Set Theory

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October 26, 2020

1 Introduction

Axiom of Extensionality (for sets)

For two sets a, b, we say $a = b$ iff:

$$\forall x, [x \in a \Rightarrow x \in b]$$

Axiom of Extensionality (for classes)

For two classes A, B, we say $A = B$ iff:

$$\forall x, [x \in A \Rightarrow x \in B]$$

Axiom of Pair Set

For any sets x, y there is a set $z = \{x, y\}$ with elements just x and y. z is called the *(unordered) pair set of* x, y. *NB: If* $x = y$ *then we have* $\{x, y\} = \{x, x\} = \{x\}$

Definition 1.1

Let $\mathcal{P}(x)$ denote the class $\{y | y \subseteq x\}$, called the *Power set of* x.

Definition 1.2

The *Empty Set*, denoted $\emptyset$, is the unique set with no elements.

- We can define $\emptyset$ as $\{x | x \neq x\}$.
- For any set/class A, we have $\emptyset \subset A$.

2 Classes

Theorem 1.4

The collection $R = \{x|x \notin x\}$ does not define a set.

Proof. Suppose R was a set z. Assume $z \in R$, then by the definition of R, $z \notin z$. However if $z \notin R$ then we should have $z \in z$. This is a contradiction. □

NB: This is an example of a class that is not a set. Any class which is not, or cannot be, a set is called a Proper Class.

Axiom of Subsets

Let $\Phi(x)$ be a definite, welldefined property. Let x be a set. Then

$$\{y \in x|\Phi(y)\} \quad \text{is a set.}$$

Corollary 1.5

Let V denote the class of all sets. Then V is a proper class.

Proof. If V were a set then we should have, $R = \{y \in V|y \notin y\}$ is a set by the Axiom of Subsets, however we have just shown R is not a set. □

Definition 1.6

For any set Z there is a class, $\cup Z$, which consists of the members of members of Z.

$$\cup Z = \{x|\exists t(x \in t \in Z)\}$$

Axiom of Unions

For any set Z, $\cup Z$ is a set.

Definition 1.8

For any non-empty set Z, there is another set, $\cap Z$, which consists of the members of all members of Z.

$$\cap Z = \{x|\forall t \in Z(x \in t)\}$$

or using index sets we write

$$x \in \cap_{j \in I} A_j \iff (\forall j \in I)(x \in A_j)$$

3 Relations and Functions

For two sets X, Y, there are relations R that hold between some elements of X and of Y, denoted xRy. The types of relation are:

- Reflexive: $x \in X \Rightarrow xRx$
- Irreflexive: $x \in X \Rightarrow \neg(xRx)$
- Symmetric: $(x, y \in X \wedge xRy) \Rightarrow yRx$
- Antisymmetric: $(x, y \in X \wedge xRy \wedge yRx) \Rightarrow x = y$
- Connected: $(x, y \in X) \Rightarrow (x = y \vee xRy \vee yRx)$
- Transiative: $(x, y, z \in X \wedge xRy \wedge yRz) \Rightarrow xRz$

NB: recall that an equivalence relation *is that R should satify symmetry, reflixivity and transiativity.*

Definition 1.10

A relation $\prec$ on a set X is a (strict) partial ordering if it is irreflexitve and transitive. I.e.

i) $x \in X \Rightarrow \neg(x \prec x)$.

ii) $(x, y, z \in X \wedge x \prec y \wedge y \prec z) \Rightarrow (x \prec z)$

Definition 1.11

i) If $\prec$ is a partial ordering of a set X, and $\emptyset \neq Y \subseteq X$, then $z \in X$ is a lower bound for Y in X if:
$$\forall Y(y \in Y \Rightarrow z \preceq y)$$

ii) $z \in X$ is an infimum or greatest lower bound (glb) for Y if it is a lower bound for Y and if z' is a lower bound for Y then $z' \preceq z$.

iii) The concepts of upper bound and supremum (least upper bound (lub)) are defined analogously.

