Section 3.7: Derivatives of Inverse Functions

This section will explore the relationship between the derivative of a function and the derivative of its inverse.

The Derivative of an Inverse Function

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x)) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Media: Watch these video1 and video 2 examples on the derivative of inverse functions.

Examples

1) Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$

$f'(x) = \frac{(x-1)(0) - 2(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$ and $f'(g(x)) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x} - 1\right)^2} = -\frac{x^2}{2}$

$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{-\frac{x^2}{2}} = \boxed{\frac{2}{-x^2}}$ *verify by applying quotient rule to $g(x)$.

2) Find the derivative of $g(x) = \sqrt[3]{x}$ by using the inverse function theorem.

Inverse of $g(x) = \sqrt[3]{x}$ is $f(x) = x^3$

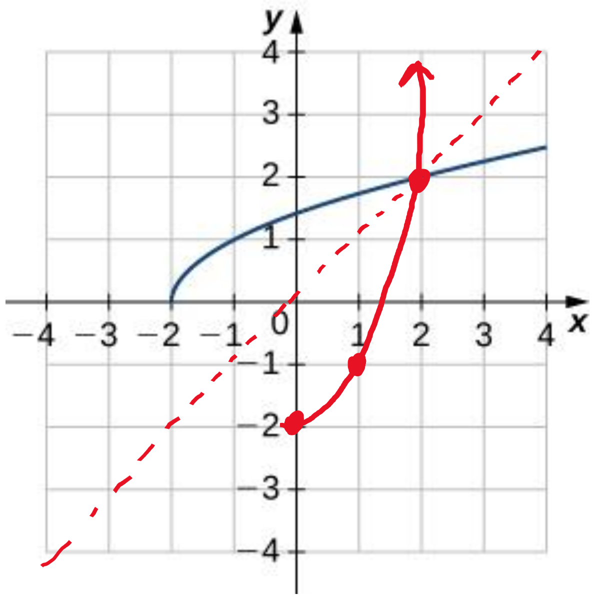
$f'(x) = 3x^2$ and $f'(g(x)) = 3(\sqrt[3]{x})^2 = 3x^{2/3}$

$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3x^{2/3}} = \boxed{\frac{1}{3}x^{-2/3}}$

3) Use the graph of $y = f(x)$ to

   a. Sketch the graph of $y = f^{-1}(x)$, and → reflects across $y=x$

   b. Use part a. to estimate $(f^{-1})'(1)$. $= \boxed{-1}$



Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

$$\frac{d}{dx}\left(x^{1/n}\right) = \frac{1}{n}x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then

$$\frac{d}{dx}\left(x^{m/n}\right) = \frac{m}{n}x^{(m/n)-1}.$$

Media: Watch this video example on derivatives of functions with rational exponents.

Example: Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

find slope → $\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$ at $x=8$: $\frac{2}{3}(8)^{-\frac{1}{3}} = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}$ ← slope

at $x=8$ $y=(8)^{2/3} = 4$ } point: $(8,4)$

$y-y_1 = m(x-x_1)$ $\quad$ $y-4 = \frac{1}{3}x - \frac{8}{3}$

$\boxed{y-4 = \frac{1}{3}(x-8)}$ or $\boxed{y = \frac{1}{3}x + \frac{4}{3}}$

Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}}$

$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-(x)^2}}$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+(x)^2}$

$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+(x)^2}$

$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{(x)^2-1}}$

$\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{(x)^2-1}}$

Media: Watch these video1 and video2 examples on derivatives of inverse trig functions.

Examples

1) Find the derivative of the following functions:

a. $f(x) = \tan^{-1}(x^2)$

$f'(x) = \frac{1}{1+(x^2)^2} \cdot (2x)$

$\boxed{f'(x) = \frac{2x}{1+x^4}}$

b. $y = \sec^{-1}\left(\frac{1}{x}\right)$ x^{-1}

$y' = \frac{1}{\left|\frac{1}{x}\right|\sqrt{\left(\frac{1}{x}\right)^2-1}} \cdot -x^{-2}$

$y' = \frac{1}{\frac{1}{x}\sqrt{\frac{1}{x^2}-1}} \cdot -\frac{1}{x^2}$

$y' = \frac{-1}{x\sqrt{\frac{1}{x^2}-1}} = \boxed{\frac{-1}{\sqrt{1-x^2}}}$

c. $y = \cot^{-1}\sqrt{4-x^2}$

$y' = \frac{-1}{1+(\sqrt{4-x^2})^2} \cdot \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$

$y' = \frac{-1}{1+(4-x^2)} \cdot \frac{-x}{\sqrt{4-x^2}}$

$y' = \frac{-1}{5-x^2} \cdot \frac{-x}{\sqrt{4-x^2}} = \boxed{\frac{x}{(5-x^2)(\sqrt{4-x^2})}}$

2) The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

t^{-1}

$$s'(t) = v(t) = \frac{1}{1+\left(\frac{1}{t}\right)^2} \cdot -t^{-2}$$

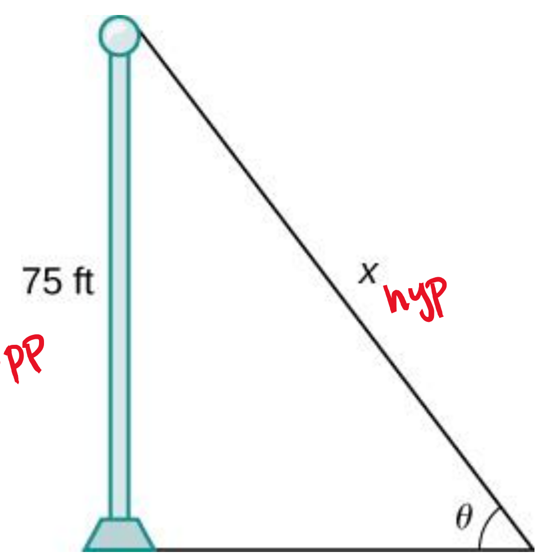
$$= \frac{1}{\left(1+\frac{1}{t^2}\right)} \cdot \frac{-1}{t^2} = \frac{-1}{t^2+1}$$

$$v(1) = \frac{-1}{(1)^2+1} = \frac{-1}{2}$$

$$\boxed{\text{The velocity of the particle at time } t=1 \text{ is } \frac{-1}{2}}$$

Media: Watch this video example on applications of derivatives of inverse trig functions.

3) A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



opp

hyp

$$\sin\theta = \frac{75}{x} \qquad \text{when } x = 90$$

$$\sin^{-1}\left(\frac{75}{x}\right) = \theta$$

$$\text{so } \theta = \sin^{-1}(75x^{-1})$$

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-(75x^{-1})^2}} \cdot -75x^{-2}$$

$$\text{at } x=90: \frac{d\theta}{dx} = \frac{1}{\sqrt{1-(75(90)^{-1})}} \cdot -75(90)^{-2}$$

$$\boxed{\frac{d\theta}{dx} = -0.02 \text{ radians/ft}}$$