MATH 250A: Groups, Rings, and Fields

Jad Damaj

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Contents

Chapter 1

Groups

1.1 August 25

1.1.1 Groups

Two ways to define groups

- concrete: group = symmetries of an object X. Here a symmetry is a bijection $X \to X$ with inverse that preserves "structure" (topology, order, binary operation, ...)

Example 1.1.1. The rectangle has 4 symmetries.
The icossahedron has 20×3 symmetries since after fixing the first face there are 3 possible rotations.
Vector space $\mathbb{R}^k$: $n \times n$ matrices with det $\neq 0$, denoted $GL_n(K)$

- abstract definition:

Definition 1.1.2. A group is a set G with a binary operation $G \times G \to G$ by $(a, b) \mapsto ab, a\times, a+b, \ldots$ with "Inverse" : $G \to G$ by $a \mapsto a^{-1}$ and "Identity": $1, 0, e, I, \ldots$ satsifying the axioms:
$1x = x1 = x \quad x(x^{-1}) = (x^{-1})x = 1 \quad (xy)z = x(yz)$

We can go from the concrete definition to the abstract one: the binary operation is composition, the identity is the trivial symmetry, inverses given y "undoing' a symmetry.

Is an abstract group the symmetries of something?

Theorem 1.1.3 (Cayley's Theorem)**.** Any abstract group is the group of symmetries of some mathematical object.

Recall group actions :

Definition 1.1.4. Given a group G, a set S, a (left) gtroup action is a map $G \times S \to S$ by $(g, s) \mapsto g(s), gs$ satisfying $g(h(s)) = gh(s)$, $1s = s$.

To prove Cayley's theorem we need to find :

1. a set S acted on by G

2. structure on S so that $G =$ all symmetries.

What is S? Take $S = G$.

Need to define the action of $GonG$. There are 8 natural ways to do this.
First 4, we defin4 $G \times S \to S$ by

- $g(s) = s$ trivial action
- $g(s) = gs$ group product
- Try $g(s) = sg$ Fails since G not necesarily commutative: $g(h(s)) = (sh)g \neq s(gh) = gh(s)$
- $g(s) = sg^{-1}$ works since $g(h(s)) = g(sh^{-1}) = sh^{-1}g^{-1} = s(gh)^{-1} = gh(s)$
- $g(s) = gsg^{-1}$ adjoint action

The above group action is known as a left group action, We define a right group action in a similar way :
$S \times G \to S$ by$(s, g) \mapsto (s)g, s^g$ satisfying (sg)h = s(gh), $s1 = s$.

We now define right group actions of G on G: $S \times G \to G$ by

- $(s, g) \mapsto s$
- $(s, g) \mapsto sg$
- $(s, g) \mapsto g^{-1}s$
- $(s, g) \mapsto g^{-1}sg$

Now we have $S = G$, S=set acted on by G using left action $g(s) = gs$ - left translation. So we have shown $G \subseteq$ symmetries of S.

Want : G =symmetries of S + "structure". Let structure on S= right action of G on S.
We now have 3 copies of G:

1. set $S = G$
2. G acts on left on S (G = symmetries of S)
3. G acts o the right on S (Structure of S)

Object $S = S$ + right G action

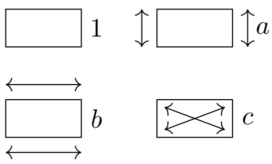
What are the symmetries of this?
Bijection $f : S \to S$ preserving the right G-action. eg. $f(sg) = f(s)g$
Need to check:

1. Left G-action of G preserves the right G-action
2. Anything that preserves the right G-action is given by left multiplication of an element of G

Check (1): For $g \in G$ need $(gs)h = g(sh)$, follows by commutativity
Note: left G-action does not preserve right G-action: $g(hs) \neq h(gs)$ in general

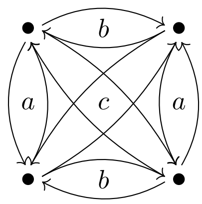
Check (2): Suppose $f : S \to S$ preserves the right G-action, $f(sh) = f(s)h$ for all $h \in G$. Need to find $g \in G$ such that $f(s) = gs$. Take $s = 1$, $f(1) = g1 = g$ so $g = f(1)$. If $g = f(1)$, then $f(s) = gs$ since $gs = (f(1))s = f(1s) = f(s)$.
So we have G = symmetries of (Set G + right G action)

Example 1.1.5. G=symmetries of rectange, set $S = G$



	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

We get the graph:



Cayley graph: Point for each $g \in G$ Draw a line from g to h with $gf = h$.

Goal of Group theory

1. Classify all groups
   - Hard but can do special cases: Groups of order 60, finite subgroups of rotations in $\mathbb{R}^3$, all finite simple groups, symmetries of crystals
2. Given a group G, classify all ways G can act on something (called a representation of G)
   - Permutation representation : G acts on a set S
   - Linear representation : G acts on a vector space

Example 1.1.6. Poncaire group = symmetries of space time
elementary particle: space of states = vector space acted on by G = linear group of G

1.1.2 Review of homomorphisms, isomorphims

Definition 1.1.7. A homomorphism is a map $f : G \to H$ that preserves structure
eg. $f(gh) = f(g)f(h)$, $f(1) = 1$, $f(g^{-1}) = f(g)^{-1}$

Note: last two properties can be derived from the first.

