MATH 135: Introduction to the Theory of Sets

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Fall 2022

Contents

Chapter 1

Introduction

1.1 August 25

1.1.1 Introduction

Foundations of Mathematics: language, axioms, formal proofs

- We focus on the axioms in set theory
- We use ZFC (Zermelo-Fraenkel + Choice)
- There is only one primitive notion : $\in$
- Within the ZFC universe, everything is a set

Course Outline:

- Basic axioms
- Operations, relations, functions
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- carindals
- AC
- ordinals

1.1.2 Basics

Principle of Extensionality: Two sets A, B are the same $\leftrightarrow$ they have the same elements $\forall x(x \in A \leftrightarrow x \in B)$
Example 1.1.1. $2, 3, 5 = \{5, 2, 4\} = \{2, 5, 2, 3, 3, 2\}$

Definition 1.1.2. There is a set with no elements, denoted $\varnothing$

- $\varnothing \neq \{\varnothing\}$
- $A \subseteq B$: A is a subset of $B \leftrightarrow$ each element of A is in B (use $\subsetneq$ to denote proper subset)

- $\{2\} \subseteq \{2,3,5\}$ but $\{2\} \notin \{2,3,5\}$
- Power set opertaion: $\mathcal{P}(A) = \{B \mid B \subseteq A\}$

We can define a hierarchy:
$V_0 = \varnothing$, $V_1 = \mathcal{P}(\varnothing) = \{\varnothing\}$, $V_2 = \mathcal{P}\mathcal{P}(\varnothing) = \{\varnothing, \{\varnothing\}\}$
$V_3 = \mathcal{P}(V_2) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$, $V_4, \ldots$
$V_\omega = \bigcup_{n\in\mathbb{N}} V_n$, $\mathcal{P}(V_\omega)$, $\mathcal{P}\mathcal{P}(V_\omega), \ldots, V_{\omega+\omega}, \ldots, V_{\omega+\omega+\cdots}, \ldots, V_{\omega\times\omega}, \ldots, V_{\omega^\omega}$

Chapter 2

Axioms and Operations

2.1 August 30

2.1.1 Zermelo Fraenkel Axioms of Set Theory

Setting: in ZFC all objects are sets
Language: contains vocabulary ($\in$), logical symbols ($=, \wedge, \vee \exists, \forall, \neg$), variables ($x, y, A, B$, etc.)

Axiom 2.1.1 (Extensionality Axiom)**.** Two sets are the same if they have the same elements
$\forall A, B(\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B)$

Axiom 2.1.2 (Empty Set Axiom)**.** There is a set with no members, denoted $\varnothing$
$\exists A \forall x(x \notin A)$

Axiom 2.1.3 (Pairing Axiom)**.** For any sets u, v there is a est whose elements are u and v, denoted $\{u, v\}$
$\forall u, v \exists A \forall x(x \in A \leftrightarrow x = u \vee x = v)$

Axiom 2.1.4 (Union Axiom (Preliminary Form))**.** For any sets a, b there is a set whose elements are elements of a and elements of b, denoted $a \cup b$
$\forall a, b \exists A \forall x(x \in Ax \in u \vee x \in v)$

Axiom 2.1.5 (Powerset Axiom)**.** Each set A, has a power set $\mathcal{P}(A)$.
$\forall A \exists B \forall x(x \in B \iff x \subseteq A)$ where $x \subseteq A$ stands for $\forall y(y \in x \rightarrow y \in A)$

Axiom 2.1.6 (Union Axiom)**.** For any set A, there is a set $\bigcup A$ whose members are members of the members of A.
$\forall A \exists B \forall x(x \in B \leftrightarrow \exists y \in A(x \in y))$

Idea for the subset axiom: For any set A, there is a set B whose memebrs are members of A satisfying some property.

eg. $B = \{x \in A \mid x \text{ satisfies property } P\} \subseteq A$

Example 2.1.7. $B = \{n \in \mathbb{N} \mid n \text{cannot be described in less that 20 words}\}$

- let b be the smallest element in B, then b is the smallest element that cannot be described in 20 words.
- Paradox : need to use formal language to express property P.

Example 2.1.8. Let $B = \{x \mid x \notin x\}$
Question: $B \in B$? $B \in B \leftrightarrow B \notin B$: need to have property be contained in some larger set.

We can now restate the axiom more formally:

Axiom 2.1.9 (Subset Axiom (Scheme))**.** For each formula $\phi(x)$, there is an axiom:
$\forall A \exists B \forall x (x \in B \leftrightarrow x \in A \wedge \phi(x))$

Example 2.1.10. Suppose there is a set of all sets A. Consider $B = \{x \in A \mid x \notin x\}$. Then $B \in B \leftrightarrow B \notin B$, contradiction. So there can be no such set A.

