Measure Theory and Integration

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1 The Riemann Integral

1.1 Definition

Let $f : [a,b] \to \mathbb{R}$ s.t. $a < b$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ with $a = x_0 < x_1 < \cdots < x_n = b$. Set

$$L(P,f) = \sum_{i=1}^{n} \inf\{f(x) : x_{i-1} \leq x < x_i\}(x_i - x_{i-1})$$

$$U(P,f) = \sum_{i=1}^{n} \sup\{f(x) : x_{i-1} \leq x < x_i\}(x_i - x_{i-1})$$

as the lower and upper integral sums respectivly.
Then f is integrable iff

$$\sup_P L(P,f) = \inf_P U(P,f)$$

where the supremum and infinum are taken over all possible partitions.
Hence the Rieman Integral is

$$\int_a^b f(x)dx = \sup_P L(P,f) = \inf_P U(P,f)$$

1.2 Lemma

Let $a, b \in \mathbb{R}$ s.t. $a < b$, $f : [a,b] \to \mathbb{R}$ a function. Then

$$\inf_P U(P,f) \geq \sup_P L(P,f)$$

Proof. Let P_1, P_2 be partitions, define $\mathbb{Q} = P_1 \cap P_2$. Since Q is a 'finer' partition than both P_1 and P_2, it is clear

$$L(P_1,f) \leq L(Q,f) \text{and } U(P_2,f) \geq U(Q,f)$$

Therefore

$$L(P_1,f) \leq L(Q,f) \leq U(Q,f) \leq U(P_2,f)$$

Now as P_1, P_2 are arbitrary, the equality is true for all P, so we can take the supremum of the LHS over all partitions, and the infinum of the RHS over all partitions, leaving us with...

$$\inf_P U(P,f) \geq \sup_P L(P,f)$$

□

1.3 Theorem

A function $f : [a, b] \to \mathbb{R}$ is *Riemann Integrable* $\iff \forall \epsilon > 0, \exists P_*$ s.t.

$$U(P_*, f) - L(P_*, f) < \epsilon$$

Where P_* is any partition on $[a.b]$

Proof. Assume $\forall \epsilon > 0, \exists P_*$ s.t. $U(P_*, f) - L(P_*, f) < \epsilon$, wts f is Riemann Integrable. It is clear

$$\inf_P U(P, f) \leq U(P_*, f) \text{ and } \sup_P L(P, f) \geq L(P_*, f)$$

Now by subtracting and using lemma 1.2 we obtain

$$0 \leq \inf_P U(P, f) - \sup_P L(P, f) \leq U(P_*, f) - L(P_*, f) < \epsilon$$

Since ϵ arbitrary take $\epsilon \to \infty$ and thus

$$\sup_P L(P, f) = \inf_P U(P, f)$$

which is the definition of Riemann Integrability

Now Assume f is Riemann Integrable,
Let $\epsilon > 0$, then $\exists P_1, P_2$ s.t

$$U(P_1, f) < \inf_P U(P, f) + \frac{\epsilon}{2} \text{ and } L(P_2, f) > \sup_P L(P, f) + \frac{\epsilon}{2}$$

Define $P_* = P_1 \cup P_2$ and obtain

$$\begin{aligned} U(P_*, f) - L(P_*, f) &\leq U(P_1, f) - L(P_2, f) \\ &< \inf_P U(P, f) - \sup_P L(P, f) + \epsilon \\ &= \epsilon \end{aligned}$$

$\square$

1.4 Theorem

$f : [a, b] \to \mathbb{R}$ is Riemann Integrable iff the set of discontinuities of f has *Lebasque Measure Zero.*

1.5 Definition

The Lebasque Measure of an open interval $I = (a, b)$ is $\mu(I) = b - a$.
A set $N \subset \mathbb{R}$ has Lebasque Measure Zero if $\forall \epsilon > 0$ there exists a countable collection of open intervals $\{I_1, I_2, \ldots\}$ s.t. $N \subset \cup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} \mu(I_i) < \epsilon$