Definition 1.12

i) We say $f : (X, \prec_1) \to (Y, \prec_2)$ is an order preserving map of the partial orders $(X, \prec_1), (Y, \prec_2)$ iff:

$$\forall x, z \in X(x, \prec_1 z \Rightarrow f(x), \prec_2 f(z))$$

ii) Orderings $(X, \prec_1), (Y, \prec_2)$ are (order) isomorphic, written $(X, \prec_1) \cong (Y, \prec_2)$, if there is an order preserving map between them which is also a bijection.

iii) There are completely analogous definitions between nonstrics orders $\preceq_1, \preceq_2$.

Theorem 1.13 *(Representation Theorem for partially ordered sets)*

If $\prec$ partially orders X, then there is a set Y of subsets of X which is such that $(X, \preceq)$ is order isomorphic to $(Y, \subseteq)$.

Proof. Given any $x \in X$, let $X^x = \{z \in X | z \preceq x\}$. Notice if $x \neq y$ then $X^x \neq X^y$. So the assignemnt of x to X^x is 1:1. Let $Y = \{X^x | x \in X\}$. Then we have

$$x \preceq y \iff X^x \subseteq X^y$$

Coonsequently, setting $f(x) = X^x$ we have an order isomorphism. □

NB: Often we deal with roderings where every element is comparable with every other, known as strong connectivity *and we call the rdering* total.

Definition 1.14

A relation $\prec$ on X is a string total ordering if it is a partial ordering which is connected:

$$\forall x, y(x, y \in X \Rightarrow (x = y \vee x \prec y \vee y \prec x))$$

NB: For $\preceq$ we call the ordering non-strict.

Definition 1.15

i) $(A, \prec)$ is a wellordering if it is a string total orderings and for any subset $Y \subseteq A$, $Y \neq \emptyset \Rightarrow Y$ has a $\prec$-least element. We write $(A, \prec) \in WO$

ii) A partial ordering R on a set A, (A, R) is a wellfounded relation if for any subset $Y \subseteq A$, $Y \neq \emptyset \Rightarrow Y$ has an R-minimal element.

Lemma 1.16

A strict total ordering $(A, \prec)$ is a wellordering iff any non-empty *end segment* $C \subseteq A$, has a $\prec$-least element. *We say $C \subseteq A$ is an end segment of the strict total order $(A, \prec)$, if whenever $a \in C$ and $a \prec b$, then $b \in C$.*

Proof. □

Definition 1.17 (Kuratowski)

Let x, y sets. The ordered pair set of x and y is the set

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$

Lemma 1.18 (Uniqueness theorem for ordered pairs)

$$\langle x, y\rangle = \langle u, v\rangle \iff x = u \wedge y = v$$

Proof. ($\Leftarrow$) is trivial. So Suppose $\langle x, y\rangle = \langle u, v\rangle$.

Case 1 $x = y$. Then $\langle x, y\rangle = \langle x, x\rangle = \{\{x\}, \{x, x\}\} = \{x\{x\}, \{x\}\} = \{\{x\}\}$. If this equals $\langle x, v\rangle$ then we must have $u = v$. So $\langle u, v\rangle = \{\{u\}\} = \{\{x\}\}$. Hence by Extensionality $\{u\} = \{x\}$, and so again by Extensionality $u = x = y = v$.

Case 2 $x \neq y$. Then $\langle x, y\rangle, \langle u, v\rangle$ have the same two elements, $(u \neq v)$. Hence one of these elements has one member and the other two, so we cannot have $\{x\} = \{u, v\}$. So $\{x\} = \{u\}$ and $x = u$. But that means $\{x, y\} = \{u, y\} = \{u, v\}$. So of these last two sets, if they are the same then $y = v$.

□

Definition 1.20

We define ordered k-tuple by induction: $\langle x_1, x_2\rangle$ has been defined; if $\langle x_1, x_2 . \ldots, x_k\rangle$ has been defined then $\langle x_1, \ldots, x_k, x_{k+1}\rangle = \langle\langle x_1, \ldots, x_k\rangle, x_{k+1}\rangle$.

