MATH 142: Elementary Algebraic Topology

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Contents

Chapter 1

Topology

1.1 August 24

1.1.1 What is Algebraic Topology

Recall Metric Spaces: (X, d), X is a set, d is a metric on X (ie. $d : X \times X \to \mathbb{R}$)

1. $d(x, y) = 0$ exactly if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leqslant d(x, y) + d(y, z)$

Let V be a vector space, let $||\cdot||$ be a norm on V, let $d(v, w) = ||v - w||$

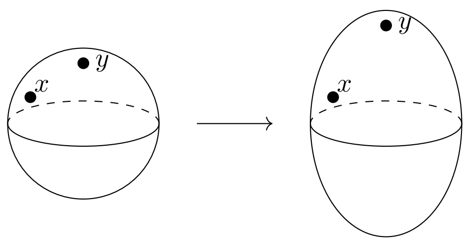
- $\mathbb{R}^n$: $||(r_j)||_2 = (\Sigma|r_j|^2)^{\frac{1}{2}}$ - Euclidean Norm, $||(r_j)||_1 = \Sigma|r_j|$, $||(r_j)| = \max|r_j|$

If (X,d) is a metric space and if $Y \subseteq X$, let d^Y be the restriction of d to $Y \times Y$. Then (Y, d^Y) is a metric space.

Metric spaces $\leftrightarrow$ geometry: length, area, size of angles.

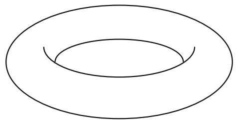
Let X be a balloon on $\mathbb{R}^3$

- Two natural metrics: inherited metric from $\mathbb{R}^3$, path-lenght metric (eg. length of shortest path on surface between two points)
- Consider a deformation:



the shapes have different Euclidean distances but still have an underlying commonality

- We also observe that the balloon cannot be continuously deformed into the shape below:



We want to be able to prove such things without embedding into a metric space. This is done by attaching algebraic objects to topological spaces such that their isomorphism classes dont change under continuous deformation.

1.1.2 Continuity

Let (X, d^X) and (Y, d^Y) be two metric spaces. Let $f : X \to Y$ be a function. Let $x_0 \in X$. We say f is continuous at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d^X(x, x_0) < \delta$ then $d^Y(f(x), f(x_0)) < \varepsilon$.

- Let (X, d) be a metric space. By the open ball of radius r about x_0, we mean $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ (closed ball is $\{x \in X : d(x, x_0) \leqslant r\}$)
- the above definition can be rephrased as: for any $B(f(x_0), \varepsilon)$ there is an open ball $B(x_0, \delta)$ such that if $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \varepsilon)$.
  eg. For every open ball B_1 about $f(x_0)$ there is an open ball B_2 about x_0 such that if $x \in B_2$ then $f(x) \in B_1$

Definition 1.1.1. For (X, d) a metric space, by a neighborhood of a point $x \in X$, we mean any subset of X that contains an open ball about x.

- rephrasing the definition again we get: For any neighborhood $N_{f(x_0)}$ of $f(x_0)$ there is a neighborhood N_{x_0} of x_0 such that if $x \in N_{x_0}$ then $f(x) \in N_{f(x_0)}$

Definition 1.1.2. $f : X \to Y$ is continuous if it is continuous at each points of X.

1.2 August 26

1.2.1 Continuity

Recall: Given (X, d^X), (Y, d^Y) and $f : X \to Y$, f is continuous at x_0 if for any open ball B_1 about $f(x_0)$ there is an open ball B_2 about x_0 such that if $x \in B_2$ then $f(x) \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1)$

Definition 1.2.1. Let (X, d) be a metric space. Let $U \subseteq X$. We say that U is open if for every $x \in U$ ther is an open ball B about x such that $B \subseteq U$, ie. U is a neighborhood of each point it contains.

We say $f : X \to Y$ is continuous if it is continuous at each point of X.
Let U be an open set in Y, $x \in X$ with $f(x) \in U$. For each ball B_1 in U about $f(x)$, there is an open ball about x $B_2 \subseteq X$ such that if $x' \in B_2$ then $f(x') \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1) \subseteq f^{-1}(U)$
ie. if $x \in f^{-1}(U)$ then there is an open ball B_2 about x with $B_2 \subseteq f^{-1}(U)$

ie. $f^{-1}(U)$ is open
Conversely, if the preimage $f^{-1}(U)$ of every open set U in Y is open, then f is continuous. This is because if $x_0 \in X$, B_1 an open ball about $f(x_0)$, then $f^{-1}(B_1)$ is open in X. $f(x_0) \in B_1$ so we have an open ball $B_2 \subseteq X$ about x_0 such that $B_2 \subseteq f^{-1}(B_1)$ so f is continuous at x_0.
Thus, $f : X \to Y$ is continuous exactly if for any open U in Y, $f^{-1}(U)$ is open in X.

