

MATH 225A: Metamathematics

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Fall 2022

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Chapter 1

Structures and Theories

1.1 August 25

1.1.1 Review

Definition 1.1.1. A language \mathcal{L} consists of $\{\mathcal{C}, \mathcal{R}, \mathcal{F}\}$ where \mathcal{C} is the set of constant symbols, \mathcal{R} is the set of relation symbols, \mathcal{F} is the set of function symbols, and an arity function $n : \mathcal{R} \cup \mathcal{F} \rightarrow \mathbb{N}$. For $R \in \mathcal{R}$, n_R is the arity of R , for $f \in \mathcal{F}$, n_f is the number of inputs f takes.

Definition 1.1.2. An \mathcal{L} -structure consist of

- a set M called the domain
- an element $c^{\mathcal{M}}$ for each $c \in \mathcal{C}$
- a subset $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
- a function $f^{\mathcal{M}} : M^{n_f} \rightarrow M$ for each $f \in \mathcal{F}$

denoted $\mathcal{M} = (M : \{c^{\mathcal{M}} : c \in \mathcal{C}\}, \{R^{\mathcal{M}} : R \in \mathcal{R}\}, \{f^{\mathcal{M}} : f \in \mathcal{F}\})$

Definition 1.1.3. An \mathcal{L} -embedding $\eta : \mathcal{M} \rightarrow \mathcal{N}$ is a one to one function $M \rightarrow N$ that preserves interpretation

eg. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$, $\eta(f^{\mathcal{M}})(m_1, \dots, m_{n_f}) = f^{\mathcal{N}}(\eta(m_1), \dots, \eta(m_{n_f}))$,
 $(m_1, \dots, m_{n_R}) \in R^{\mathcal{M}} \iff (\eta(m_1), \dots, \eta(m_{n_R})) \in R^{\mathcal{N}}$

Definition 1.1.4. An \mathcal{L} -isomorphism is an \mathcal{L} -embedding that is onto.

Definition 1.1.5. \mathcal{M} is a substructure of \mathcal{N} , written $\mathcal{M} \subseteq \mathcal{N}$ if:
 $c^{\mathcal{M}} = c^{\mathcal{N}}$, $f^{\mathcal{M}} = f^{\mathcal{N}} \upharpoonright M^{n_f}$, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^{n_R}$

First Order language:

- Use symbols :

- \mathcal{L}
- Logical symbols: connectives (\wedge, \vee, \neg), quantifiers (\forall, \exists), equality ($=$), variables (v_0, v_1, \dots)
- paranthesis and commas
- terms
 - c : constants
 - v_i : variables
 - $f(t_1, \dots, t_{n_f})$ for terms t_1, \dots, t_{n_f}
- given an \mathcal{L} -structure \mathcal{M} , a term $t(v_0, \dots, v_n)$, and $m_0, \dots, m_n \in M$ we inductively define $t^{\mathcal{M}}(m_0, \dots, m_n)$
- atomic formulas: $t_1 = t_2$ and $R(t_1, \dots, t_{n_R})$
- \mathcal{L} -formulas: If ϕ and ψ are \mathcal{L} -formulas, then so are: $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $\exists v\phi$, $\forall v\phi$

Definition 1.1.6. We say a variable v occurs freely in ψ when it is not in a quantifier $\forall v$ or $\exists v$

- an \mathcal{L} -sentence is an \mathcal{L} -formula with no free variables

Definition 1.1.7. A theory is a set of \mathcal{L} -sentences

Definition 1.1.8. Given an \mathcal{L} -formula $\psi(v_1, \dots, v_k)$, \mathcal{L} -structure \mathcal{M} , $m_1, \dots, m_k \in M$ we can define $\mathcal{M} \models \psi(m_1, \dots, m_k)$ inductively. We say (m_1, \dots, m_k) satisfies ϕ in \mathcal{M} or ϕ is true in $\mathcal{M}, m_1, \dots, m_k$.

- A theory T is satisfiable if it has a model \mathcal{M} , eg. \mathcal{M} such that $\mathcal{M} \models \phi$ for $\phi \in T$

Proposition 1.1.9. If $\mathcal{M} \subseteq \mathcal{N}$, $\phi(\bar{v})$ is quantifier free, $\bar{m} \in M$, then $\mathcal{M} \models \phi(\bar{m}) \leftrightarrow \mathcal{N} \models \phi(\bar{m})$.

