Introduction to Linear Programming

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Optimization is a very broad and complex topic; in this note, we introduce you to the most common class of optimization models- linear programming (LP).

Developing any optimization model consists of four basic steps:

1) Identify the decision variables.
2) Identify the objective function.
3) Identify all appropriate constraints.
4) Write the objective function and constraints as mathematical expressions.

Decision variables are the *unknown* values that the model seeks to determine. Depending on the application, decision variables might be the quantities of different products to produce, amount of money spent on R&D projects, the amount of shelf space to devote to a product, and so on.

The quantity that we seek to minimize or maximize is called the **objective function**; for example, we might wish to maximize profit or minimize cost or some measure of risk. The objective function is a function of decision variables.

Constraints are limitations, requirements, or other restrictions that are imposed on any solution, either from practical or technological considerations or by management policy.

- The presence of constraints along with a large number of variables usually makes identifying an optimal solution considerably more difficult and necessitates the use of powerful software tools.

- The essence of building an optimization model is to first identify these model components, and then translate the objective function and constraints into **mathematical expressions**.

Characteristics of Linear Programming Models:

- The objective function and all constraints are linear functions of the decision variables.

- All variables are continuous, meaning that they may assume any real value (typically, nonnegative). Of course, this assumption may not be realistic for a practical business problem (you can not produce half a refrigerator).

Example: WYNDOR GLASS CO. Resource Allocation

The WYNDOR GLASS CO. produces high-quality glass products, including *windows* and *glass doors*. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earnings, top management has decided to revamp the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

- Product 1: A glass door with aluminum framing
- Product 2: A double-hung wood-framed window

The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity, it is not clear which mix of the two products would be most profitable (Each product will be produced in batches of 20, so the production rate is defined as the number of batches produced per week.) Therefore, an OR (operations research) team has been formed to study this question. The OR team gathered the following data:

- Product 1 requires one hour production time in Plants 1 and three hours of production time in Plant 3.

- Product 2 requires two hours of production time in Plants 2 and two hours of production time in Plant 3.

- Plant 1, 2, and 3 are available for 4, 12, and 18 hours of production time per week respectively.

- Profit per batch produced of product 1 and 2 are $3,000 and $5,000 respectively.

The formulation as a Linear Programming Problem:

$x_1 =$ number of batches of product 1 produced per week

$x_2 =$ number of batches of product 2 produced per week

$Z =$ total profit per week (in thousands of dollars)

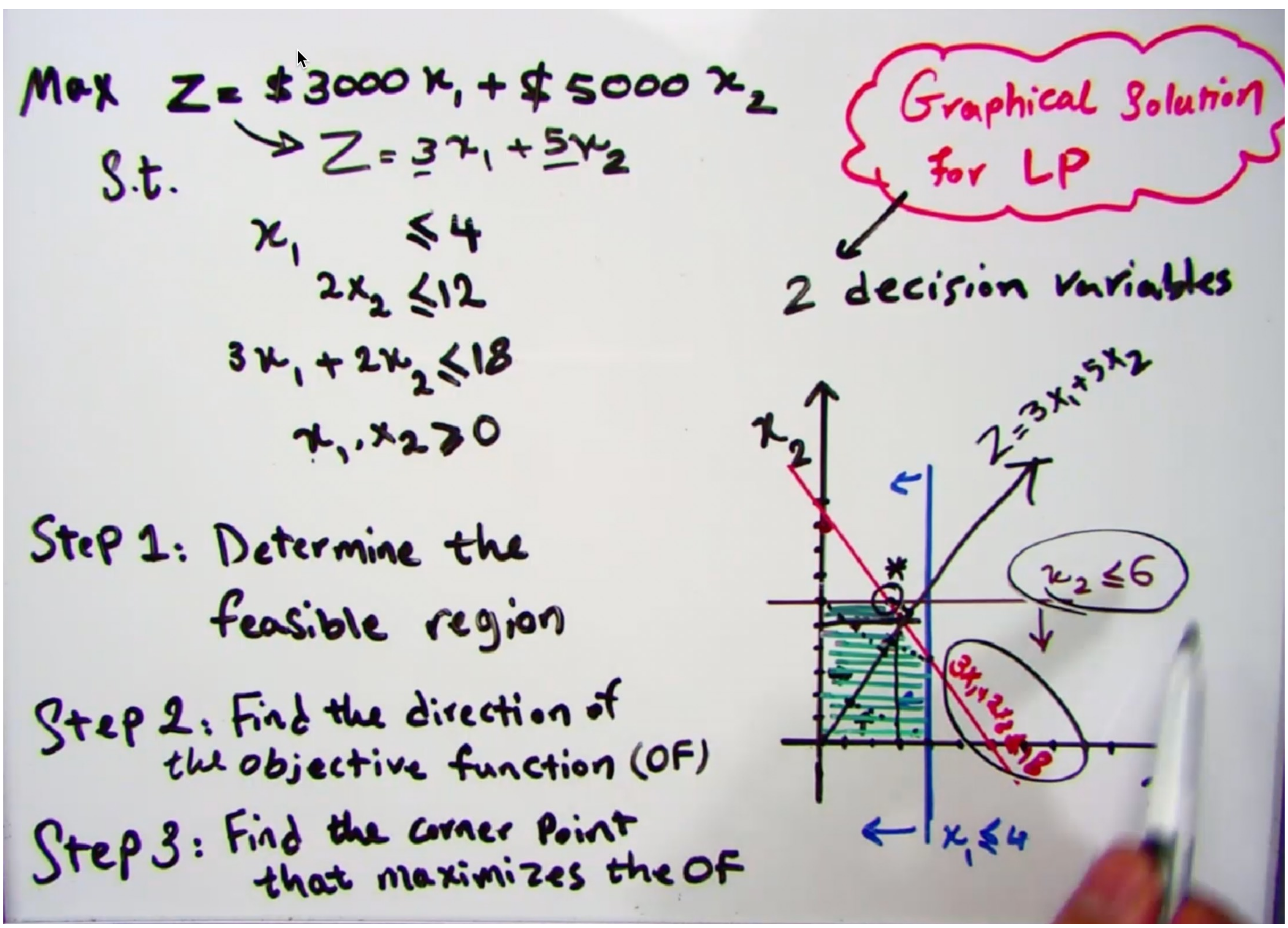
To summarize, in the mathematical language of linear programming, the problem is to choose values of x_1 and x_2 so as to:

This problem is a classic example of a *resource-allocation problem*, the most common type of linear programming problem.

LP Graphical Solution

If our LP problem has only two decision variables and therefore only two dimensions, so a graphical procedure can be used to solve it. This procedure involves constructing a two-dimensional graph with x_1 and x_2 as the axes. The first step is to identify the values of (x_1, x_2) that are permitted by the restrictions. This is done by drawing each line that borders the range of permissible values for one restriction. Then, we pick out the point in this feasible region that maximizes the value of the objective function.

Example: LP Graphical Solution for WYNDOR GLASS CO. Resource Allocation Problem



Graphical Solution for LP

2 decision variables

$$\text{Max } Z = \$3000\,x_1 + \$5000\,x_2 \;\rightarrow\; Z = \underline{3}x_1 + \underline{5}x_2$$

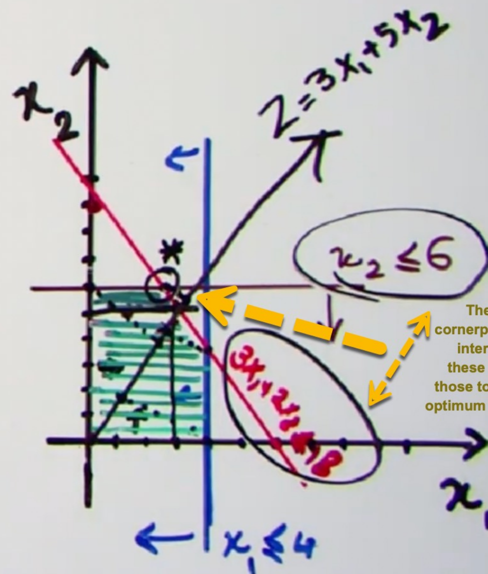
s.t.

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

Solve for intersection of these two lines to determine optimum values. Plug those values into the objective function (Z)

$$\boxed{\begin{aligned} x_2 &= 6 \\ 3x_1 + 2x_2 &= 18 \end{aligned}} \;\rightarrow\; \boxed{\begin{cases} x_1^* = 2 \\ x_2^* = 6 \end{cases} \quad Z^* = \$36000}$$

$$3x_1 + 2(6) = 18$$
$$3x_1 = 6 \Rightarrow x_1 = 2$$



The optimum cornerpoint is the intersection of these lines. Use those to calculate optimum values on the left.

Class Assignment- Intro to LP

3.1-9. The Primo Insurance Company is introducing two new product lines: special risk insurance and mortgages. The expected profit is \$5 per unit on special risk insurance and \$2 per unit on mortgages.

Management wishes to establish sales quotas for the new product lines to maximize total expected profit. The work requirements are as follows:

	Work-Hours per Unit		
Department	**Special Risk**	**Mortgage**	**Work-Hours Available**
Underwriting	3	2	2400
Administration	0	1	800
Claims	2	0	1200

a) Formulate a linear programming model for this problem.

b) Solve the problem. Only provide the final answer for the optimal decision variables and objective value.

c) Given you solved this problem graphically; the final answer is based on the intersection of what constraints.