MATH 225A: Metamathmatics

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Contents

Chapter 1

Structures and Theories

1.1 August 25

1.1.1 Review

Definition 1.1.1. A language $\mathcal{L}$ consists of $\{\mathcal{C}, \mathcal{R}, \mathcal{F}\}$ where $\mathcal{C}$ is the set of constant symbols, $\mathcal{R}$ is the set of relation symbols, $\mathcal{F}$ is the set of function symbols, and and arity function $n : \mathcal{R} \cup \mathcal{F} \to \mathbb{N}$. For $R \in \mathcal{R}$, n_R is the arity of R, for $f \in \mathcal{F}$, n_f is the number of inputs f takes.

Definition 1.1.2. An $\mathcal{L}$-structure consist of

- a set M called the domain
- an element $c^{\mathcal{M}}$ for each $c \in \mathcal{C}$
- a subset $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
- a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$

denoted $\mathcal{M} = (M \, : \, \{c^{\mathcal{M}} : c \in \mathcal{C}\}, \{R^{\mathcal{M}} : R \in \mathcal{R}\}, \{f^{\mathcal{M}} : f \in \mathcal{F}\})$

Definition 1.1.3. An $\mathcal{L}$-embedding $\eta : \mathcal{M} \to \mathcal{N}$ is a one to one function $M \to N$ that preserves interpretation
eg. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$, $\eta(f^{\mathcal{M}})(m_1, \ldots, m_{n_f}) = f^{\mathcal{N}}(\eta(m_1), \ldots, \eta(m_{n_f}))$,
$(m_1, \ldots, m_{n_R}) \in R^{\mathcal{M}} \iff (\eta(m_1), \ldots, \eta(m_n)) \in R^{\mathcal{N}}$

Definition 1.1.4. An $\mathcal{L}$-isomorphim is an $\mathcal{L}$-embedding that is onto.

Definition 1.1.5. $\mathcal{M}$ is a substructure if $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$ if:
$c^{\mathcal{M}} = c^{\mathcal{N}}$, $f^{\mathcal{M}} = f^{\mathcal{N}} \restriction M^{n_f}$, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^{n_R}$

First Order language:

- Use symbols :

- $\mathcal{L}$
- Logical symbols: connectives $(\wedge, \vee, \neg)$, quantifiers $(\forall, \exists)$, equality $(=)$, variables $(v_0, v_1, \ldots)$
- paranthesis and commas

- terms
  - c : constants
  - v_i : variables
  - $f(t_1, \ldots, t_{n_f})$ for terms $t_1, \ldots, t_{n_f}$
- given an $\mathcal{L}$-structure $\mathcal{M}$, a term $t(v_0, \ldots, v_n)$, and $m_0, \ldots, m_n \in M$ we inductively define $t^{\mathcal{M}}(m_0, \ldots, m_n)$
- atomic formulas: $t_1 = t_2$ and $R(t_1, \ldots, t_{n_R})$
- $\mathcal{L}$-formulas: If $\phi and \psi$ are $\mathcal{L}$-formulas, then so are: $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $\exists v\phi$, $\forall v\phi$

Definition 1.1.6. We say a variable v occurs freely in ψ when it is not in a quantifier $\forall v$ or $\exists v$

- an $\mathcal{L}$-sentence is an $\mathcal{L}$-formula with no free variables

Definition 1.1.7. A theory is a set of $\mathcal{L}$-sentences

Definition 1.1.8. Given an $\mathcal{L}$-formla $\psi(v_1, \ldots, v_k)$, $\mathcal{L}$-structure $\mathcal{M}$, $m_1, \ldots, m_k \in M$ we can define $\mathcal{M} \models \phi(m_1, \ldots, m_k)$ inductively. We say $(m_1, \ldots, m_k)$ satisfies ϕ in $\mathcal{M}$ or ϕ is true in $\mathcal{M}, m_1, \ldots, m_k$.

- A theory T is satisfiable if it has a model $\mathcal{M}$, eg. $\mathcal{M}$ such that $\mathcal{M} \models \phi$ for $\phi \in T$

Proposition 1.1.9. If $\mathcal{M} \subseteq \mathcal{N}$, $\phi(\overline{v})$ is quantifier free, $\overline{m} \in M$, then $\mathcal{M} \models \phi(\overline{m}) \leftrightarrow \mathcal{N} \models \phi(\overline{m})$.

