MATH 225A: Metamathmatics

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Contents

Chapter 1

Structures and Theories

1.1 August 25

1.1.1 Review

Definition 1.1.1. A language $\mathcal{L}$ consists of $\{\mathcal{C}, \mathcal{R}, \mathcal{F}\}$ where $\mathcal{C}$ is the set of constant symbols, $\mathcal{R}$ is the set of relation symbols, $\mathcal{F}$ is the set of function symbols, and and arity function $n : \mathcal{R} \cup \mathcal{F} \to \mathbb{N}$. For $R \in \mathcal{R}$, n_R is the arity of R, for $f \in \mathcal{F}$, n_f is the number of inputs f takes.

Definition 1.1.2. An $\mathcal{L}$-structure consist of

- a set M called the domain
- an element $c^{\mathcal{M}}$ for each $c \in \mathcal{C}$
- a subset $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
- a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$

denoted $\mathcal{M} = (M \ : \{c^{\mathcal{M}} : c \in \mathcal{C}\}, \{R^{\mathcal{M}} : R \in \mathcal{R}\}, \{f^{\mathcal{M}} : f \in \mathcal{F}\})$

Definition 1.1.3. An $\mathcal{L}$-embedding $\eta : \mathcal{M} \to \mathcal{N}$ is a one to one function $M \to N$ that preserves interpretation
eg. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$, $\eta(f^{\mathcal{M}})(m_1, \ldots, m_{n_f}) = f^{\mathcal{N}}(\eta(m_1), \ldots, \eta(m_{n_f}))$,
$(m_1, \ldots, m_{n_R}) \in R^{\mathcal{M}} \iff (\eta(m_1), \ldots, \eta(m_n)) \in R^{\mathcal{N}}$

Definition 1.1.4. An $\mathcal{L}$-isomorphim is an $\mathcal{L}$-embedding that is onto.

Definition 1.1.5. $\mathcal{M}$ is a substructure if $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$ if:
$c^{\mathcal{M}} = c^{\mathcal{N}}$, $f^{\mathcal{M}} = f^{\mathcal{N}} \upharpoonright M^{n_f}$, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^{n_R}$

First Order language:

- Use symbols :

  - $\mathcal{L}$
  - Logical symbols: connectives $(\wedge, \vee, \neg)$, quantifiers $(\forall, \exists)$, equality $(=)$, variables $(v_0, v_1, \ldots)$
  - paranthesis and commas

- terms
  - c : constants
  - v_i : variables
  - $f(t_1, \ldots, t_{n_f})$ for terms $t_1, \ldots, t_{n_f}$

- given an $\mathcal{L}$-structure $\mathcal{M}$, a term $t(v_0, \ldots, v_n)$, and $m_0, \ldots, m_n \in M$ we inductively define $t^{\mathcal{M}}(m_0, \ldots, m_n)$
- atomic formulas: $t_1 = t_2$ and $R(t_1, \ldots, t_{n_R})$
- $\mathcal{L}$-formulas: If $\phi and \psi$ are $\mathcal{L}$-formulas, then so are: $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $\exists v\phi$, $\forall v\phi$

Definition 1.1.6. We say a variable v occurs freely in ψ when it is not in a quantifier $\forall v$ or $\exists v$

- an $\mathcal{L}$-sentence is an $\mathcal{L}$-formula with no free variables

Definition 1.1.7. A theory is a set of $\mathcal{L}$-sentences

Definition 1.1.8. Given an $\mathcal{L}$-formla $\psi(v_1, \ldots, v_k)$, $\mathcal{L}$-structure $\mathcal{M}$, $m_1, \ldots, m_k \in M$ we can define $\mathcal{M} \models \phi(m_1, \ldots, m_k)$ inductively. We say $(m_1, \ldots, m_k)$ satisfies ϕ in $\mathcal{M}$ or ϕ is true in $\mathcal{M}, m_1, \ldots, m_k$.

- A theory T is satisfiable if it has a model $\mathcal{M}$, eg. $\mathcal{M}$ such that $\mathcal{M} \models \phi$ for $\phi \in T$

Proposition 1.1.9. If $\mathcal{M} \subseteq \mathcal{N}$, $\phi(\overline{v})$ is quantifier free, $\overline{m} \in M$, then $\mathcal{M} \models \phi(\overline{m}) \leftrightarrow \mathcal{N} \models \phi(\overline{m})$.

Definition 1.1.10. $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$ if for all $\mathcal{L}$-sentences ϕ, $\mathcal{M} \models \phi \leftrightarrow \mathcal{N} \models \phi$, denoted $\mathcal{M} \equiv \mathcal{N}$

- $\mathrm{Th}(\mathcal{M})$, the full theory of $\mathcal{M}$, is $\{\phi\ \mathcal{L} - \text{sentence } \mid \mathcal{M} \models \phi\}$
- $\mathcal{M} \equiv \mathcal{N} \iff \mathrm{TH}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$
- A class of $\mathcal{L}$-structures $\mathcal{K}$ is elementary if there is a theory T such that $\mathcal{K}$ is the class of all $\mathcal{M}$ such that $\mathcal{M} \models T$.

Logical implication: $T \models \phi$ if for every $\mathcal{M} \models T$, $\mathcal{M} \models \phi$
Gödels Completeness Theorem: $T \models \phi \leftrightarrow$ there is a formal proof for $T \vdash \phi$

1.1.2 Definable Sets

Definition 1.1.11. $X \subseteq M^n$ is definable if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $b_1, \ldots, b_m \in M$ such that $\forall \overline{a}$, $\overline{a} \in X \leftrightarrow \mathcal{M} \models \phi(\overline{a}, \overline{b})$ (definable over $\overline{b}$)

- Given $A \subseteq M$, X is definable over A, or A-definable, if it is definable over $\overline{b}$ for some $\overline{b} \in A$.

Proposition 1.1.12. Suppose $\mathcal{D} = (D_n : n \in \omega)$ is the smallest collection of subsets $D_n \subseteq \mathcal{P}(M^n)$ such that

- $M^n \in D_n$
- D_n is closed under union, intersection, complement, permutation
- if $X \in D_{n+1}$, then $\pi(X) \in D_n$ where $\pi(m_1, \ldots, m_{n+1}) = (m_1, \ldots, m_n)$
- $\{\overline{b}\} \in D_n$ for $\overline{b} \in M^n$
- $R^{\mathcal{M}} \in D_{n_R}$, graph$(f) \in D_{n_f+1}$
- if $X \in D_n$, $M \times X \in D_{n+1}$
- $\{(m_1, \ldots, m_n) : m_i - m_j\} \in D_n$

Then $X \subseteq \mathcal{M}^n$ is definable $\leftrightarrow X \in D_n$

Chapter 2

Basic Techniques

2.1 August 30

2.1.1 Compactness Theorem

Theorem 2.1.1 (Compactness)**.** If T is finitely satisfiable, then T has a model $\mathcal{M}$. Furthermore, $|\mathcal{M}| \leqslant |\mathcal{L}| + \aleph_0$

- T is finitely satisfiable if every finite subset is satisfiable

Compactness II: if $T \models \phi$, then there is finite $T_0 \subset T$ such that $T_0 \models \phi$
$T \models \phi \leftrightarrow T \cup \{\neg\phi\}$ is not satisfiable

Proposition 1: If T is finitely satisfiable, maximal, and has the witness property, then T has a model $\mathcal{M}$ with $|\mathcal{M}| \leqslant |\mathcal{L}|$

Proposition 2: If T is finitely satisfiable, then there is $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-theory $T^* \supseteq T$ such that T^* is finite;y satisfiable, maximal, and has the witness property. Further, $|\mathcal{L}^*| \leqslant |\mathcal{L}| + \aleph_0$

Definition 2.1.2.

- T is maximal if for any sentence ϕ, either $\phi \in T$ or $\neg\phi \in T$
- T has the witness property if for all $\mathcal{L}$-formulas $\phi(v)$ there is a constant c_ϕ such that $\exists v\phi(v) \rightarrow \phi(c_\phi) \in T$

Lemma 1: If T is maximal and finitely satisfiable, if there is finite $\Delta \subseteq T$ such that $\Delta \models \phi$, then $\phi \in T$.

Proof. If $\phi \notin T$, $\neg\phi \in T$. Since $\Delta \models \phi$, $\Delta \cup \{\neg\phi\}$ is not satisfiable, contradicting our assumption.