Example 1.1.8. $\exp(x) = e^x : (\mathbb{R}, +) \to (\mathbb{R}, \times)$
$\exp(x + y) = \exp(x)\exp(y)$, $\exp(0) = 1$, $\exp(-x) = \exp(x)^{-1}$

Definition 1.1.9. The kernel of a homomorphism f is the set of elements with image the identity.

Example 1.1.10. $\mathbb{R} \to$ rotation is the plane by $\theta \mapsto$ rotation by angle θ.
nontrivial kernel : multiples of 2π.
We get the short exact sequence: $0 \to 2\pi\mathbb{Z} \to \mathbb{R} \to \text{rotations} \to 0$

Definition 1.1.11. A sequence of homomorphisms $A \to B \to C$ is exact if Image $A \to B$ = Kernel $B \to C$

$0 \to A \to B$ means $A \to B$ is injective
$A \to B \to 0$ means $A \to B$ is surjective

Definition 1.1.12. $f : A \to B$ is an isomorphim if it is a homomorphism with an inverse. We say A, B are isomorphic. "basically the same"

Example 1.1.13. $2\pi\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.

Example 1.1.14. $\mathbb{Z}/4\mathbb{Z}$, integers mod 4 with addition: $\{0, 1, 2, 3\}$ and $(\mathbb{Z}/5\mathbb{Z})^{\times}$, under multiplcation: $\{1, 2, 3, 4\}$ are isomorphic.
We map $0 \to 1 = 2^0$, $1 \to 2 = 2^1$, $2 \to 4 = 2^2$, $3 \to 3 = 2^3$ eg. $x \mapsto 2^x$

1.1.3 Classify all finite groups up to isomorphim

Definition 1.1.15. The order of a group G = number of elements in G

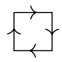
Order 1: $e \times e = e$ 1 group - trivial group
Order 2: 1 group - e, f with $f^2 = e \cong \mathbb{Z}/2\mathbb{Z}$
Order p for p prime: only one group $\mathbb{Z}/p\mathbb{Z}$ (integers modulo p)

Definition 1.1.16. For $g \in G$ the order of g is the smallest $n \geqslant 1$ with $g^n = 1$

Theorem 1.1.17 (Lagrange's Theorem)**.** If $g \in G$, the roder of g divides the order of G.

Example 1.1.18. Suppose $|G| = p$, (p prime). Pick $g \in G$ with $g \neq e$. Order of g divides $|G| = p$ so is either 1 or p. Can't be one since $g \neq e$. So elemenets of G $1, g, \ldots, g^{p-1}$ are all distinct since $g^p = 1$, $g^x \neq 1$ for $0 \leqslant x < p$ and if $g^i = g^j, g^{i-j} = 1$. Thus, these must be all elements of G.

Order 4:

- Ex : $\mathbb{Z}/4\mathbb{Z}$, symmetries of rectangle, $(\mathbb{Z}/5\mathbb{Z})^{\times}$, $(\mathbb{Z}/8\mathbb{Z})^{\times}$, symmetries of 
- only 2 groups of order 4

1.2 August 30

1.2.1 Langrange's Theorem

Order 4: $\mathbb{Z}/4\mathbb{Z}$, symmetries of rectangle
How to show not isomorphic?
Find some property (preserved by isomorphism) that one group has but the other does not.

Property: Order of elements

- in $\mathbb{Z}/4\mathbb{Z}$, 0, 1, 2, 3 have orders 1, 4, 2, 4 respectively
- all nontrivial elements of the group of symmetries of the rectangle have order 2

Note: counting elements of each order works for small gorups but 2 groups of order 16 with same number of elements of each order

Classification: By Lagrange's theorem, each element has order 1, 2, or 4

1. Have an element of order 4: g, group $= \{1, g, g^2, g^3\} \cong \mathbb{Z}/4\mathbb{Z}$
   In general, if a group of n elements has an element of order n, it is $\cong \mathbb{Z}/4\mathbb{Z}$
2. All elements have order 1 or 2.
   Suppose G is finite and has this property. Then G commutes since $(gh)^2 = ghgh = 1 = g^2g^2$ so $gh = hg$.
   Note: only true for prime 2, there is a group of order 27 such that all elements have order 1 or 3 but is not commutative
   Write group operation as $+$. G is a vector space over $\mathbb{F}_2$ (field of 2 elements). So $G \cong \mathbb{F}_2^k$ for osme set $|G| = 2^k$. We get 1 group of order 4 with all elements of order 1 or 2.

Group of order 4 is product of 2 groups, $\mathbb{F}_2^2 = \mathbb{F}_2 \oplus \mathbb{F}_2$.
Suppose G, H are gorups, $G \times H$ is a gorup under operation $(g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, h_1h_2)$

Example 1.2.1. $\mathbb{C}^\times \cong \mathbb{R}_{\geqslant 0} \times S^1$, $z = |z| \cdot e^{i\theta}$

Chinese Remainder Theorem: (m, n) coprime, $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.
We have maps $f : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $g : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. This gives $h : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. If $(m, n) = 1$, then the map is injecitve since if $h(k) = 0$, $k \equiv 0 \mod m, \mod n$

Infinite Products: $G_1 \times G_2 \times G_3 \times \cdots$, set of all elements $(g_1, g_2, g_3, \ldots,)$
Infinite Sums: Like infinite products but all but finitely many of g_1 are 1.

