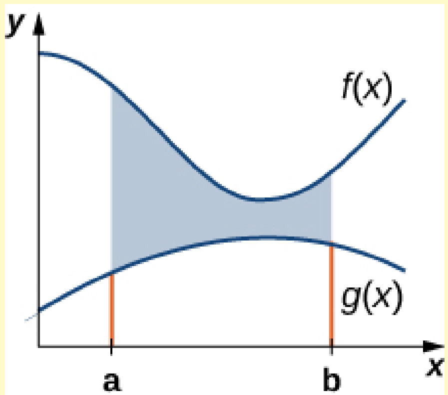
Section 6.1: Areas Between Curves

We have seen how to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions.

Area of a Region between Two Curves

Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$.



Let R denote the region bounded above the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

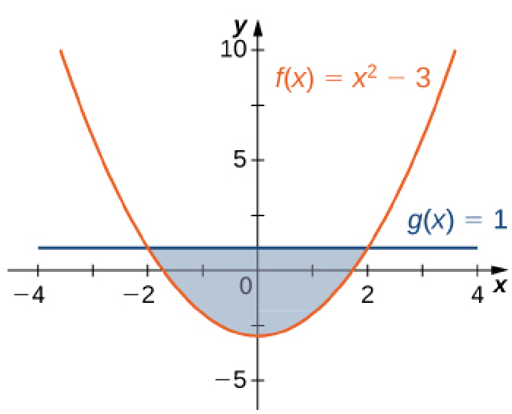
$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Media: Watch this video to learn more about how to find the area between two curves.

Media: Watch these video1 and video2 examples on finding areas between curves.

Examples

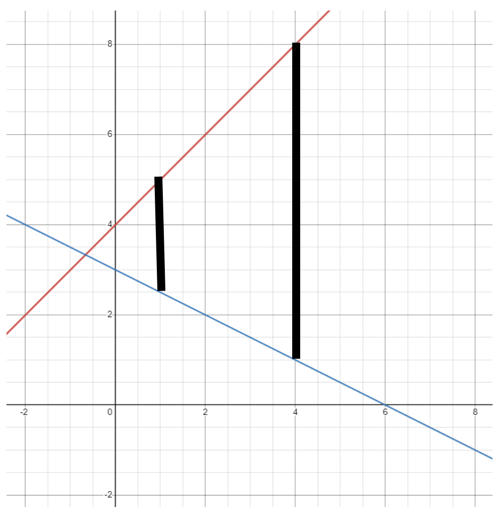
1) Determine the area of the region between the two curves in the given figure by integrating over the x-axis.



The graphs intersect at $x = -2$ and $x = 2$, so integrate from -2 to 2. We also know that $g(x) \geq f(x)$ since the graph of $g(x)$ is above the graph of $f(x)$ within that interval.

$$\begin{aligned} A &= \int_{-2}^{2} [g(x) - f(x)]dx \\ &= \int_{-2}^{2} [1 - (x^2 - 3)]dx \\ &= \int_{-2}^{2} [-x^2 + 4]dx \\ &= -\frac{x^3}{3} + 4x\Big|_{-2}^{2} = \frac{32}{3} \end{aligned}$$

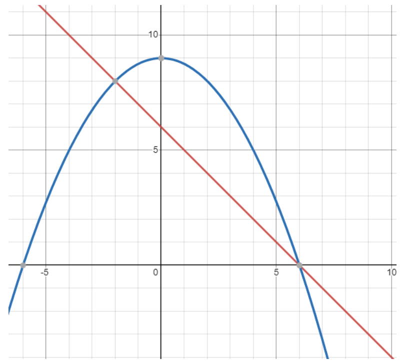
2) If R is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1,4]$, find the area of region R.



Notice in the graph that $f(x) \geq g(x)$.

$$\begin{aligned} A &= \int_{a}^{b} [f(x) - g(x)]dx \\ &= \int_{1}^{4} \left[(x + 4) - \left(3 - \frac{x}{2}\right)\right] dx \\ &= \int_{1}^{4} \left[\frac{3x^2}{2} + 1\right] dx \\ &= \frac{3x^2}{4} + x\Big|_{1}^{4} = \frac{57}{4} \end{aligned}$$

3) If R is the region bounded above by the graph of the function $f(x) = 9 - \left(\frac{x}{2}\right)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.



