

Problem:

Consider a deck of m cards. Suppose you turn over the first card, record what it is, put it back in the deck and shuffle. If you repeat this process, how many times do you expect you have to do this to see every card in the deck?

Solution:

Let N_n be a random variable denoting the number of times you need to perform this action until you are left with n cards you have yet to see in the deck of size m . Then, the probability of turning over a card you have seen already is $(m - n)/m$ and the probability of turning over a new card is n/m . Thus, the expected value of this game can be calculated by conditioning on whether or not the next card turned over has been seen or not. This is an application of the partition theorem and yields

$$\begin{aligned}
\mathbb{E}(N_n) &= \frac{m-n}{m} \left[1 + \mathbb{E}(N_n) \right] + \frac{n}{m} \left[1 + \mathbb{E}(N_{n-1}) \right] \\
&= 1 + \frac{m-n}{m} \mathbb{E}(N_n) + \frac{n}{m} \mathbb{E}(N_{n-1}) \\
\implies m\mathbb{E}(N_n) &= m + (m-n)\mathbb{E}(N_n) + n\mathbb{E}(N_{n-1}) \\
\implies \mathbb{E}(N_n) &= \frac{m}{n} + \mathbb{E}(N_{n-1}) \\
&= \frac{m}{n} + \frac{m}{n-1} + \mathbb{E}(N_{n-2}) \\
&= m \left[\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 \right] + \mathbb{E}(N_0) \\
&= m \sum_{i=1}^n \frac{1}{i} \\
&\sim m \left[\log n + \gamma + \frac{1}{2n} \right] \\
\implies \mathbb{E}(N_m) &\sim m \left[\log m + \gamma + \frac{1}{2m} \right] \\
\implies \mathbb{E}(N_m) &= 1 + \mathbb{E}(N_{m-1}),
\end{aligned}$$

where $\gamma \approx 0.577$ and the final line is derived from having to initiate the game by drawing the first card so that it has been seen by the player. Therefore, when $m = 52$, the asymptotic approximation for the expected value is given by

$$\mathbb{E}(N_{52}) \sim 1 + 52 \left[\log 51 + \gamma + \frac{1}{102} \right] \approx 235,$$

which is in agreement with Monte Carlo simulations I have run for this problem.