Complex Function Theorem

jdRanda

October 12, 2020

1 Introduction

1.1 Complex Numbers

Defined as $\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\}$ subject to conditions, for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$

- Addition (+):
  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
- Multiplication (·):
  $(x_1, y_1) + (x_2, y_2) = (x_1x_2 - y_1y_2, x_2y_1 + x_1y_2)$

$(\mathbb{C}, +), (\mathbb{C}, \cdot)$ are albelian groups, with units $(0, 0)$ and $(1, 0)$ respectivly

Lemma 1.1

$(\mathbb{C}, +, \cdot)$ is a field with multiplicative inverse

$$z \in \mathbb{C}\backslash\{0, 0\}, \quad z^{-1} = (\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2})$$

such that $z \cdot z^{-1} = (1, 0)$

We will define as follows:

$$1 := (1, 0), \quad i := (0, 1), \quad 0 := (0, 0)$$

Allowing us to write complex numbers as

$$z = 1 \cdot x + i \cdot y$$

We also define the real and imaginary parts of complex numbers as

$$Re(z) := x = \frac{z + \overline{z}}{2}, \quad Im(z) = y = \frac{z - \overline{z}}{2i}$$

The modulo and argument are

$$|z| = \sqrt{z\overline{z}} = \sqrt{x_2 + y^2}$$

$$arg(z) = \phi(mod 2\pi), \text{ so that } \cos\phi = \frac{Re(z)}{|z|}, \sin\phi = \frac{Im(z)}{|z|}$$

And all z have a polar co-ordiante representation

$$z = |z|(\cos(arg(z)) + i\sin(arg(z)) = r(\cos\phi + i\sin\phi)$$

Then by eulers formula

$$z = |z|(\cos\phi + i\sin\phi) = |z|e^{i\phi}$$

1.2 Sequences, Series and Convergence

The space $(\mathbb{C}, |.|)$ is a *metric space* which can be identified with $\mathbb{R}^2$ with the euclidean distance.

Definition 1.5

A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges, $\lim_{n\to\infty} z_n = w$ iff

$$\lim_{n\to\infty} |z_n - w| = 0$$

Since $\mathbb{R}^2$ is complete, so is $\mathbb{C}$. Namely every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$.

Proposition 1.6

A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges iff

$$\forall\epsilon > 0, \exists N > 0, \text{s.t. } n, m > N \Rightarrow |z_n - z_m| < \epsilon$$

A series $\sum_{n=0}^{\infty} z_n$ converges iff the sequence of partial sums (s_n) converges

$$\text{where } s_n = \sum_{j=0}^{n} z_j$$

The series of real numbers $\sum_{n=0}^{\infty} b_n$, for $b_n \geq 0$ converges if $\limsup_{n\to\infty} |b_n|^{1/n} < 1$, and diverges to $+\infty$ if $\limsup_{n\to\infty} |b_n|^{1/n} > 1$

Theorem

If a series $\sum_{n=0}^{\infty} z_n$ converges absolutely, then it too converges.

Proof. Let $T_n = \sum_{j=0}^{n} |z_j|, \quad S_n = \sum_{j=0}^{n} z_j$. Since the series converges absolutley, we know for some $N \in \mathbb{N}$

$$|T_n - T_m| = |a_n| + \cdots + |a_{m+1}| < \epsilon \text{ for } n > m > N, \epsilon > 0$$

and by the triangle inequality we can show

$$\begin{aligned} |S_n - S_m| &= |a_n + \cdots + a_{m+1}| \\ &\leq |a_n| + \cdots + |a_{m+1}| \\ &= |T_n - T_m| < \epsilon \end{aligned}$$

Hence the sequence (S_n) is cauchy and must converge $\square$

2 Holomorphic Functions

2.1 Differentiation of Complex functions

Definition 2.1

A subset $G \subset \mathbb{C}$ is called a *domain* if it is open and connected.

Definition 2.2

A function $f : G \to \mathbb{C}$ has a limit c at a point $z_0 \in G$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |z - z_0| < \delta \Rightarrow |f(z) - c| < \epsilon$$

Definition 2.3

The function f is continuous at a point $z_0 \in G$ if the limit of f at z_0 exists and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

If f is continuous at every point $z \in G$ then f is continuous in G.

We use the notation

$$z = x + iy, \quad f(z) = u(x, y) + iv(x, y)$$

Where $u(x, y), v(x, y)$ are functions from $\mathbb{R}^2 \to \mathbb{R}$. From this we can write

$$Re(f) = u, \quad Im(f) = v$$

So we can say f is continious in $G \iff u, v$ are continious in G.

