Section 4.10: Antiderivatives

Given a function f, how do we find a function with derivative f and why would we be interested in such a function?

The Reverse of Differentiation

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Knowing the rules of differentiation helps begin to find the antiderivatives of functions. For instance, consider the function $f(x) = 2x$. An antiderivative of f could be $F(x) = x^2$ since $F'(x) = 2x$. However, that is not the only function that has a derivative of $2x$. Since the derivative of any constant is zero, other functions, such as $x^2 + 5$ or $x^2 - 2$ could also have a derivative of $2x$.

General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

i. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;

ii. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of an antiderivative of f over I is $F(x) + C$.

Media: Watch this video example on antiderivatives of monomials.

Media: Watch this video example on antiderivatives of rational functions.

Media: Watch this video example on antiderivatives of trigonometric functions.

Examples: For each of the following functions, find all antiderivatives.

1) $f(x) = 3x^2$

Since $\frac{d}{dx}(x^3) = 3x^2$

then $F(x) = x^3$ is an antiderivative of $3x^2$

Every antiderivative of $3x^2$ is of the form $\boxed{x^3 + C}$

2) $f(x) = \frac{1}{x}$

Since $\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$

and $\frac{d}{dx}(\ln(-x)) = \frac{1}{x}$ for $x < 0$

therefore, $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

so every antiderivative is of the form $\boxed{\ln|x| + C}$

3) $f(x) = \cos x$

$\frac{d}{dx}(\sin x) = \cos x$

So every antiderivative is of the form $\boxed{\sin x + C}$

4) $f(x) = e^x$

$\frac{d}{dx}(e^x) = e^x$

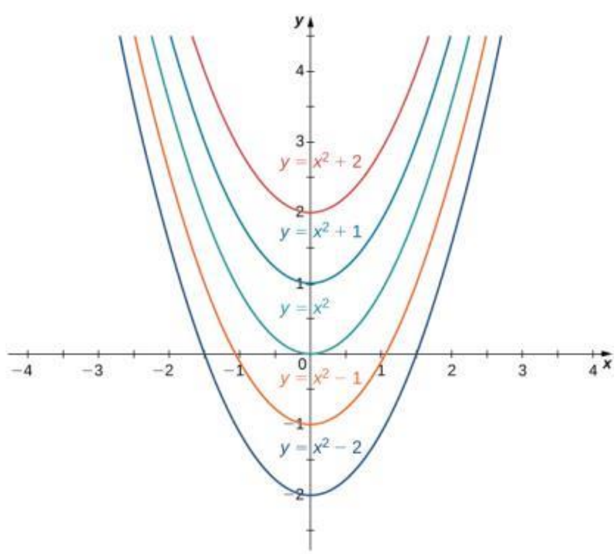
So every antiderivative is of the form $\boxed{e^x + C}$

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. Recall that when given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f.

> Given a function f, the **indefinite integral** of f, denoted
>
> $$\int f(x)\ dx,$$
>
> Is the most general antiderivative of f. If F is an antiderivative of f, then
>
> $$\int f(x)\ dx = F(x) + C.$$
>
> The expression $f(x)$ is called the **integrand** and the variable x is the **variable of integration**. The act of finding the antiderivatives of a function f is usually referred to as **integrating** f.

For a function f and an antiderivative F, the functions $F(x) + C$, where C is any real number, are often referred to as the **family of antiderivatives** of f. For example, the collection of all functions of the form $x^2 + C$, where C is any real number, is known as the family of antiderivatives of $2x$. The figure below shows a graph of this family.



The following table lists the indefinite integrals for several common functions.

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k\,dx = \int kx^0\,dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{1}{\sqrt{1+x^2}}dx = \tan^{-1} x + C$
$\frac{d}{dx}(\sec^{-1}\lvert x\rvert) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + C$

We can evaluate indefinite integrals for more complicated functions by using different properties of indefinite integrals.

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int (f(x) \pm g(x))\,dx = F(x) \pm G(x) + C$$

Constant Multiplies

$$\int kf(x)\,dx = kF(x) + C$$

Examples: Evaluate each of the following indefinite integrals.

1) $\int (5x^3 - 7x^2 + 3x + 4)\,dx$

$= \int 5x^3\,dx - \int 7x^2\,dx + \int 3x\,dx + \int 4\,dx$

$= 5\int x^3\,dx - 7\int x^2\,dx + 3\int x\,dx + 4\int dx$

$= 5\left(\frac{x^4}{4}\right) - 7\left(\frac{x^3}{3}\right) + 3\left(\frac{x^2}{2}\right) + 4(x) = \boxed{\frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C}$

2) $\int \frac{x^2+4\sqrt[3]{x}}{x}\,dx$ $\qquad \frac{x^2+4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = x + \frac{4}{x^{2/3}}$

$\int x + \frac{4}{x^{2/3}}\,dx = \int x\,dx + \int \frac{4}{x^{2/3}}\,dx$

$= \int x\,dx + 4\int x^{-2/3}\,dx$

$= \frac{1}{2}x^2 + 4\left(\frac{1}{-2/3+1}\right)x^{-2/3+1} = \boxed{\frac{1}{2}x^2 + 12x^{1/3} + C}$

3) $\int \frac{4}{1+x^2}\,dx$

$= 4\int \frac{1}{1+x^2}\,dx$

$= 4\left(\tan^{-1}x\right)$

$= \boxed{4\tan^{-1}x + C}$

4) $\int \tan x \cos x\,dx$ $\qquad \tan x \cos x = \frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x} = \sin x$

$= \int \sin x$

$= \boxed{-\cos x + C}$

Initial Value Problems

One common use for antiderivatives that arises often in many applications is solving differential equations. A **differential equation** is an equation that relates an unknown function and one or more of its derivatives.

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) – that is, $y(x_0) = y_0$ (the initial condition). The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

With the additional condition

$$y(x_0) = y_0$$

Is an example of an **initial-value problem**.

Media: Watch this video example on solving initial value problems.

Examples

1) Solve the initial-value problem $\frac{dy}{dx} = \sin x\,, y(0) = 5$.

$y = \int \sin x \, dx = -\cos x + C$

since $y(0)=5$, then $-\cos(0)+C=5$

$-1+C=5$

$C=6$

so $\boxed{y=-\cos x+6}$

2) A car is traveling at the rate of $88\,\frac{\text{ft}}{\text{sec}}$ (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of $15\,\frac{\text{ft}}{\text{sec}^2}$.

$v(0) = 88$ ft/sec

$a(t) = -15$ ft/sec^2

$v'(t) = -15$

a. How many seconds elapse before the car stops?

so $v(t) = \int -15\,dt = -15t + C$

$-15(0)+C = 88$

$C = 88$

$v(t) = -15t+88$

$-15t+88=0$

$-15t=-88$

$\boxed{t \approx 5.87 \text{ sec}}$

b. How far does the car travel during that time?

$v(t) = s'(t) = -15t+88$ and $s(0)=0$

$s(t) = \int -15t+88\,dt = -\frac{15}{2}t^2+88t$

$s(5.87) \approx \boxed{258.133 \text{ ft}}$