The language of 1rst order logic for ZFC:
The following are formulas:

- $x = y$, $x \in y$ atomic formulas
- $(\varphi \wedge \psi), (\varphi \vee \psi), \neg\varphi$ where φ, ψ are formulas
- $\exists v \varphi, \forall x \varphi$

Example 2.1.11. $\varphi(v, w) := (\exists v (v \in x \wedge \neg v = w)) \to (\forall y (\neg y \in y))$ is a formula

Chapter 3

Relations and Functions

3.1 September 1

3.1.1 Relations and Functions

Ordered Pair: $\langle a, b\rangle = \langle c, d\rangle \leftrightarrow a = c, b = d$

Definition 3.1.1. $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$

Cartesian product of A and B, denoted $A \times B = \{\langle x, y\rangle x \in A, y \in B\}$
Using the subset axiom $A \times B = \{z \in \mathcal{P}\mathcal{P}(A \cup B) \mid \exists x \in A \exists y \in B z = \langle x, y\rangle\}$
Observation: $\langle x, y\rangle \in \mathcal{P}\mathcal{P}(C)$ for $x, y \in C$
$\{x\}, \{x, y\} \in \mathcal{P}(C)$ so $\{\{x\}, \{x, y\}\} \subseteq \mathcal{P}(C)$ so $\{\{x\}, \{x, y\}\} \in \mathcal{P}\mathcal{P}(C)$

Definition 3.1.2. A binary relation is a set R whose elements are ordered pairs.

If $R \subset A \times B$ then R is a relation from $A \to B$.

Definition 3.1.3. Given a relation R, dom $R = \{x \in \bigcup\bigcup R \mid \exists y \langle x, y\rangle \in R\}$,
range $R = \{y \in \bigcup\bigcup R \mid \exists x \langle x, y\rangle \in R\}$, field (R) = dom$(R)\cup$range(R)

Example 3.1.4. $R = \{\langle a, b\rangle, \langle c, d\rangle, \langle e, f\rangle\} = \{\{\{a\}, \{a, b\}\}, \{\{c\}, \{c, d\}\}, \{\{e\}, \{e, f\}\}\}$
$\bigcup R = \{\{a\}, \{a, b\}, \{c\}, \{c, d\}, \{e\}, \{e, f\}\}$
$\bigcup\bigcup R = \{a, b, c, d, e, f\}$

n-ary relations: define n-tuple by $\langle a, b, c\rangle = \langle\langle a, b, \rangle, c\rangle$ etc.

Definition 3.1.5. A function is a relation F such that $\forall x, y, z\ \langle x, y\rangle \in F$ and $\langle x, z\rangle \in F \to y = z$

$\forall x\ \in$dom (F) there is y such that $\langle x, y\rangle \in F$. If $A =$ dom(F), $B \supseteq$ range(F) then F is said to a funtion from A to B, $f : A \to B$
We say that $f : A \to B$ is onto if $B =$ range(F)

Definition 3.1.6. F is injective if $\forall x, y, z\ \langle x, z\rangle \in F \wedge \langle y, z\rangle inF \to x = y$.

Definition 3.1.7. For a set A, relations F, G

(a) inverse $F^{-1} = \{\langle y, x\rangle \mid \langle x, y\rangle \in F\}$

(b) composition: $G \circ F = \{\langle x, z\rangle \mid \exists y \langle x, y\rangle \in F, \langle y, z\rangle \in G\}$

(c) restriction: $F \restriction_A - \{\langle x, y\rangle \in F \mid x \in A\}$

(d) image of A under F, $F[\![A]\!] = \{y \mid \exists x \in A \langle x, y\rangle \in F\} = \text{range}(F \restriction_A)$

Example 3.1.8. If F is a function, F^{-1} may not be a function. F^{-1} is a function $\leftrightarrow$ F is one to one.

Example 3.1.9. $F^{-1} \circ F = \{\langle x, x\rangle \mid x \in \text{dom}(F)\}$ if F is one to one
More generally, $F^{-1} \circ F = \{\langle x, z\rangle \mid \exists y \in \text{range } F \langle x, y\rangle, \langle z, y\rangle \in F\}$.

3.2 September 6

3.2.1 Functions and Relations

Theorem 3.2.1. Let $F : A \to B$ with $A \neq \varnothing$

(a) There is a function $G : B \to A$ such that $G \circ F = \text{id}_A \leftrightarrow F$ is one to one.

(b) There is a function $G : B \to A$ such that $F \circ F = \text{id}_B \leftrightarrow F$ is onto.

