Linear Perturbation Series Documentation

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March 20, 2024

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1 Exact Solution Of 2x2 Case

For $E_1, E_2, a \in \mathbb{R}$ let

$$H := \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

and

$$V := \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Define $T : \mathbb{C} \to \mathbb{C}^{2\times 2}$ by $T\ x = H + x \cdot V$. Assume that $E_1 \neq E_2$, then the two eigenvalues $E_\pm$ of $T\ x$ (the roots of the characteristic polynomial) can be found as

$$E_\pm(x) = \frac{E_1 + E_2}{2} \pm \frac{|E_1 - E_2|}{2}\sqrt{1 + \left(\frac{2ax}{E_1 - E_2}\right)^2}. \tag{1}$$

Now upon potentially relabeling $E_\pm$ we obtain

$$E_\pm(x) = \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2}\sqrt{1 + \left(\frac{2ax}{E_1 - E_2}\right)^2}. \tag{2}$$

Now the binomial series says that for $x \in \mathbb{C}$ with $|x| < 1$:

$$(1 + x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k, \tag{3}$$

where the radius of convergence of the power series is 1. Therefore

$$
\begin{aligned}
E_{\pm}(x) &= \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\frac{2ax}{E_1 - E_2} \right)^{2k} \\
&= E_{1/2} \pm \frac{1}{2} \sum_{k=1}^{\infty} \binom{1/2}{k} \frac{(2a)^{2k}}{(E_1 - E_2)^{2k-1}} x^{2k}
\end{aligned} \tag{4}
$$

where the radius of convergence r of the power series is

$$
r = \frac{|E_1 - E_2|}{2|a|}. \tag{5}
$$