PROBLEM SET # 2

Astro 512 – Spring 2017
Extragalactic Astronomy

Problem 1: The Vertical Structure of Disks

The notes for Lecture 2 show the expanded derivation of the vertical structure for an isothermal (i.e., uniform velocity dispersion), self-gravitating (i.e., no other sources of mass) slab of stars. In that derivation, it was implicitly assumed the slab is infinite (i.e., that we can ignore the radial variation in the surface density); this assumption is valid using the definition of $\nabla \cdot \vec{g}$ in cylindrical coordinates, because the term involving a gradient in the circular velocity is small for a flattish rotation curve. The resulting density as a function of distance z from the midplane was found to be $\rho(z) = \rho_0 \operatorname{sech}^2(\frac{z}{2z_0})$. where z_0 is a characteristic scale height.

In the following problem, after reviewing the derivation in the notes, you will solve for how the characteristic scale height z_0 depends on the vertical velocity dispersion σ_z and the surface density Σ of the disk.

a) First, calculate the surface density Σ in terms of ρ_0 and z_0. Check your units and scalings to make sure your answer is sensible.

b) Solve for the value of z_0 in terms of the vertical velocity dispersion σ_z and the surface density Σ of the disk. (No need to substitute in actual numbers for various physical constants). In other words, you want to find the value of z_0 such that the differential equation for the equilibrium distribution of $\rho(z)$ holds.

c) Use the definition of sech^2 to (1) show that the distribution of stars becomes exponential with scale length h_z far above the midplane; and (2) calculate the relationship between h_z and z_0.

d) Given reasonable values for σ_z and z_0 for comparable old stars in the Milky Way's thin disk (see, for example, Mackereth et al 2019 and Bovy 2017), what is your predicted value for the surface density of the Milky Way's disk near the solar circle? Does this agree with currently accepted values that you find in the literature? If they don't agree, what are plausible reasons why (i.e., question the assumptions you've made)

e) In the outskirts of spiral disks, the mass density within the stellar disk is probably dominated by the dark matter halo, not the stars themselves. Show that the resulting vertical profile of the stars is not sech^2, but is instead gaussian with a scale height $z_0^2 = \sigma_z^2/4\pi G\rho_{halo}$, where ρ_{halo} is the local density of the dark matter halo. You may assume that the density of the halo varies little with height within the disk, and thus that ρ_{halo} is approximately constant with z. You may assume that $\rho_{stars}(z)$ is everywhere negligible compared with ρ_{halo}. This is the complete opposite of the case for a self-gravitating disk, since here, we are concerned with the distribution of stars when the gravitational forces come from an entirely different population. In such situations, we refer to the stars as "tracer particles" – i.e, something that traces the underlying potential, but does not actively participate in setting that potential.

Problem 2: Scaling Laws for Galaxy Structure

This problem introduces some of the basic scaling relations among galaxies, and their variation with redshift. You will use these relations later, and they're good basic numbers and relations to have at your disposal to understand scales of galaxies. They are also frequently invoked in the literature.

We haven't yet talked about the *internal* structure of dark matter halos, but in class we've introduced the idea that (1) dark matter halos exist; and (2) that within some radius, the dark matter halos are essentially virialized. These will be covered in more detail later in the quarter and in Cosmology, but for now, we will assume that regions of the larger dark matter distribution that have collapsed and virialized are "overdense" compared to the background average dark matter density. A careful treatment shows that when a region is more than about 200 times the critical density (i.e. $\rho_c(z)$, which divides open and closed universes in a zero-Λ cosmology), it is probably virialized.

As such, a common way to define the size of a galaxy is the radius within which the mean interior density $\langle\rho(< r_{200})\rangle$ is some large multiple of the critical density $\rho_c(z)$. This definition has the nice properties that it is independent of the luminosity of a galaxy and it easy to calculate for numerical simulations. A standard choice for "size" of the dark matter halo is r_{200}, the radius where $\rho_{200} = \langle\rho(< r_{200})\rangle = 200\rho_c$, which is sometimes called the "virial radius". With this notation, V_{200} is the circular velocity at r_{200} and M_{200} is the mass contained within r_{200} (i.e. $M_{200} = M(< r_{200})$). This mass is always dominated by dark matter, so in what follows the baryons and their effects are ignored.

