Linear Perturbation Series Documentation

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1 Perturbation Series

1.1 Laurent-Expansion of the Resolvent

Let X be a complex Banach space and $T : X \to X$ a bounded linear operator and R the resolvent of T. Let $\lambda_0 \in \sigma(T)$ be an isolated point of the spectrum of T. Let P be the spectral projection associated to λ_0 and $Q := I - P$ the complementary projection. Then the Laurent-expansion of R around λ_0 is:

$$\begin{aligned} R(\lambda) = &\sum_{n=1}^{\infty}(\lambda - \lambda_0)^{-n} \cdot (T - \lambda_0 I)^{n-1} P \\ &+ \sum_{n=0}^{\infty}(\lambda - \lambda_0)^{n} \cdot (-1)^n \cdot S^{n+1} Q, \end{aligned} \tag{1}$$

where S is the inverse of $(\lambda_0 I - T)|_{Q(X)} : Q(X) \to Q(X)$. The inner radius of convergence of the Laurent-series is 0 and the outer $d(\lambda_0, \sigma(T))$. For $n \in \mathbb{Z}$ let V_n be the coefficient of $(\lambda - \lambda_0)^z$ in the Laurent expansion.

1.2 Perturbation-series of the Resolvent

$$T(x) = \sum_{n=0}^{\infty} T_n x^n. \tag{2}$$

Let $U := \{(\lambda, x) \in \mathbb{C}^2 : \lambda I - T(x) \text{ invertible}\}$ and define $R : U \to \mathscr{B}(X)$ by

$$R(\lambda, x) := (\lambda I - T(x))^{-1}. \tag{3}$$

Let $A(x) := T(x) - T_1$. Now

$$\lambda I - T(x) = \lambda I - T - A(x) = (I - A(x)R(\lambda, 0))(\lambda I - T) \tag{4}$$

and so

$$R(\lambda, x) = R(\lambda, 0)(I - A(x)R(\lambda, 0))^{-1} = R(\lambda, 0) \sum_{n=0}^{\infty} \big(A(x)R(\lambda, 0)\big)^n. \tag{5}$$

From this it follows, by collecting the terms that have the same power of x, that

$$R(\lambda, x) = R(\lambda, 0) + \sum_{n=1}^{\infty} R_n(\lambda) x^n \tag{6}$$

with

$$R_n(\lambda) = \sum_{\substack{(i_1,\dots,i_n)\in(\mathbb{N}\setminus\{0\})^n \\ i_1+\cdots+i_n=n}} T_{i_1} R(\lambda, 0) \cdots T_{i_n} R(\lambda, 0). \tag{7}$$

1.3 Perturbation-series for the Eigenvalues

Let λ_0 be an isolated element of $\sigma(T_0)$ with finite dimensional generalized eigenspace (which is equivalent to the Laurent-expansion having only finitely many non-zero coefficients to negative power). Let m be the dimension of the generalized eigenspace. Now for $x \in \mathbb{C}$ "small enough"

$$P(x) = \frac{1}{2\pi i} \int_{\gamma} R(\lambda, x) d\lambda \tag{8}$$

is the projection onto sum of all the generalized eigenspaces to the eigenvalues that have split from λ_0 ("split eigenvalues"). Therefore

$$\alpha(x) := \operatorname{tr} T(x)P(x) \tag{9}$$

is the weighted (by the dimension of the generalized eigenspaces) sum of all the split eigenvalues.

