Section 5.2: The Definite Integral

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

> If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by
>
> $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$
>
> Provided the limit exits. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an **integrable function**.

Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions.

On the definite integral, above and below the summation symbol are the boundaries of the interval $[a, b]$. The numbers a and b are x −values and are called the limits of integration; specifically, a is the lower limit and b is the upper limit.

> **Continuous Functions Are Integrable**
>
> If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Media: Watch this video example on using Reimann sum to evaluate a definite integral.

Media: Watch this video to understand the definition of the definite integral.

Example: Use the definition of the definite integral to evaluate $\int_0^2 x^2\,dx$. Use a right-endpoint approximation to generate the Riemann sum.

$a = 0, b = 2$

For $i = 1, 2, 3, \ldots., n$

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Since using right-endpoint approximation to generate Riemann sum, we need to calculate the function value at the right endpoint of the interval $[x_{i-1}, x_i]$.

$$x_i = x_0 + i\Delta x = 0 + i\left[\frac{2}{n}\right] = \frac{2i}{n}$$

$$f(x_i) = {x_i}^2 = \left(\frac{2i}{n}\right)^2 = \frac{4i^2}{n^2}$$

So,

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{8i^2}{n^3} = \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2\right)$$

Note that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, so

$$\sum_{i=1}^{n} f(x_i)\Delta x = \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{16n^3 + 24n^2 + 8n}{6n^3} = \frac{8}{3} + \frac{4}{n} + \frac{8}{6n^2}$$

So,

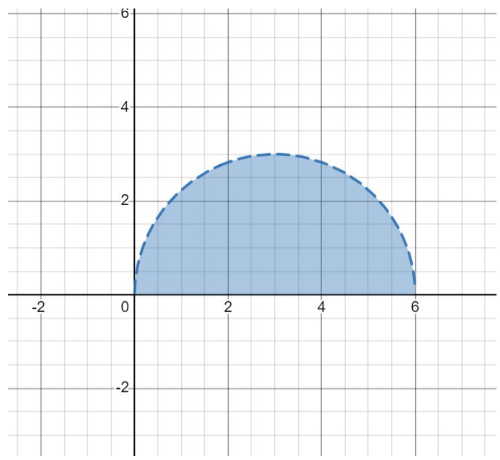
$$\int_0^2 x^2 dx = \lim_{n\to\infty}\left(\frac{8}{3} + \frac{4}{n} + \frac{8}{6n^2}\right) = \frac{8}{3} + 0 + 0 = \frac{8}{3}$$

Evaluating Definite Integrals

Evaluating definite integrals with Riemann sums can be tedious due to the complexities of the calculations (evaluating definite integrals without taking limits of Riemann sums will be shown later). For now, know that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area.

Media: Watch this video example on using shapes to find area.

Example: Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.



Note that this graph is a semicircle with a radius of 3 and you want to find the area of this semicircle from when $x = 3$ to when $x = 6$ (which is ¼ of the circle). Since the area of a circle is πr^2 and we want to find the area of ¼ of the circle, then

$$\int_3^6 \sqrt{9-(x-3)^2}dx = \frac{1}{4}\pi(3)^2 = \frac{9\pi}{4} \approx 7.069$$

Area and the Definite Integral

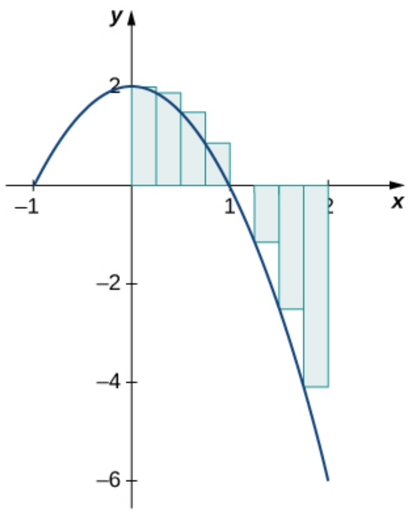
When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

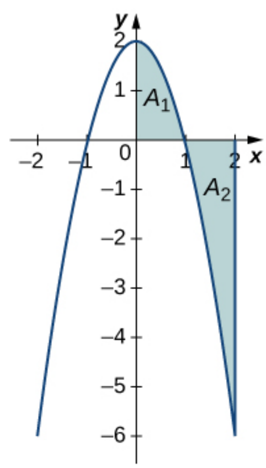
Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x —axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x —axis that lies below the axis. Then, the **net signed area** between $f(x)$ and the x —axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2$$

Consider the example below. For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x —axis.





