Section 3.4: Derivatives as Rates of Change

This section looks at applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include acceleration and velocity in physics, population growth rates in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the point.

If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h) - f(a).$$

The **average rate of change** of the function f over the same interval is the ratio of the amount of change over that interval to the corresponding change in the x values.

$$\frac{f(a+h)-f(a)}{h}$$

The **instantaneous rate of change** of $f(x)$ at a is its derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

For small enough values of h, $f'(a) \approx \frac{f(a+h)-f(a)}{h}$. Solving for $f(a+h)$ gives the **amount of change formula**:

$$f(a+h) \approx f(a) + f'(a)h.$$

Examples

1) If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

$h = 3.2 - 3 = 0.2$

$f(3.2) = f(3+0.2) \approx f(3) + 0.2f'(3)$

$= 2 + 0.2(5) = \boxed{3}$

2) Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

$h = 10.1 - 10 = 0.1$

$f(10.1) = f(10+0.1) \approx f(10) + 0.1f'(10)$

$= -5 + 0.1(6) = \boxed{-4.4}$

3) The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

$P(0)=10$
$P(5)=30$

$$P'(0) \approx \frac{P(5)-P(0)}{5-0} = \frac{30-10}{5} = 4.$$

$$P(2) \approx P(0) + 2P'(0) \approx 10 + 2(4) = 18$$

In 2 years, the population will be 18,000.

4) The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3{,}000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

$P(0)=3000$
$P'(0)=100$

$$P(3) \approx P(0) + 3P'(0) \approx 3000 + 3(100)$$
$$= 3300$$

In 3 days, the mosquito colony will have 3300

Motion along a Line

Another use of the derivative is to analyze the motion along a line.

Let $s(t)$ be a function giving the **position** of an object at time t.

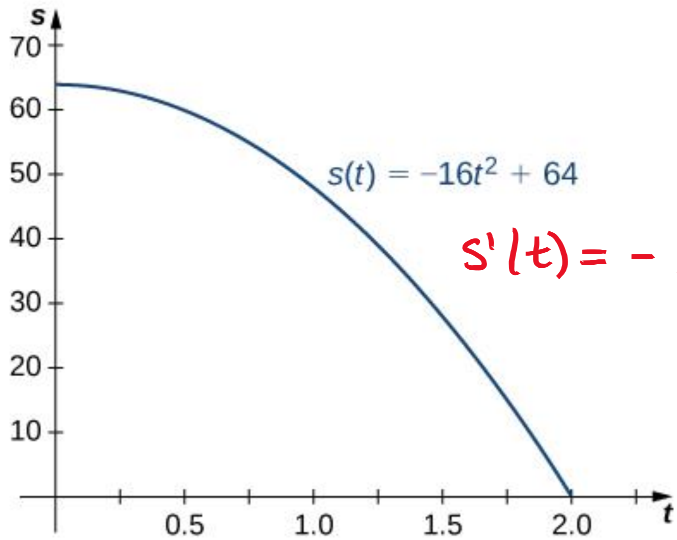
The **velocity** of the object at time t is given by $v(t) = s'(t)$.

The **speed** of the object at time t is given by $|v(t)|$.

The **acceleration** of the object at time t is given by $a(t) = v'(t) = s''(t)$.

Examples

1) A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



$s'(t) = -32t$

a. What is the instantaneous velocity of the ball when it hits the ground?

$v(2) = s'(2) = -32(2) = \boxed{-64 \text{ ft/s}}$ ↳ hits ground at $t=2$

b. What is the average velocity during its fall?

$$\text{average velocity} = \frac{s(2)-s(0)}{2-0} = \frac{0-64}{2} = \boxed{-32 \text{ ft/s}}$$

2) A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions:

$s(t) = t^3 - 4t + 2$

$v(t) = 3t^2 - 4$

$a(t) = 6t$

$v(1) = 3(1)^2 - 4 = -1$

$a(1) = 6(1) = 6$

a. Is the particle moving from left to right or from right to left at time $t = 1$?

Since $v(1) < 0$, the particle is moving from right to left

b. Is the particle speeding up or slowing down at time $t = 1$?

Since $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions.

The particle is slowing down

3) The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4$, $t \geq 0$.

a. Find $v(t)$.

$$\boxed{v(t) = s'(t) = 3t^2 - 18t + 24}$$

b. At what time(s) is the particle at rest?

particle is at rest when $v(t) = 0$

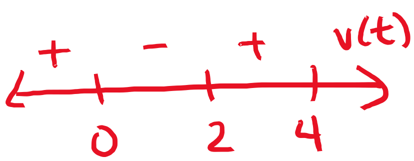
$$3t^2 - 18t + 24 = 0$$
$$3(t^2 - 6t + 8) = 0$$
$$3(t-2)(t-4) = 0$$
$$t = 2 \qquad t = 4$$

particle is at rest at $t=2$ and $t=4$

c. On what time intervals is the particle moving from left to right? From right to left?

moves left to right when $v(t) > 0$: $[0,2) \cup (2,\infty)$

moves right to left when $v(t) < 0$: $(2,4)$

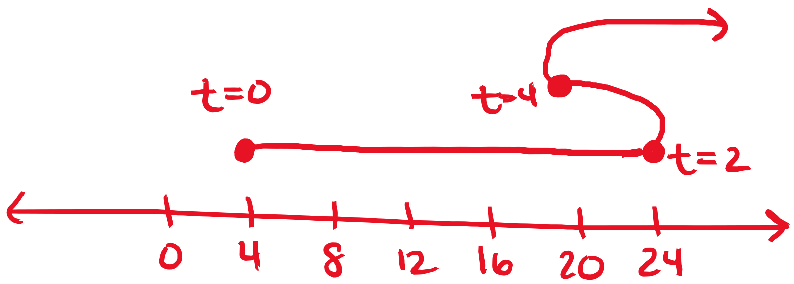


d. Use the information obtained to sketch the path of the particle along a coordinate axis.

$t=0$ $s(0) = 4$

$t=2$ $s(2) = 24$

$t=4$ $s(4) = 20$



Change in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives.

> If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
>
> If $R(x)$ is the revenue obtained from selling x items, then the **marginal revenue** $MR(x)$ is $MR(x) = R'(x)$.
>
> If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

Example

Assume that the number of barbeque dinner that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x$, $0 \leq x \leq 300$. In this case, the revenue in dollars obtained by selling x barbeque dinners is given by $R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x$ for $0 \leq x \leq 300$. Use the marginal revenue function to estimate the revenue obtained from selling the 101^{st} barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

$$MR(x) = R'(x) = -0.06x + 9$$

$$R(101) - R(100) = 602.97 - 600$$

$$= 2.97 \text{ or } \boxed{\$2.97}$$

The marginal revenue is a fairly good estimate.