Joseph Dowling.

Exercise 1

Exercise 4

Let $\epsilon > 0$. We know by definition $f = f^+ - f^-$, and $\int f d\mu = \sup\{\int s d\mu | o \leq s \leq f, s \text{simple and measurable}\}$. So we have that

$$\exists \gamma^+, \gamma^- \text{ simple,measurable s.t. } \int |f^+ - \gamma^+| d\mu < \epsilon/2, \int |f^- - \gamma^-| d\mu < \epsilon/2$$

Then define $\gamma := \gamma^+ - \gamma^-$, so that

$$\begin{aligned}\int |f - \gamma| d\mu &= \int |(f^+ - \gamma^+) - (f^- - \gamma^-)| d\mu \\ &\leq \int |f^+ - \gamma^+| d\mu + \int |f^- - \gamma^-)| d\mu \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

Exercise 6

We know for fixed $\epsilon > 0$, we can choose N s.t. for $n \geq N$, $\sup_{x \in \Omega} |f_n(x) - f(x)| < \epsilon/\mu(\Omega)$. So then

$$\int f_n d\mu \leq \int (f + \epsilon/\mu(\Omega)) d\mu, \int f d\mu \leq \int (f_n + \epsilon/\mu(\Omega)) d\mu$$

then since we have that $\int \epsilon/\mu(\Omega) d\mu = \epsilon$,

$$\int f_n d\mu - \int f d\mu \leq \epsilon, \int f d\mu - \int f_n d\mu \leq \epsilon$$

Thus we get

$$\int |f_n - f| d\mu \leq \epsilon$$

Hence

$$\int f d\mu = \lim_n \int f_n d\mu$$

In the case when $\mu(\Omega) = \infty$ we cannot take our ϵ over $\mu(\Omega)$, so the equality may fail.