# 1 Intrduction

## 1.1 Complex Numbers

Defined as $\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$ subject to conditions, for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$

- Addition $(+)$:
  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
- Multiplication $(\cdot)$:
  $(x_1, y_1) + (x_2, y_2) = (x_1x_2 - y_1y_2, x_2y_1 + x_1y_2)$

$(\mathbb{C}, +), (\mathbb{C}, \cdot)$ are albelian groups, with units $(0, 0)$ and $(1, 0)$ respectivly

### 1.1.1 Lemma

$(\mathbb{C}, +, \cdot)$ is a field with multiplicative inverse

$$z \in \mathbb{C}\backslash\{0, 0\}, \quad z^{-1} = (\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2})$$

such that $z * z^{-1} = (1, 0)$

We will define as follows:

$$1 := (1, 0), \quad i := (0, 1), \quad 0 := (0, 0)$$

Allowing us to write complex numbers as

## 1.2 Sequences, Series and Convergence

The space $(\mathbb{C}, |.|)$ is a *metric space* which can be identified with $\mathbb{R}^2$ with the euclidean distance.

### 1.2.1 Definition

A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges, $\lim_{n\to\infty} z_n = w$ iff

$$\lim_{n\to\infty} |z_n - w| = 0$$

Since $\mathbb{R}^2$ is complete, so is $\mathbb{C}$. Namely every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$.

### 1.2.2 Proposition

A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges iff

$$\forall\epsilon > 0, \exists N > 0, \text{s.t. } n, m > N \Rightarrow |z_n - z_m| < \epsilon$$

A series $\sum_{n=0}^{\infty} z_n$ converges iff the sequence of partial sums $(s_n)$ converges

$$\text{where } s_n = \sum_{j=0}^{n} z_j$$

The series of real numbers $\sum_{n=0}^{\infty} b_n$, for $b_n \geq 0$ converges if $\limsup_{n\to\infty} |b_n|^{1/n} < 1$ , and diverges to $+\infty$ if $\limsup_{n\to\infty} |b_n|^{1/n} > 1$

### 1.2.3 Theorem

If a series $\sum_{n=0}^{\infty} z_n$ converges absolutely, then it too converges.
proof
Let $T_n = \sum_{j=0}^{n} |z_j|, \quad S_n = \sum_{j=0}^{n} z_j$. Since the series converges absolutley, we know for some $N \in \mathbb{N}$

$$|T_n - T_m| = |a_n| + \cdots + |a_{m+1}| < \epsilon \text{ for } n > m > N, \epsilon > 0$$

and by the triangle inequality we can show

$$\begin{aligned} |S_n - S_m| &= |a_n + \cdots + a_{m+1}| \\ &\leq |a_n| + \cdots + |a_{m+1}| \\ &= |T_n - T_m| < \epsilon \end{aligned}$$

Hence the sequence $(S_n)$ is cauchy and must converge $\square$

# 2 Holomorphic Functions

## 2.1 Differentiation of Complex functions

### 2.1.1 Definition

A subset $G \subset \mathbb{C}$ is called a *domain* if it is open and connected.

### 2.1.2 Definition

A function $f : G \to \mathbb{C}$ has a limit $c$ at a point $z_0 \in G$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |z - z_0| < \delta \Rightarrow |f(z) - c| < \epsilon$$

### 2.1.3 Definition

The function $f$ is continuous at a point $z_0 \in G$ if the limit of $f$ at $z_0$ exists and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

If $f$ is continuous at every point $z \in G$ then f is continuous in $G$.

We use the notation

$$z = x + iy, \quad f(z) = u(x, y) + iv(x, y)$$

Where $u(x, y), v(x, y)$ are functions from $\mathbb{R}^2 \to \mathbb{R}$. From this we can write

$$Re(f) = u, \quad Im(f) = v$$

So we can say $f$ is continious in $G \iff u, v$ are continious in $G$.

### 2.1.4 Definition

Let $G$ be a domain in $\mathbb{C}$ and $f : G \to \mathbb{C}$ a complex function on $G$. The function $f$ is (complex) differentiable at a point $z_0 \in G$ iff the limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad [= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}]$$

exists and is finite. The limit is independant from the direction in the complex plane in which $h$ tends to zero.

## 2.2 Basic Properties of Complex Differentiation

Complex differentiability shares several properties with real differentiability. Those being it is linear and obeys the product rule, quotient rule, and chain rule.

Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}, g : G \to \mathbb{C}$ complex functions on $G$, with $z_0 \in G$.

i) Suppose $f, g$ are differentiable at $z_0$ then $f + g, af, fg (a \in \mathbb{C})$ are also differentiable at said point and

$$(f + g)'(z_0) = f'(z_0) + g'(z_0), (af)'(z_0) = af'(z_0)$$

$$(fg)'(z_0) = f'(z_0)g(z_0) + g'(z_0)f'(z_0)$$

ii) Suppose $f, g$ are differentiable at $z_0$ and $(g)'(z_0) \neq 0$. Then $f/g$ is differentiable at the point and

$$(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g^2(z_0)}$$

iii) Suppose $f$ is differentiable at $z_0$, then $f$ is continuous at $z_0$.
proof
As $f$ is differentiable at $z_0$ then we know $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$ and we wts $\lim_{z \to z_0} f(z) = f(z_0)$. So

$$\begin{aligned}
\lim_{z \to z_0} (f(z) - f(z_0)) &= \lim_{z \to z_0} (z - z_0) \frac{(f(z) - f(z_0))}{z - z_0} \\
&= [\lim_{z \to z_0} (z - z_0)][\lim_{z \to z_0} \frac{(f(z) - f(z_0))}{z - z_0}] \\
&= 0 \cdot f'(z_0) = 0
\end{aligned}$$