Find the points of intersection:

$$9 - \left(\frac{x}{2}\right)^2 = 6 - x$$
$$9 - \frac{x^2}{4} = 6 - x$$
$$36 - x^2 = 24 - 4x$$
$$x^2 - 4x - 12 = 0$$
$$(x - 6)(x + 2) = 0$$
$$x = 6, -2$$

Since we know the points of intersection, we now know that we should integrate from -2 to 6. So,

$$A = \int_{-2}^{6} [f(x) - g(x)]dx$$
$$= \int_{-2}^{6} \left[\left(9 - \left(\frac{x}{2}\right)^2 - (6 - x)\right)\right] dx$$
$$= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right] dx$$
$$= 3x - \frac{x^3}{12} + \frac{x^2}{2}\Bigg|_{-2}^{6}$$
$$= \frac{64}{3}$$

Areas of Compound Regions

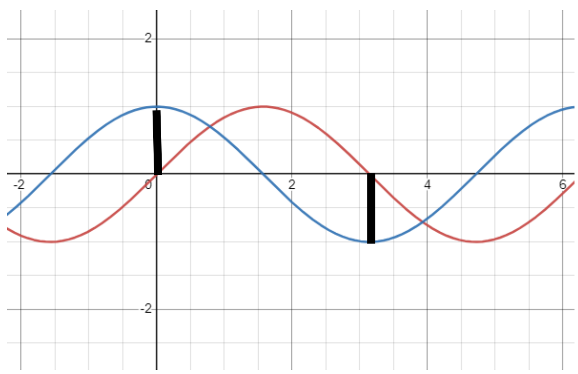
So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but often times the regions of interest are not simple.

> **Finding the Area of a Region between Curves That Cross**
>
> Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by
>
> $$A = \int_a^b |f(x) - g(x)|\, dx.$$

Media: Watch this video example on finding areas with multiple regions.

Examples

1) If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R.



Note that there are two regions you will need to find the area for.

For $\left[0, \frac{\pi}{4}\right]$: $|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x$

For $\left[\frac{\pi}{4}, \pi\right]$: $|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x$

So,

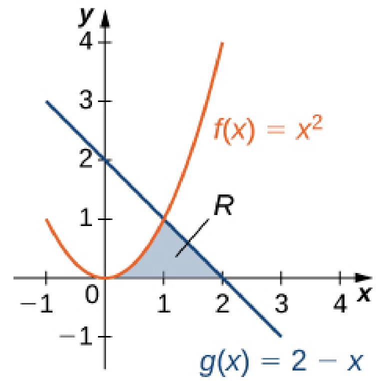
$$A = \int_0^{\pi} |\sin x - \cos x| dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x) dx$$

$$= (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\pi}$$

$$2\sqrt{2}\ units^2$$

2) Consider the region shown below. Find the area of R.



Note that the two graphs intersect at $x = 1$ and that there are two intervals to find the area of: $[0, 1]$ and $[1,2]$.

Let A_1 represent the area under the curve in the interval $[0,1]$

Let A_2 represent the area under the curve in the interval $[1,2]$

So,

$$A_1 = \int_0^1 x^2 \, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$A_2 = \int_1^2 (2 - x) dx = \left.\left(2x - \frac{x^2}{2}\right)\right|_1^2 = \frac{1}{2}$$

So, the total area $A = A_1 + A_2$

$$A = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \; units^2$$

Regions Defined with Respect to y

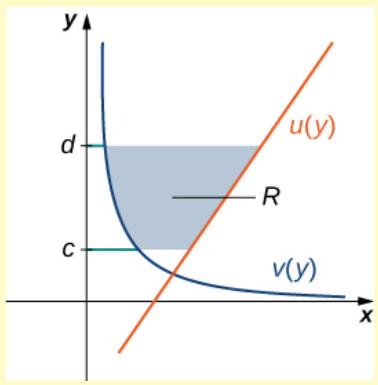
We can also find the area between two graphs with respect to y. Sometimes this method is easier to evaluate rather than evaluating multiple integrals to calculate the area of a region.

Media: Watch this video to help you determine when to find the area with respect to

x or to y .

Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively.

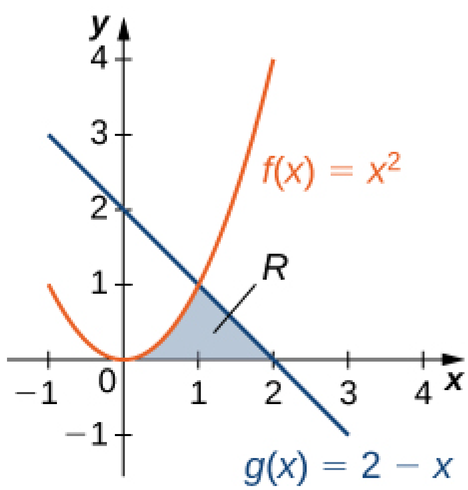


Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)]\, dy.$$

Media: Watch this video example on finding areas with respect to y.

Example: Consider the region shown below. Integrate with respect to y to find the area of R.



First, express the graphs in respect to y:

$$f(x) = x^2 \ \rightarrow x = \sqrt{y}$$

$$g(x) = 2 - x \ \rightarrow x = 2 - y$$

Notice that the limits of integration also change. If we are looking at it with respect to y, we need to look at the starting and ending y-values. From the graph, we can see that the shaded area starts at $y = 0$ and ends at $y = 1$.

So,

$$A = \int_0^1 \left[(2 - y) - \left(\sqrt{y}\right)\right] dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{2}{3} y^{\frac{3}{2}}\right]\Bigg|_0^1$$

$$= \frac{5}{6} \ units^2$$