RC Circuit

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Contents

1 Theory

1.1 RC Circuit

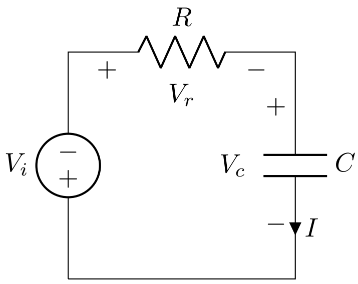


Figure 1: RC circuit

We consider the RC circuit (see fig. 1) as a system with the voltage V_i as the input and the voltage across the capacitor V_c as the output. At the capacitor

$$Q = CV_c,$$

where C is the capacity (a constant) and Q is the charge stored in the capacitor. At the resistor (Ohms law)

$$RI = V_r,$$

where V_r is the voltage across the resistor and R is the resistance (a constant). The loop rule implies that

$$V_i = V_c + V_r.$$

The current rule implies that the current I at the capacitor is the current at the resistor. The current at the capacitor is the derivative of the stored charge. Therefore combining the equations:

$$V_i = Q/C + R\dot{Q}.$$

This can be written in the standard form with Q as the state variable:

$$\dot{Q} = -Q/(RC) + V_0/R.$$

Therefore the RC circuit is equivalent to the system $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, where

$$\mathcal{A} := -1/(RC),\ \mathcal{B} := 1/R,\ \mathcal{C} := 1/C.$$

From now on system theory notation, that is for $u \in L^1_{\mathrm{loc}}$ and $x_0 \in \mathbb{R}$ we denote by $x(\bullet, x_0, u)$ the state of the system (that is the charge of the capacitor as a function of time) with initial charge x_0 and input voltage u.

In the case of no input: $\forall t \in [0, \infty)$ and $x_0 \in \mathbb{R}$:

$$x(t, x_0, 0) = x_0 \exp(-t/(RC)).$$

1.2 Transfer function

The transfer function G is given by

$$G(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} = \frac{1}{RC}\frac{1}{s + 1/(RC)} = \frac{1}{1 + \omega_0^{-1}s},$$

where $\omega_0 := 1/(RC)$. Let $\omega \in (0, \infty)$, $G(i\omega) \neq 0, u_0 \in \mathbb{R}$ and $u(t) := u_0 \sin(\omega t)$. Let $r := |G(i\omega)|$ and $\varphi := \arg G(i\omega)$. Then asymptotically (for $t \to \infty$) the output of the system with input u and arbitary initial value is given by y_a, where

$$y_a(t) := u_0 r \sin(\omega t + \varphi).$$

Now

$$|G(i\omega)| = \frac{1}{|1 + RCi\omega|} = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

and

$$\arg(G(i\omega)) = -\arctan(\omega/\omega_0).$$

2 Experiment

Used $C = 10^5$ pF and $R = 1.6 \times 10^3\,\Omega$ (including the output impedance of the signal generator). Therefore we expect $\tau := RC \approx 1.6 \times 10^{-4}$ s. The input voltage can be set with a signal generator that is part of the oscilloscope. The input and the output (the voltage across the capacitor) of the system are monitored with an oscilloscope, which can log the voltage as a function of time.

2.1 Discharge Curve

To obtain τ experimentally: We charge the capacitor first by applying a constant 1 V input. The input is then set to 0 and the output is measure for some time. In practice this is implemented by using as input a periodic step function with large period compared to τ. The exponential decay law can then be fitted to the discharge data. The result of the fit is $\tau = (5.41 \pm 0.01) \times 10^{-4}$ s. The measured data and the fit are visualised in figure 2.

2.2 Transfer Function

Let $f_0 := 2\pi/\tau$. For $f \in (0, \infty)$ and $U_{\mathrm{in}} = 2$ V we apply