Henekin Construction:
We want to define $\mathcal{M} = (M, c^{\mathcal{M}}, R^{\mathcal{M}}, f^{\mathcal{M}})$

- Let $M = \mathcal{C}/\sim$ where $\mathcal{C}$ is the set of constant symbols and $\sim$ is the equivalence relation defined by $c_1 \sim c_2 \leftrightarrow c_1 = c_2 \in T$
- $R^{\mathcal{M}} \subseteq M^{n_R}$ by $(c_1^*, \ldots, c_{n_R}^*) \in R^{\mathcal{M}} \leftrightarrow R(c_1, \ldots, c_n) \in T$ where c^* equivalence class of c
  This is well defined since if we have $c_1' \sim c_1, \ldots, c_n' \sim c_n, R(c_1, \ldots, c_n) \in T$ then $R(c_1', \ldots, c_n') \in T$

- $f^{\mathcal{M}}$ by $f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \leftrightarrow f(c_1, \ldots, c_n) = d \in T$. SUch a d^* exists since T has the witness property: $\exists v f(c_1, \ldots, c_n) = v \to f(c_1, \ldots, c_n) \in T$

- $c^{\mathcal{M}} := c^*$

Claim: For every formula $\phi(v_1, \ldots, v_k)$ and constant symbols $c_1, \ldots, c_k$, $\mathcal{M} \models \phi(c_1^*, \ldots, c_n^*) \leftrightarrow \phi(c_1, \ldots, c_n) \in T$
This implies $\mathcal{M} \models T$

Proof. By induction on formulas $\phi(v_1, \ldots, v_l)$

- atomic formulas: $\phi(v_1, \ldots, v_k)$ is $t_1(v_1, \ldots, v_k) = t_2(v_1, \ldots, v_k)$
  Subclaim: $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = c^* \leftrightarrow t(c_1 \ldots, c_n) = c \in T$
  Proved by induction on terms

- $\phi(v_1, \ldots, v_k)$ is $R(v_1, \ldots, v_k)$. Follows by deifnition of $R^{\mathcal{M}}$

- Suppose $\phi(\overline{v})$ is $\psi_1(\overline{v}) \wedge \psi_2(\overline{v})$, then
  $\mathcal{M} \models \psi_1 \wedge \psi_2(\overline{v}) \leftrightarrow \mathcal{M} \models \psi_1(\overline{v})$ and $\mathcal{M} \models \psi_2(\overline{v}) \overset{\text{IH}}{\leftrightarrow} \psi_1(\overline{c}) \in T$ and $\psi_2(\overline{c}) \in T \overset{\text{lemma}}{\leftrightarrow} \psi_1 \wedge \psi_2(\overline{c}) \in T$

- Suppose $\phi(\overline{v})$ is $\neg\psi(\overline{v})$, then
  $\mathcal{M} \models \neg\psi(\overline{c}^*) \leftrightarrow \mathcal{M} \not\models \psi(\overline{c}^*) \overset{\text{IH}}{\leftrightarrow} \varphi(\overline{c}) \notin T \overset{\text{maximality}}{\leftrightarrow} \neg\psi(\overline{(c)}) \in T$

- Suppose $phi(\overline{v})$ is $\exists w \varphi(\overline{v}, w)$, then
  $\mathcal{M} \models \exists w \varphi(\overline{c}^*, w) \leftrightarrow \exists d \in M$ such that $\mathcal{M} \models \phi(\overline{c}^*, d) \leftrightarrow \exists d \in M$ such that $\varphi(\overline{c}, d) \in T \overset{\text{witness principle}}{\leftrightarrow}$ $\exists w \varphi(\overline{c} w) \in T$

2.2 September 1

2.2.1 Compactness

Proof of Compactness continued:
We now prove proposition 2
Lemma 1: If T is finitely satisfiable then there is $\mathcal{L}^* \supset \mathcal{L}$, $T^* \supset T$ such that T^* has the witness property and is finitely satisfiable

Proof. For each $\mathcal{L}$-formula define a new constant symbol c_ϕ. Let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v)\, \mathcal{L} - \text{formula}\}$, $T_1 = T \cup \{\exists v \phi(v) \to \phi(c_\phi) : \phi(v)\, \mathcal{L} - \text{formula}\}$.
Claim: T_1 is finitely satisfiable.
Take $\Delta \subseteq T_1$ finite. $\Delta = T' \cup \{\exists v \phi_i(v) \to c_{\phi_i} : i = 1, \ldots, k\}$ for finite T' in T. We make an $\mathcal{L}_1$-structure $\mathcal{M}_1$ that satisfies Δ. Take $\mathcal{M} \models T'$, $\mathcal{M}$ $\mathcal{L}$-structure. Make $\mathcal{M}$ an $\mathcal{L}_1$-structure by defining $c_\phi^{\mathcal{M}_1}$ for each c_ϕ. If $\mathcal{M} \models \exists v \phi(v)$ let $c^{\mathcal{M}_1}$ be such a v otherwise let $c^{\mathcal{M}_1}$ be anything.
We repeat this process, defining $\mathcal{L}_{n+1}$ from $\mathcal{L}_n$ similarly.
We have $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \cdots$, $T \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq \cdots$ such that each T_i is finitely satisfiable and for $\phi(v)$ an $\mathcal{L}_{i-1}$-formula, there is c_ϕ in $\mathcal{L}_i$ such that $\exists v \phi(v) \to \phi(c_\phi) \in T_i$.
Let $\mathcal{L}^* = \bigcup_{n \in \omega} \mathcal{L}_n$, $T^* = \bigcup_{n \in \omega} T_n$. We see T^* has the witness property.
Sub-claim: If $T_0 \subset T_1 \subset T_2 \subset \cdots$ all finitely satisfiable, then $U_{n \in \omega} T_n$ is finitely satisfiable.

Lemma 2: If T is finitely satisfiable and ϕ a sentence, one of $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable.

Proof. Assume that both $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are not finitely satisfiable. Then there are $T_0, T_1 \subseteq T$ such that $T_0 \cup \{\phi\}$ and $T_1 \cup \{\neg\phi\}$ are not satisfiable. Let $\mathcal{M} \models T_0 \cup T_1$, then $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$ so $T_0 \cup \{\phi\}$ or $T_1 \cup \{\neg\phi\}$ is satisfiable, contradicting our assumption.

Zorn's Lemma: Let $\mathcal{A}$ be a collection of sets such that for any chain $\mathcal{C} \in \mathcal{A}$. $\bigcup \mathcal{C} \in \mathcal{A}$ where $\mathcal{C}$ is a chain if for $A, B \in \mathcal{C}$ either $A \subseteq B$ or $B \subseteq A$, then $\mathcal{A}$ has a maximal element, eg. $A \in \mathcal{A}$ such that there is not $B \in \mathcal{A}$ with $A \subsetneq B$.

Lemma: For every T, finitely satisfiable, there is $T' \supseteq T$ that is maximal and finitely satisfiable.

Proof. Let $\mathcal{A} = \{S\ \mathcal{L}\text{-theory} \mid S \supseteq T,\ S \text{ finitely satisfiable}\}$. Can apply zorns lemma since for any $\mathcal{C} \subseteq A$, $\bigcup \mathcal{C} \in \mathcal{A}$ so we have a maximal S.

Example 2.2.1. Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. In a group G, $g \in G$, ord$g$ = least n such that $\overbrace{g \cdots g}^{n \text{ times}} = e$, if it exists.
Observation: If T is an $\mathcal{L}$-theory extending the axioms of groups, $\phi(v)$ such that for every n there is $G_n \models T$, $g_n \in G_n$ of order greater than n such that $G_n \models \phi(g_n)$. Then there is $G \models T$ and $g \in G$, $\text{ord}(g) = \infty$ such that $G \models \phi(g)$.

Proof. Let $\mathcal{L}' = \{\cdot, e, c\}$. Let $T^* = T \cup \phi(c) \cup \{\psi_n\}$ where ψ_n is $\underbrace{c \cdot c}_{n \text{ times}} \neq e$. T^* finitely satisfiable so follows by compactness.

This tells us that there is not sentence that axiomatizes when an element is torsion.

Lemma 2.2.2. Let κ be a carindal $\kappa \geqslant |\mathcal{L}|$. Let T be a satisfiable theory such that $\forall n \in \mathbb{N}$, there is $\mathcal{M} \models T$ such that $|\mathcal{M}| > n$. Then T has a model of size κ.

Proof. Extend the language by adding κ may new constant symbols c_i for $i \in \kappa$. $T^* = T \cup \{c_i \neq c_j \mid i \neq j\}$. If $\mathcal{M} \models T^*$, $|\mathcal{M}| \geqslant \kappa$. T^* is finitely satisfiable so by compactness T^* has a model $\mathcal{M}$, $|\mathcal{M}| \leqslant |\mathcal{L}^*| + \aleph_0 = \kappa$. Thus, $|M| = \kappa$.

2.3 September 6

2.3.1 Complete Theories

Definition 2.3.1. Let κ be an infinite cardinal. A theory T is κ-categorical if all models of T of size κ are isomorphic (and there is at least one).

Example 2.3.2. The theory of torsion free abelian division groups (TFADG) is κ categorical for all uncountable κ.

Language $= \{\cdot, e\}$, TFADG = group axioms, commutativity, torsion free - $\forall a \neq e\ \overbrace{a \cdot a \cdots a}^{n} \neq e$ for $n \in \omega$,

divisible - $\forall a \exists b \, \overbrace{b + b + \cdots + b}^{n}$ for each $n \in \omega$
Observation: TFADG are essentialy $\mathbb{Q}$-vector spaces
For $a \in G$, $n \in \mathbb{N}$ $a \cdot n = \overbrace{a + \cdots + a}^{n \text{ times}}$ $\frac{a}{n}$ is b such that $b \cdot n = a$. Such a b exists since the group is division and is uniquely defined since if $b \cdot n = a = b' \cdot n$, $(b - b') \cdot n = 0$ so since the group is torsion free, $b - b' = 0$. For $a \in G$, $\frac{p}{q} \in \mathbb{Q}$ we define $a \cdot \frac{p}{q} = \frac{a}{q} \cdot p$
Two vector $\mathbb{Q}$-vector spaces are isomorphic $\leftrightarrow$ they have the same dimension. A $\mathbb{Q}$ vector space of size κ must have dimension κ so two $\mathbb{Q}$ vector spaces of size κ must be isomorphic.

