Section 3.3: Differentiation Rules

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

Let c be a constant.

If $f(x) = c$, then $f'(c) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example: Find the derivative of $g(x) = -3$.

$$\boxed{g'(x) = 0}$$

The Power Rule

Let n be a postive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Example: Find the derivative of $f(x) = x^7$.

$$f'(x) = 7x^{7-1}$$

$$\boxed{f'(x) = 7x^6}$$

The Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equation holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

That is,

$$\big(f(x)+g(x)\big)' = f'(x) + g'(x)$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

That is,

$$\big(f(x)-g(x)\big)' = f'(x) - g'(x)$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the cosntant multiplied by the derivative.

$$\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx}(f(x));$$

That is,

$$\big(kf(x)\big)' = kf'(x).$$

Examples

1) Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

$g'(x) = 3(2x^1) = 6x$ $\quad f'(x) = 2x$

the derivative of $g(x)$ is 3 times the derivative of $f(x)$.

2) Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

$f'(x) = 2(3x^2) - 6(2x) + 0$

$\boxed{f'(x) = 6x^2 - 12x}$

3) Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

$f(1) = (1)^2 - 4(1) + 6 = 3$ point: $(1,3)$

$f'(x) = 2x - 4$

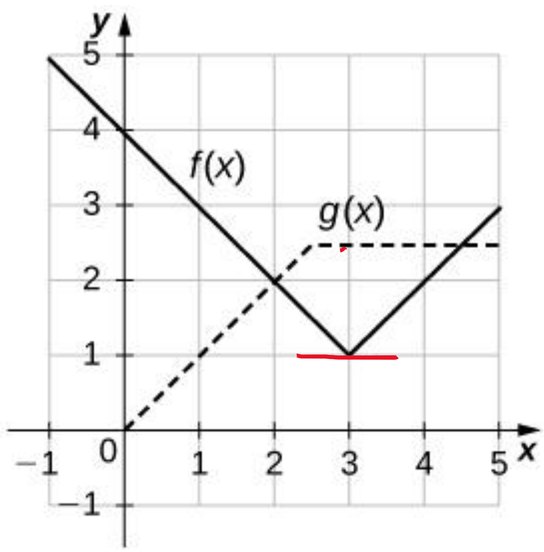
$f'(1) = 2(1) - 4 = -2$ ← slope

$y - y_1 = m(x - x_1)$

$y - 3 = -2x + 2$

$\boxed{y - 3 = -2(x-1)}$ or $\boxed{y = -2x + 5}$

Use the figure below to find the indicated derivatives.



4) Let $h(x) = f(x) + g(x)$. Find

$h'(x) = f'(x) + g'(x)$

a. $h'(1)$

$h'(1) = f'(1) + g'(1)$

$h'(1) = -1 + 1$

$\boxed{h'(1) = 0}$

b. $h'(3)$

$h'(3) = f'(3) + g'(3)$

$h'(3) = 0 + 0$

$\boxed{h'(3) = 0}$

c. $h'(4)$

$h'(4) = f'(4) + g'(4)$

$h'(4) = 1 + 0$

$\boxed{h'(4) = 1}$

The Product Rule

> Let $f(x)$ and $g(x)$ be differentiable functions. Then
>
> $$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x);$$
>
> That is,
>
> $$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x).$$
>
> This means that the derivative of a product of two function is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Examples

1) For $j(x) = f(x)g(x)$, use the product rule to find $j'(2)$ if $f(2) = 3, f'(2) = -4, g(2) = 1$, and $g'(2) = 6$.

$$j(x) = f(x)g(x)$$
$$j'(x) = f(x)g'(x) + g(x)f'(x)$$
$$j'(2) = f(2)g'(2) + g(2)f'(2)$$
$$j'(2) = 3 \cdot 6 + 1 \cdot -4 = 18 - 4 = \boxed{14}$$

2) For $j(x) = (x^2 + 2)(3x^3 - 5x)$, find $j'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

