Section 5.7: Integrals Resulting in Inverse Trigonometric Functions

In this section we focus on integrals that result in inverse trigonometric functions. Recall that trigonometric functions are not one-to-one unless the domains are restricted.

The following integration formulas yield inverse trigonometric functions:

1) $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{|a|} + C$

2) $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

3) $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{|a|}\sec^{-1}\frac{u}{|a|} + C$

Media: Watch these video1, video2, and video3 examples on integrals involving inverse trig functions.

Examples

1) Evaluate the definite integral $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^{\frac{1}{2}}$$
$$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$
$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

2) Evaluate the integral $\int \frac{dx}{\sqrt{4-9x^2}}$.

Let $u = 3x$
$du = 3\,dx$
$\frac{1}{3}du = dx$

So,

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C$$

3) Evaluate the definite integral $\int_0^{\frac{\sqrt{3}}{2}} \frac{du}{\sqrt{1-u^2}}$.

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u\Big|_0^{\frac{\sqrt{3}}{2}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(0)$$
$$= \frac{\pi}{3}$$

4) Find an antiderivative of $\int \frac{1}{1+4x^2}\,dx$.

Let $u = 2x$
$du = 2\,dx$
$\frac{1}{2}du = dx$

So,

$$\int \frac{1}{1+4x^2}\,dx = \frac{1}{2}\tan^{-1} u + C$$
$$= \frac{1}{2}\tan^{-1}(2x) + C$$

5) Find an antiderivative of $\int \frac{1}{9+x^2}\,dx$.

$$\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{\pi}{3} + C$$

6) Evaluate the definite integral $\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$.

$$\int_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x|_{\frac{\sqrt{3}}{3}}^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}\frac{\sqrt{3}}{3}$$

$$= \frac{\pi}{6}$$