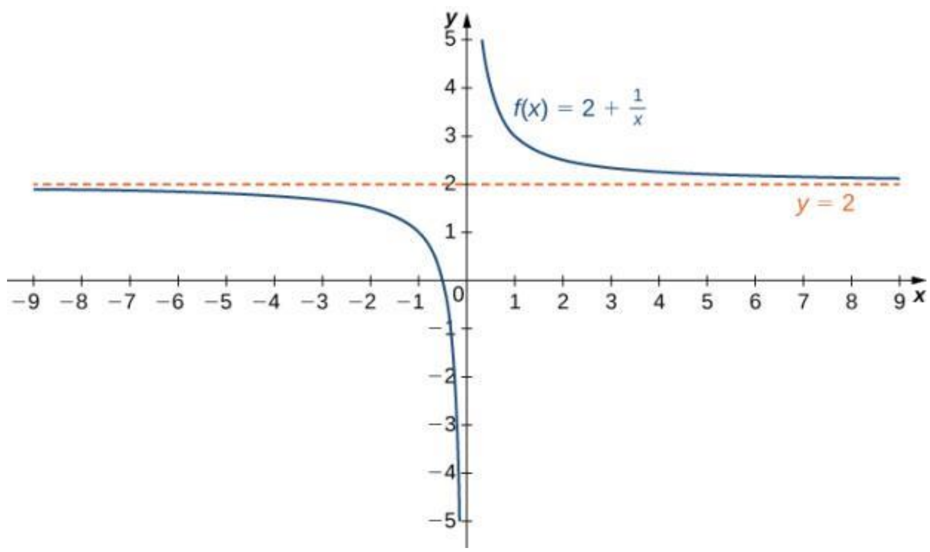
Section 4.6: Limits at Infinity and Asymptotes

Limits at Infinity

In this section, we focus on the behavior of a function at the extreme values of x and look at horizontal asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim_{x \to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as x gets closer to a. We extend this idea to limits at infinity. For example, in the graph below, as x gets larger (moving to the right in the graph), the values of $f(x)$ get closer to 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim_{x \to \infty} f(x) = 2$.



Similarly, as x gets smaller (moving to the left in the graph), the values of $f(x)$ get closer to 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim_{x \to -\infty} f(x) = 2$.

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, the function f has a limit at infinity, written

$$\lim_{x \to \infty} f(x) = L.$$

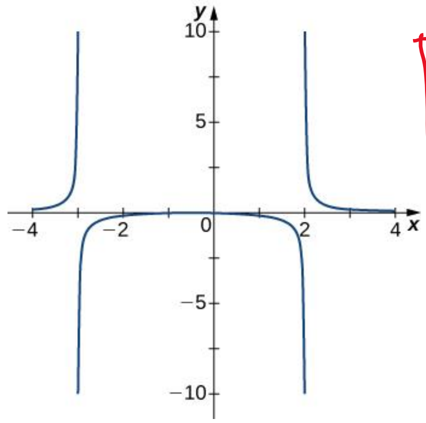
If the values of $f(x)$ become arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, the function f has a limit at negative infinity, written

$$\lim_{x \to -\infty} f(x) = L.$$

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, the line $y = L$ is a **horizontal asymptote** of f.

Examples:

1) For the graph below, identify where the vertical and horizontal asymptotes are located.



$$\boxed{\begin{array}{l} \text{vertical asymptotes: } x = -3 \\ \qquad\qquad\qquad\qquad\quad\;\; x = 2 \\ \text{horizontal asymptotes: } y = 0 \end{array}}$$

2) For each of the following functions f, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Determine the horizontal asymptote(s) for f.

   a. $f(x) = 5 - \frac{2}{x^2}$

$\lim_{x\to\infty} 5 - \frac{2}{x^2} = 5 \qquad \lim_{x\to-\infty} 5 - \frac{2}{x^2} = 5 \Rightarrow \boxed{y=5}$

   b. $f(x) = \frac{\sin x}{x} = \sin x \cdot \frac{1}{x} \qquad \lim_{x\to\infty} \frac{1}{x} = 0$

* squeeze thm.

so $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ & $\lim_{x\to\infty} \frac{\sin x}{x} = 0 \Rightarrow \boxed{y=0}$