Proof. (a) Suppose there is such a G. Take a_1, a_2 such that $F(a_1) = F(a_2)$, then $a_1 = G \circ F(a_1) = G \circ F(a_2) = a_2$
Conversely, suppose F is one to one. We want to define $G : B \to A$ given $b \in B$, let $G(b)$=the unique $a \in A$ such that $F(a) = b$ if $b \in$range(F). If $b \notin \text{range}(F)$, let $G(b) = a_0$ with $a_0 \in A$ arbitrary (exists since A nonempty)

(b) Supppose that $G : B \to A$, with $F \circ G = \text{id}_B$ Want to show $\forall b \in B \exists a\, F(a) = b$ Take $a = G(b) \to F(a) = F(G(b)) = b$
Conversely, suppose F is onto. We want to define G, given $b \in B$ want to define $G(b)$ such that $F(G(b)) = b$, equivalently, want $G(b) \in F^{-1}(\{b\})$. Since F is onto $F^{-1}(\{b\})$ is nonempty. Let $G(b)$ be any element of $F^{-1}(b)$, equivalently $G \subseteq F^{-1}$ and $\text{dom}(G) = B =$dom(F^{-1}).

Example 3.2.2. Suppose $A = \mathbb{N}$, let $G = \{(b, a) \in B \times \mathbb{N} : a \text{ is least satisfying } f(a) = b\}$

- Don't have a method to specify such elements in gneral.

Axiom 3.2.3 (Axiom of Choice - Form I)**.** For every relation R, there is a function $G \subseteq R$ with dom(G) =dom(R)

3.2.2 Infinite Cartesion Products

$A \times B = \{\langle x, y\rangle \in \mathcal{PP}(A \cup B) \mid x \in A \wedge y \in B\}$

Definition 3.2.4. Let M be a function with domain I such that for every $i \in I$, $H(i)$ is a set. Let

$$\bigtimes_{i \in I} H(i) - \{f : I \to \bigcup H(i) \mid f(i) \in H9 = (i)\}$$

Example 3.2.5. Let ω_g be $\{G \subseteq \mathbb{R} \mid 0 \neq G,\ G \cup \{0\}$ is closed under addition $\}$
$\bigtimes_{G \in \omega_g} = \bigtimes_{G \in \omega_g} H(G)$ is a function such that for each $G \in \omega_g$, you get an element of G.
Observation: If one of the $H(i)$ is $\varnothing$, then $\bigtimes_{i \in I} H(i) = \varnothing$

Axiom 3.2.6 (Axiom of Choice - Form II)**.** If H is a function with domain I such that $H(i) \neq \varnothing\ \forall i \in I$, then $\bigtimes_{i \in I} H(i) \neq \varnothing$

(ACI) $\to$ (ACII): We are given H with $H(i) \neq \varnothing$ for all i. Want $f : I \to H(i)$ with $f(i) \in H(i)\ \forall i \in I$. Let $R = \{\langle i, h \rangle \in I \times \bigcup_{i \in I} H(i) \mid h \in H(i)\}$. $\mathrm{dom}(R) = I$, since $H(i) \neq \varnothing$ there is $h \in H(i)$ so $\langle i, h \rangle \in R$. BY ACI, there is $F \subseteq R$ with $\mathrm{dom}(F)$=$\mathrm{dom}(R) = I$. $\forall i$, $\langle i, f(i) \rangle \in R$ so $f(i) \in H(i)$

Chapter 4

Naturals, Rationals, Reals

4.1 September 8

4.1.1 Natural Numbers

Idea: each natural number is the set of all the previous numbers.
$0 = \varnothing, 1 = \{\varnothing\}, 2 = \{\varnothing, \{\varnothing\}\}, \ldots, n = \{0, 1, \ldots, n-1\}, \ldots$

Definition 4.1.1. The succesor of a set a is defined as $a^+ = a \cup \{a\}$

Definition 4.1.2. A set I is inductive if $\varnothing \in I$ and $\forall a \in I, a^+ \in I$

Definition 4.1.3. a is a natural number if it belongs to all inductive sets, $\forall I(I \text{ inductive} \rightarrow a \in I)$

If I is any inductive set, let $\omega = \{a \in I \mid a \text{ belongs to all inductive sets}\}$=the minimal inductive set.
Observation: ω is inductive because $\varnothing$ is in all inductive sets and if n belongs to all inductive sets then so does n^+

Axiom 4.1.4 (Ifinity Axiom)**.** There is an inductive set.