Definition 1.21

i) Let A, B be sets. $A \times B = \{\langle x, y\rangle | x \in A \wedge y \in B\}$. If $A = B$ this is written as A^2.

ii) If $A_1, \ldots, A_{k+1}$ sets we define $A_1 \times \cdots \times A_{k+1} = (A_1 \times \cdots \times A_k) \times A_{k+1} = \{\langle \ldots \langle\langle\langle x_1, x_2\rangle, x_3\rangle, \ldots, x_k\rangle, x_{k+1}\rangle | \forall i(1 \leq i \leq k+1 \Rightarrow x_i \in A_i).\}$

iii) In general $A \times B \neq B \times A$, and further $\times$ operation is not associative.

Definition 1.22

i) A (binary) relation R is a class of ordered pairs. R is thus any subset of some $A \times B$.

ii) We write $R^{-1} = \{\langle y, x\rangle | \langle x, y\rangle \in R$.

Definition 1.24

If R is a relation, then

$$dom(R) = \{x | \exists y \langle x, y\rangle \in R\}, ran(R) = \{y | \exists y \langle x, y\rangle \in R\}$$

The field of a relation R, $Field(R) = dom(R) \cup ran(R)$.

Definition 1.25

i) A relation F is a function ("$Func(F)$") if $\forall x \in dom(F)$ there is a unique y s.t $\langle x, y\rangle \in F$.

ii) If F is a function then F is $(1-1)$ iff $\forall x, x'(\langle x, y\rangle \in F \wedge \langle x', y\rangle \in F \Rightarrow x = x')$

Definition 1.27

If X, Y sets, then ${}^X Y = \{F | F : X \to Y$.

Definition 1.28 (Indexed Cartesia Product)

Let I be a set, and for each $i \in I$ let $A_i = \emptyset$ be a set; then

$$\prod_{i \in I} A_i = \{f | Func(f), dom(f) = I \wedge \forall i \in I(f(i) \in A_i)\}$$

This allows us to take cartesian product indexed by any set, not just some finite n. *NB: Our function f can be seen as a 'choice' function that choose some $f(i) \in A_i$ for each i.*

Definition 1.30

A set x is *transiative*, $Trans(x)$, iff $\forall y \in x(y \subseteq x)$. *NB: We also equivalently abbreviate* $Trans(x) = \cup x \subseteq x$

Definition 1.32 (The successor function)

Let x a set. Then $S(x) = x \cup \{x\}$.

Examples 1.33

If x is transitive then so too is $S(x)$.

Proof. Assume x is transiative. Let $y \in S(x)$. If $y \in x$ then $y \subseteq x$ as x is transitive. Then $y \subseteq S(x)$. Else $y = x$, trivially $y \subseteq s(x)$, Hence transitive. $\square$

For X a class of transitive sets. Then $\cup X$ is transitive.

Proof. Let $y \in X$. Want to show $y \subseteq \cup X$. So let $z \in y$, as $y \in \cup X$ theres some $t \in X, Trans(t)$, with $y \in t$. Then $z \in y \subseteq t$. So $z \in t \in X$. So $zin \cup X$. Hence $y \subseteq \cup X$. $\square$

Lemma 1.33

$Trans(x) \iff \cup S(x) = x$

Proof. First note that $\cup S(x) = \cup(x \cup \{x\}) = (\cup x) \cup (\cup\{x\}) = (\cup x) \cup x$. For $(\rightarrow)$, assume $Trans(x)$; then $\cup x \subseteq x$. Hence by the above $\cup S(x) \subseteq x$. Hence $\cup S(x) = x$. For $(\leftarrow)$, assume $\cup S(x) = x$. We have from above $\cup x \subseteq (\cup x) \cup x = x$ by assumption. Hence transitive. $\square$

Definition 1.34 (Transiative Closure)

We define by recursion on n:

$$\cup^0 x = x; \cup^{n+1} x = \cup(\cup^n x); TC(x) = \cup\{\cup^n x | n \in \mathbb{N}\}$$

Lemma 1.35

For any set x

i) $x \subseteq TC(x)$, $Trans(TC(x))$.

ii) $Trans(t) \wedge x \subseteq t$ then $TC(x) \subseteq t$. Hence $TC(x)$ is the smallest transitive set containing x.

iii) $Trans(x) \iff TC(x) = x$.

