预积分

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1 IMU 的测量值及其积分

IMU 测量加速度 $\alpha_{\mathbf{m}}^{\mathbf{b}}$ 和角速度 $\omega_{\mathbf{m}}^{\mathbf{b}}$，重力加速度定义为 $[0\ 0\ \ -9.81]^T$。真实的加速度和角速度在世界坐标系下为 $\alpha^{\mathbf{w}}$ 和角速度 $\omega^{\mathbf{w}}$。白噪声和 Bias 分别为 n 和 b:

$$
\begin{aligned}
\omega_{\mathbf{m}}^{\mathbf{b}} &= \omega^{\mathbf{b}} + \mathbf{b^g} + \mathbf{n^g} \\
\alpha_{\mathbf{m}}^{\mathbf{b}} &= \mathbf{q_{bw}}(\alpha^{\mathbf{w}} - \mathbf{g}) + \mathbf{b^a} + \mathbf{n^a}
\end{aligned}
$$

因此，body 系在世界坐标系下真实的角速度和加速度为:

$$
\begin{aligned}
\omega^{\mathbf{w}} &= \omega^{\mathbf{b}} = \omega_{\mathbf{m}}^{\mathbf{b}} - \mathbf{b^g} - \mathbf{n^g} \\
\alpha^{\mathbf{w}} &= \mathbf{q_{wb}}(\alpha_{\mathbf{m}}^{\mathbf{b}} - \mathbf{b^a} - \mathbf{n^a}) + \mathbf{g}
\end{aligned}
$$

在这里需要注意的是，角速度是无所谓世界坐标系和 body 系的。

测量量角速度和加速度与位姿的关系为:

$$
\begin{aligned}
&\dot{q}_{wb_t} = q_{wb_t} \bigotimes \begin{bmatrix} 0 \\ \frac{1}{2}\omega^{b_t} \end{bmatrix} \text{! 这里的 q 和 w 是不同的!} \\
&\dot{v}_t^w = a_t^w \\
&\dot{p}_{wb_t} = v_t^w
\end{aligned}
$$

对上式积分，就可以从测量量得到位姿:

$$
\begin{aligned}
&q_{wb_j} = \int_i^j q_{wb_t} \bigotimes \begin{bmatrix} 0 \\ \frac{1}{2}\omega^{b_t} \end{bmatrix} dt \\
&v_j^w = v_i^w + \int_i^j a_t^w dt = v_i^w + \int_i^j (q_{wb_t} a_t^b + g) dt \\
&p_{wb_j} = p_{wb_i} + \int_i^j v_t^w dt = p_{wb_i} + v_i^w \Delta t + \iint_i^j (q_{wb_t} a_t^b + g) dt^2
\end{aligned}
$$

以上是在连续时间下的测量量到位姿量的变换。i 和 j 是任意两个时刻。接下来将 i 和 j 定义为 IMU 两个连续测量时刻，即实际情况下 IMU 的两个连续测量值 k 和 k+1, 则可利用离散中值积分来计

算测量量到位姿量的变换：

$$\text{角速度测量量：}\omega=\frac{1}{2}((\omega_k^b-b_k^g)+(\omega_{k+1}^b-b_k^g))$$

$$q_{wb_{k+1}}=q_{wb_k}\bigotimes\begin{bmatrix}1\\ \frac{1}{2}\omega\Delta t\end{bmatrix}$$

$$\text{加速度测量量：}\alpha=\frac{1}{2}((q_{wb_k}(\alpha_k^b-b_k^a)+g)+(q_{wb_{k+1}}(\alpha_{k+1}^b-b_k^a)+g))$$

$$v_{k+1}^w=v_k^w+a\Delta t$$

$$p_{wb_{k+1}}=p_{wb_k}+v_k^w\Delta t+\frac{1}{2}a\Delta t^2$$

2 预积分

在 vio 中，相机的 BA 也会影响位姿。在这里，我们将与相机采样周期同步的两个时间称作 i 和 j。从之前的连续积分可以看到，一旦 i 时刻 q_{wb_i} 发生了改变，例如由于 BA 之后发生了更新，我们需要重新积分，才能得到新的 q_{wb_j}。这将会耗费大量运算，因为 LM 算法每迭代一步，就会更新 q_{wb_i}，从而影响到后面所有的位姿。于是，预积分的概念应运而生。预积分，本质上是计算两个相机时刻之间的相对位姿。对预积分，我认为有两种理解：

1) 预积分将 i 和 j 之间的 IMU 测量值积分 $q_{b_ib_j}$，将其固定，从而每次更新 q_{wb_i}，直接右乘 $q_{b_ib_j}$，就可以得到 q_{wb_j}。