Inductivion Principle: If $A \subseteq \omega$ is inductive set $A = \omega$

Example 4.1.5. Every natural number is 0 or the succesor of some natural number.
Let $A = \{n \in \omega \mid n = 0 \vee \exists m \in \omega\, n = m^+\}$. A is inductive so $A = \omega$

Definition 4.1.6. A set A is transitive if one of the following equivalent conditions holds:

- if $x \in a \in A$, then $x \in A$
- $\bigcup A \subseteq A$
- if $a \in A$, then $a \subseteq A$
- $A \in \mathcal{P}(A)$

Example 4.1.7. Transitive sets includ $\varnothing$, each natural number, ω, V_ω
Claim: $A = \{n \in \omega \mid n \text{ is transitive }\}$ is inductive (implies all nautrual numbers are transitiev)

- Base: $0 \in A$ since $\varnothing$ is transitive
- Inductive Step: Suppose $n \in A$ transitive, want to show n^+ is transitive.
  Consider $x \in a \in n^+ = n \cup \{n\}$. If $a = n$, $x \in n \subseteq n^+$. If $a \in n$, $x \in a \in$ so by transitivity $x \in \subseteq n^+$ so $x \in n^+$

Theorem 4.1.8. If a is tansitive, then $\bigcup a^+ = a$

Proof. ($\supseteq$) $a = \bigcup\{a\} \subseteq \bigcup(a \cup \{a\} = \bigcup a^+)$ ($a \in a^+$ so $a \subseteq \bigcup a^+$)
($\subseteq$) Take $x \in \bigcup a^+$, then let $b \in a^+$ with $x \in b$. If $b = a$, $x \in a$. If $b \in a$, $x \in b \in a$ so $x \in a$.

- If a, b transitive and $a^+ = b^+$ then $a = \bigcup a^+ = \bigcup b^+ = b$ so succesor function is one to one on transitive sets, more specifically ω.

Fix a number $k \in \omega$. Consdier the following functions:

- $A_k : \omega \to \omega$ by $A_k(0) = 0$, $A_k(n^+) = A_k(n)^+$
- $M_k : \omega \to \omega$ by $M_k(0) = 0$, $M_k(n^+) = A_k(M_k(n))$

4.2 September 13

4.2.1 Operations on the Natural Numbers

Theorem 4.2.1. Let A be a set, $a \in A$ and $F : A \to A$. Then there is a unique function $h : \omega \to A$ such that:

1. $h(0) = a$
2. $h(n^+) = F(h(n))$ for all $n \in \omega$

Proof. Let $h = \{\langle n, b\rangle \in \omega \times A \mid \text{there is } g : n^+ \to A \text{ such that } g(0) = a, g(i^+) = F(g(i)) \wedge g(n) = b\}$
Claim 1: For all n there is a $g : \{0, \ldots, n\} \to A$ such that $g(0) = a$, $g(i^+) = F(g(i))$
Claim 2: Such a g is unique.

Proof of Claim 1. Let $I = \{n \in \omega \mid \text{ such a } g \text{ exists}\}$. Want to show that I is inductive.

1. $0 \in I$: let $g : \{0\} \to A$ be such that $g(0) = a$ eg. $g = \{\langle 0, a\rangle\}$
2. Suppose $n \in I$, we know such a g exists for n, $g : \{0, \ldots, n\} \to A$. We want $\tilde{g} : \{0, \ldots, n, n^+\} \to A$. Let $\tilde{g} = g \cup \{\langle n^+, F(g(n))\rangle\}$

□

Proof of Claim 2. Suppose $g, \tilde{g} : \{0, \ldots, n\} \to A$ such that $g(0) = a = \tilde{g}(0)$, $g(i^+) = F(g(i))$, $\tilde{g}(i^+) = F(\tilde{g}(i^+)), i < n$. We want to show $g(i) = \tilde{g}(i)$ $\forall i \leqslant n$. $g(0) = \tilde{g}(0)$, $g(i^+) = F(g(i)) = F(\tilde{g}(i)) = \tilde{g}(i^+)$

Can formally show this by induction using $I = \{i \in \omega \,|\, i \in n^+ \wedge g(i) = \tilde{g}(i) \vee i \notin n^+\}$ □

Claim 3: $\forall n \in \omega,\ h(n^+) = F(H(n))$

Definition 4.2.2. Given $k \in \omega$, define $A_k : \omega \to \omega$ by $A_k(0) = k$, $A_k(n^+) = (A_k(n))^+$. Define $n+k = A_k(n)$
Define $M_k : \omega \to \omega$ by $M_k(0) = 0$, $M_k(n^+) = A_k(M_k(n))$, let $n \times k = M_k(n)$.
Let $m < n$ if $m \in n$

Theorem 4.2.3. We can show the associativity of addition: $\forall a, b, v \in \omega((a + b) + c = a + (b + c))$, commutativity of addition: $\forall a, b \in \omega a + b = b + a$, etc.