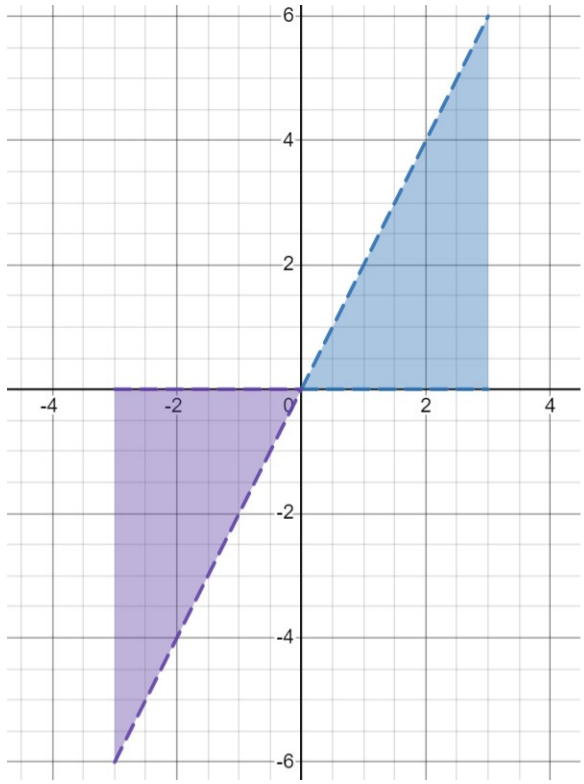
In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Notice that the net area can be positive, negative, or zero. If the area above the x —axis is larger, the net signed area is positive. If the area below the x —axis is larger, the net signed area is negative. If the areas above and below the x —axis are equal, the net signed area is zero.

Media: Watch this video example on net signed area.

Examples

1) Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3,3]$.



Notice that you are looking for the area of two different triangles. To find the area of a triangle, we can use the formula $A = \frac{1}{2}bh$. So,

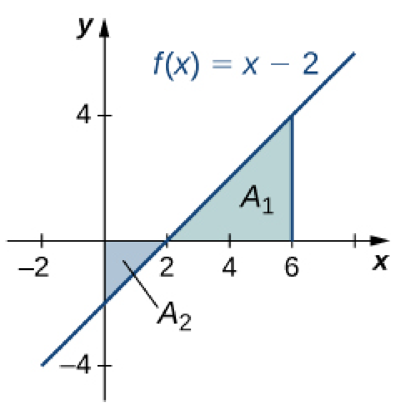
$A_1 = \frac{1}{2}(3)(6) = 9$ (where A_1 represents the shaded region in the interval $[0,3]$)
$A_2 = \frac{1}{2}(3)(6) = 9$ (where A_2 represents the shaded region in the interval $[-3,0]$)

So,

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0$$

Thus, the net area is 0 (since the area above the x-axis is equal to the area below the x-axis)

2) Find the net signed area of $f(x) = x - 2$ over the interval $[0,6]$, illustrated in the image below.



Recall that the area of a triangle can be represented by $A = \frac{1}{2}bh$.

$A_1 = \frac{1}{2}(4)(4) = 8$ (where A_1 represents the shaded region in the interval [2,6])
$A_2 = \frac{1}{2}(2)(2) = 2$ (where A_2 represents the shaded region in the interval [0,2])

So,

$$\int_0^6 (x-2)dx = A_1 - A_2 = 8 - 2 = 6$$

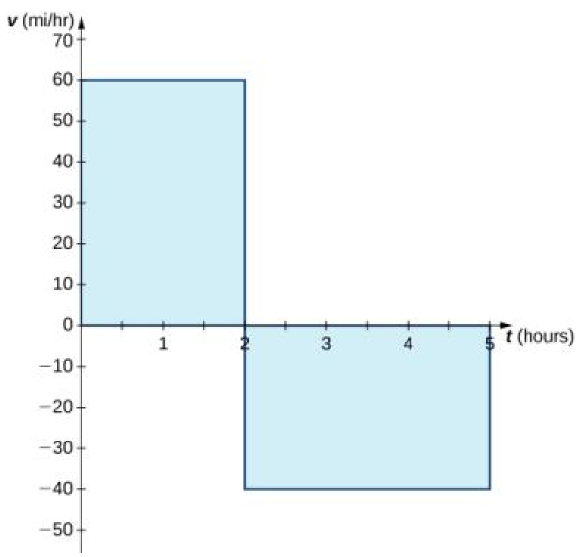
Total Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x −axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x −axis that lies below the axis. Then, the **total area** between $f(x)$ and the x −axis is given by

$$\int_a^b |f(x)|\, dx = A_1 + A_2$$

One application of the definite integral is finding displacement when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position.

