Exercise 1.2

Prove for $\alpha < 3, V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha)$.

Proof. We know, by definition, $V_{n+1} = \mathcal{P}(V_n), \forall n \in \mathbb{N}$. So want to show $V_\alpha \cup \mathcal{P}(V_\alpha) = \mathcal{P}(V_\alpha)$. We have that $\mathcal{P}(V_\alpha) = \{v | v \subseteq V_\alpha\}$. Everything is a subset of itself so,

$$V_\alpha \subseteq \mathcal{P}(V_\alpha)$$

Hence

$$V_\alpha \cup \mathcal{P}(V_\alpha) = \mathcal{P}(V_\alpha)$$

□

Exercise 1.4

Let $x, y = \{\{0,1\},\{2\}\}, \{\{0,1,2\}\}$ respectivly. Then $x \neq y$ but

$$\cup x = \{0,1,2\} \ = \cup y$$

Exercise 1.11

If $(A, \prec)$ is a total ordering and A finite, show it is a well ordering.

Proof. Want to show (i) it is a string total ordering and (ii) $\forall Y \subseteq A, Y \neq \emptyset \Rightarrow Y$ has a $\prec$-least element.
(i) is true by initial assumption, so wts (ii).
Let $a_0, \dots, a_k \in A$ be the ordered elements of A. Let $Y \subseteq A, Y \neq \emptyset$. Suppose to the contrary that Y has no $\prec$-least element, so $a_0 \notin Y$ as it would be a $\prec$-least element. Let $B = Y^c \subseteq A$ be the set of elements of A not in Y, so $a_0 \in B$. Now suppose $a_0, \dots, a_n \in B$, for some $n \in \mathbb{N}$, then $a_{n+1} \notin Y$ as $a_0, \dots, a_n \notin Y$ so a_{n+1} would be a $\prec$-least element. So $a_{n+1} \in B$. By induction, (as A finite), $B = A$ so $Y = \emptyset$ which is a contradiction.
Hence (i) and (ii) hold and $(A, \prec)$ is a wellordering. □