Section 3.9: Derivatives of Exponential and Logarithmic Functions

Derivative of the Exponential Function

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'(x).$$

Media: Watch these video1 and video2 examples on the derivative of an exponential function.

Examples

1) Find the derivative of each of the following:
   a. $f(x) = e^{\tan(2x)}$

$f'(x) = e^{\tan(2x)} \cdot \sec^2(2x)(2)$

$\boxed{f'(x) = 2\sec^2(2x)e^{\tan(2x)}}$

   b. $y = \frac{e^{x^2}}{x}$

$y' = \frac{x(e^{x^2})(2x) - e^{x^2}(1)}{x^2}$

$\boxed{y' = \frac{e^{x^2}(2x^2-1)}{x^2}}$

2) A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

$A'(t) = 1000e^{0.3t}(0.3) = 300e^{0.3t}$

ratio of $A'(t)$ to $A(t) \Rightarrow \frac{300e^{0.3t}}{1000e^{0.3t}} = \boxed{0.3}$

Thus the ratio of the rate of change of the population to the population is constant

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

> **The Derivative of the Natural Logarithmic Function**
>
> If $x > 0$ and $y = \ln x$, then
>
> $$\frac{dy}{dx} = \frac{1}{x}$$
>
> More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by
>
> $$h'(x) = \frac{1}{g(x)} g'(x)$$

Media: Watch these video1 and video2 examples on the derivative of logarithmic functions.

Examples: Find the derivative of each of the following:

1) $f(x) = \ln(x^3 + 3x - 4)$

$$f'(x) = \frac{1}{x^3+3x-4} \cdot (3x^2+3)$$

$$\boxed{f'(x) = \frac{3x^2+3}{x^3+3x-4}}$$

2) $f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right)$

$$f(x) = 2\ln x + \ln(\sin x) - \ln(2x+1)$$

$$f'(x) = \frac{2}{x} + \frac{1}{\sin x} \cdot \cos x - \frac{1}{2x+1} \cdot 2$$

$$\boxed{f'(x) = \frac{2}{x} + \cot x - \frac{2}{2x+1}}$$

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0, b \neq 1$.

> **Derivatives of General Exponential and Logarithmic Functions**
>
> Let $b > 0, b \neq 1$, and let $g(x)$ be a differentiable function.
>
> i. If, $y = \log_b x$, then
>
> $$\frac{dy}{dx} = \frac{1}{x \ln b}.$$
>
> ii. If $y = b^x$, then
>
> $$\frac{dy}{dx} = b^x \ln b.$$

Media: Watch these video1 and video2 examples on derivatives of exponentials and logs with other bases.

Examples

1) Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

$$\frac{dy}{dx} = \frac{1}{(3x+1)\ln 2} \cdot 3 = \frac{3}{(3x+1)\ln 2} \leftarrow \text{slope}$$

$$\text{at } x = 1: \quad \frac{3}{(3\cdot 1+1)\ln 2} = \frac{3}{4\ln 2} = \boxed{\frac{3}{\ln 16}}$$

2) Find the derivative of $h(x) = \frac{3^x}{3^x+2}$.

$$h'(x) = \frac{(3^x+2)(3^x \ln 3) - 3^x(3^x \ln 3)}{(3^x+2)^2}$$

$$\boxed{h'(x) = \frac{2 \cdot 3^x \ln 3}{(3^x+2)^2}}$$

Logarithmic Differentiation

Logarithmic differentiation allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one.

How to Solve Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Media: Watch these video1 and video2 examples on logarithmic differentiation.

Examples: Find the derivative of each of the following.

1) $y = (2x^4+1)^{\tan x}$

$$\ln y = \ln(2x^4+1)^{\tan x}$$

$$\ln y = \tan x \ln(2x^4+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \tan x\left(\frac{1}{2x^4+1}\right)(8x) + \ln(2x^4+1)(\sec^2 x)$$

$$\frac{dy}{dx} = y \cdot \left[\tan x\left(\frac{8x}{2x^4+1}\right) + \sec^2 x \ln(2x^4+1)\right]$$

$$\boxed{\frac{dy}{dx} = (2x^4+1)^{\tan x}\left[\tan x\left(\frac{8x}{2x^4+1}\right) + \sec^2 x \ln(2x^4+1)\right]}$$

2) $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$

$$\ln y = \ln\left(\frac{x\sqrt{2x+1}}{e^x \sin^3 x}\right)$$

$$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x\ln e - 3\ln(\sin x)$$

$$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x - 3\ln(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\cdot\left(\frac{1}{2x+1}\right)(2) - 1 - 3\left(\frac{1}{\sin x}\right)\cos x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$$

$$\boxed{\frac{dy}{dx} = \left(\frac{x\sqrt{2x+1}}{e^x \sin^3 x}\right)\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)}$$