

MATH 142: Elementary Algebraic Topology

Jad Damaj

Fall 2022

Contents

1	Topology	3
1.1	August 24	3
1.1.1	What is Algebraic Topology	3
1.1.2	Continuity	4
1.2	August 26	4
1.2.1	Continuity	4
1.2.2	Topology	5
1.3	August 29	5
1.3.1	Bases and Subbases	5
1.4	August 31	6
1.4.1	Initial Topologies	6
1.5	September 2	7
1.5.1	Quotient Topologies	7
1.6	September 7	8
1.6.1	Group Actions on Topological Spaces	8
1.6.2	Connectedness	8
1.7	September 9	9
1.7.1	Connectedness	9
1.7.2	Connected Components	10
1.8	September 12	10
1.8.1	Connected Components	10
1.9	September 14	11
1.9.1	Compactness	11
1.10	September 16	12
1.10.1	Compactness	12
1.11	September 19	12
1.11.1	Compactness	12
2	Algebraic Topology	14
2.1	September 19	14
2.1.1	Homotopy	14
2.2	September 21	14
2.2.1	Path Homotopy	14
2.3	September 23	15
2.3.1	The Fundamental Group	15

Chapter 1

Topology

1.1 August 24

1.1.1 What is Algebraic Topology

Recall Metric Spaces: (X, d) , X is a set, d is a metric on X (ie. $d : X \times X \rightarrow \mathbb{R}$)

1. $d(x, y) = 0$ exactly if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Let V be a vector space, let $\|\cdot\|$ be a norm on V , let $d(v, w) = \|v - w\|$

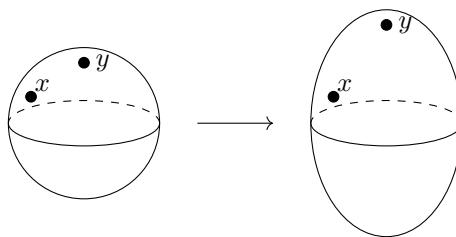
- \mathbb{R}^n : $\|(r_j)\|_2 = (\sum |r_j|^2)^{\frac{1}{2}}$ - Euclidean Norm, $\|(r_j)\|_1 = \sum |r_j|$, $\|(r_j)\|_\infty = \max |r_j|$

If (X, d) is a metric space and if $Y \subseteq X$, let d^Y be the restriction of d to $Y \times Y$. Then (Y, d^Y) is a metric space.

Metric spaces \leftrightarrow geometry: length, area, size of angles.

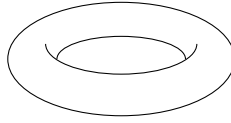
Let X be a balloon on \mathbb{R}^3

- Two natural metrics: inherited metric from \mathbb{R}^3 , path-length metric (eg. length of shortest path on surface between two points)
- Consider a deformation:



the shapes have different Euclidean distances but still have an underlying commonality

- We also observe that the balloon cannot be continuously deformed into the shape below:



We want to be able to prove such things without embedding into a metric space. This is done by attaching algebraic objects to topological spaces such that their isomorphism classes don't change under continuous deformation.

1.1.2 Continuity

Let (X, d^X) and (Y, d^Y) be two metric spaces. Let $f : X \rightarrow Y$ be a function. Let $x_0 \in X$. We say f is continuous at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d^X(x, x_0) < \delta$ then $d^Y(f(x), f(x_0)) < \varepsilon$.

- Let (X, d) be a metric space. By the open ball of radius r about x_0 , we mean $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ (closed ball is $\{x \in X : d(x, x_0) \leq r\}$)
- the above definition can be rephrased as: for any $B(f(x_0), \varepsilon)$ there is an open ball $B(x_0, \delta)$ such that if $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \varepsilon)$.
eg. For every open ball B_1 about $f(x_0)$ there is an open ball B_2 about x_0 such that if $x \in B_2$ then $f(x) \in B_1$

Definition 1.1.1. For (X, d) a metric space, by a neighborhood of a point $x \in X$, we mean any subset of X that contains an open ball about x .

- rephrasing the definition again we get: For any neighborhood $N_{f(x_0)}$ of $f(x_0)$ there is a neighborhood N_{x_0} of x_0 such that if $x \in N_{x_0}$ then $f(x) \in N_{f(x_0)}$

Definition 1.1.2. $f : X \rightarrow Y$ is continuous if it is continuous at each point of X .

1.2 August 26

1.2.1 Continuity

Recall: Given (X, d^X) , (Y, d^Y) and $f : X \rightarrow Y$, f is continuous at x_0 if for any open ball B_1 about $f(x_0)$ there is an open ball B_2 about x_0 such that if $x \in B_2$ then $f(x) \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1)$

Definition 1.2.1. Let (X, d) be a metric space. Let $U \subseteq X$. We say that U is open if for every $x \in U$ there is an open ball B about x such that $B \subseteq U$, ie. U is a neighborhood of each point it contains.

We say $f : X \rightarrow Y$ is continuous if it is continuous at each point of X .

Let U be an open set in Y , $x \in X$ with $f(x) \in U$. For each ball B_1 in U about $f(x)$, there is an open ball about x $B_2 \subseteq X$ such that if $x' \in B_2$ then $f(x') \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1) \subseteq f^{-1}(U)$
ie. if $x \in f^{-1}(U)$ then there is an open ball B_2 about x with $B_2 \subseteq f^{-1}(U)$

ie. $f^{-1}(U)$ is open

Conversely, if the preimage $f^{-1}(U)$ of every open set U in Y is open, then f is continuous. This is because if $x_0 \in X$, B_1 an open ball about $f(x_0)$, then $f^{-1}(B_1)$ is open in X . $f(x_0) \in B_1$ so we have an open ball $B_2 \subseteq X$ about x_0 such that $B_2 \subseteq f^{-1}(B_1)$ so f is continuous at x_0 .

