Measure Theory and Integration

jdRanda

October 14, 2020

1 The Riemann Integral

1.1 Definition

Let $f : [a, b] \to \mathbb{R}$ s.t. $a < b$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ with $a = x_0 < x_1 < \cdots < x_n = b$. Set

$$L(P, f) = \sum_{i=1}^{n} \inf\{f(x) : x_{i-1} \leq x < x_i\}(x_i - x_{i-1})$$

$$U(P, f) = \sum_{i=1}^{n} \sup\{f(x) : x_{i-1} \leq x < x_i\}(x_i - x_{i-1})$$

as the lower and upper integral sums respectivly.
Then f is integrable iff

$$\sup_P L(P, f) = \inf_P U(P, f)$$

where the supremum and infinum are taken over all possible partitions.
Hence the Rieman Integral is

$$\int_a^b f(x)dx = \sup_P L(P, f) = \inf_P U(P, f)$$

1.2 Lemma

Let $a, b \in \mathbb{R}$ s.t. $a < b$, $f : [a, b] \to \mathbb{R}$ a function. Then

$$\inf_P U(P, f) \geq \sup_P L(P, f)$$

Proof. Let P_1, P_2 be partitions, define $\mathbb{Q} = P_1 \cap P_2$. Since Q is a 'finer' partition than both P_1 and P_2, it is clear

$$L(P_1, f) \leq L(Q, f) \text{and } U(P_2, f) \geq U(Q, f)$$

Therefore

$$L(P_1, f) \leq L(Q, f) \leq U(Q, f) \leq U(P_2, f)$$

Now as P_1, P_2 are arbitrary, the equality is true for all P, so we can take the supremum of the LHS over all partitions, and the infinum of the RHS over all partitions, leaving us with...

$$\inf_P U(P, f) \geq \sup_P L(P, f)$$

□

1.3 Theorem

A function $f : [a, b] \to \mathbb{R}$ is *Riemann Integrable* $\iff \forall \epsilon > 0, \exists P_*$ s.t.

$$U(P_*, f) - L(P_*, f) < \epsilon$$

Where P_* is any partition on $[a.b]$

Proof. Assume $\forall \epsilon > 0, \exists P_*$ s.t. $U(P_*, f) - L(P_*, f) < \epsilon$, wts f is Riemann Integrable. It is clear

$$\inf_P U(P, f) \leq U(P_*, f) \text{ and } \sup_P L(P, f) \geq L(P_*, f)$$

Now by subtracting and using lemma 1.2 we obtain

$$0 \leq \inf_P U(P, f) - \sup_P L(P, f) \leq U(P_*, f) - L(P_*, f) < \epsilon$$

Since ϵ arbitrary take $\epsilon \to \infty$ and thus

$$\sup_P L(P, f) = \inf_P U(P, f)$$

which is the definition of Riemann Integrability

Now Assume f is Riemann Integrable,
Let $\epsilon > 0$, then $\exists P_1, P_2$ s.t

$$U(P_1, f) < \inf_P U(P, f) + \frac{\epsilon}{2} \text{ and } L(P_2, f) > \sup_P L(P, f) + \frac{\epsilon}{2}$$

Define $P_* = P_1 \cup P_2$ and obtain

$$\begin{aligned} U(P_*, f) - L(P_*, f) &\leq U(P_1, f) - L(P_2, f) \\ &< \inf_P U(P, f) - \sup_P L(P, f) + \epsilon \\ &= \epsilon \end{aligned}$$

□

1.4 Theorem

$f : [a, b] \to \mathbb{R}$ is Riemann Integrable iff the set of discontinuities of f has *Lebasque Measure Zero*.

