MATH 135: Introduction to the Theory of Sets

Jad Damaj

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Contents

Chapter 1

Introduction

1.1 August 25

1.1.1 Introduction

Foundations of Mathematics: language, axioms, formal proofs

- We focus on the axioms in set theory
- We use ZFC (Zermelo-Fraenkel + Choice)
- There is only one primitive notion : $\in$
- Within the ZFC universe, everything is a set

Course Outline:

- Basic axioms
- Operations, relations, functions
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- carindals
- AC
- ordinals

1.1.2 Basics

Principle of Extensionality: Two sets A, B are the same $\leftrightarrow$ they have the same elements $\forall x(x \in A \leftrightarrow x \in B)$
Example 1.1.1. $2, 3, 5 = \{5, 2, 4\} = \{2, 5, 2, 3, 3, 2\}$

> **Definition 1.1.2.** There is a set with no elements, denoted $\varnothing$

- $\varnothing \neq \{\varnothing\}$
- $A \subseteq B$: A is a subset of $B \leftrightarrow$ each element of A is in B (use $\subsetneq$ to denote proper subset)

- $\{2\} \subseteq \{2,3,5\}$ but $\{2\} \notin \{2,3,5\}$
- Power set opertaion: $\mathcal{P}(A) = \{B \mid B \subseteq A\}$

We can define a hierarchy:
$V_0 = \varnothing$, $V_1 = \mathcal{P}(\varnothing) = \{\varnothing\}$, $V_2 = \mathcal{P}\mathcal{P}(\varnothing) = \{\varnothing, \{\varnothing\}\}$
$V_3 = \mathcal{P}(V_2) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$, $V_4, \dots$
$V_\omega = \bigcup_{n\in\mathbb{N}} V_n$, $\mathcal{P}(V_\omega)$, $\mathcal{P}\mathcal{P}(V_\omega), \dots, V_{\omega+\omega}, \dots, V_{\omega+\omega+\cdots}, \dots, V_{\omega\times\omega}, \dots, V_{\omega^\omega}$

Chapter 2

Axioms and Operations

2.1 August 30

2.1.1 Zermelo Fraenkel Axioms of Set Theory

Setting: in ZFC all objects are sets
Language: contains vocabulary ($\in$), logical symbols ($=, \wedge, \vee\ \exists, \forall, \neg$), variables ($x, y, A, B$, etc.)

Axiom 2.1.1 (Extensionality Axiom)**.** Two sets are the same if they have the same elements
$\forall A, B(\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B)$

Axiom 2.1.2 (Empty Set Axiom)**.** There is a set with no members, denoted $\varnothing$
$\exists A \forall x(x \notin A)$

Axiom 2.1.3 (Pairing Axiom)**.** For any sets u, v there is a est whose elements are u and v, denoted $\{u, v\}$
$\forall u, v \exists A \forall x(x \in A \leftrightarrow x = u \vee x = v)$

Axiom 2.1.4 (Union Axiom (Preliminary Form))**.** For any sets a, b there is a set whose elements are elements of a and elements of b, denoted $a \cup b$
$\forall a, b \exists A \forall x(x \in Ax \in u \vee x \in v)$

Axiom 2.1.5 (Powerset Axiom)**.** Each set A, has a power set $\mathcal{P}(A)$.
$\forall A \exists B \forall x(x \in B \iff x \subseteq A)$ where $x \subseteq A$ stands for $\forall y(y \in x \rightarrow y \in A)$

Axiom 2.1.6 (Union Axiom)**.** For any set A, there is a set $\bigcup A$ whose members are members of the members of A.
$\forall A \exists B \forall x(x \in B \leftrightarrow \exists y \in A(x \in y))$

Idea for the subset axiom: For any set A, there is a set B whose memebrs are members of A satisfying some property.

eg. $B = \{x \in A \mid x \text{ satisfies property } P\} \subseteq A$

Example 2.1.7. $B = \{n \in \mathbb{N} \mid n\text{cannot be described in less that 20 words}\}$

- let b be the smallest element in B, then b is the smallest element that cannot be described in 20 words.
- Paradox : need to use formal language to express property P.

Example 2.1.8. Let $B = \{x \mid x \notin x\}$
Question: $B \in B$? $B \in B \leftrightarrow B \notin B$: need to have property be contained in some larger set.

We can now restate the axiom more formally:

Axiom 2.1.9 (Subset Axiom (Scheme))**.** For each formula $\phi(x)$, there is an axiom:
$\forall A \exists B \forall x (x \in B \leftrightarrow x \in A \wedge \phi(x))$

Example 2.1.10. Suppose there is a set of all sets A. Consider $B = \{x \in A \mid x \notin x\}$. Then $B \in B \leftrightarrow B \notin B$, contradiction. So there can be no such set A.

The language of 1rst order logic for ZFC:
The following are formulas:

- $x = y$, $x \in y$ atomic formulas
- $(\varphi \wedge \psi), (\varphi \vee \psi), \neg\varphi$ where φ, ψ are formulas
- $\exists v \varphi, \forall x \varphi$

Example 2.1.11. $\varphi(v, w) := (\exists v (v \in x \wedge \neg v = w)) \rightarrow (\forall y (\neg y \in y))$ is a formula

Chapter 3

Relations and Functions

3.1 September 1

3.1.1 Relations and Functions

Ordered Pair: $\langle a, b\rangle = \langle c, d\rangle \leftrightarrow a = c, b = d$

Definition 3.1.1. $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$

Cartesian product of A and B, denoted $A \times B = \{\langle x, y\rangle x \in A, y \in B\}$
Using the subset axiom $A \times B = \{z \in \mathcal{P}\mathcal{P}(A \cup B) \mid \exists x \in A \exists y \in B z = \langle x, y\rangle\}$
Observation: $\langle x, y\rangle \in \mathcal{P}\mathcal{P}(C)$ for $x, y \in C$
$\{x\}, \{x, y\} \in \mathcal{P}(C)$ so $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}(C)$ so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}(C)$

Definition 3.1.2. A binary relation is a set R whose elements are ordered pairs.

If $R \subset A \times B$ then R is a relation from $A \to B$.

Definition 3.1.3. Given a relation R, dom $R = \{x \in \bigcup\bigcup R \mid \exists y \langle x, y\rangle \in R\}$,
range $R = \{y \in \bigcup\bigcup R \mid \exists x \langle x, y\rangle \in R\}$, field (R) = dom$(R)\cup$range(R)

Example 3.1.4. $R = \{\langle a, b\rangle, \langle c, d\rangle, \langle e, f\rangle\} = \{\{\{a\}, \{a, b\}\}, \{\{c\}, \{c, d\}\}, \{\{e\}, \{e, f\}\}\}$
$\bigcup R = \{\{a\}, \{a, b\}, \{c\}, \{c, d\}, \{e\}, \{e, f\}\}$
$\bigcup\bigcup R = \{a, b, c, d, e, f\}$

n-ary relations: define n-tuple by $\langle a, b, c\rangle = \langle\langle a, b, \rangle, c\rangle$ etc.

Definition 3.1.5. A function is a relation F such that $\forall x, y, z\ \langle x, y\rangle \in F$ and $\langle x, z\rangle \in F \to y = z$

$\forall x\ \in$dom (F) there is y such that $\langle x, y\rangle \in F$. If $A =$ dom(F), $B \supseteq$ range(F) then F is said to a funtion from A to B, $f : A \to B$
We say that $f : A \to B$ is onto if $B =$ range(F)

Definition 3.1.6. F is injective if $\forall x, y, z\ \langle x, z\rangle \in F \wedge \langle y, z\rangle inF \to x = y$.

Definition 3.1.7. For a set A, relations F, G

(a) inverse $F^{-1} = \{\langle y, x\rangle \mid \langle x, y\rangle \in F\}$

(b) composition: $G \circ F = \{\langle x, z\rangle \mid \exists y \langle x, y\rangle \in F, \langle y, z\rangle \in G\}$

(c) restriction: $F \restriction_A - \{\langle x, y\rangle \in F \mid x \in A\}$

(d) image of A under F, $F[\![A]\!] = \{y \mid \exists x \in A \langle x, y\rangle \in F\} = \text{range}(F \restriction_A)$

Example 3.1.8. If F is a function, F^{-1} may not be a function. F^{-1} is a function $\leftrightarrow$ F is one to one.

Example 3.1.9. $F^{-1} \circ F = \{\langle x, x\rangle \mid x \in \text{dom}(F)\}$ if F is one to one
More generally, $F^{-1} \circ F = \{\langle x, z\rangle \mid \exists y \in \text{range } F \langle x, y\rangle, \langle z, y\rangle \in F\}$.

3.2 September 6

3.2.1 Functions and Relations

Theorem 3.2.1. Let $F : A \to B$ with $A \neq \varnothing$

(a) There is a function $G : B \to A$ such that $G \circ F = \text{id}_A \leftrightarrow F$ is one to one.

(b) There is a function $G : B \to A$ such that $F \circ F = \text{id}_B \leftrightarrow F$ is onto.

Proof. (a) Suppose there is such a G. Take a_1, a_2 such that $F(a_1) = F(a_2)$, then $a_1 = G \circ F(a_1) = G \circ F(a_2) = a_2$
Conversely, suppose F is one to one. We want to define $G : B \to A$ given $b \in B$, let $G(b)$=the unique $a \in A$ such that $F(a) = b$ if b $\in$range(F). If $b \notin \text{range}(F)$, let $G(b) = a_0$ with $a_0 \in A$ arbitrary (exists since A nonempty)

(b) Supppose that $G : B \to A$, with $F \circ G = \text{id}_B$ Want to show $\forall b \in B \exists a\, F(a) = b$ Take $a = G(b) \to F(a) = F(G(b)) = b$
Conversely, suppose F is onto. We want to define G, given $b \in B$ want to define $G(b)$ such that $F(G(b)) = b$, equivalently, want $G(b) \in F^{-1}(\{b\})$. Since F is onto $F^{-1}(\{b\})$ is nonempty. Let $G(b)$ be any element of $F^{-1}(b)$, equivalently $G \subseteq F^{-1}$ and $\text{dom}(G) = B$ =dom(F^{-1}).

