} \;\middle|\; \begin{array}{l} \hat{f}_i \in \mathbb{R}[\hat{x}] \\ \hat{h}_i \mid \theta(\hat{h}_i) \end{array} \right\}$$

derivation

Def: logarithmic derivation module of $\hat{\mathcal{A}}$:

polynomial

$$D(\hat{\mathcal{A}}) = \left\{ \theta = \sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i} \;\middle|\; \begin{array}{l} \hat{f}_i \in \mathbb{R}[\hat{x}] \\ \hat{h}_i \mid \theta(\hat{h}_i) \end{array} \right\}$$

derivation

tangent condition

$$\hat{h}_i \mid \theta(\hat{h}_i) \Longleftrightarrow \langle (\hat{f}_0(\hat{x}), \dots, \hat{f}_d(\hat{x})), \hat{\alpha}_i \rangle = 0, \ \forall \hat{x} \in \hat{H}_i$$

tangent condition $\longleftrightarrow$ slip boundary condition

$$\begin{tikzcd}
\text{polytope } P \subseteq \mathbb{R}^d \arrow[r, leftrightarrow] \arrow[d] & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
\mathrm{Poly}_d(P) \arrow[r, leftrightarrow, "?"] & D(\hat{\mathcal{A}})
\end{tikzcd}$$

$$
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\text{polytope } P \subseteq \mathbb{R}^d \arrow[r, leftrightarrow] \arrow[d] & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
\mathrm{Poly}_\partial(P) \arrow[r, leftrightarrow, "?"] & D(\hat{\mathcal{A}})
\end{tikzcd}
$$

$$D(\hat{\mathcal{A}}) = \bigoplus_{k \in \mathbb{Z}} D(\hat{\mathcal{A}})_k$$

graded $R[x_i]$-module

$$
\begin{tikzcd}
\text{polytope } P \subseteq \mathbb{R}^d \arrow[rr, leftrightarrow] \arrow[d] & & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
\mathrm{Poly}_d(P)_k \arrow[rr, leftrightarrow, "?"] & & D(\hat{\mathcal{A}})_k \\
\xi = (f_1, \cdots, f_d) \arrow[u, phantom, sloped, "\in"] & & \theta = \sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}
$$

$$
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\text{polytope } P \subseteq \mathbb{R}^d \arrow[rr, leftrightarrow] \arrow[d] & & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
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\end{tikzcd}
$$

degree at most k

$\mathbb{R}[x_1, \dots, x_d]$

polytope $P \subseteq \mathbb{R}^d$ $\longleftrightarrow$ central arrangement $\hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1}$

degree at most k

homogenous of degree k

$\mathrm{Poly}_d(P)_k$

$D(\hat{\mathcal{A}})_k$

$\rotatebox{90}{$\in$}$

$\mathbb{R}[x_1, \dots, x_d]$

$\xi = (f_1, \dots, f_d)$

$\mathbb{R}[x_0, x_1, \dots, x_d]$

$\theta = \sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i}$

$$\begin{tikzcd} \text{polytope } P \subseteq \mathbb{R}^d \arrow[r, leftrightarrow] \arrow[d] & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\ \mathrm{Poly}_d(P)_k \arrow[r, mapsto] & D(\hat{\mathcal{A}})_k \end{tikzcd}$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & \boxed{\text{Homogenization}} & \rotatebox{90}{$\in$} \\ \xi = (f_1, \cdots, f_d) & \xmapsto[x_0^k f_i(x_1/x_0, \cdots, x_d/x_0) = \hat{f}_i(x_0, x_1, \cdots, x_d)]{} & \theta = \sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i} \end{array}$$

$$\begin{tikzcd}
\text{polytope } P \subseteq \mathbb{R}^d \arrow[rr, leftrightarrow] \arrow[d] & & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
\mathrm{Poly}_d(P)_k \arrow[rr, leftrightarrow, "\text{iso. ?}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & D(\hat{\mathcal{A}})_k \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\mathfrak{f} = (f_1, \dots, f_d) \arrow[rr, "\text{Homogenization}", "x_0^k f_i(x_1/x_0, \dots, x_d/x_0) = \hat{f}_i(x_0, x_1, \dots, x_d)"'] & & \theta = \sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i}
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=huge]
\text{polytope } P \subseteq \mathbb{R}^d \arrow[r, leftrightarrow] \arrow[d] & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
\mathrm{Poly}_d(P)_k \arrow[r, leftrightarrow, "\text{iso. ?}"] & D(\hat{\mathcal{A}})_k \\
\xi = (f_1, \cdots, f_d) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, "x_0^k f_i(x_1/x_0, \cdots, x_d/x_0) = \hat{f}_i(x_0, x_1, \cdots, x_d)"] & \theta = \sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i} \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}
$$