Proof. i) trivial.

ii) $x \subseteq t$ then $\cup^0 \subseteq t$. By induction on k, assume $\cup^k \subseteq t$. Now use the fact $A \subseteq B \wedge Trans(B) \Rightarrow \cup A \subseteq B$ to deduce $\cup^{k+1} \subseteq t$. So it follows $TC(x) \subset t$. But t was arbitrary.

iii) $x \subseteq TC(x)$, if $Trans(x)$ then substitude x for t in the above, concluding $TC(x) \subseteq x$.

$\square$

4 Number Systems

Definition 2.1

A set Y is called inductive if $\emptyset \in Y$ and $\forall x \in Y(S(x) \in Y)$. *Axiom of Infinity: There exists an inductive set,* $\exists Y(\emptyset \in Y \wedge \forall x \in Y(S(x) \in Y))$.

Definition 2.2

i) x is a natural number if $\forall Y[Y$ is an inductive set $\to x \in Y]$.

ii) ω is the class of natural numbers.

NB: $\omega = \cap\{Y | Y \text{ an inductive set}\}$

Proposition 2.3

ω is a set.

Proof. Let z be any inductive set. By the Axiom of subsets: there is a set N so that:

$$N = \{x \in z | \forall Y[Y \text{ an inductive set} \to x \in Y]$$

$\square$

Proposition 2.4

i) ω is an inductive set.

ii) It is the smallest inductive set.

Proof. We have proven ω is a set. To show inductivity, not by definition $\emptyset$ is in any inductive set Y so $\emptyset \in \omega$. Moreover, if $x \in \omega$, then for any inductive set Y, we have both $x, S(x) \in Y$. Hence $S(x) \in \omega$. So ω closed under the S function. (ii) then follows. $\square$

Theorem 2.5 (Principle of Mathematical Induction)

Suppose Φ is a welldefined definite propery of sets. Then

$$[\Phi(0) \wedge \forall x \in \omega(\Phi(x) \rightarrow \Phi(S(x))) \rightarrow \forall x \in \omega \quad \Phi(x)]$$

Proof. Assume the antecedent here, then it suffices to show that the set of $x \in \omega$ for which $\Phi(x)$ holds is inductive. Let $Y = \{x \in \omega | \Phi(x)\}$. However the antecedent then says $0 \in Y$; and moreover if $x \in Y$ then $S(x) \in Y$. That Y is inductive is then simply the antededent assumption. Hence $\omega \subseteq Y$. And so $\omega = Y$. $\square$

Proposition 2.6

Every natural number y is either 0 or is $S(x)$ for some natural number x.

Proof. Let $Z = \{y \in \omega | y = 0 \vee \exists x \in \omega(S(x) = y)\}$. Then $0 \in Z$ and if $u \in Z$, then $u \in \omega$. Hence $S(u) \in \omega$ as ω inductive. Hence $S(u) \in Z$, so Z is inductive and thus ω. $\square$

Exercise 2.1

Every natural number is transitive

Proof. Wts $Z = \{x \in \omega | Trans(x)\}$ is inductive. $\square$

Lemma 2.7

ω is transiative.

