MEF

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1 Preliminaries

Definition 1.1 (Topological dynamical system). A *topological dynamical system* (TDS) is a triplet $(X, T, \cdot)$, where

- X is a non-empty, compact topological Hausdorff space,
- T is a topological group,
- $\cdot : X \times T \to X$ is a continuous group action, that is $\forall x \in X,\ s, t \in T$:
  1. $x \cdot e = x$,
  2. $(x \cdot s) \cdot t = x \cdot (s \cdot t)$.

Remark 1.2. The map $\cdot$ is usually supressed in the notation.

Definition 1.3 (Morphism of TDS). Let (X, T), (Y, T) be topological dynamical systems. A map $\pi : X \to Y$ is called a *homomorphism* from $(X, T) \to (Y, T)$ if π is continuous and $\forall x \in X, t \in T$:

$$\pi(x)t = \pi(xt).$$

Definition 1.4 (Factor of TDS). Let (X, T), (Y, T) be topological dynamical systems and $\pi : X \to Y$ a homomorphism. Then π is called a *factor map* if it is surjective. If there exists a factor map $X \to Y$ then Y is called a *factor* of X.

Proposition 1.5 (Quotient system). *Let (X, T) be a TDS. Let $R \subset X \times X$ be a T-invariant*[1] *and closed equivalence relation (ICER for short). Let $\pi : X \to X/R$ be the quotient map. Endow X/R with the quotient topology, that is $U \subset X/R$ open $:\Leftrightarrow \pi^{-1}(U)$ is open. Define the T-action by $[x]t := [xt]$. Then $(X/R, T)$ is a TDS and π is a factor map.*

Proof. The quotient R/F is a Hausdorff space, because X is a compact Hausdorff space and R is closed (not trivial). The compactness of R/F follows from $R/F = \pi(X)$ and the compactness of X together with the continuity of π. The T-invariance of R implies the well-definedness of the T-action and that π is a factor map. **Warum ist die T-Wirkung stetig?!** □

Proposition 1.6 (ICER generated by factor). *Let (X, T) and (Y, T) be TDS and $\pi : X \to Y$ a factor map. Then*

$$R := \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$$

is an ICER.

Definition 1.7 (Product system). Let $(X_\alpha, T, \cdot_\alpha)_{\alpha \in A}$ be a family of topological dynamical systems. Then the *product system* $(X, T, \cdot)$ of the family is defined by

$$X := \prod_{\alpha \in A} X_\alpha, \quad (x \cdot t)_\alpha := x \cdot_\alpha t.$$

[1] that is $\forall t \in T, \forall (x, x') \in R : (xt, x't) \in R$

Definition 1.8 (Equicontinuous TDS). Let (X,T) be a TDS. Then (X,T) is called *equicontinuous*, if for any $\alpha \in \mathcal{U}_X$ there is $\beta \in \mathcal{U}_X$ with $(x,x') \in \beta \Rightarrow \forall t \in T : (xt, x't) \in \alpha$.

Proposition 1.9. *products of eq. cont. flows are eq. cont., subflows are eq. cont.*

Proposition 1.10 (Subsystem). *Let (X,T) be a TDS and $A \subset X$ a closed, T-invariant set. Then (A,T) is also a TDS with the inherited T-action called the* subsystem *of (X,T) induced by A.*

Proof. Since A is a closed subset of a compact Hausdorff space it is a compact Hausdorff space. The invariance of A implies that the T action is well defined, continuity is clear. □

Theorem 1.11. *Let (X,T) be a TDS. Then (X,T) has an equicontinuous factor that is maximal in the sense that the ICER generated by it is the smallest with respect to set inclusion.*

Proof. Let

$$\mathcal{R} := \{R \subset X \times X : R \text{ ICER with } X/R \text{ equicontinuous}\}.$$

Then $X \times X \in \mathcal{R}$ and so the set is non empty. Now let

$$R := \cap \mathcal{R}.$$

Then R is an ICER (because intersection of ICERs are ICERs). It remains to show that X/R is equicontinuous. The product system $(\prod_{R' \in \mathcal{R}} X/R', T)$ is equicontinuous, since products of equicontinuous systems are equicontinuous. Consider the subset

$$A := \left\{ (y_{R'})_{R' \in \mathcal{R}} \in \prod_{R' \in \mathcal{R}} X/R' : \bigcap_{R' \in \mathcal{R}} y_{R'} \neq \varnothing \right\}.$$

Then A is closed (**Warum?**) and T-invariant. Therefore we can consider the subsystem (A,T) which is an equicontinuous system (subsystems of equicontinuous systems are equicontinuous). Consider the map $\pi : X \to A,\ x \mapsto ([x]_{R'})_{R' \in \mathcal{R}}$. Then π is a factor and the ICER generated by π is equal to R. □