Example 1.2.2. Roots of $1 = e^{2\pi q}$, $q \in \mathbb{Q}$.
Infinite sum $G_2 + G_3 + G_5 + G_7 + G_11 + \cdots$ $\quad$ (G_p = roots of order p^n for some $n \geqslant 1$)

Symmetry of Platonic Solids

Faces	Name	Rotations	Rotations + Reflections	
4	tetrahedron	$12 = 4 \times 3$	24	$\to$ not a product
6	hexahedron (cube)	$24 = 6 \times 4$	48	product $\mathbb{Z}/2\mathbb{Z}$ × rotations
8	octahedron	$24 = 8 \times 3$	48	
12	dodecahedron	$60 = 12 \times 5$	120	
20	icosahedron	$60 = 20 \times 3$	120	

All except tetrahedron have

symmetry $\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ fo reflections in $\mathbb{R}^3$, so it commutes with everything

For the tetrahedron, we have $\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

Order 5: $\mathbb{Z}/5\mathbb{Z}$

Exercise 1.2.3. Find a graph as small as possible with symmetries $\mathbb{Z}/5\mathbb{Z}$

Order 6: 3 obvious examples: $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, symmetries of the triangle

- $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
- group of symmetries of the triange is not abelian
  Permutation Notation: $(5\,2\,1\,3)$ = function sending $5 \to 2$, $2 \to 1$, $1 \to 3$, $3 \to 5$
  (Insert Figure)
  $(1\,2)(2\,3) = (1\,2\,3)$ but $(2\,3)(1\,2) = (1\,3\,2)$

Definition 1.2.4. A subgroup of a group G, is a subset closed under group operations.

Theorem 1.2.5 (Lagrange's Theorem)**.** If H is a subgroup of G, $|H|$ divides $|G|$.

Special Case: If H = powers of g, $1, g, g^2, \dots, g^{n-1}$, $|H| = |g|$

Construction of subgorups: Pick a set S acted on by G, pick $s \in S$.
H: elements g with $gs = s$ (elements fixing s). Then H is a subgroup.
Lagrange (Converse to Cayley's Thm): If H is a subgroup of G we can find a set acted on by G, such that H=elements fixing $s \in S$.

Given a gorup G, subgroup H. We want to construct: a set S acted on by G.
Consider G=symmetries of triangle, $H = \{(1)(2)(3), (2\,3)\}$ fixing 1.
How do we write 1, 2, 3 in terms of G, H?
Left cosets of H: $1 \leftrightarrow$ elements g with $g(1) = 1$ (H), $2 \leftrightarrow$ elements g with $g(1) = 2$ ($(1\,2)H$), $3 \leftrightarrow$ elements g with $g(1) = 3$ ($(1\,3)H$)

Left cosets of H are sets of the from aH (some fixed $a \in G$).
Define $g_1 \approx g_2$ if $g_1 = g_2h$ for some $h \in H$. This is an equivalence relation:
Reflexivity: $g_1 \approx g_1$ group identity, 1
Symmetry: $g_1 \approx g_2 \to g_2 \approx g_1$ group inverses, h^{-1}
Transitivity: $g_1 \approx g_2, g_2 \approx g_3 \to g_1 \approx g_3$ group operation, h_1h_2
G = disjoint union of cosets (equivalence classes of $\approx$) and any two cosets have the same same $|H|$ since we have a bijection $H \to aH$ byb $h \mapsto ah$ with inverse $h \mapsto a^{-1}h$.
So $G = \#$ cosets $\times$ size of cosets $= \#$ elements of S $\times$ |subgroup of elements fixing s|
Note: We assume S is transisitve - if $s_1, s_2 \in S$. $g(s_1) = s_2$ for some g

Rotations of a dodecahedron: 12 (faces) $\times$ 5 = 20 (vertices) $\times$ 3 = 30 (edges) $\times$ 2 = 60

Conways Group: has order 8315553613086720000
Acting on Frames: # 8252375 Group fixing each frame: 1002795171840

Special Cases of Lagrange:

- Fermat: $a^p \equiv a \mod p$ (p prime), $a^{p-1} \equiv 1 \mod p$ $(a, p) = 1$
  Group $(\mathbb{Z}/p\mathbb{Z})^\times$ integers modulo p under $\times$ has order $p - 1$.
  Lagrange: order of a divides $p - 1$ so $a^{p-1} \equiv 1$
- Euler: $a^{\varphi(m)} \equiv 1 \mod n$ $(a, m) = 1$
  $(\mathbb{Z}/m\mathbb{Z})^\times$= group of elements coprime to m, mod m, order = $\varphi(m)$

$m = 8$: $\varphi(m) = 4$, $(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\}$. Euler $a^4 \equiv 1 \mod 8$ (a odd) but we see $a^2 \equiv 1 \mod 8$

Right Cosets: $Ha \leftrightarrow$ elements of a set acted on, on the right by G. $S \times G \to S$
Are left cosets the same as right cosets? sometimes

Example 1.2.6. Take G = symmetries of triangle. $H = \{1, (2\,3)\}$. Find the left, right costs of H in G.
Left: $H = \{1(2\,3)\}, (3\,1)H = \{(3\,1), (3\,2\,1)\}, (1\,2)H = \{(1\,2), (1\,2\,3)\}$
Right: $H = \{1(2\,3)\}, (3\,1)H = \{(3\,1), (1\,2\,3)\}, (1\,2)H = \{(1\,2), (3\,2\,1)\}$
so left cosets $\neq$ right cosets

Definition 1.2.7. Index of H in G, $[G : H]$= # cosets of H in G.

