Section 3.7: Derivatives of Inverse Functions

This section will explore the relationship between the derivative of a function and the derivative of its inverse.

The Derivative of an Inverse Function

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x)) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Media: Learn more about the derivatives of inverse functions here.

Media: Watch these video1 and video 2 examples on the derivative of inverse functions.

Examples

1) Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$

$f'(x) = \frac{(x-1)(0)-2(1)}{(x-1)^2}$ and $f'\left(g(x)\right) = -\frac{2}{(g(x)-1)^2} = -\frac{2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}$

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{-\frac{x^2}{2}} = \frac{2}{-x^2}$$

You should verify by applying the quotient rule to $g(x)$.

2) Find the derivative of $g(x) = \sqrt[3]{x}$ by using the inverse function theorem.

Inverse of $g(x) = \sqrt[3]{x}$ is $f(x) = x^3$

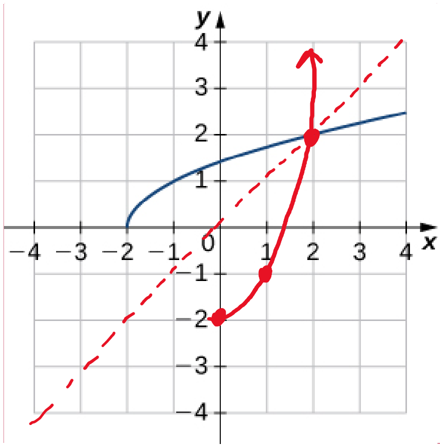
$f'(x) = 3x^2$ and $f;\left(g(x)\right) = 3\left(\sqrt[3]{x}\right)^2 = 3x^{\frac{2}{3}}$

$$g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3}x^{-\frac{2}{3}}$$

3) Use the graph of $y = f(x)$ to

a. Sketch the graph of $y = f^{-1}(x)$, and (**reflects across the y-axis)

b. Use part a. to estimate $(f^{-1})'(1)$. (equals -1)



Commented [TD1]: I don't know how to do this part....

Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.$$

Media: Watch this video example on derivatives of functions with rational exponents.

Example: Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Find slope first: $\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$

At $x = 8$:

The slope is: $\frac{2}{3}(8)^{-\frac{1}{3}} = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}$

To find the point: $y = (8)^{\frac{2}{3}} = 4$, which gives (8.4)

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{3}(x - 8)$$

$$y - 4 = \frac{1}{3}x - \frac{8}{3}$$

$$y = \frac{1}{3}x + \frac{4}{3}$$

Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-(x)^2}}$
$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-(x)^2}}$
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+(x)^2}$
$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+(x)^2}$
$\frac{d}{dx}\sec^{-1} x = \frac{1}{\lvert x\rvert\sqrt{(x)^2-1}}$
$\frac{d}{dx}\csc^{-1} x = \frac{-1}{\lvert x\rvert\sqrt{(x)^2-1}}$

Media: Watch these video1 and video2 examples on derivatives of inverse trig functions.

Examples

1) Find the derivative of the following functions:

a. $f(x) = \tan^{-1}(x^2)$

$$f'(x) = \frac{1}{1+(x^2)^2} \cdot (2x)$$

$$f'(x) = \frac{2x}{1+x^4}$$

b. $y = \sec^{-1}\left(\frac{1}{x}\right)$

$$y' = \frac{1}{\left(\frac{1}{x}\right)\left(\sqrt{\left(\frac{1}{x}\right)^2-1}\right)} \cdot -x^{-2}$$

$$y' = \frac{1}{\left(\frac{1}{x}\right)\left(\sqrt{\frac{1}{x^2}-1}\right)} \cdot -\frac{1}{x^2}$$

$$y' = \frac{-1}{x\left(\sqrt{\frac{1}{x^2}}-1\right)} = -\frac{1}{\sqrt{1-x^2}}$$

c. $y = \cot^{-1}\sqrt{4-x^2}$

$$y' = -\frac{1}{1+\left(\sqrt{4-x^2}\right)^2} \cdot \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$y' = -\frac{1}{1+(4-x^2)} \cdot -\frac{x}{\sqrt{4-x^2}}$$

$$y' = -\frac{1}{5-x^2} \cdot -\frac{x}{(5-x^2)\left(\sqrt{4-x^2}\right)}$$

2) The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

$$s'(t) = v(t) = \frac{1}{1+\left(\frac{1}{t}\right)^2} \cdot -t^{-2}$$

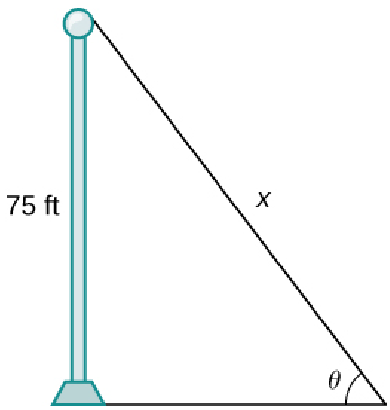
$$s'(t) = \frac{1}{1+\frac{1}{t^2}} \cdot -\frac{1}{t^2} = -\frac{1}{(t^2+1)}$$

$$v(1) = -\frac{1}{(1)^2+1} = -\frac{1}{2}$$

The velocity of the particle at time $t = 1$ is $-\frac{1}{2}$.

Media: Watch this video example on applications of derivatives of inverse trig functions.

3) A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



$$\sin\theta = \frac{75}{x} \text{ when } x = 90$$

$$\sin^{-1}\left(\frac{75}{x}\right) = \theta$$

$$\text{So, } \theta = \sin^{-1}(75x^{-1})$$

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-(75x^{-1})^2}} \cdot -75x^{-2}$$

At $x = 90$:

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1 - (75x(90)^{-1})^2}} \cdot -75(90)^{-2}$$

$$\frac{d\theta}{dx} = -0.02\,\frac{radians}{ft}$$