Section 4.4: The Mean Value Theorem

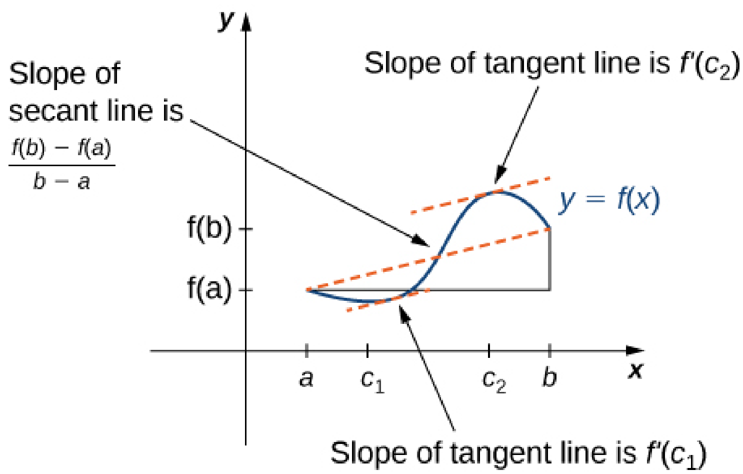
The Mean Value Theorem is one of the most important theorems in calculus. The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

The Mean Value Theorem and Its Meaning

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Media: Watch this video to learn more about the Mean Value Theorem.

Media: Watch this video example on the Mean Value Theorem of a quadratic function.

Media: Watch this video example on the Mean Value Theorem of a rational function.

Examples

1) For $f(x) = \sqrt{x}$ over the interval $[0,9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0,9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find the values c guaranteed by the Mean Value Theorem.

   Since $f(x) = \sqrt{x}$ is continuous over [0, 9] and differentiable over (0, 9), f satisfies the hypotheses of the Mean Value Theorem.

   So $f(x) = x^{\frac{1}{2}}$ and $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

   The slope connecting (0, f(0)) and (9, f(9)) is $\frac{f(9)-f(0)}{9-0} = \frac{1}{3}$
   Now we find c such that $f'(c) = \frac{1}{3}$

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}$$
$$2\sqrt{c} = 3$$
$$\sqrt{c} = \frac{3}{2}$$
$$c = \frac{9}{4}$$

2) If a rock is dropped from a height of $100\ ft$, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.
   a. Determine how long it takes before the rock hits the ground.

      Note that "hits the ground" is when s(t) = 0, so we solve
      $$-16t^2 = -100$$
      $$t^2 = 6.25$$
      $$t = \pm 2.5$$
      The ball will hit the ground 2.5 seconds after it is dropped.

   b. Find the average velocity v_{avg} of the rock for when the rock is released, and the rock hits the ground.

      Note that "released" is when t = 0 and it hits the ground when t = 2.5

      So $v_{avg} = \frac{s(2.5)-s(0)}{2.5-0}$= -40 ft/sec

   c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}

Instantaneous velocity = *s'(t)*
We need to find a t such that *v(t) = s'(t) =* v_{avg} = -40 *ft/sec*
Since s(t) is continuous over the interval [0,2.5] and differentiable over (0, 2.5), the Mean Value Theorem guarantees a point $c \in (0, 2.5)$ such that $s'(c) = \frac{s(2.5)-s(0)}{2.5-0} = -49 = 0$

We have that $s'(t) = -32t$, so $-32c = -40$ and $c = 1.25$
That is, 1.25 seconds after the rock is dropped the instantaneous velocity is the same as the average