1.5 Definition

The Lebasque Measure of an open interval $I = (a, b)$ is $\mu(I) = b - a$.
A set $N \subset \mathbb{R}$ has Lebasque Measure Zero if $\forall \epsilon > 0$ there exists a countable collection of open intervals $\{I_1, I_2, \ldots\}$ s.t. $N \subset \cup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} \mu(I_i) < \epsilon$

1.5.1 Examples

Some examples of sets with a Lebasque Measure of Zero:

- A single point $\{a\}$
- Any countable set of points $E = \{a_1, \ldots\}$
- Any countable union of sets of measure 0
- Any subset of a set of measure 0

2 Measurable sets and Integrals

2.1 Definition

Let $X \neq \emptyset$ be a set. A family $\mathbb{X}$ of subsets X is a σ-algebra if

i) $\emptyset \in \mathbb{X}$, $X \in \mathbb{X}$

ii) $A \in \mathbb{X} \to A^c \in \mathbb{X}$

iii) $A_1, A_2, \cdots \in \mathbb{X} \to \cup_{n=1}^{\infty} A_n \in \mathbb{X}$

Now $(X, \mathbb{X})$ is called a *Measurable Space*, and each $S \in \mathbb{X}$ are called *Measurable Sets*.

2.2 Definition

Let $X \neq \emptyset$ and $\mathbb{A}$ be a non-empty collections of subsets of X. Let $\mathbb{Y}$ be the collection of all σ-algebras containing $\mathbb{A}$. Then $\beta(\mathbb{A}) = \cap_{\mathbb{X} \in \mathbb{Y}} \mathbb{X}$ is the σ-algebra generated by $\mathbb{A}$. This is the smallest σ-algebra generated by $\mathbb{A}$.

2.3 Definition

Let $X = \mathbb{R}$, $\mathbb{A} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a < b\}$. The σ-algebras generated by $\mathbb{A}$ is *Borel Algebra*, denoted $\mathbb{B}$. This is the smallest σ-algebras contianing all open sets. A set $B \in \mathbb{B}$ is a *Borel Set*. *NB:* $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

2.4 Definition

Let $(X, \mathbb{X})$ be a measurable space. Then $f : X \to \mathbb{R}$ is a $\mathbb{X}$-measurable function if for any Borel set $A \in \mathbb{B}$ we have $f^{-1}(A) \in \mathbb{X}$.

2.5 Definition

Let $(X, \mathbb{X})$, $(Y, \mathbb{Y})$ be measurable spaces. Then $f : X \to Y$ is an $\mathbb{X}$-measurable function if for any set $A \in \mathbb{Y}$ we have $f^{-1}(A) \in \mathbb{X}$.

2.6 Lemma

A function $f : X \to \mathbb{R}$ is measurable $\iff \forall \alpha \in \mathbb{R}$ the set

$$\{x \in X : f(x) < \alpha\} \equiv f^{-1}((-\infty, \alpha))$$

is measurable (i.e belongs to $\mathbb{X}$).
Proof
Assuming the set is measurable, as $(-\infty, \alpha) \in \mathbb{B}$, then by definition $f^{-1}((-\infty, \alpha)) \in \mathbb{X}$

Now assume $f^{-1}((-\infty, \alpha)) \in \mathbb{X} \forall \alpha \in \mathbb{R}$, wts for any Borel set $A \in \mathbb{B}$ we have $f^{-1}(A) \in \mathbb{X}$
Define $\mathbb{A}$ to be a collection of sets of the form $(-\infty, \alpha)$ with $\alpha \in \mathbb{R}$. Now we can see that sets in $\mathbb{A}$ generate Borel σ-algebra $\mathbb{B}$.
Let $\mathbb{Y}$ be the smallests σ-algebra contianing $\mathbb{A}$. Obviously $\mathbb{A} \subset \mathbb{B}$ and therefore $\mathbb{Y} = \beta(\mathbb{A}) \subset \mathbb{B}$.
Now wts $\mathbb{Y}$ contians intervals (a, b) for $a < b \in \mathbb{R}$. Take the sets $(-\infty, a), (-\infty, b) \in \mathbb{Y}$. Then we have

$$[a, b) = (-\infty, b) \cap (-\infty, a)^c \in \mathbb{Y} \forall a < b \in \mathbb{R}$$