Thus, $f : X \rightarrow Y$ is continuous exactly if for any open U in Y , $f^{-1}(U)$ is open in X .

1.2.2 Topology

Let (X, d) be a metric space. Let J be the collection of open subsets in X of d . J has the following properties:

1. $X \in J$, $\emptyset \in J$
2. an arbitrary, maybe infinite, union of open sets is open
3. a finite intersection of open sets is open.

Proof (of (3)). If U_1, \dots, U_n are open sets and $x \in U_1 \cap \dots \cap U_n$ then there are $r_1, \dots, r_n \in \mathbb{R}$ such that $B(x, r_j) \subseteq U_j$ for $j = 1, \dots, n$. Let $r = \min\{r_1, \dots, r_n\}$, then $B(x, r) \subseteq U_j$ for each j so $B(x, r) \subseteq U_1 \cap \dots \cap U_n$. Thus, $U_1 \cap \dots \cap U_n$ is open.

Note: This does not hold for infinite intersections, consider $\bigcap_{i \in \mathbb{N}} B(x, \frac{1}{n}) = \{x\}$ in the plane.

This motivates the following definition:

Definition 1.2.2. Let X be a set. By a topology on X we mean a collection, \mathcal{T} , of subsets of X (called the open sets of the topology) satisfying **1**, **2**, and **3** above.

Definition 1.2.3. If (X, \mathcal{T}^X) , (Y, \mathcal{T}^Y) are topological spaces, $f : X \rightarrow Y$ is continuous if for every $U \in \mathcal{T}^Y$, $f^{-1}(U) \in \mathcal{T}^X$.

Example 1.2.4. Given X , let \mathcal{T}_X be all subsets of X . This is called the discrete topology on X .

- This topology can also be given by the metric $d(x, y) = 1$ if $x \neq y$

Definition 1.2.5. If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X , we say \mathcal{T}_1 is bigger, or finer, than \mathcal{T}_2 if $\mathcal{T}_1 \supseteq \mathcal{T}_2$.

- the discrete topology is the biggest topology on X .

Example 1.2.6. $\mathcal{T} = \{X, \emptyset\}$, called the indiscrete topology on X .

Note: this topology can not be given by a metric if X has 2 or more points.

1.3 August 29

1.3.1 Bases and Subbases

Let (X, \mathcal{T}) be a topological space.

Definition 1.3.1. A subset A of X is said to be closed if $A' (X - A)$ is open.

Let \mathcal{C} be the collection of closed subsets

1. $X, \emptyset \in \mathcal{C}$
2. any (maybe infinite) intersection of closed sets is closed
3. A finite union of closed sets is closed

Let X be a set, any (maybe infinite) intersection of topologies on X is a topology on X .

Thus, for any \mathcal{S} , a subset of X , there is a smallest topology that contains \mathcal{S} , namely the intersection of all topologies that contain \mathcal{S} . We say that \mathcal{S} generates this topology.

Definition 1.3.2. If \mathcal{S} has the property that $\bigcup(U \in \mathcal{S}) = X$, then \mathcal{S} is called a subbasis of the topology it generates.

Let $\mathcal{I}^{\mathcal{S}}$ be the collection of all finite intersection of elements of \mathcal{S} , then the intersection of a finite number of elements of $\mathcal{I}^{\mathcal{S}}$ is in $\mathcal{I}^{\mathcal{S}}$.

Let \mathcal{I} be a collection of subsets of X (union of elements of \mathcal{I} is X) with the property that the intersection of a finite number of elements of \mathcal{I} is in \mathcal{I} . Then the collection, \mathcal{T} , of arbitrary unions of elements of \mathcal{I} is a topology (the smallest topology containing \mathcal{I})

Why is a finite intersection of elements of \mathcal{T} in \mathcal{T} ?

Suppose $\mathcal{O}_1 = \bigcup_{\alpha} U_{\alpha}^1$, $\mathcal{O}_2 = \bigcup_{\beta} U_{\beta}^2$ with $U_{\alpha}^1, U_{\beta}^2 \in \mathcal{I}$, then $\mathcal{O}_1 \cap \mathcal{O}_2 = (\bigcup U_{\alpha}^1) \cap (\bigcup U_{\beta}^2) = \bigcup_{\alpha, \beta} (U_{\alpha}^1 \cap U_{\beta}^2)$.

Definition 1.3.3. Given a topological space (X, \mathcal{T}) , a base for it is a set of subsets, \mathcal{B} , of \mathcal{T} , with the property that every element of \mathcal{T} is a (maybe infinite) union of elements of \mathcal{B} .

If \mathcal{S} is a subbase for \mathcal{T} , then $\mathcal{I}^{\mathcal{S}}$ is a base for \mathcal{T} .

Note: definition does not require \mathcal{B} to be closed under finite intersection

(X, d) is a metric space, let \mathcal{B} be the set of open balls. Then \mathcal{B} is a base for the metric topology but usually the intersection of two open balls is not an open ball.

The intersection of finitely many elements of \mathcal{B} is the union of elements of \mathcal{B} .

Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces, and \mathcal{S} a subbase of \mathcal{T}^Y . Let $f : X \rightarrow Y$, then f is continuous if for every $U \in \mathcal{S}$, $f^{-1}(U) \in \mathcal{T}^X$.

