First Ace

Problem:
Given a standard shuffled deck of 52 cards, if we continue to remove cards from the top of the deck without replacement, then how many times do you need to perform this action on average before coming across the first Ace?

Solution:
Let N_n be the random event denoting the number of cards we draw from a deck of n cards to get the first Ace. Then, if we condition on the first card drawn from the deck, the probability of getting an Ace is $4/n$ and the probability of not getting an Ace is $(n-4)/n$. Therefore, if V represents the first outcome, it follows that

$$\begin{aligned}
\mathbb{E}(N_n) &= \mathbb{E}(N_n|V=A)\mathbb{P}(V=A) + \mathbb{E}(N_n|V=A')\mathbb{P}(V=A') \\
&= \frac{4}{n}\cdot 1 + \frac{n-4}{n}\Big[1+\mathbb{E}(N_{n-1})\Big] \\
\implies \quad n\mathbb{E}(N_n) &= n + (n-4)\mathbb{E}(N_{n-1}),
\end{aligned}$$

which we can proceed to solve by by finding a complementary function and particular integral. If we set $e_n = \mathbb{E}(N_n)$ and use the superscripts CF and PI to refer to the complementary function and particular integral, respectively, then

$$e_n^{\text{CF}} = A\left(\frac{n-4}{n}\right)^n,$$

where A is an arbitrary constant to be determined. Moreover, if we substitute $e_n^{PI} = \alpha n + \beta$, we find that

$$\begin{aligned}
& n(\alpha n+\beta) + (4-n)\Big[\alpha(n-1)+\beta\Big] = n \\
\implies\ & \alpha n^2 + \beta n + 4\alpha n - 4\alpha - \alpha n^2 + \alpha n + 4\beta - \beta n = n \\
\implies\ & 5\alpha = 1 \ \text{ and } \ 4(\beta-\alpha) = 0 \\
\implies\ & \alpha = \frac{1}{5} \ \text{ and } \ \beta = \frac{1}{5}.
\end{aligned}$$

Therefore, the general solution is given by

$$\begin{aligned}
e_n &= e_n^{\text{CF}} + e_n^{\text{PI}} \\
&= A\left(\frac{n-4}{n}\right)^n + \frac{n+1}{5} \\
&= -\frac{1}{5}\left(\frac{n-4}{n}\right)^n + \frac{n+1}{5},
\end{aligned}$$

where the value of the constant A has been derived from the condition $e_0 = 0$. Thus,

$$e_{52} = \frac{53}{5} - \frac{1}{5}\left(\frac{48}{52}\right)^{52} \approx 10.6.$$