$$U_{\mathrm{in}} \sin(2\pi f \bullet)$$

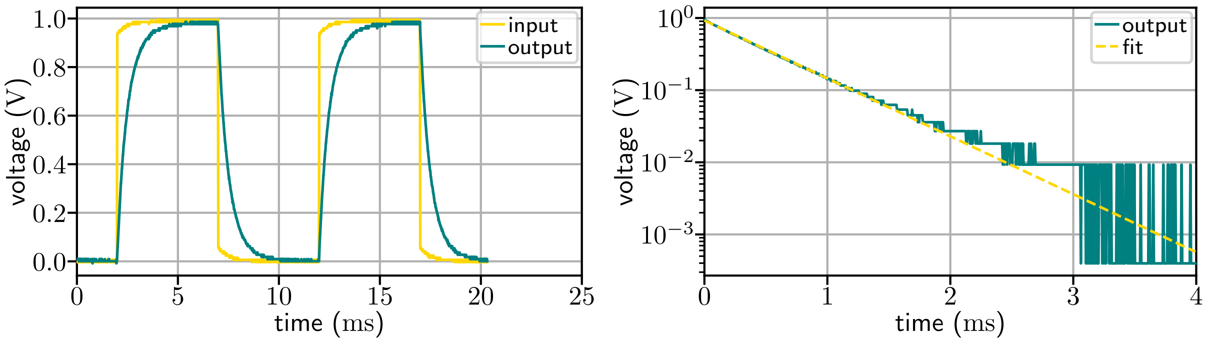
as input. We wait for 2 s so that the transient settles and then measure the output. From the theory section: The output has the (asymptotic) form

$$U_{\mathrm{out}} \sin(2\pi f \bullet + \varphi),$$

where

$$U_{\mathrm{out}} = \frac{1}{\sqrt{1 + f/f_0}} U_{\mathrm{in}}, \quad \varphi = -\arctan(f/f_0).$$

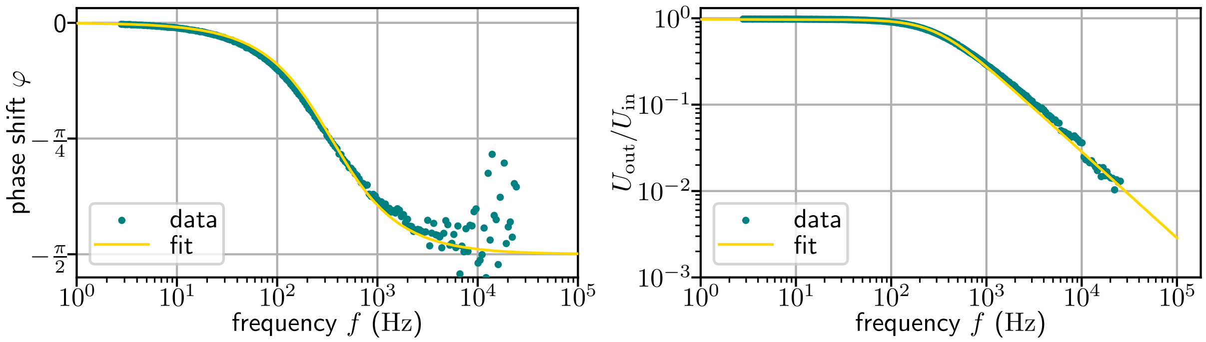
To obtain U_{out} and φ we do a least squares fit of the output signal. We then repeat this procedure for different frequencies and fit the above relation (with f_0 as the parameter). The results can be found in figure 3. For the phase shift fit the optimal τ was $\tau = (4.6 \pm 0.2) \times 10^{-4}$ s. For the amplitude quotient fit the optimal τ was $\tau = (5.39 \pm 0.03) \times 10^{-4}$ s.



(a) capacitor charging and discharging measurement

(b) fit of the first discharge in figure 2a (time shifted)

Figure 2: capacitor charging and discharging



(a) Phase shift fit

(b) Amplitude quotient fit

Figure 3: Transfer function fit

3 Inhomogeneous Solutions

Let G be the transfer function (from above) and $a \in (0, \infty)$. Consider $u : [0, \infty) \to \mathbb{R}$, $u(t) := at$ as input. Let y be the output of the system with initial value 0 and input u. Then (in some right half plane) $\hat{y} = G\hat{u}$, where ˆ denotes Laplace transform. Now $\hat{u}(s) = as^{-2}$ for all $s \in \mathbb{C}_{\Re>0}$ and so for all $s \in \mathbb{C}_{\Re>0}$:

$$\hat{y}(s) = as^{-2}\frac{1}{1 + \tau s}.$$

According to Wolfram-alpha (inverse Laplace) this implies for all $t \in [0, \infty)$:

$$y(t) = a(\tau(e^{-t/\tau} - 1) + t).$$

Measured by applying periodic signal with ramp first and then nothing (so the capacitor can fully discharge): The results can be found in figure 4.

