Section 2.3: The Limit Laws

Evaluating Limits with the Limit Laws

In the previous section we estimated limits using a table and graph. In addition to estimating, we look at some properties of limits that will allow us to evaluate limits of many types of algebraic functions.

Basic Limits

For any real number a and any constant c,

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} c = c$$

Examples: Evaluate each of the following limits using the Basic Limits.

1) $\lim_{x \to 2} x = 2$

2) $\lim_{x \to 2} 5 = 5$

We now look at the limit laws, the individual properties of limits, and practice using them.

Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M.$$

Let c be a constant. Then, the following statements hold:

Sum law for limits: $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} c\, f(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every integer $n > 0$

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$

Media: Watch this video example on determining a limit analytically.

Media: Watch this video example on using the limit laws.

Examples

1) Use limit laws to evaluate each of the following. In each step, indicate the limit law applied.

*Make sure you know which laws were applied where!

a. $\lim_{x \to -3} (4x + 2)$

$= \lim_{x \to -3} 4x + \lim_{x \to -3} 2$ (sum)

$= 4 \lim_{x \to -3} x + \lim_{x \to -3} 2$ (constant multiple)

$= 4(-3) + 2 = 10$

b. $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$

$$= \frac{2(4) - 3(2) + 1}{2^3 + 4} = \frac{1}{4}$$

c. $\lim_{x \to 6} (2x - 1)\sqrt{x + 4}$

$= (2(6) - 1)\sqrt{(6 + 4)} = 11\sqrt{10}$

2) For each of the following, assume that $\lim_{x \to 6} f(x) = 4$, $\lim_{x \to 6} g(x) = 9$, and $\lim_{x \to 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

a. $\lim_{x \to 6} \frac{g(x) - 1}{f(x)}$

$\frac{\lim_{x \to 6} g(x) - 1}{\lim_{x \to 6} f(x)} = \frac{9 - 1}{4} = 2$

b. $\lim_{x \to 6} [(x + 1) \cdot f(x)]$

$\lim_{x \to 6} (x + 1) \cdot \lim_{x \to 6} f(x) = 7 \cdot 4 = 28$

c. $\lim_{x \to 6} (f(x) \cdot g(x) - h(x))$

$= \lim_{x \to 6} f(x) \cdot \lim_{x \to 6} g(x) - \lim_{x \to 6} h(x) = 4 \cdot 9 - 4 = 30$

Limits of Polynomial and Rational Functions

In the previous examples, it has been the case that $\lim_{x \to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x \to a} p(x) = p(a) \text{ and}$$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

Examples: Evaluate the following limits.

1) $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$

$$\frac{2(3)^2 - 3(3) + 1}{5(3) + 4} = \frac{10}{19}$$

2) $\lim_{x \to -2} (3x^3 - 2x + 7)$

$$3(-2)^3 - 2(-2) + 7 = 13$$

Additional Limit Evaluation Techniques

We can evaluate the limits of polynomials and limits of some rational functions by direct substitution. However, it is possible for $\lim_{x \to a} f(x)$ to exist when $f(a)$ is undefined.

Calculating a Limit When $\frac{f(x)}{g(x)}$ has the Indeterminate Form $\frac{0}{0}$

1) First, make sure that the function has the appropriate form and cannot be evaluated immediately using the limit laws.

2) Then find a function that is equal to $h(x) = \frac{f(x)}{g(x)}$ for all $x \neq a$ over some interval containing a. To do this, try one or more of the following steps:

   a. If $f(x)$ and $g(x)$ are polynomials, factor each function and cancel out any common factors.

   b. If the numerator or denominator contains a difference involving square root, try multiplying the numerator and denominator by the conjugate of the expression involving a square root.

   c. If $\frac{f(x)}{g(x)}$ is a complex fraction, begin by simplifying it.

3) Last, apply the limit laws.

Media: Watch these video1 and video2 examples on limits of rational functions by factoring.

Media: Watch these video1 and video2 examples on finding limits by rationalizing.

Examples: Evaluate each of the following limits.

