Section 3.6: The Chain Rule

We have seen the techniques for differentiating basic functions as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}$.

The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$h'(x) = f'\big(g(x)\big)g'(x).$$

Alternatively, if y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Applying the Chain Rule

1. To differentiate $h(x) = f(g(x))$, begin by indentifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$.

Media: Watch this video example on using the chain rule with the quotient rule.

Media: Watch this video example on the chain rule with trigonometric functions.

Examples

1) Find the derivative of each of the following:

   a. $h(x) = \frac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$

$$h'(x) = -2(3x^2+1)^{-3}(6x)$$

$$\boxed{h'(x) = -12x(3x^2+1)^{-3} \quad \text{or} \quad h'(x) = \frac{-12x}{(3x^2+1)^3}}$$

b. $h(x) = \sin^3 x$ $= (\sin x)^3$

$h'(x) = 3(\sin x)^2 (\cos x)$

$\boxed{h'(x) = 3\sin^2 x \cos x}$

Media: Watch this video example on using the chain rule to find the equation of a tangent line.

2) Find the equation of a line tangent to the graph of $f(x) = (x^2 - 2)^3$ at $x = -2$.

slope → $f'(x) = 3(x^2-3)^2(2x) = 6x(x^2-3)^2$

at $x=-2$ $f'(-2) = 6(-2)((-2)^2-3)^2 = -12$ ← slope

$f(-2) = ((-2)^2-2)^3 = 8$ point : $(-2, 8)$

$y - y_1 = m(x - x_1)$ $y - 8 = -12x - 24$

$\boxed{y - 8 = -12(x+2)}$ OR $\boxed{y = -12x - 16}$

Media: Watch this video example on using the chain rule with a table of values.

3) Use the information in the following table to find $h'(a)$ at the given value of a.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	**0**	2
1	1	−2	3	**0**
2	4	4	1	−1
3	3	−3	2	3

a. $h(x) = f(g(x)); a = 0$

$h'(x) = f'(g(x))g'(x)$ at $a=0$ $h'(0) = f'(g(0))g'(0)$

$= f'(0)g'(0) = 5 \cdot 2 = \boxed{10}$

b. $h(x) = f(x + f(x)); a = 1$

$h'(x) = f'(x + f(x))(1 + f'(x))$

$h'(1) = f'(1 + f(1))(1 + f'(1)) = f'(1+1)(1+ -2) = 4 \cdot -1 = \boxed{-4}$

c. $h(x) = \left(\frac{f(x)}{g(x)}\right)^2; a = 3$

$h'(x) = 2\left(\frac{f(x)}{g(x)}\right)^1\left(\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}\right)$

$h'(3) = 2\left(\frac{f(3)}{g(3)}\right)\left(\frac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2}\right) = 2\left(\frac{3}{2}\right)\left(\frac{2 \cdot -3 - (3 \cdot 3)}{2^2}\right)$

$= 3\left(\frac{-15}{4}\right) = \boxed{-\frac{45}{4}}$

Media: Watch this video example on the chain rule with other rules.

4) Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

$h'(x) = (2x+1)^5(7(3x-2)^6(3)) + (3x-2)^7(5(2x+1)^4(2))$

$h'(x) = (2x+1)^5(21(3x-2)^6) + (3x-2)^7(10(2x+1)^4)$

$h'(x) = 21(2x+1)^5(3x-2)^6 + 10(3x-2)^7(2x+1)^4$

$h'(x) = (2x+1)^4(3x-2)^6(21(2x+1) + 10(3x-2))$

$h'(x) = (2x+1)^4(3x-2)^6(42x+21+30x-20)$

$\boxed{h'(x) = (2x+1)^4(3x-2)^6(72x+1)}$

Composites of Three or More Functions

When differentiating the composition of three or more function, we need to apply the chain rule more than once.

Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h(f(g(x))),$$

Then

$$k'(x) = h'\left(f(g(x))\right) f'(g(x))g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, remember, always work from the outside in, taking one derivative at a time.

Media: Watch this video example on using chain rule with multiple functions.

Examples

1) Find the derivative of $k(x) = \cos^4(7x^2+1)$. $= (\cos(7x^2+1))^4$

$k'(x) = 4(\cos(7x^2+1))^3(-\sin(7x^2+1))(14x)$

$\boxed{k'(x) = -56x\cos^3(7x^2+1)\sin(7x^2+1)}$

2) A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

$\hookrightarrow s'(t)$

$$s'(t) = v(t) = \cos(2t)(2) - \sin(3t)(3)$$

$$v(t) = 2\cos(2t) - 3\sin(3t)$$

$$v\left(\frac{\pi}{6}\right) = 2\cos\left(2\cdot\frac{\pi}{6}\right) - 3\sin\left(3\cdot\frac{\pi}{6}\right)$$

$$v\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) - 3\sin\left(\frac{\pi}{2}\right)$$

$$v\left(\frac{\pi}{6}\right) = 2\cdot\frac{1}{2} - 3\cdot 1 = -2$$

$\boxed{\text{The velocity of the particle at time } t = \frac{\pi}{6} \text{ is } -2.}$

3) Let $h(x) = f(g(x))$. If $g(1) = 4$, $g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

$$h'(x) = f'(g(x))g'(x)$$

$$h'(1) = f'(g(1))g'(1)$$

$$h'(1) = f'(4)\cdot 3$$

$$h'(1) = 7\cdot 3 = \boxed{21}$$