

Linear Perturbation Series Documentation

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1 Exact Solution Of 2x2 Case

For $E_1, E_2, a \in \mathbb{R}$ let

$$H := \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

and

$$V := \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Define $T : \mathbb{C} \rightarrow \mathbb{C}^{2 \times 2}$ by $T x = H + x \cdot V$.

The two eigenvalues E_{\pm} of $T x$ (the roots of the characteristic polynomial) can be found as

$$E_{\pm}(x) = \frac{E_1 + E_2}{2} \pm \frac{|E_1 - E_2|}{2} \sqrt{1 + \left(\frac{2ax}{E_1 - E_2} \right)^2}. \quad (1)$$

Now upon potentially relabeling E_{\pm} we obtain

$$E_{\pm}(x) = \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \sqrt{1 + \left(\frac{2ax}{E_1 - E_2} \right)^2}. \quad (2)$$

Now the binomial series says that for $x \in \mathbb{C}$ with $|x| < 1$:

$$(1 + x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k, \quad (3)$$

where the radius of convergence of the power series is 1. Therefore

$$\begin{aligned}
 E_{\pm}(x) &= \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\frac{2ax}{E_1 - E_2} \right)^{2k} \\
 &= E_{1/2} \pm \frac{1}{2} \sum_{k=1}^{\infty} \binom{1/2}{k} \frac{(2a)^{2k}}{(E_1 - E_2)^{2k-1}} x^{2k}
 \end{aligned} \tag{4}$$

where the radius of convergence is

$$\frac{|E_1 - E_2|}{2a}. \tag{5}$$