4.2.2 Integers

Let $\sim$ be the following equivalence relation on $\omega \times \omega$ by $\langle a, b\rangle \sim \langle c, d\rangle \leftrightarrow a + d = b + c$
Define $\mathbb{Z} = \omega \times \omega / \sim$. $0_{\mathbb{Z}} = [\langle 0, 0\rangle]$, $1_{\mathbb{Z}} = [\langle 1, 0\rangle]$
Let $[\langle a, b\rangle] +_{\mathbb{Z}} [\langle c, d\rangle] = [\langle a+c, b+d\rangle]$. One needs to show this is well defined eg. if $\langle a, b\rangle \sim \langle a', b'\rangle$, $\langle c, d\rangle \sim \langle c', d'\rangle$ then $\langle a + c, b + d\rangle \sim \langle a' + c', b' + d'\rangle /$
Let $[\langle a, b\rangle] \times_{\mathbb{Z}} [\langle c, d\rangle] = [\langle ac + bd, ad + bc\rangle]$
Let $E : \omega \to \mathbb{Z}$ by $E(n) = [\langle n, 0\rangle]$

4.2.3 Rationals

Let $\sim$ be the following equivalence relation on $\mathbb{Z} \times \mathbb{Z} \backslash \{0\}$. $\langle a, b\rangle \sim \langle c, d\rangle \iff a \times_{\mathbb{Z}} d = b \times_{\mathbb{Z}} c$
Define $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z} \backslash \{0\} / \sim$. $0_{\mathbb{Q}} = [\langle 0, 1\rangle]$, $1_{\mathbb{Q}} = [\langle 1, 1, \rangle]$
Let $[\langle a, b\rangle] \times_{\mathbb{Q}} [\langle c, d\rangle] = [\langle a \times c, b \times d\rangle]$
Let $[\langle a, b\rangle] +_{\mathbb{Q}} [\langle c, d\rangle] = [\langle ad + bc, bd\rangle]$
$E : \mathbb{Z} \to \mathbb{Q}$ by $E(z) = [\langle z, 1\rangle]$

4.3 September 15

4.3.1 Reals (Dedekind Cuts)

Definition 4.3.1. A dedekind cut is a subset $D \subseteq \mathbb{Q}$ such that

- $\varnothing \neq D \neq \mathbb{Q}$
- D is closed downwards, if $d \in D$, $c < d \to c \in D$
- D has no greatest element.

Let $\mathbb{R} = \{D \in \mathcal{P}(\mathbb{Q}) \,|\, D \text{ is a dedekind cut }\}$
$\sqrt{2} = \{q \in \mathbb{Q} \,|\, q \times_{\mathbb{Q}} q < 2\}$, $e = \{q \in \mathbb{Q} \,|\, exn \in \omega\, q <_{\mathbb{Q}} (1 + \frac{1}{N})^N\}$ For $r \in \mathbb{R}$, $-r = \{q \in \mathbb{Q} \,|\, -q \in r\} \backslash \{-\sup(r)\}$
For $r_1, r_2 \in \mathbb{R}$, $r_1 \leqslant_{\mathbb{R}} r_2 \iff r_1 \subseteq r_2$
$r_1 \times r_2 = \{q \in \mathbb{Q} \,|\, \exists q \leqslant 0 \in r \exists b \leqslant 0 \in r_2\, q, a \times_{\mathbb{Q}} b$ if $r_1, r_2 > 0, \ldots$

Theorem 4.3.2. $(\mathbb{R}, 0, 1, +, \times, \leqslant)$ is an ordered field.

$E : \mathbb{Q} \to \mathbb{R}$ is a field embedding.

Chapter 5

Cardinal Numbers and the Axiom of Choice

5.1 September 15

5.1.1 Cardinality

Definition 5.1.1. A is equinumerous to B (written $A \approx B$) if there is a bijection $f : A \to B$

Theorem 5.1.2. For every A, B, C

- $A \approx A$
- If $A \approx B$, $B \approx B$
- If $A \approx B$, $B \approx C$ then $A \approx C$

Lemma 5.1.3. $\mathbb{Z} \approx \omega$

Proof. For $z \in Z$, $f(z) = \begin{cases} -2z & z \leqslant 0 \\ 2z+1 & z > 0 \end{cases}$

Lemma 5.1.4. $\mathbb{Q} \approx \omega$

Proof. $f : \omega \to \mathbb{Z} \times \mathbb{Z}^+$, $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}^+ / \sim$
$f' : \omega \to \mathbb{Q}$, $f'(n) =$ least $i \in \omega$ $g(i) \notin \{f(1), \ldots, f(n-1)\}$

Lemma 5.1.5. $\mathbb{R} \approx (0, 1)_{\mathbb{R}}$

5.2 September 20

5.2.1 Cardinality

Lemma 5.2.1. 1. $\mathbb{N} \not\approx \mathbb{R}$

2. For any set A, $A \not\approx \mathcal{P}(A)$

Proof. 1. Let $f : \omega \to \mathbb{R}$, claim f is not onto. Want $r \notin \text{ran}(f)$, $\forall n \in \omega r \neq f(n)$. Choose A_0 such that $f(0) \notin A_0$. Given A_n such that $f(0), \ldots, f(n) \notin A_n$. Divide A_n by 2, take half that does not contain $f(n+1)$ to be A_{n+1}, then $A_0 \supset A_1 \supset A_2 \supset \cdots$, $\bigcap_{n\in\omega} A_n \neq \varnothing$ and for each n, $f(n) \notin A_n$ so $f(n) \notin \bigcap A_n$

2. let $f : A \to A$. Claim f is not onto. Let $B = \{b \in A \,|\, b \notin f(b)\}$. Claim $B \notin \text{range}(f)$. Suppose for contradiction that $B = f(b)$ for $b \in A$, $b \in B \leftrightarrow b \notin f(b) \iff b \notin B$, contradiction.