Let ACF_p be the theory of algebraicly closed fields of characteristic p.
Language $= \{0, 1, +, \times\}$. ACF_P: field axioms, char p - $\underbrace{1 + \cdots + 1}_{p} = 0$, char 0 - $\underbrace{1 + \cdots + 1}_{n} \neq 0$ for $n \in \omega$, algebraicly closed - every non-constant polynomial has at least one root: for degree n, $\forall z_0, z_1, \ldots, z_n \, z_n \neq 0 \, \exists x (z_n x^n + z_{n-1} x^{n-1} + \cdots + z_0 = 0)$. For each $n \in \omega$

Proposition 2.3.3. ACF is κ categorical for all uncountable κ.

Facts and Definitions

- Every fielf F has a prime subfield $P = \{\frac{\overbrace{1 + \cdots + 1}^{p}}{\underbrace{1 + \cdots + 1}_{q}} : p \in \mathbb{Z}, q \in \mathbb{N}\}$
  - if F has char $p > 0$, then the prime subfield is $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$
  - If F has char $o = 0$, then the prime subfield in $\mathbb{Q}$
- An element $a \in F$ is algebraic if there is a polynomial $p(x) \in P[x]$ such that $p(x) = 0$. (Can think of as a polynomial in $\mathbb{Z}[x]$)
- Otherwise a is transcendental
- A tuple $\overline{a}$ is algebraicly independent if there is no nontrivial polynomial $p(\overline{x}) \in P[x]$ such that $p(\overline{x}) = 0$.
- the transcendence degree of a field F is the size fo a maximal algebraicly independent set.
- Algebraicly closed fields are isomorphic $\leftrightarrow$ they have the same transcendence degree.

Observation: an ACF_p of size κ must have transcendence degree κ
If $M \subset F$ is a maximal algebraicly independent set, $\forall a \in F$ there is a polynomial $p(\overline{x}, y) \in P[\overline{x}, y]$ and $\overline{m} \in M$ such that $p(\overline{m}, a) = 0$.

Definition 2.3.4. A theory T is complete if for all $\mathcal{L}$-sentences, ϕ either $T \models \phi$ or $T \models \neg\phi$

Theorem 2.3.5 (Vaught's Test)**.** If T is satisfiable and has no finite models and is κ-categorical for $\kappa > |\mathcal{L}|$, then T is complete.

Corollary 2.3.6. ALL ACF_p satisfy the same sentences.

Proof. Suppose not. There is ϕ such taht $T \not\models \phi$, $T \not\models \neg\phi$ so $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are satisfiable. Both have models of size κ, contradicting κ-categoricity.

Definition 2.3.7. T is decidable if there is an algorithm to decdide $T \models \phi$ given ϕ

Observation: If T is computably enumerable and complete then T is decidable

Corollary 2.3.8. $\mathrm{Th}(\mathbb{C}; 0, 1, +, \times)$ is decidable.

2.4 September 8

2.4.1 Complete Theories

Observation: Let f be a function : $k \to k$. If f is one to one then f is onto, provided k is finite.

Theorem 2.4.1. Every injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective.
(A polynomial map consists of n polynomials $p_1[x_1, \ldots, x_n], \ldots, p_n[x_1, \ldots, x_n] \in \mathbb{C}[x]$)

Lemma 2.4.2. Let *phi* be a senctence in the language $\{0, 1, +, \times\}$. TFAE

1. $C \models \phi$
2. ϕ is true in any algebraically closed field of characteristic 0.
3. ϕ is true in some algebraically closed field of characteristic 0.
4. There are arbitrarily large primes p such that ϕ is true in some $F \models \mathrm{ACF}_p$
5. There is an $m \in \mathbb{N}$ such that for all $p \geqslant n$ and all $F \models \mathrm{ACF}_p$, $F \models \phi$

Proof. (1), (2), (3) equivalent since ACF_0 is complete. (4) $\to$ (5) clear.
(2) $\to$ (5) $\mathrm{ACF}_0 \models \phi$. There is finite $\Delta \subseteq \mathrm{ACF}_0$ such that $\Delta \models \phi$. If $p \geqslant n$ for an all n such that "$\overbrace{1 + \cdot + 1} \neq 0$" shows up in Δ, then if $F \models \mathrm{ACF}_p$, $F \models \Delta$ so $f \models \phi$
(4) $\to$ (3) If (3) was false, $\mathrm{ACF}_0 \models\neq \phi$ and for some n, all $p > n$, if $F \models \mathrm{ACF}_p$ then $F \models \neg\phi$ so (4) is false.

Claim: Every injective polynomial function $f : (\mathbb{F}_p^{\mathrm{alg}})^n \to (\mathbb{F}_p^{\mathrm{alg}})^n$ is onto
where $\mathbb{F}_p^{\mathrm{alg}}$ is the algebraic closure of $\mathbb{F}_p : \mathbb{Z}/p\mathbb{Z}$. $\mathbb{F}_p^{\mathrm{alg}} = \bigcup_{n \in \mathbb{N}} \mathbb{F}_{p^n}$ where $\mathbb{F}_{p^n}$ is the unique field of size p^n.

For every polynomial $p(\overline{x}) \in F$ there is an atomic $t(\overline{x}, \overline{z})$ and parameters $\overline{c} \in F$ such that $p(\overline{x}) = t(\overline{x}, \overline{c})$ so $t_1(\overline{x}, \overline{c}), \ldots, t_n(\overline{x}, \overline{c})$ for $\overline{c} \in \mathbb{F}_p^{\mathrm{alg}}$, $\overline{x} = x_1, \ldots, x_n$
Claim states $\forall \overline{z}(\forall \overline{x} \forall \overline{y} \bigwedge_{i=1}^n t_i(x_i, z) = t_i(y_1, z) \to \overline{x} = \overline{y}) \to (\forall \overline{w} \exists \overline{x} \bigwedge_{i=1}^n t_i(\overline{x}, z) = w_i)$

Proof (Pf of Claim)**.** Take $\overline{b} \in (\mathbb{F}_p^{\text{alg}})^n$, want to show $\overline{b}$ is in the range of f
Let k be the finite subfield of $\mathbb{F}_p^{\text{alg}}$ generated by $\overline{c}$ and $\overline{b}$. $\mathbb{F}_p(\overline{c}, \overline{d})$
Restricting f to k^n, we get a one to one function from k^n to k^n so $f \upharpoonright k^n$ is onto so $\overline{b}$ is in the range of f

2.4.2 Up and Down

Definition 2.4.3. A map $j : \mathcal{M} \to \mathcal{N}$ is an elementary embedding if for all formulas $\phi(\overline{x})$, all $m \in M$

$$\mathcal{M} \models \phi(\overline{m}) \leftrightarrow \mathcal{N} \models \phi(j(\overline{m}))$$

Definition 2.4.4. If for $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is an elementary subset of $\mathcal{N}$ if $i : M \hookrightarrow N$ is elementary ($\mathcal{M} \preceq \mathcal{N}$)

- $\mathbb{Q}^{\text{alg}} \preceq \mathbb{C}$, $(\mathbb{Q}, 0, +) \preceq (\mathbb{R}, 0, +)$

Definition 2.4.5. Given $\mathcal{M}$, let $\mathcal{L}_M = \mathcal{L} \cup \{c_m \,|\, m \in M\}$. $\mathcal{M}$ can be made into an $\mathcal{L}_m$-structure $\mathcal{M}^*$ by letting $c_m^{\mathcal{M}^*} = m$

Definition 2.4.6. Diag($\mathcal{M}$) the atomic diagram of $\mathcal{M} = \{\phi \,|\, \phi$ atomic $\mathcal{L}_M$ sentence such that $\mathcal{M} \models \phi\} \cup \{\neg\phi \,|\, \phi$ atomic $\mathcal{L}_M$ sentence such that $\mathcal{M} \models \neg\phi\}$
This is equivalent to $\{\phi \,|\, \phi$ is an $\mathcal{L}$-formula $\mathcal{M} \models \phi\}$
$\text{Diag}_{\text{el}}(\mathcal{M})$, the elementary diagram of $\mathcal{M}$ is $\{\phi \,|\, \phi$ is an $\mathcal{L}$formula $\mathcal{M} \models \phi\}$

Lemma 2.4.7. (i) if $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$ then there is an $\mathcal{L}$-embedding $\mathcal{M} \to \mathcal{N}$ (where $\mathcal{N}$ the restriction of $\mathcal{N}^*$ to $\mathcal{L}$)

Proof. Suppose $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$. If $\phi(\overline{x})$ is an $\mathcal{L}$ formula and $\overline{c_m}$ new constants, we can give an embedding by $m \mapsto c_m^{\mathcal{M}^*}$
$\mathcal{M} \models \phi(\overline{m}) \leftrightarrow \mathcal{M}^* \models \phi(\overline{c_m}) \leftrightarrow \mathcal{N}^* \models \phi(\overline{c_m}) \leftrightarrow \mathcal{N} \models \phi(\overline{m})$