Definition 1.1.10. \mathcal{M} is elementarily equivalent to \mathcal{N} if for all \mathcal{L} -sentences ϕ , $\mathcal{M} \models \phi \leftrightarrow \mathcal{N} \models \phi$, denoted $\mathcal{M} \equiv \mathcal{N}$

- $\text{Th}(\mathcal{M})$, the full theory of \mathcal{M} , is $\{\phi \text{ } \mathcal{L}\text{-sentence} \mid \mathcal{M} \models \phi\}$
- $\mathcal{M} \equiv \mathcal{N} \iff \text{Th}(\mathcal{M}) = \text{Th}(\mathcal{N})$
- A class of \mathcal{L} -structures \mathcal{K} is elementary if there is a theory T such that \mathcal{K} is the class of all \mathcal{M} such that $\mathcal{M} \models T$.

Logical implication: $T \models \phi$ if for every $\mathcal{M} \models T$, $\mathcal{M} \models \phi$

Gödel's Completeness Theorem: $T \models \phi \leftrightarrow$ there is a formal proof for $T \vdash \phi$

1.1.2 Definable Sets

Definition 1.1.11. $X \subseteq M^n$ is definable if there is an \mathcal{L} -formula $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $b_1, \dots, b_m \in M$ such that $\forall \bar{a}, \bar{a} \in X \leftrightarrow \mathcal{M} \models \phi(\bar{a}, \bar{b})$ (definable over \bar{b})

- Given $A \subseteq M$, X is definable over A , or A -definable, if it is definable over \bar{b} for some $\bar{b} \in A$.

Proposition 1.1.12. Suppose $\mathcal{D} = (D_n : n \in \omega)$ is the smallest collection of subsets $D_n \subseteq \mathcal{P}(M^n)$ such that

- $M^n \in D_n$
- D_n is closed under union, intersection, complement, permutation
- if $X \in D_{n+1}$, then $\pi(X) \in D_n$ where $\pi(m_1, \dots, m_{n+1}) = (m_1, \dots, m_n)$
- $\{\bar{b}\} \in D_n$ for $\bar{b} \in M^n$
- $R^{\mathcal{M}} \in D_{n_R}$, $\text{graph}(f) \in D_{n_f+1}$
- if $X \in D_n$, $M \times X \in D_{n+1}$
- $\{(m_1, \dots, m_n) : m_i = m_j\} \in D_n$

Then $X \subseteq M^n$ is definable $\leftrightarrow X \in D_n$

Chapter 2

Basic Techniques

2.1 August 30

2.1.1 Compactness Theorem

Theorem 2.1.1 (Compactness). If T is finitely satisfiable, then T has a model \mathcal{M} . Furthermore, $|\mathcal{M}| \leq |\mathcal{L}| + \aleph_0$

- T is finitely satisfiable if every finite subset is satisfiable

Compactness II: if $T \models \phi$, then there is finite $T_0 \subset T$ such that $T_0 \models \phi$

$T \models \phi \leftrightarrow T \cup \{\neg\phi\}$ is not satisfiable

Proposition 1: If T is finitely satisfiable, maximal, and has the witness property, then T has a model \mathcal{M} with $|\mathcal{M}| \leq |\mathcal{L}|$

Proposition 2: If T is finitely satisfiable, then there is $\mathcal{L}^* \supseteq \mathcal{L}$ and an \mathcal{L}^* -theory $T^* \supseteq T$ such that T^* is finitely satisfiable, maximal, and has the witness property. Further, $|\mathcal{L}^*| \leq |\mathcal{L}| + \aleph_0$

Definition 2.1.2.

- T is maximal if for any sentence ϕ , either $\phi \in T$ or $\neg\phi \in T$
- T has the witness property if for all \mathcal{L} -formulas $\phi(v)$ there is a constant c_ϕ such that $\exists v\phi(v) \rightarrow \phi(c_\phi) \in T$

Lemma 1: If T is maximal and finitely satisfiable, if there is finite $\Delta \subseteq T$ such that $\Delta \models \phi$, then $\phi \in T$.

Proof. If $\phi \notin T$, $\neg\phi \in T$. Since $\Delta \models \phi$, $\Delta \cup \{\neg\phi\}$ is not satisfiable, contradicting our assumption.