Left or right cosets? $[G : H][H] = |G|$ when G finite so # left cosets = # right cosets.
In gernal, right cosets → left cosets by $Ha \mapsto a^{-1}H$ so # left cosets = # right cosets

1.2.2 Normal Subgroups

G/H = set of left coset of G. Is G/H a group?
How to definte $(g_1H) \times (g_2H)$? g_1g_2H
Problem: not well defined - suppose we have g_1, g_2, g_1h_1, g_2h_2. Want $g_1g_2H = g_1h_1g_2h_2H$
Is $h_1g_2 = g_2(h \in H)$? not in general
Want: $ghg^{-1} \in H$ for all $g \in G$. If this holds, then we can turn G/H into a group.

Definition 1.2.8. If H satisfies the above property, H is called a normal subgroup of G.

Example 1.2.9. G = symmetries of triangle. $H = \{(2\,3), 1\}$. Is H normal?
$(1\,2)(2\,3)(1\,2)^{-1} = (1\,3) \notin H$ so H is not normal

What about $H = \{1, (1\,2\,3), (1\,3\,2)\}$. Is H normal?
H has index 2 in G. $[G : H] = \frac{|G|}{|H|} = 2$. We claim any subset of order 2 is normal.
There are only 2 left cosets: H, things not in H. Similarly for right cosets. So right cosets = left cosets. So H is normal.

Classifying Groups of Order 6

- orders of elements 1, 2, 3, 6
- If element of order 6, group must be cyclic
- Want element of order 3

Lagrange: order of element divides order of group
Converse: If n divides $|G|$, does G have a subgroup of order n?
No: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has no element of order 4
Yes: if n is prime (Cauchy)
So G has elements a, b of order 2,3 and subset $(1, b, b^2)$ has order 2 so it is normal.

1.3 September 1

1.3.1 Semidirect Products

Groups of Order 6:
2 subgroups A, B of order 2,3 $\quad |A| \cdot |B| = |G|$, $A \cap B = \{e\}$

In general, suppose that for a group G, subgroups A, B

1. $|G| = |A| \cdot |B|$
2. $A \cap B = \{e\}$

Want to reconstruct G from A, B
$G = AB = \{ab \mid a \in A, b \in B\}$, # pairs $(a, b) = |G|$

If $a_1b_1 = a_2b_2$, $a_2^{-1}a_1 = b_2b_1^{-1} \in A \cap B = \{e\}$ so $a_1 = a_2, b_1 = b_2$
Every element of G can be written uniquely as a product of $a \in A$, $b \in B$

Problem: What is $a_1b_1 \cdot a_2b_2$? $\quad = a_3b_3$
Easy case: $ab = ba$ for all $a \in A$, $b \in B$ $\quad (a_1b_1)(a_2b_2) = (a_1a_2)(b_1b_2)$
We can view G as the product of $A, B \to G = A \times B$
Slightly less easy case: A is a normal subgroup of G. We get an action of the group B on the group A.
Define the action of B on A by $b(a) = bab^{-1} \in A$ (A normal)
This determines the product on G. $(a_1b_1)(a_2b_2) = a_1(b_1a_2b^{-1})b_1b_2 = \underbrace{a_1b_1(a_2)}_{\in A} \times \underbrace{b_1b_2}_{\in B}$.

Suppose given groups A, B action of V on A. We construct the semidirect product of A and B, $A \rtimes B$ on the set $A \times B$ with the product given by : $(a_1, b_1)(a_2, b_2) = (a_1b_1(a_2), b_1b_2)$. We can check this is a group.

Order 6
So $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ defined by the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/3\mathbb{Z}$.
$\mathrm{Sym}(\mathbb{Z}/3\mathbb{Z})$: either $f(1) = 1$ or $f(1) = 2$ so only two possible homomorphisms $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Sym}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$: identity and trivial homomorphisms
So groups of order 6:

- $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ trivial action $\cong \mathbb{Z}/6\mathbb{Z}$
- $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ nontrivial action $\cong S_3$

1.3.2 Cauchy's Theorem

Theorem 1.3.1 (Cauchy's Theorem)**.** If $p \,|\, |G|$ (p prime), G has an element of order p.

Proof. We use induction on the size of the group: can assume true for any peroper subgroups and quotient groups
G abelian: pick $g \in G$. If $p|\,|g|$, g has order pn so g^n has order p.
If $p \nmid |g|$, look at $G/\langle g\rangle$. $\langle g\rangle$ normal since G is ableian, p divides $|G/\langle g\rangle|$. Pick $h \in G/\langle g\rangle$, order divisible by p. Lift h_1 in G. Then $p|\,|h_1$.

