Section 5.6: Integrals Involving Exponential and Logarithmic Functions

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Media: Watch these video1 and video 2 examples on integrals involving exponentials.

Examples

1) For each of the following, find the antiderivative.

a. e^{-x}

let $u = -x$

$du = -dx$

so $-du = dx$

$\int e^{-x}\,dx = -\int e^u\,du$

$= -e^u + C$

$= \boxed{-e^{-x} + C}$

b. $e^x\sqrt{1+e^x}$

$\int e^x\sqrt{1+e^x}\,dx = \int e^x(1+e^x)^{1/2}\,dx$

let $u = 1+e^x$

$du = e^x dx$

$= \int u^{1/2}\,du$

$= \dfrac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}(1+e^x)^{3/2} + C}$

2) Use substitution to evaluate the indefinite integral $\int 3x^2 e^{2x^3}\,dx$.

let $u = 2x^3$

$du = 6x^2\,dx$

$\frac{1}{2}du = 3x^2\,dx$

$\int 3x^2 e^{2x^3}\,dx = \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^u + C$

$= \boxed{\frac{1}{2}e^{2x^3} + C}$

$*$ change limits of integration

3) Evaluate the definite integral $\int_1^2 e^{1-x}\,dx$. ← when $x=1$, $u=1-1=0$

$x=2$, $u=1-2=-1$

let $u=1-x$

$du=-dx$

$-du=-dx$

$$\int_1^2 e^{1-x}\,dx = -\int_0^{-1} e^u\,du$$

$$= \int_{-1}^{0} e^u\,du = e^u \Big|_{-1}^{0}$$

$$= e^0-(e^{-1}) = \boxed{-e^{-1}+1}$$

4) Evaluate the definite integral using substitution: $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}$ ⇒ rewrite as $\int_1^2 e^{x^{-1}}x^{-2}\,dx$

let $u=x^{-1}$

$du=-x^{-2}\,dx$

$-du=x^{-2}\,dx$

$$\int_1^2 e^{x^{-1}}x^{-2}\,dx = -\int_1^{1/2} e^u\,du$$

$*$ change limits of integration

$$= \int_{1/2}^{1} e^u\,du = e^u\Big|_{1/2}^{1} = e-e^{1/2}$$

$$= \boxed{e-\sqrt{e}}$$

5) Find the price-demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at \$2.35 per tube, given that the marginal price – demand function, $p'(x)$, for x number of tubes per week, is given as $p'(x) = -0.015e^{-0.01x}$. If the supermarket chain sells 100 tubes per week, what price should it set?

$$p(x) = \int -0.015e^{-0.01x}\,dx = -0.015\int e^{-0.01x}\,dx = 1.5\int e^u\,du = 1.5e^u + C$$

let $u=-0.01x$

$du=-0.01\,dx$

$$= 1.5e^{-0.01x}+C$$ ⊛ need to find C

we know $p(50)=2.35$

so $2.35 = 1.5e^{-0.01(50)}+C$

$C = 2.35-1.5e^{-0.01(50)}$

$C \simeq 1.44$

so, $p(x) = 1.5e^{-0.01x}+1.44$

If the supermarket sells 100 tubes of tooth paste per week, the price would be

$p(100) = 1.5e^{-0.01(100)}+1.44 \approx 1.99$

The supermarket could charge \$1.99 per tube if it is selling 100 tubes per week.

6) Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in a Petri dish at any time t. How many bacteria are in the dish after 2 hours?

$$Q(t) = \int 3^t \, dt = \frac{3^t}{\ln 3} + C$$

at $t=0$, $Q(0) = 10 = \frac{1}{\ln 3} + C$

so $C \approx 9.090$

so $Q(t) = \frac{3^t}{\ln 3} + 9.090$

at $t=2$, $Q(2) = \frac{3^2}{\ln 3} + 9.090$

≈ 17.282

$\boxed{\text{After 2 hours, there are 17,282 bacteria}}$

Integrals Involving Logarithmic Functions

The following formulas can be used to evaluate integrals involving logarithmic functions.

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int \ln x\,dx = x\ln x - x + C = x(\ln x - 1) + C$$

$$\int \log_a x = \frac{x}{\ln a}(\ln x - 1) + C$$

Media: Watch these video1, video 2 and video 3 examples on integrals involving logarithms.

Examples

1) Find the antiderivative of each of the following.

a. $\frac{3}{x-10}$

$\int \frac{3}{x-10} = 3\int \frac{1}{x-10}\,dx$

$u = x-10$
$du = dx$

$\rightarrow$ use u^{-1} rule $\rightarrow$ $= 3\int \frac{du}{u} = 3\ln|u| + C$

$= \boxed{3\ln|x-10| + C}$

b. $\frac{2x^3+3x}{x^4+3x^2}$

$\int \frac{2x^3+3x}{x^4+3x^2}\,dx = \int (2x^3+3x)(x^4+3x^2)^{-1}\,dx$

let $u = x^4+3x^2$
$du = 4x^3+6x\,dx$
$\frac{1}{2}du = 2x^3+3x\,dx$

$= \frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\ln|u| + C$

$= \boxed{\frac{1}{2}\ln|x^4+3x^2| + C}$

c. $\log_2 x$

use formula above $\rightarrow$ $\int \log_2 x\,dx = \boxed{\frac{x}{\ln 2}(\ln x - 1) + C}$

2) Find the definite integral of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\,dx$.

let $u = 1+\cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

when $x=0,\ u=2$
$x=\pi/2,\ u=1$

$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx = -\int_2^1 u^{-1}\,du = \int_1^2 u^{-1}\,du$

$= \ln|u|\Big|_1^2 = \ln 2 - \ln 1 = \boxed{\ln 2}$