2) 预积分值 $q_{b_ib_j}$ 被当作测量量来构造残差，就好比 BA 中的特征点像素值。

接下来我们进行预积分的具体推导，最后用其构建残差：

$$\text{预积分的出发点公式：}q_{wb_t}=q_{wb_i}\bigotimes q_{b_ib_t}$$

$$\text{用上式替换 IMU 测量量到位姿量积分公式中的}q_{wb_t}\rightarrow$$

$$q_{wb_j}=\int_i^j q_{wb_t}\bigotimes\begin{bmatrix}0\\ \frac{1}{2}\omega^{b_t}\end{bmatrix}dt=q_{wb_i}\bigotimes\underbrace{\int_i^j q_{b_ib_t}\bigotimes\begin{bmatrix}0\\ \frac{1}{2}\omega^{b_t}\end{bmatrix}dt}_{\text{旋转的预积分：}q_{b_ib_j}}$$

$$v_j^w=v_i^w+\int_i^j(q_{wb_t}a_t^b+g)dt=v_i^w+g\Delta t+q_{wb_i}\bigotimes\underbrace{\int_i^j q_{b_ib_t}a_t^b dt}_{\text{速度的预积分：}\beta_{b_ib_j}}$$

$$p_{wb_j}=p_{wb_i}+v_i^w\Delta t+\iint_i^j(q_{wb_t}a_t^b+g)dt^2=p_{wb_i}+v_i^w\Delta t+\frac{1}{2}g\Delta t^2+q_{wb_i}\bigotimes\underbrace{\iint_i^j q_{b_ib_t}a_t^b dt^2}_{\text{位置的预积分：}\alpha_{b_ib_j}}$$

上式可以简写为：

$$q_{wb_j}=q_{wb_i}\bigotimes q_{b_ib_j}$$

$$v_j^w=v_i^w+g\Delta t+q_{wb_i}\bigotimes\beta_{b_ib_j}$$

$$p_{wb_j}=p_{wb_i}+v_i^w\Delta t+\frac{1}{2}g\Delta t^2+q_{wb_i}\bigotimes\alpha_{b_ib_j}$$

接着我们利用离散中值积分来计算预积分，此过程与测量值的离散积分类似，不同点在于，加速度变换到世界坐标系后不需要去除重力加速度，以及积分相对于 b_i 坐标系：

$$\omega = \frac{1}{2}((w_k^b - b_k^g) + (w_{k+1}^b - b_k^g))$$

$$q_{b_i b_{k+1}} = q_{b_i b_k} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega\Delta t \end{bmatrix}$$

$$\alpha = \frac{1}{2}(q_{b_i b_k}(\alpha_k^b - b_k^a) + q_{b_i b_{k+1}}(\alpha_{k+1}^b - b_k^a))$$

$$\beta_{b_i b_{k+1}} = \beta_{b_i b_k} + \alpha\Delta t$$

$$\alpha_{b_i b_{k+1}} = \alpha_{b_i b_k} + \beta_{b_i b_k}\Delta t + \frac{1}{2}\alpha\Delta t^2$$

同时我们考虑 bias 的随机游走：

$$b_{k+1}^g = b_k^g + n_{b_k^g}\Delta t$$

$$b_{k+1}^a = b_k^a + n_{b_k^a}\Delta t$$

这里需要提一下，对于 k+1 时刻的测量量 Bias，我们使用 k 时刻的 Bias。

3 构建残差

正如前面所说，为了构建残差，我们使用预积分作为“测量值”。那么什么是“理论值”呢？正如在 BA 中将 3D 点的反向投影作为“理论值”，两个相机时刻 i 和 j 之间的相对位姿作为“理论值”。因为在理想情况下，“理论值” 和 “测量值” 之间有如下关系：

$$q_{b_i w} \bigotimes q_{w b_j} = q_{b_i b_j}$$

$$q_{b_i w}(v_j^w - v_i^w - g\Delta t) = \beta_{b_i b_j}$$

$$q_{b_i w}(p_{w b_j} - p_{w b_i} - v_i^w - \frac{1}{2}g\Delta t^2) = \alpha_{b_i b_j}$$

由此，可以得到残差。这里需要注意的是，对于角度的残差不是加减，而是左乘角度预积分的逆或者转置。取虚部的意义是角度等于 $sin(\frac{\theta}{2}) = 0$：

$$\begin{bmatrix} r_p \\ r_q \\ r_v \\ r_{b_a} \\ r_{b_g} \end{bmatrix}_{15\times 1} = \begin{bmatrix} q_{b_i w}(p_{w b_j} - p_{w b_i} - v_i^w - \frac{1}{2}g\Delta t^2) - \alpha_{b_i b_j} \\ 2[q_{b_j b_i} \bigotimes (q_{b_i w} \bigotimes q_{w b_j})]_{xyz} \\ q_{b_i w}(v_j^w - v_i^w - g\Delta t) - \beta_{b_i b_j} \\ b_j^a - b_i^a \\ b_j^g - b_i^g \end{bmatrix}$$

4 预积分量的方差传递

在之前的分析中，我们没有考虑 IMU 测量的白噪声和 Bias 的随机游走。由于是随机误差，我们无法简单通过加减法将其消除，因此我们需要分析其方差以及方差随着时间的传递，从而分析得到白噪声对我们的“测量量”预积分量的影响。例如，假设在某一时刻，预积分量的随机误差来源仅仅是上一时刻的测量随机噪声，可以认为预积分量的方差等于测量白噪声的方差。但是到下一个时间点，预积分量的不仅与上一时刻的随机测量噪声有关，也同时与上一时刻预积分量的方差有关。

4.1 简单系统方差传递

对于简单的线性系统 $y = Ax$，x 和 y 都为高斯白噪声。若 x 的方差为 σ_x，则 y 的方差 $\sigma_y = A\sigma_x A^T$：

$$\begin{aligned}\sigma_y = E\{yy^T\} &= E\{Ax(Ax)^T\} \\ &= E\{Axx^T A^T\} \\ &= A\sigma_x A^T\end{aligned}$$