Henekin Construction:

We want to define $\mathcal{M} = (M, c^\mathcal{M}, R^\mathcal{M}, f^\mathcal{M})$

- Let $M = \mathcal{C} / \sim$ where \mathcal{C} is the set of constant symbols and \sim is the equivalence relation defined by $c_1 \sim c_2 \leftrightarrow c_1 = c_2 \in T$
- $R^\mathcal{M} \subseteq M^{n_R}$ by $(c_1^*, \dots, c_{n_R}^*) \in R^\mathcal{M} \leftrightarrow R(c_1, \dots, c_n) \in T$ where c^* equivalence class of c
This is well defined since if we have $c'_1 \sim c_1, \dots, c'_n \sim c_n, R(c_1, \dots, c_n) \in T$ then $R(c'_1, \dots, c'_n) \in T$

- $f^{\mathcal{M}}$ by $f^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^* \leftrightarrow f(c_1, \dots, c_n) = d \in T$. Such a d^* exists since T has the witness property:
 $\exists v f(c_1, \dots, c_n) = v \rightarrow f(c_1, \dots, c_n) \in T$
- $c^{\mathcal{M}} := c^*$

Claim: For every formula $\phi(v_1, \dots, v_k)$ and constant symbols c_1, \dots, c_k , $\mathcal{M} \models \phi(c_1^*, \dots, c_k^*) \leftrightarrow \phi(c_1, \dots, c_k) \in T$
 This implies $\mathcal{M} \models T$

Proof. By induction on formulas $\phi(v_1, \dots, v_l)$

- atomic formulas: $\phi(v_1, \dots, v_k)$ is $t_1(v_1, \dots, v_k) = t_2(v_1, \dots, v_k)$
 Subclaim: $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = c^* \leftrightarrow t(c_1, \dots, c_n) = c \in T$
 Proved by induction on terms
- $\phi(v_1, \dots, v_k)$ is $R(v_1, \dots, v_k)$. Follows by definition of $R^{\mathcal{M}}$
- Suppose $\phi(\bar{v})$ is $\psi_1(\bar{v}) \wedge \psi_2(\bar{v})$, then
 $\mathcal{M} \models \psi_1 \wedge \psi_2(\bar{v}) \leftrightarrow \mathcal{M} \models \psi_1(\bar{v})$ and $\mathcal{M} \models \psi_2(\bar{v}) \xrightarrow{\text{IH}} \psi_1(\bar{c}) \in T$ and $\psi_2(\bar{c}) \in T \xrightarrow{\text{lemma}} \psi_1 \wedge \psi_2(\bar{c}) \in T$
- Suppose $\phi(\bar{v})$ is $\neg\psi(\bar{v})$, then
 $\mathcal{M} \models \neg\psi(\bar{c}^*) \leftrightarrow \mathcal{M} \not\models \psi(\bar{c}^*) \xrightarrow{\text{IH}} \varphi(\bar{c}) \notin T \xrightarrow{\text{maximality}} \neg\psi(\bar{c}) \in T$
- Suppose $\phi(\bar{v})$ is $\exists w \varphi(\bar{v}, w)$, then
 $\mathcal{M} \models \exists w \varphi(\bar{c}^*, w) \leftrightarrow \exists d \in M$ such that $\mathcal{M} \models \varphi(\bar{c}^*, d) \leftrightarrow \exists d \in M$ such that $\varphi(\bar{c}, d) \in T \xrightarrow{\text{witness principle}} \exists w \varphi(\bar{c}, w) \in T$

2.2 September 1

2.2.1 Compactness

Proof of Compactness continued:

We now prove proposition 2

Lemma 1: If T is finitely satisfiable then there is $\mathcal{L}^* \supset \mathcal{L}$, $T^* \supset T$ such that T^* has the witness property and is finitely satisfiable

Proof. For each \mathcal{L} -formula define a new constant symbol c_ϕ . Let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v) \text{ } \mathcal{L}\text{-formula}\}$, $T_1 = T \cup \{\exists v \phi(v) \rightarrow \phi(c_\phi) : \phi(v) \text{ } \mathcal{L}\text{-formula}\}$.

Claim: T_1 is finitely satisfiable.