Example 3.2.2. Suppose $A = \mathbb{N}$, let $G = \{(b, a) \in B \times \mathbb{N} : a \text{ is least satisfying } f(a) = b\}$

- Don't have a method to specify such elements in gneral.

Axiom 3.2.3 (Axiom of Choice - Form I)**.** For every relation R, there is a function $G \subseteq R$ with dom(G) =dom(R)

3.2.2 Infinite Cartesion Products

$A \times B = \{\langle x, y\rangle \in \mathcal{P}\mathcal{P}(A \cup B) \mid x \in A \wedge y \in B\}$

Definition 3.2.4. Let M be a function with domain I such that for every $i \in I$, $H(i)$ is a set. Let

$$\bigtimes_{i \in I} H(i) - \{f : I \to \bigcup H(i) \mid f(i) \in H9 = (i)\}$$

Example 3.2.5. Let ω_g be $\{G \subseteq \mathbb{R} \mid 0 \neq G,\, G \cup \{0\}$ is closed under addition $\}$
$\bigtimes_{G \in \omega_g} = \bigtimes_{G \in \omega_g} H(G)$ is a function such that for each $G \in \omega_g$, you get an element of G.
Observation: If one of the $H(i)$ is $\varnothing$, then $\bigtimes_{i \in I} H(i) = \varnothing$

Axiom 3.2.6 (Axiom of Choice - Form II)**.** If H is a function with domain I such that $H(i) \neq \varnothing$ $\forall i \in I$, then $\bigtimes_{i \in I} H(i) \neq \varnothing$

(ACI) $\to$ (ACII): We are given H with $H(i) \neq \varnothing$ for all i. Want $f : I \to H(i)$ with $f(i) \in H(i)$ $\forall i \in I$. Let $R = \{\langle i, h \rangle \in I \times \bigcup_{i \in I} H(i) \mid h \in H(i)\}$. $\mathrm{dom}(R) = I$, since $H(i) \neq \varnothing$ there is $h \in H(i)$ so $\langle i, h \rangle \in R$. BY ACI, there is $F \subseteq R$ with $\mathrm{dom}(F)$=$\mathrm{dom}(R) = I$. $\forall i$, $\langle i, f(i) \rangle \in R$ so $f(i) \in H(i)$

Chapter 4

Naturals, Rationals, Reals

4.1 September 8

4.1.1 Natural Numbers

Idea: each natural number is the set of all the previous numbers.
$0 = \varnothing, 1 = \{\varnothing\}, 2 = \{\varnothing, \{\varnothing\}\}, \ldots, n = \{0, 1, \ldots, n-1\}, \ldots$

Definition 4.1.1. The succesor of a set a is defined as $a^+ = a \cup \{a\}$

Definition 4.1.2. A set I is inductive if $\varnothing \in I$ and $\forall a \in I, a^+ \in I$

Definition 4.1.3. a is a natural number if it belongs to all inductive sets, $\forall I(I \text{ inductive} \rightarrow a \in I)$

If I is any inductive set, let $\omega = \{a \in I \mid a \text{ belongs to all inductive sets}\}$=the minimal inductive set.
Observation: ω is inductive because $\varnothing$ is in all inductive sets and if n belongs to all inductive sets then so does n^+

Axiom 4.1.4 (Ifinity Axiom)**.** There is an inductive set.

Inductivion Principle: If $A \subseteq \omega$ is inductive set $A = \omega$

Example 4.1.5. Every natural number is 0 or the succesor of some natural number.
Let $A = \{n \in \omega \mid n = 0 \vee \exists m \in \omega\, n = m^+\}$. A is inductive so $A = \omega$

Definition 4.1.6. A set A is transitive if one of the following equivalent conditions holds:

- if $x \in a \in A$, then $x \in A$
- $\bigcup A \subseteq A$
- if $a \in A$, then $a \subseteq A$
- $A \in \mathcal{P}(A)$

Example 4.1.7. Transitive sets includ $\varnothing$, each natural number, ω, V_ω
Claim: $A = \{n \in \omega \mid n \text{ is transitive } \}$ is inductive (implies all nautrual numbers are transitiev)

- Base: $0 \in A$ since $\varnothing$ is transitive
- Inductive Step: Suppose $n \in A$ transitive, want to show n^+ is transitive.
  Consider $x \in a \in n^+ = n \cup \{n\}$. If $a = n$, $x \in n \subseteq n^+$. If $a \in n$, $x \in a \in$ so by transitivity $x \in\subseteq n^+$ so $x \in n^+$

Theorem 4.1.8. If a is tansitive, then $\bigcup a^+ = a$

Proof. ($\supseteq$) $a = \bigcup\{a\} \subseteq \bigcup(a \cup \{a\} = \bigcup a^+)$ ($a \in a^+$ so $a \subseteq \bigcup a^+$)
($\subseteq$) Take $x \in \bigcup a^+$, then let $b \in a^+$ with $x \in b$. If $b = a$, $x \in a$. If $b \in a$, $x \in b \in a$ so $x \in a$.

- If a, b transitive and $a^+ = b^+$ then $a = \bigcup a^+ = \bigcup b^+ = b$ so succesor function is one to one on transitive sets, more specifically ω.

Fix a number $k \in \omega$. Consdier the following functions:

- $A_k : \omega \to \omega$ by $A_k(0) = 0$, $A_k(n^+) = A_k(n)^+$
- $M_k : \omega \to \omega$ by $M_k(0) = 0$, $M_k(n^+) = A_k(M_k(n))$

4.2 September 13

4.2.1 Operations on the Natural Numbers

Theorem 4.2.1. Let A be a set, $a \in A$ and $F : A \to A$. Then there is a unique function $h : \omega \to A$ such that:

1. $h(0) = a$
2. $h(n^+) = F(h(n))$ for all $n \in \omega$

Proof. Let $h = \{\langle n, b\rangle \in \omega \times A \mid \text{there is } g : n^+ \to A \text{ such that } g(0) = a, g(i^+) = F(g(i)) \wedge g(n) = b\}$
Claim 1: For all n there is a $g : \{0, \ldots, n\} \to A$ such that $g(0) = a$, $g(i^+) = F(g(i))$
Claim 2: Such a g is unique.

Proof of Claim 1. Let $I = \{n \in \omega \mid \text{ such a } g \text{ exists}\}$. Want to show that I is inductive.

1. $0 \in I$: let $g : \{0\} \to A$ be such that $g(0) = a$ eg. $g = \{\langle 0, a\rangle\}$
2. Suppose $n \in I$, we know such a g exists for n, $g : \{0, \ldots, n\} \to A$. We want $\tilde{g} : \{0, \ldots, n, n^+\} \to A$.
   Let $\tilde{g} = g \cup \{\langle n^+, F(g(n))\rangle\}$

□

Proof of Claim 2. Suppose $g, \tilde{g} : \{0, \ldots, n\} \to A$ such that $g(0) = a = \tilde{g}(0)$, $g(i^+) = F(g(i))$, $\tilde{g}(i^+) = F(\tilde{g}(i^+)), i < n$. We want to show $g(i) = \tilde{g}(i)$ $\forall i \leqslant n$. $g(0) = \tilde{g}(0)$, $g(i^+) = F(g(i)) = F(\tilde{g}(i)) = \tilde{g}(i^+)$

Can formally show this by induction using $I = \{i \in \omega \mid i \in n^+ \wedge g(i) = \tilde{g}(i) \vee i \notin n^+\}$ $\square$

Claim 3: $\forall n \in \omega,\ h(n^+) = F(H(n))$

Definition 4.2.2. Given $k \in \omega$, define $A_k : \omega \to \omega$ by $A_k(0) = k$, $A_k(n^+) = (A_k(n))^+$. Define $n+k = A_k(n)$
Define $M_k : \omega \to \omega$ by $M_k(0) = 0$, $M_k(n^+) = A_k(M_k(n))$, let $n \times k = M_k(n)$.
Let $m < n$ if $m \in n$

Theorem 4.2.3. We can show the associativity of addition: $\forall a, b, v \in \omega((a + b) + c = a + (b + c))$, commutativity of addition: $\forall a, b \in \omega a + b = b + a$, etc.

4.2.2 Integers

Let $\sim$ be the following equivalence relation on $\omega \times \omega$ by $\langle a, b\rangle \sim \langle c, d\rangle \leftrightarrow a + d = b + c$
Define $\mathbb{Z} = \omega \times \omega/\sim$. $0_{\mathbb{Z}} = [\langle 0, 0\rangle]$, $1_{\mathbb{Z}} = [\langle 1, 0\rangle]$
Let $[\langle a, b\rangle] +_{\mathbb{Z}} [\langle c, d\rangle] = [\langle a+c, b+d\rangle]$. One needs to show this is well defined eg. if $\langle a, b\rangle \sim \langle a', b'\rangle$, $\langle c, d\rangle \sim \langle c', d'\rangle$ then $\langle a + c, b + d\rangle \sim \langle a' + c', b' + d'\rangle$/
Let $[\langle a, b\rangle] \times_{\mathbb{Z}} [\langle c, d\rangle] = [\langle ac + bd, ad + bc\rangle]$
Let $E : \omega \to \mathbb{Z}$ by $E(n) = [\langle n, 0\rangle]$

4.2.3 Rationals

Let $\sim$ be the following equivalence relation on $\mathbb{Z} \times \mathbb{Z}\backslash\{0\}$. $\langle a, b\rangle \sim \langle c, d\rangle \iff a \times_{\mathbb{Z}} d = b \times_{\mathbb{Z}} c$
Define $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}\backslash\{0\}/\sim$. $0_{\mathbb{Q}} = [\langle 0, 1\rangle]$, $1_{\mathbb{Q}} = [\langle 1, 1, \rangle]$
Let $[\langle a, b\rangle] \times_{\mathbb{Q}} [\langle c, d\rangle] = [\langle a \times c, b \times d\rangle]$
Let $[\langle a, b\rangle] +_{\mathbb{Q}} [\langle c, d\rangle] = [\langle ad + bc, bd\rangle]$
$E : \mathbb{Z} \to \mathbb{Q}$ by $E(z) = [\langle z, 1\rangle]$

4.3 September 15

4.3.1 Reals (Dedekind Cuts)

Definition 4.3.1. A dedekind cut is a subset $D \subseteq \mathbb{Q}$ such that

- $\varnothing \neq D \neq \mathbb{Q}$
- D is closed downwards, if $d \in D$, $c < d \to c \in D$
- D has no greatest element.