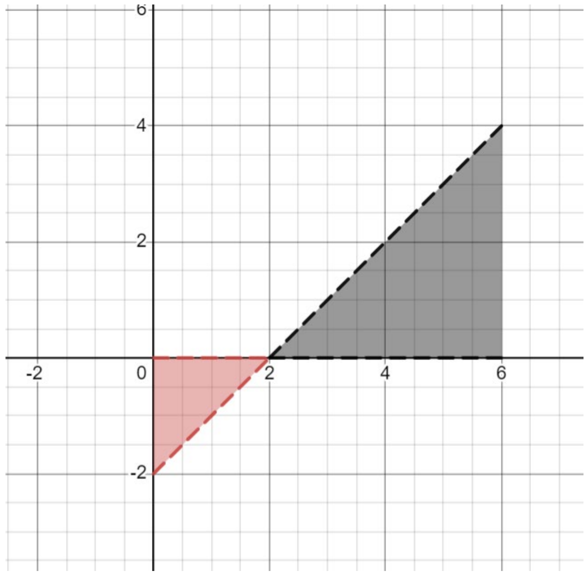
When velocity is constant, the area under the curve is just velocity times time. In the context of displacement, net signed area allows us to take direction into account. For example, if a car travels straight north at a speed of 60 mph for 2 hours, it is 120 mi north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at its starting position (a displacement of zero).



However, if we want to know how far the car travels overall (regardless of direction), we want to know the area between the curve and the x −axis, regardless of whether that area is above or below the axis (i.e. total area).

Media: Watch this video example on finding total area.

Example: Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0,6]$.



To find the total area, find the area of each triangle and then add them together.

Recall from previous example:

$A_1 = \frac{1}{2}(4)(4) = 8$ (where A_1 represents the shaded region in the interval [2,6])
$A_2 = \frac{1}{2}(2)(2) = 2$ (where A_2 represents the shaded region in the interval [0,2])

So,

$$\int_0^6 |(x-2)|\, dx = A_2 + A_1 = 2 + 8 = 10$$

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These help manipulate expressions to evaluate definite integrals.

Properties of the Definite Integral

1. If the limits of integration are the same, the integral is just a line and contains no area.

$$\int_a^a f(x)\,dx = 0$$

2. If the limits are reversed, then place a negative sign in front of the integral.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

3. The integral of a sum is the sum of integrals.

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

4. The integral of a difference is the difference of the integrals.

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

5. For constant c, the integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

6. Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Media: Watch this video example on properties of definite integrals.

Examples

1) Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2,1]$ as the sum of three definite integrals.

$$\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx = \int_{-2}^{1} -3x^3 dx + \int_{-2}^{1} 2x\,dx + \int_{-2}^{1} 2\,dx$$

$$= -3\int_{-2}^{1} x^3 dx + 2\int_{-2}^{1} x\,dx + \int_{-2}^{1} 2\,dx$$

2) Suppose that $\int_0^4 f(x)\,dx = 5$ and $\int_0^2 f(x)\,dx = -3$, and $\int_0^4 g(x)\,dx = -1$ and $\int_0^2 g(x)\,dx = 2$. Compute the following integrals:

a. $\int_2^4 (f(x) + g(x))\,dx$

Note: $\int_0^4 f(x)dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx$

So $\int_2^4 f(x)dx = 5 - (-3) = 8$

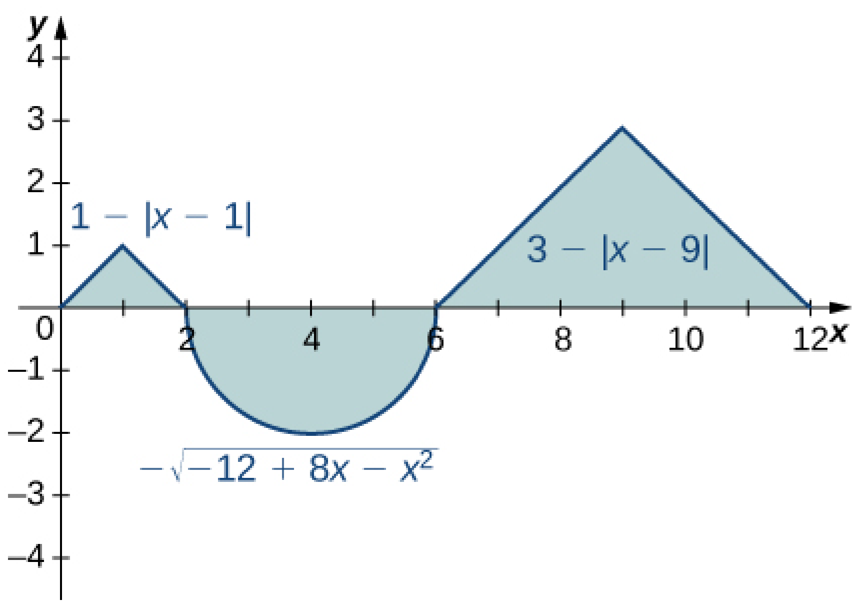
Similarly, for $\int_2^4 g(x)dx$

$$\int_2^4 (f(x) + g(x))\,dx = \int_2^4 f(x)dx + \int_2^4 g(x)dx$$
$$= 8 - 3 = 5$$

