Degrees of the logarithmic vector fields
for close-to-free hyperplane arrangements

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1. INTRODUCTION

QUESTION 1. *[2]*
Show that p cuts can divide a cheese into as many as $\frac{(p+1)(p^2-p+6)}{6}$ pieces.

Let planes be in a 'general position', we can maximize the number of pieces. For example, $p = 4$ planes divide $\mathbb{R}^3$ into at most

$$1 + p + \binom{p}{2} + \binom{p}{3} = \frac{(p+1)(p^2-p+6)}{6} = 15$$

regions. This leads us to a fundamental and intriguing question:

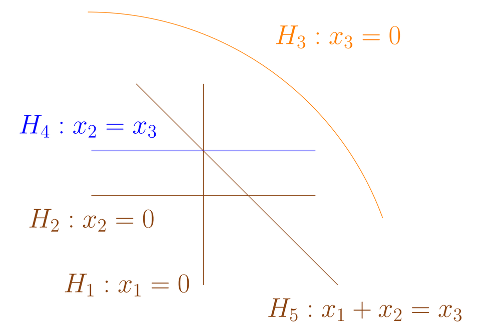
QUESTION 2.
How can we determine # {regions formed by a non-generic arrangement of p planes in $\mathbb{R}^\ell$}?

2. DEFINITIONS

1. $S := \mathbb{K}[x_1, \dots, x_\ell]$: polynomial ring over a field $\mathbb{K}$,
2. $\mathscr{A} := \{H_i : \sum_{j=1}^{\ell} a_{ij}x_j = 0 \mid i = 1, \dots, p\}$: hyperplane arrangement in a vector space $V = \mathbb{K}^\ell$,
3. $L(\mathscr{A}) := \{\bigcap_{H \in \mathscr{B}} H \mid \mathscr{B} \subset \mathscr{A}\}$: intersection lattice,
4. $\mathscr{A}_X := \{H \in \mathscr{A} \mid X \subset H\}$: localization of $\mathscr{A}$ at $X \in L(\mathscr{A})$,
5. $\mathscr{A}^H := \{L \cap H \mid L \in \mathscr{A} \setminus \{H\}\}$: restriction of $\mathscr{A}$ onto $H \in L(\mathscr{A})$,

EXAMPLE 1.
The figure shows the projectivized form of $\mathscr{A}$ in $\mathbb{R}^3$.

- *When $X = H_3 \cap H_4$, we have $\mathscr{A}_X = \{H_2, H_3, H_4\}$;*
- *When $X = H_3$, we have:*
  $\mathscr{A}^{H_3} = \{x_1 = 0, x_2 = 0, x_1 + x_2 = 0 \mid x_3 = 0\}$.



6. $D(\mathscr{A}) := \{\theta = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i} \mid f_i \in S, \theta(\alpha_H) \in \alpha_H S, H \in \mathscr{A}\}$: log derivation module of $\mathscr{A}$,

EXAMPLE 2.
For any arrangement $\mathscr{A}$, we have $\theta_E = x_1\frac{\partial}{\partial x_1} + \cdots + x_\ell\frac{\partial}{\partial x_\ell} \in D(\mathscr{A})$.

7. $\mathrm{DS}(\mathscr{A}) := (1, d_2 \dots, d_\ell)$: *degrees* of the minimal homogeneous generators of $D(\mathscr{A})$,
8. $\mathrm{pd}_S D(\mathscr{A})$: projective dimension of $D(\mathscr{A})$.

3. MOTIVATION

We say that $\mathscr{A}$ is a free arrangement if $D(\mathscr{A})$ is a free module.
When $\mathscr{A}$ is free, we can compute # {(bounded) regions of $\mathscr{A}$ projected on to $H \in \mathscr{A}$}.

THEOREM 1. *[4]*
If $\mathscr{A}$ is free with $\mathrm{DS}(\mathscr{A}) = (1, d_2, \dots, d_\ell)$, then the number of (bounded) regions of $\mathscr{A}$ projected on to $H \in \mathscr{A}$ are as follows:

$$\#regions = \prod_{i=2}^{\ell}(1 + d_i).$$

$$\#bounded\ regions = \prod_{i=2}^{\ell}(d_i - 1).$$

Now, let us compute the number of (bounded) regions of $\mathscr{A}$ in Example 1 projected on to H_3.