Let $\mathbb{R} = \{D \in \mathcal{P}(\mathbb{Q}) \mid D \text{ is a dedekind cut }\}$
$\sqrt{2} = \{q \in \mathbb{Q} \mid q \times_{\mathbb{Q}} q < 2\}$, $e = \{q \in \mathbb{Q} \mid exn \in \omega\, q <_{\mathbb{Q}} (1 + \frac{1}{N})^N\}$ For $r \in \mathbb{R}$, $-r = \{q \in \mathbb{Q} \mid -q \in r\}\backslash\{-\sup(r)\}$
For $r_1, r_2 \in \mathbb{R}$, $r_1 \leqslant_{\mathbb{R}} r_2 \iff r_1 \subseteq r_2$
$r_1 \times r_2 = \{q \in \mathbb{Q} \mid \exists q \leqslant 0 \in r \exists b \leqslant 0 \in r_2\, q, a \times_{\mathbb{Q}} b$ if $r_1, r_2 > 0, \ldots$

Theorem 4.3.2. $(\mathbb{R}, 0, 1, +, \times, \leqslant)$ is an ordered field.

$E : \mathbb{Q} \to \mathbb{R}$ is a field embedding.

Chapter 5

Cardinal Numbers and the Axiom of Choice

5.1 September 15

5.1.1 Cardinality

Definition 5.1.1. A is equinumerous to B (written $A \approx B$) if there is a bijection $f : A \to B$

Theorem 5.1.2. For every A, B, C

- $A \approx A$
- If $A \approx B$, $B \approx B$
- If $A \approx B$, $B \approx C$ then $A \approx C$

Lemma 5.1.3. $\mathbb{Z} \approx \omega$

Proof. For $z \in Z$, $f(z) = \begin{cases} -2z & z \leqslant 0 \\ 2z+1 & z > 0 \end{cases}$

Lemma 5.1.4. $\mathbb{Q} \approx \omega$

Proof. $f : \omega \to \mathbb{Z} \times \mathbb{Z}^+$, $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^+ / \sim$
$f' : \omega \to \mathbb{Q}$, $f'(n) =$ least $i \in \omega$ $g(i) \notin \{f(1), \ldots, f(n-1)\}$

Lemma 5.1.5. $\mathbb{R} \approx (0, 1)_{\mathbb{R}}$

5.2 September 20

5.2.1 Cardinality

Lemma 5.2.1. 1. $\mathbb{N} \not\approx \mathbb{R}$

2. For any set A, $A \not\approx \mathcal{P}(A)$

Proof. 1. Let $f : \omega \to \mathbb{R}$, claim f is not onto. Want $r \notin \operatorname{ran}(f)$, $\forall n \in \omega r \neq f(n)$. Choose A_0 such that $f(0) \notin A_0$. Given A_n such that $f(0), \dots, f(n) \notin A_n$. Divide A_n by 2, take half that does not contain $f(n+1)$ to be A_{n+1}, then $A_0 \supset A_1 \supset A_2 \supset \cdots$, $\bigcap_{n\in\omega} A_n \neq \varnothing$ and for each n, $f(n) \notin A_n$ so $f(n) \notin \bigcap A_n$

2. let $f : A \to A$. Claim f is not onto. Let $B = \{b \in A \mid b \notin f(b)\}$. Claim $B \notin \operatorname{range}(f)$. Suppose for contradiction that $B = f(b)$ for $b \in A$, $b \in B \leftrightarrow b \notin f(b) \iff b \notin B$, contradiction.

Definition 5.2.2. A set A is finite if $\exists n \in omega(A \approx n)$ eg. $\exists n\, ex f : n \to A$ bijection. $A = \{f(0), f(1), \dots, f(n-1)\}$

Lemma 5.2.3 (Pigeonhole Principle)**.** No finite set is equinumerous to a finite subset of itself.

Lemma 5.2.4. If B is a proper subset of $n \in \omega$ ther is $m < n$ such that $B \approx m$

Proof. Use induction on n. Let $A = \{n \in \omega \mid \forall B \in n \exists m \in n\, B \approx n\}$.
Claim A is inductive. $0 \in A$ trivial, $1 \in A$. $B \subsetneq \{\varnothing\} \to B = \varnothing \to B \approx 0$.
Suppose $n \in A$, want to show $n^+ \in A$. Take $B \subsetneq n^+ = n \cup \{n\}$. If $n \in B$, $B \cap n \subseteq n$ so $\exists m < n\, B \cap n \approx m$ so $B \approx m^+ < n^+$. If $n \notin B$, either $B \cap n = n$ so $B \approx n < n^+$ of $B \cap n \subsetneq n$ so $\exists m < n\, B = B \cap n \approx m$.

Proof (Pigeonhole Principle)**.** Take n, $B \subsetneq n$, $B \approx n$. THen $B \approx m$ for some $m < n$ so $m \approx n$. Let $A = \{n \mid Am < n\, m \not\approx n\}$. Claim A is inductive. $0 \in A$, suppose $n \in A$, want to show $n^+ \in A$. Idea: turn a bijection for $n^+ \approx m^+$ so a bijection $n \approx m/$

Corollary 5.2.5.
- No finite set is equinumerous to a proper subset
- ω is not finite ($\omega \approx \omega\backslash\{0\}$ by $n \mapsto n+1$)
- Every finite set is equinumerous to a unique natural number.
  We call that number the cardinality of A, $\operatorname{card}(A)$
- A subset of a finite subset is finite

Definition 5.2.6. A set κ is said to be a cardinal if

- κ is transitive (if $x \in a, a \in \kappa \to x \in \kappa$)
- $\in$ is a linear order on κ ($\forall x, y\, x \in y$ or $y \in x$ or $x = y$)
- $\forall x \in \kappa\, x \not\approx \kappa$

Theorem 5.2.7. For every set A, there is a unique cardinal κ such that $A \approx \kappa$. We call this κ card(A)

Example 5.2.8.
- $n = \{0 \in 1 \in 2 \in \cdots \in n-1\}$ is a cardinal
- $\omega = \{0 \in 1 \in 2 \in \cdot\}$ is a cardinal
- $\omega^+ = \{0, 1, 2, \ldots\} \cup \{\omega\} \approx \omega$ is not a carinal

Notation: $\omega - \aleph_0$, card$(\mathbb{R}) = 2^{\aleph_0}$, smallest cardinal greater than $\aleph_0 = \aleph_1$

5.3 September 22

5.3.1 Cardinals

Definition 5.3.1. Given carindals κ and λ let

- $\kappa + \lambda =$ card$(K \cup L)$ where K and L are disjoint sets of carindality κ and λ
- $\kappa \cdot \lambda =$ card$(K \times L)$ where K and L are sets of carindality κ and λ
- $\kappa^\lambda = \{f$ function $L \to K\} =$ card$({}^L K)$ were K and L are sets of carindality κ and λ

Notation: ${}^A B = \{f : f$ is a function $A \to B\}$

Theorem 5.3.2. Let κ, λ, μ be carindals

- $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

  Proof. Let K, L, M be disjoint sets of size κ, λ, μ. $K \cup (L \cup M) = (K \cup L) \cup M$

- $\kappa + \lambda = \lambda + \kappa$
- $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$

  Proof. $(K \times L) \times M \to K \times (L \times M)$ by $\langle\langle k, l\rangle, m\rangle \to \langle k, \langle l, m\rangle\rangle$

- $\kappa \cdot \lambda = \lambda \cdot \kappa$
- $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$

  Proof. $K \times (L \cup M) \approx (K \times L) \cup (K \times M)$

- $\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu$

- $\kappa^{\lambda\cdot\mu} = (\kappa^{\lambda})^{\mu}$

Proof. $F : {}^{L\times M}K \to {}^{M}{}^{L}K$, $f : {}^{L\times M}K$, $F(g)$ = the function that maps m to $g_m : L \to K$ where $g_m(l) = g(l,m)$
$F^{-1}(h)$ with $h : M \to ({}^{L}K)$ is g such that $g(l,m) = h(m)(l)$

Definition 5.3.3. A is dominated by B (written $A \preceq B$) if there is a one to one function from $A \to B$

$A \preceq B \iff \text{card}(A) \leqslant \text{card}(B)$

Example 5.3.4.
- $A \subseteq B \iff A \preceq B$
- $\mathbb{N} \approx \mathbb{N} \approx \mathbb{Q} \preceq \mathbb{R} \preceq \mathbb{C}$

Example 5.3.5. $\mathbb{R} \approx (0,1)_{\mathbb{R}} \preceq {}^{\omega}2 \preceq \mathbb{R}$

- $(0,1)_{\mathbb{R}} \preceq {}^{\omega}2$. Given r, let $f_r : \omega \to \{0,1\}$ be $f_r(n) = n$th digit of binary representation of r avoiding representations that end in all 1s.
- ${}^{\omega}2 \preceq \mathbb{R}$, $f : \omega \to 2 \mapsto \sum_{i\in\omega} f(i)\cdot 10^{-1}$

Observation: ${}^{2}\omega \approx \mathcal{P}(\omega)$ $\text{card}({}^{2}\omega) = 2^{\aleph_0}$

5.4 September 27

5.4.1 Schroder-Bernstein Theorem

Example 5.4.1. Show that $\mathbb{R} \cup \{*\}$ and $\mathbb{R}$ are equinumerous.

