Average Center Distance

Problem:
A point is chosen uniformly from the unit disk. What is the expected value and variance of the distance between the point and the center of the disk?

Solution:
Since we are concerned with the unit disk, we shall work in polar coordinates, i.e., $0 \leq r \leq 1$ and $0 \leq \vartheta \leq 2\pi$. Then, if $\boldsymbol{x}$ denotes the random point, it follows that

$$\begin{aligned}
\mathbb{E}\big(|\boldsymbol{x}|\big) &= \int_A |\boldsymbol{x}| \cdot f_{XY}(x, y)\,\mathrm{d}A \\
&= \int_A \sqrt{x^2 + y^2} \cdot \frac{1}{\pi}\,\mathrm{d}A \\
&= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r \cdot r\,\mathrm{d}r\,\mathrm{d}\vartheta \\
&= 2 \cdot \frac{r^3}{3}\bigg|_0^1 = \frac{2}{3}.
\end{aligned}$$

In a similar fashion,

$$\mathbb{E}\big(|\boldsymbol{x}|^2\big) = 2 \cdot \frac{r^4}{4}\bigg|_0^1 = \frac{1}{2},$$

and so the variance is given by

$$\mathbb{V}\mathrm{ar}\big(|\boldsymbol{x}|\big) = \mathbb{E}\big(|\boldsymbol{x}|^2\big) - \mathbb{E}\big(|\boldsymbol{x}|\big)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}.$$