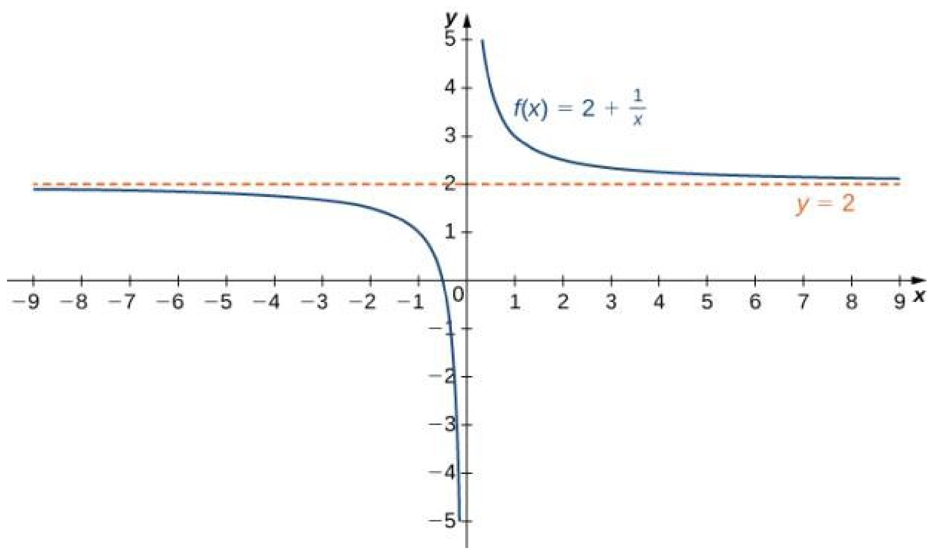
Section 4.6: Limits at Infinity and Asymptotes

Limits at Infinity

In this section, we focus on the behavior of a function at the extreme values of x and look at horizontal asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim_{x\to a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as x gets closer to a. We extend this idea to limits at infinity. For example, in the graph below, as x gets larger (moving to the right in the graph), the values of $f(x)$ get closer to 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim_{x\to\infty} f(x) = 2$.



Similarly, as x gets smaller (moving to the left in the graph), the values of $f(x)$ get closer to 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim_{x\to-\infty} f(x) = 2$.

If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, the function f has a limit at infinity, written

$$\lim_{x\to\infty} f(x) = L.$$

If the values of $f(x)$ become arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, the function f has a limit at negative infinity, written

$$\lim_{x\to-\infty} f(x) = L.$$

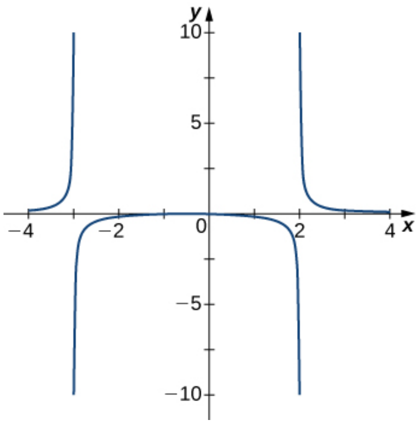
If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, the line $y = L$ is a **horizontal asymptote** of f.

Media: Watch this video to learn more about horizontal asymptotes.

Media: Watch these video1 and video2 examples on limits at infinity and horizontal asymptotes.

Examples:

1) For the graph below, identify where the vertical and horizontal asymptotes are located.



Vertical asymptotes: $x = -3$ and $x = 2$

Horizontal asymptote: $y = 0$

2) For each of the following functions f, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Determine the horizontal asymptote(s) for f.

a. $(x) = 5 - \frac{2}{x^2}$

$\lim_{x\to\infty} 5 - \frac{2}{x^2} = 5$ and $\lim_{x\to-\infty} 5 - \frac{2}{x^2} = 5$ so there is a horizontal asymptote at $y = 5$

b. $f(x) = \frac{\sin x}{x} = \sin x \cdot \frac{1}{x}$ and $\lim_{x\to\infty} \frac{1}{x} = 0$

We use the squeeze theorem.
$\lim_{x\to\infty} \frac{\sin x}{x} = 0$ and $\lim_{x\to-\infty} \frac{\sin x}{x} = 0$ so there is a horizontal asymptote at $y = 0$.

c. $f(x) = \tan^{-1}(x)$ We look at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
Since $\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$, $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$

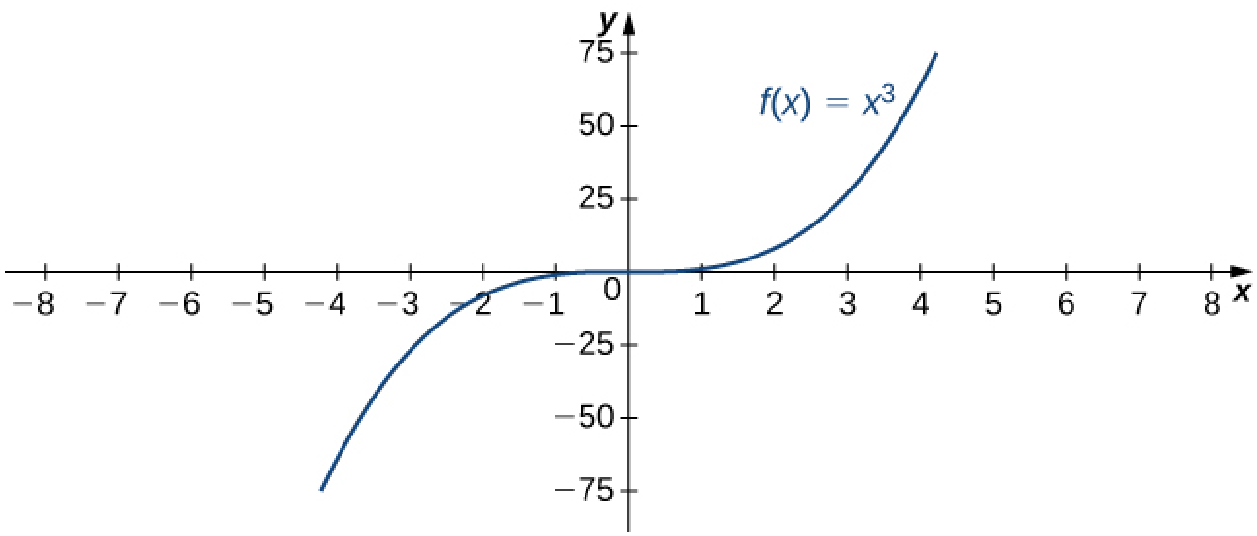
And since $\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$, $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

