Midterm

aaa →
bb →

ba → ab
ab → ba

$\boxed{\text{Make enough examples}}$

Not part of the answer
but part of the work:

[]	does not reduce normal form	nf
a	"	nf
b	"	nf
ab ba	ab → ba → ab ... ab not a n.f.	~~nf~~

	ab $\leftrightarrow$ ba		
aa	n.f.		nf
bb	bb $\to$ [ ]	bb $\leftrightarrow$ [ ]	
aaa	aaa $\to$ [ ]	aaa $\leftrightarrow$ [ ]	
aab aba baa	not a n.f.		~~nf~~
abb	abb $\to$ a	abb $\leftrightarrow$ a	
bbb	bbb $\to$ b	bbb $\leftrightarrow$ b	

How many equi. classes?

Guess from the data: 6

Every word is in some equivalence class.

$\boxed{\text{By now, we know } \exists \text{ at least } 6 \text{ equivalence classes}}$

assuming that we can establish that the 6 are all different

Invariants

(can we show that $\exists$ <u>exactly</u> 6 equ. class ?)

Can we describe the equ. class of [ ] ?

$aca \to$

$bb \to$

$\boxed{\text{triple } a \text{, double } b \text{ reduce to } [\,]}$

$w \overset{*}{\longleftrightarrow} [\,]$ if and only if

$$\boxed{\begin{array}{l}\#a \text{ is a multiple of } 3 \\ \#b \text{ is a multiple of } 2\end{array}} \quad (*)$$

We characterised $\{ w \mid w \overset{*}{\longleftrightarrow} [\,] \}$.

This should give us the idea of the invariant.

Look at $(*)$

$\#a \bmod 3$

$\#b \bmod 2$

To put them in one invariant

$$(\#a \bmod 3,\ \#b \bmod 2)$$

So far we only <u>guessed</u> the invariant.

(Method: Guess and Verify)

So we need to verify that

$$(\#a \bmod 3,\ \#b \bmod 2)$$

is an invariant.

	before	after
$ab \to ba$	$(1,1)$	$(1,1)$
$ba \to ab$	$(1,1)$	$(1,1)$
$baaa \to b$	$(0,1)$	$(0,1)$

aabb → aa	$(\underset{2}{0}, 0)$	$(\underset{2}{0}, 0)$

Now we know that all 6 equ. classes are different. In a picture:

$\#b \bmod 2 =$ \ $\#a \bmod 3 =$	0	1	2
1	b aaab bbb	ab	aab
0	bb aaa []	a	aa

How can we show that these 6 equ classes are the only ones?

If we start with an arbitrary word w, it is in one of the 6 equ. classes.

Why? Because: We can always delete 'aaa' and 'bb'. Hence the longest words we can end up with are: [ ], a, b, ab, aa, aab (and ba, aba, baa are equivalent to the others).

<u>Confluence:</u>

$$
\begin{tikzcd}
 & aaaabbb \arrow[dl, "f"'] \arrow[dr, "\alpha"] & \\
abbb & & aaaab
\end{tikzcd}
$$

$\boxed{\text{rules don't overlap}}$

(sufficient but not necessary for confluence)

If the rules do overlap you need to check that there is a diamond for each way two rules can overlap.

n.f.s: []
a
b
aa

n. f.s.: ab
aab

termination: drop ab → ba
keep ba → ab
aaa →
bb →

now ab
aab are also n.f.s.

