Section 4.1: Related Rates

For quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related.

Setting up Related-Rates Problems

How to Solve a Related Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if possible.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in Step 1.
4. Using the chain rule, differentiate both sides of the equation found in Step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from Step 4, then solve for the unknown rate of change.

Note: when solving a related-rates problem, it is crucial not to substitute known values too soon.

Media: Watch this video example on a growing sphere.

Media: Watch this video example on a falling ladder.

Media: Watch this video example on an increasing cone.

Examples

1) Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.

$$\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(1)(4) = \boxed{8}$$

2) A spherical balloon is being filled with air at the constant rate of 2 $\frac{\text{cm}^3}{\text{sec}}$. How fast is the radius increasing when the radius is 3 cm?

Volume of sphere $= \frac{4}{3}\pi r^3$

$V(t) = 2 \text{ cm}^3/\text{sec}$

$r = 3\text{cm} \qquad \frac{dr}{dt} = ?$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

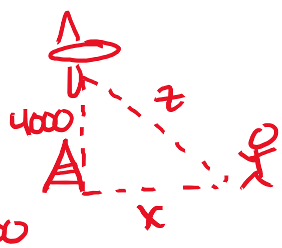
$$2 = 4\pi(3)^2\frac{dr}{dt}$$

$$2 = 36\pi\frac{dr}{dt}$$

$$\frac{2}{36\pi} = \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{18\pi} \text{ cm/sec}}$$

3) An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 $\frac{\text{ft}}{\text{sec}}$, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?



at $x = 3000$

$$x^2 + (4000)^2 = z^2$$

$$(3000)^2 + (4000)^2 = z^2$$

$$25{,}000{,}000 = z^2$$

$$z = 5000$$

$$x^2 + (4000)^2 = z^2$$

$$x^2 + 16{,}000{,}000 = z^2$$

$$2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$2(3000)(600) = 2(5000)\frac{dz}{dt}$$

$$3{,}600{,}000 = 10{,}000\frac{dz}{dt}$$

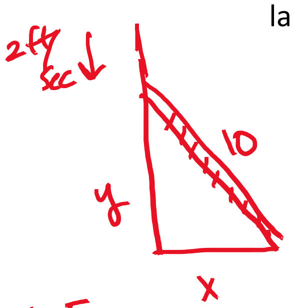
$$\boxed{\frac{dz}{dt} = 360 \text{ ft/sec}}$$

$x = 3000$

$\frac{dx}{dt} = 600 \text{ ft/sec}$

$\frac{dz}{dt} = ?$

4) A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 $\frac{\text{ft}}{\text{sec}}$, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



$$\frac{dy}{dt} = -2 \text{ ft/sec}$$

$$x = 5$$

$$\frac{dx}{dt} = ?$$

$$x^2 + y^2 = 10^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

at $x = 5$

$$x^2 + y^2 = 10^2$$

$$5^2 + y^2 = 10^2$$

$$25 + y^2 = 100$$

$$y^2 = 75$$

$$y = \sqrt{75}$$

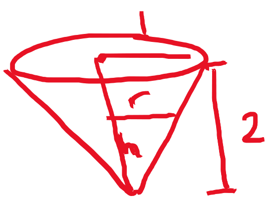
$$2(5)\frac{dx}{dt} + 2(\sqrt{75})(-2) = 0$$

$$10\frac{dx}{dt} - 4\sqrt{75} = 0$$

$$10\frac{dx}{dt} = 4\sqrt{75}$$

$$\frac{dx}{dt} = \frac{4\sqrt{75}}{10} = \frac{20\sqrt{3}}{10} = \boxed{2\sqrt{3} \text{ ft/sec}}$$

5) Water is draining from the bottom of a cone-shaped funnel at the rate of 0.03 $\frac{\text{ft}^3}{\text{sec}}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?



$$\frac{dV}{dt} = -0.03 \text{ ft}^3/\text{sec}$$

$$h = \frac{1}{2}$$

$$\frac{dh}{dt} = ?$$

$$\frac{r}{h} = \frac{1}{2}$$

$$2r = h$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h^3}{4}\right)$$

$$V = \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt}$$

$$-0.03 = \frac{\pi}{12}\left(3\left(\frac{1}{2}\right)^2\right)\cdot\frac{dh}{dt}$$

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} \approx -0.153 \text{ ft/sec}}$$