No Consecutive Heads

Problem:
What is the probability that there are no consecutive heads in a string of n coin tosses, assuming the coin is fair? What if the coin is not fair? What if we want the probability of no three consecutive heads?

Solution:
Let X_n be the random variable denoting a string of n coin outcomes that do not include consecutive heads. Then, if V is the outcome of the first coin toss, we can condition on this event to derive the following:

$$\begin{aligned}\mathbb{P}(X_n) &= \mathbb{P}(X_n|V=H)\mathbb{P}(V=H) + \mathbb{P}(X_n|V=T)\mathbb{P}(V=T) \\ &= \frac{1}{4}\mathbb{P}(X_{n-2}) + \frac{1}{2}\mathbb{P}(X_{n-1}),\end{aligned}$$

where we have noted that if the first outcome is heads, then we require the next coin toss to be tails to avoid the event of consecutive heads. If for compactness we let $p_n = \mathbb{P}(X_n)$, then

$$4p_{n+2} - 2p_{n+1} - p_n = 0,$$

where we have shifted indices. Solving the corresponding auxillary equation for ϑ, we find that

$$\vartheta_\pm = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)} = \frac{1 \pm \sqrt{5}}{4},$$

and so

$$p_n = A\left(\frac{1+\sqrt{5}}{4}\right)^n + B\left(\frac{1-\sqrt{5}}{4}\right)^n,$$

for arbitrary constants A and B. Since there are no consecutive heads for a single coin toss, and only one occurrence of consecutive heads for two coins, we impose the conditions $p_1 = 1$ and $p_2 = 3/4$. Therefore, the constants can be found to be

$$A = \frac{3 - 4\vartheta_-}{4\vartheta_+(\vartheta_+ - \vartheta_-)} \text{ and } B = \frac{4\vartheta_+ - 3}{4\vartheta_-(\vartheta_+ - \vartheta_-)}.$$

As an example, if we consider 10 coin tosses, then the probability is given by $p_{10} = 9/64$. Now, if we turn our attention to a biased coin, where the probability of heads is p, it follows that

$$\begin{aligned}& p_n = p(1-p)p_{n-2} + (1-p)p_{n-1} \\ \implies & p_{n+2} + (p-1)p_{n+1} + p(p-1)p_n = 0.\end{aligned}$$

This can be solved for different value of p between 0 and 1. Finally, if we are instead concerned with not getting 3 consecutive heads, our probability becomes

$$p_n = p^2(1-p)p_{n-3} + p(1-p)p_{n-2} + (1-p)p_{n-1}.$$