□

iv) Let $B \subset \mathbb{C}$ also be an open set. Suppose $f$ is differentiable at $z_0, f(G) \subset B$ and $g : B \to \mathbb{C}$ is differentiable at $f(z_0) \in B$. Then we can say the composition $g \circ f : G \to \mathbb{C}$ is differentiable at $z_0$ and

$$(g(f))'(z_0) = g'(f(z_0))f'(z_0)$$

proof
We know $g, f$ are differentiable at $z_0, f(z_0)$ respectivly. Thus we can write

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0), \quad \lim_{h \to 0} \frac{g(f(z_0) + h)) - g(f(z_0))}{h} = g'(f(z_0))$$

Now lets define

$$v := \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0), \quad u := \frac{g(f(z_0) + h)) - g(f(z_0))}{h} - g'(f(z_0))$$

This allows us to write

$$f(z_0 + h) = f(z_0) + [f'(z_0) + v]h, \quad g(f(z_0) + k) = g(f(z_0)) + [g'(f(z_0)) + u]k$$

So we can say

$$g(f(z_0 + h)) = g(f(z_0) + [f'(z_0) + v]h)$$

and then

$$\begin{aligned}
\frac{g(f(z_0 + h)) - g(f(z_0))}{h} &= \frac{g(f(z_0)) + [g'(f(z_0)) + u] \cdot [f'(z_0) + v] \cdot h - g(f(z_0))}{h} \\
&= [g'(f(z_0)) + u] \cdot [f'(z_0) + v] \\
\lim_{h \to 0} \frac{g(f(z_0 + h)) - g(f(z_0))}{h} &= \lim_{h \to 0} [g'(f(z_0)) + u] \cdot [f'(z_0) + v] \\
&= [\lim_{h \to 0} g'(f(z_0)) + \lim_{h \to 0} u] \cdot [\lim_{h \to 0} f'(z_0) + \lim_{h \to 0} v] \\
&= g'(f(z_0))f'(z_0)
\end{aligned}$$

*as both* $u, v \to 0$ *as* $h \to 0$.
Hence the limit exists and is finite so we can say $g \circ f$ is differentiable at $z_0$ and

$$(g(f))'(z_0) = g'(f(z_0))f'(z_0)$$

□

### 2.2.1 Definition

A function $f : G \to \mathbb{C}$ defined on a domain $G$ is called *holomorphic* in $G$ if it has a complex derivative at all points in $G$.

## 2.3 Derivative as a linear approximation

### 2.3.1 Definition

Let $w(z)$ be a complex function in a neighbourhood of $z = 0$. The function $w(z) = o(z)$ if

$$\lim_{z \to 0} \frac{w(z)}{z} = 0$$

$w(z) = O(z)$ if

$$\lim_{z \to 0} \frac{w(z)}{z} = c, \quad c \neq 0$$

### 2.3.2 Lemma

Let $G$ be a domain in $\mathbb{C}$ and $f : G \to \mathbb{C}$ a compelx function on $G$. $f$ is complex differentiable at $z_0 \in G$ iff there exists a constant $A \in \mathbb{C}$ s.t.

$$f(z_0 + h) - f(z_0) = Ah + o(h), \quad \forall h \in G_{z_0}$$

*where $G_{z_0}$ is a neighbourhood of $z_0$ in $G$ and $A = f'(z_0)$.*

Proof

Suppose $f$ differentiable at $z_0$. Let $A = f'(z_0)$. Define $w : G_{z_0} \to \mathbb{C}; h \to f(z_0 + h) - f(z_0) - Ah$. Then

$$\lim_{h \to 0} \frac{w(h)}{h} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - f'(z_0)h}{h} = f'(z_0) - f'(z_0) = 0$$

Hence $w(h) = o(h)$ is satified.

Now supppose that $A \in \mathbb{C}, w : G_{z_0} \to \mathbb{C}; h \to f(z_0 + h) - f(z_0) - Ah = o(h)$. Then

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{Ah + w(h)}{h} = A + \lim_{h \to 0} \frac{w(h)}{h} = A$$

It follows that $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists and $f'(z_0) = A$. $\square$

## 2.4 Real versus Complex Differentiation

We consider arbitrary maps of the form

$$F : \mathbb{R}^2 \to \mathbb{R}^2; (x, y) \to (u(x, y), v(x, y))$$

### 2.4.1 Theorem

The function $F$ is real differentiable at $(x_0, y_0)$ if there exists a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$F(x, y) - F(x_0, y_0) = (x - x_0, y - y_0)A + o(\sqrt{(x - x_0)^2 + (y - y_0)^2})$$

*The matrix $A$ is the transpose of the Jacobian of $F$ calculated in $(x_0, y_0)$, namely*

$$A^t = JF(x_0, y_0) = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix}$$

and

$$\lim_{x \to x_0, y \to y_o} \frac{o(\sqrt{(x - x_0)^2 + (y - y_0)^2})}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$