Definition 1.1.10. $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$ if for all $\mathcal{L}$-sentences ϕ, $\mathcal{M} \models \phi \leftrightarrow \mathcal{N} \models \phi$, denoted $\mathcal{M} \equiv \mathcal{N}$

- Th$(\mathcal{M})$, the full theory of $\mathcal{M}$, is $\{\phi\ \mathcal{L} - \text{sentence } \mid \mathcal{M} \models \phi\}$
- $\mathcal{M} \equiv \mathcal{N} \iff \text{TH}(\mathcal{M}) = \text{Th}(\mathcal{N})$
- A class of $\mathcal{L}$-structures $\mathcal{K}$ is elementary if there is a theory T such that $\mathcal{K}$ is the class of all $\mathcal{M}$ such that $\mathcal{M} \models T$.

Logical implication: $T \models \phi$ if for every $\mathcal{M} \models T$, $\mathcal{M} \models \phi$
Gödels Completeness Theorem: $T \models \phi \leftrightarrow$ there is a formal proof for $T \vdash \phi$

1.1.2 Definable Sets

Definition 1.1.11. $X \subseteq M^n$ is definable if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $b_1, \ldots, b_m \in M$ such that $\forall \overline{a}$, $\overline{a} \in X \leftrightarrow \mathcal{M} \models \phi(\overline{a}, \overline{b})$ (definable over $\overline{b}$)

- Given $A \subseteq M$, X is definable over A, or A-definable, if it is definable over $\overline{b}$ for some $\overline{b} \in A$.

Proposition 1.1.12. Suppose $\mathcal{D} = (D_n : n \in \omega)$ is the smallest collection of subsets $D_n \subseteq \mathcal{P}(M^n)$ such that

- $M^n \in D_n$
- D_n is closed under union, intersection, complement, permutation
- if $X \in D_{n+1}$, then $\pi(X) \in D_n$ where $\pi(m_1, \ldots, m_{n+1}) = (m_1, \ldots, m_n)$
- $\{\overline{b}\} \in D_n$ for $\overline{b} \in M^n$
- $R^{\mathcal{M}} \in D_{n_R}$, graph$(f) \in D_{n_f+1}$
- if $X \in D_n$, $M \times X \in D_{n+1}$
- $\{(m_1, \ldots, m_n) : m_i - m_j\} \in D_n$

Then $X \subseteq \mathcal{M}^n$ is definable $\leftrightarrow X \in D_n$

Chapter 2

Basic Techniques

2.1 August 30

2.1.1 Compactness Theorem

Theorem 2.1.1 (Compactness)**.** If T is finitely satisfiable, then T has a model $\mathcal{M}$. Furthermore, $|\mathcal{M}| \leq |\mathcal{L}| + \aleph_0$

- T is finitely satisfiable if every finite subset is satisfiable

Compactness II: if $T \models \phi$, then there is finite $T_0 \subset T$ such that $T_0 \models \phi$
$T \models \phi \leftrightarrow T \cup \{\neg\phi\}$ is not satisfiable

Proposition 1: If T is finitely satisfiable, maximal, and has the witness property, then T has a model $\mathcal{M}$ with $|\mathcal{M}| \leq |\mathcal{L}|$

Proposition 2: If T is finitely satisfiable, then there is $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-theory $T^* \supseteq T$ such that T^* is finite;y satisfiable, maximal, and has the witness property. Further, $|\mathcal{L}^*| \leq |\mathcal{L}| + \aleph_0$

Definition 2.1.2.

- T is maximal if for any sentence ϕ, either $\phi \in T$ or $\neg\phi \in T$
- T has the witness property if for all $\mathcal{L}$-formulas $\phi(v)$ there is a constant c_ϕ such that $\exists v\phi(v) \to \phi(c_\phi) \in T$

Lemma 1: If T is maximal and finitely satisfiable, if there is finite $\Delta \subseteq T$ such that $\Delta \models \phi$, then $\phi \in T$.

Proof. If $\phi \notin T$, $\neg\phi \in T$. Since $\Delta \models \phi$, $\Delta \cup \{\neg\phi\}$ is not satisfiable, contradicting our assumption.