4 Linear Quadratic Optimal Control

Let $\Pi \in \mathbb{R}$ solve the CARE. That is

$$\Pi^2\mathcal{B}^2 - 2\Pi\mathcal{A} - \mathcal{C}^2 = 0.$$

Which has the solutions

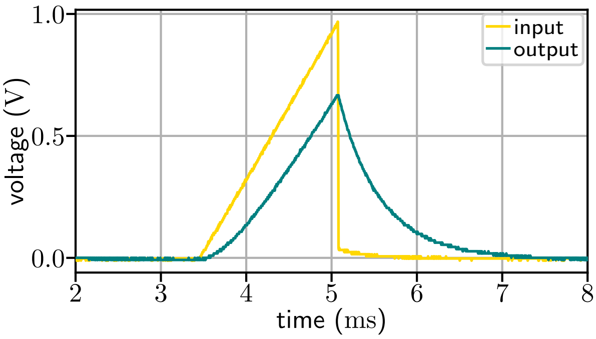
$$\Pi = \mathcal{B}^{-2}(\mathcal{A} \pm \sqrt{\mathcal{A}^2 + \mathcal{B}^2\mathcal{C}^2}).$$

Only the $+$ case results in a non-negative Π. Inserting the values:

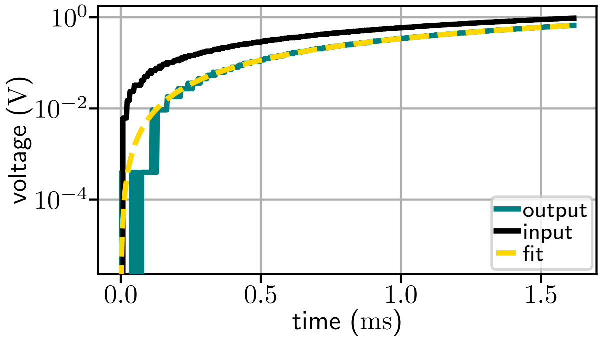
$$\Pi = R^2(\sqrt{2} - 1)\tau^{-1}.$$

Let $x_0 \in \mathbb{R}$ (the initial charge). And $u_0 := C^{-1}x_0$ the initial voltage. The optimal control $u : [0, \infty) \to \mathbb{R}$ with initial value x_0 satisfies for all $t \in [0, \infty)$:

$$u(t) = -\mathcal{B}\Pi \exp(t(\mathcal{A} - \mathcal{B}^2\Pi))x_0 = -(\sqrt{2} - 1)\exp(-\sqrt{2}t/\tau)u_0.$$



(a) measured data (periodic ramp)



(b) fit of ramp on the left (time shifted)

Figure 4: Inhomogeneous RC circuit

The optimal cost is $\Pi x_0^2 = C^2R^2/\tau(\sqrt{2}-1)u_0^2 = \tau(\sqrt{2}-1)u_0^2 \approx 0.0002237u_0^2$. Experiments: Start with $u_0 = 1\,$V. Then: No input vs. LQ-optimal control vs. optimal input to 0 in time 2τ and then no input. Experiment in figure 5. Cost for no input: 0.00248. Cost for LQ-optimal control: 0.00210. Cost for optimal input to 0 in time 2τ then no input: 0.00248.

5 Minimal Energy Control

Let $t_1 \in (0,\infty)$ and $x_0, x_1 \in \mathbb{R}$. Then the unique (in the a.e. sense) input $u : [0,\infty) \to \mathbb{R}$ with $x(t_1, x_0, u) = x_1$ is given by

$$u(t) := \mathcal{B}\exp((t_1 - t)\mathcal{A})W^{-1}(x_1 - \exp(\mathcal{A}t_1)x_0),$$

where

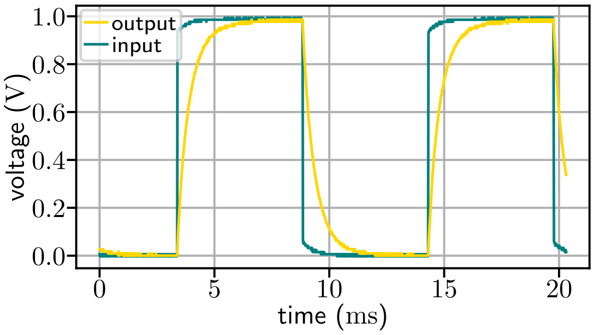
$$W := \int_0^{t_1} \exp(2\mathcal{A}t)\mathcal{B}^2 dt.$$

