

# MATH 142: Elementary Algebraic Topology

Jad Damaj

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# Chapter 1

## Topology

### 1.1 August 24

#### 1.1.1 What is Algebraic Topology

Recall Metric Spaces:  $(X, d)$ ,  $X$  is a set,  $d$  is a metric on  $X$  (ie.  $d : X \times X \rightarrow \mathbb{R}$ )

1.  $d(x, y) = 0$  exactly if  $x = y$
2.  $d(x, y) = d(y, x)$
3.  $d(x, z) \leq d(x, y) + d(y, z)$

Let  $V$  be a vector space, let  $\|\cdot\|$  be a norm on  $V$ , let  $d(v, w) = \|v - w\|$

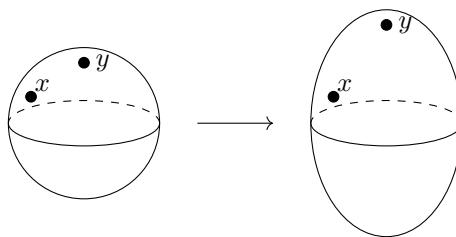
- $\mathbb{R}^n$ :  $\|(r_j)\|_2 = (\sum |r_j|^2)^{\frac{1}{2}}$  - Euclidean Norm,  $\|(r_j)\|_1 = \sum |r_j|$ ,  $\|(r_j)\|_\infty = \max |r_j|$

If  $(X, d)$  is a metric space and if  $Y \subseteq X$ , let  $d^Y$  be the restriction of  $d$  to  $Y \times Y$ . Then  $(Y, d^Y)$  is a metric space.

Metric spaces  $\leftrightarrow$  geometry: length, area, size of angles.

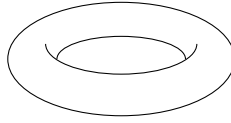
Let  $X$  be a balloon on  $\mathbb{R}^3$

- Two natural metrics: inherited metric from  $\mathbb{R}^3$ , path-length metric (eg. length of shortest path on surface between two points)
- Consider a deformation:



the shapes have different Euclidean distances but still have an underlying commonality

- We also observe that the balloon cannot be continuously deformed into the shape below:



We want to be able to prove such things without embedding into a metric space. This is done by attaching algebraic objects to topological spaces such that their isomorphism classes don't change under continuous deformation.

### 1.1.2 Continuity

Let  $(X, d^X)$  and  $(Y, d^Y)$  be two metric spaces. Let  $f : X \rightarrow Y$  be a function. Let  $x_0 \in X$ . We say  $f$  is continuous at  $x_0$  if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $d^X(x, x_0) < \delta$  then  $d^Y(f(x), f(x_0)) < \varepsilon$ .

- Let  $(X, d)$  be a metric space. By the open ball of radius  $r$  about  $x_0$ , we mean  $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$  (closed ball is  $\{x \in X : d(x, x_0) \leq r\}$ )
- the above definition can be rephrased as: for any  $B(f(x_0), \varepsilon)$  there is an open ball  $B(x_0, \delta)$  such that if  $x \in B(x_0, \delta)$  then  $f(x) \in B(f(x_0), \varepsilon)$ .  
eg. For every open ball  $B_1$  about  $f(x_0)$  there is an open ball  $B_2$  about  $x_0$  such that if  $x \in B_2$  then  $f(x) \in B_1$

**Definition 1.1.1.** For  $(X, d)$  a metric space, by a neighborhood of a point  $x \in X$ , we mean any subset of  $X$  that contains an open ball about  $x$ .

- rephrasing the definition again we get: For any neighborhood  $N_{f(x_0)}$  of  $f(x_0)$  there is a neighborhood  $N_{x_0}$  of  $x_0$  such that if  $x \in N_{x_0}$  then  $f(x) \in N_{f(x_0)}$

**Definition 1.1.2.**  $f : X \rightarrow Y$  is continuous if it is continuous at each point of  $X$ .

## 1.2 August 26

### 1.2.1 Continuity

Recall: Given  $(X, d^X)$ ,  $(Y, d^Y)$  and  $f : X \rightarrow Y$ ,  $f$  is continuous at  $x_0$  if for any open ball  $B_1$  about  $f(x_0)$  there is an open ball  $B_2$  about  $x_0$  such that if  $x \in B_2$  then  $f(x) \in B_1$ , ie.  $B_2 \subseteq f^{-1}(B_1)$

**Definition 1.2.1.** Let  $(X, d)$  be a metric space. Let  $U \subseteq X$ . We say that  $U$  is open if for every  $x \in U$  there is an open ball  $B$  about  $x$  such that  $B \subseteq U$ , ie.  $U$  is a neighborhood of each point it contains.