Henekin Construction:
We want to define $\mathcal{M} = (M, c^{\mathcal{M}}, R^{\mathcal{M}}, f^{\mathcal{M}})$

- Let $M = \mathcal{C}/\sim$ where $\mathcal{C}$ is the set of constant symbols and $\sim$ is the equivalence relation defined by $c_1 \sim c_2 \leftrightarrow c_1 = c_2 \in T$
- $R^{\mathcal{M}} \subseteq M^{n_R}$ by $(c_1^*, \ldots, c_{n_R}^*) \in R^{\mathcal{M}} \leftrightarrow R(c_1, \ldots, c_n) \in T$ where c^* equivalence class of c
  This is well defined since if we have $c_1' \sim c_1, \ldots, c_n' \sim c_n, R(c_1, \ldots, c_n) \in T$ then $R(c_1', \ldots, c_n') \in T$

- $f^{\mathcal{M}}$ by $f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \leftrightarrow f(c_1, \ldots, c_n) = d \in T$. SUch a d^* exists since T has the witness property: $\exists v f(c_1, \ldots, c_n) = v \to f(c_1, \ldots, c_n) \in T$
- $c^{\mathcal{M}} := c^*$

Claim: For every formula $\phi(v_1, \ldots, v_k)$ and constant symbols $c_1, \ldots, c_k$, $\mathcal{M} \models \phi(c_1^*, \ldots, c_n^*) \leftrightarrow \phi(c_1, \ldots, c_n) \in T$
This implies $\mathcal{M} \models T$

Proof. By induction on formulas $\phi(v_1, \ldots, v_l)$

- atomic formulas: $\phi(v_1, \ldots, v_k)$ is $t_1(v_1, \ldots, v_k) = t_2(v_1, \ldots, v_k)$
  Subclaim: $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = c^* \leftrightarrow t(c_1 \ldots, c_n) = c \in T$
  Proved by induction on terms
- $\phi(v_1, \ldots, v_k)$ is $R(v_1, \ldots, v_k)$. Follows by deifnition of $R^{\mathcal{M}}$
- Suppose $\phi(\overline{v})$ is $\psi_1(\overline{v}) \wedge \psi_2(\overline{v})$, then
  $\mathcal{M} \models \psi_1 \wedge \psi_2(\overline{v}) \leftrightarrow \mathcal{M} \models \psi_1(\overline{v})$ and $\mathcal{M} \models \psi_2(\overline{v}) \overset{\text{IH}}{\leftrightarrow} \psi_1(\overline{c}) \in T$ and $\psi_2(\overline{c}) \in T \overset{\text{lemma}}{\leftrightarrow} \psi_1 \wedge \psi_2(\overline{c}) \in T$
- Suppose $\phi(\overline{v})$ is $\neg\psi(\overline{v})$, then
  $\mathcal{M} \models \neg\psi(\overline{c}^*) \leftrightarrow \mathcal{M} \not\models \psi(\overline{c}^*) \overset{\text{IH}}{\leftrightarrow} \varphi(\overline{c}) \notin T \overset{\text{maximality}}{\leftrightarrow} \neg\psi(\overline{(c)}) \in T$
- Suppose $phi(\overline{v})$ is $\exists w \varphi(\overline{v}, w)$, then
  $\mathcal{M} \models \exists w \varphi(\overline{c}^*, w) \leftrightarrow \exists d \in M$ such that $\mathcal{M} \models \phi(\overline{c}^*, d) \leftrightarrow \exists d \in M$ such that $\varphi(\overline{c}, d) \in T \overset{\text{witness principle}}{\leftrightarrow}$ $\exists w \varphi(\overline{c} w) \in T$

2.2 September 1

2.2.1 Compactness

Proof of Compactness continued:
We now prove proposition 2
Lemma 1: If T is finitely satisfiable then there is $\mathcal{L}^* \supset \mathcal{L}$, $T^* \supset T$ such that T^* has the witness property and is finitely satisfiable

