Linear Perturbation Series Documentation

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1 Perturbation Series

1.1 Laurent-Expansion of the Resolvent

Let X be a complex Banach space and $T_0 : X \to X$ a bounded linear operator and R the resolvent of T_0. Let $\lambda_0 \in \sigma(T_0)$ be an isolated point of the spectrum of T_0. Let P be the spectral projection associated to λ_0 and $Q := I - P$ the complementary projection. Then the Laurent-expansion of R around λ_0 is:

$$\begin{aligned} R(\lambda) = &\sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} \cdot (T_0 - \lambda_0 I)^{n-1} P \\ &+ \sum_{n=0}^{\infty} (\lambda - \lambda_0)^{n} \cdot (-1)^n \cdot S^{n+1} Q, \end{aligned} \tag{1}$$

where S is the inverse of $(\lambda_0 I - T_0)|_{Q(X)} : Q(X) \to Q(X)$. The inner radius of convergence of the Laurent-series is 0 and the outer $d(\lambda_0, \sigma(T))$.

1.2 Perturbation-series of the Resolvent

Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}(X)$. Let $T : D \to \mathscr{B}(X)$ defined by

$$T(x) := \sum_{n=0}^{\infty} T_n x^n, \tag{2}$$

where D is the open disk centered at 0 whose radius is the radius of convergence of the power series (assumed > 0).

Let $U := \{(\lambda, x) \in \mathbb{C}^2 : \lambda I - T(x) \text{ invertible}\}$ and define $R : U \to \mathscr{B}(X)$ by

$$R(\lambda, x) := (\lambda I - T(x))^{-1}. \tag{3}$$

Let $A(x) := T(x) - T_1$. Now

$$\lambda I - T(x) = \lambda I - T_0 - A(x) = (I - A(x)R(\lambda, 0))(\lambda I - T_0) \tag{4}$$

and so

$$R(\lambda, x) = R(\lambda, 0)(I - A(x)R(\lambda, 0))^{-1} = R(\lambda, 0) \sum_{n=0}^{\infty} \big(A(x)R(\lambda, 0)\big)^n \tag{5}$$

with a > 0 radius of convergence. From this it follows, by collecting the terms that have the same power of x, that $R(\lambda, x)$ is analytic in x at 0. Similarly one can show that R is bi-analytic on U as well.

1.3 Eigenvalue Perturbation-series

Let λ_0 be an isolated element of $\sigma(T_0)$ with finite dimensional generalized eigenspace (which is equivalent to the Laurent-expansion having only finitely many non-zero coefficients to negative power). Let m be the dimension of the generalized eigenspace. Let γ be a circular path surrounding λ_0 once in the positiv sense. Now for $x \in \mathbb{C}$ "small enough"

$$P(x) = \frac{1}{2\pi i} \int_\gamma R(\lambda, x) d\lambda \tag{6}$$

is the projection onto sum of all the generalized eigenspaces to the eigenvalues that have split from λ_0 ("split eigenvalues"). Therefore

$$\alpha(x) := \operatorname{tr} T(x)P(x) \tag{7}$$

is the weighted (by the dimension of the generalized eigenspaces) sum of all the split eigenvalues.

Therefore

$$\alpha(x) - m\lambda_0 = \operatorname{tr}(T(x) - \lambda_0 I)P(x) = \frac{1}{2\pi i}\operatorname{tr}\int_\gamma (\lambda - \lambda_0)R(\lambda, x)d\lambda \tag{8}$$

Inserting the expansion for R (equation 5):

$$\begin{aligned}\alpha(x) - m\lambda_0 &= \frac{1}{2\pi i}\operatorname{tr}\int_\gamma (\lambda - \lambda_0)R(\lambda, 0)\sum_{n=1}^{\infty}\big(A(x)R(\lambda, 0)\big)^n d\lambda \\ &= -\frac{1}{2\pi i}\operatorname{tr}\int_\gamma (\lambda - \lambda_0)\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\partial}{\partial z}(A(x)R(z, 0))^n\right)(\lambda)d\lambda \\ &= \frac{1}{2\pi i}\operatorname{tr}\int_\gamma \sum_{n=1}^{\infty}\frac{1}{n}(A(x)R(\lambda, 0))^n d\lambda.\end{aligned} \tag{9}$$

