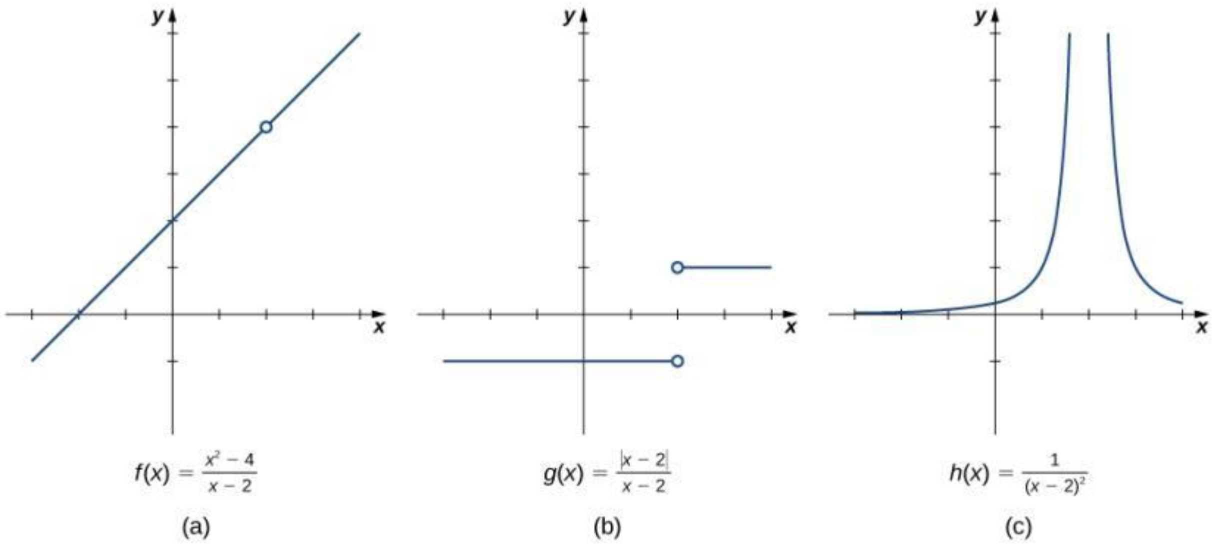
Section 2.2: The Limit of a Function

To understand the concept of a limit, first look at graphs of functions. For example, looking at the three functions below, we can see that at $x = 2$, all these functions are undefined.



Yet, simply stating they are undefined does not give an accurate picture of what is happening around $x = 2$.

Intuitive Definition of a Limit

Looking at the graphs above, we see that the behavior of the function as x approaches 2 can be very different depending on the function.

> Let $f(x)$ be a function defined at all values in an open interval containing a, except maybe at a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of x ($\neq a$) approach the number a, then the **limit of $f(x)$ as x approaches a is L**. Symbolically,
>
> $$\lim_{x \to a} f(x) = L.$$
>
> In other words, as x gets closer to a, $f(x)$ gets closer to L.

For example, in the first function $f(x) = \frac{x^2-4}{x-2}$, as x approaches 2 from either side, the values of $y = f(x)$ approach 4. So, mathematically,

$$\lim_{x \to 2} f(x) = 4.$$

One way to approximate a limit is to use a table. Choose sets of x −values – one set approaching a from the left (values slightly smaller than a) and another set approaching a from the right (values slightly larger than a).