We say  $f : X \rightarrow Y$  is continuous if it is continuous at each point of  $X$ .

Let  $U$  be an open set in  $Y$ ,  $x \in X$  with  $f(x) \in U$ . For each ball  $B_1$  in  $U$  about  $f(x)$ , there is an open ball about  $x$   $B_2 \subseteq X$  such that if  $x' \in B_2$  then  $f(x') \in B_1$ , ie.  $B_2 \subseteq f^{-1}(B_1) \subseteq f^{-1}(U)$   
ie. if  $x \in f^{-1}(U)$  then there is an open ball  $B_2$  about  $x$  with  $B_2 \subseteq f^{-1}(U)$

ie.  $f^{-1}(U)$  is open

Conversely, if the preimage  $f^{-1}(U)$  of every open set  $U$  in  $Y$  is open, then  $f$  is continuous. This is because if  $x_0 \in X$ ,  $B_1$  an open ball about  $f(x_0)$ , then  $f^{-1}(B_1)$  is open in  $X$ .  $f(x_0) \in B_1$  so we have an open ball  $B_2 \subseteq X$  about  $x_0$  such that  $B_2 \subseteq f^{-1}(B_1)$  so  $f$  is continuous at  $x_0$ .

Thus,  $f : X \rightarrow Y$  is continuous exactly if for any open  $U$  in  $Y$ ,  $f^{-1}(U)$  is open in  $X$ .

### 1.2.2 Topology

Let  $(X, d)$  be a metric space. Let  $J$  be the collection of open subsets in  $X$  of  $d$ .  $J$  has the following properties:

1.  $X \in J$ ,  $\emptyset \in J$
2. an arbitrary, maybe infinite, union of open sets is open
3. a finite intersection of open sets is open.

**Proof** (of (3)). If  $U_1, \dots, U_n$  are open sets and  $x \in U_1 \cap \dots \cap U_n$  then there are  $r_1, \dots, r_n \in \mathbb{R}$  such that  $B(x, r_j) \subseteq U_j$  for  $j = 1, \dots, n$ . Let  $r = \min\{r_1, \dots, r_n\}$ , then  $B(x, r) \subseteq U_j$  for each  $j$  so  $B(x, r) \subseteq U_1 \cap \dots \cap U_n$ . Thus,  $U_1 \cap \dots \cap U_n$  is open.

Note: This does not hold for infinite intersections, consider  $\bigcap_{i \in \mathbb{N}} B(x, \frac{1}{n}) = \{x\}$  in the plane.

This motivates the following definition:

**Definition 1.2.2.** Let  $X$  be a set. By a topology on  $X$  we mean a collection,  $\mathcal{T}$ , of subsets of  $X$  (called the open sets of the topology) satisfying **1**, **2**, and **3** above.

**Definition 1.2.3.** If  $(X, \mathcal{T}^X)$ ,  $(Y, \mathcal{T}^Y)$  are topological spaces,  $f : X \rightarrow Y$  is continuous if for every  $U \in \mathcal{T}^Y$ ,  $f^{-1}(U) \in \mathcal{T}^X$ .

**Example 1.2.4.** Given  $X$ , let  $\mathcal{T}_X$  be all subsets of  $X$ . This is called the discrete topology on  $X$ .

- This topology can also be given by the metric  $d(x, y) = 1$  if  $x \neq y$

**Definition 1.2.5.** If  $\mathcal{T}_1, \mathcal{T}_2$  are topologies on  $X$ , we say  $\mathcal{T}_1$  is bigger, or finer, than  $\mathcal{T}_2$  if  $\mathcal{T}_1 \supseteq \mathcal{T}_2$ .

- the discrete topology is the biggest topology on  $X$ .

**Example 1.2.6.**  $\mathcal{T} = \{X, \emptyset\}$ , called the indiscrete topology on  $X$ .

Note: this topology can not be given by a metric if  $X$  has 2 or more points.

## 1.3 August 29

### 1.3.1 Bases and Subbases

Let  $(X, \mathcal{T})$  be a topological space.

**Definition 1.3.1.** A subset  $A$  of  $X$  is said to be closed if  $A' (X - A)$  is open.

Let  $\mathcal{C}$  be the collection of closed subsets

1.  $X, \emptyset \in \mathcal{C}$
2. any (maybe infinite) intersection of closed sets is closed
3. A finite union of closed sets is closed

Let  $X$  be a set, any (maybe infinite) intersection of topologies on  $X$  is a topology on  $X$ .