Definition 2.4

Let G be a domain in $\mathbb{C}$ and $f : G \to \mathbb{C}$ a complex function on G. The function f is (complex) differentiable at a point $z_0 \in G$ iff the limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad [= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}]$$

exists and is finite. The limit is independant from the direction in the complex plane in which h tends to zero.

2.2 Basic Properties of Complex Differentiation

Complex differentiability shares several properties with real differentiability. Those being it is linear and obeys the product rule, quotient rule, and chain rule.

Let $G \subset \mathbb{C}$ be an open set and $f : G \to \mathbb{C}, g : G \to \mathbb{C}$ complex functions on G, with $z_0 \in G$.

i) Suppose f, g are differentiable at z_0 then $f + g, af, fg (a \in \mathbb{C})$ are also differentiable at said point and

$$(f+g)'(z_0) = f'(z_0) + g'(z_0), (af)'(z_0) = af'(z_0)$$

$$(fg)'(z_0) = f'(z_0)g(z_0) + g'(z_0)f'(z_0)$$

ii) Suppose f, g are differentiable at z_0 and $(g)'(z_0) \neq 0$. Then f/g is differentiable at the point and

$$(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g^2(z_0)}$$

iii) Suppose f is differentiable at z_0, then f is continuous at z_0.
proof
As f is differentiable at z_0 then we know $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$ and we wts $\lim_{z \to z_0} f(z) = f(z_0)$. So

$$\begin{aligned}
\lim_{z \to z_0} (f(z) - f(z_0)) &= \lim_{z \to z_0} (z - z_0) \frac{(f(z) - f(z_0))}{z - z_0} \\
&= [\lim_{z \to z_0} (z - z_0)][\lim_{z \to z_0} \frac{(f(z) - f(z_0))}{z - z_0}] \\
&= 0 \cdot f'(z_0) = 0
\end{aligned}$$

$\square$

iv) Let $B \subset \mathbb{C}$ also be an open set. Suppose f is differentiable at $z_0, f(G) \subset B$ and $g : B \to \mathbb{C}$ is differentiable at $f(z_0) \in B$. Then we can say the composition $g \circ f : G \to \mathbb{C}$ is differentiable at z_0 and

$$(g(f))'(z_0) = g'(f(z_0))f'(z_0)$$

proof

We know g, f are differentiable at $z_0, f(z_0)$ respectivly. Thus we can write

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0), \quad \lim_{h\to 0} \frac{g(f(z_0)+h)) - g(f(z_0))}{h} = g'(f(z_0))$$

Now lets define

$$v := \frac{f(z_0+h) - f(z_0)}{h} - f'(z_0), \quad u := \frac{g(f(z_0)+h)) - g(f(z_0))}{h} - g'(f(z_0))$$

This allows us to write

$$f(z_0+h) = f(z_0) + [f'(z_0) + v]h, \quad g(f(z_0)+k) = g(f(z_0)) + [g'(f(z_0)) + u]k$$

So we can say

$$g(f(z_0+h)) = g(f(z_0) + [f'(z_0) + v]h)$$

and then

$$\begin{aligned}
\frac{g(f(z_0+h)) - g(f(z_0))}{h} &= \frac{g(f(z_0)) + [g'(f(z_0)) + u] \cdot [f'(z_0) + v] \cdot h - g(f(z_0))}{h} \\
&= [g'(f(z_0)) + u] \cdot [f'(z_0) + v] \\
\lim_{h\to 0} \frac{g(f(z_0+h)) - g(f(z_0))}{h} &= \lim_{h\to 0} [g'(f(z_0)) + u] \cdot [f'(z_0) + v] \\
&= [\lim_{h\to 0} g'(f(z_0)) + \lim_{h\to 0} u] \cdot [\lim_{h\to 0} f'(z_0) + \lim_{h\to 0} v] \\
&= g'(f(z_0))f'(z_0)
\end{aligned}$$

as both $u, v \to 0$ *as* $h \to 0$.

Hence the limit exists and is finite so we can say $g \circ f$ is differentiable at z_0 and

$$(g(f))'(z_0) = g'(f(z_0))f'(z_0)$$

□

Definition 2.11

A function $f : G \to \mathbb{C}$ defined on a domain G is called *holomorphic* in G if it has a complex derivative at all points in G.

2.3 Derivative as a linear approximation

Definition 2.12

Let $w(z)$ be a complex function in a neighbourhood of $z = 0$. The function $w(z) = o(z)$ if

$$\lim_{z\to 0} \frac{w(z)}{z} = 0$$

$w(z) = O(z)$ if

$$\lim_{z\to 0} \frac{w(z)}{z} = c, \quad c \neq 0$$

Lemma 2.13

Let G be a domain in $\mathbb{C}$ and $f : G \to \mathbb{C}$ a compelx function on G. f is complex differentiable at $z_0 \in G$ iff there exists a constant $A \in \mathbb{C}$ s.t.

$$f(z_0 + h) - f(z_0) = Ah + o(h), \quad \forall h \in G_{z_0}$$

where G_{z_0} is a neighbourhood of z_0 in G and $A = f'(z_0)$.