b. $\int_0^2 (3f(x) - 4g(x))\,dx$

$$= \int_0^2 3f(x)dx - \int_0^2 4g(x)dx$$
$$= 3\int_0^2 f(x)dx - 4\int_0^2 g(x)dx$$
$$= 3(-3) - 4(2) = -9 - 8 = -17$$

3) Evaluate the integrals of the function graphed using the formulas for areas of triangles and circles, and subtracting the areas below the x-axis.



$A_1 = \frac{1}{2}(2)(1) = 1$ (where A_1 represents the shaded region in the interval [0,2])

$A_2 = \frac{1}{2}(\pi)(2)^2 = 2\pi$ (where A_2 represents the shaded region in the interval [2,6])

$A_3 = \frac{1}{2}(6)(3) = 9$ (where A_3 represents the shaded region in the interval [6,12])

So, the total area can be represented as

$$A_1 - A_2 + A_3$$
$$= 1 - 2\pi + 9$$
$$= 10 - 2\pi$$

Comparison Properties of Integrals

Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x −axis is greater than the area between $g(x)$ and the x −axis. This is true depending on the interval over which the comparison is made.

Comparison Theorem

If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0$$

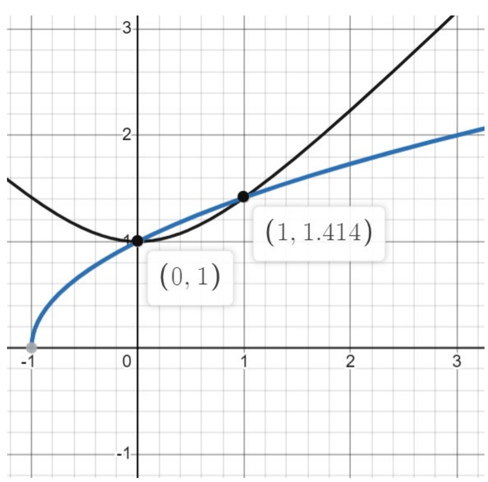
If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx$$
$$\leq M(b-a)$$

Example: Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0,1]$.



$g(x)$ is above $f(x)$ in the interval $[0, 1]$ and intersect at $x = 0$ and $x = 1$.

$$g(x) \geq f(x)$$

$$\int_0^1 g(x)dx \geq \int_0^1 f(x)\, dx$$

Average Value of a Function

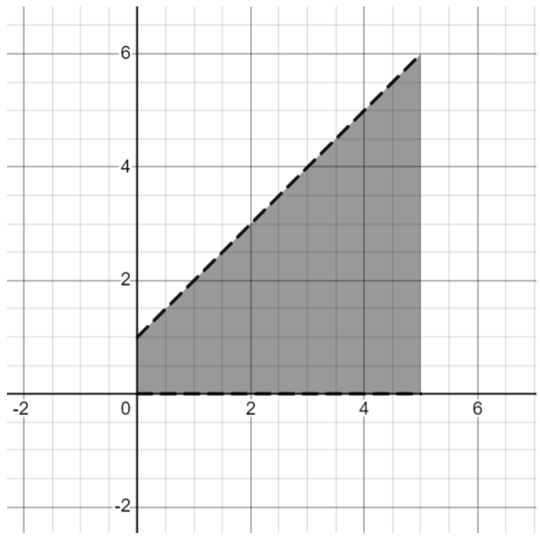
We often need to find the average of a set of numbers, such as an average test grade. However, if a function takes on an infinite number of values, we can't use the process we normally use to find the average.

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the **average value of the function** $f(x)$ $(or\ f_{ave})$ on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Media: Watch this video example on average value.

Example: Find the average value of $f(x) = x + 1$ over the interval $[0,5]$.



Notice that the shape is that of a trapezoid (on its side). So, the area of a trapezoid is

$$A = \frac{1}{2}h(b_1 + b_2)$$

So,

$$\int_0^5 (x+1)dx = \frac{1}{2}h(b_1 + b_2)$$

$$= \frac{1}{2}(5)(1+6)$$

$$= \frac{35}{2}$$

So, the average value of the function is

$$\frac{1}{5-0}\int_0^5 (x+1)dx = \frac{1}{5}\cdot\frac{35}{2}$$

$$= \frac{7}{2}$$