Example 1.3.4. For $X = \mathbb{R}$, $\mathcal{S} = \{(-\infty, a), (b, +\infty) : a, b \in \mathbb{Q}\}$ generates the usual topology.

1.4 August 31

1.4.1 Initial Topologies

Definition 1.4.1. Let X be a set. Suppose we have a collection of topologies $(Y_{\alpha}, \mathcal{T}_{\alpha})$, and for each α a function $f_{\alpha} : X \rightarrow Y_{\alpha}$. The smallest topology \mathcal{T} such that each f_{α} is continuous is called the initial topology.

For each α , $U \in \mathcal{T}_{\alpha}$, must have $f_{\alpha}^{-1}(U) \in \mathcal{T}$ so a subbase of \mathcal{T} is $\{f_{\alpha}^{-1}(U) : \text{for all } \alpha, U \in \mathcal{T}_{\alpha}\}$

Example 1.4.2. Have (Y, \mathcal{T}^Y) , let X be a subset of Y . $f : X \hookrightarrow Y$ by $f(x) = x$.

Initial topology on X has subbase $f^{-1}(U) = U \cap X \subseteq X$ for $U \in \mathcal{T}^Y$. Further, $\{U \cap X : U \in \mathcal{T}^Y\}$ is a topology. This topology is called the relative topology on X .

Example 1.4.3. $Y = \mathbb{R}$, $X = [0, 1]$, relative topology contains $[0, \frac{1}{2})$, not in the original topology

Example 1.4.4. Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces. Form set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

We have projections $p^X : X \times Y \rightarrow X$ and $p^Y : X \times Y \rightarrow Y$. The initial topology has basis $(p^X)^{-1}(U) = U \times Y$ for $U \in \mathcal{T}^X$, $(p^Y)^{-1}(V) = X \times V$ for $V \in \mathcal{T}^Y$.

Further, $(U \times Y) \cap (X \times V) = U \times V$ (rectangle) so the open sets of the initial topology consist of all arbitrary unions of rectangles $U \times V$ for $U \in \mathcal{T}^X, V \in \mathcal{T}^Y$, called the product topology on $X \times Y$.

Example 1.4.5. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The product topology contains rectangles $(a, b) \times (c, d)$

Gives same topology as the euclidean metric

- Given $(X_1, \mathcal{T}^{X_1}), (X_2, \mathcal{T}^{X_2}), \dots, (X_n, \mathcal{T}^{X_n})$ can form $X_1 \times X_2 \times \dots \times X_n$ with projections $p_i : X_1 \times X_2 \times \dots \times X_n \rightarrow X_i$. The product topology is generated by “rectangles” $U_1 \times U_2 \times \dots \times U_n$ with $U_i \in \mathcal{T}^{X_i}$
- Suppose for $n \in \mathbb{N}$ we have (X_n, \mathcal{T}^n) , can form $\prod X_n$ with $p_j : \prod X_n \rightarrow X_j, \forall j$.
Only needs to contain finite intersections so we have a base of $U_1 \times U_2 \times \dots \times U_m \times X_{m+1} \times X_{m+2} \times \dots$ with $U_j \in \mathcal{T}^j$.

Example 1.4.6. $X_j = \{0, 1\}$ with discrete topology. $\prod_{j=1}^{\infty} X_j$ not discrete, also compact.

Example 1.4.7. $C([0, 1])$, set of continuous functions on $[0, 1]$, $\|f\|_{\infty} = \sup\{f(t) : t \in [0, 1]\} \rightarrow$ metric $d(f, g) = \|f - g\|_{\infty}$

Given an normed vector space $(V, \|\cdot\|)$, let $V' =$ all continuous linear functionals on V .

eg. for $g \in C([0, 1])$ we have $\varphi_g(f) = \int_0^1 f(t)g(t)dt$

For $C([0, 1]) \xrightarrow{\varphi_g} \mathbb{R}$, given topology not the smallest that makes each φ_g continuous.

1.5 September 2

1.5.1 Quotient Topologies

Definition 1.5.1. Let Y be a set. Let $(X_{\alpha}, \mathcal{T}_{\alpha})$ be a topology with, for each α , a function $f_{\alpha} : Y_{\alpha} \rightarrow Y$. The final topology is the largest topology that makes each f_{α} continuous.

So for $A \subset Y$, in order for A to be in \mathcal{T} need $f_{\alpha}^{-1}(A) \in \mathcal{T}_{\alpha}$ for all α .

For fixed α , we want $\{A \in Y : f_{\alpha}^{-1}(A) \in \mathcal{T}_{\alpha}\}$. This is a topology, denote it \mathcal{T}_{α}^Y . It follows that $T = \bigcap_{\alpha} \mathcal{T}_{\alpha}^Y$

Let Y be a set (X, \mathcal{T}^X) , $f : X \rightarrow Y$, we require f is onto Y . Then $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}^X\}$ is the smallest topology that makes f continuous. It is called the quotient topology.

Other view: Let X, Y be sets, $f : X \rightarrow Y$ onto. Then f defines an equivalence relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$.

If we have an equivalence relation on a set, it defines a partition of the set.

If you have a partition, P , of a set X , then a set P is a set where the elements are nonempty subsets of X . Then define $f : X \rightarrow P$, where $f(x)$ is the element, A , of P such that $x \in A$. Then $f : X \rightarrow P$ onto.

Definition 1.5.2. (X, \mathcal{T}^X) and (Y, \mathcal{T}^Y) are homeomorphic if their $f : X \rightarrow Y$, one to one, onto such that f and f^{-1} are continuous.

