# MATH 142: Elementary Algebraic Topology

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# Contents

# Chapter 1

# Topology

## 1.1 August 24

### 1.1.1 What is Algebraic Topology

Recall Metric Spaces: $(X, d)$, $X$ is a set, $d$ is a metric on $X$ (ie. $d : X \times X \to \mathbb{R}$)

1. $d(x, y) = 0$ exactly if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Let $V$ be a vector space, let $||\cdot||$ be a norm on $V$, let $d(v, w) = ||v - w||$

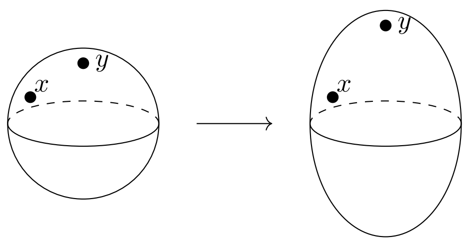
- $\mathbb{R}^n$: $||(r_j)||_2 = (\Sigma|r_j|^2)^{\frac{1}{2}}$ - Euclidean Norm, $||(r_j)||_1 = \Sigma|r_j|$, $||(r_j)|| = \max|r_j|$

If (X,d) is a metric space and if $Y \subseteq X$, let $d^Y$ be the restriction of $d$ to $Y \times Y$. Then $(Y, d^Y)$ is a metric space.

Metric spaces $\leftrightarrow$ geometry: length, area, size of angles.

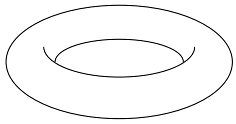
Let $X$ be a balloon on $\mathbb{R}^3$

- Two natural metrics: inherited metric from $\mathbb{R}^3$, path-lenght metric (eg. length of shortest path on surface between two points)
- Consider a deformation:




the shapes have different Euclidean distances but still have an underlying commonality

- We also observe that the balloon cannot be continuously deformed into the shape below:



We want to be able to prove such things without embedding into a metric space. This is done by attaching algebraic objects to topological spaces such that their isomorphism classes dont change under continuous deformation.

### 1.1.2 Continuity

Let $(X, d^X)$ and $(Y, d^Y)$ be two metric spaces. Let $f : X \to Y$ be a function. Let $x_0 \in X$. We say $f$ is continuous at $x_0$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d^X(x, x_0) < \delta$ then $d^Y(f(x), f(x_0)) < \varepsilon$.

- Let $(X, d)$ be a metric space. By the open ball of radius $r$ about $x_0$, we mean $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ (closed ball is $\{x \in X : d(x, x_0) \leq r\}$ )
- the above definition can be rephrased as: for any $B(f(x_0), \varepsilon)$ there is an open ball $B(x_0, \delta)$ such that if $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \varepsilon)$.
  eg. For every open ball $B_1$ about $f(x_0)$ there is an open ball $B_2$ about $x_0$ such that if $x \in B_2$ then $f(x) \in B_1$

**Definition 1.1.1.** For $(X, d)$ a metric space, by a neighborhood of a point $x \in X$, we mean any subset of $X$ that contains an open ball about $x$.

- rephrasing the definition again we get: For any neighborhood $N_{f(x_0)}$ of $f(x_0)$ there is a neighborhood $N_{x_0}$ of $x_0$ such that if $x \in N_{x_0}$ then $f(x) \in N_{f(x_0)}$

**Definition 1.1.2.** $f : X \to Y$ is continuous if it is continuous at each points of $X$.

## 1.2 August 26

### 1.2.1 Continuity

Recall: Given $(X, d^X)$, $(Y, d^Y)$ and $f : X \to Y$, f is continuous at $x_0$ if for any open ball $B_1$ about $f(x_0)$ there is an open ball $B_2$ about $x_0$ such that if $x \in B_2$ then $f(x) \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1)$

**Definition 1.2.1.** Let $(X, d)$ be a metric space. Let $U \subseteq X$. We say that $U$ is open if for every $x \in U$ ther is an open ball $B$ about x such that $B \subseteq U$, ie. $U$ is a neighborhood of each point it contains.

We say $f : X \to Y$ is continuous if it is continuous at each point of $X$.
Let $U$ be an open set in $Y$, $x \in X$ with $f(x) \in U$. For each ball $B_1$ in $U$ about $f(x)$, there is an open ball about $x$ $B_2 \subseteq X$ such that if $x' \in B_2$ then $f(x') \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1) \subseteq f^{-1}(U)$
ie. if $x \in f^{-1}(U)$ then there is an open ball $B_2$ about $x$ with $B_2 \subseteq f^{-1}(U)$

ie. $f^{-1}(U)$ is open
Conversely, if the preimage $f^{-1}(U)$ of every open set $U$ in $Y$ is open, then $f$ is continuous. This is because if $x_0 \in X$, $B_1$ an open ball about $f(x_0)$, then $f^{-1}(B_1)$ is open in $X$. $f(x_0) \in B_1$ so we have an open ball $B_2 \subseteq X$ about $x_0$ such that $B_2 \subseteq f^{-1}(B_1)$ so $f$ is continuous at $x_0$.
Thus, $f : X \to Y$ is continuous exactly if for any open $U$ in $Y$, $f^{-1}(U)$ is open in $X$.

### 1.2.2 Topology

Let $(X, d)$ be a metric space. Let $J$ be the collection of open subsets in $X$ of $d$. $J$ has the following properties:

1. $X \in J$, $\varnothing \in J$
2. an arbitrary, maybe infinite, union of open sets is open
3. a finite intersection of open sets is open.