Take $\Delta \subseteq T_1$ finite. $\Delta = T' \cup \{\exists v \phi_i(v) \rightarrow c_{\phi_i} : i = 1, \dots, k\}$ for finite T' in T . We make an \mathcal{L}_1 -structure \mathcal{M}_1 that satisfies Δ . Take $\mathcal{M} \models T'$, \mathcal{M} \mathcal{L} -structure. Make \mathcal{M} an \mathcal{L}_1 -structure by defining $c_{\phi}^{\mathcal{M}_1}$ for each c_ϕ . If $\mathcal{M} \models \exists v \phi(v)$ let $c^{\mathcal{M}_1}$ be such a v otherwise let $c^{\mathcal{M}_1}$ be anything.

We repeat this process, defining \mathcal{L}_{n+1} from \mathcal{L}_n similarly.

We have $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \dots$, $T \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq \dots$ such that each T_i is finitely satisfiable and for $\phi(v)$ an \mathcal{L}_{i-1} -formula, there is c_ϕ in \mathcal{L}_i such that $\exists v \phi(v) \rightarrow \phi(c_\phi) \in T_i$.

Let $\mathcal{L}^* = \bigcup_{n \in \omega} \mathcal{L}_n$, $T^* = \bigcup_{n \in \omega} T_n$. We see T^* has the witness property.

Sub-claim: If $T_0 \subset T_1 \subset T_2 \subset \dots$ all finitely satisfiable, then $\bigcup_{n \in \omega} T_n$ is finitely satisfiable.

Lemma 2: If T is finitely satisfiable and ϕ a sentence, one of $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable.

Proof. Assume that both $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are not finitely satisfiable. Then there are $T_0, T_1 \subseteq T$ such that $T_0 \cup \{\phi\}$ and $T_1 \cup \{\neg\phi\}$ are not satisfiable. Let $\mathcal{M} \models T_0 \cup T_1$, then $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$ so $T_0 \cup \{\phi\}$ or $T_1 \cup \{\neg\phi\}$ is satisfiable, contradicting our assumption.

Zorn's Lemma: Let \mathcal{A} be a collection of sets such that for any chain $\mathcal{C} \in \mathcal{A}$, $\bigcup \mathcal{C} \in \mathcal{A}$ where \mathcal{C} is a chain if for $A, B \in \mathcal{C}$ either $A \subseteq B$ or $B \subseteq A$, then \mathcal{A} has a maximal element, eg. $A \in \mathcal{A}$ such that there is not $B \in \mathcal{A}$ with $A \subsetneq B$.

Lemma: For every T , finitely satisfiable, there is $T' \supseteq T$ that is maximal and finitely satisfiable.

Proof. Let $\mathcal{A} = \{S \text{ } \mathcal{L}\text{-theory} \mid S \supseteq T, S \text{ finitely satisfiable}\}$. Can apply zorns lemma since for any $\mathcal{C} \subseteq \mathcal{A}$, $\bigcup \mathcal{C} \in \mathcal{A}$ so we have a maximal S .

Example 2.2.1. Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. In a group G , $g \in G$, $\text{ord } g = \text{least } n \text{ such that } \underbrace{g \cdots g}_{n \text{ times}} = e$, if it exists.

Observation: If T is an \mathcal{L} -theory extending the axioms of groups, $\phi(v)$ such that for every n there is $G_n \models T$, $g_n \in G_n$ of order greater than n such that $G_n \models \phi(g_n)$. Then there is $G \models T$ and $g \in G$, $\text{ord}(g) = \infty$ such that $G \models \phi(g)$.

Proof. Let $\mathcal{L}' = \{\cdot, e, c\}$. Let $T^* = T \cup \phi(c) \cup \{\psi_n\}$ where ψ_n is $\underbrace{c \cdot c}_{n \text{ times}} \neq e$. T^* finitely satisfiable so follows by compactness.

This tells us that there is not sentence that axiomatizes when an element is torsion.

Lemma 2.2.2. Let κ be a cardinal $\kappa \geq |\mathcal{L}|$. Let T be a satisfiable theory such that $\forall n \in \mathbb{N}$, there is $\mathcal{M} \models T$ such that $|\mathcal{M}| > n$. Then T has a model of size κ .

Proof. Extend the language by adding κ many new constant symbols c_i for $i \in \kappa$. $T^* = T \cup \{c_i \neq c_j \mid i \neq j\}$. If $\mathcal{M} \models T^*$, $|\mathcal{M}| \geq \kappa$. T^* is finitely satisfiable so by compactness T^* has a model \mathcal{M} , $|\mathcal{M}| \leq |\mathcal{L}^*| + \aleph_0 = \kappa$. Thus, $|\mathcal{M}| = \kappa$.