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$\boldsymbol{a-0.1}$	$f(a-0.1)$
$\boldsymbol{a-0.01}$	$f(a-0.01)$
$\boldsymbol{a-0.001}$	$f(a-0.001)$
$\boldsymbol{a-0.0001}$	$f(a-0.0001)$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$\boldsymbol{a+0.1}$	$f(a+0.1)$
$\boldsymbol{a+0.01}$	$f(a+0.01)$
$\boldsymbol{a+0.001}$	$f(a+0.001)$
$\boldsymbol{a+0.0001}$	$f(a+0.0001)$

Then, look at each of the $f(x)$ columns, determine whether the outputs seem to be approaching a single value. If both columns approach a common y −value L, then we say

$$\lim_{x\to a} f(x) = L$$

Examples:

1) Evaluate each of the following limits using a table of function values. Then use a graph to confirm your estimate.

a. $\lim_{x\to 0}\frac{\sin x}{x}$

x	$\frac{\sin x}{x}$
-0.1	0.998334
-0.01	0.999983
-0.001	0.999999
-0.0001	0.999999

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.999999
0.0001	0.999999

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1}$$

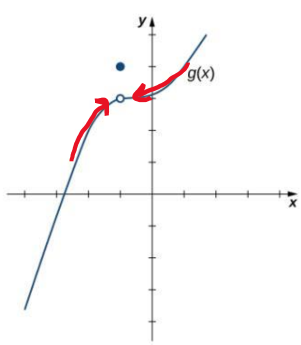
b. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582
3.99	0.250156
3.999	0.250015
3.9999	0.250001

x	$\frac{\sqrt{x}-2}{x-4}$
4.1	0.248456
4.01	0.249843
4.001	0.249984
4.0001	0.249998

$$\boxed{\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = 0.25}$$

2) For $g(x)$ shown below, evaluate $\lim_{x\to -1} g(x)$.



$$\boxed{\lim_{x\to -1} g(x) = 3}$$

The Existence of a Limit

For a limit of a function to exist at a point, the function values must approach a single real-number value at that point. If the function values do not approach a single value, then the limit does not exist.

Example: Evaluate $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ using a table of values.

x	$\sin \frac{1}{x}$
-0.1	0.544021
-0.001	0.506365
-0.0001	-0.826879
-0.00001	0.305614

x	$\sin \frac{1}{x}$
0.1	-0.544021
0.01	-0.506365
0.001	0.826879
0.0001	-0.305614

y-values do not approach a single value

$$\boxed{\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = DNE}$$

One-Sided Limits

Indicating that the limit of a function fails to exist at a point does not always provide us with enough information about the behavior of the function at that point. Instead, we look at what happens as we approach the point from the left and right sides.

We define two types of **one-sided limits**.

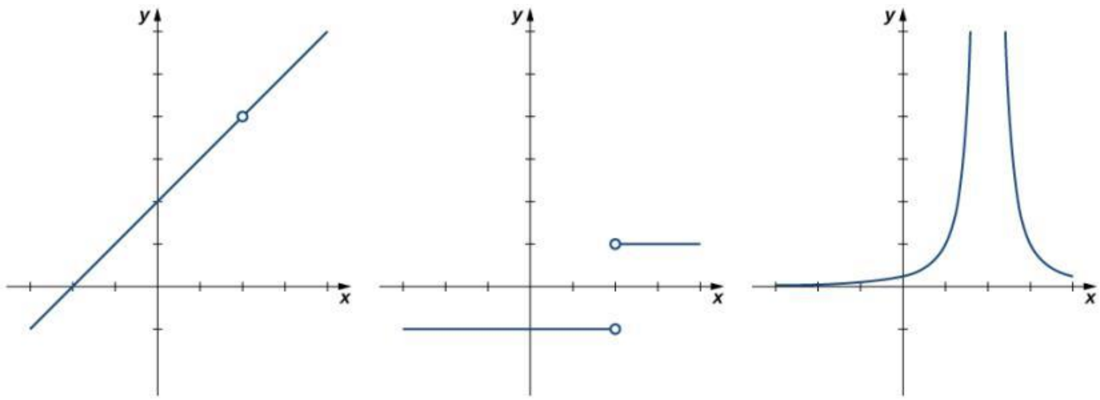
Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

So, looking back at the three graphs from the beginning, we can see that the y −values of the second graph approach different values as we approach $x = 2$ from the left and right.



