MATH 135: Introduction to the Theory of Sets

Jad Damaj

Fall 2022

Contents

Chapter 1

Introduction

1.1 August 25

1.1.1 Introduction

Foundations of Mathematics: language, axioms, formal proofs

- We focus on the axioms in set theory
- We use ZFC (Zermelo-Fraenkel + Choice)
- There is only one primitive notion : $\in$
- Within the ZFC universe, everything is a set

Course Outline:

- Basic axioms
- Operations, relations, functions
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- carindals
- AC
- ordinals

1.1.2 Basics

Principle of Extensionality: Two sets A, B are the same $\leftrightarrow$ they have the same elements $\forall x(x \in A \leftrightarrow x \in B)$
Example 1.1.1. $2, 3, 5 = \{5, 2, 4\} = \{2, 5, 2, 3, 3, 2\}$

Definition 1.1.2. There is a set with no elements, denoted $\varnothing$

- $\varnothing \neq \{\varnothing\}$
- $A \subseteq B$: A is a subset of $B \leftrightarrow$ each element of A is in B (use $\subsetneq$ to denote proper subset)

- $\{2\} \subseteq \{2,3,5\}$ but $\{2\} \notin \{2,3,5\}$
- Power set opertaion: $\mathcal{P}(A) = \{B \mid B \subseteq A\}$

We can define a hierarchy:
$V_0 = \varnothing$, $V_1 = \mathcal{P}(\varnothing) = \{\varnothing\}$, $V_2 = \mathcal{P}\mathcal{P}(\varnothing) = \{\varnothing, \{\varnothing\}\}$
$V_3 = \mathcal{P}(V_2) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$, $V_4, \ldots$
$V_\omega = \bigcup_{n\in\mathbb{N}} V_n$, $\mathcal{P}(V_\omega)$, $\mathcal{P}\mathcal{P}(V_\omega), \ldots, V_{\omega+\omega}, \ldots, V_{\omega+\omega+\cdots}, \ldots, V_{\omega\times\omega}, \ldots, V_{\omega^\omega}$

Chapter 2

Axioms and Operations

2.1 August 30

2.1.1 Zermelo Fraenkel Axioms of Set Theory

Setting: in ZFC all objects are sets
Language: contains vocabulary ($\in$), logical symbols ($=, \wedge, \vee \exists, \forall, \neg$), variables ($x, y, A, B$, etc.)

Axiom 2.1.1 (Extensionality Axiom)**.** Two sets are the same if they have the same elements
$\forall A, B(\forall x(x \in A \leftrightarrow x \in B) \to A = B)$

Axiom 2.1.2 (Empty Set Axiom)**.** There is a set with no members, denoted $\varnothing$
$\exists A \forall x(x \notin A)$

Axiom 2.1.3 (Pairing Axiom)**.** For any sets u, v there is a est whose elements are u and v, denoted $\{u, v\}$
$\forall u, v \exists A \forall x(x \in A \leftrightarrow x = u \vee x = v)$

Axiom 2.1.4 (Union Axiom (Preliminary Form))**.** For any sets a, b there is a set whose elements are elements of a and elements of b, denoted $a \cup b$
$\forall a, b \exists A \forall x(x \in Ax \in u \vee x \in v)$

Axiom 2.1.5 (Powerset Axiom)**.** Each set A, has a power set $\mathcal{P}(A)$.
$\forall A \exists B \forall x(x \in B \iff x \subseteq A)$ where $x \subseteq A$ stands for $\forall y(y \in x \to y \in A)$

Axiom 2.1.6 (Union Axiom)**.** For any set A, there is a set $\bigcup A$ whose members are members of the members of A.
$\forall A \exists B \forall x(x \in B \leftrightarrow \exists y \in A(x \in y))$

Idea for the subset axiom: For any set A, there is a set B whose memebrs are members of A satisfying some property.

eg. $B = \{x \in A \,|\, x \text{ satisfies property } P\} \subseteq A$

Example 2.1.7. $B = \{n \in \mathbb{N} \,|\, n\text{cannot be described in less that 20 words}\}$

- let b be the smallest element in B, then b is the smallest element that cannot be described in 20 words.
- Paradox : need to use formal language to express property P.

Example 2.1.8. Let $B = \{x \,|\, x \notin x\}$
Question: $B \in B$? $B \in B \leftrightarrow B \notin B$: need to have property be contained in some larger set.

We can now restate the axiom more formally:

Axiom 2.1.9 (Subset Axiom (Scheme))**.** For each formula $\phi(x)$, there is an axiom:
$\forall A \exists B \forall x (x \in B \leftrightarrow x \in A \wedge \phi(x))$

Example 2.1.10. Suppose there is a set of all sets A. Consider $B = \{x \in A \,|\, x \notin x\}$. Then $B \in B \leftrightarrow B \notin B$, contradiction. So there can be no such set A.

The language of 1rst order logic for ZFC:
The following are formulas:

- $x = y,\ x \in y$ atomic formulas
- $(\varphi \wedge \psi), (\varphi \vee \psi), \neg\varphi$ where φ, ψ are formulas
- $\exists v \varphi, \forall x \varphi$

Example 2.1.11. $\varphi(v, w) := (\exists v(v \in x \wedge \neg v = w)) \rightarrow (\forall y(\neg y \in y))$ is a formula