Thus, for any  $\mathcal{S}$ , a subset of  $X$ , there is a smallest topology that contains  $\mathcal{S}$ , namely the intersection of all topologies that contain  $\mathcal{S}$ . We say that  $\mathcal{S}$  generates this topology.

**Definition 1.3.2.** If  $\mathcal{S}$  has the property that  $\bigcup(U \in \mathcal{S}) = X$ , then  $\mathcal{S}$  is called a subbasis of the topology it generates.

Let  $\mathcal{I}^{\mathcal{S}}$  be the collection of all finite intersection of elements of  $\mathcal{S}$ , then the intersection of a finite number of elements of  $\mathcal{I}^{\mathcal{S}}$  is in  $\mathcal{I}^{\mathcal{S}}$ .

Let  $\mathcal{I}$  be a collection of subsets of  $X$  (union of elements of  $\mathcal{I}$  is  $X$ ) with the property that the intersection of a finite number of elements of  $\mathcal{I}$  is in  $\mathcal{I}$ . Then the collection,  $\mathcal{T}$ , of arbitrary unions of elements of  $\mathcal{I}$  is a topology (the smallest topology containing  $\mathcal{I}$ )

Why is a finite intersection of elements of  $\mathcal{T}$  in  $\mathcal{T}$ ?

Suppose  $\mathcal{O}_1 = \bigcup_{\alpha} U_{\alpha}^1$ ,  $\mathcal{O}_2 = \bigcup_{\beta} U_{\beta}^2$  with  $U_{\alpha}^1, U_{\beta}^2 \in \mathcal{I}$ , then  $\mathcal{O}_1 \cap \mathcal{O}_2 = (\bigcup U_{\alpha}^1) \cap (\bigcup U_{\beta}^2) = \bigcup_{\alpha, \beta} (U_{\alpha}^1 \cap U_{\beta}^2)$ .

**Definition 1.3.3.** Given a topological space  $(X, \mathcal{T})$ , a base for it is a set of subsets,  $\mathcal{B}$ , of  $\mathcal{T}$ , with the property that every element of  $\mathcal{T}$  is a (maybe infinite) union of elements of  $\mathcal{B}$ .

If  $\mathcal{S}$  is a subbase for  $\mathcal{T}$ , then  $\mathcal{I}^{\mathcal{S}}$  is a base for  $\mathcal{T}$ .

Note: definition does not require  $\mathcal{B}$  to be closed under finite intersection

$(X, d)$  is a metric space, let  $\mathcal{B}$  be the set of open balls. Then  $\mathcal{B}$  is a base for the metric topology but usually the intersection of two open balls is not an open ball.

The intersection of finitely many elements of  $\mathcal{B}$  is the union of elements of  $\mathcal{B}$ .

Let  $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$  be topological spaces, and  $\mathcal{S}$  a subbase of  $\mathcal{T}^Y$ . Let  $f : X \rightarrow Y$ , then  $f$  is continuous if for every  $U \in \mathcal{S}$ ,  $f^{-1}(U) \in \mathcal{T}^X$ .

**Example 1.3.4.** For  $X = \mathbb{R}$ ,  $\mathcal{S} = \{(-\infty, a), (b, +\infty) : a, b \in \mathbb{Q}\}$  generates the usual topology.

## 1.4 August 31

### 1.4.1 Initial Topologies

**Definition 1.4.1.** Let  $X$  be a set. Suppose we have a collection of topologies  $(Y_{\alpha}, \mathcal{T}_{\alpha})$ , and for each  $\alpha$  a function  $f_{\alpha} : X \rightarrow Y_{\alpha}$ . The smallest topology  $\mathcal{T}$  such that each  $f_{\alpha}$  is continuous is called the initial topology.

For each  $\alpha$ ,  $U \in \mathcal{T}_{\alpha}$ , must have  $f_{\alpha}^{-1}(U) \in \mathcal{T}$  so a subbase of  $\mathcal{T}$  is  $\{f_{\alpha}^{-1}(U) : \text{for all } \alpha, U \in \mathcal{T}_{\alpha}\}$

**Example 1.4.2.** Have  $(Y, \mathcal{T}^Y)$ , let  $X$  be a subset of  $Y$ .  $f : X \hookrightarrow Y$  by  $f(x) = x$ .

Initial topology on  $X$  has subbase  $f^{-1}(U) = U \cap X \subseteq X$  for  $U \in \mathcal{T}^Y$ . Further,  $\{U \cap X : U \in \mathcal{T}^Y\}$  is a topology. This topology is called the relative topology on  $X$ .

