Uniform Distance

Problem:
Consider the independent random variables $X \sim U(0,1)$ and $Y \sim U(0,2)$. Evaluate the expectation and variance of $|X-Y|$.

Solution:
Clearly, the probability density functions of X and Y are given by

$$f_X(x) = \begin{cases} 1 & \text{for } x \in (0,1), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 1/2 & \text{for } y \in (0,2), \\ 0 & \text{otherwise.} \end{cases}$$

Since the random variables are independent, they have the joint density function

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 1/2 & \text{for } (x,y) \in (0,1) \times (0,2), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, it follows that

$$\begin{aligned}
\mathbb{E}\big(|X-Y|\big) &= \int_A |x-y| \cdot \frac{1}{2} \, \mathrm{d}A \\
&= \frac{1}{2}\left(\int_0^1 \int_y^1 x - y \, \mathrm{d}x \, \mathrm{d}y + \int_0^1 \int_0^y y - x \, \mathrm{d}x \, \mathrm{d}y + \int_1^2 \int_0^1 y - x \, \mathrm{d}x \, \mathrm{d}y \right) \\
&= \frac{1}{2}\left(\int_0^1 \frac{x^2}{2} - xy \bigg|_y^1 \mathrm{d}y + \int_0^1 xy - \frac{x^2}{2} \bigg|_0^y \mathrm{d}y + \int_1^2 xy - \frac{x^2}{2} \bigg|_0^1 \mathrm{d}y \right) \\
&= \frac{1}{2}\left(\int_0^1 \frac{1}{2} - y + \frac{y^2}{2} \, \mathrm{d}y + \int_0^1 \frac{y^2}{2} \, \mathrm{d}y + \int_1^2 y - \frac{1}{2} \, \mathrm{d}y \right) \\
&= \frac{1}{2}\left(\frac{y}{2} - \frac{y^2}{2} + \frac{y^3}{6} \bigg|_0^1 + \frac{y^3}{6} \bigg|_0^1 + \frac{y^2}{2} - \frac{y}{2} \bigg|_1^2 \right) = \frac{2}{3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\big(|X-Y|^2\big) &= \frac{1}{2} \int_0^2 \int_0^1 (x-y)^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{2} \int_0^2 \int_0^1 x^2 - 2xy + y^2 \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{2} \int_0^2 \frac{x^3}{3} - \frac{2x^2 y}{2} + xy^2 \bigg|_0^1 \mathrm{d}y \\
&= \frac{1}{2} \int_0^2 \frac{1}{3} - y + y^2 \, \mathrm{d}y \\
&= \frac{1}{2}\left(\frac{y}{3} - \frac{y^2}{2} + \frac{y^3}{3} \bigg|_0^2 \right) = \frac{2}{3}.
\end{aligned}$$

Therefore,

$$\mathbb{V}\mathrm{ar}\big(|X-Y|\big) = \mathbb{E}\big(|X-Y|^2\big) - \mathbb{E}\big(|X-Y|\big)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}.$$