Example 2.4.8. $\mathcal{M} = (\mathbb{Z}, +)$, $\mathcal{L} = \{*\}$, $\mathcal{L}_M = \{*, c_0, c_1, c_2, \ldots, c_{-1}, c_{-2}, \ldots\}$, in $\mathcal{M}^*$, $c_n^{\mathcal{M}^*} = n$
$\mathcal{N} = (\mathbb{R}, \times)$, define $\mathcal{N}^*$ by $c_n^{\mathcal{N}^*} = 2^n$. $\mathcal{N}^* = (\mathbb{R}, \times, c_n \mapsto 2^n)$
$\mathcal{N}^* \models \text{Diag}(\mathcal{M})$ size $(\mathbb{Z}, +) \to (\mathbb{R}, \times)$ by $n \mapsto 2^n$ is an embedding.
If $j : \mathcal{M} \to \mathcal{N}$ is an embedding, let $c_m^{\mathcal{M}^*} = j(m)$. Then $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$

2.5 September 13

2.5.1 Up and Down

Definition 2.5.1. $\mathcal{L}^- \subseteq \mathcal{L}$, $\mathcal{M}$ is an $\mathcal{L}$-stucture, then $\mathcal{L}^-$ reduct of $\mathcal{M}$ is the $\mathcal{L}^-$ stucture with the same domain and $\mathcal{L}^-$ interpretations of $\mathcal{M}$. We say that $\mathcal{M}^-$ is a reduction of $\mathcal{M}$, $\mathcal{M}$ is an expansion of $\mathcal{M}^-$

Lemma 2.5.2. Consider $\mathcal{L}$ structures $\mathcal{M}, \mathcal{N}$

1. there is an embedding $\mathcal{M} \to \mathcal{N} \leftrightarrow$ there is an $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ such that $\mathcal{N}^* \models \mathrm{Diag}(\mathcal{M})$
2. there is an elementary embedding $\mathcal{M} \to \mathcal{N} \leftrightarrow$ there is an $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ such that $\mathcal{N}^* \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$

Here $\mathcal{N}^* = (\mathcal{N}, c_m^{\mathcal{N}} \in N \text{ for } m \in M)$

Proof. $\to$) Suppose $f : \mathcal{M} \to \mathcal{N}$ is an embeeding. We need to find a $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ by defining $c_m^{\mathcal{N}}$ for $m \in M$ such that for all $\mathcal{L}$-formulas $\varphi(\overline{x})$, all $\overline{m} \in M$, if $\varphi(\overline{c_m}) \in \mathrm{Diag}(M) \to \mathcal{N}^* \models \varphi(\overline{c_m})$. Let $c_m^{\mathcal{N}} = f(m)$ so $\varphi(\overline{c_m}) \in \mathrm{Diag}(\mathcal{M}) \leftrightarrow \mathcal{M} \models \varphi(\overline{m}) \leftrightarrow \mathcal{N} \models \varphi(f(\overline{m})) \leftrightarrow \mathcal{N}^* \models \varphi(c_m^{\mathcal{N}})$
$\leftarrow$) Given the $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ such that $\mathcal{N}^* \models \mathrm{Diag}(\mathcal{M})$. Let $f : \mathcal{M} \to \mathcal{N}$ by $f(m) = c_m^{\mathcal{N}^*}$

Theorem 2.5.3 (Upwards Lowenheim-Skolem). Let $\mathcal{M}$ be an infinite $\mathcal{L}$-strucutre. For every $\kappa \geqslant |M| + |\mathcal{L}|$ there is an $\mathcal{L}$-strucure $\mathcal{N}$ such that $|\mathcal{N}| = \kappa$ and $\mathcal{M} \preceq \mathcal{N}$.

Proof. It suffices to show there is an elementary embedding $j : \mathcal{M} \to \mathcal{N}$ as $\mathcal{M}$ can be identified with its image. Lt $\mathcal{N}^*$ be a model of $\mathrm{Diag}(\mathcal{M})$ of size κ. Let $\mathcal{N}$ be the $\mathcal{L}$-reduct of $\mathcal{N}^*$

Example 2.5.4. $(\mathbb{Q}, 0, +) \preceq (\mathbb{R}, 0,_)$ κ-categorical so there is only structure of size $2^{\aleph_0}$ up to isomorphism

Example 2.5.5. $(\mathbb{Q}^{\deg}, 0, 1, +, \times) \preceq (\mathbb{C}, 0, 1, +, \times)$

Theorem 2.5.6 (Downward Lowenheim-Skolem). Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. For all $X \subseteq M$, there is an $\mathcal{L}$ structure $\mathcal{N} \subseteq \mathcal{M}$, $|\mathcal{N}| = |X| + |\mathcal{L}| + \aleph_0$ and $\mathcal{N} \preceq \mathcal{M}$

Proposition 2.5.7 (Tarski-Vaught Test). Suppose $\mathcal{N} \subseteq \mathcal{M}$. Then $\mathcal{N} \preceq \mathcal{M} \leftrightarrow$ formulas $\phi(\overline{v}, w)$ and all $\overline{n} \in N$ if $\mathcal{M} \models \exists \phi(\overline{n}, w)$ then there is $c \in N$ such that $\mathcal{M} \models \phi(\overline{n}, c)$.

Proof. $\to$) Assume $\mathcal{N} \preceq \mathcal{M}$, $\mathcal{M} \models \exists w \phi(\overline{n}, w)$ then $\mathcal{N} \models \exists w \phi(\overline{n}, w)$ so there is $c \in N$ such taht $\mathcal{N} \models \phi(\overline{n}, c)$ so $\mathcal{M} \models \phi(\overline{n}, c)$
$\leftarrow$) We use induction on $\mathcal{L}$-formulas to show that for all formulas $\psi(\overline{x})$ and all $\overline{n}$, $\mathcal{N} \models \psi(\overline{n}) \leftrightarrow \mathcal{M} \models \psi(\overline{n})$

- For ψ atomic, this follows since $\mathcal{N} \subseteq \mathcal{M}$
- For $\psi = \psi_1 \wedge \psi_2$, $\neg \psi_1$ clear by applying IH
- For $\psi(\overline{x})$ of the form $\exists \phi(\overline{x}, w)$, pick $\overline{n} \in N$, $\mathcal{M} \models \psi(\overline{n}) \leftrightarrow \mathcal{M} \models \exists w \phi(\overline{n}, w) \leftrightarrow$ there is $c \in N$ such that $\mathcal{M} \models \phi(\overline{n}, c) \overset{\text{IH}}{\leftrightarrow}$ there is $c \in N$ such that $\mathcal{N} \models \phi(\overline{n}, c) \leftrightarrow \mathcal{N} \models \exists w \phi(\overline{n}, w) \leftrightarrow \mathcal{N} \models \psi(\overline{n})$.

Proof (Proof of Lowenheim Skolem). Let $X = X_0$. For any $\overline{n} \in X$ and $\varphi(\overline{v}, w)$ if $\mathcal{M} \models \exists w \varphi(\overline{n}, w)$. let $c_{\overline{n}, \varphi} \in m$ such that $\mathcal{M} \models \phi(\overline{n}, c_{\overline{n}, \varphi})$. Let $X_1 = \{c_{\overline{n}, \varphi} | \varphi\, \mathcal{L} \text{ forumula }, \overline{n} \in X_0, \mathcal{M} \models \exists w \varphi(\overline{n}, w)\} \cup X_0$
We can define X_{n+1} from X_n similiarly and let $N = \bigcup_{i \in \omega} X_i$

$|X_1| = (\#\ \mathcal{L}$ forumas$) \times (\#$ terms $X_0) = (|\mathcal{L}| + \aleph_0) \times (|X_0|)$
Since $|\mathcal{N}| \leqslant |\mathcal{L}| + |\aleph_0| + |X_0|$, then $|X| \leqslant |\mathcal{N}| \leqslant |X| + |\mathcal{L}| + \aleph_0$.
We define $\mathcal{N}$ with domain N by restricting functions, relations, and constants from $\mathcal{M}$. If $\varphi(\overline{x}, w)$ is the formula $f(\overline{x}) = w$ and $\overline{n} \in X$, $\mathcal{M} \models \exists w f(\overline{m}) = w$ in X_{i+1} so $c_{\varphi,n}$ satisfies $f(\overline{n}) = c_{\varphi.n}$

2.6 September 15

2.6.1 Universal Axiomatizations

Example 2.6.1. Consider $\mathcal{M} = (\mathbb{Z}, 0, +)$, $\mathcal{N} = (2\mathbb{Z}, 0, +)$, $\mathcal{N} \subset \mathcal{M}$, $\mathcal{N} \equiv \mathcal{M}$ but $\mathcal{N} \npreceq \mathcal{M}$. Consider $\varphi(x) = \exists y(y + y = z)$. $\mathcal{M} \models \varphi(2)$, $\mathcal{N} \models \neg\varphi(2)$.
We have $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$ but $\mathcal{N} \not\models \mathrm{Diag}_{\mathrm{el}}(\mathcal{N})$.

Definition 2.6.2. A universal formula is of the form $\forall x_1, \ldots, x_n \varphi(x_1, \ldots, x_n, \overbrace{\ y\ })$ where ψ is quantifier free.

Observation: If $\mathcal{M} \subseteq \mathcal{N}$ and $\varphi(\overline{x})$ is a univeral formulas, $\overline{m} \in M$, if $\mathcal{N} \models \varphi(\overline{m})$, then $\mathcal{M} \models \varphi(\overline{n})$

Definition 2.6.3. T has a univeral axiomatization if there is a set of universal sentences Γ such that $T \models \Gamma$ and $\Gamma \models T$

Observation: If T has a universalthen if $\mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \models T$

Example 2.6.4. Group axioms, if $\mathcal{L} = \{\cdot, e\}$, not universal, $(\mathbb{N}, 0, +) \subseteq (\mathbb{Z}, 0, +)$ but is not a group.
If we consider $\mathcal{L} = \{\cdot, e, (\cdot)^{-1}\}$, universal, $\forall x(x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e)$

Theorem 2.6.5. If T is such that $\forall \mathcal{M} \subseteq \mathcal{N}(\mathcal{N} \models T \to \mathcal{M} \models T)$, then T has a universal axiomatization.