**Example 1.4.3.**  $Y = \mathbb{R}$ ,  $X = [0, 1]$ , relative topology contains  $[0, \frac{1}{2})$ , not in the original topology

**Example 1.4.4.** Let  $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$  be topological spaces. Form set  $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ .

We have projections  $p^X : X \times Y \rightarrow X$  and  $p^Y : X \times Y \rightarrow Y$ . The initial topology has basis  $(p^X)^{-1}(U) = U \times Y$  for  $U \in \mathcal{T}^X$ ,  $(p^Y)^{-1}(V) = X \times V$  for  $V \in \mathcal{T}^Y$ .

Further,  $(U \times Y) \cap (X \times V) = U \times V$  (rectangle) so the open sets of the initial topology consist of all arbitrary unions of rectangles  $U \times V$  for  $U \in \mathcal{T}^X, V \in \mathcal{T}^Y$ , called the product topology on  $X \times Y$ .

**Example 1.4.5.**  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ . The product topology contains rectangles  $(a, b) \times (c, d)$

Gives same topology as the euclidean metric

- Given  $(X_1, \mathcal{T}^{X_1}), (X_2, \mathcal{T}^{X_2}), \dots, (X_n, \mathcal{T}^{X_n})$  can form  $X_1 \times X_2 \times \dots \times X_n$  with projections  $p_i : X_1 \times X_2 \times \dots \times X_n \rightarrow X_i$ . The product topology is generated by “rectangles”  $U_1 \times U_2 \times \dots \times U_n$  with  $U_i \in \mathcal{T}^{X_i}$
- Suppose for  $n \in \mathbb{N}$  we have  $(X_n, \mathcal{T}^n)$ , can form  $\prod X_n$  with  $p_j : \prod X_n \rightarrow X_j, \forall j$ .  
Only needs to contain finite intersections so we have a base of  $U_1 \times U_2 \times \dots \times U_m \times X_{m+1} \times X_{m+2} \times \dots$  with  $U_j \in \mathcal{T}^j$ .

**Example 1.4.6.**  $X_j = \{0, 1\}$  with discrete topology.  $\prod_{j=1}^{\infty} X_j$  not discrete, also compact.

**Example 1.4.7.**  $C([0, 1])$ , set of continuous functions on  $[0, 1]$ ,  $\|f\|_{\infty} = \sup\{f(t) : t \in [0, 1]\} \rightarrow$  metric  $d(f, g) = \|f - g\|_{\infty}$

Given an normed vector space  $(V, \|\cdot\|)$ , let  $V' =$  all continuous linear functionals on  $V$ .

eg. for  $g \in C([0, 1])$  we have  $\varphi_g(f) = \int_0^1 f(t)g(t)dt$

For  $C([0, 1]) \xrightarrow{\varphi_g} \mathbb{R}$ , given topology not the smallest that makes each  $\varphi_g$  continuous.

## 1.5 September 2

### 1.5.1 Quotient Topologies

**Definition 1.5.1.** Let  $Y$  be a set. Let  $(X_{\alpha}, \mathcal{T}_{\alpha})$  be a topology with, for each  $\alpha$ , a function  $f_{\alpha} : Y_{\alpha} \rightarrow Y$ . The final topology is the largest topology that makes each  $f_{\alpha}$  continuous.

So for  $A \subset Y$ , in order for  $A$  to be in  $\mathcal{T}$  need  $f_{\alpha}^{-1}(A) \in \mathcal{T}_{\alpha}$  for all  $\alpha$ .

For fixed  $\alpha$ , we want  $\{A \in Y : f_{\alpha}^{-1}(A) \in \mathcal{T}_{\alpha}\}$ . This is a topology, denote it  $\mathcal{T}_{\alpha}^Y$ . It follows that  $T = \bigcap_{\alpha} \mathcal{T}_{\alpha}^Y$

Let  $Y$  be a set  $(X, \mathcal{T}^X)$ ,  $f : X \rightarrow Y$ , we require  $f$  is onto  $Y$ . Then  $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}^X\}$  is the smallest topology that makes  $f$  continuous. It is called the quotient topology.

Other view: Let  $X, Y$  be sets,  $f : X \rightarrow Y$  onto. Then  $f$  defines an equivalence relation on  $X$  by  $x_1 \sim x_2$  if  $f(x_1) = f(x_2)$ .

If we have an equivalence relation on a set, it defines a partition of the set.

If you have a partition,  $P$ , of a set  $X$ , then a set  $P$  is a set where the elements are nonempty subsets of  $X$ . Then define  $f : X \rightarrow P$ , where  $f(x)$  is the element,  $A$ , of  $P$  such that  $x \in A$ . Then  $f : X \rightarrow P$  onto.