Example 1.5.3. $\mathbb{R}_d = (\mathbb{R}, \mathcal{T})$ with discrete topology.

Consider $\mathbb{R}_d \xrightarrow{f} \mathbb{R}$ by $f(t) = t$. f is one to one, onto, and continuous but f^{-1} not continuous so it is not a homeomorphism.

Example 1.5.4. Let $X = [0, 1]$, define an equivalence relation $0 \sim 1$ and $r \not\sim s$ if $r \neq s$ and $0 < r < 1$. $[0, 1]/\sim$ homeomorphic to the circle. Let $f(t) = e^{2\pi it}$, we see $f(0) = f(1)$, f is a homeomorphism.
(Insert Figure)

Example 1.5.5. $X = [0, 1] \times [0, 2]$

(Insert Figure) equivalence relation defined by $(0, r) \sim (2, r)$ for $0 \leq r \leq 1$

Quotient space is homeomorphic to a cylinder.

Suppose we define $(0, 1) \sim (2, 1 - r)$ $0 \leq r \leq 1$

(Insert Figure) Quotient space homeomorphic to the mobius strip.

Example 1.5.6. Let X be the unit sphere $\mathbb{R}^3 = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$.

Put an equivalence relation: for $v \in X$, $v \sim -v$

X/\sim is called a projective space.

1.6 September 7

1.6.1 Group Actions on Topological Spaces

For a topological spaces (X, \mathcal{T}) the set of homeomorphisms of X to X forms a group under composition, auto-homeomorphisms, $\text{Aut}((X, \mathcal{T}))$

Then if G is a group, then of an action of G on a topological space is a group homomorphism $\alpha, \alpha : G \rightarrow \text{Aut}((X, \mathcal{T}))$, so for each $g \in G$, α_g is a homeomorphism if (X, \mathcal{T})

$$\alpha_{g_1} \circ \alpha_{g_2} = \alpha_{g_1 g_2}, \alpha_{g_1^{-1}} = (\alpha_{g_1})^{-1}$$

Definition 1.6.1. For an action α , of G on some set X , given $x_0 \in X$, the orbit of x_0 for the action α is $\{\alpha_g(x_0) : g \in G\}$. The orbits form a partition of X . (orbits of $\alpha_g(x_0)$ same as x_0 , $\alpha_{g_1^{-1}}(\alpha_g(x_0)) = x_0$)

Let X/α be the set of orbits. Have “quotient map” $X \rightarrow X/\alpha$ by $x \mapsto \text{orbit of } x$.

If X has a topology and α acts by homeomorphism, puts quotient topology on X/α

Example 1.6.2. Symmetry of letters:

$X = A$ given $Z_2 = \mathbb{Z}/2\mathbb{Z}$ act by reflection. $X/\alpha =$ (Insert Figure)

$X = H$, $Z_2 \times Z_2$, $X/\alpha =$ (Insert Figure)

Example 1.6.3. Let $G = \mathbb{Z}$, let $X = \mathbb{R}$, let α be an action of \mathbb{Z} on \mathbb{R} by translation, $\alpha_n(t) = t + n$

each of $\{\dots, t_0 - 1, t_0, t_0 + 1, \dots\}$. What is \mathbb{R}/α

Example 1.6.4. A fundamental domain for α is a subset of X that contains exactly one element of each orbit.

- For the above example, fundamental domain $[0, 1)$ with open subsets “wrapped around edges” so \mathbb{R}/α is homeomorphic to the circle. Homeomorphism given by $t = e^{2\pi it}$, constant on equivalence classes.

Example 1.6.5. The antipodal relation on the unit sphere with $v \sim -v$ acted on by $Z_2 = (0, 1)$ by $\alpha_1(v) = -v$

Let Y be set of all lines in \mathbb{R}^3 through 0. Let $\mathbb{R} - \{0\}$, have an action on \mathbb{R}^3 by $\alpha_t(r, s, v) = (tr, ts, tv)$

Orbits in $\mathbb{R}^3 - \{0\}$, set of all lines through 0, (with 0 removed). Each line intersects the unit sphere in 2 antipodal points. Quotient topology gives a topology on the set of lines.

1.6.2 Connectedness

Definition 1.6.6. A topological space (X, \mathcal{T}) is connect if it does have two, nonempty, disjoint open sets A, B with $A \cup B = X$

- If this is the acse, A, B also closed - called “clopen”

Theorem 1.6.7. If (X, \mathcal{T}) is connected, $f : X \rightarrow Y$ is continuous, $f(X) = \text{range}(f)$ with the inherited topology is connected.

1.7 September 9

1.7.1 Connectedness

(X, \mathcal{T}) is connected if the only clopen sets are X, \emptyset

Proposition 1.7.1. If (X, \mathcal{T}) , $A \subseteq X$, give A the relative topology, then if A is connected then so is \bar{A}

Proof. Suppose that C is a clopen subset of \bar{A} , then $C \cap A$ is a clopen subset of A so either $C \cap A = A$ or $C \cap A = \emptyset$. If $C \cap A = \emptyset$, $C \cap \bar{A} = \emptyset$ since C is open. If $C \cap A = A$ so $C \supseteq A$ since C is closed. So $C = \emptyset$ or \bar{A}

Proposition 1.7.2. Given (X, \mathcal{T}) a collection of $\{F_\alpha\}$ of subsets of X , let $Y = \bigcup_\alpha F_\alpha$. Suppose that each F_α is connected. If $\exists p \in \bigcap F_\alpha$ then Y is connected.