Definition 5.2.2. A set A is finite if $\exists n \in omega(A \approx n)$ eg. $\exists n\, ex f : n \to A$ bijection. $A = \{f(0), f(1), \ldots, f(n-1)\}$

Lemma 5.2.3 (Pigeonhole Principle)**.** No finite set is equinumerous to a finite subset of itself.

Lemma 5.2.4. If B is a proper subset of $n \in \omega$ ther is $m < n$ such that $B \approx m$

Proof. Use induction on n. Let $A = \{n \in \omega \,|\, \forall B \in n \exists m \in n\, B \approx n\}$.
Claim A is inductive. $0 \in A$ trivial, $1 \in A$. $B \subsetneq \{\varnothing\} \to B = \varnothing \to B \approx 0$.
Suppose $n \in A$, want to show $n^+ \in A$. Take $B \subsetneq n^+ = n \cup \{n\}$. If $n \in B$, $B \cap n \subseteq n$ so $\exists m < n\, B \cap n \approx m$ so $B \approx m^+ < n^+$. If $n \notin B$, either $B \cap n = n$ so $B \approx n < n^+$ of $B \cap n \subsetneq n$ so $\exists m < n\, B = B \cap n \approx m$.

Proof (Pigeonhole Principle)**.** Take n, $B \subsetneq n$, $B \approx n$. THen $B \approx m$ for some $m < n$ so $m \approx n$. Let $A = \{n \,|\, Am < n\, m \not\approx n\}$. Claim A is inductive. $0 \in A$, suppose $n \in A$, want to show $n^+ \in A$. Idea: turn a bijection for $n^+ \approx m^+$ so a bijection $n \approx m/$

Corollary 5.2.5.
- No finite set is equinumerous to a proper subset
- ω is not finite ($\omega \approx \omega \backslash \{0\}$ by $n \mapsto n+1$)
- Every finite set is equinumerous to a unique natural number.
  We call that number the cardinality of A, card(A)
- A subset of a finite subset is finite

Definition 5.2.6. A set κ is said to be a cardinal if

- κ is transitive (if $x \in a, a \in \kappa \to x \in \kappa$)
- $\in$ is a linear order on κ ($\forall x, y\, x \in y$ or $y \in x$ or $x = y$)
- $\forall x \in \kappa\, x \not\approx \kappa$

Theorem 5.2.7. For every set A, there is a unique cardinal κ such that $A \approx \kappa$. We call this κ $\text{card}(A)$

Example 5.2.8.
- $n = \{0 \in 1 \in 2 \in \cdots \in n-1\}$ is a cardinal
- $\omega = \{0 \in 1 \in 2 \in \cdot\}$ is a cardinal
- $\omega^+ = \{0, 1, 2, \ldots\} \cup \{\omega\} \approx \omega$ is not a carinal

Notation: $\omega - \aleph_0$, $\text{card}(\mathbb{R}) = 2^{\aleph_0}$, smallest cardinal greater than $\aleph_0 = \aleph_1$

5.3 September 22

5.3.1 Cardinals

Definition 5.3.1. Given carindals κ and λ let

- $\kappa + \lambda = \text{card}(K \cup L)$ where K and L are disjoint sets of carindality κ and λ
- $\kappa \cdot \lambda = \text{card}(K \times L)$ where K and L are sets of carindality κ and λ
- $\kappa^\lambda = \{f \text{ function } L \to K\} = \text{card}({}^L K)$ were K and L are sets of carindality κ and λ

Notation: ${}^A B = \{f : f \text{ is a function } A \to B\}$

Theorem 5.3.2. Let κ, λ, μ be carindals

- $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$

  Proof. Let K, L, M be disjoint sets of size κ, λ, μ. $K \cup (L \cup M) = (K \cup L) \cup M$

- $\kappa + \lambda = \lambda + \kappa$
- $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$

  Proof. $(K \times L) \times M \to K \times (L \times M)$ by $\langle\langle k, l\rangle, m\rangle \to \langle k, \langle l, m\rangle\rangle$

- $\kappa \cdot \lambda = \lambda \cdot \kappa$
- $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$