**Definition 1.5.2.**  $(X, \mathcal{T}^X)$  and  $(Y, \mathcal{T}^Y)$  are homeomorphic if their  $f : X \rightarrow Y$ , one to one, onto such that  $f$  and  $f^{-1}$  are continuous.

**Example 1.5.3.**  $\mathbb{R}_d = (\mathbb{R}, \mathcal{T})$  with discrete topology.

Consider  $\mathbb{R}_d \xrightarrow{f} \mathbb{R}$  by  $f(t) = t$ .  $f$  is one to one, onto, and continuous but  $f^{-1}$  not continuous so it is not a homeomorphism.

**Example 1.5.4.** Let  $X = [0, 1]$ , define an equivalence relation  $0 \sim 1$  and  $r \not\sim s$  if  $r \neq s$  and  $0 < r < 1$ .  $[0, 1]/\sim$  homeomorphic to the circle. Let  $f(t) = e^{2\pi it}$ , we see  $f(0) = f(1)$ ,  $f$  is a homeomorphism.  
(Insert Figure)

**Example 1.5.5.**  $X = [0, 1] \times [0, 2]$

(Insert Figure) equivalence relation defined by  $(0, r) \sim (2, r)$  for  $0 \leq r \leq 1$

Quotient space is homeomorphic to a cylinder.

Suppose we define  $(0, 1) \sim (2, 1 - r)$   $0 \leq r \leq 1$

(Insert Figure) Quotient space homeomorphic to the mobius strip.

**Example 1.5.6.** Let  $X$  be the unit sphere  $\mathbb{R}^3 = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$ .

Put an equivalence relation: for  $v \in X$ ,  $v \sim -v$

$X/\sim$  is called a projective space.

## 1.6 September 7

### 1.6.1 Group Actions on Topological Spaces

For a topological spaces  $(X, \mathcal{T})$  the set of homeomorphisms of  $X$  to  $X$  forms a group under composition, auto-homeomorphisms,  $\text{Aut}((X, \mathcal{T}))$

Then if  $G$  is a group, then of an action of  $G$  on a topological space is a group homomorphism  $\alpha, \alpha : G \rightarrow \text{Aut}((X, \mathcal{T}))$ , so for each  $g \in G$ ,  $\alpha_g$  is a homeomorphism if  $(X, \mathcal{T})$

$$\alpha_{g_1} \circ \alpha_{g_2} = \alpha_{g_1 g_2}, \alpha_{g_1^{-1}} = (\alpha_{g_1})^{-1}$$

**Definition 1.6.1.** For an action  $\alpha$ , of  $G$  on some set  $X$ , given  $x_0 \in X$ , the orbit of  $x_0$  for the action  $\alpha$  is  $\{\alpha_g(x_0) : g \in G\}$ . The orbits form a partition of  $X$ . (orbits of  $\alpha_g(x_0)$  same as  $x_0$ ,  $\alpha_{g_1^{-1}}(\alpha_g(x_0)) = x_0$ )

Let  $X/\alpha$  be the set of orbits. Have “quotient map”  $X \rightarrow X/\alpha$  by  $x \mapsto \text{orbit of } x$ .

If  $X$  has a topology and  $\alpha$  acts by homeomorphism, puts quotient topology on  $X/\alpha$

**Example 1.6.2.** Symmetry of letters:

$X = A$  given  $Z_2 = \mathbb{Z}/2\mathbb{Z}$  act by reflection.  $X/\alpha =$  (Insert Figure)

$X = H$ ,  $Z_2 \times Z_2$ ,  $X/\alpha =$  (Insert Figure)

**Example 1.6.3.** Let  $G = \mathbb{Z}$ , let  $X = \mathbb{R}$ , let  $\alpha$  be an action of  $\mathbb{Z}$  on  $\mathbb{R}$  by translation,  $\alpha_n(t) = t + n$

each of  $\{\dots, t_0 - 1, t_0, t_0 + 1, \dots\}$ . What is  $\mathbb{R}/\alpha$

**Example 1.6.4.** A fundamental domain for  $\alpha$  is a subset of  $X$  that contains exactly one element of each orbit.

- For the above example, fundamental domain  $[0, 1)$  with open subsets “wrapped around edges” so  $\mathbb{R}/\alpha$  is homeomorphic to the circle. Homeomorphism given by  $t = e^{2\pi it}$ , constant on equivalence classes.