Standard Error: Can't always lift h to element of the same order
$G \cong \mathbb{Z}/4\mathbb{Z}$, $g = 2$. $G/\langle g\rangle$ has order 2 so take nontrivial element. Its lift does not have order 2 in G

Definition 1.3.2. The center of G is the elements that commute with all elements of G.

Lemma 1.3.3. Suppose G is nonotrivial, all proper subgroups have index divisible by p. Then the center of G is divisible by p.

Proof. Look at left action of G on itself by conjugation. $G =$ union of orbuts where a, b in the same orbit if there is some g such that $g(a) = b$. $|G| = \sum(\text{size of orbits})$
Size of orbit $= |G|/$subgroup of elements fixing a point. Either 1 or divisble by p so
$G = \underbrace{1+1+1}_{\text{size } 1} + \cdots + \underbrace{pn_1 + pn_2}_{\text{size } >1} + \cdots$. Since G divisible by p # orbits with one element is. Theorem follows

since Center of G = elements with orbit of size 1.

Proof (Cauchy's Theorem (Cont))**.** Case 1: Some proper subgorup has order dvisible by p.
Such a subgroup has an element of order divisble by p by induction.
Casse 2: All proper subgroups have index divisible by p. By lemma, center of G has order divisble by p
Center of G is abelian so it has an element of order p.

Order 7: $\mathbb{Z}/7\mathbb{Z}$

Order 8: Obvious examples: Producst $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}/8\mathbb{Z}$, symmetries of a square (D_8) - dihedral group.
Orders of elements: $1, 2, 4, 8$

- If element has order 8, group is cylic
- If all elements have order 1 or 2, group is vector field over $\mathbb{F}^2$ so is $(\mathbb{Z}/2\mathbb{Z})^2$

So can assume G has an element a, of order 4. $a^4 = 1$. Subgroup $A = \{1, a, a^2, a^3\}$ has index 2 so is normal.
Quotient group has order 2 so $\cong \mathbb{Z}/2\mathbb{Z}$
We have an exact sequence $1 \to \mathbb{Z}/4\mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$

Problem: Given $1 \to A \to G \to B \to 1$ How to construct G form A, B?
Possibilities: $G = A \times B$, or $A \rtimes B$, not always the case:

- $1 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 1$ not a semidirect product
- $1 \to \mathbb{Z}/3\mathbb{Z} \to S_3 \to \mathbb{Z}/2\mathbb{Z} \to 1$ $S_3 = \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$

We get an action of B on A by conjugation so considering $1 \to \mathbb{Z}/4\mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 1$ we can take the nontrivial element b of $\mathbb{Z}/2\mathbb{Z}$. Cant say $b^2 = 1$, but $b^2 \in A$. Also B acts on A by conjugation.
So we have $\mathbb{Z}/4\mathbb{Z} = \{1.a, a^2, a^3\}$ $a \mapsto bab^{-1}$: $a \mapsto a$ or $a \mapsto a^{-1}$
Possibilities:

	$bab^{-1} = a$	$bab^{-1} = a^{-1}$
$b^2 = 1$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	D_8
$b^2 = a$ $b^2 = a^3$	$\mathbb{Z}/8\mathbb{Z}$ $(a = 1, b = 2)$	Impossible
$b^2 = a^2$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	Quaternions

Semidirect Products
$a = b^2$, $ab = ba \to a^2 = 1$

Quaternion group: generated by a, b with $a^4 = 1$, $b^2 = a^2$, $bab^{-1} = a^{-1}$
Does it exst? Yes: have be viewed in $M_2(\mathbb{C})$- $a = \begin{pmatrix} i & \\ & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Usually denote elements: $I = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $K = IJ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
Quaternions $Q_8 = \{i, I, J, J, -1, -I, -J, -K\}$ satisfying $I^2 = j^2 = K^2 = 1$, $IJ = K$, $JK = 1$, $KI = J$
Hamilton's Quaternions(H) = all numbers $a + bi + cj + dk$ a, b, c, d real
Nonzero elements of H form a gorup. Problem: Show inverses exist.
$(a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^+ d^2 > 0$ so
$(a + bI + cj + dk)^{-1} = \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}$
Can also look at $S^3 \subset H = \{a + bi + cj + dk \,|\, a^2 + b^2 + c^2 = d^2 = 1\}$
For $z = a + bi + cj + dK$, $\overline{z} = a - bi - cj - dk$ let $z\overline{z} = N(z)$
We see $N(z_1 z_2) = N(z_1)N(z_2)$ so if $N(z) = 1$ closed under $\times$ so is a group.

Only spheres that are a group are S^0, S^1, S^3. Elements of $\mathbb{R}, \mathbb{C}, H$ with absolute value 1.

