Section 5.3: The Fundamental Theorem of Calculus

The only tools available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, both of which are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

The Mean Value Theorem for Integrals

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If $f(x)$ is continuous over an interval $[a,b]$, then there is at least one point $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This formula can also be stated as

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Essentially, this theorem states that a continuous function on a closed interval takes on its average value at some point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a,b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a,b]$.

Media: Watch this video example on the Mean Value Theorem with area.

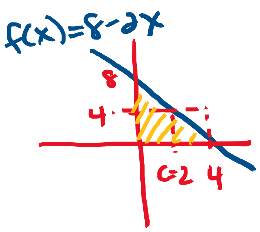
Examples

1) Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0,4]$ and find c such that $f(c)$ equals the average value of the function over $[0,4]$.

mean value → $\frac{1}{4-0}\int_0^4 (8-2x)\,dx$

Area of $\Delta = \frac{1}{2}bh$

Area of $\Delta = \frac{1}{2}(4)(8) = 16$



makes Δ w/ x and y-axes

average value is $\frac{1}{4}(16) = 4$

so $8 - 2c = 4$

$\boxed{c = 2}$

2) Given $\int_0^3 (2x^2 - 1)\,dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0,3]$.

find c such that $f(c) = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}(9) = 3$

Replace $f(c)$ with c^2: $c^2 = 3$

$c = \pm\sqrt{3}$

Since $-\sqrt{3}$ is outside the interval, we only take the positive value. So $\boxed{c = \sqrt{3}}$

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

The Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives a way to evaluate definite integrals without using Riemann sums or calculating areas.

> **Fundamental Theorem of Calculus, Part 1**
>
> If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by
>
> $$F(x) = \int_a^x f(t)\, dt,$$
>
> Then $F'(x) = f(x)$ over $[a, b]$.

Not only does the Fundamental Theorem of Calculus establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Media: Watch this video example on the Fundamental Theorem of Calculus Part 1.

Examples

1) Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(x) = \int_1^x \frac{1}{t^3+1}\, dt$.

$$\boxed{g'(x) = \frac{1}{x^3+1}}$$

2) Let $F(x) = \int_1^{\sqrt{x}} \sin t\, dt$. Find $F'(x)$.

let $u(x) = \sqrt{x}$, then $F(x) = \int_1^{u(x)} \sin t\, dt$

$$F'(x) = \sin(u(x))\frac{du}{dx}$$
$$= \sin(u(x)) \cdot \left(\tfrac{1}{2}x^{-1/2}\right) = \boxed{\frac{\sin\sqrt{x}}{2\sqrt{x}}}$$

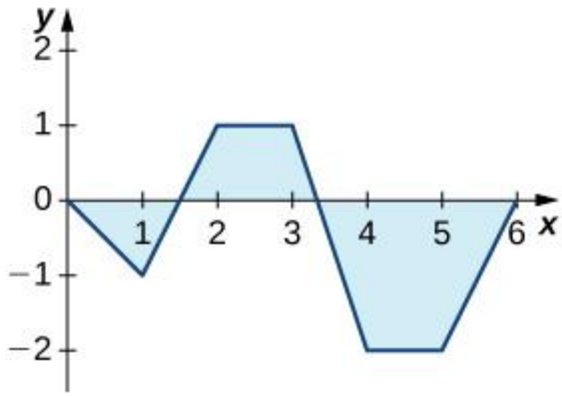
3) Let $F(x) = \int_x^{2x} t^3\, dt$. Find $F'(x)$.

$$F(x) = \int_x^{2x} t^3 dt = \int_x^0 t^3 dt + \int_0^{2x} t^3 dt$$
$$= -\int_0^x t^3 dt + \int_0^{2x} t^3 dt$$
$$F'(x) = \frac{d}{dx}\left[-\int_0^x t^3 dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3 dt\right]$$
$$= -x^3 + 16x^3 = \boxed{15x^3}$$

$$\frac{d}{dx}\left[\int_0^{2x} t^3 dt\right]$$
$$= \frac{d}{dx}\left[\int_0^{u(x)} t^3 dt\right]$$
$$= (u(x))^3 \frac{du}{dx}$$
$$= (2x)^3 \cdot 2$$
$$= 16x^3$$

Media: Watch this video example on properties of a function given a graph of the integral.

4) The graph of $y = \int_0^x f(t)\,dt$, where f is a piecewise constant function, is shown below.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?

positive: [1, 2] and [5, 6]

negative: [0, 1] and [3, 4]

zero: [2, 3] and [4.5]

b. What are the maximum and minimum values of f?

The max value is 2.

The min value is -3.

c. What is the average value of f?

The average value is 0.

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. The area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

> **The Fundamental Theorem of Calculus, Part 2**
>
> If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus, Part 2 (also known as the evaluation theorem) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Media: Watch this video example on the Fundamental Theorem of Calculus Part 2.

Examples

1) Use the Fundamental Theorem of Calculus, Part 2 to evaluate the following integrals:

   a. $\int_{-2}^{2}(t^2-4)\,dt$

$$\int_{-2}^{2}(t^2-4)\,dt = \frac{t^3}{3}-4t\Big|_{-2}^{2} = \left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right]$$

$$= \boxed{-\frac{32}{3}}$$

   b. $\int_{1}^{9}\frac{x-1}{\sqrt{x}}\,dx$

$$\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = \int_1^9 \frac{x-1}{x^{1/2}}\,dx = \int_1^9 \left(x^{1/2}-x^{-1/2}\right)dx$$

$$=\left(\frac{2}{3}x^{3/2}-2x^{1/2}\right)\Big|_1^9 = \left[\frac{2}{3}(9)^{3/2}-2(9)^{1/2}\right]-\left[\frac{2}{3}(1)^{3/2}-2(1)^{1/2}\right]$$

$$= \boxed{40/3}$$

2) James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t)=5+2t\ \frac{\text{ft}}{\text{sec}}$ and Kathy can skate at a velocity of $g(t)=10+\cos\left(\frac{\pi}{2}t\right)\frac{\text{ft}}{\text{sec}}$, who is going to win the race?

James: $\int_0^5 (5+2t)\,dt = \left(5t+\cancel{2}\cdot\frac{t^2}{\cancel{2}}\right)\Big|_0^5 = \left[5(5)+(5)^2\right]-0 = \boxed{50}$

Kathy: $\int_0^5 10+\cos\left(\frac{\pi}{2}t\right)dt = \left(10t+\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\right)\Big|_0^5$

$$= \left(50+\frac{2}{\pi}\right)-\left(0-\frac{2}{\pi}\sin 0\right) \approx \boxed{50.6}$$

$\boxed{\text{Kathy will win the race}}$