**Example 1.6.5.** The antipodal relation on the unit sphere with  $v \sim -v$  acted on by  $Z_2 = (0, 1)$  by  $\alpha_1(v) = -v$

Let  $Y$  be set of all lines in  $\mathbb{R}^3$  through 0. Let  $\mathbb{R} - \{0\}$ , have an action on  $\mathbb{R}^3$  by  $\alpha_t(r, s, v) = (tr, ts, tv)$

Orbits in  $\mathbb{R}^3 - \{0\}$ , set of all lines through 0, (with 0 removed). Each line intersects the unit sphere in 2 antipodal points. Quotient topology gives a topology on the set of lines.

### 1.6.2 Connectedness



**Definition 1.6.6.** A topological space  $(X, \mathcal{T})$  is connect if it does have two, nonempty, disjoint open sets  $A, B$  with  $A \cup B = X$

- If this is the acse,  $A, B$  also closed - called “clopen”

**Theorem 1.6.7.** If  $(X, \mathcal{T})$  is connected,  $f : X \rightarrow Y$  is continuous,  $f(X) = \text{range}(f)$  with the inherited topology is connected.

## 1.7 September 9

### 1.7.1 Connectedness

$(X, \mathcal{T})$  is connected if the only clopen sets are  $X, \emptyset$

**Proposition 1.7.1.** If  $(X, \mathcal{T})$ ,  $A \subseteq X$ , give  $A$  the relative topology, then if  $A$  is connected then so is  $\bar{A}$

**Proof.** Suppose that  $C$  is a clopen subset of  $\bar{A}$ , then  $C \cap A$  is a clopen subset of  $A$  so either  $C \cap A = A$  or  $C \cap A = \emptyset$ . If  $C \cap A = \emptyset$ ,  $C \cap \bar{A} = \emptyset$  since  $C$  is open. If  $C \cap A = A$  so  $C \supseteq A$  since  $C$  is closed. So  $C = \emptyset$  or  $\bar{A}$

**Proposition 1.7.2.** Given  $(X, \mathcal{T})$  a collection of  $\{F_\alpha\}$  of subsets of  $X$ , let  $Y = \bigcup_\alpha F_\alpha$ . Suppose that each  $F_\alpha$  is connected. If  $\exists p \in \bigcap F_\alpha$  then  $Y$  is connected.

**Proof.** Let  $C$  be a clopen subset of  $Y$ . We can assume that  $p \in C$ , then for each  $\alpha$ ,  $C \cap F_\alpha \neq \emptyset$ ,  $C \cap F_\alpha$  is clopen so  $C \cap F_\alpha = F_\alpha$  so  $F_\alpha \subseteq C$ . Thus  $C \supseteq \bigcup F_\alpha = Y$ , so  $C = Y$ .

**Proposition 1.7.3.** Let  $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$  be topological spaces and suppose that each is connected. Then  $X \times Y$  with the product topology is connected.

**Proof.** Choose a point  $b \in Y$  (a “basepoint”). Then  $E = \{b\} \times Y$  is homeomorphic to  $Y$  and so is connected. For each  $y \in Y$ , let  $H_y = X \times \{y\}$ . Homeomorphic to  $X$  so connected. For each  $y \in Y$ , let  $T_y = H_y \cup E$ , connected since  $(y, b)$  is in both. Choose a basepoint  $c \in Y$  so  $(b, c) \in E$  and  $(b, c)$  is in each  $T_y$  so  $X \times Y = \bigcup_{y \in Y} T_y$  is connected.

Follows that if  $X_1, \dots, X_n$  are topological spaces and each is connected then  $X_1 \times \dots \times X_n$  is connected.

Any open interval  $(a', b')$  in  $\mathbb{R}$  is connected. (False for  $(a, b)$  in  $\mathbb{Q}$ )

Suppose  $C \subseteq (a', b')$  is clopen and  $\neq \emptyset$  and suppose we have  $a \in C, b \in C', a < b$ . Consider  $A = \{r \in C : r < b\}$ .  $a \in A$  and  $b$  is an upper bound. Let  $c$  be its least upper bound then  $c \in A$  since if  $c \in C'$  then there is an open ball about  $c$  contained in  $C$  (since  $C$  is open), but  $c \notin C'$  for a similar reason.

### 1.7.2 Connected Components

Given  $(X, \mathcal{T})$  define an equivalence relation on  $X$  by  $x \sim y$  if there is a connected subset that contains both of them.

Reflexivity, symmetry clear. If  $x \sim y, y \sim z$ , then  $x, y \in C, y, z \in D$  so  $y \in C \cap D$  so  $C \cup D$  is connected.