Note: $Q_8 \subseteq S^3$

1.3.3 Burnside's Lemma

Problem: How many ways to arrange 8 rooks on a chess board so that no 2 attack each other?
8 ways for first row, 7 for second, ..., so 8! = 40320 total
Suppose we want to count them up to symmetry:

- For 3×3: (Insert Figure)
  can only have 2

Approximate number $= \frac{\text{total \# of elements}}{\text{order of group}} = \frac{8!}{8} = 7! = 5050$

General problem: Suppose we have a group G acting on a set S. How many orbits? $\geqslant \frac{|S|}{|G|}$
Answer:

Lemma 1.3.4 (Burnside's Lemma)**.** # of orbits = average number of fixed points of $g \in G$, eg. $s \in S$ with $g(s) = s$

Proof. Count number of pairs $(g, s) \in G \times S$ with $g(s) = s$ in 2 ways:

1. Sum over G: $\sum_{g \in G}$ (# fixed by g)
2. Sum over S: Each orbit contributes (size of orbit) $\times$ (# of elements fixing a point) $= |G|$
   so sum $= |G| \times$ # of orbits

So # of orbits $= \frac{1}{|G|} \sum_g$ # fixed points = avg # fixed points

1.4 September 6

1.4.1 Burnside's Lemma

Example 1.4.1. Find the number of ways to arrange 8 nonattacking rooks on a chessboard up to symmetry.
Recall - # of orbits of a set = average number of fixed points $= \frac{1}{|G|} \sum_{g \in G}$ # fixed points of g.
G = dihedral group D_8, acting on 8! = 40320 ways to arrange 8 rooks
Elements of D_8:

- Trivial (Insert Figure): 8! = 40320
- 180° rotation (Insert Figure) : 8 options for 1rst, 6 options for 2cnd, ... so $8 \times 6 \times 4 \times 2$
- 90° rotation (Insert Figure): 6 options for 1rst, 2 options for 2cnd so 6×2

2 elements g_1, g_2 are called conjugate if $g_1 = g g_2 g^{-1}$ for some g (Formalizes notion of "looks the same")
g_1 =(Insert Figure) g_2 =(Insert Figure) g = (Insert Figure) exchanging g_1, g_2.
If two elements are conjugate then they have the same number of fixed points.
$g_1(s) = s \to g_2(gs) = g g_1 g^{-1} g s = gs$

- (Insert Figure): conjugate with 90° rotation so 6×2

- (Insert Figure): conjugate and have 0 since rotates rook to the same column/row
- (Insert Figure): conjugate. $C_n = \#$ ways to place rooks on $n \times n$ chessboard invariant under transformation. $c_0 = 1, c_1 = 1$.
  Case 1 : (Insert Figure) Case 2: (Insert Figure)
  so $c_n = c_{n-1} + (n-1)c_{n-2}$ and $c_n = 1, 1, 2, 4, 10, 26, 76, 232, 764$

So # of ways to place rooks $= \frac{1}{8}(1 \times 8! + 1 \times 384 + 2 \times 12 + 2 \times 0 + 2 \times 764) = 5282$
Slighly more than original guess $\frac{40320}{8} = 5040$

Example 1.4.2. Find the number of ways to color a cube with n different colors up to symmetry.

1.4.2 Groups of order p^2

Order 9: Obvious examples = $\mathbb{Z}/9\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

Classify all groups of order p^2 (p prime): only ex are $\mathbb{Z}/p^2\mathbb{Z}$, $(\mathbb{Z}/p\mathbb{Z})^2$
(1): Every group of order p^n (p prime, $n > 0$) has nontrivial order

Proof. Recall, if all proper subgroups have index divisible by p, $p \mid |G|$ then G has nontrivial center. So if $|G| = p^n$, $n > 0$, we see G has nontrivial center. $\square$

Implies that if $|G| = p^n$, G is nilpotent. ie. repeatedly modding out by the center gives you the trivial group.
$G_0 = G$, $G_1 = G_0/Z(G_0)$, $G_2 = G_1/Z(G_1)$, $\ldots$ If G_n is trivial for some n, G is caleld nilpotent.
This gives an exact sequence: $1 \to Z(G_i) \to G_i \to G_{i+1} \to 1$
Note: A group may still have nontrivial center even after modding out by the original center: $G = D_8$, $G/Z(G) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
S_3 (order 6) is not nilpotent

(2): If $G/Z(G)$ is cyclic then G is abelian.

Proof. Consider $1 \to Z(G) \to G/Z(G) \to 1$. $Z/(G)$ is powers of g_1, lift g_1 to g in G.
Every element in G is of the form zg^n ($z \in$ center) so all commute $z_1 g^{n_1}, z_2 g^{n_2}$:
z_1 commutes with $z_2 g^2 n$, g^{n_1} commutes with z_2, and g^{n_1} commutes with g^{n_2} $\square$

(3): Every group of order p^2 is abelian.
Note: not true for p^3, consider D_8, Q_8 of order 2^3

Proof. Center is nontrivial so has order $\geqslant p$. $G/Z(G)$ has order 1 or p so it is cyclic so G is abelian. $\square$

(4): Every group of order p^2 is $(\mathbb{Z}/p^2\mathbb{Z})$ or $(\mathbb{Z}/p\mathbb{Z})^2$

Proof. Case 1 : elements of order $p^2 \to G$ is cyclic $\cong \mathbb{Z}/p^2\mathbb{Z}$
Case 2: all elements have order p or $1 + G$ abelian. G is really a vector field over $\mathbb{F}_p$ the field with p elements so $G = \mathbb{F}_p \oplus \mathbb{F}_p$. $\square$

1.4.3 Dihedral Groups

Order 10: $\mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, $D_{10} = (\mathbb{Z}/5\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$

Groups of Order $2p$: G has a subgroup of order p, index 2 so is normal. G has a subgroup of order 2 so $G \cong \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, determined by action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}$.
Symmetries of $\mathbb{Z}/p\mathbb{Z}$: map generator $1 \to$ elment of order p. $n \mapsto na\ p\ \ |a$
Symmetries $= (\mathbb{Z}/p\mathbb{Z})^{\times}$ nonzero integers mod p under $\times$. Only elements of order 2 are $\pm a \mod p$
$G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}2\mathbb{Z}$ (trivial action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/p\mathbb{Z}$)
$G \cong \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}2\mathbb{Z}$ ($\mathbb{Z}/2\mathbb{Z}$ acting by -1 on $\mathbb{Z}/p\mathbb{Z}$) = dihedral group.

