Section 4.7: Applied Optimization Problems

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume.

Solving Optimization Problems

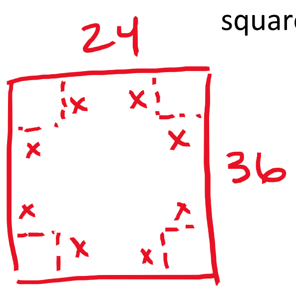
Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from Step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in Step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from Step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Media: Watch this video example on maximizing volume of a box.

Examples:

1) An open-top box is to be made from a 24 in by 36 in piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?



let x = side length of square to be removed

$V = l \cdot w \cdot h$

$V = (36-2x)(24-2x)(x) = 4x^3 - 120x^2 + 864x$

$V' = 12x^2 - 240x + 864 = 0$

$x^2 - 20x + 72 = 0$

$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2(1)} = \frac{20 \pm \sqrt{112}}{2} = 10 \pm 2\sqrt{7}$

Domain: $[0, 12]$

$10 + 2\sqrt{7}$ not in domain, so check $10 - 2\sqrt{7}$ ← max

$V(0) = 0$

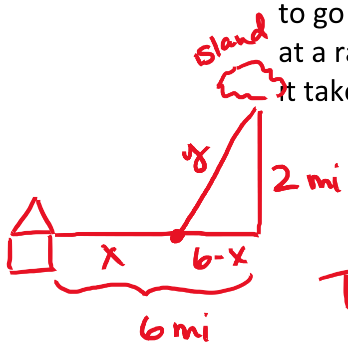
$V(12) = 0$

$V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx \boxed{1825 \text{ in}^3 \leftarrow \text{max volume}}$

Media: Watch this video example on a minimizing total time.

Media: Watch this video example on a maximizing profit.

2) An island is 2 miles due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 miles west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?



Let x = distance running

y = distance swimming

(minimize T)

T = time it takes to get to island

$D = RT$

$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8}$ $\quad T_{swimming} = \frac{D_{swim}}{R_{swim}} = \frac{y}{3}$

$2^2 + (6-x)^2 = y^2$

$y = \sqrt{(6-x)^2+4}$

$T = \frac{x}{8} + \frac{y}{3} = \frac{x}{8} + \frac{\sqrt{(6-x)^2+4}}{3}$

Domain: $[0,6]$ $\quad T' = \frac{1}{8} - \frac{1}{2}\,\frac{[(6-x)^2+4]^{-1/2}}{3} \cdot 2(6-x) = \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2+4}}$

If $T' = 0$, then $\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}} \Rightarrow 3\sqrt{(6-x)^2+4} = 8(6-x)$

$55(6-x)^2 + 4 = 64(6-x)^2$

$(x-6)^2 = \frac{36}{35}$

exclude $6 + \frac{6}{\sqrt{55}}$ $\longrightarrow$ $x = 6 \pm \frac{6}{\sqrt{55}}$

$T(0) \approx 2.108$

$T(6) \approx 1.417$

$T\left(6 - \frac{6}{\sqrt{55}}\right) \approx 1.368$ min

$\boxed{\text{local min at } x = 5.19 \text{ miles}}$

3) Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?

let p = price charged per car per day

n = # of cars rented per day

R = revenue per day

Domain: $[50, 200]$

$R = n \cdot p = (1000 - 5p)p = -5p^2 + 1000p$

$R' = -10p + 1000$

$-10p + 1000 = 0$

$p = 100$: $R = \$50{,}000$ ← max

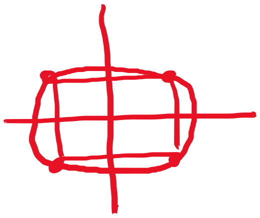
$p = 50$: $R = \$37{,}500$

$p = 200$: $R = \$0$

$\boxed{\text{Rental car company should charge \$100 per day per car.}}$

Media: Watch this [video](#) example on a maximizing area or a rectangle inscribed on a parabola.

4) A rectangle is to be inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$. What should the dimensions of the rectangle be to maximize its area? What is the maximum area?



let L = length of rectangle, W = width of rectangle
A = area of rectangle

$L = 2x$

$W = 2y$

$\frac{x^2}{4} + y^2 = 1$

$y = \sqrt{1 - \frac{x^2}{4}}$

$A = L \cdot W = 2x \cdot 2y = 4x\left(\sqrt{1 - \frac{x^2}{4}}\right) = 2x\sqrt{4 - x^2}$

Domain: $[0, 2]$

$A(0) = 0$

$A(2) = 0$

$y = \frac{1}{\sqrt{2}}$, so

$\boxed{L = 2\sqrt{2} \quad W = \sqrt{2} \quad A = 2\sqrt{2} \cdot \sqrt{2} = 4}$

$A' = 2x\left(\frac{1}{2}(4 - x^2)^{-1/2}\right) + \sqrt{4 - x^2}\,(2)$

$= \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = \frac{8 - 4x^2}{\sqrt{4 - x^2}}$

so $8 - 4x^2 = 0$

$x = \pm\sqrt{2}$ ← exclude $-\sqrt{2}$

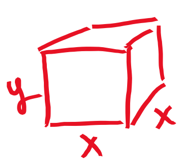
Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, the functions were defined on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Media: Watch this [video](#) example on an unbounded domain.

Example

A rectangular box with a square base, an open top, and a volume of 216 in^3 is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?



let x = length of side of square base
y = height of box S = surface area

$S = 4xy + x^2$ since volume = 216

$x^2 y = 216$

$y = \frac{216}{x^2}$

Domain: $(0, \infty)$

$S = 4x\left(\frac{216}{x^2}\right) + x^2$

$S = \frac{864}{x} + x^2$

$S' = -\frac{864}{x^2} + 2x \Rightarrow -\frac{864}{x^2} = -2x$

$x^3 = 432$

$x = 6\sqrt[3]{2}$

$y = \frac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$

$\boxed{\text{Dimensions of box: } 6\sqrt[3]{2} \times 3\sqrt[3]{2} \text{ and } S(6\sqrt[3]{2}) = 108\sqrt[3]{4} \text{ in}^2}$