For this problem, we will assume for simplicity that an individual halo has an "isothermal" density profile of $\rho(r) = V_c^2/4\pi G r^2$, where V_c is the rotational speed of the "flat" rotation curve associated with this density profile. Note that this is not a terribly realistic dark matter profile, since it has infinite mass and a constant power law slope (r^{-2}). However it is mathematically tractable, while also having about the right power law slope for the outer regions of galaxies.

a) Assuming the isothermal density profile, show that $r_{200} = V_c/10H(z)$, where $H(z)$ is the Hubble Constant at redshift z. You will need to review the definition of ρ_c at arbitrary redshift, and you may want to begin by calculating $M(< r)$, the mass interior to the radius r.

b) What is r_{200} for the Milky Way? How does it compare to the optical radius of the galaxy (usually taken to be where the surface brightness falls to $\mu_B = 25\,\mathrm{mag/arcsec^2}$)? Assume the central surface brightness of the Milky Way disk is the Freeman value, and that the outer parts of the galaxy's light are dominated by the disk.

c) Show that $M_{200} = V_c^3/10GH(z)$. Note that this also implies a general scaling $M_{200} \propto r_{200}^3 \propto V_c^3$ among halos of different mass at the same redshift. Although you derived these scalings assuming an isothermal density profile for all galaxies, they hold to a large degree for other choices of profile as well, but with slightly different numerical constants. This relation is also why people often will plot galaxy properties as a function of the galaxies' rotation speeds, rather than mass, given that they are connected to each other.

Problem 3: The Radial Structure of Disks

In this problem you will derive how the structural properties of a disk vary with the surrounding dark matter halo's mass, radius, and spin.

To begin, you will assume the following:

- That the dark matter halo can be described as a perfect isothermal sphere, as you used in the last problem ($\rho(r) = V_c^2/4\pi G r^2$). You can therefore assume that $r_{200} = V_c/10H(z)$ and $M_{200} = V_c^3/10GH(z)$, where $H(z)$ is the Hubble Constant at redshift z. This assumption is obviously an approximation, but yields the correct scaling relationships for much less work than using a more realistic dark matter profile.
- That the mass of the baryons that settle into the disk M_d is a fixed fraction m_d of the halo mass, such that $M_d = m_d M$, where M is the mass of the halo.
- That the angular momentum of the baryons that settle into the disk J_d is a fixed fraction j_d of the halo's angular momentum, such that $J_d = j_d J$, where J is the angular momentum of the halo.
- That the baryonic disk has an exponential surface density profile $\Sigma(r) = \Sigma_0 \exp(-r/h_r)$.
- That the dark matter dominates the entire rotation curve, producing a flat rotation curve no matter how the baryons are distributed. (This is obviously not true, but it certainly simplifies the calculation!)

a) Use the virial theorem to derive that the energy E of the halo is $E = -MV_c^2/2$. You can assume that all the halo particles are on circular orbits, for simplicity.

b) Derive the angular momentum J_d of the disk, in terms of M_d, h_r, and V_c. For integrating over circularly symmetric disks, it is often simplest to think about integrating in rings of width dr, each of which has an area $2\pi r\, dr$.

c) Given the relationship between J and J_d, what is the exponential scale length h_r of the disk in terms of M, V_c, E, j_d, m_d, and λ, where λ is the "dimensionless spin angular momentum" ($\lambda \equiv \frac{J|E|^{1/2}}{GM^{5/2}}$), which is a dimensionless ratio whose square λ^2 is roughly equal to the ratio of centripetal acceleration ($g_\theta \sim V_c^2/r$) to radial acceleration ($g_r \sim GM/r^2$).

d) Use your expression for E from part (a) to show that $h_r = \frac{1}{\sqrt{2}}\left(\frac{j_d}{m_d}\right)\lambda r_{200}$.

e) The calculation in (d) shows that the size of the disk should scale linearly with increasing spin parameter and linearly with increasing virial radius of the dark matter halo the disk sits in. How does h_r scale with the halo mass M?

f) Given your results in (d), how does the central surface density of the baryonic disk scale with λ, V_c, and M?

g) Now it's time to see if your work above produces numbers that are in any way reasonable. Given your results in (d), what is your predicted value of h_r for a MW-type galaxy with $V_c = 220\,\mathrm{km\,s^{-1}}$? You will need to make assumptions about the values of j_d and m_d, which you should justify with a sentence or two. You can assume $\lambda \sim 0.04$, which is where numerical simulations consistently find that the log-normal distribution of λ peaks. Please write your answer using the format
$h_r = 3\,\mathrm{kpc}(\frac{V_c}{220\,\mathrm{km\,s^{-1}}})^n$...etc, so the assumptions are clear.

h) Is answer in (g) is a reasonable match to real galaxy disks?