NO.

since $\hat{f}_0 \equiv 0$

Def: $D(\hat{A})$

$$\rotatebox{90}{$\subseteq$}$$

$$\circ \; D_{\hat{H}_0}(\hat{A}) = \left\{ \overset{\sum_{i=0}^{d} \hat{f}_i \frac{\partial}{\partial x_i}}{\overset{\shortparallel}{\theta}} \in D(\mathcal{A}) \;\middle|\; \hat{f}_0 = 0 \right\}$$

$$\rotatebox{90}{$\subseteq$}$$

$$\circ \; D_{\hat{H}_0}(\hat{A})_k = \left\{ \theta \in D(\mathcal{A})_k \;\middle|\; \hat{f}_0 = 0 \right\}$$

Main Theorem:

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{polytope } P \subseteq \mathbb{R}^d \arrow[rr, leftrightarrow] \arrow[d] & & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
\mathrm{Poly}_{\partial}(P)_k \arrow[rr, leftrightarrow, "\text{Thm [C-Kaji]}"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & D_{\hat{H}_0}(\hat{\mathcal{A}})_k \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
\xi = (f_1, \cdots, f_d) \arrow[rr, leftrightarrow, "\text{Homogenization}", "x_0^k f_i(x_1/x_0, \ldots, x_d/x_0) = \hat{f}_i(x_0, x_1, \cdots, x_d)"'] & & \theta = \sum_{i=1}^{d} \hat{f}_i \frac{\partial}{\partial x_i}
\end{tikzcd}
$$

find the basis of $\mathrm{Poly}_\partial(P)_k$

$\rightsquigarrow$ find the basis of $D_{\widehat{H_0}}(\widehat{\mathcal{A}})_k$

Prop.:

$$D_{\hat{H}_0}(\vec{A}) \triangleq \mathrm{Syz}(J(Q_{\vec{A}})),$$

where $Q_{\vec{A}} = \prod_i \hat{h}_i$ equation of $\hat{H}_i$

G. Basis

We can compute the basis of the vector space $D_{\hat{H}_0}(\vec{A})_k$, So can $\mathrm{Poly}_0(P)_k$

Say $D_{\hat{H}_0}(\hat{A})_k = \langle \theta_1, \dots, \theta_r \rangle_{\mathbb{R}}$

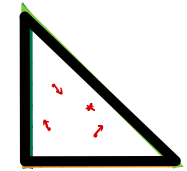
$$\begin{array}{ccc} \mathrm{Poly}_d(P)_k & \xleftrightarrow{\quad\text{Thm}\quad} & D_{\hat{H}_0}(\hat{A})_k = \langle \theta_1, \dots, \theta_r \rangle \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \xi_i & \longmapsfrom & \theta_i \end{array}$$

$$
\begin{tikzcd}[column sep=huge]
\text{polytope } P \subseteq \mathbb{R}^d \arrow[r, leftrightarrow] \arrow[d] & \text{central arrangement } \hat{\mathcal{A}} \subseteq \mathbb{R}^{d+1} \arrow[d] \\
\mathrm{Poly}_d(P)_k \arrow[r, leftrightarrow, "\text{Thm}"] & D_{\hat{H}_0}(\hat{\mathcal{A}})_k \\
\mathfrak{z} = \sum_{i=1}^{r} c_i \mathfrak{z}_i \arrow[u, phantom, "\rotatebox[origin=c]{90}{$\in$}"] \arrow[r, leftrightarrow] & \theta = \sum_{i=1}^{r} c_i \theta_i \arrow[u, phantom, "\rotatebox[origin=c]{90}{$\in$}"]
\end{tikzcd}
$$

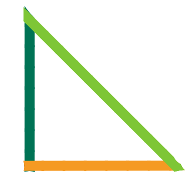
↳ Exactly satisfy the boundary condition.

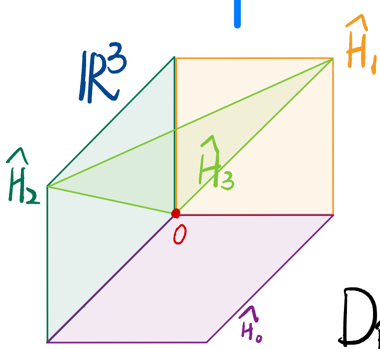
Summarize:

Given data points in P.