对于非线性系统 $x_k = f(x_{k-1}, u_{k-1})$，可以使用泰勒一阶展开来推导误差传递方程。令 $x_k = \hat{x}_k + \delta x_k$，$u_k = \hat{u}_k + \delta u_k$。其中 δx、δu 是高斯白噪声：

$$\begin{aligned}\hat{x}_k + \delta x_k &= f(\hat{x}_{k-1} + \delta_{k-1}, \hat{u}_{k-1} + \delta u_{k-1}) \\ \hat{x}_k + \delta x_k &\approx f(\hat{x}_{k-1}, \hat{u}_{k-1}) + \frac{\partial f}{\partial x}\delta x_{k-1} + \frac{\partial f}{\partial u}\delta u_{k-1} \rightarrow \\ \delta x_k &= \frac{\partial f}{\partial x}\delta x_{k-1} + \frac{\partial f}{\partial u}\delta u_{k-1}\end{aligned}$$

4.2 预积分方差传递方程

为了推导预积分的方差传递方程，我们首先需要明确的是预积分的传递方程：

$$\begin{aligned}&\omega = \frac{1}{2}((w_k^b - b_k^g) + (w_{k+1}^b - b_k^g)) \\ &q_{b_i b_{k+1}} = q_{b_i b_k} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega\Delta t \end{bmatrix} \\ &\alpha = \frac{1}{2}(q_{b_i b_k}(\alpha_k^b - b_k^a) + q_{b_i b_{k+1}}(\alpha_{k+1}^b - b_k^a)) \\ &\beta_{b_i b_{k+1}} = \beta_{b_i b_k} + \alpha\Delta t \\ &\alpha_{b_i b_{k+1}} = \alpha_{b_i b_k} + \beta_{b_i b_k}\Delta t + \frac{1}{2}\alpha\Delta t^2 \\ &b_{k+1}^g = b_k^g + n_{b_k^g}\Delta t \\ &b_{k+1}^a = b_k^a + n_{b_k^a}\Delta t\end{aligned}$$

接着，需要确定在预积分的传递方程中，哪些是对应的 x_{k+1}，哪些是 x_k, 哪些是 u_k：

$$x_{k+1} = \begin{bmatrix} \alpha_{b_i b_{k+1}} \\ q_{b_i b_{k+1}} \\ \beta_{b_i b_{k+1}} \\ b_{k+1}^a \\ b_{k+1}^g \end{bmatrix} x_k = \begin{bmatrix} \alpha_{b_i b_k} \\ q_{b_i b_k} \\ \beta_{b_i b_k} \\ b_k^a \\ b_k^g \end{bmatrix} u_k = \begin{bmatrix} \alpha_k^b \\ \omega_k^b \\ \alpha_{k+1}^b \\ \omega_{k+1}^b \\ \dot{b}_k^a \\ \dot{b}_k^g \end{bmatrix}$$

对于 x_{k+1} 和 x_k 是显而易见的。但是对于 u_k，我们可以发现，输入量其实是 k 和 k+1 时刻的总计 4 个测量值以及随机游走 $\dot{b}_k^a$ 和 $\dot{b}_k^g$，它们俩的理想值应该是 0。接下来需要分析的是 δx 和 δu。δx 很简单，就是预积分的误差值。而 δu 代表了理想测量量与实际测量量之间的误差，这个误差就是测量白噪声和 Bias 的随机游走造成的。α 和 ω 理想量与实际测量量之间就差一个白噪声 n。Bias 的

随机游走理想量认为是零，所以理想量与实际量之间的差值就是实际游走本身 n_b:

$$\delta x = \begin{bmatrix} \delta\alpha_{b_ib_k} \\ \delta q_{b_ib_k} \\ \delta\beta_{b_ib_k} \\ \delta b_k^a \\ \delta b_k^g \end{bmatrix} \delta u = \begin{bmatrix} n_k^a \\ n_k^g \\ n_{k+1}^a \\ n_{k+1}^g \\ n_{b_k^a} \\ n_{b_k^g} \end{bmatrix}$$

从上面的分析我们可以看到，对动系统不断向前的是一次次测量，所以系统的输入是所有的测量量。总的来说系统中产生方差的根本原因在于噪声，包括测量噪声和随机游走。因此从方差传递的角度来看这个系统，系统的输入就是这些测量噪声。

$$\delta x_{k+1} = \mathbf{F}_{5\times 5}\delta x + \mathbf{G}_{5\times 6}\delta u$$

接下来，为了将状态传递方程转换为误差传递方程，我们需要计算 $\frac{\partial f}{\partial x}$ 和 $\frac{\partial f}{\partial u}$。