Proof. Let C be a clopen subset of Y . We can assume that $p \in C$, then for each α , $C \cap F_\alpha \neq \emptyset$, $C \cap F_\alpha$ is clopen so $C \cap F_\alpha = F_\alpha$ so $F_\alpha \subseteq C$. Thus $C \supseteq \bigcup F_\alpha = Y$, so $C = Y$.

Proposition 1.7.3. Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces and suppose that each is connected. Then $X \times Y$ with the product topology is connected.

Proof. Choose a point $b \in Y$ (a “basepoint”). Then $E = \{b\} \times Y$ is homeomorphic to Y and so is connected. For each $y \in Y$, let $H_y = X \times \{y\}$. Homeomorphic to X so connected. For each $y \in Y$, let $T_y = H_y \cup E$, connected since (y, b) is in both. Choose a basepoint $c \in Y$ so $(b, c) \in E$ and (b, c) is in each T_y so $X \times Y = \bigcup_{y \in Y} T_y$ is connected.

Follows that if X_1, \dots, X_n are topological spaces and each is connected then $X_1 \times \dots \times X_n$ is connected.

Any open interval (a', b') in \mathbb{R} is connected. (False for (a, b) in \mathbb{Q})

Suppose $C \subseteq (a', b')$ is clopen and $\neq \emptyset$ and suppose we have $a \in C, b \in C', a < b$. Consider $A = \{r \in C : r < b\}$. $a \in A$ and b is an upper bound. Let c be its least upper bound then $c \in A$ since if $c \in C'$ then there is an open ball about c contained in C (since C is open), but $c \notin C'$ for a similar reason.

1.7.2 Connected Components

Given (X, \mathcal{T}) define an equivalence relation on X by $x \sim y$ if there is a connected subset that contains both of them.

Reflexivity, symmetry clear. If $x \sim y, y \sim z$, then $x, y \in C, y, z \in D$ so $y \in C \cap D$ so $C \cup D$ is connected.

1.8 September 12

1.8.1 Connected Components

(X, \mathcal{T}) a topological space. Define an equivalence relation on X by $x \sim y$ if there is a connected subset of X containing both x and y .

Transitivity: If $x \sim y$ and $y \sim z$, there is connected A with $x, y \in A$ and connected B with $y, z \in B$ then $A \cup B$ is connected since $y \in A, y \in B, x, z \in A \cup B$.

The equivalence classes for this equivalence relation are called the connected components of X . Given $x \in X$, the equivalence class of x is the union of all connected sets containing x . So the equivalence class is the largest connected set containing x .

Since the closure of a connected set is connected, the equivalence classes are closed subsets of X .

Example 1.8.1. $X = \mathbb{Q}$, the connected components we get are the one point subsets.

(\mathbb{Q} is totally disconnected, as is $\prod_{m=1}^{\infty} \{0, 1\}$, “0 dimensional”)

Definition 1.8.2. By a parametrized path in X we mean a continuous function, f , from some interval $[a, b] \subseteq \mathbb{R}$. This path connects $f(a)$ to $f(b)$.

Define an equivalence relation on (X, \mathcal{T}) by $x \sim y$ if there is a path in X connecting x to y .

Reflexive: Assume $f : [0, 1] \rightarrow X, f(0) = x, f(1) = y$ set $g(t) = f(1 - t)$, then $g(0) = y, g(1) = x$

Transitive: If $f : [a, b] \rightarrow X, f(a) = x, f(b) = y$ and $g : [c, d] \rightarrow X, g(c) = y, g(d) = z$ change interval such that

$g : [b, e]$ with $g(b) = y, g(e) = z$. $[a, e] = [a, b] \cup [b, e]$ so define $h : [a, e] \rightarrow X$ by $h(t) = \begin{cases} f(t) & t \in [a, b] \\ g(t) & t \in [b, e] \end{cases}$

The equivalence classes are called path components of (X, \mathcal{T})

Note: path connected \rightarrow connected.

Example 1.8.3. Let $f : (0, 1] \rightarrow X, f(t) = (t, \sin(\frac{1}{t}))$, graph of $\sin(\frac{1}{t})$.

Subset is path connected but not closed. Closure is graph $\cup \{0\} \times [0, 1]$. Closure consists of 2 path connected components but only 1 connected component. In closure, 1 path connected component is not closed, while the other is closed but not open.

Definition 1.8.4. (X, \mathcal{T}) is locally connected if $\forall x \in X \forall$ open \mathcal{O} if $x \in \mathcal{O}$ there is an open $U, x \in U \subseteq \mathcal{O}$ with U connected.

- If (X, \mathcal{T}) is locally connected then all connected components are open, and hence clopen.

Definition 1.8.5. (X, \mathcal{T}) is locally path connected if $\forall x \in X \forall$ open \mathcal{O} if $x \in \mathcal{O}$ there is an open $U, x \in U \subseteq \mathcal{O}$ with U path connected.

- If (X, \mathcal{T}) is locally path connected, then all path connected components are clopen. path components = connected components.

Definition 1.8.6. A topological manifold of dimension n is a topological space (X, \mathcal{T}) with the property that every $x \in X$ has an open set \mathcal{O} such that $x \in \mathcal{O}$ with \mathcal{O} homeomorphic to an open set in \mathbb{R}^n (open ball in \mathbb{R}^n , all of \mathbb{R}^n).

1.9 September 14

1.9.1 Compactness

Definition 1.9.1. Let (X, \mathcal{T}) be a topological space. By an open cover of X we mean a subset \mathcal{C} of \mathcal{T} , ie. a family of open sets such that $\bigcup\{\mathcal{O} \in \mathcal{C}\} = X$. By a subcover of \mathcal{C} we mean a subset \mathcal{D} of \mathcal{C} such that \mathcal{D} is a cover of X .