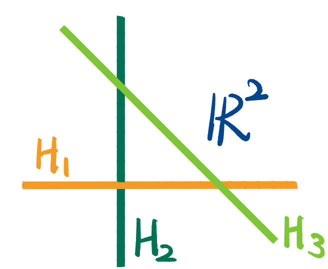


→ fitting a polynomial vector field tangent to ∂P



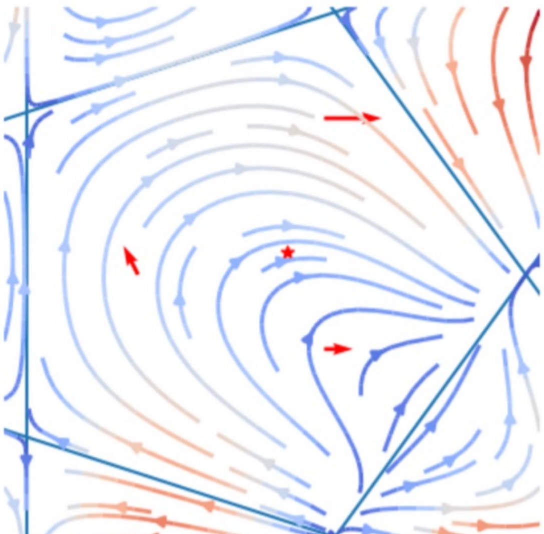


$D_{\hat{H}_0}(\mathcal{A})_R$ ← tangent to H_i

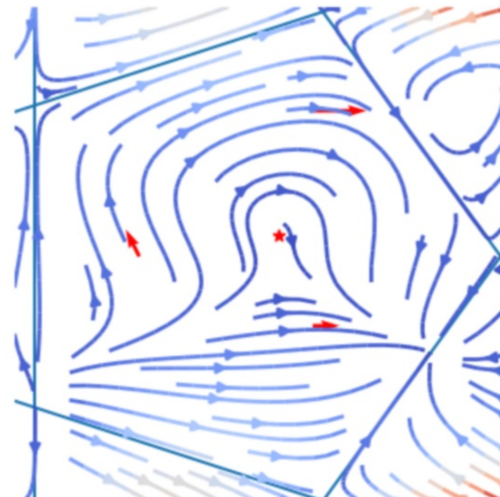


Example A:

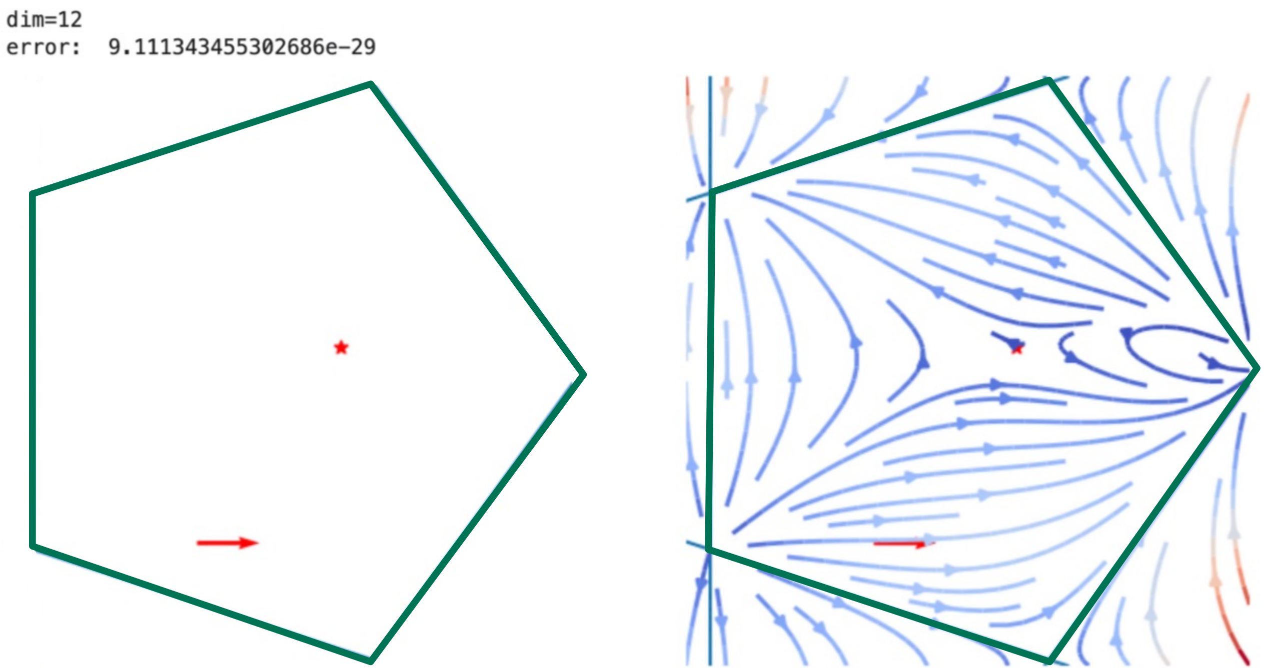
Degree: 4
Error: 2.8



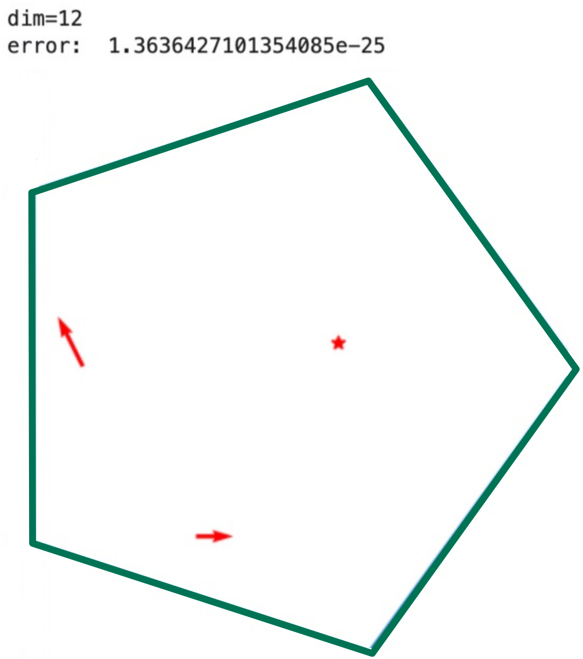
Degree: 5
Error: 3.0×10^{-27}

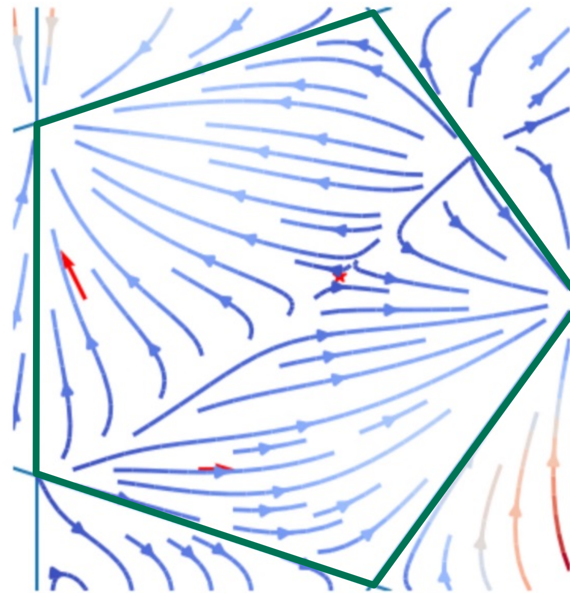


Example B: Given degree=5 $\longrightarrow$ Error: 9.1×10^{-29}



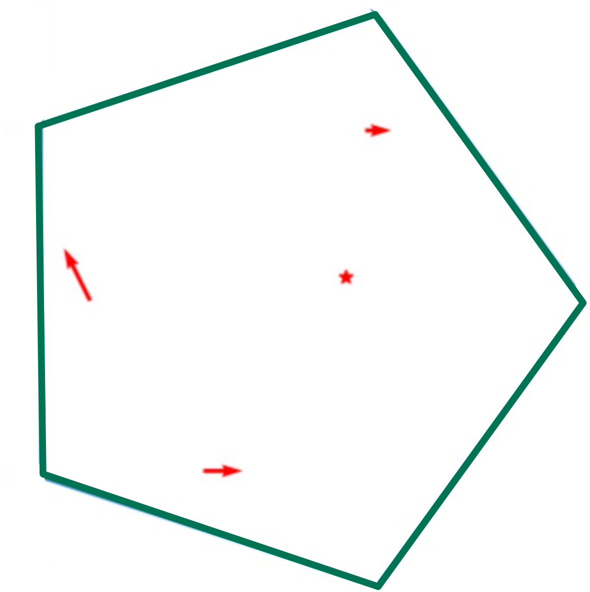
Example B: Given degree=5 $\longrightarrow$ Error: 1.4×10^{-25}