4.3 F

$$\mathbf{f}_{11} = \frac{\partial\alpha_{b_ib_{k+1}}}{\partial\alpha_{b_ib_k}} = \frac{\partial\ \alpha_{b_ib_k} + \beta_{b_ib_k}\Delta t + \frac{1}{2}\alpha\Delta t^2}{\partial\alpha_{b_ib_k}} = \mathbf{I}_{3\times 3}$$

$$\mathbf{f}_{12} = \frac{\partial\alpha_{b_ib_{k+1}}}{\partial q_{b_ib_k}} = \frac{\partial\ \alpha_{b_ib_k} + \beta_{b_ib_k}\Delta t + \frac{1}{2}\alpha\Delta t^2}{\partial q_{b_ib_k}} = \frac{1}{2}\Delta t^2\frac{\partial\alpha}{\partial q_{b_ib_k}} = \frac{1}{2}\Delta t^2\frac{\partial\frac{1}{2}(q_{b_ib_k}(\alpha_k^b - b_k^a) + q_{b_ib_{k+1}}(\alpha_{k+1}^b - b_k^a))}{\partial q_{b_ib_k}}$$

$$= \frac{1}{4}\Delta t^2(-R_{b_ib_k}[a_k^b - b_k^a]_\times - R_{b_ib_{k+1}}[a_{k+1}^b - b_k^a]_\times(\mathbf{I} - [\omega]_\times\Delta t))$$

- 第一个重要的公式:

$$\begin{aligned}
\frac{\partial q_{b_ib_k}(\alpha_k^b - b_k^a)}{\partial q_{b_ib_k}} &= \lim_{\delta\theta\to 0}\frac{q_{b_ib_k}\otimes\begin{bmatrix}1\\ \frac{1}{2}\delta\theta\end{bmatrix}(\alpha_k^b - b_k^a) - q_{b_ib_k}(\alpha_k^b - b_k^a)}{\delta\theta} \\
&= \lim_{\delta\theta\to 0}\frac{R_{b_ib_k}\exp([\delta\theta]_\times)(\alpha_k^b - b_k^a) - R_{b_ib_k}(\alpha_k^b - b_k^a)}{\delta\theta} \\
&\approx \lim_{\delta\theta\to 0}\frac{R_{b_ib_k}[\delta\theta]_\times(\alpha_k^b - b_k^a)}{\delta\theta} \\
&= \lim_{\delta\theta\to 0}\frac{-R_{b_ib_k}[\alpha_k^b - b_k^a]_\times\delta\theta}{\delta\theta} \\
&= -R_{b_ib_k}[\alpha_k^b - b_k^a]_\times
\end{aligned}$$

- 第二个重要的公式:

$$\begin{aligned}
\frac{\partial q_{b_ib_{k+1}}(\alpha_{k+1}^b - b_k^a)}{\partial q_{b_ib_k}} &= \lim_{\delta\theta\to 0}\frac{q_{b_ib_k}\otimes\begin{bmatrix}1\\ \frac{1}{2}\delta\theta\end{bmatrix}\otimes\begin{bmatrix}1\\ \frac{1}{2}\omega\Delta t\end{bmatrix}(a_{k+1}^b - b_k^a) - q_{b_ib_k}\otimes\begin{bmatrix}1\\ \frac{1}{2}\omega\Delta t\end{bmatrix}(a_{k+1}^b - b_k^a)}{\delta\theta} \\
&\approx \lim_{\delta\theta\to 0}\frac{q_{b_ib_k}[\theta]_\times\exp([\omega\Delta t]_\times)(a_{k+1}^b - b_k^a)}{\delta\theta} \\
&= \lim_{\delta\theta\to 0}\frac{-q_{b_ib_k}[\exp([\omega\Delta t]_\times)(a_{k+1}^b - b_k^a)]_\times\delta\theta}{\delta\theta} \\
&= -q_{b_ib_k}\exp([\omega\Delta t]_\times)[a_{k+1}^b - b_k^a]_\times\exp([-\omega\Delta t]_\times) \\
&\approx -q_{b_ib_{k+1}}[a_{k+1}^b - b_k^a]_\times(\mathbf{I} - [\omega\Delta t]_\times)
\end{aligned}$$

在这个公式中，使用了这样一个性质：一个旋转，在自身旋转了 $\omega\Delta t$ 之后，再去旋转一个角度。等于这个角度旋转 $-\omega\Delta t$，再被这个旋转矩阵旋转，最后再整体旋转 $\omega\Delta t$。

$$
\begin{aligned}
&\mathbf{f}_{13} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \beta_{b_i b_k}} = \mathbf{I}_{3\times3}\Delta t \\
&\mathbf{f}_{14} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial b_k^a} = \frac{1}{2}\Delta t^2 \frac{\partial \alpha}{\partial b_k^a} = -\frac{1}{4}\Delta t^2 (q_{b_i b_k} + q_{b_i b_{k+1}}) \\
&\mathbf{f}_{15} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial b_k^g} = \frac{1}{2}\Delta t^2 \frac{\partial \alpha}{\partial b_k^g} = \frac{1}{2}\Delta t^2 \frac{\partial \alpha}{\partial q_{b_i b_{k+1}}} \frac{\partial q_{b_i b_{k+1}}}{\partial \omega\Delta t} \frac{\partial \omega\Delta t}{\partial b_k^g} \\
&= \frac{1}{2}\Delta t^2 (-\frac{1}{2} R_{b_i b_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times)(\mathbf{I}_{3\times3})(\Delta t \cdot -1) = \frac{1}{4}\Delta t^2 (R_{b_i b_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times)\Delta t
\end{aligned}
$$