Definition 1.9.2. (X, \mathcal{T}) is said to be compact if every open cover of X has a finite subcover.

- $[0, 1] \subseteq \mathbb{R}$ is compact
- Heine - Borel Property: any bounded closed subset of \mathbb{R}^n is compact.

Let (X, \mathcal{T}) be a topological space. Let A be a subset of X , give A the relative topology. Then A is compact iff for any $\mathcal{C} \subseteq \mathcal{T}$ such that $\bigcup\{\mathcal{O} \in \mathcal{C}\} = A$ there is a finite subcover \mathcal{D} of \mathcal{C} such that $\bigcup\{\mathcal{O} \in \mathcal{D}\} \supseteq A$

Proposition 1.9.3. Let (X, \mathcal{T}) be compact. If $A \subseteq X$ is closed, then A is compact.

Proof. If $\mathcal{C} \subseteq \mathcal{T}$ such that $\bigcup\{\mathcal{O} \in \mathcal{C}\} \supseteq A$, since A closed, A' open so $\mathcal{C} \cup \{A'\}$ is an open cover of X . Since X is compact, there is a finite subcover, \mathcal{D} . If we remove A' from \mathcal{D} (if $A' \in \mathcal{D}$) we get a finite subcover \mathcal{C} covering A .

Any set with the indiscrete topology is compact and any subset of it is compact but not necessarily closed.

Proposition 1.9.4. Given (X, \mathcal{T}) and $A \subseteq X$ compact. If (X, \mathcal{T}) is Hausdorff then for any $x \in X$, $x \notin A$ there are disjoint open sets U, V with $A \subseteq U$, $x \in V$

Proof. For any $a \in A$, by Hausdorff, there are open sets U_a, V_a disjoint with $a \in U_a$, $x \in V_a$. The collection of sets $\{U_a : a \in A\}$ covers A . Since A is compact there is a finite subcover U_{a_1}, \dots, U_{a_n} . Let $U = U_{a_1} \cup \dots \cup U_{a_n} \supseteq A$, let $V = V_{a_1} \cap \dots \cap V_{a_n}$ so we get $x \in V$, U, V disjoint.

Corollary 1.9.5. Given (X, \mathcal{T}) Hausdorff, $A \subseteq X$ compact, then A is closed.

Proof. A' open since for $x \in A'$ can find open set containing x , disjoint from A .

Theorem 1.9.6. Given (X, \mathcal{T}) compact, and $f : X \rightarrow Y$ continuous, then $f(X)$ is compact.

Proof. Let \mathcal{C} be an open cover of $f(X)$. Since for $\mathcal{O} \in \mathcal{T}^Y$, $f^{-1}(\mathcal{O}) \in \mathcal{T}^X$, then $\{f^{-1}(\mathcal{O}) : \mathcal{O} \in \mathcal{C}\}$ is an open cover of X . Since X is compact, there is a finite subcover $f^{-1}(\mathcal{O}_1), \dots, f^{-1}(\mathcal{O}_n)$. Then $\mathcal{O}_1, \dots, \mathcal{O}_n$ is an open cover of $f(X)$

Example 1.9.7. Given $f : [0, 1] \rightarrow \mathbb{R}$ continuous, $f([0, 1])$ is connected, compact so is some $[a, b]$. So f attains its supremum = $\text{lub}\{f(t) : t \in [a, b]\}$

Theorem 1.9.8. Given (X, \mathcal{T}) , (Y, \mathcal{T}) , $f : X \rightarrow Y$ continuous, assume f is continuous, one to one, onto, X is compact, Y is Hausdorff. Then f is homeomorphism.

Proof. Need to show f^{-1} continuous, so need $f(\mathcal{O}) \in \mathcal{T}^Y$ for $\mathcal{O} \in \mathcal{T}^X$, equivalently, if A is closed in X , then $f(A)$ is closed in Y . If A closed, A compact so $f(A)$ is compact, but Y is Hausdorff so $f(A)$ is closed. \square

1.10 September 16

1.10.1 Compactness

Proposition 1.10.1 (The Tube Lemma). Given (X, \mathcal{T}^X) , (Y, \mathcal{T}^Y) and assume Y is compact. Given $x_0 \in X$ and some \mathcal{O} open set in $X \times Y$ such that $\{x_0\} \times Y$ is contained in \mathcal{O} . Then there is an open neighborhood, U , of x_0 such that $U \times Y \subseteq \mathcal{O}$, called the tube about $\{x_0\} \times Y$

Proof. Note that $\{x_0\} \times Y$ is homeomorphic to Y so $\{x_0\} \times Y$ is compact. For $y \in Y$, $(x_0, y) \in \mathcal{O}$ so there is some $U_y \subseteq X, V_y \subseteq Y$ such that $(x_0, y) \in U_y \times V_y$. The V_y 's cover Y so since Y is compact there is a finite subcover, $V_{y_1}, V_{y_2}, \dots, V_{y_n}$. Then, let $U = \bigcap_{i=1}^n U_{y_i}$, U is open and we claim $U \times Y \subseteq \mathcal{O}$. Given $(x, y) \in U \times Y$, $\exists j$ such that $y \in V_j$ and $U_j \times V_j \subseteq \mathcal{O}$ so $U \times V_j \subseteq \mathcal{O}$ so $U \times Y \subseteq \mathcal{O}$.