termination measure

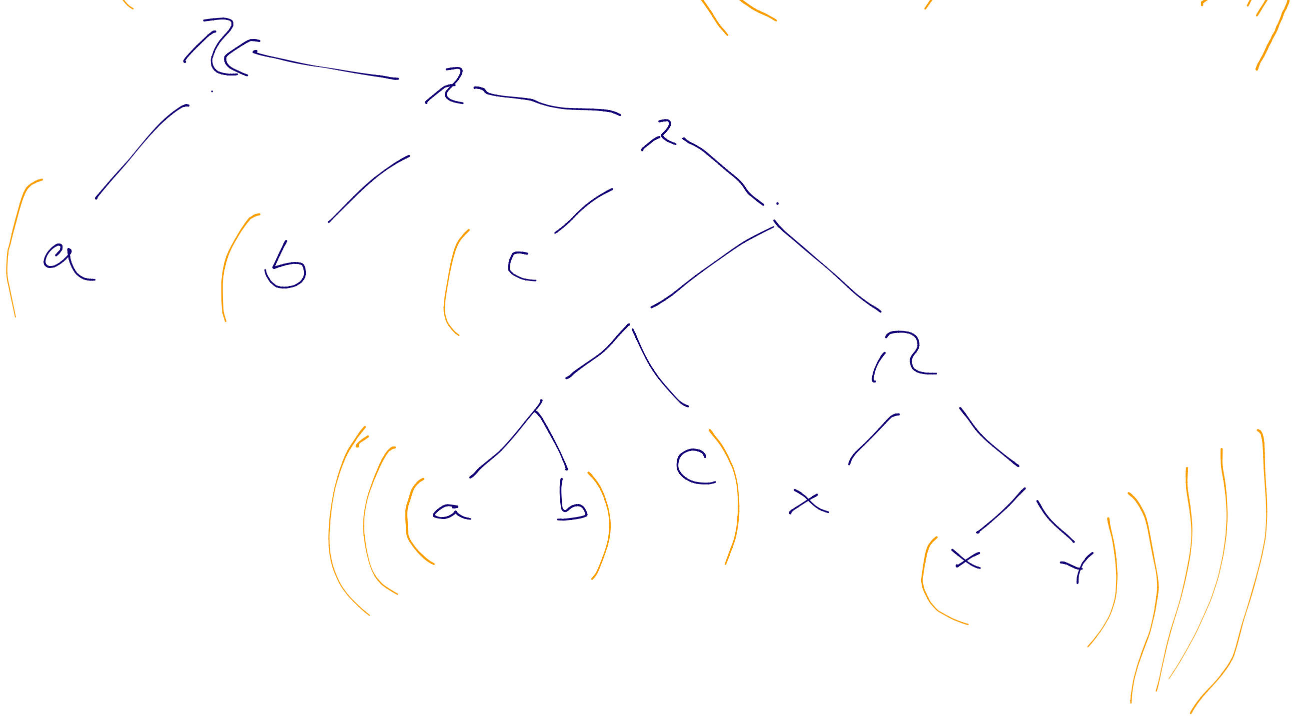
2 1	>	1 2
ba	→	ab
1 1 1	>	0
aaa	→	[ ]
2 2	>	0
bb	→	[ ]

Q3

$$\lambda a\ \lambda b\ \lambda c\ a\ b\ c\ \lambda x.x\ y$$

$$Exp ::= (\lambda x.Exp) \mid (Exp\ Exp)$$

$$(\lambda a\ (\lambda b\ (\lambda c\ (((a\ b)\ c)\ (\lambda x.(x\ y))))))$$



Q4

$$(\lambda x.\lambda y.y)((\lambda x.xx)(\lambda x.xx))(\lambda x.x)$$

have nf $\iff$ $\exists$ terminating reduction sequence

$$(\lambda x.\lambda y.y)((\lambda x.xx)(\lambda x.xx))(\lambda x.x) \circlearrowleft_{\beta}$$

$\swarrow$ CBN $\qquad\qquad\qquad\qquad$ $\Uparrow$ CBV

$$(\lambda y.y)(\lambda x.x)$$

$$\downarrow$$

$\lambda x.x$ $\qquad$ n.f.

QUESTIONS AFTER THE LECTURE

What does termination mean?

$\exists$ terminating computation	$\forall$ computation terminate
$\boxed{\text{w has a n.f.}}$	$\boxed{\text{ARS terminating}}$

$w \to \;- -\; \to nf$

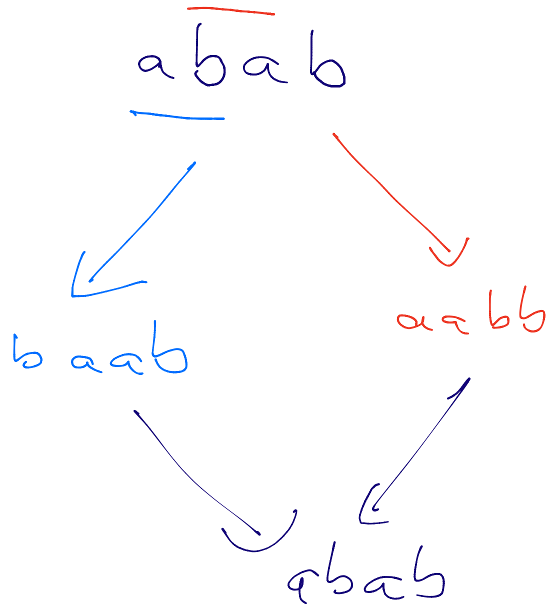
CONFLUENCE

$ab \to ba$

$ba \to ab$

$aa \to$

$bb \to$



For more check out

$$\boxed{\text{Knuth-Bendix Algorithm}}$$

https://en.wikipedia.org/wiki/Knuth%E2%80%93Bendix_completion_algorithm

The rules $ab \to ba$, $ba \to ab$ mean that we work with bags

https://www.youtube.com/watch?v=xwGvfkVpNi0

strings	bags
aabab	$3a + 2b$
$\downarrow$	$=$
aaabb	$3a + 2b$
$\downarrow$	$-3a$
bb	$2b$