$$j(x) = \underbrace{(x^2+2)}_{f(x)}\underbrace{(3x^3-5x)}_{g(x)}$$

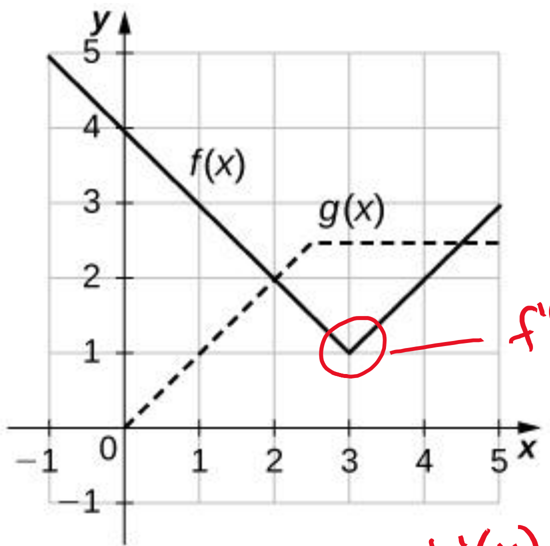
$$f'(x) = 2x$$
$$g'(x) = 9x^2 - 5$$

$$j'(x) = f(x)g'(x) + g(x)f'(x)$$
$$j'(x) = (x^2+2)(9x^2-5) + (3x^3-5x)(2x)$$
$$j'(x) = 9x^4 - 5x^2 + 18x^2 - 10 + 6x^4 - 10x^2$$
$$\boxed{j'(x) = 15x^4 + 3x^2 - 10}$$

same

$$j(x) = (x^2+2)(3x^3-5x) = 3x^5 - 5x^3 + 6x^3 - 10x$$
$$j(x) = 3x^5 + x^3 - 10x \qquad \boxed{j'(x) = 15x^4 + 3x^2 - 10}$$

Use the figure below to find the indicated derivatives.



$h'(x) = f(x)g'(x) + g(x)f'(x)$

3) Let $h(x) = f(x)g(x)$. Find

a. $h'(1)$

$h'(1) = f(1)g'(1) + g(1)f'(1)$

$= 3 \cdot 1 + 1 \cdot -1$

$\boxed{h'(1) = 2}$

b. $h'(3)$

$h'(3) = f(3)g'(3) + g(3)f'(3)$

$h'(3) = 1 \cdot 0 + 2.5 \cdot \text{DNE}$

$\boxed{\text{does not exist}}$

c. $h'(4)$

$h'(4) = f(4)g'(4) + g(4)f'(4)$

$h'(4) = 2 \cdot 0 + 2.5 \cdot 1$

$\boxed{h'(4) = 2.5}$

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

That is,

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

Examples

1) Use the quotient rule to find the derivative of $k(x) = \frac{5x^2}{4x+3}$.

$5x^2 \leftarrow f(x)$ $\quad f'(x) = 10x$

$4x+3 \leftarrow g(x)$ $\quad g'(x) = 4$

$$k'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

$$k'(x) = \frac{(10x)(4x+3) - 4(5x^2)}{(4x+3)^2} = \frac{40x^2 + 30x - 20x^2}{(4x+3)^2}$$

$$\boxed{k'(x) = \frac{20x^2 + 30x}{(4x+3)^2}}$$

2) Find the derivative of $h(x) = \frac{3x+1}{4x-3}$. $\leftarrow f(x)$ $\leftarrow g(x)$

$f'(x) = 3$

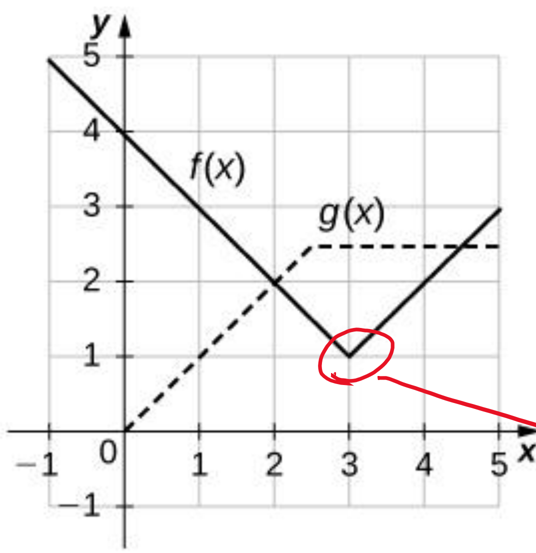
$g'(x) = 4$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

$$h'(x) = \frac{3(4x-3) - 4(3x+1)}{(4x-3)^2} = \frac{12x-9-12x-4}{(4x-3)^2}$$

$$\boxed{h'(x) = \frac{-13}{(4x-3)^2}}$$

Use the figure below to find the indicated derivatives.



$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$f'(3)$ does not exist

3) Let $h(x) = \frac{f(x)}{g(x)}$. Find

a. $h'(1)$

$$h'(1) = \frac{1(-1) - 3(1)}{1^2}$$

$$h'(1) = \frac{-4}{1}$$

$$\boxed{h'(1) = -4}$$

b. $h'(3)$

does not exist

c. $h'(4)$

$$h'(4) = \frac{2(0) - 2.5(0)}{0^2}$$

does not exist

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

> **Extended Power Rule**
>
> If k is a negative integer, then
>
> $$\frac{d}{dx}(x^k) = kx^{k-1}.$$

Examples

1) Find $\frac{d}{dx}(x^{-4})$.