$$(a, b) = \cup_{n=N}^{\infty} [a - \frac{1}{n}, b] \text{ for large enough N}$$

and hence $(a, b) \in \mathbb{Y}$.
Since $\mathbb{B}$ is the smallest σ-algebra contianing all open intervals (a, b) we have $\mathbb{Y} = \mathbb{B}$.
Now we define the smallest σ-algebra contianing sets $f^{-1}(\mathbb{A})$, i.e. $\beta(f^{-1}(\mathbb{A}))$. Since $f^{-1}(\mathbb{A}) \subset \mathbb{X}$ we also have $\beta(f^{-1}(\mathbb{A})) \subset \mathbb{X}$.
But $\beta(f^{-1}(\mathbb{A})) = f^{-1}(\beta(\mathbb{A})) = f^{-1}(\mathbb{B})$ and so $f^{-1}(\mathbb{B}) \subset \mathbb{X}$ $\square$

2.7 Lemma

Let $(X, \mathbb{X})$ be a measurable space, let $f : X \to \mathbb{R}$. Then the following are equivalent:

i) $\forall \alpha \in \mathbb{R}, A_\alpha = \{x \in X : f(x) > \alpha\} \in \mathbb{X}$

ii) $\forall \alpha \in \mathbb{R}, B_\alpha = \{x \in X : f(x) \leq \alpha\} \in \mathbb{X}$

iii) $\forall \alpha \in \mathbb{R}, C_\alpha = \{x \in X : f(x) \geq \alpha\} \in \mathbb{X}$

iv) $\forall \alpha \in \mathbb{R}, D_\alpha = \{x \in X : f(x) < \alpha\} \in \mathbb{X}$

2.8 Lemma

Let $(X, \mathbb{X})$ be a measurable space, $f, g : X \to \mathbb{R}$ be measurable. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then $h : X \to \mathbb{R}; h(x) = F(f(x), g(x))$ is measurable.

Proof. Fix $\alpha \in \mathbb{R}$ and we want to show:

$$X_\alpha := \{x : F(f(x), g(x)) < \alpha\} = \{x : (f(x), g(x)) \in F^{-1}((-\infty, \alpha))\}$$

is measurable.
Let $A = F^{-1}((-\infty, \alpha))$ and note it is an open set in $\mathbb{R}^2$ as F is continuous, thus it is a countable union of open rectangles, so it is sufficient to show for any set $(a, b) \times (c, d)$ we have that

$$\{x : (f(x), g(x)) \in (a, b) \times (c, d)\} = \{x : a < f(x) < b\} \cap \{x : c < g(x) < d\}$$

is measurable. We can now use the fact f, g are measurable. $\square$

2.9 Corollary

Let f, g be measurable and $c \in \mathbb{R}$. Then

$$f + g, fg, |f|, \frac{f}{g} \quad g \neq 0, \max\{f, g\}, \min\{f, g\}, cf$$

are all measurable by above lemma.

2.10 Definition

Let $(X, \mathbb{X})$ measurable space. Then $f : X \to \overline{\mathbb{R}}$ is $\mathbb{X}$-measurable if for any $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) > \alpha\} \in \mathbb{X}$.
The collection of all $\overline{\mathbb{R}}$-valued, $\mathbb{X}$-measurable functions on X is denoted $M(X, \mathbb{X})$.

2.11 Lemma

Let $(X, \mathbb{X})$ be a measurable space. $f : X \to \overline{\mathbb{R}}$ is measurable iff:

i) $A = \{x \in X : f(x) = +\infty\}$
   $B = \{x \in X : f(x) = -\infty\}$

ii) the function $f_1(x) = \begin{cases} f(x), & x \in (A \cup B)^c \\ 0, & x \in A \cup B \end{cases}$ is measurable.

Proof. Suppose f_1, A, B are measurable. For $\alpha \geq 0$ we have

$$\{x : f(x) > \alpha\} = \{x : f_1(x) > \alpha\} \cup A$$

And thus is measurable, analogous for $\alpha < 0$.
Now suppose f is measurable. $\square$

Definition 2.14

A *simple function* is a finite linear combination of characteristic functions of measurable sets, i.e. $f : X \to \mathbb{R}$ is simple if $f(x) = \sum_{i=1}^{n} a_i \chi(A_i)$, were $a_i \in \mathbb{R}, A_i \in \mathbb{X}$.

