Set Theory

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1 Introduction

Axiom of Extensionality (for sets)

For two sets a, b, we say $a = b$ iff:

$$\forall x, [x \in a \Rightarrow x \in b]$$

Axiom of Extensionality (for classes)

For two classes A, B, we say $A = B$ iff:

$$\forall x, [x \in A \Rightarrow x \in B]$$

Axiom of Pair Set

For any sets x, y there is a set $z = \{x, y\}$ with elements just x and y. z is called the *(unordered) pair set of* x, y. *NB: If* $x = y$ *then we have* $\{x, y\} = \{x, x\} = \{x\}$

Definition 1.1

Let $\mathcal{P}(x)$ denote the class $\{y | y \subseteq x\}$, called the *Power set of* x.

Definition 1.2

The *Empty Set*, denoted $\emptyset$, is the unique set with no elements.

- We can define $\emptyset$ as $\{x | x \neq x\}$.
- For any set/class A, we have $\emptyset \subset A$.

2 Classes

Theorem 1.4

The collection $R = \{x | x \notin x\}$ does not define a set.

Proof. Suppose R was a set z. Assume $z \in R$, then by the definition of R, $z \notin z$. However if $z \notin R$ then we should have $z \in z$. This is a contradiction. $\square$

NB: This is an example of a class that is not a set. Any class which is not, or cannot be, a set is called a Proper Class.

Axiom of Subsets

Let $\Phi(x)$ be a definite, welldefined property. Let x be a set. Then

$$\{y \in x | \Phi(y)\} \quad \text{is a set.}$$

Corollary 1.5

Let V denote the class of all sets. Then V is a proper class.

Proof. If V were a set then we should have, $R = \{y \in V | y \notin y\}$ is a set by the Axiom of Subsets, however we have just shown R is not a set. $\square$

Definition 1.6

For any set Z there is a class, $\cup Z$, which consists of the members of members of Z.

$$\cup Z = \{x | \exists t(x \in t \in Z)\}$$

Axiom of Unions

For any set Z, $\cup Z$ is a set.

Definition 1.8

For any non-empty set Z, there is another set, $\cap Z$, which consists of the members of all members of Z.

$$\cap Z = \{x | \forall t \in Z(x \in t)\}$$

or using index sets we write

$$x \in \cap_{j \in I} A_j \iff (\forall j \in I)(x \in A_j)$$

3 Relations and Functions

For two sets X, Y, there are relations R that hold between some elements of X and of Y, denoted xRy. The types of relation are:

- Reflexive: $x \in X \Rightarrow xRx$
- Irreflexive: $x \in X \Rightarrow \neg(xRx)$
- Symmetric: $(x, y \in X \wedge xRy) \Rightarrow yRx$
- Connected: $(x, y \in X) \Rightarrow (x = y \vee xRy \vee yRx)$
- Transiative: $(x, y, z \in X \wedge xRy \wedge yRz) \Rightarrow xRz$

NB: recall that an equivalence relation *is that R should satify symmetry, reflixivity and transiativity.*

Definition 1.10

A relation $\prec$ on a set X is a (strict) partial ordering if it is irreflexitve and transitive. I.e.

i) $x \in X \Rightarrow \neg(x \prec x)$.

ii) $(x, y, z \in X \wedge x \prec y \wedge y \prec z) \Rightarrow (x \prec z)$

Definition 1.11

i) If $\prec$ is a partial ordering of a set X, and $\emptyset \neq Y \subseteq X$, then $z \in X$ is a lower bound for Y in X if:
$$\forall Y(y \in Y \Rightarrow z \preceq y)$$

ii) $z \in X$ is an infimum or greatest lower bound (glb) for Y if it is a lower bound for Y and if z' is a lower bound for Y then $z' \preceq z$.

iii) The concepts of upper bound and supremum (least upper bound (lub)) are defined analogously.

Definition 1.12

i) We say $f : (X, \prec_1) \to (Y, \prec_2)$ is an order preserving map of the partial orders $(X, \prec_1), (Y, \prec_2)$ iff:

$$\forall x, z \in X(x, \prec_1 z \Rightarrow f(x), \prec_2 f(z))$$

ii) Orderings $(X, \prec_1), (Y, \prec_2)$ are (order) isomorphic, written $(X, \prec_1) \cong (Y, \prec_2)$, if there is an order preserving map between them which is also a bijection.

iii) There are completely analogous definitions between nonstrics orders $\preceq_1, \preceq_2$.

Theorem 1.13 *(Representation Theorem for partially ordered sets)*

If $\prec$ partially orders X, then there is a set Y of subsets of X which is such that $(X, \preceq)$ is order isomorphic to $(Y, \subseteq)$.

Proof. Given any $x \in X$, let $X^x = \{z \in X | z \preceq x\}$. Notice if $x \neq y$ then $X^x \neq X^y$. So the assignemnt of x to X^x is 1:1. Let $Y = \{X^x | x \in X\}$. Then we have

$$x \preceq y \iff X^x \subseteq X^y$$

Coonsequently, setting $f(x) = X^x$ we have an order isomorphism. □

NB: Often we deal with roderings where every element is comparable with every other, known as strong connectivity *and we call the rdering* total*.*

Definition 1.14

A relation $\prec$ on X is a string total ordering if it is a partial ordering which is connected:

$$\forall x, y(x, y \in X \Rightarrow (x = y \vee x \prec y \vee y \prec x))$$

NB: For $\preceq$ we call the ordering non-strict.

Definition 1.15

i) $(A, \prec)$ is a wellordering if it is a string total orderings and for any subset $Y \subseteq A$, $Y \neq \emptyset \Rightarrow Y$ has a $\prec$-least element. We write $(A, \prec) \in WO$

ii) A partial ordering R on a set A, (A, R) is a wellfounded relation if for any subset $Y \subseteq A$, $Y \neq \emptyset \Rightarrow Y$ has an R-minimal element.