$f(x) = \frac{x^2 - 4}{x - 2}$ (a) $g(x) = \frac{|x - 2|}{x - 2}$ (b) $h(x) = \frac{1}{(x - 2)^2}$ (c)

In this case, the $\lim_{x \to 2^-} g(x) = -1$ since as x approaches 2 from the left, $f(x)$ (y −values) approach -1.

Similarly, the $\lim_{x \to 2^+} g(x) = 1$ since as since as x approaches 2 from the right, $f(x)$ (y −values) approach 1.

So, if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. If the limit from the left and the limit from the right take on different values, the limit of the function does not exist at that point.

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Examples

1) Use a table of values to estimate the following limits, if possible.

a. $\lim_{x \to 2^-} \frac{|x^2-4|}{x-2}$

X	$\frac{\lvert x^2-4 \rvert}{x-2}$
1.9	-3.9
1.99	-3.99
1.999	-3.999
1.9999	-3.9999

$$\boxed{\lim_{x \to 2^-} \frac{|x^2-4|}{x-2} = -4}$$

b. $\lim_{x \to 2^+} \frac{|x^2-4|}{x-2}$

X	$\frac{\lvert x^2-4 \rvert}{x-2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

$$\boxed{\lim_{x \to 2^+} \frac{|x^2-4|}{x-2} = 4}$$

2) For the function $f(x) = \begin{cases} x+1 & if \;\; x < 2 \\ x^2 - 4 & if \; x \geq 2 \end{cases}$ evaluate each of the following limits.

a. $\lim\limits_{x \to 2^-} f(x)$

x	$x+1$
1.9	2.9
1.99	2.99
1.999	2.999
1.9999	2.9999

$$\boxed{\lim_{x \to 2^-} f(x) = 3}$$

b. $\lim\limits_{x \to 2^+} f(x)$

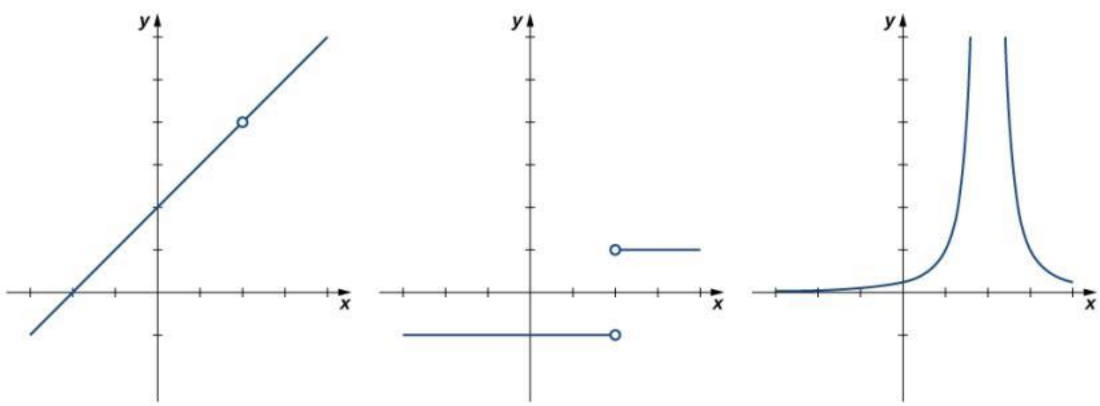
x	x^2-4
2.1	0.41
2.01	0.0401
2.001	0.004001
2.0001	0.00040001

$$\boxed{\lim_{x \to 2^+} f(x) = 0}$$

Infinite Limits

Evaluating the limit (or right and left limit) of a function at a point helps us to characterize the behavior of a function around a given value. We can also describe the behavior of functions that do not have finite limits.

Looking back at the original three functions, we see that as x approaches 2, the values of $h(x)$ becomes larger and larger. Hence, we say $\lim\limits_{x \to 2} h(x) = +\infty$. This is called an **infinite limit**.



