Exercise 1.b)

$$\begin{aligned} |z+w|^2 &= |z^2| + 2zw + |w^2| \\ &\leq |z|^2 + |w|^2 \\ &\leq (|z|+|w|)^2 \\ |z+w| &\leq |z| + |w| \end{aligned}$$

Exercise 4

a) Suppose $z \in \mathbb{C}, z \neq 0$, then $f(z) = \frac{z^4}{|z|^2} = \frac{z^4}{\bar{z}z}$. Hence applying lemma 2.22 we have

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{-z^3}{\bar{z}^2} \neq 0 \text{ for } z \neq 0$$

Now suppose $z = 0 \in \mathbb{C}$, let $h \in \mathbb{C}$, then

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{\bar{h}}}{h} = \lim_{h \to 0} \frac{h^2}{\bar{h}} = 0$$

Hence f is complex differentiable at $z = 0$.

b) From our function $f(z) = 2xy + i(x + \frac{2}{3}y^3)$ we have $u(x,y) = 2xy, v(x,y) = x + \frac{2}{3}y^3$ which are both trivially continious as they are polynomials. So we have

$$u_x = 2y, u_y = 2x, v_x = 1, v_y = 2y^2$$

Hence for

$$\begin{aligned} u_x &= 2y = v_y = 2y^2 \\ u_y &= 2x = -v_x = -1 \end{aligned}$$

The only solutions are $z = 0, z = -\frac{1}{2}$.

Exercise 7,a)

We prove directly from the definition of the derivative. We already know

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

and so we show

$$\begin{aligned}
\lim_{h \to 0} \frac{g(z+h) - g(z)}{h} &= \lim_{h \to 0} \frac{\overline{f(\overline{z+h})} - \overline{f(\overline{z})}}{h} \\
&= \lim_{h \to 0} \frac{\overline{f(\overline{z} + \overline{h}) - f(\overline{z})}}{h} \\
&= \lim_{h \to 0} \overline{\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}}} \\
&= \overline{f'(\overline{z})} < \infty
\end{aligned}$$

Hence g is complex differentiable at z.

Exercise 9

Exercise 10

Let $f(z) = z^3$, $z = x + iy, x, y \in \mathbb{R}$. Then

$$f(z) = (x+iy)^3 = x^3 + 3x^2 iy - 3xy^2 - iy^3$$

and so we have that

$$Re(f) = x^3 - 3xy^2$$