Theorem 1.10.2. If (X, \mathcal{T}^X) and (Y, \mathcal{T}^Y) are both compact then $X \times Y$ is compact.

Proof. If \mathcal{C} is an open cover of $X \times Y$, for each x , \mathcal{C} covers $\{x\} \times Y$ so there is a finite cover \mathcal{C}_x , take the union to get an open set \mathcal{O}_x containing $\{x\} \times Y$, so there is an open neighborhood $U_x \times Y$ such that $U_x \times Y \subseteq \mathcal{O}$. The U_x 's form an open cover of X , since X is compact there is a finite subcover U_{x_1}, \dots, U_{x_n} . The $(U_{x_j} \times Y)$ cover $X \times Y$. \mathcal{C}_{x_j} is a cover of $(U_{x_j} \times Y)$ so $\bigcup_{j=1}^n \{\mathcal{O} \in \mathcal{C}_{x_j}\}$ cover $X \times Y$.

By induction, given $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \dots, (X_n, \mathcal{T}_n)$ all compact, then $X_1 \times X_2 \times \dots \times X_n$ is compact.

Let \mathcal{F} is an infinite collection of topologies such that $(X_\alpha, \mathcal{T}_\alpha)$ each compact, then is $\prod X_\alpha$ compact?

1.11 September 19

1.11.1 Compactness

If X is any set, and if \mathcal{C} is a collection of closed subsets of X , then $\bigcup\{A \in \mathcal{C}\} = X$ iff $\bigcap\{A' : A \in \mathcal{C}\} = \emptyset$. So (X, \mathcal{T}) is compact if whenever \mathcal{C} is a collection of subsets such that $\bigcap\{C \in \mathcal{C}\} = \emptyset$ then there is a finite subset $\mathcal{F} \subseteq \mathcal{C}$ such that $\bigcap\{A \in \mathcal{F}\} \neq \emptyset$

Definition 1.11.1. A collection \mathcal{C} of subsets of a set X has the finite intersection property (FIP), if for any finite $\mathcal{F} \subseteq \mathcal{C}$ we have $\bigcap \{A \in \mathcal{F}\} \neq \emptyset$

Then (X, \mathcal{T}) is compact if for any collection \mathcal{C} of closed subsets with FIP, $\bigcap \{A \in \mathcal{C}\} \neq \emptyset$

Definition 1.11.2. (X, \mathcal{T}) is locally compact if each point $x \in X$ has a compact neighborhood, ie. $\mathcal{O}, x \in \mathcal{O}$ and $\overline{\mathcal{O}}$ compact.

- \mathbb{R}, \mathbb{R}^n locally compact

Proposition 1.11.3. Let (X, \mathcal{T}) be locally compact and Hausdorff. Then for any $x \in X$ and $\mathcal{O} \in \mathcal{T}$ with $x \in \mathcal{O}$ there is $U \in \mathcal{T}$, $x \in U$, $\overline{U} \subseteq \mathcal{O}$ is compact.

Proof. By local compactness, there is open V , $x \in V$, \overline{V} compact. Then $V \cap \mathcal{O}$ is open, $x \in V \cap \mathcal{O}$ so we can replace \mathcal{O} with $V \cap \mathcal{O}$, $\overline{V \cap \mathcal{O}}$ is compact. Thus we can assume that $\overline{\mathcal{O}}$ is compact. Let $C = \overline{\mathcal{O}} \setminus \mathcal{O}$, closed, compact, $x \notin C$. By Hausdorff, $\exists U, V \in \mathcal{T}$ disjoint $x \in U$, $C \subseteq V$, $U \subseteq \mathcal{O}$, $C' \supseteq V'$, $U \subseteq V'$ closed so $\overline{U} \subseteq V'$ so $\overline{U} \cap V = \emptyset$ so $\overline{U} \cap C = \emptyset$ so $\overline{U} \subseteq \mathcal{O}$

Chapter 2

Algebraic Topology

2.1 September 19

2.1.1 Homotopy

Definition 2.1.1. Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$. Let $f_0, f_1 : X \rightarrow Y$ continuous, then f_0 and f_1 are homotopic if $F : X \times [0, 1] \rightarrow Y$ continuous such that $F(x, 0) = f_0(x), F(x, 1) = f_1(x)$. F is called a homotopy from f_0 to f_1 .

Proposition 2.1.2. Homotopy is an equivalence relation on the set of continuous functions from X to Y

Proof. 1. $f \sim f$ by constant homotopy

2. If $f_0 \sim f_1$, set $F'(x, t) = F(x, 1 - t)$, $f_1 \sim f_0$

3. $f \sim g$ and $g \sim h$ with homotopies F, G . Define $H : X \times [0, 2] \rightarrow Y$. $H(x, t) = \begin{cases} F(x, t) & t \in [0, 1] \\ G(x, t - 1) & t \in [1, 2] \end{cases}$.
If $t = 1$, $F(x, 1) = g(x), G(x, 1 - 1) = g(x)$

Lemma 2.1.3 (Pasting Lemma). If (X, \mathcal{T}) , $X = A \cup B$, A, B closed and if $\varphi : X \rightarrow Y$ and if $\varphi|_A$ is continuous and if $\varphi|_B$ is continuous then φ on X is continuous.

Proof. If $C \subseteq Y$ closed $\varphi^{-1}(C) = (\varphi|_A)^{-1}(C) \cup (\varphi|_B)^{-1}(C)$. $(\varphi|_B)^{-1}(C)$ closed in B so closed in X . Similarly, for $(\varphi|_A)^{-1}(C)$ so $\varphi^{-1}(C)$ is closed.