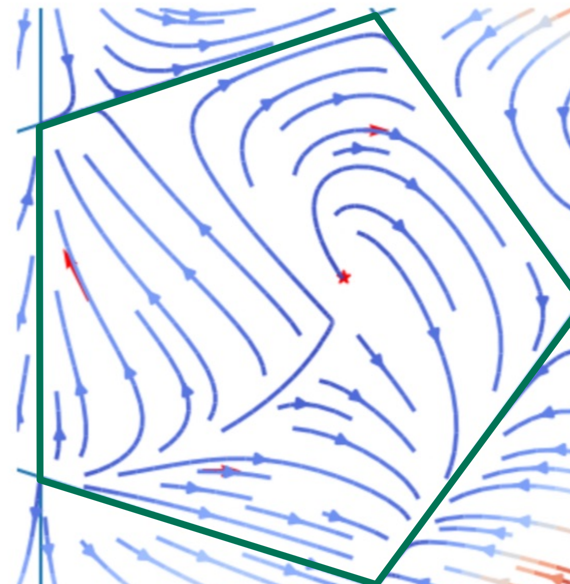




Example B: Given degree=5 $\longrightarrow$ Error: 2.8×10^{-26}

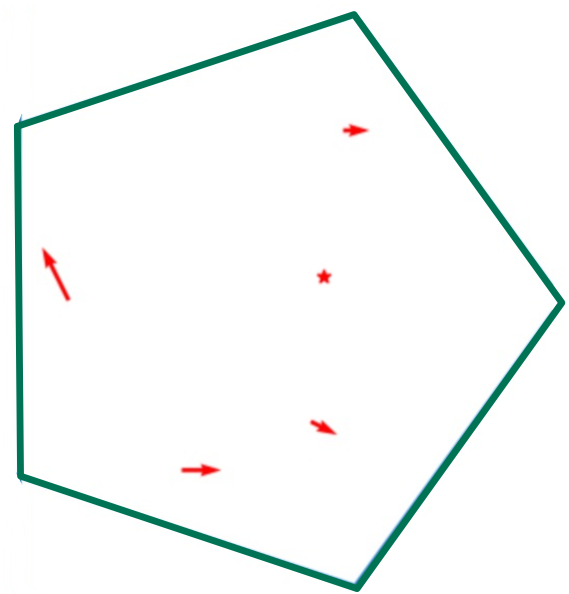
```
dim=12
error:  2.833903915915963e-26
```