We define f by $f(*) = 0$, $f(r) = \begin{cases} r+1 & r \in \mathbb{N} \\ r & r \in \mathbb{R}\backslash\mathbb{N} \end{cases}$

Lemma 5.4.2. If A is finite, then $\omega \preceq A$

Proof. $A \neq 0$ so $\exists a_0 \in A$. Let $f(0) = a_0$, $A\backslash\{a_0\} \neq \varnothing$ since $A \not\approx 1$ so $a_1 \in A\backslash\{a_1\}$ Let $f(1) = a_1$.
We want $G : \{\text{finite subsets of } A\} \to A$ such that $G(F) \in A\backslash F$. Let $R = \{\langle F, a\rangle | F \text{ finite } a \in A\backslash F\}$
Observation: dom(R)= all finite subsets of A. Since A is not finite $A\backslash F \neq \varnothing$ for all finite sets, $F \subseteq A$. Use AC to get a function $G \subseteq R$ such that dom (G) = dom(R).
Define $f : \omega \to A$ by recusrion. $f(0) = a_0$, $f(n) = G(\{f(0), \ldots, f(n-1)\}) \in A\backslash\{f(0), \ldots, f(n-1)\}$.

Corollary 5.4.3. A set A is infinite $\leftrightarrow$ A is equinumerous to some proper subset of itself.

If A is infinite, there is 1 to 1 $f : \omega \to A$. We define a bijection $h : A \to A\{f(0)\}$ by $h(a) = \begin{cases} a & a \notin \text{dom}(f) \\ f(n+1) & a = f(n) \end{cases}$

Theorem 5.4.4 (SChroder Bernstein Theorem). If $A \preceq B$, $B \preceq A$, then $A \approx B$

Proof. Let $f : A \to B$ 1 to 1, $g : A \to B$ 1 to 1. We want $h : A \to B$ bijection.
Let $C_0 = A\backslash\text{ran}(g)$, let $D_0 = f[\![C_0]\!]$, $C_1[\![D_0]\!]$. $C_0 \cap C_1 = \varnothing$ because $C_0 = A\backslash\text{ran}g$ and $C_1 \subseteq \text{ran}(g)$. We recursivley define $C + n + 1 = g[\![D_n]\!]$, $D_{n+1} = [\![C_{n+1}]\!]$. We see that C_n disjoint, D_n disjoint. Define $h(a) = \begin{cases} g(a) & a \in \bigcup_{n\in\omega} C_n \\ g^{-1} & a \in A\backslash\bigcup_{n\in\omega} C_n \end{cases}$. $f \rightharpoonup \bigcup_{n\in\omega}$ is a bijection $\bigcup C_n \to \bigcup D_n$. $g \rightharpoonup \bigcup_{n\in\omega} D_n$ is a bijection $B\backslash\bigcup_{n\in\omega} D_n \to A\backslash A\backslash\bigcup_{n\in\omega} C_n$

- Follows that $\mathbb{R} \approx \mathcal{P}(\omega)$

5.5 September 29

5.5.1 Zorn's Lemma

Theorem 5.5.1. For every A, B either $A \preceq B$ or $B \preceq A$.

Zorn's Lemma: Let $\mathcal{A}$ be a collection of sets such that for every chain $\mathcal{C} \subseteq \mathcal{A}$, $\bigcup\mathcal{C} \in \mathcal{A}$. Then $\mathcal{A}$ has a maximal element.

Definition 5.5.2. $\mathcal{C}$ is a chain if for every $C, D \in \mathcal{C}$ either $C \subseteq D$ or $D \subseteq C$
$B \in \mathcal{A}$ is maximal if ther is no $C \in \mathcal{A}$ with $B \subsetneq C$

We prove the following theorem to get some practice with Zorn's Lemma

Theorem 5.5.3. Every vector space has a basis.

Proof. Let V be a vector space over a field k. $B \subseteq V$ is linearly independent if for every $v_1, \ldots, v_n \in B$, distinct, $k_1, \ldots, k_n$ such that $\sum k_i v_i = 0$, $k_1 = k_2 = \cdots = 0$. B is a basis if B is linearly indpendent and $\langle B\rangle = V$ where $\langle B\rangle = \{\sum_{i=1}^n k_i v_i \mid v_1, \ldots, v_n \in B,\ k_1, \ldots, k_n \in k\}$
Let $\mathcal{A} = \{B \subseteq V \mid B \text{ is linearly independent}\}$. W need to showt that if $\mathcal{C} \subseteq \mathcal{A}$ is a chain then $\bigcup\mathcal{C} \in \mathcal{A}$. Consider a chain $\mathcal{C}$ consisting of linearly independent sets. To prove that $\bigcup\mathcal{C}$ is linearly independent assume we have $v_1, \ldots, v_n \in \bigcup\mathcal{C}$, $k_1, \ldots, k_n \in k$ with $\sum_{i=1}^n v_i k_i = 0$. For each v_i, there is $C_i \in \mathcal{C}$ with $v_i \in C_i$. One C_i contains all the others, say C_{i_0}. $v_1, \ldots, v_n \in C_{i_0}$. C_{i_0} is linearly independent so all $k_i = 0$. Now we apply Zorns Lemma to get a maximal element $B \in \mathcal{A}$. B is a maximal linearly independent set in V. $\langle B\rangle = V$ since if there is some $v \in V\backslash\langle B\rangle$ then $B \cup \{v\}$ is linearly indpendent, contradicting the maximality of B.

Lemma 5.5.4. Let $\mathcal{C}$ be a collection of functions. Then

(i) $\bigcup\mathcal{C}$ is a function

(ii) dom $(\bigcup\mathcal{C}) = \bigcup\{\text{dom } f : f \in \mathcal{C}\}$

(iii) ran $(\bigcup\mathcal{C}) = \bigcup\{\text{ran } f : f \in \mathcal{C}\}$

(iv) if all functions in $\mathcal{C}$ are 1 to 1, then $\bigcup\mathcal{C}$ is one to one.

Proof. (ii): dom $(\bigcup \mathcal{C}) = \{a \mid \exists b \langle a,b\rangle \in \bigcup \mathcal{C}\} = \{a \mid \exists b \exists f \in \mathcal{C} \langle a,b\rangle \in f\} = \{a \mid \exists f(\exists b \langle a,b\rangle \in f)\} = \{a \mid \exists f \in \mathcal{C}\, a \in \text{dom } f\} = \bigcup\{\text{dom } f : f \in \mathcal{C}\}$
(i): $\bigcup \mathcal{C}$ is a relation. Want to show it is a function. Suppose $\langle a,b\rangle \in \bigcup \mathcal{C}$ and $\langle a,c\rangle \in \bigcup \mathcal{C}$. $\exists f \in \mathcal{C}$, $\langle a,b\rangle \in f$, $\exists g \in \mathcal{C}$ $\langle a,c\rangle \in g$. Since $\mathcal{C}$ a chain, either $f \subseteq g$ or $g \subseteq f$. If $f \subseteq g$, $\langle a,b\rangle, \langle a,c\rangle \in g$, a function, $b = c$.
(iv): $\bigcup \mathcal{C}$ is a function. Want to show it is one to one. Suppose $\langle a,b\rangle \in \bigcup \mathcal{C}$ and $\langle c,b\rangle \in \bigcup \mathcal{C}$. $\exists f \in \mathcal{C}$, $\langle a,b\rangle \in f$, $\exists g \in \mathcal{C}$ $\langle c,b\rangle \in g$. Since $\mathcal{C}$ a chain, either $f \subseteq g$ or $g \subseteq f$. If $f \subseteq g$, $\langle a,b\rangle, \langle c,b\rangle \in g$, a one to one, $a = c$.

5.6 October 4

5.6.1 Axiom of Choice

Theorem 5.6.1. For all set C and D, we have $C \preceq D$ or $D \preceq C$

Proof. Let $\mathcal{A} = \{f \subseteq C \times D \mid f \text{ is a one to one function }\}$. If $\mathcal{C} \subseteq A$ is a chain $\bigcup \mathcal{C}$ is a function with dom $(\bigcup \mathcal{C}) = \bigcup\{\text{dom } f : f \in \mathcal{C}\} \subseteq C$, ran $(\bigcup \mathcal{C}) = \bigcup\{\text{ran } f : f \in \mathcal{C}\} \subseteq D$ so $\bigcup \mathcal{C} \in \mathcal{A}$. By Zorn's lemma, $\mathcal{A}$ has a maximal element, call it F, a one to one function with dom $F \subseteq C$, ran $F \subseteq D$
Claim dom $F = C$ or ran $F = D$. If not there is $c \in C \backslash \text{dom } F$ and $d \in D \backslash \text{ran } F$. Let $G = F \cup \{\langle c,d\rangle\}$. We see that G is a one to one function, $G \subseteq C \times D$ so $G \in \mathcal{A}$, $F \subsetneq G$, contradicting the maximality of F.
If dom $F = C$, we have $F : C \to D$ and $C \preceq D$.
If dom $F = D$, we have $F : D \to C$ and $D \preceq C$

Theorem 5.6.2. The following are equivalent:

1. For any relation R, there is a function $F \subseteq R$ and dom $F =$ dom R
2. If H is a function, $I =$ dom H, $\forall i \in I\, H(i) \neq \varnothing$, then $\times_{i \in I} H(i) \neq \varnothing$
3. For every set A ther eis a function $F : \mathcal{P}(A) \backslash \{\varnothing\} \to A$ such that $\forall B \subseteq A$, $F(B) \in B$
4. For every set $\mathcal{A}$ of nonempty disjoint sets, there is a set C such that $\forall A \in \mathcal{A}$, $\text{ord}(C \cap A) = 1$
5. Cardinal comparibility: For any sets C, D, $C \preceq D$ or $D \preceq C$
6. Zorn's Lemma

Proof. $1 \to 2$) Let H be a function such that $\forall i \in I$, $H(i) \neq \varnothing$. Let $R = \{\langle i,h\rangle \in I \times \bigcup H(i) \quad i \in I, h \in H(I)\}$. By (1) there is a function $F \subseteq R$ with dom $F =$ dom $R = I$. $\forall i \in I$, $\langle i, F(i)\rangle \in F \subseteq R \to F(i) \in H(i)$ so $F \in \times_{i \in I} H(i)$
$2 \to 4$) We have a colleciton $\mathcal{A}$ of disjoint nonempty subsets. We want to define H such that $H(A)$ is nonempty for $A \in \mathcal{A}$. Let $I = \mathcal{A}$, for $A \in I, H(A) = A$. Then $\times_{A \in I} H(A) = \times_{A \in \mathcal{A}} A$, by (2) there is $f \in \times_{A \in \mathcal{A}} A$. We claim that $C =$ ran f is as wanted. For all $A \in \mathcal{A}$, $f(A) \in A$ and if $A' \neq A$, $F(A') \in A'$ disjoint from A so ran $(f) \cap A = \{f(A)\}$
$6 \to 1$) Let R be a relation. Let $\mathcal{A} = \{f \subseteq R \mid f \text{ is a function }\}$. $\mathcal{A} \neq \varnothing$ since $\varnothing \in \mathcal{A}$. If $\mathcal{C} \subseteq \mathcal{A}$ is chain, $\bigcup \mathcal{C}$ is a function, $\bigcup \mathcal{C} \subseteq R$ so $\bigcup \mathcal{C} \in \mathcal{A}$. By (6) there is a maximal $F \in \mathcal{A}$, $F \subseteq R$ is a function.

