MATH 142: Elementary Algebraic Topology

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Chapter 1

Introduction

1.1 August 24

1.1.1 What is Algebraic Topology

Recall Metric Spaces: (X, d), X is a set, d is a metric on X (ie. $d : X \times X \to \mathbb{R}$)

1. $d(x, y) = 0$ exactly if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Let V be a vector space, let $||\cdot||$ be a norm on V, let $d(v, w) = ||v - w||$

- $\mathbb{R}^n$: $||(r_j)||_2 = (\Sigma|r_j|^2)^{\frac{1}{2}}$ - Euclidean Norm, $||(r_j)||_1 = \Sigma|r_j|$, $||(r_j)| = \max|r_j|$

If (X,d) is a metric space and if $Y \subseteq X$, let d^Y be the restriction of d to $Y \times Y$. Then (Y, d^Y) is a metric space.

Metric spaces $\leftrightarrow$ geometry: length, area, size of angles.

Let X be a balloon on $\mathbb{R}^3$

- Two natural metrics: inherited metric from $\mathbb{R}^3$, path-lenght metric (eg. length of shortest path on surface between two points)
- Consider a deformation: (Insert Figure)
  the shapes have different Euclidean distances but still have an underlying commonality
- We also observe that the baloon cannot be continuously deformed into: (Insert Figure)

We want to be able to prove such things without embedding into a metric space. This is done by attaching algebraic objects to topological spaces such that their isomorphism classes dont change under continuous deformation.

1.1.2 Continuity

Let (X, d^X) and (Y, d^Y) be two metric spaces. Let $f : X \to Y$ be a function. Let $x_0 \in X$. We say f is continuous at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d^X(x, x_0) < \delta$ then $d^Y(f(x), f(x_0)) < \varepsilon$.

- Let (X, d) be a metric space. By the open ball of radius r about x_0, we mean $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ (closed ball is $\{x \in X : d(x, x_0) \leq r\}$)
- the above definition can be rephrased as: for any $B(f(x_0), \varepsilon)$ there is an open ball $B(x_0, \delta)$ such that if $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \varepsilon)$.
  eg. For every open ball B_1 about $f(x_0)$ there is an open ball B_2 about x_0 such that if $x \in B_2$ then $f(x) \in B_1$

Definition 1.1.1. For (X, d) a metric space, by a neighborhood of a point $x \in X$, we mean any subset of X that contains an open ball about x.

- rephrasing the definition again we get: For any neighborhood $N_{f(x_0)}$ of $f(x_0)$ there is a neighborhood N_{x_0} of x_0 such that if $x \in N_{x_0}$ then $f(x) \in N_{f(x_0)}$

Definition 1.1.2. $f : X \to Y$ is continuous if it is continuous at each points of X.