Re-rolling Payoff

Problem:
Consider a game where we throw a die and receive the value of the outcome in pounds. If we are given the option to re-roll the dice up to n times, what is the expected payoff?

Solution:
Let X_n be a random variable denoting the value of the dice roll given you are allowed to re-roll up to n times. Clearly, when $n = 0$, the expected value of this game is given by

$$\mathbb{E}(X_0) = \sum_{i=1}^{6} i\mathbb{P}(X_1 = i) = \frac{1}{6} \cdot \frac{1}{2} \cdot 6 \cdot 7 = \frac{7}{2}.$$

If we now allow one re-roll of the die, then if V is the value of the first roll, it follows by applying the Partition theorem that

$$\begin{aligned}
\mathbb{E}(X_1) &= \mathbb{E}\big[X_1 | V < \mathbb{E}(X_0)\big]\mathbb{P}\big[V < \mathbb{E}(X_0)\big] + \mathbb{E}\big[X_1 | V \geq \mathbb{E}(X_0)\big]\mathbb{P}\big[V \geq \mathbb{E}(X_0)\big] \\
&= \frac{1}{2}\mathbb{E}(X_0) + \frac{1}{2} \cdot \frac{4+5+6}{3} = \frac{7}{4} + \frac{5}{2} = \frac{17}{4}.
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
\mathbb{E}(X_2) &= \mathbb{E}\big[X_2 | V < \mathbb{E}(X_1)\big]\mathbb{P}\big[V < \mathbb{E}(X_1)\big] + \mathbb{E}\big[X_2 | V \geq \mathbb{E}(X_1)\big]\mathbb{P}\big[V \geq \mathbb{E}(X_1)\big] \\
&= \frac{2}{3}\mathbb{E}(X_1) + \frac{1}{3} \cdot \frac{5+6}{2} = \frac{17}{6} + \frac{11}{6} = \frac{14}{3} < 5, \\
\mathbb{E}(X_3) &= \mathbb{E}\big[X_3 | V < \mathbb{E}(X_2)\big]\mathbb{P}\big[V < \mathbb{E}(X_2)\big] + \mathbb{E}\big[X_3 | V \geq \mathbb{E}(X_2)\big]\mathbb{P}\big[V \geq \mathbb{E}(X_2)\big] \\
&= \frac{2}{3}\mathbb{E}(X_2) + \frac{1}{3} \cdot \frac{5+6}{2} = \frac{28}{9} + \frac{11}{6} = \frac{89}{18} < 5, \\
\mathbb{E}(X_4) &= \mathbb{E}\big[X_4 | V < \mathbb{E}(X_3)\big]\mathbb{P}\big[V\big] < \mathbb{E}(X_3)\big] + \mathbb{E}\big[X_4 | V \geq \mathbb{E}(X_3)\big]\mathbb{P}\big[V \geq \mathbb{E}(X_3)\big] \\
&= \frac{2}{3}\mathbb{E}(X_3) + \frac{1}{3} \cdot \frac{5+6}{2} = \frac{89}{27} + \frac{11}{6} = \frac{277}{54} > 5.
\end{aligned}$$

Now that we have arrived at a number of re-rolls that gives us an expected value greater than 5, if we are offered the option of 5 re-rolls or more, then we will only stop if the outcome of the die roll is a 6. In other words, for $n \geq 5$,

$$\begin{aligned}
\mathbb{E}(X_n) &= \mathbb{E}\big[X_n | V < \mathbb{E}(X_{n-1})\big]\mathbb{P}\big[V < \mathbb{E}(X_{n-1})\big] + \mathbb{E}\big[X_n | V \geq \mathbb{E}(X_{n-1})\big]\mathbb{P}\big[V \geq \mathbb{E}(X_{n-1})\big] \\
&= \frac{5}{6}\mathbb{E}(X_{n-1}) + \frac{1}{6} \cdot 6 = \frac{7}{2}\left(\frac{5}{6}\right)^n + \sum_{i=0}^{n-1}\left(\frac{5}{6}\right)^i,
\end{aligned}$$

which tends to 6 as n tends to infinity, by noting that

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r},$$

provided $|r| < 1$. To identify what value of n we need to generate an expected value between 5 and 6, we can rewrite the expectation as

$$\mathbb{E}(X_n) = \frac{7}{2}\left(\frac{5}{6}\right)^n + \frac{1-(5/6)^n}{1-5/6} = 6 - \frac{5}{2}\left(\frac{5}{6}\right)^n,$$

and set it equal to a target value and solve for n.