## 1.8 September 12

### 1.8.1 Connected Components

$(X, \mathcal{T})$  a topological space. Define an equivalence relation on  $X$  by  $x \sim y$  if there is a connected subset of  $X$  containing both  $x$  and  $y$ .

Transitivity: If  $x \sim y$  and  $y \sim z$ , there is connected  $A$  with  $x, y \in A$  and connected  $B$  with  $y, z \in B$  then  $A \cup B$  is connected since  $y \in A, y \in B, x, z \in A \cup B$ .

The equivalence classes for this equivalence relation are called the connected components of  $X$ . Given  $x \in X$ , the equivalence class of  $x$  is the union of all connected sets containing  $x$ . So the equivalence class is the largest connected set containing  $x$ .

Since the closure of a connected set is connected, the equivalence classes are closed subsets of  $X$ .

**Example 1.8.1.**  $X = \mathbb{Q}$ , the connected components we get are the one point subsets.

( $\mathbb{Q}$  is totally disconnected, as is  $\prod_{m=1}^{\infty} \{0, 1\}$ , “0 dimensional”)

**Definition 1.8.2.** By a parametrized path in  $X$  we mean a continuous function,  $f$ , from some interval  $[a, b] \subseteq \mathbb{R}$ . This path connects  $f(a)$  to  $f(b)$ .

Define an equivalence relation on  $(X, \mathcal{T})$  by  $x \sim y$  if there is a path in  $X$  connecting  $x$  to  $y$ .

Reflexive: Assume  $f : [0, 1] \rightarrow X, f(0) = x, f(1) = y$  set  $g(t) = f(1 - t)$ , then  $g(0) = y, g(1) = x$

Transitive: If  $f : [a, b] \rightarrow X, f(a) = x, f(b) = y$  and  $g : [c, d] \rightarrow X, g(c) = y, g(d) = z$  change interval such that

$g : [b, e]$  with  $g(b) = y, g(e) = z$ .  $[a, e] = [a, b] \cup [b, e]$  so define  $h : [a, e] \rightarrow X$  by  $h(t) = \begin{cases} f(t) & t \in [a, b] \\ g(t) & t \in [b, e] \end{cases}$

The equivalence classes are called path components of  $(X, \mathcal{T})$

Note: path connected  $\rightarrow$  connected.

**Example 1.8.3.** Let  $f : (0, 1] \rightarrow X, f(t) = (t, \sin(\frac{1}{t}))$ , graph of  $\sin(\frac{1}{t})$ .

Subset is path connected but not closed. Closure is graph  $\cup \{0\} \times [0, 1]$ . Closure consists of 2 path connected components but only 1 connected component. In closure, 1 path connected component is not closed, while the other is closed but not open.

**Definition 1.8.4.**  $(X, \mathcal{T})$  is locally connected if  $\forall x \in X \forall$  open  $\mathcal{O}$  if  $x \in \mathcal{O}$  there is an open  $U, x \in U \subseteq \mathcal{O}$  with  $U$  connected.

- If  $(X, \mathcal{T})$  is locally connected then all connected components are open, and hence clopen.

**Definition 1.8.5.**  $(X, \mathcal{T})$  is locally path connected if  $\forall x \in X \forall$  open  $\mathcal{O}$  if  $x \in \mathcal{O}$  there is an open  $U, x \in U \subseteq \mathcal{O}$  with  $U$  path connected.

- If  $(X, \mathcal{T})$  is locally path connected, then all path connected components are clopen. path components = connected components.

**Definition 1.8.6.** A topological manifold of dimension  $n$  is a topological space  $(X, \mathcal{T})$  with the property that every  $x \in X$  has an open set  $\mathcal{O}$  such that  $x \in \mathcal{O}$  with  $\mathcal{O}$  homeomorphic to an open set in  $\mathbb{R}^n$  (open ball in  $\mathbb{R}^n$ , all of  $\mathbb{R}^n$ ).

## 1.9 September 14

### 1.9.1 Compactness

**Definition 1.9.1.** Let  $(X, \mathcal{T})$  be a topological space. By an open cover of  $X$  we mean a subset  $\mathcal{C}$  of  $\mathcal{T}$ , ie. a family of open sets such that  $\bigcup\{\mathcal{O} \in \mathcal{C}\} = X$ . By a subcover of  $\mathcal{C}$  we mean a subset  $\mathcal{D}$  of  $\mathcal{C}$  such that  $\mathcal{D}$  is a cover of  $X$ .