Proof

Suppose f differentiable at z_0. Let $A = f'(z_0)$. Define $w : G_{z_0} \to \mathbb{C}; h \to f(z_0 + h) - f(z_0) - Ah$. Then

$$\lim_{h\to 0} \frac{w(h)}{h} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0) - f'(z_0)h}{h} = f'(z_0) - f'(z_0) = 0$$

Hence $w(h) = o(h)$ is satified.

Now supppose that $A \in \mathbb{C}, w : G_{z_0} \to \mathbb{C}; h \to f(z_0 + h) - f(z_0) - Ah = o(h)$. Then

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h\to 0} \frac{Ah + w(h)}{h} = A + \lim_{h\to 0} \frac{w(h)}{h} = A$$

It follows that $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists and $f'(z_0) = A$. $\square$

2.4 Real versus Complex Differentiation

We consider arbitrary maps of the form

$$F : \mathbb{R}^2 \to \mathbb{R}^2; (x, y) \to (u(x, y), v(x, y))$$

Theorem 2.14

The function F is real differentiable at (x_0, y_0) if there exists a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$F(x, y) - F(x_0, y_0) = (x - x_0, y - y_0)A + o(\sqrt{(x - x_0)^2 + (y - y_0)^2})$$

The matrix A is the transpose of the Jacobian of F calculated in (x_0, y_0), namely

$$A^t = JF(x_0, y_0) = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix}$$

and

$$\lim_{x \to x_0, y \to y_o} \frac{o(\sqrt{(x - x_0)^2 + (y - y_0)^2})}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

2.5 Cauchy-Riemann Equations

Theorem 2.17 ***(Cauchy-Riemann Theorem)***

Suppose $f : G \to \mathbb{C}; z \to u(x,y) + iv(x,y)$ is complex differentiable in $z_0 \in G$. Then $u, (x,y)$ and $v(x,y)$ satisfy the following equations:

$$\frac{\delta u}{\delta x}(x_0, y_0) = \frac{\delta v}{\delta y}(x_0, y_0)$$
$$\frac{\delta u}{\delta y}(x_0, y_0) = -\frac{\delta v}{\delta x}(x_0, y_0)$$

Moreover the derivative of f at z can be represented as:

$$f'(z_0) = \frac{\delta u}{\delta x}(x_0, y_0) + i\frac{\delta v}{\delta x}(x_0, y_0)$$

Proof. Since f is differentiable at z_0 we can chose $h = t \in \mathbb{R}$ s.t.

$$\begin{aligned} f'(z_0) &= \lim_{t \to 0} \frac{f(z_0 + t) - f(z_0)}{t} \\ &= \lim_{t \to 0} \frac{u(x_0 + t, y_0) - u(x_0, y_0) + i(v(x_0 + t, y_0) - v(x_0, y_0))}{t} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

We could also chose $h = it \in i\mathbb{R}$ s.t.

$$\begin{aligned} f'(z_0) &= \lim_{t \to 0} \frac{f(z_0 + it) - f(z_0)}{it} \\ &= \lim_{t \to 0} \frac{u(x_0, y_0 + t) - u(x_0, y_0) + i(v(x_0, y_0) - v(x_0, y_0 + t))}{it} \\ &= -iu_y(x_0, y_0) + v_y(x_0, y_0) \end{aligned}$$

Comparing the above two expressions we obtain the Cauchy-Riemann Equations. □

Theorem 2.18

The function $f : G \to \mathbb{C}; z \to u(x,y) + iv(x,y)$ is complex differentiable at $z_0 \in G$ if the functions u, v are differentiable at $(x_0, y_0) \in G$ and satisfy the Cauchy-Riemann Equations.

Proof. □

Lemma 2.22

If the functin $f = u + iv$ is holoomorphic in a domain $G \subseteq \mathbb{C}$, the Cauchy-Riemann equations can be written in the form:

$$\frac{\partial}{\partial \overline{z}} f = 0$$

Proof. Let us calculate:

$$\frac{\partial}{\partial \overline{z}} f = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(u + iv) = \frac{1}{2}(u_x + iu_y) + \frac{i}{2}(v_x + iv_y) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y)$$

Hence the Cauchy-Riemann equations are equivalent to $\frac{\partial}{\partial \overline{z}} f = 0$. □

3 Integration of complex functions

3.1 Paths and contours on the plane