Dihedral Groups: symmetries of a regular n-gon ($n \geqslant 3$). Order $2n$
(Insert Figure)

What is the center of D_{2n}? ($n \geqslant 2$)? Order 2 if even, order 1 if odd.

Why does D_{12} split as a product?
(Insert Figure) $D_12 = D_6 \times \mathbb{Z}/2\mathbb{Z}$ = symmetries of triangels $\times$ 180° rotation commutes with elements and flips the two triangles
D_{10} (Insert Figure) Problem: 180° does not flip two squares.
D_{2n} can be split $D_{2n} \times \mathbb{Z}/2\mathbb{Z}$ for $D_4, D_{12}, D_{20}, D_{28}$ ($\equiv 2 \mod 4$)

Involutions in dihedral groups (elements of order 2)
D_{2n} (Insert Figure)

Reflection Groups (generated by relations)
(Insert Figure) Suppose g and h are relations. If $g^2 = 1$, $h^2 = 1$, $(gh)^n = 1$

- Fid property of all finite groups that doesn't hold for all infinite groups, in the language of groups.

Property: If g, h are involutions, either g, h are conjugates or some involution commutes with g, h
$g^2 = 1$, $h^2 = 1$, $(gh)^n = 1$ for some n (since group finite)
n even: D_{2n} has nontrivoal element in center
n odd: All involutions commute
Fails for ∞ dihedral group $g^2 = 1$, $h^2 = 1$ (Insert Figure)
Order 12: $\mathbb{Z}/12\mathbb{Z}$, products - $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $S_3 \times \mathbb{Z}/2\mathbb{Z}$, rotations of tetrahedrons, semidirect products- $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/4\mathbb{Z}$.

Binary Dihedral: S^3(= unit quaternions) is a group acting on $\mathbb{R}^3 = bi + cj + dk$ - rotations in $\mathbb{R}^3$
$1 \to \pm 1 \to S^3 \to$ rotaitons on $\mathbb{R}^3 \to 1$ where ± 1 act trivially on $\mathbb{R}^3$
$1 \to \pm 1 \to \hat{G} \to G$ = finite reflecction group. Ex: group over D_{2n}
Binary dihedral groups of order $4n$ so binary dihedral group of order 12. (Q_8 binary dihedral group of order 8)
5 groups of order 12.

1.5 September 8

1.5.1 Sylow Theorems

Order 12: $\mathbb{Z}/12\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $S_3 \times \mathbb{Z}/2\mathbb{Z}$, A_4, $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$

Sylow Theorems:

- Lagrange: if $H \subseteq G$, $|H| \mid |G|$
- If $m \mid |G|$ can we find a subgroup of order G?
  No: A_4=reflections of tetrahedron has no subgroup of order 6

Theorem 1.5.1 (Sylow's Theorems)**.**
1. If $p^n \mid |G|$ (p prime) then G has a subgroup of order p^n if n is maximal, called p-Sylow subgroup.
2. Number is $1 \mod p$, divides $|G|$
3. All p-sylow subgroup are conjugate (so all isomorphic)
4. Any p-subgroup is contained in some sylow p-subgroup.

Example 1.5.2. $G = D_8$, contains two non-conjugate elements of order 2 - (Insert Figure)

Example 1.5.3. $G = D_8$, has nonisomorphic subgroups of order 4
(Insert Figure)

Proof.
1. Existence. We proceed by induction on the order of the group.
   Case 1: G has some proper subgroup H,index not divisible by p.
   - Pick sylow p-subgroup of H. This is a sylow p-subgroup of G.

   Case 2: All Sylow p-subgroups have index divisble by $p \to$ center if G has order divisible by p.
   - pick $g \in$ center, $g^p = 1$. Look at $G/\langle g\rangle$. Pick p-sylow subgroup. Inverse image in G is a sylow p-subgroup.
2. Number of Sylow subroups is $1 \mod p$
   Key idea: look at action of Sylow p-subgroup S on set of sylow p-subgroups by conjugation
   All orbits have size power of p. Orbit $\{S\}$ has size 1. No other orbits of size 1. if $\{T\}$ orbit of size 1, then S normalizes T so ST of order p^m, $m > n$. impossible.
   1 orbit of size 1, all other orbits have size p^k, $k > 0$. Divisible by p so total is $1 \mod p$
3. All Sylow p-subgroups are conjugate
   Suppose not, then if S is a p-sylow subgroup, number of conjugates is divisble by $p - 1$. Suppose T is a non-conjugate p-subgroup and let T act on the set of p-sylow subgroups conjugate to S. T can have no fixed points so the total number of p-sylow subgroups conjugate to S is divisble by p, contradiction.
4. Numberof Sylow p-subgroups divides the order of G
   Look at action of G on sylow p-subgroups. Transitive so $\#$ subgroups $= \frac{|G|}{|\text{subgroup fixing1}|}$ which divides G.
5. Any subgroup with order power of $p \subseteq$ some sylow p-subgroup

Apply to groups of order $12 = 2^2 \times 3$
We know that G has subgroups of order 3 and 4.
Case 1: subgroup of order 3 is normal.

