Exercise 1

i) Want to show $[a, b] = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$.
We have $a - \frac{1}{n} > a, b + \frac{1}{n} < b, \forall n$. Which implies $[a, b] \subseteq (a - \frac{1}{n}, b + \frac{1}{n}), \forall n$ which implies $[a, b] \subseteq \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$.
Now take $x \notin [a, b]$, consider the case $x > b$. As $n \to \infty, b + \frac{1}{n} \to b$. By A.P. we can say $\exists N \in \mathbb{N}$ s.t. $b + \frac{1}{N} < x$, so $x \notin (a - \frac{1}{N}, b + \frac{1}{N})$. Analogous for the case $x < a$. Finally $x \notin [a, b] \Rightarrow x \notin \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$ giving us $\cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n}) \subseteq [a, b]$.

i) Want to show $(a, b) = \cup_{n=1}^{\infty}(a + \frac{1}{n}, b - \frac{1}{n})$.
We have $a + \frac{1}{n} > a, b - \frac{1}{n} < b, \forall n$. Which implies $(a + \frac{1}{n}, b - \frac{1}{n}) \subseteq (a, b), \forall n$ which implies $\cap_{n=1}^{\infty}(a + \frac{1}{n}, b - \frac{1}{n}) \subseteq (a, b)$.
Now take $x \in (a, b)$. By A.P. $\exists N \in \mathbb{N}$ s.t. $x > (a + \frac{1}{N}), x < (b - \frac{1}{N})$. So $x \in (a + \frac{1}{N}, b - \frac{1}{N}) \Rightarrow x \in \cup_{n=1}^{\infty}(a + \frac{1}{n}, b - \frac{1}{n})$, giving us $(a, b) \subseteq \cup_{n=1}^{\infty}(a + \frac{1}{n}, b - \frac{1}{n})$.

Exercise 6

Take $\Omega = \{a, b, c\}, \mathbb{A} = \{\emptyset, \Omega, \{a, b\}, \{c\}\}$. Let $f : \Omega \to \mathbb{R}$ such that $f(a) = f(c) = 1, f(b) = -1$. Then we have that f is not measurable as, when considering $A_\alpha = \{x \in \Omega : f(x) < \alpha\}$, take $\alpha = 1$ then $A_\alpha = \{b\} \notin \mathbb{A}$.
Now we have $|f(a)| = |f(b)| = |f(c)| = 1, f^2(a) = f^2(b) = f^2(c) = 1$, so for both

$$A_\alpha = \begin{cases} \Omega, & \alpha > 1 \\ \emptyset, & \alpha \leq 1 \end{cases} \Rightarrow \forall \alpha, A_\alpha \in \mathbb{A}$$

Hence both $|f|, f^2$ are measurable.

Exercise 10

i) Want to show $f^{-1}(\emptyset) = \emptyset$
Suppose to the contrary that $\exists x \in f^{-1}(\emptyset)$. This would imply $f(x) \in \emptyset$ which is a contradiction. Hence $f^{-1}(\emptyset)$ has no elements and thus $f^{-1}(\emptyset) = \emptyset$.

ii) Want to show $f^{-1}(\Omega_2) = \Omega_1$
We have $f^{-1}(\Omega_2) = \{w \in \Omega_1 : f(w) \in \Omega_2\}$, and by the definition of f, $\forall w \in \Omega_1, f(w) \in \Omega_2$. So trivially we have $f^{-1}(\Omega_2) = \Omega_1$.

iii) Want to show $f^{-1}E \backslash F = f^{-1}(E) \backslash f^{-1}(F)$.

$$\begin{aligned} x \in f^{-1}(E \backslash F) &\iff f(x) \in E \wedge f(x) \notin F \\ &\iff x \in f^{-1}(E) \wedge x \notin f^{-1}(F) \\ &\iff x \in f^{-1}(E) \backslash f^{-1}(F) \end{aligned}$$

iv) Want to show $f^{-1}(\cup_\alpha E_\alpha) = \cup_\alpha f^{-1}(E_\alpha)$

$$\begin{aligned} x \in f^{-1}(\cup_\alpha E_\alpha) &\iff f(x) \in \cup_\alpha E_\alpha \\ &\iff f(x) \in E_{\alpha_1} \quad \text{for some } \alpha_1 \\ &\iff x \in f^{-1}(E_{\alpha_1}) \\ &\iff x \in \cup_\alpha f^{-1}(E_\alpha) \end{aligned}$$

v) Want to show $f^{-1}(\cap_\alpha E_\alpha) = \cap_\alpha f^{-1}(E_\alpha)$

$$\begin{aligned} x \in f^{-1}(\cap_\alpha E_\alpha) &\iff f(x) \in \cap_\alpha E_\alpha \\ &\iff f(x) \in E_\alpha \quad \forall \alpha \\ &\iff x \in f^{-1}(E_\alpha) \\ &\iff x \in \cap_\alpha f^{-1}(E_\alpha) \end{aligned}$$