2.2 September 21

2.2.1 Path Homotopy

Definition 2.2.1. (X, T) (usually path connected). Two paths $f, g : [0, 1] \rightarrow X$ are path homotopic if $f(0) = g(0), f(1) = g(1)$ and if they are homotopic via a homotopy $F : [0, 1] \times [0, 1] \rightarrow X$ with $F(t, 0) = f(0), F(t, 1) = f(1)$ for all t .

Path homotopy is an equivalence relation.

- Can compose equivalence classes. If f and g are paths, $f(1) = g(0)$ can compose them viewing g as a path on $[1, 2]$ (instead of $[0, 1]$). Define $(f * g)$ on $[0, 2]$ by $(f * g)(t) = \begin{cases} f(t) & t \in [0, 1] \\ g(t) & t \in [1, 2] \end{cases}$

Proposition 2.2.2. If $f \sim f', g \sim g'$ then $f * g \sim f' * g'$

Proof. Show first that $f * g \sim f' * g$. If F is a homotopy from f to f' , let $\tilde{F}(r, t) = \begin{cases} F(r, t) & t \in [0, 1] \\ g(r) & t \in [1, 2] \end{cases}$.

Similarly, $f' * g \sim f' * g'$

- let \mathcal{G} be the collection of path-homotopy classes of X . Then $*$ is a partially defined product. It is associative (when it makes sense). So for path-homotopic equivalence classes it is associative.
- Each $x \in X$ provides an equivalence class $e_x : [0, 1] \rightarrow X$ by $e_x(t) = x$. If F is a path from x to y then $e_x * f \sim f, f * e_y \sim f$ so have an identity element for $x \in X$
- Each element has an inverse. Given f from x to y , let $f^{-1}(t) = f(1 - t)$, $f^{-1}(0) = f(1), f^{-1}(1) = f(0)$, $f * f^{-1} \sim e_x, f^{-1} * f \sim e_y$. So equivalence classes in \mathcal{G} has inverses.
- This is an example of a groupoid. \mathcal{G} path groupoid for X . In fact, \mathcal{G} is a topological groupoid.

Given $x_0 \in X$, consider all paths from x_0 to x_0 . Path homotopic equivalence classes form a group $\pi_1(X, x_0)$. This is the fundamental group of X for the basepoint x_0 .

If we change base point from x_0 to x'_0 , f a path from x_0 to x'_0 , from a loop α based at x'_0 $f * \alpha * f^{-1}$ is a loop based at x_0 . This gives an isomorphism from $\pi_1(X, x'_0)$ to $\pi_1(X, x_0)$. Isomorphism depends on f up to homotopy.

2.3 September 23

2.3.1 The Fundamental Group

By a pointed set (or space) we mean a set together with a selected special point.

(X, x_0) path connected $x_0 \in X$, can attach to (X, x_0) the group $\pi_1(X, x_0)$ (= the set of homotopy classes of loops on X based at x_0)

Given $(X, x_0) (Y, y_0)$, $\varphi : X \rightarrow Y$ continuous. Let f be a path in X , then $f \circ \varphi$ is a path in Y . If $\varphi(x_0) = y_0$, we map loops based on x_0 to loops based at y_0

Let F be a homotopy between a path f and a path g on X , then $\varphi \circ F$ is a homotopy from $\varphi \circ f$ to $\varphi \circ g$.

So $\varphi : X \rightarrow Y$, $\varphi(x_0) = y_0$ gives a function $\tilde{\varphi}$ from homotopy classes of loops based at x_0 to homotopy classes of loops based at y_0 . $\tilde{\varphi} : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$

Theorem 2.3.1. $\tilde{\varphi}$ is a group homomorphism.

Proof. Let f and g be paths in X . $f * g$, view f as defined on $[0, 1]$, g as defined on $[1, 2]$. $(f * g)(r) = \begin{cases} f(r) & r \in [0, 1] \\ g(r-1) & r \in [1, 2] \end{cases}$, then $(\varphi \circ f) * (\varphi \circ g)(r) = \begin{cases} \varphi \circ f(r) & r \in [0, 1] \\ \varphi \circ g(r-1) & r \in [1, 2] \end{cases} = \varphi(f * g)$. Passes to homotopy classes.

Theorem 2.3.2. $(X, x_0), (Y, y_0), (Z, z_0), X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z, \pi_1(X) \xrightarrow{\tilde{\varphi}} \pi_1(Y) \xrightarrow{\tilde{\psi}} \pi_1(Z)$, we have $\hat{\psi} \circ \hat{\varphi} = \widehat{\psi \circ \varphi}$

Proof. If f path on X , $(\hat{\psi} \circ \hat{\varphi})(f) = \hat{\psi}(\varphi \circ f) = \psi \circ (\varphi \circ f) = (\psi \circ \varphi) \circ f = \widehat{\psi \circ \varphi}(f)$

$(X, x_0), (Y, y_0), \varphi : X \rightarrow Y$. Assume φ is a homeomorphism. $\varphi^{-1} \circ \varphi = \text{id}_X, \varphi \circ \varphi^{-1} = \text{id}_Y$. Then $\pi_1(\varphi^{-1})\pi_1(\varphi) = \pi_1(\text{id}_X) = \text{id}_{\pi_1(X)}, \pi_1(\varphi)\pi_1(\varphi^{-1}) = \pi_1(\text{id}_Y) = \text{id}_{\pi_1(Y)}$ ie, $\pi_1(\varphi)$ is a group isomorphism of $\pi_1(X)$ and $\pi_1(Y)$.