Now

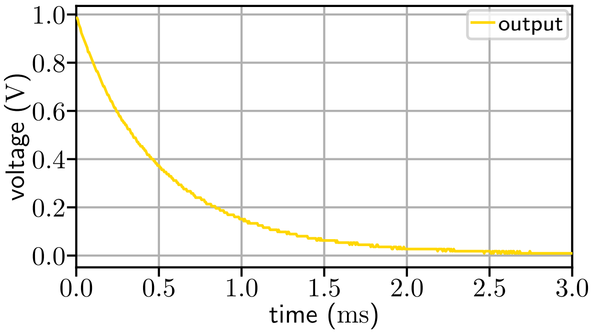
$$W = \frac{C}{2R}(1 - \exp(-2t_1/\tau)) \Rightarrow W^{-1} = \frac{2R}{C}(1 - \exp(-2t_1/\tau))^{-1}.$$

Let $u_0 := C^{-1}x_0,\ u_1 := C^{-1}x_1$. Then for all $t \in [0,\infty)$

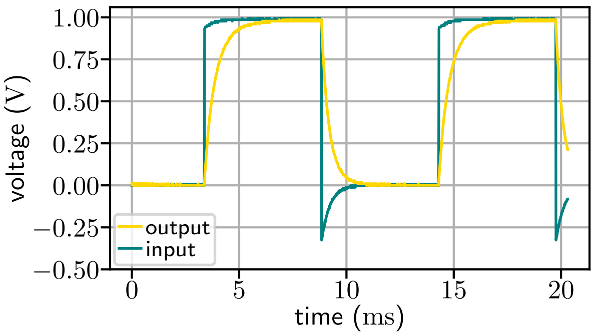
$$u(t) = 2\exp(-(t_1 - t)/\tau)(1 - \exp(-2t_1/\tau))^{-1}(u_1 - \exp(-t_1/\tau)u_0).$$



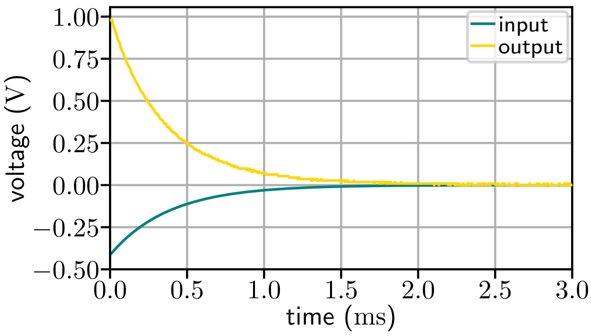
(a) no control discharge



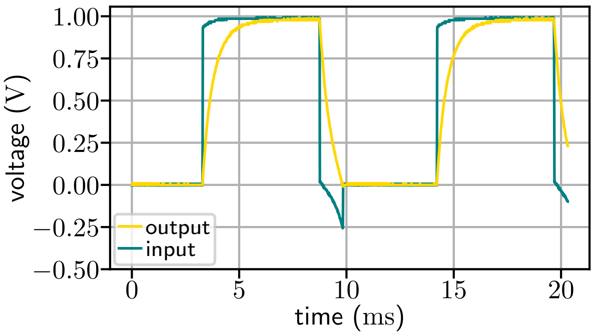
(b) no control discharge, zoomed and time shifted



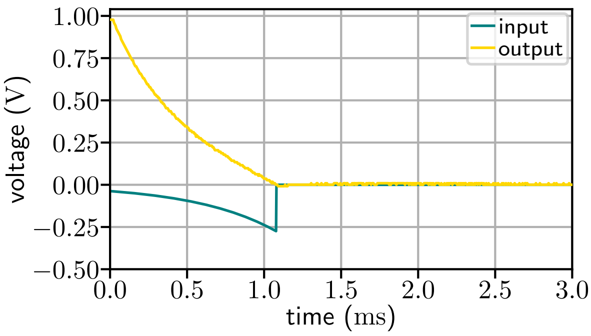
(c) LQ-optimal control discharge



(d) LQ-optimal control discharge, zoomed and time shifted



(e) optimal input to zero control



(f) optimal input to zero control, zoomed and shifted

Figure 5: LQR