

Problem:

Consider a deck of m cards. Suppose you turn over the first card, record what it is, put it back in the deck and shuffle. If you repeat this process, how many times do you expect you have to do this to see every card in the deck?

Solution:

Let N_n be a random variable denoting the number of times you need to perform this action until you are left with n cards you have yet to see in a deck of size m . Then, the probability of turning over a card you have seen already is $(m - n)/m$ and the probability of turning over a new card is n/m . Thus, the expected value of this game can be calculated by conditioning on whether or not the next card turned over has been seen or not. This is an application of the partition theorem, and yields

$$\begin{aligned}
\mathbb{E}(N_n) &= \frac{m-n}{m} \left[1 + \mathbb{E}(N_n) \right] + \frac{n}{m} \left[1 + \mathbb{E}(N_{n-1}) \right] \\
&= 1 + \frac{m-n}{m} \mathbb{E}(N_n) + \frac{n}{m} \mathbb{E}(N_{n-1}) \\
\implies m\mathbb{E}(N_n) &= m + (m-n)\mathbb{E}(N_n) + n\mathbb{E}(N_{n-1}) \\
\implies \mathbb{E}(N_n) &= \frac{m}{n} + \mathbb{E}(N_{n-1}) \\
&= \frac{m}{n} + \frac{m}{n-1} + \mathbb{E}(N_{n-2}) \\
&= m \left[\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 \right] + \mathbb{E}(N_0) \\
&= m \sum_{i=1}^n \frac{1}{i} \\
&\sim m \left[\log n + \gamma + \frac{1}{2n} \right] \\
\implies \mathbb{E}(N_m) &\sim m \left[\log m + \gamma + \frac{1}{2m} \right] \\
\implies \mathbb{E}(N_m) &= 1 + \mathbb{E}(N_{m-1}),
\end{aligned}$$

where we have: used $\mathbb{E}(N_0) = 0$, introduced an approximation for the harmonic series involving $\gamma \approx 0.577$, and derived the final line by recalling that you initiate the game by drawing the first card so that you have information recorded before you proceed. Therefore, when $m = 52$, the asymptotic estimate for the expected value is given by

$$\mathbb{E}(N_{52}) \sim 1 + 52 \left[\log 51 + \gamma + \frac{1}{102} \right] \approx 235,$$

which is consistent with Monte Carlo simulations for the problem.