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Exercise 1

Want to show for fixed $A \in \mathbb{U}, \forall E \in \mathbb{U}$, $\lambda(E) = \mu(A \cap E)$ is a measure.

I) $\lambda(\emptyset) = \mu(A \cap \emptyset) = \mu(\emptyset) = 0$

II) Let $E \in \mathbb{U}$. $\lambda(E) = \mu(E \cup A) \geq 0$ as μ is a measure and $E \cup A \subseteq A \in \mathbb{U}$.

III) Let $E_1, E_2, \dots$ be disjoint subsets of $\mathbb{U}$ such that $\cup_i E_i = E \in \mathbb{U}$.

$$\begin{aligned}
\lambda(\cup_{n=1}^{\infty} E_n) &= \lambda(E) \\
&= \mu(A \cap E) = \mu(\cup_{n=1}^{\infty}(A \cap E_n)) \\
&= \sum_{n=1}^{\infty} \mu(A \cap E_n) \\
&= \sum_{n=1}^{\infty} \lambda(E_n)
\end{aligned}$$

Hence we have that λ is a measure on $\mathbb{U}$.

Exercise 8

Let X be a set of at least two elements. Set $\mathbb{X} = \{\emptyset, X\}$. Then trivially $(X, \mathbb{X})$ is a measure space. Define $\mu : \mathbb{X} \to \overline{\mathbb{R}}$ by $\mu(X) = \mu(\emptyset) = 0$. Then $(X, \mathbb{X}, \mu)$ is not complete as $\exists A \subset X$ with $A \notin \mathbb{X}$ but $\mu(X) = 0$.

Exercise 12

a) Want to show $f_n \to f$ uniform.
Let $\epsilon > 0, N > \frac{1}{\epsilon}$. Then for $n \geq N$ there are two cases. Case 1 for $x \in [0, n]$

$$|f_n(x) - f(x)| = |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Case 2 for $x \notin [0, n]$

$$|f_n(x) - f(x)| = 0 < \epsilon$$

Now want to show $\int f dm \neq \lim \int f_n dm$.
We know trivially that $\int f dm = 0$ as f is the zero function. So Now

$$\lim_n \int f_n dm = \lim_n \frac{1}{n} m([0, n]) = \lim_n \frac{n}{n} = 1$$

So we have that
$$\int f dm = 0 \neq 1 = \lim_n \int f_n dm$$
Fatous lemma is applicable here but, this does not contradict the Monotone Convergence Theorem as (f_n) is not monotone non-decreasing.

b) Want to show $\int gdm \neq \lim \int g_n dm$.
We know trivially that $\int gdm = 0$ as g is the zero function. So Now
$$\lim_n \int g_n dm = \lim_n nm([\frac{1}{n}, \frac{2}{n}]) = n(\frac{2}{n} - \frac{1}{n}) = 1$$
So we have that
$$\int gdm = 0 \neq 1 = \lim_n \int g_n dm$$
(g_n) converges uniform to g, Fatous lemma is applicable here, and (g_n) is monotone increasing so MCT applies also.