  Proof. $K \times (L \cup M) \approx (K \times L) \cup (K \times M)$

- $\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu$

- $\kappa^{\lambda\cdot\mu} = (\kappa^\lambda)^\mu$

Proof. $F : {}^{L\times M}K \to {}^{M}{}^{L}K$, $f : {}^{L\times M}K$, $F(g)$ = the function that maps m to $g_m : L \to K$ where $g_m(l) = g(l,m)$
$F^{-1}(h)$ with $h : M \to ({}^{L}K)$ is g such that $g(l,m) = h(m)(l)$

Definition 5.3.3. A is dominated by B (written $A \preceq B$) if there is a one to one function from $A \to B$

$A \preceq B \iff \text{card}(A) \leqslant \text{card}(B)$

Example 5.3.4.
- $A \subseteq B \iff A \preceq B$
- $\mathbb{N} \approx \mathbb{N} \approx \mathbb{Q} \preceq \mathbb{R} \preceq \mathbb{C}$

Example 5.3.5. $\mathbb{R} \approx (0,1)_{\mathbb{R}} \preceq {}^{\omega}2 \preceq \mathbb{R}$

- $(0,1)_{\mathbb{R}} \preceq {}^{\omega}2$. Given r, let $f_r : \omega \to \{0,1\}$ be $f_r(n) = n$th digit of binary representation of r avoiding representations that end in all 1s.
- ${}^{\omega}2 \preceq \mathbb{R}$, $f : \omega \to 2 \mapsto \sum_{i\in\omega} f(i) \cdot 10^{-1}$

Observation: ${}^{2}\omega \approx \mathcal{P}(\omega)$ $\text{card}({}^{2}\omega) = 2^{\aleph_0}$

5.4 September 27

5.4.1 Schroder-Bernstein Theorem

Example 5.4.1. Show that $\mathbb{R} \cup \{*\}$ and $\mathbb{R}$ are equinumerous.
We define f by $f(*) = 0$, $f(r) = \begin{cases} r+1 & r \in \mathbb{N} \\ r & r \in \mathbb{R}\backslash\mathbb{N} \end{cases}$

Lemma 5.4.2. If A is finite, then $\omega \preceq A$

Proof. $A \neq 0$ so $\exists a_0 \in A$. Let $f(0) = a_0$, $A\backslash\{a_0\} \neq \varnothing$ since $A \not\approx 1$ so $a_1 \in A\backslash\{a_1\}$ Let $f(1) = a_1$.
We want $G : \{\text{finite subsets of } A\} \to A$ such that $G(F) \in A\backslash F$. Let $R = \{\langle F, a\rangle | F \text{ finite } a \in A\backslash F\}$
Observation: dom(R)= all finite subsets of A. Since A is not finite $A\backslash F \neq \varnothing$ for all finite sets, $F \subseteq A$. Use AC to get a function $G \subseteq R$ such that dom (G) = dom(R).
Define $f : \omega \to A$ by recusrion. $f(0) = a_0$, $f(n) = G(\{f(0), \ldots, f(n-1)\}) \in A\backslash\{f(0), \ldots, f(n-1)\}$.

Corollary 5.4.3. A set A is infinite $\leftrightarrow$ A is equinumerous to some proper subset of itself.

If A is infinite, there is 1 to 1 $f : \omega \to A$. We define a bijection $h : A \to A\{f(0)\}$ by $h(a) = \begin{cases} a & a \notin \text{dom}(f) \\ f(n+1) & a = f(n) \end{cases}$

Theorem 5.4.4 (SChroder Bernstein Theorem)**.** If $A \preceq B$, $B \preceq A$, then $A \approx B$

Proof. Let $f : A \to B$ 1 to 1, $g : A \to B$ 1 to 1. We want $h : A \to B$ bijection.
Let $C_0 = A \backslash \mathrm{ran}(g)$, let $D_0 = f[\![C_0]\!]$, $C_1[\![D_0]\!]$. $C_0 \cap C_1 = \varnothing$ because $C_0 = A \backslash \mathrm{rang}$ and $C_1 \subseteq \mathrm{ran}(g)$. We recursivley define $C + n + 1 = g[\![D_n]\!]$, $D_{n+1} = [\![C_{n+1}]\!]$. We see that C_n disjoint, D_n disjoint. Define $h(a) = \begin{cases} g(a) & a \in \bigcup_{n\in\omega} C_n \\ g^{-1} & a \in A \backslash \bigcup_{n\in\omega} C_n \end{cases}$. $f \rightharpoonup \bigcup_{n\in\omega}$ is a bijection $\bigcup C_n \to \bigcup D_n$. $g \rightharpoonup \bigcup_{n\in\omega} D_n$ is a bijection $B \backslash \bigcup_{n\in\omega} D_n \to A \backslash A \backslash \bigcup_{n\in\omega} C_n$

- Follows that $\mathbb{R} \approx \mathcal{P}(\omega)$

5.5 September 29

5.5.1 Zorn's Lemma

Theorem 5.5.1. For every A, B either $A \preceq B$ or $B \preceq A$.