$f(x) = \frac{x^2 - 4}{x - 2}$ (a) $g(x) = \frac{|x - 2|}{x - 2}$ (b) $h(x) = \frac{1}{(x - 2)^2}$ (c)

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty$$

If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty$$

If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a.

If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty$$

If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty$$

*Note, when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$, we are NOT stating that the limit exists. We are simply describing the behavior of the function.

Examples: Evaluate each of the following limits, if possible. Use a table of values and graph $f(x) = \frac{1}{x}$ to confirm your conclusion.

1) $\lim_{x \to 0^-} \frac{1}{x} = \boxed{-\infty}$

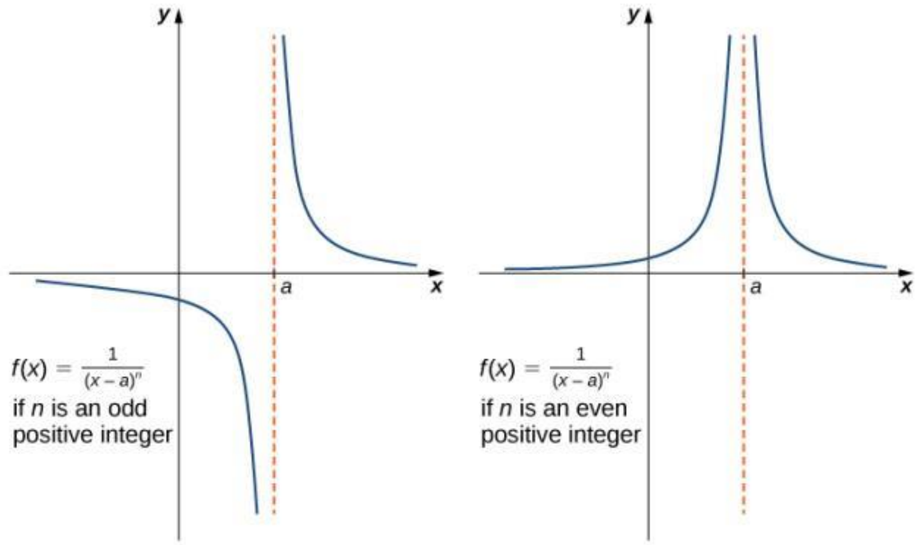
x	$\frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
-0.0001	-10000

2) $\lim_{x \to 0^+} \frac{1}{x} = \boxed{\infty}$

x	$\frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10000

3) $\lim_{x \to 0} \frac{1}{x} = \boxed{\text{DNE}}$

Functions of the form $f(x) = \frac{1}{(x-a)^n}$ where n is a positive integer, have infinite limits as x approaches a from either the left or the right.



If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

Notice, that for these types of graphs, we also have vertical asymptotes at $x = a$. We can determine whether a function has vertical asymptotes by looking at limits.

Let $f(x)$ be a function. The line $x = a$ is a **vertical asymptote** of $f(x)$, if any of the following conditions hold:

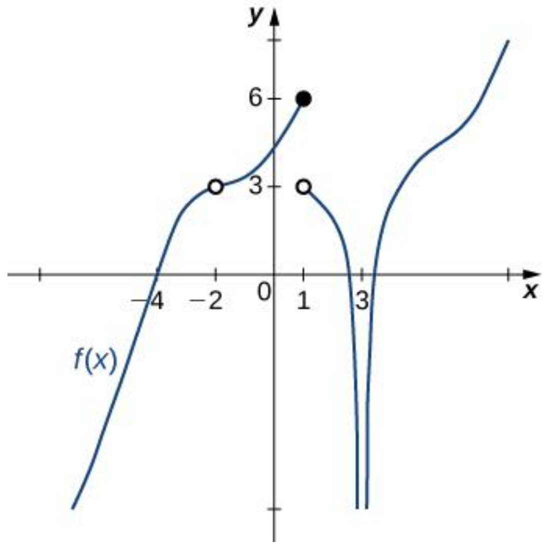
$$\lim_{x \to a^-} f(x) = +\infty \text{ or} -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty \text{ or} -\infty$$

