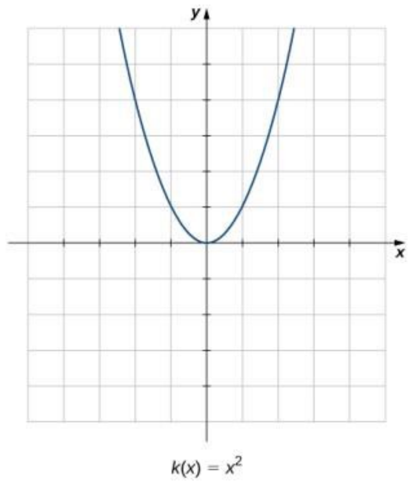
Section 2.1: A Preview of Calculus

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. Recall that for lines, the rate of change is constant, called the slope. This is because as we move from left to right, the function will either increase or decrease at a constant rate.

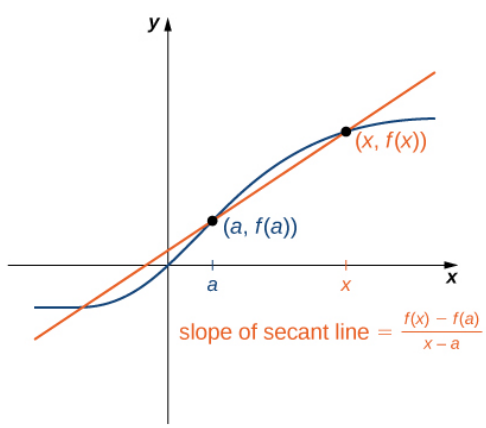
However, for nonlinear functions, the rate of change can vary. For example, in the graph below, if we were to move from left to right, the graph decreases rapidly, decreases more slowly and then levels off, before increasing (slowly at first and then more rapidly). Unlike for a linear function, we cannot represent the rate of change of this function by a single number.



We can, however, approximate the rate of change of a function of $f(x)$ by looking at the slope of the secant line.

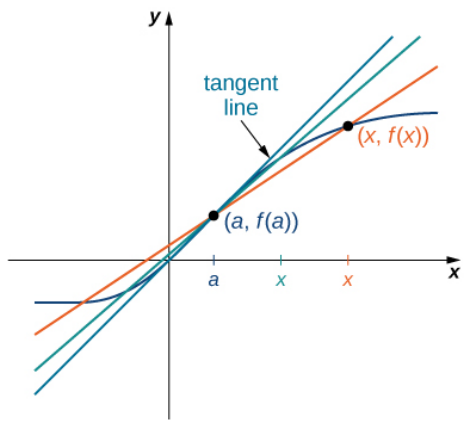
The **secant to the function** $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. It's slope is given by

$$m_{sec} = \frac{f(x)-f(a)}{x-a}$$



However, secant lines are not always the best approximation for the rate of change of the function. It will depend on how close x is to a. As x gets closer to a, the secant lines approach the tangent line.

The secant lines approach a line that is called the **tangent to the function** $f(x)$. The slope of the tangent line is a more accurate measure of the rate of change of the function at a and represents the derivative of the function $f(x)$ at a. The derivative is denoted $f'(a)$.



Media: Learn more about secant and tangent lines and how they relate to the derivative. Then, watch this video example on estimating the slope of the tangent line.

Examples:

1) Estimate the slope of the tangent line to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1,1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$

$$m_{sec} = \frac{4-1}{2-1} = 3$$

b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

$$m_{sec} = \frac{\frac{9}{4}-1}{\frac{3}{2}-1} = \frac{5}{2} = 2.5$$

2) Points $P(1{,}2)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^2 + 1$. Complete the following table with the appropriate values: y −coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to six decimal places.

x	y	$Q\ (x, y)$	m_{sec}
1.1	2.21	(1.1, 2.21)	2.100000
1.01	2.201	(1.01, 2.201)	2.01000
1.001	2.002001	(1.001, 2.002002)	2.001000
1.0001	2.00020001	(1.0001, 2.00020001)	2.000100

Use the values in the right column of the table to guess the value of the slope of the line tangent to f at $x = 1$.

Since values appear to get closer to 2, the slope of the line tangent to f at x = 1 is 2.

Differential calculus is the field of calculus concerned with the study of derivatives and their applications. For instance, velocity can be thought of as the rate of change of position, $s(t)$, which represents the position of an object along a coordinate axis at any given time t. Using ideas about rate of change (with secant and tangent lines), we can approximate the instantaneous velocity with an average velocity.

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{\text{average}} = \frac{s(t) - s(a)}{t - a}$$

Finding the average velocity of a position function over a time interval is the same as finding the slope of the secant line to a function.

Media: Watch this video to review average rate of change of a function.

As t gets closer to a, the average velocity becomes closer to the instantaneous velocity just as when the secant lines approach the tangent line. This process of letting x or t approach a in an expression is called taking a **limit**.

For a position function $s(t)$, the instantaneous velocity at a time $t = a$ is the value that the average velocities approach on intervals $[a, t]$ and $[t, a]$ as the value of t becomes closer to a, provided such a value exists.

Finding the instantaneous velocity of a position function over a time interval is the same as finding the slope of the tangent line to a function.

Media: Watch this video example on estimating instantaneous velocity.

Example

A rock is dropped from a height of 64 feet. It is determined that its height (in feet) above ground t seconds later (for $0 \leq t \leq 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49{,}0.5]$

$$\text{average velocity} = \frac{s(0.5) - s(0.49)}{0.5 - 0.49} = \frac{60 - 60.1584}{0.5 - 0.49} = -15.84\,\frac{ft}{\text{sec}}$$

b. $[0.5, 0.51]$

$$\text{average velocity} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = \frac{59.8384 - 60}{0.51 - 0.5} = -16/16\,\frac{\text{ft}}{\text{sec}}$$

The instantaneous veloctiy is somewhere between -15.84 and -16.16 ft/sec.
A good guess might be -16 ft/sec.