Zorn's Lemma: Let $\mathcal{A}$ be a collection of sets such that for every chain $\mathcal{C} \subseteq \mathcal{A}$, $\bigcup \mathcal{C} \in \mathcal{A}$. Then $\mathcal{A}$ has a maximal element.

Definition 5.5.2. $\mathcal{C}$ is a chain if for every $C, D \in \mathcal{C}$ either $C \subseteq D$ or $D \subseteq C$
$B \in \mathcal{A}$ is maximal if ther is no $C \in \mathcal{A}$ with $B \subsetneq C$

We prove the following theorem to get some practice with Zorn's Lemma

Theorem 5.5.3. Every vector space has a basis.

Proof. Let V be a vector space over a field k. $B \subseteq V$ is linearly independent if for every $v_1, \ldots, v_n \in B$, distinct, $k_1, \ldots, k_n$ such that $\sum k_i v_i = 0$, $k_1 = k_2 = \cdots = 0$. B is a basis if B is linearly indpendent and $\langle B \rangle = V$ where $\langle B \rangle = \{\sum_{i=1}^n k_i v_i \mid v_1, \ldots, v_n \in B,\ k_1, \ldots, k_n \in k\}$
Let $\mathcal{A} = \{B \subseteq V \mid B \text{ is linearly independent}\}$. W need to showt that if $\mathcal{C} \subseteq \mathcal{A}$ is a chain then $\bigcup \mathcal{C} \in \mathcal{A}$. Consider a chain $\mathcal{C}$ consisting of linearly independent sets. To prove that $\bigcup \mathcal{C}$ is linearly independent assume we have $v_1, \ldots, v_n \in \bigcup \mathcal{C}$, $k_1, \ldots, k_n \in k$ with $\sum_{i=1}^n v_i k_i = 0$. For each v_i, there is $C_i \in \mathcal{C}$ with $v_i \in C_i$. One C_i contains all the others, say C_{i_0}. $v_1, \ldots, v_n \in C_{i_0}$. C_{i_0} is linearly independent so all $k_i = 0$. Now we apply Zorns Lemma to get a maximal element $B \in \mathcal{A}$. B is a maximal linearly independent set in V. $\langle B \rangle = V$ since if there is some $v \in V \backslash \langle B \rangle$ then $B \cup \{v\}$ is linearly indpendent, contradicting the maximality of B.

Lemma 5.5.4. Let $\mathcal{C}$ be a collection of functions. Then

(i) $\bigcup \mathcal{C}$ is a function

(ii) dom $(\bigcup \mathcal{C}) = \bigcup \{\text{dom } f : f \in \mathcal{C}\}$

(iii) ran $(\bigcup \mathcal{C}) = \bigcup \{\text{ran } f : f \in \mathcal{C}\}$

(iv) if all functions in $\mathcal{C}$ are 1 to 1, then $\bigcup \mathcal{C}$ is one to one.

Proof. (ii): dom $(\bigcup \mathcal{C}) = \{a \mid \exists b \langle a, b\rangle \in \bigcup \mathcal{C}\} = \{a \mid \exists b \, \exists f \in \mathcal{C} \, \langle a, b\rangle \in f\} = \{a \mid \exists f(\exists b \langle a, b\rangle \in f)\} = \{a \mid \exists f \in \mathcal{C} \, a \in \text{dom } f\} = \bigcup\{\text{dom } f : f \in \mathcal{C}\}$
(i): $\bigcup \mathcal{C}$ is a relation. Want to show it is a function. Suppose $\langle a, b\rangle \in \bigcup \mathcal{C}$ and $\langle a, c\rangle \in \bigcup \mathcal{C}$. $\exists f \in \mathcal{C}$, $\langle a, b\rangle \in f$, $\exists g \in \mathcal{C}$ $\langle a, c\rangle \in g$. Since $\mathcal{C}$ a chain, either $f \subseteq g$ or $g \subseteq f$. If $f \subseteq g$, $\langle a, b\rangle, \langle a, c\rangle \in g$, a function, $b = c$.
(iv): $\bigcup \mathcal{C}$ is a function. Want to show it is one to one. Suppose $\langle a, b\rangle \in \bigcup \mathcal{C}$ and $\langle c, b\rangle \in \bigcup \mathcal{C}$. $\exists f \in \mathcal{C}$, $\langle a, b\rangle \in f$, $\exists g \in \mathcal{C}$ $\langle c, b\rangle \in g$. Since $\mathcal{C}$ a chain, either $f \subseteq g$ or $g \subseteq f$. If $f \subseteq g$, $\langle a, b\rangle, \langle c, b\rangle \in g$, a one to one, $a = c$.