EXAMPLE 3.
Arrangement $\mathscr{A}$ is free with $\mathrm{DS}(\mathscr{A}) = (1, 2, 2)$.

$$\#regions = (1 + 2)^2 = 9.$$
$$\#bounded\ regions = (2 - 1)^2 = 1.$$

• Connections

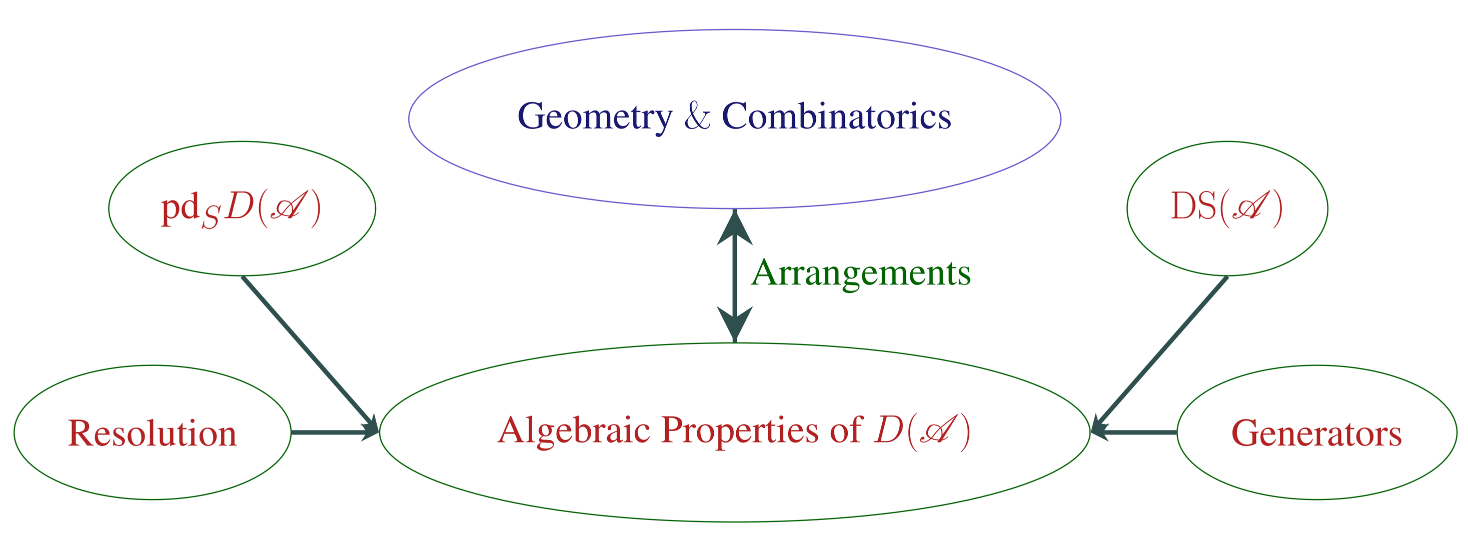
The free arrangement $\mathscr{A}$ uncovers significant connections between topology and combinatorics [3]. In contrast, the counterparts for non-free arrangements have received limited attention.

To address this gap, our aim is to establish a theorem analogous to Theorem 1 for non-free cases.

To begin, let's refine the initial question:

QUESTION 3.
Can we characterize the algebraic structure $D(\mathscr{A})$ when $\mathscr{A}$ is "close to free"?

Absolutely! Our investigation focuses on identifying the following:



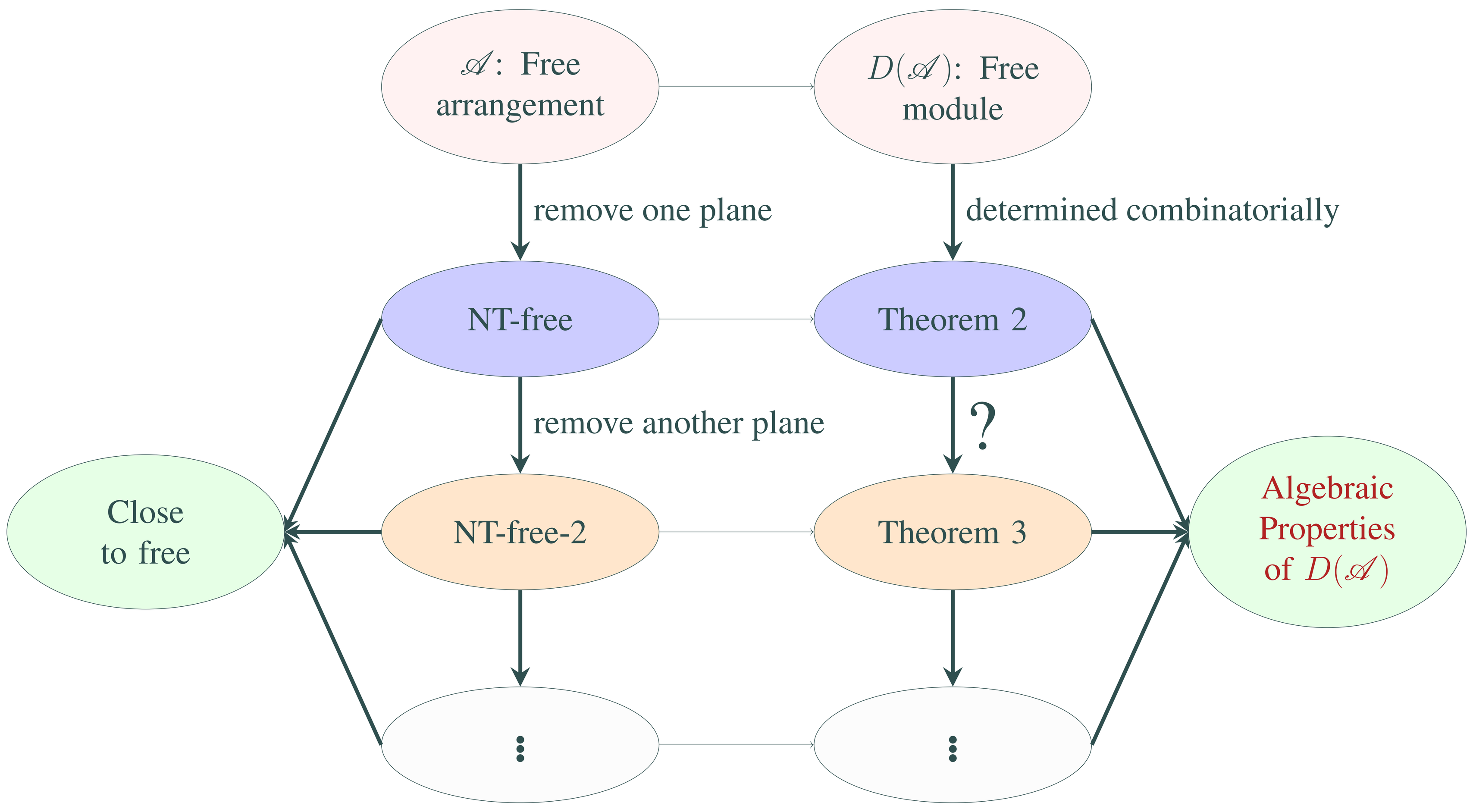
Close-to-free Arrangements:

- $\mathscr{A}_j := \mathscr{A} \setminus \{H_j\}$
  We say $\mathscr{A}_j$ is a NT-free arrangement if $\mathscr{A}$ is free but $\mathscr{A}_j$ is not.
- $\mathscr{A}_{i,j} := \mathscr{A} \setminus \{H_i, H_j\}$
  We say $\mathscr{A}_{i,j}$ is a NT-free-2 arrangement if both $\mathscr{A}_i$ and $\mathscr{A}_j$ are NT-free.

Abe shows that $\mathscr{A}_j$ is determined combinatorially.

THEOREM 2 (Theorem 1.4 in [1]).
Let $\mathscr{A}$ be free with $\mathrm{DS}(\mathscr{A}) = (1, d_2, \cdots, d_\ell)$ and $H \in \mathscr{A}$. Then $\mathscr{A}_j$ is free, or $\mathrm{DS}(\mathscr{A}) = (1, d_2, \cdots, d_\ell, |\mathscr{A}_j| - |\mathscr{A}^H|)$ with $pd_S D(\mathscr{A}) = 1$.