1.5.1 Examples

Some examples of sets with a Lebasque Measure of Zero:

- A single point $\{a\}$
- Any countable set of points $E = \{a_1, \ldots\}$
- Any countable union of sets of measure 0
- Any subset of a set of measure 0

2 Measurable sets and Integrals

2.1 Definition

Let $X \neq \emptyset$ be a set. A family $\mathbb{X}$ of subsets X is a σ-algebra if

i) $\emptyset \in \mathbb{X}$, $X \in \mathbb{X}$

ii) $A \in \mathbb{X} \to A^c \in \mathbb{X}$

iii) $A_1, A_2, \cdots \in \mathbb{X} \to \cup_{n=1}^{\infty} A_n \in \mathbb{X}$

Now $(X, \mathbb{X})$ is called a *Measurable Space*, and each $S \in \mathbb{X}$ are called *Measurable Sets*.

2.2 Definition

Let $X \neq \emptyset$ and $\mathbb{A}$ be a non-empty collections of subsets of X. Let $\mathbb{Y}$ be the collection of all σ-algebras containing $\mathbb{A}$. Then $\beta(\mathbb{A}) = \cap_{\mathbb{X} \in \mathbb{Y}} \mathbb{X}$ is the σ-algebra generated by $\mathbb{A}$. This is the smallest σ-algebra generated by $\mathbb{A}$.

2.3 Definition

Let $X = \mathbb{R}$, $\mathbb{A} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a < b\}$. The σ-algebras generated by $\mathbb{A}$ is *Borel Algebra*, denoted $\mathbb{B}$. This is the smallest σ-algebras contianing all open sets. A set $B \in \mathbb{B}$ is a *Borel Set*. *NB:* $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

2.4 Definition

Let $(X, \mathbb{X})$ be a measurable space. Then $f : X \to \mathbb{R}$ is a $\mathbb{X}$-measurable function if for any Borel set $A \in \mathbb{B}$ we have $f^{-1}(A) \in \mathbb{X}$.

2.5 Definition

Let $(X, \mathbb{X})$, $(Y, \mathbb{Y})$ be measurable spaces. Then $f : X \to Y$ is an $\mathbb{X}$-measurable function if for any set $A \in \mathbb{Y}$ we have $f^{-1}(A) \in \mathbb{X}$.

2.6 Lemma

A function $f : X \to \mathbb{R}$ is measurable $\iff \forall \alpha \in \mathbb{R}$ the set

$$\{x \in X : f(x) < \alpha\} \equiv f^{-1}((-\infty, \alpha))$$

is measurable (i.e belongs to $\mathbb{X}$).
<u>Proof</u>
Assuming the set is measurable, as $(-\infty, \alpha) \in \mathbb{B}$, then by definition $f^{-1}((-\infty, \alpha)) \in \mathbb{X}$

Now assume $f^{-1}((-\infty, \alpha)) \in \mathbb{X} \forall \alpha \in \mathbb{R}$, wts for any Borel set $A \in \mathbb{B}$ we have $f^{-1}(A) \in \mathbb{X}$
Define $\mathbb{A}$ to be a collection of sets of the form $(-\infty, \alpha)$ with $\alpha \in \mathbb{R}$. Now we can see that sets in $\mathbb{A}$ generate Borel σ-algebra $\mathbb{B}$.
Let $\mathbb{Y}$ be the smallests σ-algebra contianing $\mathbb{A}$. Obviously $\mathbb{A} \subset \mathbb{B}$ and therefore $\mathbb{Y} = \beta(\mathbb{A}) \subset \mathbb{B}$.
Now wts $\mathbb{Y}$ contians intervals (a, b) for $a < b \in \mathbb{R}$. Take the sets $(-\infty, a), (-\infty, b) \in \mathbb{Y}$. Then we have

$$[a, b) = (-\infty, b) \cap (-\infty, a)^c \in \mathbb{Y} \forall a < b \in \mathbb{R}$$