Proof. For each $\mathcal{L}$-formula define a new constant symbol c_ϕ. Let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v)\, \mathcal{L} - \text{formula}\}$, $T_1 = T \cup \{\exists v \phi(v) \to \phi(c_\phi) : \phi(v)\, \mathcal{L} - \text{formula}\}$.
Claim: T_1 is finitely satisfiable.
Take $\Delta \subseteq T_1$ finite. $\Delta = T' \cup \{\exists v \phi_i(v) \to c_{\phi_i} : i = 1, \ldots, k\}$ for finite T' in T. We make an $\mathcal{L}_1$-structure $\mathcal{M}_1$ that satisfies Δ. Take $\mathcal{M} \models T'$, $\mathcal{M}$ $\mathcal{L}$-structure. Make $\mathcal{M}$ an $\mathcal{L}_1$-structure by defining $c_\phi^{\mathcal{M}_1}$ for each c_ϕ. If $\mathcal{M} \models \exists v \phi(v)$ let $c^{\mathcal{M}_1}$ be such a v otherwise let $c^{\mathcal{M}_1}$ be anything.
We repeat this process, defining $\mathcal{L}_{n+1}$ from $\mathcal{L}_n$ similarly.
We have $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \cdots$, $T \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq \cdots$ such that each T_i is finitely satisfiable and for $\phi(v)$ an $\mathcal{L}_{i-1}$-formula, there is c_ϕ in $\mathcal{L}_i$ such that $\exists v \phi(v) \to \phi(c_\phi) \in T_i$.
Let $\mathcal{L}^* = \bigcup_{n\in\omega} \mathcal{L}_n$, $T^* = \bigcup_{n\in\omega} T_n$. We see T^* has the witness property.
Sub-claim: If $T_0 \subset T_1 \subset T_2 \subset \cdots$ all finitely satisfiable, then $U_{n\in\omega} T_n$ is finitely satisfiable.

Lemma 2: If T is finitely satisfiable and ϕ a sentence, one of $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable.

Proof. Assume that both $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are not finitely satisfiable. Then there are $T_0, T_1 \subseteq T$ such that $T_0 \cup \{\phi\}$ and $T_1 \cup \{\neg\phi\}$ are not satisfiable. Let $\mathcal{M} \models T_0 \cup T_1$, then $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$ so $T_0 \cup \{\phi\}$ or $T_1 \cup \{\neg\phi\}$ is satisfiable, contradicting our assumption.

Zorn's Lemma: Let $\mathcal{A}$ be a collection of sets such that for any chain $\mathcal{C} \in \mathcal{A}$. $\bigcup \mathcal{C} \in \mathcal{A}$ where $\mathcal{C}$ is a chain if for $A, B \in \mathcal{C}$ either $A \subseteq B$ or $B \subseteq A$, then $\mathcal{A}$ has a maximal element, eg. $A \in \mathcal{A}$ such that there is not $B \in \mathcal{A}$ with $A \subsetneq B$.

Lemma: For every T, finitely satisfiable, there is $T' \supseteq T$ that is maximal and finitely satisfiable.

Proof. Let $\mathcal{A} = \{S \ \mathcal{L}\text{-theory} \mid S \supseteq T,\ S \text{ finitely satisfiable }\}$. Can apply zorns lemma since for any $\mathcal{C} \subseteq A$, $\bigcup \mathcal{C} \in \mathcal{A}$ so we have a maximal S.

Example 2.2.1. Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. In a group G, $g \in G$, $\text{ord} g$ = least n such that $\overbrace{g \cdots g}^{n \text{ times}} = e$, if it exists.
Observation: If T is an $\mathcal{L}$-theory extending the axioms of groups, $\phi(v)$ such that for every n there is $G_n \models T$, $g_n \in G_n$ of order greater than n such that $G_n \models \phi(g_n)$. Then there is $G \models T$ and $g \in G$, $\text{ord}(g) = \infty$ such that $G \models \phi(g)$.

Proof. Let $\mathcal{L}' = \{\cdot, e, c\}$. Let $T^* = T \cup \phi(c) \cup \{\psi_n\}$ where ψ_n is $\underbrace{c \cdot c}_{n \text{ times}} \neq e$. T^* finitely satisfiable so follows by compactness.

This tells us that there is not sentence that axiomatizes when an element is torsion.

Lemma 2.2.2. Let κ be a carindal $\kappa \geq |\mathcal{L}|$. Let T be a satisfiable theory such that $\forall n \in \mathbb{N}$, there is $\mathcal{M} \models T$ such that $|\mathcal{M}| > n$. Then T has a model of size κ.

Proof. Extend the language by adding κ may new constant symbols c_i for $i \in \kappa$. $T^* = T \cup \{c_i \neq c_j \mid i \neq j\}$. If $\mathcal{M} \models T^*$, $|\mathcal{M}| \geq \kappa$. T^* is finitely satisfiable so by compactness T^* has a model $\mathcal{M}$, $|\mathcal{M}| \leq |\mathcal{L}^*| + \aleph_0 = \kappa$. Thus, $|M| = \kappa$.