So $f(x)$ has horizontal asymptotes at $y = \frac{\pi}{2}, y = -\frac{\pi}{2}$

Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \to \infty$ (or as $x \to -\infty$). In this case, we write $\lim_{x \to \infty} f(x) = \infty$ (or $\lim_{x \to -\infty} f(x) = \infty$). If the values of a function f are negative but become arbitrarily large in magnitude as $x \to \infty$ (or as $x \to -\infty$), we write $\lim_{x \to \infty} f(x) = -\infty$ (or $\lim_{x \to -\infty} f(x) = -\infty$).

For example, in the graph below of the function $f(x) = x^3$, as $x \to \infty$, the values of $f(x)$ become arbitrarily large. Therefore, $\lim_{x \to \infty} x^3 = \infty$.



As $x \to -\infty$, the values of $f(x)$ are negative but arbitrarily large. Therefore, $\lim_{x \to -\infty} x^3 = -\infty$.

A function f has an **infinite limit at infinity** and write

$$\lim_{x \to \infty} f(x) = \infty,$$

if $f(x)$ becomes arbitrarily large for x sufficiently large.

Aa function has a negative infinite limit at infinity and write

$$\lim_{x \to -\infty} f(x) = -\infty,$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large.

Similarly, we can define infinite limits as $x \to -\infty$.

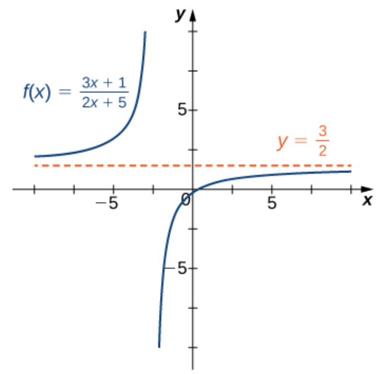
Media: Watch this video example on infinite limits at infinity.

Example: Evaluate $\lim_{x \to -\infty} \frac{x^2 - 2x + 5}{x + 2} = \infty$ by examining the graph

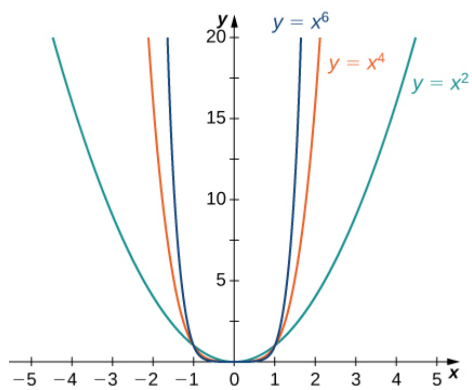
End Behavior

The behavior of a function as $x \to \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

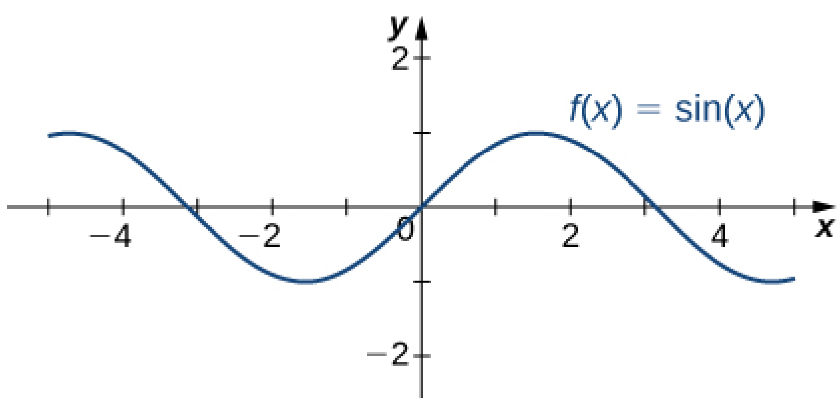
1. The function $f(x)$ approaches a horizontal asymptote $y = L$. Many rational functions have a horizontal asymptote.



2. The function $f(x) \to \infty$ or $f(x) \to -\infty$. Many power functions have this type of behavior.



3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior. Trigonometric functions, like sine and cosine, typically have this type of behavior.



Media: Watch this video example on asymptotes.

Media: Watch this video example on sketching functions with given asymptotes.

Examples

1) Find the horizontal and vertical asymptotes for the function $f(x) = \frac{1}{x^3+x^2}$.

   Notice that the denominator is equal to $x^2(x+1)$. There are vertical asymptotes at $x = 0, x = -1$. There is a horizontal asymptote at $y = 0$.
   One can also examine the graph.

2) Construct a function that has the given asymptotes:

   - $x = 1$
   - $y = 2$

Answers will vary. One example is $y = \frac{2x}{x-1}$.