Claim dom F = dom R. If not, then there is $d \in \text{dom } R \backslash \text{dom } F$.Let r be such taht $\langle d, r\rangle \in R$. Then $F \cup \{\langle d, r\rangle\} \in \mathcal{A}$, $F \subsetneq F \cup \{\langle d, r\rangle\}$, contradicting maximalilty.

5.7 October 6

5.7.1 Axiom of Choice

Proof (Pf (cont))**.** $4 \to 3)$ We have a set A. We want $f : \mathcal{P}(A)\backslash\{\varnothing\} \to A$, $F(B) \in B$. $B^* = \{\langle B, b\rangle : b \in B\} \approx B$. $B \neq C \to B^* \cap C^* = \varnothing$. Let $\mathcal{A} = \{B^* : B \subseteq A, B \neq \varnothing\}$. By (4) there is a set C such that $\forall B^* \in \mathcal{A}$, $|C \cap B^*| = 1$.
$3 \to 1)$ Let R be a relation. For $a \in$ dom R, we want to pick an element in $R_a = \{b \in \text{ran } R \,|\, \langle a, b\rangle \in R\}$. let $A =$ ran R. By (3) there is $f : \mathcal{P}(A)\backslash\{\varnothing\} \to A$, $f(B) \in B$ for all $B \subseteq A$. $F = \{\langle a, F(R_a)\rangle \,|\, a \in \text{dom } R\}$

5.7.2 Applications of Axiom of Choice

Want to define a measure on $\mathbb{R}$ with the following properties.

1. $m([0, 1]) = 1$
2. $m(A + r) = m(A)$
3. $m(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} m(A_i)$

Theorem 5.7.1. There is no $m : \mathcal{P}(\mathbb{R}) \to (\mathbb{R}^{<0} \cup \infty)$ satsifying the above conditions.

Proof. For $r, s \in [0, 1]$, let $r \sim s$ if $r - s \in \mathbb{Q}$. Let $[r] = \{s \in [0, 1) \,|\, r - s \in \mathbb{Q}\}$ Let $\mathcal{A} = \{[r] : r \in [0, 1)\}$. $\mathcal{A}$ is a family of disjoint sets so by AC there is a set C such that $|C \cap [r]| = 1$ for each $[r] \in \mathcal{A}$. Assume $C \subseteq [0, 1)$.
Consdier $C + q$ for $q \in \mathbb{Q}$

- disjoint since if $p \neq q \in \mathbb{Q}$, $(C + p) \cap (C + q) = \varnothing$
- $\bigcup_{q\in\mathbb{Q}}(C + q) = \mathbb{R}$
- $\bigcup_{q\in\mathbb{Q}\cap[0,1)}(C + q) \subseteq [0, 2)$

$m(\mathbb{R}) = m(\bigcup_{q\in\mathbb{Q}}(C + q)) = \sum_{q\in\mathbb{Q}}(c + q) = \sum_{q\in\mathbb{Q}} m(C)$, so $m(c) > 0$ since $m(\mathbb{R}) > 1$. Also, $2 = m([0, 2]) \geqslant m(\bigcup_{q\in\mathbb{Q}\cap[0,1)}(C + q)) = \sum_{q\in\mathbb{Q}\cap[0,1)} m(c + q) = \sum_{q\in\mathbb{Q}\cap[0,1)} m(C) = \infty$, a contradiction.

5.8 October 11

5.8.1 Countable Sets

Definition 5.8.1. A set is countable if $A \preceq \omega \leftrightarrow$ either $A = \varnothing$ or there is an onto function $f : \omega \to A$, ie. $A = \{f(0), f(1), f(2), \cdots\}$

Observation: $\omega \times \omega \approx \omega \sqcup \omega \approx \omega$

Theorem 5.8.2. let $\mathcal{A}$ be a countable collection of countable sets. Then $\bigcup \mathcal{A}$ is countable.

Proof. We want to define an onto function $\omega \times \omega \to \bigcup \mathcal{A}$. Since $\mathcal{A}$ is countable there is a function $g : \omega \to \mathcal{A}$. For each $A \in \mathcal{A}$, there is an onto function $f_A : \omega \to A$. Want $F : \mathcal{A} \to$ functions from ω to $\bigcup \mathcal{A}$, $A \to f_A$, such that $f_A : \omega \to A$ is onto. $R = \{\langle A, f\rangle \mid A \in \mathcal{A}\ f$ onto function $\omega \to A\}$ for each $A \in \mathcal{A}$ we know there is at least one f such that $\langle A, f\rangle \in R$ because A is countable so dom $(R) = \mathcal{A}$. By (AC1) there is a function $F \subseteq R$, dom $(F) =$ dom $(R) = \mathcal{A}$. Define $H : \omega \times \omega \to \bigcup \mathcal{A}$ by $H(n, m) = f_{g(n)}(m) = F(g(n))(m) \in \bigcup \mathcal{A}$

Observation: If A is countable $A^{<\omega} = A^1 \cup A^2 \cup A^3 \cup \cdots$ is countable

- $r \in \mathbb{R}$ is algebraic if it sis the root of a polynomial in $\mathbb{Z}[X]$ $\{r \in \mathbb{R} :$ algebraic $\}$ is countable.

Theorem 5.8.3. For every infinite cardinal κ, $\kappa + \kappa = \kappa$

Proof. Let K have size κ. Let $\mathcal{A} = \{f \in \mathcal{P}(\kappa \sqcup \kappa) \times \kappa \mid f$ is a function and there is $A \subseteq \kappa$, dom $(f) = A \sqcup A$, dom $(f) = A$, f is one to one $\}$. To check the conditions for Zorn's Lemma, take a chain $\mathcal{C} \subseteq \mathcal{A}$. By the lemma, $\bigcup \mathcal{C}$ is a one to tone function, dom $(\bigcup \mathcal{C}) = \bigcup\{$dom $f\ :\ f \in \mathcal{C}\} = \bigcup\{$ran $f \sqcup$ ran $f\ :\ f \in \mathcal{C}\} = \bigcup\{$ran $f\ :\ f \in \mathcal{C}\} \cup \bigcup\{$ran $f \sqcup$ ran $f\ :\ f \in \mathcal{C}\} =$ ran $(\bigcup \mathcal{C}) \sqcup$ ran $(\bigcup \mathcal{C})$ so $\bigcup \mathcal{C} \in \mathcal{A}$.
By Zorn's lemma, there is a maximal $F \in \mathcal{A}$. For this F there is $A \subseteq \kappa$, dom $(F) - A \sqcup A$, ran $(F) = A$.

- If $\kappa \backslash A$ is finite, card$(\kappa) =$ card(A) and $F : A \sqcup A$ is a bijection, using a bijection $\kappa \to A$, we can build a bijection $\kappa \sqcup \kappa \to \kappa$.
- If $\kappa \backslash A$ is infinite, let $D \subseteq \kappa \backslash A$ be a countable set, let $h : D \sqcup D \to D$ be a bijection. Let $G : (A \cup D) \sqcup (A \cup D) \to A \cup D$, $G \restriction_{A \sqcup A} = F$, $G \restriction_{D \sqcup D} = h$. $F \subsetneq G$ contradicitng that F was maximal.

5.9 October 13

5.9.1 Cardinal Arithmetic

Theorem 5.9.1. $\kappa \cdot \kappa = \kappa$ for all infinite cardinals κ

Proof. Let $\mathcal{A} = \{f \in \mathcal{P}((\kappa \times \kappa) \times \kappa) \mid f$ is a function dom $(f) =$ ran $f \times$ ran f, one to one$\}$. If $A =$ ran f, f is a bijection $A \times A \to A$. We need to show $\mathcal{A}$ satisifes the conditions to apply Zorn's Lemma. Let $\mathcal{C} \subseteq \mathcal{A}$ be a chain, we want to show $\bigcup \mathcal{C} \in \mathcal{A}$. By the lemma, $\bigcup \mathcal{C}$ is a function, dom $(\bigcup \mathcal{C}) = \bigcup\{$dom $f\ :\ f \in \mathcal{C}\}$, ran $(\bigcup \mathcal{C}) = \bigcup\{$ran $f\ :\ f \in \mathcal{C}\}$. By Zorn's Lemma, there is a maixmal $F \in \mathcal{A}$. Let $A =$ ran (F), $F : A \times A \to A$ bijection. Note that A must be infinite or else $A \times A \not\approx A$.
If $A \approx \kappa$, then $\kappa \times \kappa \approx A \times A \xrightarrow{F} A \approx \kappa$ so $\kappa \times \kappa \approx \kappa$, as wanted.
If not, we want to get a contradiction with the maximality of F. If $A \prec \kappa$, then $A \prec \kappa \backslash A$, otherwise $\kappa = (\kappa \backslash A) \cup A \preceq A \sqcup A \approx A \prec \kappa$. Take $D \subseteq \kappa \backslash A$ if size A. $A \times D \cup D \times D \cup D \times A \approx A \times A \sqcup A \times A \sqcup A \times A \overset{F}{\approx} A \sqcup A \sqcup A \approx A \approx D$ so there is a bijection $h : A \times D \cup D \times D \cup D \times A \to D$, then $F \cup h \in \mathcal{A}$, contradicting the maximality of F. $\square$

Corollary 5.9.2. If κ and λ are infinite cardinals, $\kappa + \lambda = \kappa \times \lambda = \max(\kappa, \lambda)$

Proof. If $\kappa = \max(\kappa, \lambda)$, $\kappa \leqslant \kappa + \lambda \leqslant \kappa + \kappa = \kappa$, $\kappa \leqslant \kappa \times \lambda \leqslant \kappa \times \kappa = \kappa$.