- 第三个重要的公式：

$$
\begin{aligned}
\frac{\partial q_{b_i b_{k+1}}}{\partial \omega\Delta t} &= \lim_{\delta\theta\to0} \frac{q_{b_i b_{k+1}}^* \bigotimes q_{b_i b_k} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega\Delta t \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta \end{bmatrix}}{\delta\theta} \\
&= \lim_{\delta\theta\to0} \frac{\mathbf{I} \cdot \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta \end{bmatrix}}{\delta\theta} = \lim_{\delta\theta\to0} \frac{\delta\theta}{\delta\theta} = \mathbf{I}_{3\times3}
\end{aligned}
$$

对于角度预积分量 $q_{b_i b_k}$，它虽然形式上是四元数，但是本质上它代表了角度，也就是说每个四元数都对应了一个角度（实际上是多个 $+2k\pi$）。如果被求导变量是 $q_{b_i b_k}$，则代表了它是一个角度（只是表达形式的问题）。特别地，当 θ 很小时（例如小扰动），$\begin{bmatrix} 1 \\ \frac{1}{2}\theta \end{bmatrix}$ 所对应的角度就是 θ。

$$
\begin{aligned}
&\mathbf{f}_{21} = \frac{\partial q_{b_i b_{k+1}}}{\partial \alpha_{b_i b_k}} = \mathbf{0}_{3\times3} \\
&\mathbf{f}_{22} = \frac{\partial q_{b_i b_{k+1}}}{\partial q_{b_i b_k}} = \lim_{\delta\theta\to0} \frac{q_{b_i b_{k+1}^*} \bigotimes q_{b_i b_k} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega\Delta t \end{bmatrix}}{\delta\theta} \\
&= \lim_{\delta\theta\to0} \frac{\begin{bmatrix} 1 \\ -\frac{1}{2}\omega\Delta t \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\omega\Delta t \end{bmatrix}}{\delta\theta} = \lim_{\delta\theta\to0} \frac{\begin{bmatrix} 1 \\ -\frac{1}{2}\omega\Delta t \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta \end{bmatrix} \bigotimes \begin{bmatrix} 1 \\ -\frac{1}{2}\omega\Delta t \end{bmatrix}^*}{\delta\theta} \\
&= \lim_{\delta\theta\to0} \frac{\begin{bmatrix} 1 \\ \frac{1}{2}\exp([-\omega\Delta t]_\times)\delta\theta \end{bmatrix}}{\delta\theta} \text{此时分母上的四元数代表了所对应的角度} \\
&= \lim_{\delta\theta\to0} \frac{\exp([-\omega\Delta t]_\times)\delta\theta}{\delta\theta} \approx \mathbf{I}_{3\times3} - [\omega]_\times \Delta t \\
&\mathbf{f}_{23} = \frac{\partial q_{b_i b_{k+1}}}{\partial \beta_{b_i b_k}} = \mathbf{0}_{3\times3} \\
&\mathbf{f}_{24} = \frac{\partial q_{b_i b_{k+1}}}{\partial b_k^a} = \mathbf{0}_{3\times3} \\
&\mathbf{f}_{25} = \frac{\partial q_{b_i b_{k+1}}}{\partial b_k^g} = \frac{\partial q_{b_i b_{k+1}}}{\partial \omega\Delta t} \frac{\partial \omega\Delta t}{\partial \omega} \frac{\partial \omega}{\partial b_k^g} = \mathbf{I}_{3\times3} \cdot \Delta t \cdot -1 = -\mathbf{I}\Delta t \text{ 见第三个重要公式}
\end{aligned}
$$

$$\mathbf{f}_{31} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha_{b_i b_k}} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{32} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial q_{b_i b_k}} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial q_{b_i b_k}} = \Delta t \cdot \frac{1}{2}(-R_{b_i b_k}[a_k^b - b_k^a]_\times - R_{b_i b_{k+1}}[a_{k+1}^b - b_k^a]_\times(\mathbf{I} - [\omega]_\times \Delta t))$$

$$= \frac{1}{2}(-R_{b_i b_k}[a_k^b - b_k^a]_\times - R_{b_i b_{k+1}}[a_{k+1}^b - b_k^a]_\times(\mathbf{I} - [\omega]_\times \Delta t))\Delta t$$

$$\mathbf{f}_{33} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \beta_{b_i b_k}} = \mathbf{I}_{3\times3}$$

$$\mathbf{f}_{34} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial b_k^a} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial b_k^a} = \Delta t \cdot -\frac{1}{2}(q_{b_i b_k} + q_{b_i b_{k+1}}) = -\frac{1}{2}(q_{b_i b_k} + q_{b_i b_{k+1}})\Delta t$$

$$\mathbf{f}_{35} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial b_k^g} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial b_k^g} = \Delta t \cdot \frac{1}{2}(R_{b_i b_{k+1}}[\alpha_{k+1}^b - b_k^a]_\times)\Delta t$$

$$= \frac{1}{2}(R_{b_i b_{k+1}}[\alpha_{k+1}^b - b_k^a]_\times \Delta t)\Delta t$$

$$\mathbf{f}_{41} = \frac{\partial b_{k+1}^a}{\partial \alpha_{b_i b_k}} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{42} = \frac{\partial b_{k+1}^a}{\partial q_{b_i b_k}} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{43} = \frac{\partial b_{k+1}^a}{\partial \beta_{b_i b_k}} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{44} = \frac{\partial b_{k+1}^a}{\partial b_k^a} = \mathbf{I}_{3\times3}$$

