Median Distance

Problem:
Consider n uniformly and identically distributed points X_i in a region defined by a unit ball in d-dimensional space, centred about the origin. If X_m denotes the point with the smallest distance from the origin, relative to all the other points, then what is the median of $|X_m|$?

Solution:
To calculate the median, we first need to derive the cumulative distribution function (CDF) for $|X_m|$. If $Y_i \sim U(0,1)$ and independent, with $1 \leq i \leq n$, then

$$\begin{aligned}
\mathbb{P}\left[\min_{1\leq i\leq n} Y_i \leq y\right] &= 1 - \mathbb{P}\left[\min_{1\leq i\leq n} Y_i > y\right] \\
&= 1 - \mathbb{P}(Y_1 > y, Y_2 > y, \cdots, Y_n > y) \\
&= 1 - \prod_{i=1}^{n} \mathbb{P}(Y_i > y) \\
&= 1 - \Big[\mathbb{P}(Y_1 > y)\Big]^n \\
&= 1 - \Big[1 - \mathbb{P}(Y_1 \leq y)\Big]^n \\
&= 1 - \Big[1 - F_Y(y)\Big]^n,
\end{aligned}$$

where F_Y is the common CDF. Extending this to our d-dimensional problem, we see that

$$\mathbb{P}(|X_m| \leq r) = 1 - \Big[1 - F(r)\Big]^n,$$

where F is the common CDF for the distances $|X_i|$. Since the probability a point lies inside the region is proportional to the radius, i.e.,

$$F(r) = \alpha r^d / \alpha \cdot 1^d = r^d,$$

we see that

$$\mathbb{P}(|X_m| \leq r) = 1 - \big(1 - r^d\big)^n.$$

Therefore, the median of the point with minimal distance can be found by setting this probability equal to $1/2$, i.e.,

$$\begin{aligned}
1 - \big(1 - r^d\big)^n &= \frac{1}{2} \\
\implies r &= \left[1 - \left(\frac{1}{2}\right)^{1/n}\right]^{1/d}.
\end{aligned}$$

As an example, if we consider 12 independent and uniformly distributed points in the unit sphere, then the desired value is $r \approx 0.38$.