$$(a, b) = \cup_{n=N}^{\infty} [a - \frac{1}{n}, b] \text{ for large enough N}$$

and hence $(a, b) \in \mathbb{Y}$.
Since $\mathbb{B}$ is the smallest σ-algebra contianing all open intervals (a, b) we have $\mathbb{Y} = \mathbb{B}$.
Now we define the smallest σ-algebra contianing sets $f^{-1}(\mathbb{A})$, i.e. $\beta(f^{-1}(\mathbb{A}))$. Since $f^{-1}(\mathbb{A}) \subset \mathbb{X}$ we also have $\beta(f^{-1}(\mathbb{A})) \subset \mathbb{X}$.
But $\beta(f^{-1}(\mathbb{A})) = f^{-1}(\beta(\mathbb{A})) = f^{-1}(\mathbb{B})$ and so $f^{-1}(\mathbb{B}) \subset \mathbb{X}$ $\square$

2.7 Lemma

Let $(X, \mathbb{X})$ be a measurable space, let $f : X \to \mathbb{R}$. Then the following are equivalent:

i) $\forall \alpha \in \mathbb{R}, A_\alpha = \{x \in X : f(x) > \alpha\} \in \mathbb{X}$

ii) $\forall \alpha \in \mathbb{R}, B_\alpha = \{x \in X : f(x) \leq \alpha\} \in \mathbb{X}$

iii) $\forall \alpha \in \mathbb{R}, C_\alpha = \{x \in X : f(x) \geq \alpha\} \in \mathbb{X}$

iv) $\forall \alpha \in \mathbb{R}, D_\alpha = \{x \in X : f(x) < \alpha\} \in \mathbb{X}$

2.8 Lemma

Let $(X, \mathbb{X})$ be a measurable space, $f, g : X \to \mathbb{R}$ be measurable. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then $h : X \to \mathbb{R}; h(x) = F(f(x), g(x))$ is measurable.

Proof. Fix $\alpha \in \mathbb{R}$ and we want to show:

$$X_\alpha := \{x : F(f(x), g(x)) < \alpha\} = \{x : (f(x), g(x)) \in F^{-1}((-\infty, \alpha))\}$$

is measurable.
Let $A = F^{-1}((-\infty, \alpha))$ and note it is an open set in $\mathbb{R}^2$ as F is continuous, thus it is a countable union of open rectangles, so it is sufficient to show for any set $(a, b) \times (c, d)$ we have that

$$\{x : (f(x), g(x)) \in (a, b) \times (c, d)\} = \{x : a < f(x) < b\} \cap \{x : c < g(x) < d\}$$

is measurable. We can now use the fact f, g are measurable. □

2.9 Corollary

Let f, g be measurable and $c \in \mathbb{R}$. Then

$$f + g, fg, |f|, \frac{f}{g} \quad g \neq 0, \max\{f, g\}, \min\{f, g\}, cf$$

are all measurable by above lemma.

2.10 Definition

Let $(X, \mathbb{X})$ measurable space. Then $f : X \to \overline{\mathbb{R}}$ is $\mathbb{X}$-measurable if for any $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) > \alpha\} \in \mathbb{X}$.
The collection of all $\overline{\mathbb{R}}$-valued, $\mathbb{X}$-measurable functions on X is denoted $M(X, \mathbb{X})$.

2.11 Lemma

Let $(X, \mathbb{X})$ be a measurable space. $f : X \to \overline{\mathbb{R}}$ is measurable iff:

i) $A = \{x \in X : f(x) = +\infty\}$
   $B = \{x \in X : f(x) = -\infty\}$

ii) the function $f_1(x) = \begin{cases} f(x), & x \in (A \cup B)^c \\ 0, & x \in A \cup B \end{cases}$ is measurable.

Proof. Suppose f_1, A, B are measurable. For $\alpha \geq 0$ we have

$$\{x : f(x) > \alpha\} = \{x : f_1(x) > \alpha\} \cup A$$

And thus is measurable, analogous for $\alpha < 0$. □