Proof. Let $\Gamma\{\varphi$ universal $\mid T \models \varphi\}$. Clearly $T \models \Gamma$, want to show $\Gamma \models T$. Suppose $\mathcal{M} \models \Gamma$, we want to show $\mathcal{M} \models T$, We want $\mathcal{N} \supseteq \mathcal{M}$ such that $\mathcal{N} \models T$. $\mathcal{N} \supseteq \mathcal{M} \leftrightarrow \mathcal{N} \models \mathrm{Diag}(\mathcal{M})$ so want $\mathrm{Diag}(\mathcal{M}) \cup T$ is satisfiable.
Claim: $T \cup \mathrm{Diag}(\mathcal{M})$ is satisfiable.
Let $\Delta \subseteq T \cup \mathrm{Diag}(\mathcal{M})$ be finite. $\Delta = T_0 \cup \{\phi_1(\overline{c_m}), \ldots, \phi_k(\overline{c_m})$. Can assume only one formula ϕ (can take the conjugation) so ϕ is quantifier free such that $\mathcal{M} \models \phi(\overline{c_m})$. $\mathcal{M} \models \phi(\overline{m}) \to \mathcal{M} \not\models \forall \overline{v} \neg \phi(\overline{v}) \to T \not\models \forall \overline{v} \neg \phi(\overline{v})$ so $T \cup \{\exists \overline{v} \phi(\overline{v})\}$ is satisfiable. Thus, $T \cup \{\phi(\overline{c_m})\}$ is satisfiable since if $\mathcal{A} \models \exists v \phi(v)$, for some $\overline{a} \in A$, $\mathcal{A} \models \phi(\overline{a})$ so let $\overline{c_m} = \overline{a}$. $(\mathcal{A}, \overline{c_m} \mapsto \overline{a}) \models \phi(\overline{c_m})$

- If $\overline{c}$ does not occur in T, ϕ, then $T \cup \{\exists \overline{v} \phi(\overline{v})$ is satisfiable $\to T \cup \phi(\overline{c})$ is satisfiable. Equivalently, $T \models \psi(\overline{c}) \to T \models \forall \overline{v} \psi(\overline{v})$

Suppose $(I, <)$ is a linear order. For each $i \in I$, $\mathcal{M}_i$ is an $\mathcal{L}$-structure, $\forall i < j\ \mathcal{M}_i \subseteq \mathcal{M}_j$ is called a chain (elementary chain if $\mathcal{M}_i \preceq \mathcal{M}_j$). Let $\mathcal{M} = \bigcup_{i \in} I \mathcal{M}_i$, $M = \bigcup_{i \in I} M_i$, $f^{\mathcal{M}} = \bigcup_{i \in I} f^{\mathcal{M}_i}$

Proposition 2.6.6. If $(\mathcal{M}_i : i \in I)$ is an elementary chain, $\forall i\, \mathcal{M}_i \preceq \mathcal{M}$

Proof. Use induction on formulas $\phi(\overline{v})$ to show that $\forall i$, $\forall m \in \mathcal{M}_i$, $\mathcal{M}_i \models \phi(\overline{m}) \leftrightarrow \mathcal{M} \models \phi(\overline{m})$

- ϕ quantifier free true since substructure
- ϕ is $\neg\psi, \psi_1 \wedge \psi_2$ clear by induction
- $\phi(\overline{x})$ is $\exists v\psi(\overline{x}, v)$ $\mathcal{M} \models \exists v\psi(\overline{x}, v) \leftrightarrow \exists n \in \mathcal{M}_j$ for some $j \in I$ such that $\mathcal{M} \models \psi(\overline{m}, n)$ $\overset{\text{IH}}{\leftrightarrow} \mathcal{M}_j \models \psi(\overline{x}, n) \leftrightarrow \mathcal{M}_j \models \exists v\phi(\overline{x}, v) \overset{M_i \preceq M_j}{\leftrightarrow} \mathcal{M}_i \models \exists v\phi(\overline{m}, v)$

2.7 September 20

2.7.1 Ultrafilters

Definition 2.7.1. A filter on I is a subset $\mathcal{D} \subseteq \mathcal{P}(I)$ such that

(i) $\varnothing \notin \mathcal{D}$, $I \in \mathcal{D}$

(ii) If $A \in cD$, $B \supseteq A \to B \in \mathcal{D}$

(iii) if $A, B \in \mathcal{D}$, then $A \cap B \in \mathcal{D}$

Example 2.7.2. (a) $I = \mathbb{R}$, $\mathcal{D} = \{X \subseteq \mathbb{R} \,|\, X \text{ has full measure }\}$ eg. $\lambda(\mathbb{R} \backslash X) = 0$

(b) $I = \mathbb{R}$, $\mathcal{D} = \{X \subseteq \,|\, X \text{ is meager }\}$

(c) For $\kappa \leqslant |I|$, $\mathcal{D} = \{X \subseteq I \,|\, |I \backslash X| < \kappa\}$
For $\kappa = \aleph_0$, $\mathcal{D}$ is called the Frechet filter or the cofinite filter

(d) For $x \in I$, $\mathcal{D} = \{X \subseteq I \,|\, x \in X\}$ called principle filter

(e) For $I = \mathbb{N}$, $\{X \subseteq N \ \lim_{n\to\infty} \frac{|X \cap n|}{n} = 1\}$

Definition 2.7.3. $\mathcal{D}$ is an ultrafilter if it is a filter and for all $X \subseteq I$, either $X \in \mathcal{D}$ or $X^C \in \mathcal{D}$

- principle filters are ultrafilters

Observataion: If $\mathcal{U}$ is an ultra filter, $A \cup B \in \mathcal{U} \leftrightarrow A \in \mathcal{U}$ or $B \in \mathcal{U}$
If $A, B \notin \mathcal{U}$, $A^C, B^C \in \mathcal{U}$ so $A^C \cap B^C \in \mathcal{U}$ so $(A^C \cap B^C)^C = A \cup B \notin \mathcal{U}$
Similarly, $C \cap D \notin \mathcal{U} \leftrightarrow C \notin \mathcal{U}$ and $D \notin \mathcal{U}$

Theorem 2.7.4. Every filter $\mathcal{D}$ on I can be extended to an ultrafilter

Proof. Let $\mathcal{A} = \{\mathcal{F} \subseteq \mathcal{P}(I) \,|\, \mathcal{F} \text{ filter and } \mathcal{D} \subseteq \mathcal{F}\}$. To apply Zorn's lemma to get a maximal $\mathcal{U}$ in $\mathcal{A}$, we need to show if $\mathcal{C} \subseteq \mathcal{A}$ is a chain then $\bigcup\mathcal{C} \in \mathcal{A}$. Clear that $\varnothing \notin \bigcup\mathcal{C}, I \in \bigcup\mathcal{C}$, and closed upwards. For $A, B \in \bigcup\mathcal{C}$, $\exists\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $A \in \mathcal{F}, B \in \mathcal{F}'$. WLOG assume $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}' \subseteq \bigcup\mathcal{C}$
Suppose $\mathcal{U}$ is a maximal filter. We show that it is an ultrafilter. Take $X \subseteq I$. We show $X \in \mathcal{U}$ or $X^C \in \mathcal{U}$. Suppose not. Let $\mathcal{D}_0$ = filter generated by $X, \mathcal{U} = \{Y \,|\, \exists V \in \mathcal{U} \, Y \supseteq V \cap X\}$, $\mathcal{D}_1 = \{Z \,|\, \exists W \in \mathcal{U} \, Z \supseteq W \cap X^C\}$. $\mathcal{D}_0, \mathcal{D}_1$ satsify all conditions except we dont know if they contain $\varnothing$. If $\varnothing \in \mathcal{D}_0, \mathcal{D}_1$, there is

$V \in \mathcal{U}$ $V \cap X = \varnothing$, $W \in \mathcal{U}$ $W \cap X^C = \varnothing$ so $V \subseteq X^C, W \subseteq X$ so $V \cap W = \varnothing$, contradicting $V, W \in \mathcal{U}$

To get a nonprinciple ultrafilter take $\mathcal{D} = \{X \subseteq I \mid I \backslash X \text{ finte }\}$ and extend to ultrafilter $\supseteq \mathcal{D}$
Observataion: $\forall x \in I$, $I \backslash \{x\} \in \mathcal{D} \subseteq \mathcal{U}$ so $\{x\} \notin \mathcal{U}$
An ultrafilter is not principle $\leftrightarrow \mathcal{U} \supseteq$ Frechet filter
Observataion: If $\mathcal{U}$ is an ultrafilter and contains a finite set $\mathcal{A} = \{a_0, \dots, a_n\}$ then $\mathcal{U}$ is principle since $\mathcal{A} = \{a_0\} \cup \{a_1\} \cup \dots \cup \{a_n\}$

Definition 2.7.5 (Ultraproduct). I an infinte set, $\mathcal{U}$ an ultafilter of I, $\{\mathcal{M}_i : i \in I\}$ a collection of cL structures. Define $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ as follows:

- Given $g, h \in \prod_{i \in I} M_I$, $g \sim h$ iff $\{i \in I \mid g(i) = h(i)\} \in \mathcal{U}$. $M = \prod_{i \in I} M_i / \sim$
- $c^{\mathcal{M}} = [i \mapsto c^{\mathcal{M}_i}]$
- $f^{\mathcal{M}}(g_1, \dots, g_n) = [i \mapsto f^{\mathcal{M}_i}(g_1(i), \dots, g_n(i))]$
- $(g_1, \dots, g_n) \in R^{\mathcal{M}} \iff \{i \in I \mid (g_1(i), \dots, g_n(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U}$

Claims:

1. $\sim$ is an equivalence relation on $\prod_{i \in I} M_i$
   Reflexivity, symmetry clear. $g \sim h, h \sim f \rightarrow g \sim f$ since $\{i \mid g(i) = f(i)\} \supseteq \{i \mid g(i) = h(i)\} \cap \{i \mid h(i) = f(i)\}$
2. $f^{\mathcal{M}}$ is well defined.
   $g_1 \sim g_1', \dots, g_n \sim g_n' \rightarrow f^{\mathcal{M}}(g_1, \dots, g_n) = f^{\mathcal{M}}(g_1', \dots, g_n')$ since $\{i \mid f^{\mathcal{M}_i}(g_1(i), \dots, g_n(i)) = f^{\mathcal{M}_i}(g_1'(i), \dots, g_n'(i))\} \supseteq \bigcap_{j=1}^{n} \{i \mid g_j(i) = g_j'(i)\}$
3. $R^{\mathcal{M}}$ well defined for a similar reason.

Definition 2.7.6. The $\mathcal{U}$ ultrapower of $\mathcal{M}$ is $\prod \mathcal{M} / \mathcal{U}$

- $\mathcal{M} \preceq \prod \mathcal{M} / \mathcal{U}$

2.8 September 22

2.8.1 Ultrafilters

Theorem 2.8.1 (Los' Theorem). For every forumla $\varphi(v_1, \dots, v_k)$ and $g_1, \dots, g_k \in \prod_{i \in I} M_i$, $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$, $\mathcal{M} \models \phi([g_1], \dots, [g_n]) \leftrightarrow \{i \in I \mid \mathcal{M}_i \models \phi(g_1(i), \dots, g_k(i))\} \in \mathcal{U}$

Corollary 2.8.2. $\mathcal{M} \preceq \mathcal{M}^I / \mathcal{U}$ by $m \mapsto g_m$ where $g_m(i) = i$ $\forall i \in I$

Proof. By induction on formulas φ

- φ atomic. $([g_1], \dots, [g_n]) \in R^{\mathcal{M}} \iff \{i \in I \mid (g_1(i), \dots, g_n(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U}$ by definition.
  Similar for $=$

- φ is $\psi_1 \wedge \psi_2$. $\mathcal{M} \models \varphi([\overline{g}]) \leftrightarrow \mathcal{M} \models \psi_1[\overline{g}]$ and $\mathcal{M} \models \psi_2[\overline{g}] \overset{\text{IH}}{\leftrightarrow} \{i \mid \mathcal{M}_i \models \psi_1(\overline{g(i)})\} \in \mathcal{U}$ and $\{i \mid \mathcal{M}_i \models \psi_2(\overline{g(i)})\} \in \mathcal{U} \leftrightarrow \{i \mid \mathcal{M}_i \models \psi_1(\overline{g(i)})\} \cap \{i \mid \mathcal{M}_i \models \psi_2(\overline{g(i)})\} \in \mathcal{U} \leftrightarrow \{i \mid \mathcal{M}_i \models \psi_1(\overline{g(i)}) \wedge \psi_2(\overline{g(i)})\} \in \mathcal{U}$
  φ is $\psi_1 \vee \psi_2$ is similar
- φ is $\neg\psi$. $\mathcal{M} \models \varphi \leftrightarrow \mathcal{M} \not\models \psi \leftrightarrow \{i\, \mathcal{M}_i \models \psi\} \notin \mathcal{U} \leftrightarrow \{i \,|\mathcal{M}_i \models \varphi\} \in \mathcal{U}$
- $\varphi(\overline{v})$ is $\exists \overline{x} \psi(x, \overline{v})$. $\mathcal{M} \models \varphi[\overline{g}] \leftrightarrow$ there is $h \in M$ such that $\mathcal{M} \models \psi([h], [\overline{g}]) \overset{\text{IH}}{\leftrightarrow} \{i \mid \mathcal{M}_i \models \psi(h(i), \overline{g(i)})\} \in \mathcal{U}$ for some $h \leftrightarrow \{i \mid \mathcal{M}_i \models \exists x \psi(x, \overline{g(i)})\} \in \mathcal{U}$

Proof (Proof of Compactness). Let T be finitely satisfiable. For every $\Delta \subseteq T$ finite, there is $\mathcal{M}_\Delta \models \Delta$. Let $I = \{\Delta \subseteq I | \Delta \text{ finite }\}$. For $\Sigma \in I$, let $X_\Sigma = \{\Delta \subseteq I \mid \Sigma \subseteq \Delta\} \subseteq I$. Let $\mathcal{D} = \{Y \subseteq I \mid$ for some Σ, $Y \supseteq X_\Sigma\}$ (filter generated by $X_\Sigma' s$). Claim $\mathcal{D}$ is a filter, $\varnothing \notin \mathcal{D}, I \in \mathcal{D}$, closed upwards. $X_\Sigma \cap X_{\Sigma'} = X_{\Sigma \cup \Sigma'}$ so closed under intersection. Let $\mathcal{U} \supseteq \mathcal{D}$ be an ultrafilter. Let $\mathcal{M} = \prod_{\Delta \in I} \mathcal{M}_\Delta \,/cU$. For $\varphi \in T$, $X_{\{\varphi\}} \in \mathcal{U}$ and for all $\Delta \in X_{\{\varphi\}}$, $\mathcal{M}_\Delta \models \varphi$ so $\{\Delta \in I \mid \mathcal{M}_\Delta \models \varphi\} \supseteq X_{\{\varphi\}} \in \mathcal{U}$ so $\mathcal{M} \models \varphi$ by Los' thm.

2.8.2 Back and Forth Proofs

Example 2.8.3. DLO - dense linear orders without endpoints, $\mathcal{L} = \{\leqslant\}$
$(\mathbb{Q}, \leqslant), (\mathbb{R}, \leqslant), (\mathbb{R}^2, \text{lex}), (2^{<\omega})$ orderd by binary tree with ends removed.

Theorem 2.8.4 (Cantor)**.** DLO is $\aleph_0$ categorical, complete, and decidable. If $A, B \models$ DLO, countable then $A \equiv B$

Proof. Given $A = \{a_0, a_1, a_2, \ldots\}, B = \{b_0, b_1, b_2, \ldots\}$ we specify an isomorphism as follows.
Choose where to send a_0 arbitrarily, choose an element in A, not already chosen, to map to b_0 such that it respects order. At each step continue ensuring a_i is in the domain, b_i is in the range while preserving order. This is possible the ordering is dense and has no endpoints.

2.9 September 27

2.9.1 Random Graphs

Let $cL = \{R\}$, $T = R$ is symmetric, irreflexive, $\{\psi_n : n \in \omega\}$ where ψ_n is $\forall x_1 \cdots x_n \forall y_1 \cdots y_n (\bigwedge i < j < n x_i \neq x_j \wedge \exists z \bigwedge_{i \leqslant n} x_i R z \wedge \bigwedge_{j \leqslant n} \neg x_j R z \wedge \bigwedge_{j \leqslant n} x_j \neq z)$.
This theory is called the Rado graph or random graph.

Theorem 2.9.1. T is satisfiable and $\aleph_0$ categorical.

Proof. To construct a model we start with some finite set of points and at each step we add a new points for each finite subset, satisfying the axioms above. Take the union of all graphs generated in this way to get a graph which satisfies the theory.
Next, given two countable graphs we construct an isomorphism between step by step, ensuring all elements of one graph are in the domain and all elements are the other are in the range. For a given set of points, suppose we want to define the image of a new point. Since the points the new point is connected to is a finite subset, by the axioms of the graphs, there is another point in the other graph satisfying with the same connections. ☐

2.9.2 EhrerFeucht-Fraise Games

Fix two structures $\mathcal{M}, \mathcal{N}$. We define a game $G_\omega(\mathcal{M}, \mathcal{N})$. On move i, player I plays $m_i \in \mathcal{M}$ or $n_i \in \mathcal{N}$, player II responds with $n_i \in \mathcal{N}$ or $m_i \in \mathcal{M}$. If there is an $i \in \omega$, atomic formula $\varphi(\overline{v})$ such that $\mathcal{M} \models \varphi(m_0 \cdots m_i) \not\leftrightarrow \mathcal{N} \models \varphi(n_0 \cdots n_i)$, player I wins. Otherwise player II wins.

Example 2.9.2. PI: $(\mathbb{Z} + \mathbb{Z}, <)$ PII: $(\mathbb{Z}, <)$,
PI has a winning strategy by choosing 0_1 then 0_2. PII will repsond with two points in $\mathbb{Z}$ that are only finitely far apart, but PI can still play infinitely many points between them.