$$\mathbf{f}_{45} = \frac{\partial b_{k+1}^a}{\partial b_k^g} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{51} = \frac{\partial b_{k+1}^g}{\partial \alpha_{b_i b_k}} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{52} = \frac{\partial b_{k+1}^g}{\partial q_{b_i b_k}} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{53} = \frac{\partial b_{k+1}^g}{\partial \beta_{b_i b_k}} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{54} = \frac{\partial b_{k+1}^g}{\partial b_k^a} = \mathbf{0}_{3\times3}$$

$$\mathbf{f}_{55} = \frac{\partial b_{k+1}^g}{\partial b_k^g} = \mathbf{I}_{3\times3}$$

4.4 G

$$\mathbf{g}_{11} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \alpha_k^b} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial \alpha_k^b} = \frac{1}{2}\Delta t^2 \cdot \frac{1}{2} q_{b_i b_k} = \frac{1}{4} q_{b_i b_k} \Delta t^2$$

$$\mathbf{g}_{12} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \omega_k^b} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial q_{b_i b_{k+1}}} \frac{\partial q_{b_i b_{k+1}}}{\partial \omega \Delta t} \frac{\partial \omega \Delta t}{\partial \omega_k^b} = \frac{1}{2}\Delta t^2 \cdot -\frac{1}{2} R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times \cdot \mathbf{I}_{3\times 3} \cdot \Delta t \frac{1}{2}$$

$$= -\frac{1}{8} (R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times) \Delta t^3$$

$$\mathbf{g}_{13} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \alpha_{k+1}^b} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial \alpha_{k+1}^b} = \frac{1}{2}\Delta t^2 \cdot \frac{1}{2} q_{b_i b_{k+1}} = \frac{1}{4} q_{b_i b_{k+1}} \Delta t^2$$

$$\mathbf{g}_{14} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \omega_{k+1}^b} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial q_{b_i b_{k+1}}} \frac{\partial q_{b_i b_{k+1}}}{\partial \omega \Delta t} \frac{\partial \omega \Delta t}{\partial \omega_{k+1}^b} = \frac{1}{2}\Delta t^2 \cdot -\frac{1}{2} R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times \cdot \mathbf{I}_{3\times 3} \cdot \Delta t \frac{1}{2}$$

$$= -\frac{1}{8} (R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times) \Delta t^3 = \mathbf{g_{12}}$$

$$\mathbf{g}_{15} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial b_k^a} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{16} = \frac{\partial \alpha_{b_i b_{k+1}}}{\partial b_k^g} = \mathbf{0}_{3\times 3}$$

- 第四个重要公式：

$$\begin{aligned}
\frac{\partial \alpha}{\partial q_{b_i b_{k+1}}} &= \frac{\partial \frac{1}{2} q_{b_i b_{k+1}} (\alpha_{k+1}^b - b_k^a)}{\partial q_{b_i b_{k+1}}} \\
&= \frac{1}{2} \lim_{\delta\theta \to 0} \frac{R_{q_i q_{k+1}} \bigotimes \begin{bmatrix} 1 \\ \frac{1}{2}\delta\theta \end{bmatrix} (\alpha_{k+1}^b - b_k^a) - R_{q_i q_{k+1}} (\alpha_{k+1}^b - b_k^a)}{\delta\theta} \\
&= \frac{1}{2} \lim_{\delta\theta \to 0} \frac{R_{q_i q_{k+1}} \exp([\delta\theta]_\times) (\alpha_{k+1}^b - b_k^a) - R_{q_i q_{k+1}} (\alpha_{k+1}^b - b_k^a)}{\delta\theta} \\
&\approx \frac{1}{2} \lim_{\delta\theta \to 0} \frac{R_{q_i q_{k+1}} (\mathbf{I} + [\delta\theta]_\times) (\alpha_{k+1}^b - b_k^a) - R_{q_i q_{k+1}} (\alpha_{k+1}^b - b_k^a)}{\delta\theta} \\
&= -\frac{1}{2} R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times
\end{aligned}$$

$$\mathbf{g}_{21} = \frac{\partial q_{b_i b_{k+1}}}{\partial \alpha_k^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{22} = \frac{\partial q_{b_i b_{k+1}}}{\partial \omega_k^b} = \frac{\partial q_{b_i b_{k+1}}}{\partial \omega \Delta t} \frac{\partial \omega \Delta t}{\partial \omega_k^b} = \mathbf{I}_{3\times 3} \cdot \Delta t \frac{1}{2} = \frac{1}{2} \mathbf{I} \Delta t$$

$$\mathbf{g}_{23} = \frac{\partial q_{b_i b_{k+1}}}{\partial \alpha_{k+1}^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{24} = \frac{\partial q_{b_i b_{k+1}}}{\partial \omega_{k+1}^b} = \frac{\partial q_{b_i b_{k+1}}}{\partial \omega \Delta t} \frac{\partial \omega \Delta t}{\partial \omega_{k+1}^b} = \mathbf{I}_{3\times 3} \cdot \Delta t \frac{1}{2} = \frac{1}{2} \mathbf{I} \Delta t$$