2.3 September 6

2.3.1 Complete Theories

Definition 2.3.1. Let κ be an infinite cardinal. A theory T is κ -categorical if all models of T of size κ are isomorphic (and there is at least one).

Example 2.3.2. The theory of torsion free abelian division groups (TFADG) is κ categorical for all uncountable κ .

Language = $\{\cdot, e\}$, TFADG = group axioms, commutativity, torsion free - $\forall a \neq e \underbrace{a \cdot a \cdots a}_n \neq e$ for $n \in \omega$,
divisible - $\forall a \exists b \underbrace{b + b + \cdots + b}_n$ for each $n \in \omega$

Observation: TFADG are essentially \mathbb{Q} -vector spaces

For $a \in G$, $n \in \mathbb{N}$ $a \cdot n = \overbrace{a + \dots + a}^{n \text{ times}} \frac{a}{n}$ is b such that $b \cdot n = a$. Such a b exists since the group is division and is uniquely defined since if $b \cdot n = a = b' \cdot n$, $(b - b') \cdot n = 0$ so since the group is torsion free, $b - b' = 0$. For $a \in G$, $\frac{p}{q} \in \mathbb{Q}$ we define $a \cdot \frac{p}{q} = \frac{a}{q} \cdot p$

Two vector \mathbb{Q} -vector spaces are isomorphic \leftrightarrow they have the same dimension. A \mathbb{Q} vector space of size κ must have dimension κ so two \mathbb{Q} vector spaces of size κ must be isomorphic.

Let ACF_p be the theory of algebraically closed fields of characteristic p .

Language = $\{0, 1, +, \times\}$. ACF_p : field axioms, $\text{char } p - \underbrace{1 + \dots + 1}_p = 0$, $\text{char } 0 - \underbrace{1 + \dots + 1}_n \neq 0$ for $n \in \omega$,

algebraically closed - every non-constant polynomial has at least one root: for degree n , $\forall z_0, z_1, \dots, z_n \ z_n \neq 0 \exists x (z_n x^n + z_{n-1} x^{n-1} + \dots + z_0 = 0)$. For each $n \in \omega$

Proposition 2.3.3. ACF is κ categorical for all uncountable κ .

Facts and Definitions

- Every field F has a prime subfield $P = \left\{ \underbrace{\frac{1 + \dots + 1}{1 + \dots + 1}}_{q}^p : p \in \mathbb{Z}, q \in \mathbb{N} \right\}$
 - if F has $\text{char } p > 0$, then the prime subfield is $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$
 - If F has $\text{char } 0$, then the prime subfield is \mathbb{Q}
- An element $a \in F$ is algebraic if there is a polynomial $p(x) \in P[x]$ such that $p(a) = 0$. (Can think of as a polynomial in $\mathbb{Z}[x]$)
- Otherwise a is transcendental
- A tuple \bar{a} is algebraically independent if there is no nontrivial polynomial $p(\bar{x}) \in P[\bar{x}]$ such that $p(\bar{a}) = 0$.
- the transcendence degree of a field F is the size of a maximal algebraically independent set.
- Algebraically closed fields are isomorphic \leftrightarrow they have the same transcendence degree.

Observation: an ACF_p of size κ must have transcendence degree κ

If $M \subset F$ is a maximal algebraically independent set, $\forall a \in F$ there is a polynomial $p(\bar{x}, y) \in P[\bar{x}, y]$ and $\bar{m} \in M$ such that $p(\bar{m}, a) = 0$.

Definition 2.3.4. A theory T is complete if for all \mathcal{L} -sentences, ϕ either $T \models \phi$ or $T \models \neg\phi$

Theorem 2.3.5 (Vaught's Test). If T is satisfiable and has no finite models and is κ -categorical for $\kappa > |\mathcal{L}|$, then T is complete.

Corollary 2.3.6. ALL ACF_p satisfy the same sentences.

Proof. Suppose not. There is ϕ such that $T \not\models \phi$, $T \not\models \neg\phi$ so $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are satisfiable. Both have models of size κ , contradicting κ -categoricity.

Definition 2.3.7. T is decidable if there is an algorithm to decide $T \models \phi$ given ϕ

Observation: If T is computably enumerable and complete then T is decidable

Corollary 2.3.8. $\text{Th}(\mathbb{C}; 0, 1, +, \times)$ is decidable.