1 Intrduction

1.1 Complex Numbers

Defined as $\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$ subject to conditions, for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$

- Addition $(+)$:
  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
- Multiplication $(\cdot)$:
  $(x_1, y_1) + (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_2 y_1 + x_1 y_2)$

$(\mathbb{C}, +), (\mathbb{C}, \cdot)$ are albelian groups, with units $(0, 0)$ and $(1, 0)$ respectivly

1.1.1 Lemma

$(\mathbb{C}, +, \cdot)$ is a field with multiplicative inverse

$$z \in \mathbb{C}\backslash\{0, 0\}, \quad z^{-1} = (\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2})$$

such that $z * z^{-1} = (1, 0)$

We will define as follows:

$$1 := (1, 0), \quad i := (0, 1), \quad 0 := (0, 0)$$

Allowing us to write complex numbers as

1.2 Sequences, Series and Convergence

The space $(\mathbb{C}, |.|)$ is a *metric space* which can be identified with $\mathbb{R}^2$ with the euclidean distance.

1.2.1 Definition

A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges, $\lim_{n\to\infty} z_n = w$ iff

$$\lim_{n\to\infty} |z_n - w| = 0$$

Since $\mathbb{R}^2$ is complete, so is $\mathbb{C}$. Namely every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$.

1.2.2 Proposition

A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges iff

$$\forall\epsilon > 0, \exists N > 0, \text{s.t. } n, m > N \Rightarrow |z_n - z_m| < \epsilon$$

A series $\sum_{n=0}^{\infty} z_n$ converges iff the sequence of partial sums (s_n) converges

$$\text{where } s_n = \sum_{j=0}^{n} z_j$$

The series of real numbers $\sum_{n=0}^{\infty} b_n$, for $b_n \geq 0$ converges if $\limsup_{n\to\infty} |b_n|^{1/n} < 1$, and diverges to $+\infty$ if $\limsup_{n\to\infty} |b_n|^{1/n} > 1$

1.2.3 Theorem

If a series $\sum_{n=0}^{\infty} z_n$ converges absolutely, then it too converges.
<u>proof</u>
Let $T_n = \sum_{j=0}^{n} |z_j|, \quad S_n = \sum_{j=0}^{n} z_j$. Since the series converges absolutley, we know for some $N \in \mathbb{N}$

$$|T_n - T_m| = |a_n| + \cdots + |a_{m+1}| < \epsilon \text{ for } n > m > N, \epsilon > 0$$

and by the triangle inequality we can show

$$\begin{aligned} |S_n - S_m| &= |a_n + \cdots + a_{m+1}| \\ &\leq |a_n| + \cdots + |a_{m+1}| \\ &= |T_n - T_m| < \epsilon \end{aligned}$$

Hence the sequence (S_n) is cauchy and must converge $\square$

2 Holomorphic Functions

2.1 Differentiation of Complex functions

2.1.1 Definition

A subset $G \subset \mathbb{C}$ is called a *domain* if it is open and connected.

2.1.2 Definition

A function $f : G \to \mathbb{C}$ has a limit c at a point $z_0 \in G$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |z - z_0| < \delta \Rightarrow |f(z) - c| < \epsilon$$

2.1.3 Definition

The function f is continuous at a point $Z_0 \in G$ if the limit of f at Z_0 exists and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

If f is continuous at every point $z \in G$ then f is continuous in G.

We use the notation

$$z = x + iy, \quad f(z) = u(x, y) + iv(x, y)$$

Where $u(x, y), v(x, y)$ are functions from $\mathbb{R}^2 \to \mathbb{R}$. From this we can write

$$Re(f) = u, \quad Im(f) = v$$

So we can say f is continious in G if u, v are continious in G.

2.1.4 Definition

Let G be a domain in $\mathbb{C}$ and $f : G \to \mathbb{C}$ a complex function on G. The function f is (complex) differentiable at a point $z_0 \in G$ iff the limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad (h \neq 0, z_0 + h \in G)$$

exists and is finite. The limit is independant from the direction in the complex plane in which h tends to zero.

2.2 Basic Properties of Complex Differentiation

Complex differentiability shares several properties with real differentiability. Those being it is linear and obeys the product rule, quotient rule, and chain rule.

Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}, g : G \to \mathbb{C}$ complex functions on G, with $z_0 \in G$.

i) Suppose f, g are differentiable at z_0 then $f + g, af, fg (a \in \mathbb{C})$ are also differentiable at said point and

$$(f+g)'(z_0) = f'(z_0) + g'(z_0), (af)'(z_0) = af'(z_0)$$
$$(fg)'(z_0) = f'(z_0)g(z_0) + g'(z_0)f'(z_0)$$

ii) Suppose f, g are differentiable at z_0 and $(g)'(z_0) \neq 0$. Then f/g is differentiable at the point and

$$(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g^2(z_0)}$$

iii) Suppose f is differentiable at z_0, then f is continuous at z_0.
<u>proof</u>
As f is differentiable at z_0 then we know $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$ and we wts $\lim_{z \to z_0} f(z) = f(z_0)$. So

$$\begin{aligned}\lim_{z \to z_0}(f(z) - f(z_0)) &= \lim_{z \to z_0}(z - z_0)\frac{(f(z) - f(z_0))}{z - z_0} \\ &= [\lim_{z \to z_0}(z - z_0)][\lim_{z \to z_0}\frac{(f(z) - f(z_0))}{z - z_0}] \\ &= 0 \cdot f'(z_0) = 0\end{aligned}$$

□

iv) Let $B \subset \mathbb{C}$ also be an open set. Suppose f is differentiable at $z_0, f(G) \subset B$ and $g : B \to \mathbb{C}$ is differentiable at $f(z_0) \in B$. Then we can say the composition $g \circ f : G \to \mathbb{C}$ is differentiable at z_0 and

$$(g(f))'(z_0) = g'(f(z_0))f'(z_0)$$

<u>proof</u>

2.2.1 Definition

A function $f : G \to \mathbb{C}$ defined on a domain G is called *holomorphic* in G if it has a complex derivative at all points in G.