1) $\lim\limits_{x \to 3} \frac{x^2-3x}{2x^2-5x-3}$

$$= \lim_{x \to 3} \frac{x(x-3)}{(2x+1)(x-3)}$$

$$= \lim_{x \to 3} \frac{x}{2x+1} = \frac{3}{2(3)+1} = \frac{3}{7}$$

2) $\lim\limits_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$

$$=\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5} \cdot \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$$

$$= \lim_{x \to 5} \frac{x-5}{(x-5)\left(\sqrt{x-1}+2\right)}$$

$$= \lim_{x \to 5} \frac{1}{\sqrt{x-1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

3) $\lim\limits_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

$$= \lim_{x \to 1} \frac{2-(x+1)}{2(x-1)(x+1}$$

$$= \lim_{x \to 1} \frac{2-x-1}{2(x-1)(x+1)}$$

$$= \lim_{x \to 1} \frac{-1}{2(x+1)} = -\frac{1}{2(1+1)} = \frac{1}{4}$$

4) $\lim\limits_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)}\right)$

$$= \lim_{x \to 0} \left(\frac{x-5}{x(x-5)} + \frac{5}{x(x-5)}\right)$$

$$= \lim_{x \to 0} \frac{x}{x(x-5)}$$

$$= \lim_{x \to 0} \frac{1}{x-5} = \frac{1}{0-5} = \frac{1}{-5} = -\frac{1}{5}$$

Evaluating One-Sided and Two-Sided Limits Using Limit Laws

We can apply the limit laws to one-sided and two-sided limits, just make sure the function is defined over a given interval. For example, to find $\lim\limits_{x \to a^-} h(x)$, we need to make sure $h(x)$ is defined over the interval (b, a). Likewise, to find $\lim\limits_{x \to a^+} h(x)$, make sure $h(x)$ is defined over the interval (a, c).

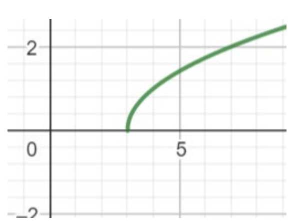
Media: Watch this [video]() example on finding limits of piecewise functions.

Examples:

1) Evaluate each of the following limits, if possible.

   a. $\lim\limits_{x \to 3^-} \sqrt{x-3}$

   b. $\lim\limits_{x \to 3^+} \sqrt{x-3}$



$\lim\limits_{x \to 3^-} \sqrt{x-3}$ Does not exist since the function is undefined left of 3

$\lim\limits_{x \to 3^+} \sqrt{x-3} = 0$ This is defined right of 3, so limit laws apply.

2) For $f(x) = \begin{cases} 4x - 3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

   a. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 4x - 3 = 4(2) - 3 = 5$

   b. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-3)^2 = (2-3)^2 = 1$

   c. $\lim_{x \to 2} f(x)$ DNE

3) Evaluate $\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)} = \lim_{x \to 2^-} \frac{x-3}{x} . \frac{1}{x-2} = \infty$

The Squeeze Theorem

Although the techniques we have been using work very well for algebraic functions, we also need to evaluate limits of trigonometric functions. The squeeze theorem can help us calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a common known limit at a.

The Squeeze Theorem

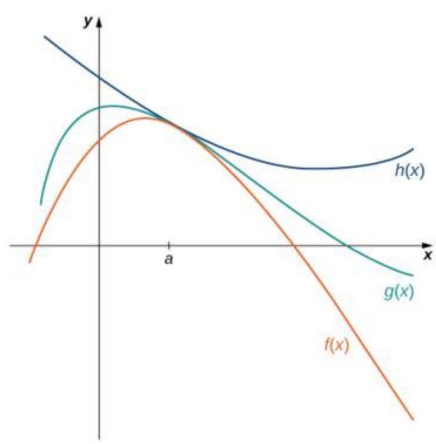
Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.



Media: Learn more about the squeeze theorem here.

Media: Watch these video1 and video2 examples on the squeeze theorem.

Examples

1) Evaluate the following limits by applying the squeeze theorem.

a. $\lim_{x \to 0} x \cos x$

$-1 \leq \cos x \leq 1$

$-|x| \leq x \cos x \leq |x|$

$\lim_{x \to 0} (-|x|) = \lim_{x \to 0} |x| = 0$, so $\lim_{x \to 0} x \cos x = 0$.

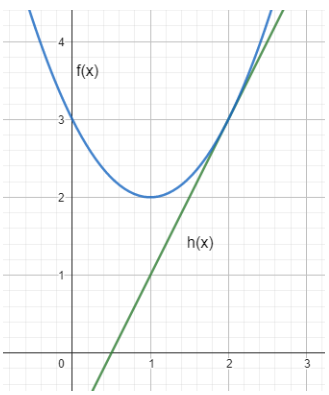
b. $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}$

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta}$$

$$= 1 \cdot \frac{0}{2} = 0$$

2) True or False? If $2x - 1 \leq g(x) \leq x^2 - 2x + 3$, then $\lim_{x \to 2} g(x) = 0$.

This is false. Let $f(x) = 2x - 1$ and $h(x) = x^2 - 2x + 3$. The graph of these functions is



$\lim_{x \to 2} 2x - 1 = 3$

$\lim_{x \to 2} x^2 - 2x + 3 = 3$

By the squeeze theorem, $\lim_{x \to 2} g(x) = 3$.