$$\lim_{x \to a} f(x) = +\infty \text{ or} -\infty$$

Examples

1) Use the graph of $f(x)$ shown below to determine each of the following:



a. $\lim_{x\to -4^-} f(x);\ \lim_{x\to -4^+} f(x);\ \lim_{x\to -4} f(x);\ f(-4)$

$= \boxed{0}$ $= \boxed{0}$ $= \boxed{0}$ $= \boxed{0}$

b. $\lim_{x\to -2^-} f(x);\ \lim_{x\to -2^+} f(x);\ \lim_{x\to -2} f(x);\ f(-2)$

$= \boxed{3}$ $= \boxed{3}$ $= \boxed{3}$ $= \boxed{\text{DNE}}$

c. $\lim_{x\to 1^-} f(x);\ \lim_{x\to 1^+} f(x);\ \lim_{x\to 1} f(x);\ f(1)$

$= \boxed{6}$ $= \boxed{3}$ $= \boxed{\text{DNE}}$ $= \boxed{6}$

d. $\lim_{x\to 3^-} f(x);\ \lim_{x\to 3^+} f(x);\ \lim_{x\to 3} f(x);\ f(3)$

$= \boxed{-\infty}$ $= \boxed{-\infty}$ $= \boxed{-\infty}$ $= \boxed{\text{DNE}}$

2) Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$

vertical ansymptotes at $x=2$

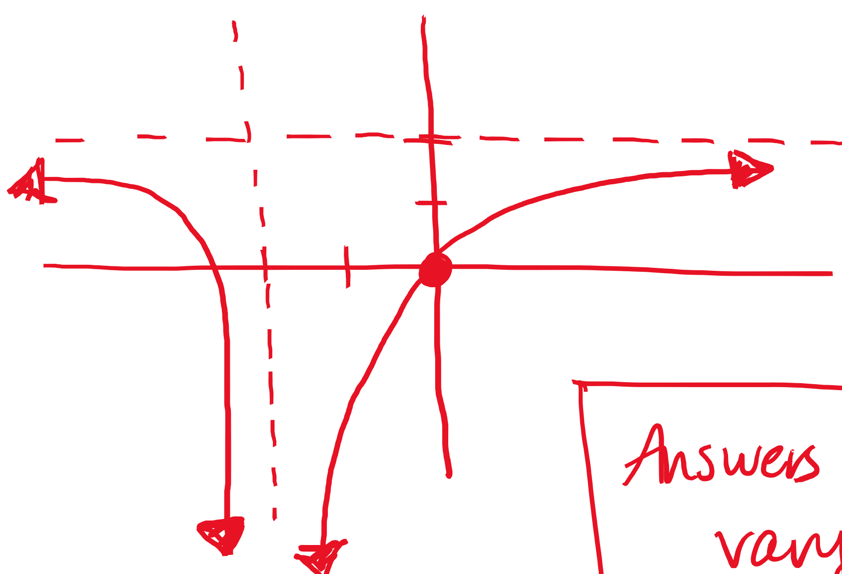
a. $\lim_{x \to 2^-} \frac{1}{(x-2)^3} = \boxed{-\infty}$

b. $\lim_{x \to 2^+} \frac{1}{(x-2)^3} = \boxed{\infty}$

c. $\lim_{x \to 2} \frac{1}{(x-2)^3} = \boxed{DNE}$

3) Sketch the graph of a function with the given properties:

- As $x \to -\infty, f(x) \to 2$
- $\lim_{x \to -2} f(x) = -\infty$
- As $x \to \infty, f(x) \to 2$
- $f(0) = 0$



Answers may vary