Proof. Let $X = \{n \in \omega | n \subseteq \omega\}$. If X were inductive, then $X \subseteq \omega \subseteq X$, and then $Trans(\omega)$. Trivially $\emptyset \in X$. Assume $n \in X$, then $n \subseteq \omega$ and $\{n\} \subseteq \omega$. Hence $S(n) \in X$. So X is inductive. $\square$

Definition 2.10

For $m, n \in \omega$ set $m < n \iff m \in n$. Set $m \leq n \iff m = n \vee m < n$

Lemma 2.11

i) $<, \leq$ are transitive.

ii) $\forall n \in \omega \forall m(m < n \iff S(m) < S(n))$.

iii) $\forall m \in \omega(m \not< m)$.

Proof. i) That $<$ is transitive comes from the fact our natural numbers are proven to be transitive sets: $n \in m \in k \Rightarrow n \in k$. The same follows for $\leq$.

ii) ($\leftarrow$) Assume $S(m) < S(n)$. $m \in S(m) = m \cup \{m\} \in S(n) = n \cup \{n\}$.
If $S(m) = n$: So $m \in n$. So $m < n$.
If $S(m) \in n$: Then $Trans(n)$, we have $m \in S(m) \subseteq n$. So again $m \in n$, and $m < n$.
($\rightarrow$) We wts by PMI. Let $\Phi(k)$ say: "$\forall m(m < k \Rightarrow S(m) < S(k))$". Then $\Phi(0)$ vacuously holds; and so we suppose $\Phi(k)$ and prove $\Phi(k+1)$.
Then let $m < S(k)$. Then $m \in k \cup \{k\}$. If $m \in k$, then by $\Phi(k)$ we have $S(m) < S(k) < S(S(k))$. If $m = k$ then $S(m) = S(k) < S(S(k))$. So we have $\Phi(S(k))$. So by PMI, $\forall k, \Phi(k)$.

iii) Note $0 \not< 0$ since $0 \notin 0$. If $k \notin k$ then $S(k) \notin S(k)$ by (ii). So $X = \{k \in \omega | k \notin k\}$ is inductive, i.e. all of ω.

□

Lemma 2.12

$<$ is a strict total ordering.

Proof. All we have to left to prove is connectivity. $\forall m, n \in \omega(m = n \vee m < n \vee n < m)$. Note that at most one may hold for m, n. Let $X = \{n/in/omega | \forall m \in \omega(m = n \vee m < n \vee n < m)\}$. If X is inductive the proof is compelte. Exercise to show. □

Theorem 2.13 *Wellordering Theorem for* ω

Let $X \subseteq \omega$. Then either $X = \emptyset$ or there is $n_0 \in X$ so that for any $m \in X$ either $n_0 = m \vee n_0 < (\in) m$.
Least Number Principle: any non-empty set of natural numbers has a least element.

Proof. Suppose $X \subseteq \omega$ but X has no least elements as above. Let

$$Z = \{k \in \omega | \forall n \in k(n \notin X)\}$$

We claim Z inductive, hence all of ω and so $X = \emptyset$. Vacuously $0 \in Z$. Suppose now $k \in Z$. Let $n < S(k)$. Hence $n \in k \cup \{k\}$. If $n \in k$ then $n \notin X$ (as $n < k \wedge k \in Z$). But if $n = k$, and so $n \notin X$, as otherwise it would be hte least element of X and X has no such element. So $S(k) \in Z$. Hence Z is inductive. □

Theorem 2.14 *Recursion Theorem on* ω

Let A be any set, $a \in A$, $f : A \to A$, any function. Then there exists a unique function $h\omega \to A$ so that

- $h(0) = a$.
- For any $n \in \omega$: $h(S(n)) = f(h(n))$.

Proof. □

Example 2.15 *Addition*

Let $n \in \omega$. We can define an "add n" function $A_n(x)$ as follows:

- $A_n(0) = n$.
- $A_n(S(k)) = S(A_n(k))$.

From now on we refer to $S(n)$ as $n + 1$. And we write $A_n(k)$ as $n + k$

Example 2.16 *Multiplication*

i) $M_n(x)$: $M_n(0) = 0; M_n(k+1) = M_n(k) + n$.

ii) $E_n(x)$: $E_n(0) = 1; E_n(k+1) = E_n(k) \cdot n$