- $\text{card}\{f : \mathbb{R} \to \mathbb{R}\} = 2^{2^{\aleph_0}}$, $\text{card}\{f : \mathbb{R} \to \mathbb{R} : f \text{ cont }\} = 2^{\aleph_0}$ since if $f, g : \mathbb{R} \to \mathbb{R}$ continuous, then $f = g \leftrightarrow f \restriction_{\mathbb{Q}} = g \restriction_{\mathbb{Q}}$ so $\leqslant {}^{\mathbb{Q}}\mathbb{R} = 2^{\aleph_0}$ and $2^{\aleph_0} \leqslant$ since have a constant function for each real number.

Theorem 5.9.3. For κ infinite and λ such that $2 \leqslant \lambda \leqslant 2^\kappa$, $\lambda^\kappa = 2^\kappa$

Proof. $2^\kappa \leqslant \lambda^\kappa \leqslant (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa$

Continuum Hypothesis (CH): Every uncountble subset of $\mathbb{R}$ is equinumerous to $\mathbb{R}$.
Thm(Godel): CH can't be refuted in ZFC
Thm(Cohen): CH can't be proved in ZFC
Generalized Continuum Hypothesis (GCH) : For every infinite cardinal κ, there is no λ with $\kappa < \lambda < 2^\kappa$

Chapter 6

Orderings and Ordinals

6.1 October 25

6.1.1 Orderings

Definition 6.1.1. A partial ordering is a pair $p = (D, <)$ where $< \subseteq D \times D$ and satisfies transitivity and irreflexivity. ie $\forall a, b, c \in D$ $a < b \wedge b < c \to c < a$ and $\forall a \nless a$

Example 6.1.2.

- $(\mathcal{P}(C), \subset)$
- $(\mathbb{N}, |)$ $a|b$ if a divides b and $a \neq b$
- $(\mathbb{R}, <)$
- $(\mathbb{N}, \lhd)$ where $m \lhd n \leftrightarrow \begin{cases} m \text{ even } n \text{ odd} \\ m, n \text{ odd } m <_{\mathbb{N}} N \\ m, n \text{ even } m <_{\mathbb{N}} n \end{cases}$
  $(0 \lhd 2 \lhd 3 \lhd \cdots) \lhd (1 \lhd 3 \lhd \cdots)$

Definition 6.1.3. A well ordering is a linear ordering $\langle A, < \rangle$ such that every nonempty set has a leasat element.

Example 6.1.4.

- $(\mathbb{N}, <)$ well ordering
- $(\mathbb{N} \times \mathbb{N}, <_{\text{lex}})$ $(a, b) <_{\text{lex}} (c, d) \leftrightarrow a < c$ or $a = c$ and $b < d$ is well ordered.

  Proof. Take $B \subseteq \mathbb{N} \times \mathbb{N}$ nonempty. Want to show B has $<_{\text{lex}}$ least element. Take $B_0 = \{a \,|\, \exists b \langle a, b \rangle \in B\} \subseteq \mathbb{N}$. Let $a_0 =$ least in B_0. Let $B_1 = \{b \,|\, \langle a_0, b \rangle \in B\}$. Let b_0 be least in B_1. $\langle a, b \rangle <_{\text{lex}}$ least in B.

- $(\mathbb{Z}, <)$ not well ordering
- $(\mathbb{N}, <) + (\mathbb{Z}, <)$ has least element but is not a well ordering since $\mathbb{Z}$ has no least element.
- $[0, 1] \cap \mathbb{Q}$ not well ordering since $\{1/n : n \in \mathbb{N}\}$ has no least element.
- $A = \{a, b, \cdots, z\}$, consider $(A^{<}, <_{\text{lex}})$. Not a well ordering

Example 6.1.5. $\mathbb{N}[x]$ the set of polynomials with coeffecients in $\mathbb{N}$, $p(x) \lhd q(x)$ if $lim_{x\to\infty} p(x) - q(x) > 0$ is a well ordering

Lemma 6.1.6. Let $(A, \leqslant)$ be a linear ordering. The following are equivalent.

1. Every nonempty subset has a least element
2. There is no infinite decreasing sequence ie. no $a_0 > a_1 > a_2 > \cdots \in A$

Proof. $1 \to 2$) If (2) is false, and there is a sequence $a_0 > a_1 > a_2 > \cdots \in A$ $\{a_0, a_1, a_2, \cdots\}$ has no least element so (1) is false.
$2 \to 1$) If (1) is false, there is nonempty $B \subseteq A$ wit on least element. Let $b_0 \in B$ since b_0 is not hte least element, have $b_1 \in B$ with $b_1 < b_0$, $b_2 \in B$ with $b_2 < b_1 < b_0$, $\cdots$ end up with $b_0 > b_1 > b_2 > \cdots$

6.2 October 27

6.2.1 Induction and Recursion

Notation: For $t \in A$, seg $t = \{s \in A \mid s < t\}$

Theorem 6.2.1 (Transfinite Induction Principle)**.** Let $(A, <)$ be a well ordering. Let $B \subseteq A$. If $\forall t \in a[\forall s < t(s \in B) \to t \in B]$ then $B = A$.

Proof. Take $B \subseteq A$, suppose $\forall t \in a[\text{seg } t \subseteq B \to t \in B]$. We want to show $B = A$. If not $A \backslash B$ is nonempty so it has a least element b. Since b is least in $A \backslash B$, $\forall s < b\, s \in B$ so seg $b \subseteq B$ so $b \in B$, a contradiction.

Example 6.2.2. If $A = \omega \times \omega, <_{\text{lex}}$. Want to define $F : \omega \times \omega \to \mathbb{R}$, $F(n, m) = \sup\{F(a, b) + 2^{-b} : (a, b) < (n, m)\}$. We get $F(0, 0) - = 0, F(0, 1) = 1, F(0, 2) = \frac{3}{2}, F(0, 3) = \frac{7}{4}, \ldots, F(1, 0) = 2, F(1, 1) = 3, \cdots, F(2, 0) = 4, \cdots$

Theorem 6.2.3 (Transfinite Recursion Principle)**.** Let $(A, <)$ be a well ordering. Given ${}^{<A}B \to B$ there is a unique function $F : A \to B$ such that $\forall t \in A\, F(t) = G(F\restriction_{\text{seg } t})$

- We define ${}^{<A}B = \{f \mid f \text{ is a function, dom } (f) = \text{seg } t \text{ for some } t \in A, \text{ ran } f \subseteq B\}$

Let $A = \omega + \omega$, $0, 1, 2 \cdots, \omega, \omega + 1, \omega + 2, \cdots$
$V_0 = \varnothing, \cdots, V_{n+1} = \mathcal{P}(V_n), \cdots, \bigcup_{n\in\omega} V_n = V_\omega, \ldots, V_{\omega+n+1} = \mathcal{P}(V_{\omega+n})$, ie $V_\alpha = \bigcup\{\mathcal{P}(V_\beta) : \beta < \alpha\}$.

Axiom 6.2.4 (Replacement Axiom)**.** For each first order formula $\varphi(x, y)$ if $\varphi(x, y)$ is function like on a set A then there is a set B such that $\forall y(y \in B \leftrightarrow \exists x \in A\varphi(x, y))$

Definition 6.2.5. $\varphi(x, y)$ is function like on A if $\forall x \in A\, \exists! y\, \varphi(x, y)$

6.3 November 1

6.3.1 The Replacement Axiom

- Every well ordering has a least element, eveyr element $t \ni a$ has a succesor $s(t) \in A$, $s(t)$ least element $\rhd t$
- Some element have a predecessor (calles succesor element) and some don't (limit elements)

Definition 6.3.1. A formula $\varphi(x, y)$ is function like if $\forall x\, \exists! y\, vp(x, y)$, function like on A if $\forall x \in A\, \exists! y\, \varphi(x, y)$

Example 6.3.2. If $\varphi(x, y)$ is $y = \mathcal{P}(x)$ ie. $(\forall z(z \in y \leftrightarrow z \subseteq x))$ or $y = \text{ran } (x)$ ie. $\forall z(z \in y \leftrightarrow \exists w\, \langle w, z\rangle \in x)$

Axiom 6.3.3. For any formula $\varphi(x, y)$ we have an axiom $\forall A$ if φ is function like on A, $\exists B$ such that $\forall x \in a\, \exists y \in B\, \varphi(x, y)$

Let $\gamma(x, y)$ be the function $y = f(x)$. For a well ordering $(A, \lhd)$ define a function E with domain A by transfinite recursion, $\forall t \in A\, E(t) = \text{ran } (E \restriction_{\text{seg } t})$
Consider this function over the polynomials in $\mathbb{N}[x]$:

- seg $(0) = \varnothing$ so $E \restriction_{\text{seg } t} \varnothing$ ie. $E(0) = 0$
- seg $(1) = \{0\}$, $E \restriction_{\text{seg } 1} = \{\langle 0, \varnothing\rangle\}$ so $E(1) = \{\varnothing\}$
- seg $(2) = \{0, 1\}$, $E \restriction_{\text{seg } 2} = \{\langle 0, \varnothing\rangle, \langle 1, \{\varnothing\}\rangle\}$ so $E(2) = \{\varnothing, \{\varnothing\}\}$
- Continuing we get $0_{\mathbb{N}}, 1_{\mathbb{N}}, 2_{\mathbb{N}}$, $E(x) = \omega$, $E(x+1) = \omega^+$, $E(x+2) = \omega^{++}, \cdots, E(2x) = \omega + \omega$

We call ran (E) the ε-image of $(A, \lhd)$

Theorem 6.3.4. Let $(A, \lhd)$ be a well ordering, let $\alpha = \text{ran } (E)$. Let $\in_\alpha = \{\langle a, b\rangle \in \alpha \times \alpha \mid a \in b\}$, then E is an isomorphism $(A, \lhd) \to (\alpha, \in_\alpha)$, ie it is a bijection and $a \lhd b \leftrightarrow E(a) \in_\alpha E(b)$. Given well orderings $(A, \lhd_A)$ and $(B, \lhd_B)$ with ε images α and β, $(A, \lhd_A) \cong (B, \lhd_B) \leftrightarrow \alpha = \beta$

6.4 November 3

6.4.1 Ordinals

Theorem 6.4.1. For $s, t \in A$, $s \lhd t \leftrightarrow E(s) \in E(t)$

Theorem 6.4.2.
- $\forall t \in A$, $E(t) \notin E(t)$
- E is one to one
- α is transitive.

