Section 5.4: Integration Formulas and the Net Change Theorem

In this section, we use some basic integration formulas studied previously to solve some key applied problems.

Basic Integration Formulas

Recall the integration formulas given in table in Antiderivatives (Section 4.10) and the rule on properties of definite integrals.

Media: Watch this video example on definite integrals.

Examples

1) Use the power rule to integrate the function $\int_1^4 \sqrt{t}\,(1+t)dt$.

$$\int_1^4 \sqrt{t}(1+t)\ dt = \int_1^4 t^{\frac{1}{2}}\,(1+t)\ dt = \int_1^4 \left(t^{\frac{1}{2}} + t^{\frac{3}{2}}\right)\ dt$$

$$= \frac{2}{3}t^{\frac{3}{2}} + \frac{2}{5}t^{\frac{5}{2}}\Big|_1^4$$

$$= \left[\frac{2}{3}(4)^{\frac{3}{2}} + \frac{2}{5}(4)^{\frac{5}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}} + \frac{2}{5}(1)^{\frac{5}{2}}\right]$$

$$= \frac{256}{15}$$

2) Find the definite integral of $f(x) = x^2 - 3x$ over the interval $[1,3]$.

$$\int_1^3 (x^2 - 3x)\,dx = \frac{1}{3}x^3 - 3\left(\frac{1}{2}x^2\right)\Big|_1^3$$

$$= \left[\frac{1}{3}(3)^3 - \frac{3}{2}(3)^2\right] - \left[\frac{1}{3}(1)^3 - \frac{3}{2}(1)^2\right]$$

$$= -\frac{10}{3}$$

The Net Change Theorem

The net change theorem considers the integral of a rate of change. Net change can be applied to area, distance, and volume, to name only a few applications.

> **Net Change Theorem**
>
> The new value of a changing quantity equals the initial value plus the integral of the rate of change:
>
> $$F(b) = F(a) + \int_a^b F'(x)\,dx$$
>
> Or
>
> $$\int_a^b F'(x)\,dx = F(b) - F(a).$$

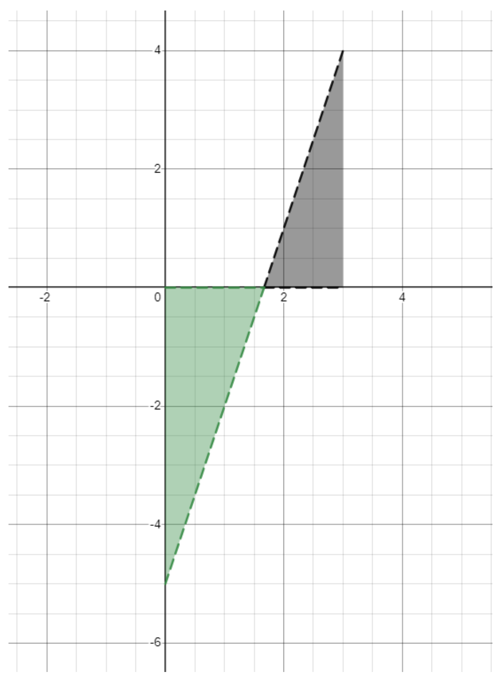
Media: Watch this video to learn about the difference between net area and total area.

Media: Watch this video example on the net change theorem.

Examples

1) Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$,

   a. find the net displacement of the particle.



$$\int_0^3 (3t - 5)\,dt = \frac{3}{2}t^2 - 5t\Big|_0^3$$

$$= \left[\frac{3}{2}(3)^2 - 5(3)\right] - 0$$

$$= -\frac{3}{2}\ meters$$

b. Find the total distance traveled by a particle.

To find the total distance, you will need to find both the positive and negative areas (i.e., absolute value). In addition, find where the velocity function crosses the x-axis by setting the function equal to 0.

$$3t - 5 = 0$$

$$t = \frac{5}{3}$$

Total Distance:

$$\int_0^3 |v(t)| dt = \int_0^{\frac{5}{3}} -v(t)\, dt + \int_{\frac{5}{3}}^3 v(t)\, dt$$

$$= \int_0^{\frac{5}{3}} (5 - 3t)\, dt + \int_{\frac{5}{3}}^3 (3t - 5)\, dt$$

$$= \left(5t - \frac{3}{2}t^2\right)\Big|_0^{\frac{5}{3}} + \left(\frac{3}{2}t^2 - 5t\right)\Big|_{\frac{5}{3}}^3$$

$$= \left[\left[5\left(\frac{5}{3}\right) - \frac{3}{2}\left(\frac{5}{3}\right)^2\right] - 0\right] + \left[\left[\frac{3}{2}(3)^2 - 5(3)\right] - \left[\frac{3}{2}\left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right)\right]\right]$$

$$= \frac{41}{6}\ meters$$

2) If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at a rate of $5 - t^3\, \frac{gal}{hr}$, how much gasoline is used in the first two hours?

$$\int_0^2 (5 - t^3) dt = \left(5t - \frac{t^4}{4}\right)\Big|_0^2$$

$$= \left[5(2) - \frac{(2)^4}{4}\right] - 0$$

$$= 6 \text{ gallons of gas in 2 hours}$$

Integrating Even and Odd Functions

Recall that the graphs of even functions are symmetric about the y −axis. An odd function is symmetric about the origin. Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas, because they are symmetric about the y −axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x −axis are equal.

Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Media: Watch this video example on integrals of an even function.

Media: Watch this video example on integrals of an odd function.

Examples

1) Integrate the even function $\int_{-2}^{2}(3x^8 - 2)\,dx$ and verify that the integration formula for even functions holds.

$$\int_{-2}^{2}(3x^8 - 2)dx = \left(3\cdot\frac{x^9}{9} - 2x\right)\Big|_{-2}^{2}$$

$$= \left[\frac{(2)^9}{3} - 2(2)\right] - \left[\frac{(-2)^9}{3} - 2(-2)\right]$$

$$= \frac{1000}{3}$$

To verify, calculate the integral from 0 to 2 and double it.

$$\int_{0}^{2}(3x^8 - 2)dx = \left(3\cdot\frac{x^9}{9} - 2x\right)\Big|_{0}^{2}$$

$$= \left[\frac{(2)^9}{3} - 2(2)\right] - 0$$

$$= \frac{500}{3}$$

And $2\cdot\frac{500}{3} = \frac{1000}{3}$

2) Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} -5\sin x\, dx = -5(-\cos x)|_{-\pi}^{\pi}$$

$$= [-5(-\cos(\pi))] - [-5(-\cos(-\pi))]$$

$$= -5 - (-5)$$

$$= 0$$