Theorem 2.9.3. For a countable $\mathcal{M}, \mathcal{N}$, PII has a winning strategy iin $G_\omega(\mathcal{M}, \mathcal{N}) \leftrightarrow \mathcal{M} \cong \mathcal{N}$

Proof. $\leftarrow$) If $\mathcal{M} \cong \mathcal{N}$ say $f : \mathcal{M} \cong \mathcal{N}$. Play following f
$\rightarrow$) Play the game such that PI makes sure to play all $m \in M$, all $n \in N$. PII responds with winning strategy.

$G_n(\mathcal{M}, \mathcal{N})$ is the same game, but with only n moves.

Theorem 2.9.4. If $\mathcal{L}$ is a finite language with no function symbols, $\forall n \in \omega$ PII has a winning strategy in $G_n(\mathcal{M}, \mathcal{N}) \leftrightarrow \mathcal{M} \equiv \mathcal{N}$

Proof. $\rightarrow$) Suppose that $\mathcal{M} \not\equiv \mathcal{N}$. Suppose s is a strategy in $G_n(\mathcal{M}, \mathcal{N})$. We want to play as PI and win. We know there is some φ true in $\mathcal{M}$ and not true in $\mathcal{N}$. Can assume WLOG starts with a quantifier (for a disjunction, $\mathcal{M}, \mathcal{N}$ must disagree on one of the disjuncts and for negations $\mathcal{M}, \mathcal{N}$ will still disagree on the formula without the negation). If $\mathcal{M} \models \forall x \psi(x), \mathcal{N} \models \exists x \neg\psi(x)$ choose n_1 such that $\mathcal{N} \models \neg\psi(n_1)$. PI plays n_1, suppose PII plays m_1 then $\mathcal{M} \models \psi(m_1)$, For the other case, if $\mathcal{M} \models \exists x \psi(x)$, $\mathcal{N} \models \forall x \neg\psi(x)$ choose m_1 sch that $\mathcal{M} \models \psi(m_1)$. PII plays n_2, $\mathcal{N} \models \neg\psi(n_2)$. Now, given $\psi(x)$ we can again assume we have a formula $\theta(x)$ that starts with a quantifier and we repeat the steps above. We continue in this fashion till we get an atomic formula that is true in one model but not the other. So we take $n=$ quantifier depth of φ to make this work.
$\leftarrow$) Follows from:
Lemma: PII has a winning strategy in $G_n(\mathcal{M}, \mathcal{N}) \leftrightarrow \mathcal{M} \equiv_n \mathcal{N}$ (sentences of quantifier depth $\leqslant n$)

Chapter 3

Quantifier Elimination

3.1 September 29

3.1.1 Quantifier Elimination

Sometimes formulas with quantifiers are equivalent to ones without quantifiers $\mathbb{R} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (v^2 - 4ac \geqslant 0 \wedge u \neq 0) \vee (u = 0 \wedge (w = 0 \vee v \neq 0))$
$\mathbb{C} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (u \neq 0 \vee v \neq 0 \vee w = 0)$

No always possible though. In $\mathbb{N}$, there are polynomials $p(x, \overline{v})$ such taht $\{n \mid \mathbb{N} \models \exists \overline{v} p(n, \overline{v})\}$ is not computable.

Definition 3.1.1. T has quantifier elimination if for every formula $\phi(\overline{v})$ ther is a quantifier free $\varphi(\overline{v})$ such that $T \models \forall \overline{v}(\phi(\overline{v}) \leftrightarrow \varphi(\overline{v}))$

Note: we allow quantifier free formulas for true and false.

Example 3.1.2. Theories with quantifier elimination

- DLO - dense linear orders
- Th(Rado Graph)
- DAG - torsion free divisible abelian groups
- ODAG - ordered divisble abelian groups
- Presburger Arithmetic - $\mathrm{Th}(\mathbb{Z}, +, -, 0, 1, <)$ adding preducates $P_n(x) \leftrightarrow \exists y(n \cdot y = x)$
- Algebraically closed fields - $\mathrm{Th}(\mathbb{C}, +, \times)$
- Real closed fields $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$

Theorem 3.1.3. DLO has quantifier elimination

Proof. Let $\phi(\overline{v})$ be a formula, $\mathcal{L} = \{\leqslant\}$

3.1.2 Quantifier Elimination

Sometimes formulas with quantifiers are equivalent to ones without quantifiers $\mathbb{R} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (v^2 - 4ac \geqslant 0 \wedge u \neq 0) \vee (u = 0 \wedge (w = 0 \vee v \neq 0))$
$\mathbb{C} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (u \neq 0 \vee v \neq 0 \vee w = 0)$

No always possible though. In $\mathbb{N}$, there are polynomials $p(x, \overline{v})$ such taht $\{n \mid \mathbb{N} \models \exists \overline{v} p(n, \overline{v})\}$ is not computable.

Definition 3.1.4. T has quantifier elimination if for every formula $\phi(\overline{v})$ ther is a quantifier free $\varphi(\overline{v})$ such that $T \models \forall \overline{v}(\phi(\overline{v}) \leftrightarrow \varphi(\overline{v}))$

Note: we allow quantifier free formulas for true and false.

Example 3.1.5. Theories with quantifier elimination

- DLO - dense linear orders
- Th(Rado Graph)
- DAG - torsion free divisible abelian groups
- ODAG - ordered divisble abelian groups
- Presburger Arithmetic - $\mathrm{Th}(\mathbb{Z}, +, -, 0, 1, <)$ adding preducates $P_n(x) \leftrightarrow \exists y(n \cdot y = x)$
- Algebraically closed fields - $\mathrm{Th}(\mathbb{C}, +, \times)$
- Real closed fields $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$

Theorem 3.1.6. DLO has quantifier elimination

Proof. Let $\phi(\overline{v})$ be a formula, $\mathcal{L} = \{\leqslant\}$
If ϕ is a sentence, then since DLO is complete either DLO $\models \phi \leftrightarrow \top$ or DLO $\models \phi \iff \bot$
Suppose ϕ not a sentence, $\overline{v} = v_1, \ldots, v_n$. Given $\sigma : \{(i,j) : i < j < n\} \to \{0,1,2\}$ define $\chi_\sigma(\overline{v}) = \left(\bigwedge_{\substack{1<j<n \\ \sigma(i,j)=0}} v_i = v_j\right) \wedge \left(\bigwedge_{\substack{1<j<n \\ \sigma(i,j)=1}} v_i < v_j\right) \wedge \left(\bigwedge_{\substack{1<j<n \\ \sigma(i,j)=2}} v_i > v_j\right)$
Observation: If $\overline{a}, \overline{b} \in \mathbb{Q}$, σ, and $\mathbb{Q} \models \chi_\sigma(\overline{a}), \chi_\sigma(\overline{b}) \to (\mathbb{Q}, \overline{a}) \cong (\mathbb{Q}, \overline{b}) \to \mathbb{Q}(a) \leftrightarrow \mathbb{Q}(b)$. Let $\Lambda_\phi = \{\sigma \mid \forall x(\chi_\sigma(\overline{x}) \to \phi(\overline{x}))\}$. Let $\psi(\overline{v}) = \bigvee_{\sigma \in \Lambda_\phi} \chi_\sigma(\overline{v})$. Claim: DLO $\models \forall v \phi(\overline{v}) \leftrightarrow \psi(v)$

Lemma 3.1.7. Suppose that for every quantifier free formula $\phi(\overline{v}, w)$ there is a quantifier free formula $\psi(\overline{v})$ such that $T \models \forall \overline{v}(\exists w \phi(\overline{v}, w) \leftrightarrow \psi(\overline{v}))$ then T has quantifier elimination.

Proof. Idea: We can construc the quantifier free formula by working from the inside out, eliminating one quantifier at a time.

Theorem 3.1.8. For T, an $\mathcal{L}$ theory, $\phi(\overline{v})$ a formula the following are equivalent

(i) There is a quantifier free formula $\varphi(\overline{v})$ such that $T \models \forall\overline{v}(\phi(\overline{v}) \leftrightarrow \varphi(\overline{v}))$

(ii) For every $\mathcal{M}, \mathcal{N}, \mathcal{A}, \overline{a} \in A$ with $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{M}, \mathcal{N}$ We have $\mathcal{M} \models \phi(\overline{a}) \leftrightarrow \mathcal{N} \models \phi(\overline{a})$

Proof. (i) $\to$ (ii): $\mathcal{M} \models \phi(\overline{a}) \leftrightarrow \mathcal{M} \models \psi(\overline{a}) \leftrightarrow \mathcal{A} \models \psi(\overline{a}) \iff \mathcal{N} \models \psi(\overline{a}) \leftrightarrow \mathcal{N} \models \phi(\overline{a})$