Lemma 2.15

Let $f \in M(X, \mathbb{X}), f \geq 0$. Then there exists a sequence (ϕ_n) in $M(X, \mathbb{X})$ such that

i) $0 \leq \phi_n(x) \leq \phi_{n+1}(x) \quad \forall x \in X, n \in \mathbb{N}$

ii) $\lim_{n\to\infty} \phi_n(x) = f(x)$

iii) Each ϕ_n is a smiple function.

Proof. Fix $n \in \mathbb{N}$ and for $k \in \{0, \ldots, n2^n - 1\}$ define

$$E_{k,n} = \{x : f(x) \in [\frac{k}{2^n}, \frac{k+1}{2^n}] \text{ and } E_{n2^n,n} = \{x : f(x) \geq n\}$$

We then define

$$\phi_n(x) = \frac{k}{2^n} \text{ if } x \in E_{k,n}$$

Giving us

$$\phi_n = \sum_{k=0}^{2^n n} \frac{k}{2^n} \chi E_{k,n}$$

which is a simple funciton.

Here we are splitting the interval $[0, n)$ into $n2^n$ parts and approximating f by its lowest value within the interval. If $f(x) \geq n$ we approximate $f(x)$ by n. Since f is measurable we know sets $E_{k,n}$ and $E_{n2^2,n} = \{x : f(x) \geq n\}$ are measurable, so ϕ_n is indeed simple.

Now we check pointwise convergence of ϕ_n to f. Let $x \in X$ and then either $f(x) = \infty, f(x) < \infty$.
Assume $f(x) = \infty$ then it is clear $\phi_n(x) = n$ and as $n \to \infty, \phi_n(x) \to f(x)$.
Assume $f(x) < \infty$ then $f(x) < N, N \in \mathbb{N}$ and $x \in E_{k,n}$ for some $k \in \mathbb{N}, n \geq N$. However in every set $E_{k,n}$ we have $|\phi_n(x) - f(x)| < \frac{1}{2^n}$. Taking $n \to \infty$ we have $\phi_n(x) \to f(x)$. Therefore pointwise convergence.

Now we check $\phi_n \leq \phi_{n+1}$.
Assume $x \in E_{(n+1)2^{n+1},n+1}$ then $\phi_{n+1}(x) = n + 1 > \phi_n(x)$.
Assume $x \in E_{k,n+1}$ for some $k \in \{0, \ldots, (n+1)2^{n+1} - 1$. Then $\phi_{n+1}(x) = \frac{k}{2^{n+1}}$. But we also have $x \in E_{[k/2],n}$ implying $\phi_n(x) = \frac{[k/2]}{n}$. Hence $\phi_{n+1}(x) \geq \phi_n(x)$. $\square$

2.12 Definition

A function $f : X \to \mathbb{R}$ is called *elementary* if it is measurable and take no more than a countable number of values, i.e.

$$f(x) = \sum_{i=1}^{\infty} a_i \chi(A_i)$$

where $a_i \in \mathbb{R}, A_i = \{x \in X : f(x) = a_i\}$.

2.13 Lemma

A function $f : X \to \mathbb{R}$ is measurable iff it is a limit of a uniformly convergent sequence of elementary functions.

Proof. Let $\{f_n\}$ be a sequence of elementary function and $f_n \to f$ uniformly on X. Uniform convergence implies pointwise convergence so we are done using lemma 2.15.

Now let f be a measurable function, define a sequence $\{f_n\}$ s.t.

$$f_n(x) = \frac{m}{n} \text{ on } A_n^m = \{x \in X : \frac{m}{n} \leq f(x) < \frac{m+1}{n}\}, m \in \mathbb{Z}, n \in \mathbb{N}$$

Clearly $f_n(x)$ is an elementary function and $|f_n(x) - f(x)| \leq \frac{1}{n}$ on X. $\square$