- Give G semiproduct $(\mathbb{Z}/3\mathbb{Z}) \rtimes$ (order 4 group)
  4 cases:

	Action trivial	Nontrivial
$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	binary dihedral
$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$	$S_3 \times \mathbb{Z}/2\mathbb{Z}$

  Case 2: Sylow 3 subgroups not normal
  \# subgroups - divides 12, 1 mod 3, not 1 $\to = 4$, call them S_1, S_2, S_3, S_4. $S_i \cap S_j = \{e\}$ so we have 8 elements of order 3, 1 element of order 1, 3 "mystery" elements.
  G has 2-sylow subgroups of order 4, at most one so must be normal. So $G =$ (group of order 4) $\rtimes \mathbb{Z}/2\mathbb{Z}$, only nontrivial action on: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong$ relfection of tetrahedron.

Example 1.5.4. Apply to groups of order 56.

Application: Nilpotent Groups
Following are equivalent:

1. Group is nilpotnent (center >1 , $G/$center is nilpotent or $|G| = 1$)
2. Any proper subgroup H has $N(H)$ strictly bigger than H.
3. ALl Sylow subgroups are normal
4. G is product of groups of prime power order.

(1) $\to$ (2): Suppose H is a subgroup.
Case 1: H does not contain $Z(G)$. $Z(G) \subseteq N(H)$.
Case 2: H contains $Z(G)$, look at $H/Z(G) \subseteq G/Z(G)$

(2) $\to$ (3): If S is a sylow p-subgroup of G. Then $N(S)$ is its own normalizer. $e \subseteq S \subseteq N(S) \subseteq G$. Suppose $g \in G$ normalizes $N(S)$ g takes S to a sylow p-subgroup of $N(S)$. This subgroup is conjugate to S in $N(S)$ so $gSg^{-1} = hSh^{-1}$ for $h \in N(S)$ so gh^{-1} normalizes S so $gh^{-1} \in N(S)$, since $h \in N(S)$, $g \in N(S)$.
Now, if $N(S)$ proper subgroup then $N(N(S)) > N(S)$ so must have $N(S) = G$ so there is only one sylow subgroup.

(3) $\to$ (4): Main step - members of different sylow subgroups comute.
S is a sylow p-subgroup, T is a sylow q-subgroup with $p \neq q$, want $st = ts$ for $s \in S$, $t \in T$
Follows from: If A, B normal subsets of G, and $A \cap B = \{e\}$ the elements of A commute with the elements of B. Look at $aba^{-1}b^{-1}$, commutator of a, b ($=1 \leftrightarrow a, b$ commute). $aba^{-1} \in B$ so $aba^{-1}b^{-1} \in B$ and $ba^{-1}b^{-1} \in A$ so $aba^{-1}b^{-1} \in A$ so $aba^{-1}b^{-1} = e$

(4) $\to$ (1): Follows since 1. p-groups are nilpotent, 2. product of nilpotent groups is nilpotent

Order 15: One group is $\mathbb{Z}/15\mathbb{Z} = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

Consider $p \neq q$, $p > q$. G has sylow p-subgroup, number is $1 \mod p$, divides pq, $q < p$ so only possibility is 1. So since p is normal $G = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$.
How doe s$\mathbb{Z}/q\mathbb{Z}$ act on $\mathbb{Z}/p\mathbb{Z}$? $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^\times$ order $p-1$ so if q does not divides $p-1$ only action is trivial so only subgroup is cylic subgroup of order pq
If $q|p-1$, $\mathbb{Z}/q\mathbb{Z}$ can act nontrivially on $\mathbb{Z}/p\mathbb{Z}$. Essentially one action $(\mathbb{Z}/p\mathbb{Z})^\times$ elements of order q forms a cyclic subgroup of order q.
Exactly two groups of order pq.
Order 16: Complete List

- 5 abelian: $\mathbb{Z}/16\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$, $(\mathbb{Z}/2\mathbb{Z})^4$
- 4 more, have subgroups of order $\mathbb{Z}/8\mathbb{Z}$: Generalized quaternion = binary dihedral, dihedral, groups generated by $a^8 = 1$ $b^2 = 1$, $bab^{-1} = a^3$ or a^5, if a^3 called semi-dihedral.
- Products: $D_8 \times \mathbb{Z}/2\mathbb{Z}$, $Q_8 \times \mathbb{Z}/2\mathbb{Z}$
- Semidirect Product: two of form $\mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/4\mathbb{Z}$
  one of form: $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$ (Pauli group)

1.5.2 Classification of Abelian Groups (finite)

All products of cylic-subgroups (not unique) eg. $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
Product is unique up to order either, $n_1, n_2, \ldots$ satisfying $n_1|n_2|n_3 \cdots$ or n_i prime powers.
eg. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}(2|6)$ or $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$ $(2^2, 3$ prime powers)