*Proof of (3).* If $U_1, \ldots, U_n$ are open sets and $x \in U_1 \cap \cdots, \cap U_n$ then there are $r_1, \ldots, r_n \in \mathbb{R}$ such that $B(x, r_j) \subseteq U_j$ for $j = 1, \ldots, j_n$. Let $r = \min\{r_1, \ldots, r_n\}$, then $B(x, r) \subseteq U_j$ for each $j$ so $B(x, r) \subseteq U_1 \cap \cdots \cap U_n$. Thus, $U_1 \cap \cdot \cap U_n$ is open. □

Note: This does not hold for infinite intersections, consider $\bigcap_{i\in\mathbb{N}} B(x, \frac{1}{n}) = \{x\}$ in the plane.

This motivates the following definition:

**Definition 1.2.2.** Let $X$ be a set. Bya topology on $X$ we mean a collection, $\mathcal{T}$, of subsets of $X$ (called the open sets of the topology) satisfuing **1**, **2**, and **3** above.

**Definition 1.2.3.** If $(X, \mathcal{T}^X)$, $(Y, \mathcal{T}^Y)$ are topological spaces, $f : X \to Y$ is continuous if for every $U \in \mathcal{T}^Y$, $f^{-1}(U) \in \mathcal{T}^X$

**Example 1.2.4.** Given $X$, let $\mathcal{T}_X$ be all subsets of $X$. This is called the discrete topology on $X$.

- This topology can also be given by the metric $d(x, y) = 1$ if $x \neq 1$

**Definition 1.2.5.** If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on $X$, we say $\mathcal{T}_1$ is bigger, or finer, than $\mathcal{T}_2$ if $\mathcal{T}_1 \supseteq \mathcal{T}_2$.

- the disrcete topology is the biggest topology on $X$.

**Example 1.2.6.** $\mathcal{T} = \{X, \varnothing\}$, called the indiscrete topology on X.
Note: this topology can not be given by a metric if $X$ has 2 or more points.

## 1.3 August 29

### 1.3.1 Bases and Subbases

Let $(X, \mathcal{T})$ be a topological space.

**Definition 1.3.1.** A subset $A$ of $X$ is said to be closed if $A'$ $(X - A)$ is open.

Let $\mathcal{C}$ be the collection of closed subsets

1. $X, \varnothing \in \mathcal{C}$
2. any (maybe infinite) intersection of closed sets is closed
3. A finite union of closed sets is closed

Let $X$ be a set, any (maybe infinite) intersection of topologies on $X$ is a topology on $X$.
Thus, for any $\mathcal{S}$, a subset of $X$, thete is a smallest topology that conatins $\mathcal{S}$, namely the intersection of all topolgies that contain $\mathcal{S}$. We sat that $\mathcal{S}$ generates this topology.

**Definition 1.3.2.** If $\mathcal{S}$ has the property that $\bigcup(U \in \mathcal{S}) = X$, then $\mathcal{S}$ is called a subbasis of the topology it generates.

Let $\mathcal{I}^{\mathcal{S}}$ be the collection of all finite intersection of elements of $\mathcal{S}$, then the intersection of a finite number of elements of $\mathcal{I}^{\mathcal{S}}$ is in $\mathcal{I}^{\mathcal{S}}$.
Let $\mathcal{I}$ be a collection of subsets of $X$ (union of elements of $\mathcal{I}$ is $X$) with the property that the intersection of a finite number of elements of $\mathcal{I}$ is in $\mathcal{I}$. Then the collection, $\mathcal{T}$, of arbitrary unions of elements of $\mathcal{I}$ is a topology (the smallest topology containing $\mathcal{I}$)
Why is a finite intersection of elements of $\mathcal{T}$ in $\mathcal{T}$?
Suppose $\mathcal{O}_1 = \bigcup_\alpha U^1_\alpha$, $\mathcal{O}_2 = \bigcup_\beta U^2_\beta$ with $U^1_\alpha, U^2_\beta \in \mathcal{I}$, then $\mathcal{O}_1 \cap \mathcal{O}_2 = (\bigcup U^1_\alpha) \cap (\bigcup U^2_\beta) = \bigcup_{\alpha,\beta}(U^1_\alpha \cap U^2_\beta)$.

**Definition 1.3.3.** Given a topological space $(X, \mathcal{T})$, a base for it is a set of subsets, $\mathcal{B}$, of $\mathcal{T}$ , with the property that every element of $\mathcal{T}$ is a (maybe infinite) union of elements of $\mathcal{B}$.

If $\mathcal{S}$ is a subbase for $\mathcal{T}$, then $\mathcal{I}^{\mathcal{S}}$ is a base for $\mathcal{T}$.
Note: definition does not require $\mathcal{B}$ to be closed under finite intersection
$(X, d)$ is a metric space, let $\mathcal{B}$ be the set of open balls. Then $\mathcal{B}$ is a base for the metric topology but usually the intersection of two open balls is not an open ball.
The intersection of finitely many elements of $\mathcal{B}$ is the union of elements of $\mathcal{B}$.

Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces, and $\mathcal{S}$ a subbase of $\mathcal{T}^Y$. Let $f : X \to Y$, then $f$ is continuous if for every $U \in \mathcal{S}$, $f^{-1}(U) \in \mathcal{T}^X$.

**Example 1.3.4.** For $X = \mathbb{R}$, $\mathcal{S} = \{(-\infty, a), (b, +\infty) \,:\, a, b \in \mathbb{Q}\}$ generates the usual topology.