• Purpose

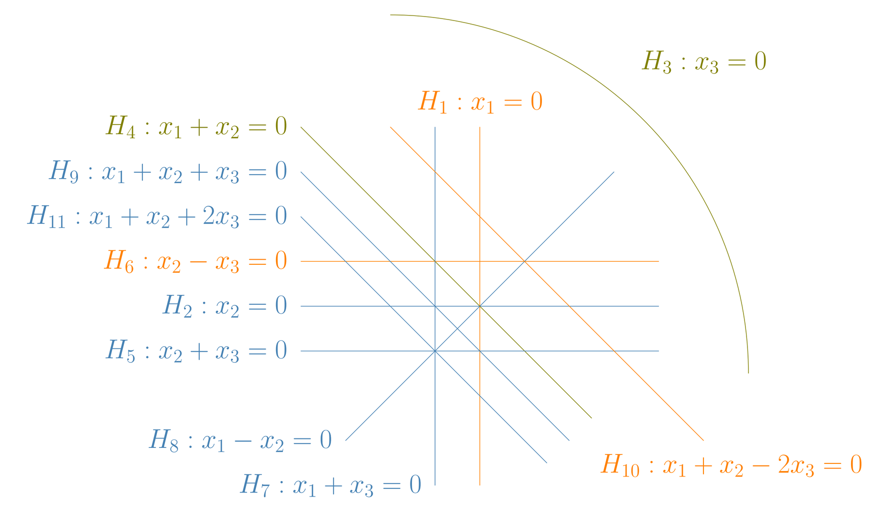


4. RESULTS

THEOREM 3 (CHU).
For an NT-free-2 arrangement $\mathscr{A}$, $DS(\mathscr{A})$ is obtained combinatorially.

Since the theorem is complicated, we explain it with an example.

EXAMPLE 4.
Let $\mathscr{A}$ be as follows. By computer, $\mathscr{A}$ is free with $\mathrm{DS}(\mathscr{A}) = (1, 5, 5)$.



$\mathscr{A}_i$:
free with $\mathrm{DS}(\mathscr{A}_i) = (1, 4, 5)$;

$\mathscr{A}_j$:
$\mathrm{DS}(\mathscr{A}_j) = (1, 5, 5, c_j = 6)$;

$\mathscr{A}_k$:
$\mathrm{DS}(\mathscr{A}_k) = (1, 5, 5, c_k = 5)$.

By Theorem 3, this implies the following results:

1. $|\mathscr{A}_{H_2 \cap H_{11}}| = 2$, then $\mathrm{DS}(\mathscr{A}_{2,11}) = (1, 5, 5, 5, 5)$.
2. $|\mathscr{A}_{H_3 \cap H_5}| > 2$ and $c_3 = 6 > c_5 = 5$, then $\mathrm{DS}(\mathscr{A}_{3,5}) = (1, 5, 5, 6 - 1, 5)$.
3. $|\mathscr{A}_{H_3 \cap H_4}| > 2$ and $c_3 = c_4 = 6 > 5$, then $\mathrm{DS}(\mathscr{A}_{3,4}) = (1, 5, 5, 6 - 1)$.
4. $|\mathscr{A}_{H_5 \cap H_7}| > 2$ and $c_5 = c_4 = 5$, then $\mathrm{DS}(\mathscr{A}_{5,7}) = (1, 5, 5, 5 - 1)$.

Note that $\mathrm{pd}_S D(\mathscr{A}_{j,k}) = 1$ and $\mathscr{A}_{j,k}$ is not NT-free.

5. FORTHCOMING RESEARCH

CONJECTURE.
1. *If $\mathscr{A}$ is NT-free-2 in $\mathbb{K}^\ell$, then $pd_s D(\mathscr{A}) \le 2$.*
2. *If # {minimal generators for $D(\mathscr{A}) \le \ell + 2$}, then $pd_s D(\mathscr{A}) \le 1$.*
3. *{NT-free} ∩ {NT-free-2} = ∅.*

References

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