It follows that E is an isomorphism $(A, \lhd) \to (\alpha, \in_\alpha)$

Theorem 6.4.3. Given well orderings $(A, \lhd_A)$ and $(B, \lhd_B)$ with ε images α and β, $(A, \lhd_A) \cong (B, \lhd_B) \leftrightarrow \alpha = \beta$.

Proof. $\leftarrow$) If $\alpha = \beta$ then $(A, \lhd_A) \cong (\alpha, \in_\alpha) = (\beta, \in_\beta) \cong (B, \lhd_B)$
$\rightarrow$) Suppose $f : A \to B$ is an isomorphism. $E_A : A \to \alpha$, $E_B : B \to \beta$. Claim $\forall t \in A$, $E_A(t) = E_B(f(t))$. Use transfinite induction. Let $T = \{t \in A \,|\, E_A(t) = E_B(f(t))\}$, want to show $T = A$. It is enough to prove that $\forall t \in A(\text{seg } t \subseteq T \to t \in T)$. $E_A(t) = \{E_A(s) : s \in A, s \lhd t\} = \{E_B(f(s)) : s \in A, s \lhd t\} = \{E_B(s) : s \in B\, s \lhd_B f(t)\} = E_B(f(t))$.

Definition 6.4.4. α is an ordinal it is the ε image of some well ordering.

Theorem 6.4.5. If α is transitive, well ordered by $\in$, then α is the ε image of some well ordering.

Proof. If α is transitive, $(\alpha, \in_\alpha)$ is a well ordering, then we claim α is the $\in$-image of itself, ie. the map $E : \alpha \to \alpha$ is the identity. Use transfinite induction to show that $\forall t \in \alpha$ $E(t) = t$. $E(t) = \{E(s) \,|\, s \in \alpha, s \in \text{seg } t\} = \{E(s) : s \in t\} = \{s \,|\, s \in t\} = t$

Theorem 6.4.6. Given well orderings $(A, \lhd_A)$ and $(B, \lhd_B)$ either

- $(A, \lhd_A) \cong (B, \lhd_B)$
- $\exists a \in A$ $(\text{seg } a, \lhd_A) \cong (B, \lhd_B)$
- $\exists b \in B$ $(A, \lhd_A) \cong (\text{seg } b, \lhd_B)$

Proof. Define $f : A \to B$, $f(a) = \min(B \backslash \text{ran } (f \restriction_{\text{seg } a})) = \min\{b \in B \,|\, \forall s \lhd_A a, f(s) \neq b\}$. f is order preserving and one to one. If $B \backslash \text{ran } (f)$ is nonempty, then it has minimal element b and f is an isomorphism from $(A, \lhd_A)$ to $(\text{seg } b, \lhd_B)$. If $B \backslash \text{ran } (f)$ is empty, dom $f = A$, then $A \cong B$. If f is not longer defined for some $a \in A$, then it is defined on seg a so $(\text{seg } a, \lhd_A) \cong (B, \lhd_B)$

Theorem 6.4.7. For any ordinals α, β, γ

- Every memeber of α is an ordinal
  If $\alpha = \text{ran } E_A$, $a \in \alpha$ then $a = E(t)$ for some $t \in A$ so $a = \text{ran } (E \restriction_{\text{seg } t})$ so a is the $\in$ image of seg t
- $\alpha \in \beta \in \gamma \to \alpha \in \gamma$
- $\alpha \notin \alpha$
- Exactly one of the following holds: $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$
- Every nonempty set of ordinals has a $\in$ least element

Proof. If S is a nonepty set of ordinals take $\alpha \in S$, $S \cap \alpha \subseteq \alpha$ has a least element if nonempty. If it has a least element, then such an element is the least element of S. If it is empty, then α is the least element of S.

Theorem 6.4.8 (Burali-Forti Paradox)**.** There is not set that contains all ordinals.

Observation:

- $\varnothing$ is an ordinal, $n \in \omega$ and ω are ordinals.
- If α is an ordinal so is $\alpha^+ = \alpha \cup \{\alpha\}$
- If S is a set of ordinals, then $\bigcup S$ is an ordinal.

6.5 November 8

6.5.1 Cumulative Hierarchy

Want to formalize hierarchy by defining $V_\alpha = \bigcup\{\mathcal{P}(V_\beta) : \beta \in \alpha\}$. Want to define this using transfinite recusion but can't do this directly. Need approximate this function since cant have a domain ORD. For $\delta \in$ ORD, $F_\delta(\alpha) = \bigcup\{\mathcal{P}(F_\delta(\beta)) : \beta \in \alpha\}$.

Theorem 6.5.1. For any ordinal δ there exists an F_δ

Proof. Transfinite recursion on $(\delta, \in_\delta)$ with $y = \bigcup\{P(z) : z \in \text{ran }(x)\}$. To check that this gives our desired function we see $F(\alpha) = \bigcup\{P(z) : z \in \text{ran }(F \restriction_{\text{seg } \alpha})\} = \bigcup\{\mathcal{P}(F(\beta)) : \beta \in \alpha\}$. Given δ_1, δ_2 with $\delta_1 \in \delta_2$ we claim that $F_{\delta_1}(\alpha) = F_{\delta_2}(\alpha)$ for $\alpha \in \delta_1$. Follows form the uniqueness of transfinite recusion since $F_{\delta_2} \restriction_{\delta_1}$ satisfies the recursive conditions so must have $F_{\delta_2} \restriction_{\delta_1} = F_{\delta_1}$

Definition 6.5.2. $V_\alpha = F_\delta(\alpha)$ for any $\delta > \alpha$

Observation

(i) $V_\alpha = \bigcup\{\mathcal{P}(V_\beta) : \beta \in \alpha\}$

(ii) V_α is a transitive set.

Proof. By induction: $V_\alpha = \bigcup\{\mathcal{P}(V_\beta) : \beta \in \alpha\}$. $x \in y \in V_\alpha$ so $y \in \mathcal{P}(V_i n \beta)$ for some $\beta \in \alpha$, $y \subseteq V_\beta$ so $x \in V_\beta$ so $x \subseteq V_\beta$ so $x \in \mathcal{P}(V_\beta)$ and so $x \in V_\alpha$

(iii) $\alpha \in \beta \to V_\alpha \subseteq V_\beta$

Proof. $V_\beta = \bigcup\{\mathcal{P}(V_\gamma) : \gamma \in \beta\}$, $V_\alpha = \bigcup\{\mathcal{P}(V_\gamma) : \gamma \in \alpha\}$

Theorem 6.5.3. (a) $V_0 = \varnothing$

(b) $V_\alpha^+ = \mathcal{P}(V_\alpha)$ for all α

(c) If λ is a limit ordinal, then $V_\lambda = \bigcup_{\beta\in\lambda} V_\beta$

Proof. (ii) $V_{\alpha^+} = \bigcup\{\mathcal{P}(V_\beta) : \beta \in \alpha^+\} = \bigcup\{\mathcal{P}(V_\beta) : \beta \in \alpha\} \cup \mathcal{P}(V_\alpha) = V_\alpha \cup \mathcal{P}(V_\alpha) = \mathcal{P}(V_\alpha)$

(iii) if $x \in V_\lambda = \bigcup\{\mathcal{P}(V_\beta) : \beta \in \lambda\}$, then $x \in \mathcal{P}(V_\beta)$ for some $\beta \in \lambda$ so $x \in V_{\beta^+}$ and $\beta^+ \in \lambda$ so $x \in \bigcup_{\beta\in\lambda} V_\beta$. If $x \in \bigcup_{x\in\lambda} V_\beta$, then $x \in V_\beta$ for $\beta \in \lambda$ so $x \subseteq V_\beta$ so $x \in \mathcal{P}(V_\beta)$ so $x \in V_{|lambda}$

Definition 6.5.4. A set S is grounded if there is some α such that $S \subseteq V_\alpha$. If S is grounded, rank(S) is the least α such that $S \subseteq V_\alpha$

Observation:

(i) If A is grounded, then so are all $a \in A$ and $\text{rank}(a) \in \text{rank}(A)$

Proof. $A \subseteq V_\alpha = \bigcup\{\mathcal{P}(V_\beta) : \beta \in \alpha\}$, $a \in A \to a \in \mathcal{P}(V_\beta)$ for $\beta \in \alpha$ so $a \subseteq V_\beta$ for $\beta \in \alpha$

(ii) $\text{rank}(A)$ = the least ordinal greater than $\text{rank}(a)$ for $a \in A$

Proof. Consider A and consider $\bigcup\{\text{rank}(a)^+ : a \in A\} = \alpha$. $\alpha \leqslant \text{rank}(A)$ since $\text{rank}(A)$ is an upper bound for $\{\text{rank}(a)^+ : a \in A\}$. Further, $\text{rank}(A) \leqslant \alpha$ since for $a \in A$, $a \subseteq V_{\text{rank}(a)}$ so $a \in V_{\text{rank}(a)^+}$ and so $a \in V_\alpha$ and $A \subseteq V_\alpha$

Theorem 6.5.5. The following are equivalent

(i) (Regularity) For any nonempty set A, there is some $m \in A$ such that $A \cap m = \varnothing$

(ii) There does not exist a function f with domain ω such that $f(n^+) \in f(n)$ for all n.