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = \boxed{-4x^{-5}}$$

2) Use the extended power rule and the constant multiple rule to find the derivative of $f(x) = \frac{4}{x^3}$. $\Rightarrow \; 4x^{-3}$

$$f'(x) = -3(4x^{-3-1}) = \boxed{-12x^{-4}} = \boxed{\frac{-12}{x^4}}$$

OR

Combining Differentiation Rules

We can find the derivative of any polynomial or rational function by combining the differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Examples

1) For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

$$k'(x) = 3h'(x) + [x^2 g'(x) + g(x)(2x)]$$

$$\boxed{k'(x) = 3h'(x) + x^2 g'(x) + 2x\,g(x)}$$

2) For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x)$, $g(x)$, $h(x)$, and their derivatives.

$$k(x) = \boxed{f(x)g(x)}\boxed{h(x)}$$

$$k'(x) = f(x)g(x) \cdot h'(x) + h(x)\left[\frac{d}{dx} f(x)g(x)\right]$$

$$k'(x) = f(x)g(x)h'(x) + h(x)[f(x)g'(x) + g(x)f'(x)]$$

$$\boxed{k'(x) = f(x)g(x)h'(x) + h(x)f(x)g'(x) + h(x)g(x)f'(x)}$$

3) For $h(x) = \frac{2x^3 k(x)}{3x+2}$, find $h'(x)$.

$$h'(x) = \frac{(3x+2)(2x^3 k'(x) + k(x)\,6x^2) - (2x^3 k(x))(3)}{(3x+2)^2}$$

$$\boxed{h'(x) = \frac{6x^4 k'(x) + 18x^3 k(x) + 4x^3 k'(x) + 12x^2 k(x) - 6x^3 k(x)}{(3x+2)^2}}$$

4) Find $\frac{d}{dx}(3f(x) - 2g(x))$.

$$= \boxed{3f'(x) - 2g'(x)}$$

5) Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

$\hookrightarrow f'(x) = 0$

$$f'(x) = 3x^2 - 14x + 8$$

$$3x^2 - 14x + 8 = 0$$

$$(3x-2)(x+4) = 0$$

$3x - 2 = 0$ $\quad$ $x + 4 = 0$

$x = 2/3$ $\quad$ $x = -4$

The function has horizontal tangent lines at $x = \frac{2}{3}$ and $x = 4$

6) The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \frac{2t^2+t}{t^3+50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C(t)$.

$$C'(t) = \frac{(t^3+50)(4t+1) - (2t^2+t)(3t^2)}{(t^3+50)^2} = \frac{4t^4 + t^3 + 200t + 50 - 6t^4 - 3t^3}{(t^3+50)^2}$$

$$\boxed{C'(t) = \frac{-2t^4 - 2t^3 + 200t + 50}{(t^3+50)^2}}$$

b. Find $C'(12)$ and briefly interpret the result.

$$C'(12) = \frac{-2(12)^4 - 2(12)^3 + 200(12) + 50}{((12)^3 + 50)^2} = \boxed{-0.0134 \text{ mg/hr}}$$

7) The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2+1}$. What is the initial velocity of the object?

$v(0) = s'(0)$

$s'(t) = \frac{(t^2+1)(1) - t(2t)}{(t^2+1)^2} = \frac{t^2+1-2t^2}{(t^2+1)^2}$

$s'(t) = \frac{1-t^2}{(t^2+1)^2}$

$s'(0) = \frac{1-(0)^2}{(0^2+1)^2} = 1$

$\boxed{\text{The initial velocity of the object is 1.}}$

8) For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the table to calculate the following derivatives.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	-1	4
2	5	3	7	1
3	-2	-4	8	2
4	0	6	-3	9

a. Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

$h'(x) = [x f'(x) + f(x)(1)] + 4g'(x)$

$h'(1) = 1 \cdot f'(1) + f(1)(1) + 4g'(1)$

$h'(1) = 1(-1) + 3(1) + 4(4) = \boxed{18}$

b. Find $h'(2)$ if $h(x) = f(x)g(x)$.

$h'(x) = f(x)g'(x) + g(x)f'(x)$

$h'(2) = f(2)g'(2) + g(2)f'(2)$

$h'(2) = 5(1) + 3(7) = \boxed{26}$

c. Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

$h(x) = x^{-1} + \frac{g(x)}{f(x)}$

$h'(x) = -1x^{-2} + \frac{f(x)g'(x) - g(x)f'(x)}{(f(x))^2}$

$h'(4) = -1(4)^{-2} + \frac{0 \cdot 9 - 6 \cdot (-3)}{0^2}$

$\boxed{\text{Does not exist}}$