**Definition 1.9.2.**  $(X, \mathcal{T})$  is said to be compact if every open cover of  $X$  has a finite subcover.

- $[0, 1] \subseteq \mathbb{R}$  is compact
- Heine - Borel Property: any bounded closed subset of  $\mathbb{R}^n$  is compact.

Let  $(X, \mathcal{T})$  be a topological space. Let  $A$  be a subset of  $X$ , give  $A$  the relative topology. Then  $A$  is compact iff for any  $\mathcal{C} \subseteq \mathcal{T}$  such that  $\bigcup\{\mathcal{O} \in \mathcal{C}\} = A$  there is a finite subcover  $\mathcal{D}$  of  $\mathcal{C}$  such that  $\bigcup\{\mathcal{O} \in \mathcal{D}\} \supseteq A$

**Proposition 1.9.3.** Let  $(X, \mathcal{T})$  be compact. If  $A \subseteq X$  is closed, then  $A$  is compact.

**Proof.** If  $\mathcal{C} \subseteq \mathcal{T}$  such that  $\bigcup\{\mathcal{O} \in \mathcal{C}\} \supseteq A$ , since  $A$  closed,  $A'$  open so  $\mathcal{C} \cup \{A'\}$  is an open cover of  $X$ . Since  $X$  is compact, there is a finite subcover,  $\mathcal{D}$ . If we remove  $A'$  from  $\mathcal{D}$  (if  $A' \in \mathcal{D}$ ) we get a finite subcover  $\mathcal{C}$  covering  $A$ .

Any set with the indiscrete topology is compact and any subset of it is compact but not necessarily closed.

**Proposition 1.9.4.** Given  $(X, \mathcal{T})$  and  $A \subseteq X$  compact. If  $(X, \mathcal{T})$  is Hausdorff then for any  $x \in X$ ,  $x \notin A$  there are disjoint open sets  $U, V$  with  $A \subseteq U$ ,  $x \in V$

**Proof.** For any  $a \in A$ , by Hausdorff, there are open sets  $U_a, V_a$  disjoint with  $a \in U_a$ ,  $x \in V_a$ . The collection of sets  $\{U_a : a \in A\}$  covers  $A$ . Since  $A$  is compact there is a finite subcover  $U_{a_1}, \dots, U_{a_n}$ . Let  $U = U_{a_1} \cup \dots \cup U_{a_n} \supseteq A$ , let  $V = V_{a_1} \cap \dots \cap V_{a_n}$  so we get  $x \in V$ ,  $U, V$  disjoint.

**Corollary 1.9.5.** Given  $(X, \mathcal{T})$  Hausdorff,  $A \subseteq X$  compact, then  $A$  is closed.

**Proof.**  $A'$  open since for  $x \in A'$  can find open set containing  $x$ , disjoint from  $A$ .

**Theorem 1.9.6.** Given  $(X, \mathcal{T})$  compact, and  $f : X \rightarrow Y$  continuous, then  $f(X)$  is compact.

**Proof.** Let  $\mathcal{C}$  be an open cover of  $f(X)$ . Since for  $\mathcal{O} \in \mathcal{T}^Y$ ,  $f^{-1}(\mathcal{O}) \in \mathcal{T}^X$ , then  $\{f^{-1}(\mathcal{O}) : \mathcal{O} \in \mathcal{C}\}$  is an open cover of  $X$ . Since  $X$  is compact, there is a finite subcover  $f^{-1}(\mathcal{O}_1), \dots, f^{-1}(\mathcal{O}_n)$ . Then  $\mathcal{O}_1, \dots, \mathcal{O}_n$  is an open cover of  $f(X)$

**Example 1.9.7.** Given  $f : [0, 1] \rightarrow \mathbb{R}$  continuous,  $f([0, 1])$  is connected, compact so is some  $[a, b]$ . So  $f$  attains its supremum =  $\text{lub}\{f(t) : t \in [a, b]\}$

**Theorem 1.9.8.** Given  $(X, \mathcal{T})$ ,  $(Y, \mathcal{T})$ ,  $f : X \rightarrow Y$  continuous, assume  $f$  is continuous, one to one, onto,  $X$  is compact,  $Y$  is Hausdorff. Then  $f$  is homeomorphism.

*Proof.* Need to show  $f^{-1}$  continuous, so need  $f(\mathcal{O}) \in \mathcal{T}^Y$  for  $\mathcal{O} \in \mathcal{T}^X$ , equivalently, if  $A$  is closed in  $X$ , then  $f(A)$  is closed in  $Y$ . If  $A$  closed,  $A$  compact so  $f(A)$  is compact, but  $Y$  is Hausdorff so  $f(A)$  is closed.  $\square$