(iii) Every set is grounded.

Proof. i $\to$ ii) Suppose (ii) is false, then look at ran $(f) = A$. For any $a \in A$, $a = f(n)$ for some n but $f(n^+) \in f(n)$ so $A \cap a \neq \varnothing$
ii $\to$ iii) Suppose there is some non grouned set a_0, a_0 must have some non grounded element a_1, similarly, there is $a_2 \in a_1$ non grounded, $\cdots$
Note: to make this more formal, need to use the transitive closure, and use choice
iii $\to$ i) For nonempty A, A is grounded. Consider $\{\text{rank}(a) : a \in A\}$, a nonempty set of ordinals so it has some least element α. Pick $m \in A$ with $\text{rank}(m) = \alpha$, then $A \cap m = \varnothing$ since any elements of m must have strictly smaller rank.

6.6 November 10

6.6.1 Transfinite Recursion

Theorem 6.6.1 (Transfinite Recursion). Let $(A, \lhd)$ be a well ordering and $\gamma(x, y)$ a function like formula, then there is a function F with domain A such that $\forall t \in A\, \gamma(F \restriction_{\text{seg } t} F(t))$. Moreover F is unique.

Proof. Let $B = \{t \in A : \exists$ a function f with dom $(f) = \text{seg } t$, $\forall s \lhd t$, $\gamma(f \restriction_{\text{seg } s}, f(s))\}$. Want to show $B = A$. Pick $t \in A$. We show that if seg $t \subseteq B$, $t \in B$
Lemma: For $r \trianglelefteq r' \in A$ if f_r has dom $f_r = \text{seg } r$ and $f_{r'}$ has dom $f_{r'}$ then $f_r = f_{r'} \restriction_{\text{seg } r}$

Proof. Let $I = \{s \in \text{seg } r \,|\, f_r(s) = f_{r'}(s)\}$. Want to show $I = \text{seg } r$ by transfinite induction. Take $s \in \text{seg } r$, if we have that $\forall s' \lhd s\; f_r(s') = f_{r'}(s')$, then $f_r \restriction_{\text{seg } s} = f_{r'} \restriction_{\text{seg } s}$. $f_r(s)$ is the unique w such that $\gamma(f_r \restriction_{\text{seg } s}, w)$ and similarly for $f_{r'}(s)$ so it follows that $f_r(s) = f_{r'}(s)$ and so $I = \text{seg } r$

Case 1: $t = \text{succ}(t')$ and $t' \in B$. Let $f_t = f_{t'} \cup \{\langle t', w\rangle\}$ where w is the unique w satisfying $\gamma(f_{t'}, \omega)$
Case 2: t is a limit. Let $f_t = \bigcup_{r \lhd t}\{f_r \,|\, r \in segt\}$ (set by replacement). Well defined since by IH since $\forall r \in \text{seg } t$, $\exists! f$ satisfying conditions.

6.7 November 15

6.7.1 Ordinals

Let ω_1 be the set of all countable ordinals. Why is this a set?
Consider $W = \{(A, R) \in (\omega + 1) \times \mathcal{P}(\omega \times \omega) \,|\, (A, R)$ is a well ordering $\}$, a set by subset axiom. For each $R \in W$ there is a unique cardinal α, $(A, R) \cong (\alpha, \in_\alpha)$, namely the $\in$ image of (A, R). If α is a countable ordinal, (say infinite), there is a bijection $f : \omega \to \alpha$. Define $R = \{\langle a, b\rangle \,|\, f(a) \in_\alpha f(b)\}$, we get the ε image of (ω, R) is α.

- If $\alpha \in \beta \in \omega_1$, α is an ordinal, countable because $\alpha \subseteq \beta$ so $\alpha \in \omega_1$
- If $\alpha, \beta \in \omega_1$, since α, β are ordinals $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$ so $\in$ is a linear ordering.

It follows that ω_1 is an ordinal and $\omega_1 \notin \omega_1$ so ω_1 is not countable. For any ordinal γ, either $\gamma \in \omega_1$ so γ is countable or $\omega_1 \in \gamma$, $\omega_1 \subseteq \gamma$ so γ is uncountable so ω_1 is the least uncountable ordinal.

Theorem 6.7.1 (Hartog's Lemma). For any set A there is an ordinal α such that $\alpha \npreceq A$

Proof. Consider $\alpha = \{\beta \;\; \beta$ carindal, $\beta \preceq A\}$. this is a set because the set of $\in$ images of $W = \{(B, R) \in \mathcal{P}(A) \times \mathcal{P}(A \times A) \,|\, (B, R)$ is a well ordering$\}$ is a set by replacement

- α is an ordinal since if $\beta \in \alpha$, β is an ordinal, if $\gamma \in \beta \in \alpha$, γ is an ordinal, $\gamma \subseteq \beta \preceq A$ so α is transitive, $\in_\alpha$ is a linear order to α is an ordinal
- $\alpha \notin \alpha \to \alpha \npreceq A$

Lemma 6.7.2. If S is a transitive set of ordinals, then S is an ordinal.

Given A, let A^+ be the least ordinal α, $\alpha \npreceq A$. $\gamma(x, y) \equiv y$ is the least ordinal such that $y \npreceq x$. γ is function like.

Theorem 6.7.3. If $\gamma(x, y)$ is a function like function, there is another function like $\theta(x, y)$ on the ordinals such that $\forall\alpha$ if $F = \{(\beta, \gamma) : \beta \in \alpha, \theta(\beta, \gamma)\}$ then the unique z such that $\theta(\alpha, z)$ satisfies $\gamma(F, z)$

Example 6.7.4. alephs, $\aleph_0 = \omega$, $\aleph_1 = \aleph_0^+, \cdots, \aleph_{\alpha+1} = \aleph_\alpha^+$, $\aleph_\lambda = \bigcup_{\alpha<\lambda} \aleph_\alpha$ if λ limit

6.8 November 17

6.8.1 Zorn's Lemma

Theorem 6.8.1. The following are equivalent

1. For every relation R, there is a function $F \subseteq R$, dom $F \subseteq$ dom R
3. For every set A, there is a function $F : \mathcal{P}(A)\backslash\{\varnothing\} \to A$. $\forall B \subseteq A$, $F(B) \in B$
5. For any sets C, D either $C \preceq D$ or $D \preceq C$
6. Zorn's Lemma
7. For every set A there is a relation $\lhd$ on A such that $(A, \lhd)$ is well ordered.

Proof. CC $\to$ WO) Take a set A, Use Hartog's Lemma to get $\alpha \npreceq A$. By CC, $A \preceq \alpha$ so there is a one to one function $f : A \to \alpha$. Define $\lhd$ on A by $a \lhd b \leftrightarrow f(a) \in f(b)$. Then $(A, \lhd) \cong (f[\![A]\!], \in)$.
WO $\to$ 3) Take A, by WO, there is a well ordering $\lhd$ on A. Define $F : \mathcal{P}(A)\backslash\{\varnothing\} \to A$ by $F = \{\langle B, b\rangle \in (\mathcal{P}(A)\backslash\{\varnothing\}) \times A : b$ is the $\lhd$-least element of $B\}$
$1 \to 6$) Consider such an $\mathcal{A}$. Suppose that $\mathcal{A}$ has no maximal element. For any set $A \in \mathcal{A}$, let $F(A)$ be a set in $\mathcal{A}$, $A \subsetneq F(A)$. The definition of F using (1) is given by $R = \{\langle A, B\rangle \in \mathcal{A} \times \mathcal{A} : A \subsetneq B\}$ since $\mathcal{A}$ has no maximal element, dom $(R) = \mathcal{A}$. Use (1) to get a function $F \subseteq R$, dom $(F) = \mathcal{A}$, $\forall A \in \mathcal{A}$ $F(A) \supsetneq A$. Now, use Hartog's theorem to get an ordinal $\alpha \npreceq \mathcal{A}$. We define a function $h : \alpha \to \mathcal{A}$ by transfinite recursion. For $\beta \in \mathcal{A}$, define $H(\beta)$ using $H \restriction_{\text{seg } \beta}$. We split into 3 cases:

- $\beta = 0$. $H(0) = A_0$ (since $\mathcal{A} \neq \varnothing$)
- $\beta = \gamma^+$, $H(\beta) = F(H(\gamma))$
- β limit, $H(\beta) = \bigcup_{\gamma\in\beta} H(\gamma) \in \mathcal{A}$ because $\{H(\gamma) : \gamma \in \beta\}$ is a chain.

Now, $\forall\gamma \in \beta \in \alpha$, $H(\gamma) \subsetneq H(\beta)$ so H is a one to one function, contradicting $\alpha \npreceq \mathcal{A}$.

Theorem 6.8.2. For every set A, there is a unique cardinal κ such that $\kappa \approx A$

Observation: If κ_1 and κ_2 are cardinals and $\kappa_1 \approx \kappa_2$, then $\kappa_1 = \kappa_2$

Proof. BY WO, there is a well ordering $\lhd$ on A. Let α be the $\in$ image of $(A, \lhd)$. α is an ordinal, $\alpha \cong A$. Let κ be the least ordinal $\cong A$.

Let γ be a formula such that $\exists$ ordinal α, $\gamma(\alpha)$. Claim: there is a least ordinal satisfying γ.
Let $G = \{\beta \in \alpha^+ : \gamma(\beta)\}$, $\alpha \in G$, $G \subseteq \alpha^+$ so G has a least element.