   c. $f(x) = \tan^{-1}(x)$ look at $(-\pi/2, \pi/2)$

$\lim_{x\to\pi/2^-} \tan x = \infty$ & $\lim_{x\to\pi/2^+} \tan x = -\infty$

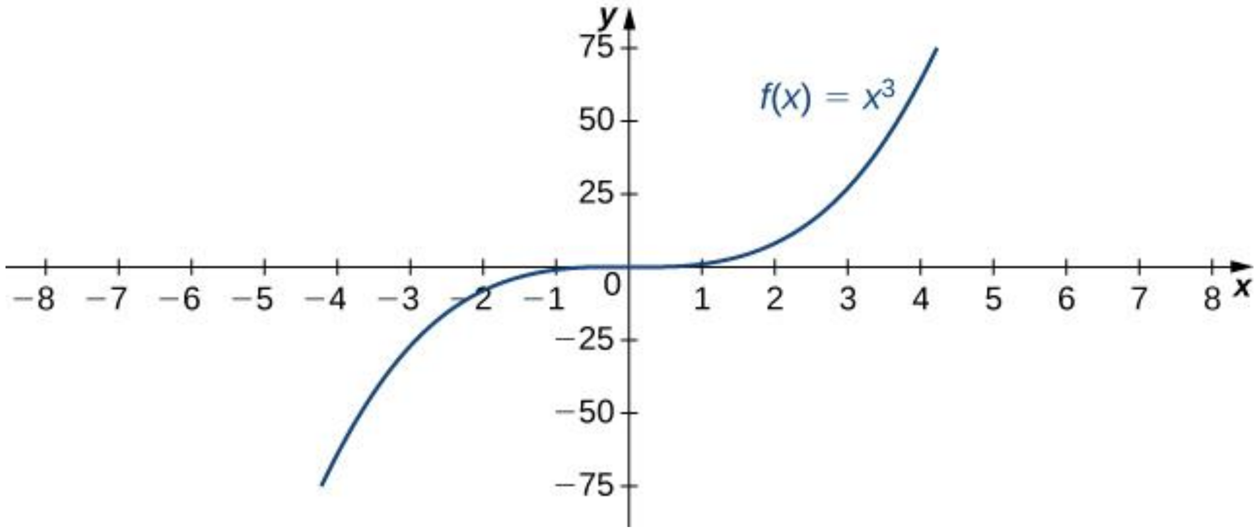
$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2} \qquad \lim_{x\to-\infty} \tan^{-1}(x) = \frac{-\pi}{2}$

$\Rightarrow \boxed{\begin{array}{l} y = \pi/2 \\ y = -\pi/2 \end{array}}$

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$ (or as $x \to -\infty$). In this case, we write $\lim_{x\to\infty} f(x) = \infty$ (or $\lim_{x\to-\infty} f(x) = \infty$). If the values of a function f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim_{x\to\infty} f(x) = -\infty$ (or $\lim_{x\to-\infty} f(x) = -\infty$).

For example, in the graph below of the function $f(x) = x^3$, as $x \to \infty$, the values of $f(x)$ become arbitrarily large. Therefore, $\lim_{x\to\infty} x^3 = \infty$.



As $x \to -\infty$, the values of $f(x)$ are negative but arbitrarily large. Therefore, $\lim_{x\to-\infty} x^3 = -\infty$.

A function f has an **infinite limit at infinity** and write

$$\lim_{x\to\infty} f(x) = \infty,$$

if $f(x)$ becomes arbitrarily large for x sufficiently large.

Aa function has a negative infinite limit at infinity and write

$$\lim_{x\to-\infty} f(x) = -\infty,$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large.

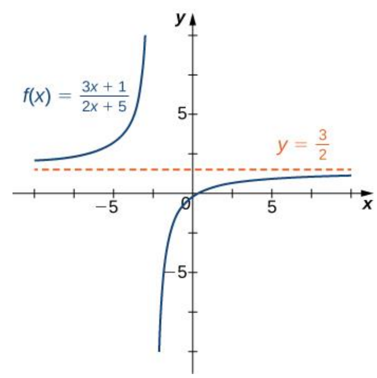
Similarly, we can define infinite limits as $x \to -\infty$.

Example: Evaluate $\lim_{x\to-\infty} \frac{x^2-2x+5}{x+2}$. $= \boxed{\infty}$ look at graph

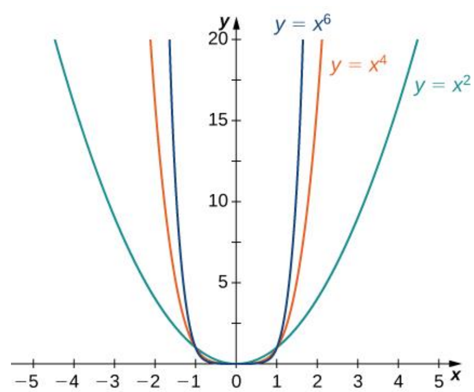
End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

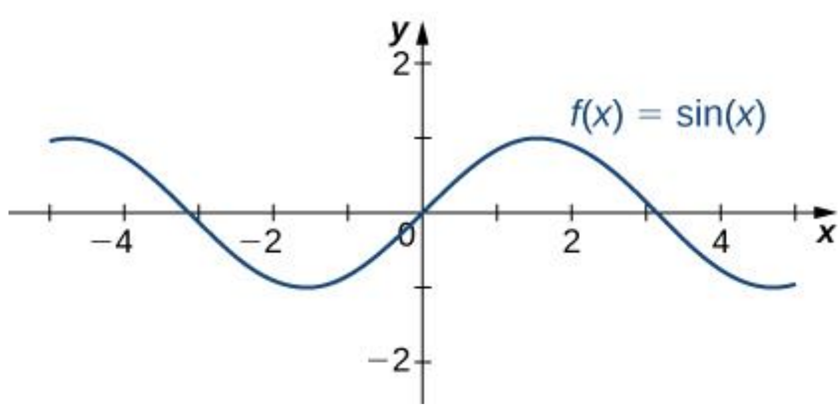
1. The function $f(x)$ approaches a horizontal asymptote $y = L$. Many rational functions have a horizontal asymptote.



2. The function $f(x) \to \infty$ or $f(x) \to -\infty$. Many power functions have this type of behavior.



3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior. Trigonometric functions, like sine and cosine, typically have this type of behavior.



Examples

1) Find the horizontal and vertical asymptotes for the function $f(x) = \frac{1}{x^3+x^2}$.

$x^2(x+1)$

$\boxed{\begin{array}{ll} \text{horizontal} & y=0 \\ \text{vertical} & x=0 \\ & x=-1 \end{array}}$

can also look at graph

2) Construct a function that has the given asymptotes:

- $x = 1$
- $y = 2$

$\boxed{\text{Answers will vary}}$

$\boxed{\underline{\text{example:}}\quad y = \frac{2x}{x-1}}$