$$\mathbf{g}_{25} = \frac{\partial q_{b_i b_{k+1}}}{\partial b_k^a} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{26} = \frac{\partial q_{b_i b_{k+1}}}{\partial b_k^g} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{31} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha_k^b} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial \alpha_k^b} = \Delta t \cdot \frac{1}{2} q_{b_i b_k} = \frac{1}{2} q_{b_i b_k} \Delta t$$

$$\mathbf{g}_{32} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \omega_k^b} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial q_{b_i b_{k+1}}} \frac{\partial q_{b_i b_{k+1}}}{\partial \omega \Delta t} \frac{\partial \omega \Delta t}{\partial \omega_k^b} = \Delta t \cdot -\frac{1}{2} R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times \cdot \mathbf{I}_{3\times 3} \cdot \Delta t \frac{1}{2}$$
$$= -\frac{1}{4} (R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times) \Delta t^2$$

$$\mathbf{g}_{33} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha_{k+1}^b} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial \alpha_{k+1}^b} = \Delta t \cdot \frac{1}{2} q_{b_i b_{k+1}} = \frac{1}{2} q_{b_i b_{k+1}} \Delta t$$

$$\mathbf{g}_{34} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \omega_{k+1}^b} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \alpha} \frac{\partial \alpha}{\partial q_{b_i b_{k+1}}} \frac{\partial q_{b_i b_{k+1}}}{\partial \omega \Delta t} \frac{\partial \omega \Delta t}{\partial \omega_{k+1}^b} = \frac{1}{2} \Delta t^2 \cdot -\frac{1}{2} R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times \cdot \mathbf{I}_{3\times 3} \cdot \Delta t \frac{1}{2}$$
$$= -\frac{1}{4} (R_{q_i q_{k+1}} [\alpha_{k+1}^b - b_k^a]_\times) \Delta t^2 = \mathbf{g_{12}}$$

$$\mathbf{g}_{35} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \dot{b}_k^a} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{36} = \frac{\partial \beta_{b_i b_{k+1}}}{\partial \dot{b}_k^g} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{41} = \frac{\partial b_{k+1}^a}{\partial \alpha_k^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{42} = \frac{\partial b_{k+1}^a}{\partial \omega_k^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{43} = \frac{\partial b_{k+1}^a}{\partial \alpha_{k+1}^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{44} = \frac{\partial b_{k+1}^a}{\partial \omega_{k+1}^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{45} = \frac{\partial b_{k+1}^a}{\partial \dot{b}_k^a} = \mathbf{I}_{3\times 3} \Delta t$$

$$\mathbf{g}_{46} = \frac{\partial b_{k+1}^a}{\partial \dot{b}_k^g} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{51} = \frac{\partial b_{k+1}^g}{\partial \alpha_k^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{52} = \frac{\partial b_{k+1}^g}{\partial \omega_k^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{53} = \frac{\partial b_{k+1}^g}{\partial \alpha_{k+1}^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{54} = \frac{\partial b_{k+1}^g}{\partial \omega_{k+1}^b} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{55} = \frac{\partial b_{k+1}^g}{\partial \dot{b}_k^a} = \mathbf{0}_{3\times 3}$$

$$\mathbf{g}_{56} = \frac{\partial b_{k+1}^g}{\partial \dot{b}_k^g} = \mathbf{I}_{3\times 3} \Delta t$$

5 残差 Jacobian

在最小二乘优化时，我们需要计算残差 $\mathbf{r}$ 对优化变量的 Jacobian。首先我们确定残差的公式。对于 vio，除了之前提到的预积分残差，还有相机反投影误差：

$$\mathbf{r_j} = \begin{bmatrix} u_{reprojection} - u_{measure} \\ v_{reprojection} - v_{measure} \end{bmatrix}_j = \begin{bmatrix} \frac{x_j}{z_j} - u_j \\ \frac{y_j}{z_j} - v_j \end{bmatrix}$$

其中：

$$\begin{aligned}
\mathbf{f}_{c_j} = \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} &= \mathbf{T}_{b_j c_j}^{-1} \mathbf{T}_{wb_j}^{-1} \mathbf{T}_{wb_i} \mathbf{T}_{b_i c_i} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \mathbf{T}_{bc}^{-1} \mathbf{T}_{wb_j}^{-1} \mathbf{T}_{wb_i} \mathbf{T}_{bc} \begin{bmatrix} \frac{1}{\lambda} u_i \\ \frac{1}{\lambda} v_i \\ \frac{1}{\lambda} \end{bmatrix} \\
&= \mathbf{R}_{bc}^T (\mathbf{R}_{wb_j}^T (\mathbf{R}_{wb_i} (\mathbf{R}_{bc} \begin{bmatrix} \frac{1}{\lambda} u_i \\ \frac{1}{\lambda} v_i \\ \frac{1}{\lambda} \end{bmatrix} + \mathbf{p}_{bc}) + \mathbf{p}_{wb_i} - \mathbf{p}_{wb_j}) - \mathbf{p}_{bc}) \\
&= \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} \mathbf{R}_{bc} \frac{1}{\lambda} \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} + \mathbf{R}_{bc}^T (\mathbf{R}_{wb_j}^T (\mathbf{R}_{wb_i} \mathbf{p}_{bc} + \mathbf{p}_{wb_i} - \mathbf{p}_{wb_j}) - \mathbf{p}_{bc})
\end{aligned}$$