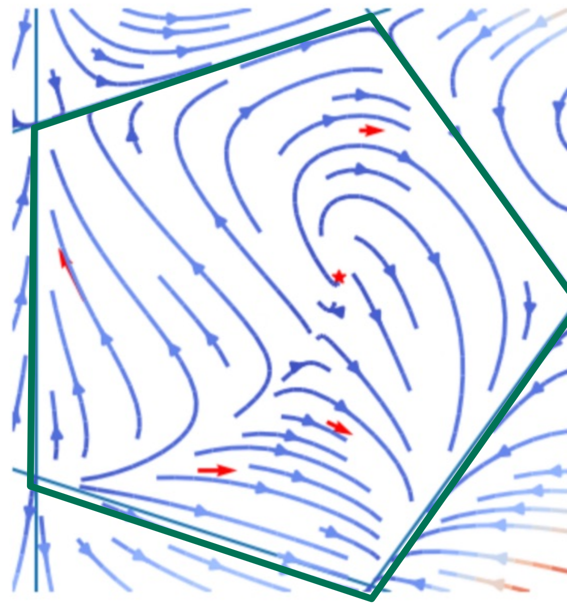




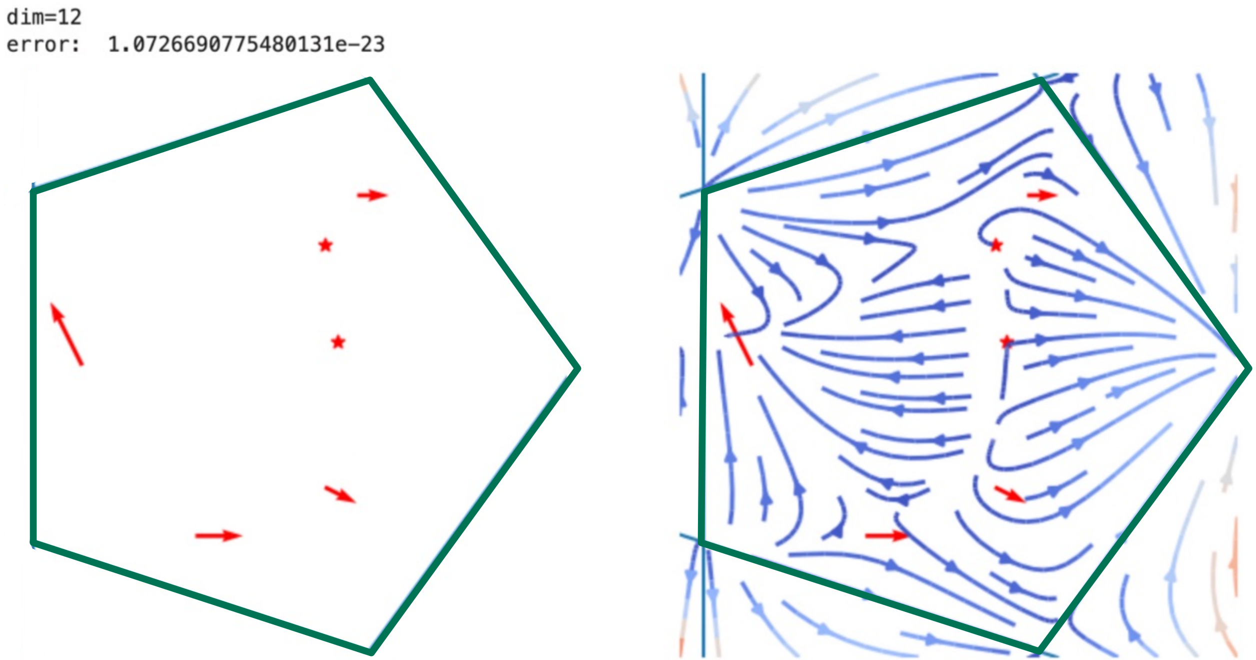
Example B: Given degree=5 $\longrightarrow$ Error: 6.1×10^{-26}

```
dim=12
error:   6.11398755982493e-26
```





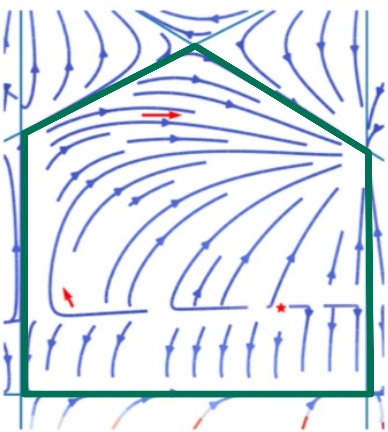
Example B: Given degree=5 $\longrightarrow$ Error: 1.1×10^{-23}



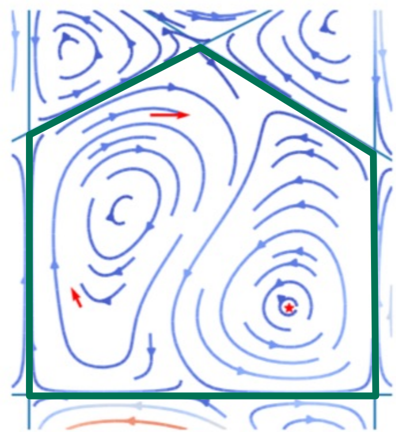
Example C:

Given degree=7

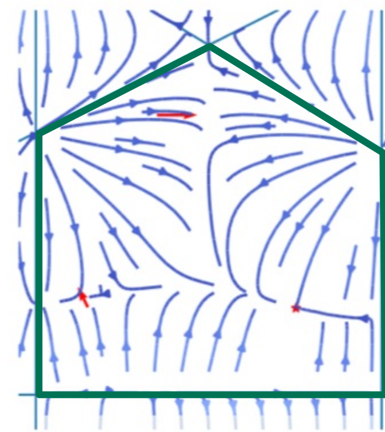
rotation free



divergence free



harmonic



Thank you

for

your attention