3.2 October 4

3.2.1 Quantifier Elimintation

Proof (Proof (Cont))**.** (ii) $\to$ (i): Define $\Gamma(\overline{v}) = \{\psi(\overline{v}) :$ quantifier free $T \models \forall\overline{v}(\phi(\overline{v}) \to \psi(\overline{v}))\}$. Let $\overline{d}$ be a tuple of new constants. Note $T \cup \phi(\overline{d}) \models \Gamma(\overline{d})$ by definition.
Claim: $T \cup \Gamma(\overline{d}) \models \phi(\overline{d})$
This would be enough to because by the compactness theorem there are $\psi_0, \ldots, \psi_k \in \Gamma$ such that $T \cup \{\psi_0(\overline{d}), \ldots, \psi_k(\overline{d})\} \models \phi(\overline{d})$ so $T \models \psi_0(\overline{d}) \wedge \ldots \wedge \psi_k(\overline{d}) \leftrightarrow \phi(\overline{d})$
Proof of claim: Suppose not, Let $\mathcal{M} \models T \cup \Gamma(\overline{d}) \cup \{\neg\phi(\overline{d})\}$, let $\mathcal{A} = \langle\overline{d}\rangle$ be the substructure of $\mathcal{M}$ generated by $\overline{d}$ (For every $a \in A$, there is a term $t(\overline{x})$ such that $a = t(\overline{d})$). We want to find $\mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{N}$ and $\mathcal{N} \models \phi(\overline{d})$ we need to show $\Sigma = T \cup \mathrm{Diag}(\mathcal{A}) \cup \phi(\overline{d})$ is satisfiable. If not there is a formula $\psi(a) \in \mathrm{Diag}(\mathcal{A})$ sich that $T \cup \psi(\overline{a}) \models \neg\phi(\overline{d})$. Let $\tilde{\psi}(\overline{d}) = \psi(\overline{t}(\overline{a}))$. $T \cup \tilde{\psi} \models \neg\phi(\overline{d})$ so $T \models \tilde{\psi}(\overline{d}) \to \neg\phi(\overline{d})$ so $T \models \phi(d) \to \neg\tilde{\psi}(d)$ so $T \models \forall\overline{v}(\phi(\overline{v}) \to \neg\tilde{\psi}(\overline{v}))$ so $\neg\tilde{\psi}(\overline{v}) \in \Gamma(\overline{v})$ so $\mathcal{M} \models \neg\psi(\overline{d})$ so $\mathcal{A} \models \neg\tilde{\psi}(\overline{d})$ contradicting $\tilde{\psi}(\overline{d}) \in \mathrm{Diag}(\mathcal{A})$
Now, let $\mathcal{N} \models \Sigma$, $\mathcal{A} \subseteq \mathcal{N}$, $\mathcal{N} \models \phi(\overline{d}), \mathcal{N} \models T$

Corollary 3.2.1. Let T be an $\mathcal{L}$ theory, suppose that for all quantifier free $\theta(\overline{v}, w)$, for all $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{N}, \mathcal{N}$, $\overline{a} \in A$, $\exists m \in M\ \mathcal{M} \models \theta(\overline{a}, m) \leftrightarrow \exists n \in N\ \mathcal{N} \models \theta(\overline{a}, n)$, then T has quantifier elimination.

DAG is the theory of torsion free divisble abelain groups. $\mathcal{L} = \{0, +\cdot\}$

Lemma 3.2.2. Let, $G, H \models$ DAG, nontrivial $G \subseteq H$. Let $\psi(\overline{v}, w)$ bea a quantifier free formula, $\overline{a} \in G$. If $H \models \exists w\psi(\overline{a}, w)$, there is $c \in G$ $H \models \psi(\overline{a}, c)$

Proof. $\psi(\overline{v}, w) = \bigvee_{i=1}^{r} \bigwedge_{j=1}^{n} \theta_{ij}(\overline{v}, w)$ where each θ_{ij} is atomic or the negation of an atomic. $H \models \exists w\psi(\overline{a}, w)$ so for some i, $H \models \exists w\psi_i(\overline{a}, w)$ so $\exists c \in G$, $H \models \psi_i(\overline{a}, c)$ where $\psi_i(v, w)$ is $\bigcap_{j=1}^{n} \theta_j(\overline{v}, w)$. θ_j is of the form $t_1(\overline{v}, w) = t_2(\overline{v}, w)$ or $t_1(\overline{v}, w) \neq t_2(\overline{v}, w)$, ie. $t_1(\overline{v}, w) - t_2(\overline{v}, w) = 0$ or $t_1(\overline{v}, w) - t_2(\overline{v}, w) \neq 0$. These can be rewritten as $t(\overline{v}, w) = 0$ or $t(\overline{v}, w) \neq 0$. These have the form $\sum n_i v_i + mw = 0$ or $\sum n_i v_i + mw \neq 0$ for $n_i, m \in \mathbb{Z}$. If it is of the first form $w = \frac{-\sum n_i a_i}{m} \in G$, given $m \neq 0$. If all formulas are of the second form, $w \neq \frac{-\sum n_i a_i}{m}$, then there is a witness since G is infinite and there are finitely many inequalities.

Definition 3.2.3. T is strongly minimal or o-minimal if for every $\mathcal{M} \models T$, every definable subset of $\mathcal{M}$ is either finite or cofinite.

We showed if $H \models$ DAG, that every set that is definable by quantifier free formulas are finite or cofinite. After proving DAG has quantifier elimination, we get that DAG is strongly minimal.

3.3 October 6

3.3.1 Quantifier Elimination

Lemma 3.3.1. Let G be a TFAG. There is DAG $H \supseteq G$ (called the divisble hull of G) such that for all DAG $H' \supseteq G$ there is an embedding $h : H \to H'$ with $h \restriction G = \mathrm{id}$.

Proof. Let $H = \{(g, n) : g \in G, n \in \mathbb{N}\}/\sim$ where $(g, n) \sim (g', n') \leftrightarrow \underbrace{g + \cdots + g}_{n'} = \underbrace{g' + \cdots + g'}_{n}$ then $[(g, n)] +_H [(h, m)] = [(m \cdot g + n \cdot h, mn)]$. We need to show if $(g, n) \sim (g', n')$ $(h, m) \sim (h', m')$ then $(m \cdot g + n \cdot h, mn) \sim (m' \cdot g' + n' \cdot h', n'm')$
Let $i : G \to H$ by $i(g) = (g/1)$
Given DAG $H' \supseteq G$, we define $h : H \to H'$ by $[(g, n)] = \frac{g}{n}$ in H'

Theorem 3.3.2. DAG has quantifier elimination.

Proof. Let $\theta(\overline{v}, w)$ be quantifier free. $G \subseteq G_1, G_2, \overline{a} \in G$. We want to show $G_1 \models \exists\theta(\overline{a}, w) \leftrightarrow G_2 \models \exists\theta(\overline{a}, w)$
Observation: G is a TFAG
Let H be the hull of G. $G_1 \models \exists w\theta(\overline{a}, w) \leftrightarrow H \models \exists w\theta(\overline{a}, w) \leftrightarrow G_2 \models exw\theta(\overline{a}, w)$

Definition 3.3.3. Given T, let $T_\forall\{\varphi\forall$-sentence : $T \models \varphi\}$

Claim: $\mathrm{DAG}_\forall$=TFAG, $\mathrm{ACF}_\forall$ = integral domains, $\mathrm{RCF}_\forall$ = ordered integral domains

Lemma 3.3.4. $\mathcal{A} \models T_\forall \leftrightarrow \exists\mathcal{M} \supseteq \mathcal{A}, \mathcal{M} \models T$

Proof. $\to$) $\mathcal{M} \models T$, $\mathcal{A} \supseteq \mathcal{M}$ and $\varphi \in T_\forall$, $T \models \varphi \to \mathcal{M} \models \varphi \to \mathcal{A} \models \varphi$
$\leftarrow$) Given $\mathcal{A} \models T_\forall$. We want to show $T \cup \mathrm{Diag}(\mathcal{A})$ is satisfiable. If $T \cup \mathrm{Diag}(\mathcal{A})$ is not satisfiable is not satisfiable then there are $\psi_1(\overline{a}), \ldots, \psi_k(\overline{a})$ such that $T \models \neg(\psi_1(\overline{a}) \wedge \cdots \wedge \psi_k(\overline{a}))$ so $T \models \neg\psi(\overline{a})$ for $\psi(\overline{a}) \in \mathrm{Diag}(\mathcal{A})$. Then $T \models \forall\overline{x}\neg\psi(\overline{x})$ so $\forall x\neg\psi(\overline{x}) \in T_\forall$ so $\mathcal{A} \models \forall\overline{x}\neg\psi(\overline{x})$ contradicting $\mathcal{A} \models \psi(\overline{a})$

Claim follows since $G \models$ TFAG $\leftrightarrow$ there is $H \supseteq G$, $H \models$ DAG $\leftrightarrow G \models \mathrm{DAG}_\forall$

Definition 3.3.5. T has algebraically prime models if for every $\mathcal{A} \models T_\forall$ there is an $\mathcal{M} \supseteq \mathcal{A}, \mathcal{M} \models T$ such that if $\mathcal{M}' \supseteq \mathcal{A}$, $\mathcal{M}' \models T$ then there is an embedding $h : \mathcal{M} \to \mathcal{M}'$ with $h \restriction \mathcal{A} = \mathrm{id}_\mathcal{A}$.

DAG has prime models. Also true for ACF, RCF.

Definition 3.3.6. $\mathcal{M} \supseteq \mathcal{N}$, we say that $\mathcal{M}$ is simply closed $\mathcal{N}$ ($\mathcal{M} \preceq_S \mathcal{N}$) if for every quantifier free formula $\theta(\overline{v}, w)$ and $\overline{a} \in M$ if $\mathcal{N} \models \exists w \theta(\overline{a}, w)$ then $\exists m \in M$, $\mathcal{N} \models \theta(\overline{a}, m)$

Theorem 3.3.7. For T an $\mathcal{L}$-theory if

(i) T has algebraically prime models

(ii) For all $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \subseteq \mathcal{N} \to \mathcal{M} \preceq_S \mathcal{N}$

Then T has quantifier elimination.