接着确定待优化变量：

$$\mathbf{X} = \begin{bmatrix} \mathbf{p}_{wb_i} \\ \theta_{wb_i} \\ \mathbf{p}_{wb_j} \\ \theta_{wb_j} \\ \mathbf{p}_{bc} \\ \theta_{bc} \\ \lambda \\ \mathbf{v}_i^w \\ \mathbf{v}_j^w \\ b_i^a \\ b_i^g \\ b_j^a \\ b_j^g \end{bmatrix}$$

5.1 重投影误差的 Jacobian

首先，计算残差 $\frac{\partial \mathbf{r}_j}{\partial \mathbf{f}_{c_j}}$：

$$\frac{\partial \mathbf{r}_j}{\partial \mathbf{f}_{c_j}} = \begin{bmatrix} \frac{1}{z_j} & 0 & -\frac{x_j}{z_j^2} \\ 0 & \frac{1}{z_j} & -\frac{y_j}{z_j^2} \end{bmatrix}$$

$$\frac{\partial \mathbf{f_{c_j}}}{\partial \mathbf{p}_{wb_i}} = \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T$$

$$\frac{\partial \mathbf{f_{c_j}}}{\partial \theta_{wb_i}} = \frac{\partial \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} (\mathbf{R}_{bc} \frac{1}{\lambda} \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} + \mathbf{p}_{bc})}{\partial \theta_{wb_i}} = \frac{\partial \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} \mathbf{f}_{b_i}}{\partial \theta_{wb_i}} = -\mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} [\mathbf{f}_{b_i}]_\times$$见第一个重要公式

$$\frac{\partial \mathbf{f_{c_j}}}{\partial \mathbf{p}_{wb_j}} = -\mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T$$

$$\frac{\partial \mathbf{f_{c_j}}}{\partial \theta_{wb_j}} = \frac{\partial \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T (\mathbf{f}_{b_i} - \mathbf{p}_{wb_j})}{\partial \theta_{wb_j}} = \lim_{\delta \to 0} \frac{\mathbf{R}_{bc}^T \exp(-\delta \theta_{wb_j}) \mathbf{R}_{wb_j}^T (\mathbf{f}_{b_i} - \mathbf{p}_{wb_j}) - \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T (\mathbf{f}_{b_i} - \mathbf{p}_{wb_j})}{\delta \theta}$$

$$\approx \lim_{\delta \to 0} \frac{\mathbf{R}_{bc}^T (\mathbf{I} - [\delta \theta]_\times) \mathbf{R}_{wb_j}^T (\mathbf{f}_{b_i} - \mathbf{p}_{wb_j}) - \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T (\mathbf{f}_{b_i} - \mathbf{p}_{wb_j})}{\delta \theta} = \mathbf{R}_{bc}^T [\mathbf{f}_{b_j}]_\times$$

$$\frac{\partial \mathbf{f_{c_j}}}{\partial \mathbf{p}_{bc}} = \mathbf{R}_{bc}^T (\mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} - \mathbf{I}_{3\times 3})$$

$$\frac{\partial \mathbf{f_{c_j}}}{\partial \theta_{bc}} = -\mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} \mathbf{R}_{bc} [\mathbf{f}_{c_i}]_\times + [\mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} \mathbf{R}_{bc} \mathbf{f}_{c_i}]_\times + [\mathbf{R}_{bc}^T (\mathbf{R}_{wb_j}^T (\mathbf{R}_{wb_i} \mathbf{p}_{bc} + \mathbf{p}_{wb_i} - \mathbf{p}_{wb_j}) - \mathbf{p}_{bc})]_\times$$

有技巧

$$\frac{\partial \mathbf{f_{c_j}}}{\partial \lambda} = -\frac{1}{\lambda} \mathbf{R}_{bc}^T \mathbf{R}_{wb_j}^T \mathbf{R}_{wb_i} \mathbf{R}_{bc} \mathbf{f}_{c_i}$$

其他导数为 0

5.2 IMU 残差的 Jacobian

对于 IMU 残差，除了 $\mathbf{b}_i$，其它求导方式都与协方差递推方程的推导类似。对于相对 $\mathbf{b}_i^{a/g}$ 的导数，我们使用如下方式推导。以 $\frac{\partial \alpha_{b_i b_j}}{\partial b_i^a}$ 为例：

$$\frac{\partial \alpha_{b_i b_j}}{\partial b_i^a} \approx \frac{\delta \alpha_{b_i b_j}}{\delta b_i^a}$$

根据之前所推出来的误差传递方程：

$$\delta x_{k+1} = \mathbf{F} \delta x_k + \mathbf{G} \delta u_k$$

我们可以通过递推：

$$\delta x_{k+1} = \mathbf{F}_k (\mathbf{F}_{k-1} \delta x_{k-1} + \mathbf{G}_{k-1} \delta u_{k-1}) + \mathbf{G}_k \delta u_k$$

最终可以得到：

$$\delta x_{k+1} = \mathbf{F}_k \mathbf{F}_{k-1} \ldots \mathbf{F}_0 \delta x_0 + \ldots$$

从而算出 $\frac{\delta \alpha_{b_i b_j}}{\delta b_i^a}$ 以及其他。