Section 5.5: Substitution

The Fundamental Theorem of Calculus gives a method to evaluate integrals without Riemann sums. The drawback of this method though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called integration by substitution, to help find antiderivatives. Specifically, this method helps find antiderivatives when the integrand is the result of a chain-rule derivative.

> **Substitution with Indefinite Integrals**
>
> Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,
>
> $$\begin{aligned} \int f[g(x)]g'(x)\,dx &= \int f(u)\,du \\ &= F(u) + C \\ &= F\big(g(x)\big) + C \end{aligned}$$

This method is called substitution because part of the integrand is substituted with the variable u and part of the integrand with du. It is also referred to as **change of variables** because the variables are changed to obtain an expression that is easier to apply the integration rules.

Media: Watch this video to learn more about substitution.

> **How to Solve Using Integration by Substitution**
>
> 1) Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$ such that $g'(x)$ is also part of the integrand.
> 2) Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral.
> 3) We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
> 4) Evalaute the integral in terms of u.
> 5) Write the result in terms of x and the expression $g(x)$.

Media: Watch video1, video2, and video3 examples on indefinite integrals using substitution.

Examples: In the following examples, use substitution to evaluate.

1) $\int 6x(3x^2+4)^4\,dx$

Let $u = 3x^2 + 4$

$du = 6x\,dx$

So,

$$\int 6x(3x^2+4)^4\,dx = \int u^4 du = \frac{u^5}{5} + C$$
$$= \frac{(3x^2+4)^5}{5} + C$$

2) $\int z\sqrt{z^2-5}\,dz$

Let $u = z^2 - 5$

$du = 2z\,dz$

So,

$$\int z\sqrt{z^2-5}\,dz = \frac{1}{2}\int u^{\frac{1}{2}} du$$

$$= \frac{1}{2}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{3}u^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(z^2-5)^{\frac{3}{2}} + C$$

3) $\int \frac{\sin t}{\cos^3 t}\,dt$

Let $u = \cos t$

$du = -\sin t\ \,dt$

$-du = \sin t\ \,dt$

So,

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{1}{u^3}\,du = -\int u^{-3}\,du$$

$$= -\left(-\frac{1}{2}\right)u^{-2} + C$$

$$= \frac{1}{2u^2} + C$$

$$= \frac{1}{2\cos^2 t} + C$$

4) $\int \frac{x}{\sqrt{x-1}}\,dx$

Let $u = x - 1$

$du = dx$

$x = u + 1$

So,

$$\int \frac{x}{\sqrt{x-1}}\,dt = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right) du = \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Media: Watch these video1, video 2, and video 3 examples on definite integrals using substitution.

Examples: In the following examples, use substitution to evaluate.

1) $\int_0^1 x^2(1 + 2x^3)^5\,dx$

Let $u = 1 + 2x^3$

$du = 6x^2 dx$

$\frac{1}{6}du = x^2 dx$

*Adjust the limits of integration. For example, when $x = 0, u = 1$ and when $x = 1, u = 3$.

So,

$$\int_0^1 x^2\left(1 + 2x^3\right)^5 dx = \frac{1}{6}\int_1^3 u^5\,du$$

$$= \frac{1}{6}\cdot\frac{u^6}{6}\Big|_1^3 = \frac{1}{6}\left[\frac{3^6}{6} - \frac{1^6}{6}\right]$$

$$= \frac{189}{9}\ or\ 20.22$$

2) $\int_0^1 xe^{4x^2+3}\, dx$

Let $u = 4x^2 + 3$

$du = 8x\, dx$

$\frac{1}{8} du = x\, dx$

*Adjust the limits of integration. For example, when $x = 0, u = 3$ and when $x = 1, u = 7$.

So,

$$\int_0^1 xe^{4x^2+3} dx = \frac{1}{8}\int_3^7 e^u\, du$$

$$= \frac{1}{8} e^u \Big|_3^7$$

$$= \frac{e^7 - e^3}{8} \approx 134.568$$

3) $\int_0^{\frac{\pi}{2}} \cos^2 \theta\, d\theta$

Let $u = 2\theta$

$du = 2\, d\theta$

$\frac{1}{2} du = d\theta$

*Adjust the limits of integration. For example, when $\theta = 0, u = 0$ and when $\theta = \frac{\pi}{2}, u = \pi$.

So,

$$\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} d\theta + \int_0^{\frac{\pi}{2}} \left(\frac{\cos 2\theta}{2}\right) d\theta\right] = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos u) d\theta\right]$$

$$= \frac{\theta}{2}\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\sin u\Big|_0^{\theta} = \left(\frac{\pi}{4} - 0\right) + (0 - 0)$$

$$= \frac{\pi}{4}$$