In the first equality the $n = 0$ term vanishes. For the second the cyclicity of the trace and the derivative of the resolvent are used. For the third partial integration is used. Collecting powers of x in the above:

$$\alpha(x) - m\lambda_0 = \sum_{n=1}^{\infty}\alpha_n x^n \tag{10}$$

with

$$\alpha_n := \sum_{k=1}^{n}\sum_{\substack{(i_1,\ldots,i_k)\in(\mathbb{N}\setminus\{0\})^k \\ i_1+\cdots+i_k=n}}\frac{1}{2\pi k i}\operatorname{tr}\int_\gamma T_{i_1}R(\lambda, 0)\cdots T_{i_k}R(\lambda, 0)d\lambda. \tag{11}$$

For $n \in \mathbb{Z}$ let $V_n \in \mathscr{B}(X)$ be the coefficient of $(\lambda - \lambda_0)^n$ in the Laurent expansion of T_0 around λ_0, then using the residue theorem and the Laurent expansion of the Resolvent around λ_0 to evaluate the integral:

$$\alpha_n = \sum_{k=1}^{n}\sum_{\substack{(i_1,\ldots,i_k)\in(\mathbb{N}\setminus\{0\})^k \\ i_1+\cdots+i_k=n}}\frac{1}{k}\sum_{\substack{(j_1,\ldots,j_k)\in\mathbb{Z}^k \\ j_1+\cdots+j_k=-1}}\operatorname{tr} T_{i_1}V_{j_1}\cdots T_{i_k}V_{j_k}. \tag{12}$$

1.4 Eigenvalue Perturbation-series for Linear Perturbation and Semi-simple Eigenvalue

In the special case of a linear perturbation ($T_n = 0$ for all $n \in \mathbb{N}$ with $n \geq 2$) and a semi simple eigenvalue (meaning that $(T_0 - \lambda_0 I)P = 0$) we obtain

$$\begin{aligned}\alpha_n &= \frac{1}{n} \sum_{\substack{(j_1,\ldots,j_n)\in\mathbb{Z}^n \\ j_1+\cdots+j_n=-1}} \operatorname{tr} T_1 V_{j_1} \cdots T_1 V_{j_k} \\ &= \frac{1}{n} \sum_{\substack{(j_1,\ldots,j_n)\in\mathbb{N}^n \\ j_1+\cdots+j_n=n-1}} \operatorname{tr} T_1 V_{j_1-1} \cdots T_1 V_{j_k-1}.\end{aligned} \tag{13}$$

α_n is the quantity that is calculated by the code (for a range of n values). To reduce the computational complexity we want to find all the index combinations $(j_1, \ldots, j_n) \in \mathbb{N}^n$ with $j_1 + \cdots + j_n = n - 1$ up to cyclic permutations. It turns out, that every orbit under the cyclic group action on these indices has n elements. The algorithm to find one element of each orbit is implemented in app/Combinatorics.hs. This algorithm is then used to calculate α_n in app/PerturbationSeries.hs.

2 Exact Solution Of 2x2 Case

For $E_1, E_2, a \in \mathbb{R}$ let

$$H := \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

and

$$V := \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

Define $T : \mathbb{C} \to \mathbb{C}^{2\times 2}$ by $T\ x = H + x \cdot V$. Assume that $E_1 \neq E_2$, then the two eigenvalues $E_\pm$ of $T\ x$ (the roots of the characteristic polynomial) can be found as

$$E_\pm(x) = \frac{E_1 + E_2}{2} \pm \frac{|E_1 - E_2|}{2} \sqrt{1 + \left(\frac{2ax}{E_1 - E_2}\right)^2}. \tag{14}$$

Now upon potentially relabeling $E_\pm$ we obtain

$$E_\pm(x) = \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \sqrt{1 + \left(\frac{2ax}{E_1 - E_2}\right)^2}. \tag{15}$$

Now the binomial series says that for $x \in \mathbb{C}$ with $|x| < 1$:

$$(1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k, \tag{16}$$

where the radius of convergence of the power series is 1. Therefore

$$\begin{aligned} E_{\pm}(x) &= \frac{E_1 + E_2}{2} \pm \frac{E_1 - E_2}{2} \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\frac{2ax}{E_1 - E_2} \right)^{2k} \\ &= E_{1/2} \pm \frac{1}{2} \sum_{k=1}^{\infty} \binom{1/2}{k} \frac{(2a)^{2k}}{(E_1 - E_2)^{2k-1}} x^{2k} \end{aligned} \tag{17}$$

where the radius of convergence r of the power series is

$$r = \frac{|E_1 - E_2|}{2|a|}. \tag{18}$$

Since the coefficients in a power series expansion are unique we can use the above equation to test the validity of the code, by comparing the exact coefficients in the above expression with the calculated ones from the code. This is implemented in the /app/test.hs file.