Section 4.4: The Mean Value Theorem

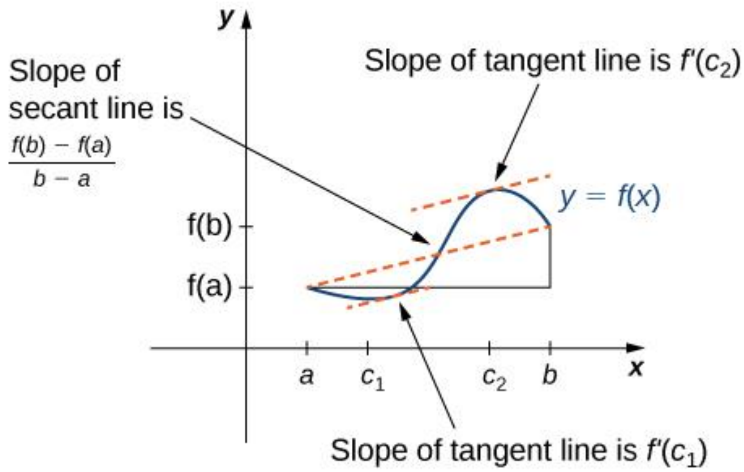
The Mean Value Theorem is one of the most important theorems in calculus. The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

The Mean Value Theorem and Its Meaning

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Media: Watch this video example on the Mean Value Theorem of a quadratic function.

Media: Watch this video example on the Mean Value Theorem of a rational function.

Examples

1) For $f(x) = \sqrt{x}$ over the interval $[0,9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0,9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find the values c guaranteed by the Mean Value Theorem.

since $f(x)=\sqrt{x}$ is continuous over $[0,9]$ and differentiable over $(0,9)$, f satisfies the hypotheses of the Mean Value Theorem.

$f(x)=x^{1/2}$ $\quad f'(x)=\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

slope connecting $(0,f(0))$ and $(9,f(9))$:

$\frac{f(9)-f(0)}{9-0} = \frac{3-0}{9-0} = \frac{1}{3}$

find c s.t. $f'(c)=\frac{1}{3}$

$\frac{1}{2\sqrt{c}} = \frac{1}{3}$

$2\sqrt{c} = 3$

$\sqrt{c} = \frac{3}{2}$

$\boxed{c = \frac{9}{4}}$

2) If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground. $s(t)=0$

$$-16t^2+100=0$$
$$-16t^2=-100$$
$$t^2=6.25$$
$$t=\pm 2.5$$

The ball will hit the ground 2.5 seconds after it is dropped.

b. Find the average velocity v_{avg} of the rock for when the rock is released, and the rock hits the ground. $\rightarrow t=2.5$ $t=0$

$$V_{average} = \frac{s(2.5)-s(0)}{2.5-0} = \frac{0-100}{2.5} = \boxed{-40 \text{ ft/sec}}$$

c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

instantaneous velocity $= s'(t)$

need to find t s.t. $v(t) = s'(t) = V_{avg} = -40$ ft/sec

* since $s(t)$ is continuous over interval $[0, 2.5]$ and differentiable over interval $(0, 2.5)$, Mean Value Theorem guarantees a point $c \in (0, 2.5)$ s.t.

$$s'(c) = \frac{s(2.5)-s(0)}{2.5-0} = -40$$

$s'(t) = -32t$

so $-32c = -40$

$c = 1.25$

1.25 seconds after the rock is dropped, the instantaneous velocity $= V_{avg}$.