Therefore

$$\alpha(x) - m\lambda_0 = \mathrm{tr}(T(x) - \lambda_0 I)P(x) = \frac{1}{2\pi i}\,\mathrm{tr}\int_\gamma (\lambda - \lambda_0)R(\lambda, x)d\lambda \tag{10}$$

and so (the $n = 1$ term vanishes)

$$\begin{aligned}
\alpha(x) - m\lambda_0 &= \frac{1}{2\pi i}\,\mathrm{tr}\int_\gamma (\lambda - \lambda_0)R(\lambda, 0)\sum_{n=1}^{\infty}\big(A(x)R(\lambda, 0)\big)^n d\lambda \\
&= -\frac{1}{2\pi i}\,\mathrm{tr}\int_\gamma (\lambda - \lambda_0)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\partial}{\partial z}(A(x)R(z, 0))^n\right)(\lambda)d\lambda \qquad (11) \\
&= \frac{1}{2\pi i}\,\mathrm{tr}\int_\gamma \sum_{n=1}^{\infty}\frac{1}{n}(A(x)R(\lambda, 0))^n d\lambda
\end{aligned}$$

upon collecting powers of x in the above:

$$\alpha(x) - m\lambda_0 = \sum_{n=1}^{\infty}\alpha_n x^n \tag{12}$$

with

$$\alpha_n := \sum_{k=1}^{n}\sum_{\substack{(i_1,\dots,i_k)\in(\mathbb{N}\setminus\{0\})^k \\ i_1+\dots+i_k=n}}\frac{1}{2\pi k i}\,\mathrm{tr}\int_\gamma T_{i_1}R(\lambda, 0)\cdots T_{i_k}R(\lambda, 0)d\lambda \tag{13}$$

using the residue theorem and the Laurent expansion of the Resolvent around λ_0 to evaluate the integral

$$\alpha_n = \sum_{k=1}^{n}\sum_{\substack{(i_1,\dots,i_k)\in(\mathbb{N}\setminus\{0\})^k \\ i_1+\dots+i_k=n}}\frac{1}{k}\sum_{\substack{(j_1,\dots,j_k)\in\mathbb{Z}^k \\ j_1+\dots+j_k=-1}}\mathrm{tr}\,T_{i_1}V_{j_1}\cdots T_{i_k}V_{j_k}. \tag{14}$$

(add more details here ...)

1.4 Eigenvalue Perturbation-series for Linear Perturbation and Semi-simple Eigenvalue

In the special case of a linear perturbation ($T_n = 0$ for all $n \in \mathbb{N}$ with $n \geq 2$) and a semi simple eigenvalue (meaning that $(T_0 - \lambda_0 I)P = 0$) we obtain

$$\alpha_n = \frac{1}{n}\sum_{\substack{(j_1,\dots,j_n)\in\mathbb{Z}^n \\ j_1+\dots+j_n=-1}}\mathrm{tr}\,T_{i_1}V_{j_1}\cdots T_{i_k}V_{j_k}. \tag{15}$$

This is the quantity that is calculated by the code.

2 Exact Solution Of 2x2 Case

For $E_1, E_2, a \in \mathbb{R}$ let

$$H := \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

and

$$V := \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Define $T : \mathbb{C} \to \mathbb{C}^{2\times 2}$ by $T\ x = H + x \cdot V$. Assume that $E_1 \neq E_2$, then the two eigenvalues $E_\pm$ of $T\ x$ (the roots of the characteristic polynomial) can be found as

$$E_\pm(x) = \frac{E_1 + E_2}{2} \pm \frac{|E_1 - E_2|}{2} \sqrt{1 + \left(\frac{2ax}{E_1 - E_2}\right)^2}. \tag{16}$$

Now upon potentially relabeling $E_\pm$ we obtain

$$E_\pm(x) = \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \sqrt{1 + \left(\frac{2ax}{E_1 - E_2}\right)^2}. \tag{17}$$

Now the binomial series says that for $x \in \mathbb{C}$ with $|x| < 1$:

$$(1 + x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k, \tag{18}$$

where the radius of convergence of the power series is 1. Therefore

$$\begin{aligned} E_\pm(x) &= \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\frac{2ax}{E_1 - E_2}\right)^{2k} \\ &= E_{1/2} \pm \frac{1}{2} \sum_{k=1}^{\infty} \binom{1/2}{k} \frac{(2a)^{2k}}{(E_1 - E_2)^{2k-1}} x^{2k} \end{aligned} \tag{19}$$

where the radius of convergence r of the power series is

$$r = \frac{|E_1 - E_2|}{2|a|}. \tag{20}$$