1.2.2 Topology

Let (X, d) be a metric space. Let J be the collection of open subsets in X of d. J has the following properties:

1. $X \in J$, $\varnothing \in J$
2. an arbitrary, maybe infinite, union of open sets is open
3. a finite intersection of open sets is open.

Proof (of (3))**.** If $U_1, \dots, U_n$ are open sets and $x \in U_1 \cap \cdots, \cap U_n$ then there are $r_1, \dots, r_n \in \mathbb{R}$ such that $B(x, r_j) \subseteq U_j$ for $j = 1, \dots, j_n$. Let $r = \min\{r_1, \dots, r_n\}$, then $B(x, r) \subseteq U_j$ for each j so $B(x, r) \subseteq U_1 \cap \cdots \cap U_n$. Thus, $U_1 \cap \cdot \cap U_n$ is open.

Note: This does not hold for infinite intersections, consider $\bigcap_{i \in \mathbb{N}} B(x, \frac{1}{n}) = \{x\}$ in the plane.

This motivates the following definition:

Definition 1.2.2. Let X be a set. Bya topology on X we mean a collection, $\mathcal{T}$, of subsets of X (called the open sets of the topology) satisfuing **1**, **2**, and **3** above.

Definition 1.2.3. If $(X, \mathcal{T}^X)$, $(Y, \mathcal{T}^Y)$ are topological spaces, $f : X \to Y$ is continuous if for every $U \in \mathcal{T}^Y$, $f^{-1}(U) \in \mathcal{T}^X$

Example 1.2.4. Given X, let $\mathcal{T}_X$ be all subsets of X. This is called the discrete topology on X.

- This topology can also be given by the metric $d(x, y) = 1$ if $x \neq 1$

Definition 1.2.5. If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X, we say $\mathcal{T}_1$ is bigger, or finer, than $\mathcal{T}_2$ if $\mathcal{T}_1 \supseteq \mathcal{T}_2$.

- the disrcete topology is the biggest topology on X.

Example 1.2.6. $\mathcal{T} = \{X, \varnothing\}$, called the indiscrete topology on X.
Note: this topology can not be given by a metric if X has 2 or more points.

1.3 August 29

1.3.1 Bases and Subbases

Let $(X, \mathcal{T})$ be a topological space.

Definition 1.3.1. A subset A of X is said to be closed if A' $(X - A)$ is open.

Let $\mathcal{C}$ be the collection of closed subsets

1. $X, \varnothing \in \mathcal{C}$
2. any (maybe infinite) intersection of closed sets is closed
3. A finite union of closed sets is closed

Let X be a set, any (maybe infinite) intersection of topologies on X is a topology on X.
Thus, for any $\mathcal{S}$, a subset of X, thete is a smallest topology that conatins $\mathcal{S}$, namely the intersection of all topolgies that contain $\mathcal{S}$. We sat that $\mathcal{S}$ generates this topology.

Definition 1.3.2. If $\mathcal{S}$ has the property that $\bigcup(U \in \mathcal{S}) = X$, then $\mathcal{S}$ is called a subbasis of the topology it generates.

Let $\mathcal{I}^{\mathcal{S}}$ be the collection of all finite intersection of elements of $\mathcal{S}$, then the intersection of a finite number of elements of $\mathcal{I}^{\mathcal{S}}$ is in $\mathcal{I}^{\mathcal{S}}$.
Let $\mathcal{I}$ be a collection of subsets of X (union of elements of $\mathcal{I}$ is X) with the property that the intersection of a finite number of elements of $\mathcal{I}$ is in $\mathcal{I}$. Then the collection, $\mathcal{T}$, of arbitrary unions of elements of $\mathcal{I}$ is a topology (the smallest topology containing $\mathcal{I}$)
Why is a finite intersection of elements of $\mathcal{T}$ in $\mathcal{T}$?
Suppose $\mathcal{O}_1 = \bigcup_\alpha U_\alpha^1$, $\mathcal{O}_2 = \bigcup_\beta U_\beta^2$ with $U_\alpha^1, U_\beta^2 \in \mathcal{I}$, then $\mathcal{O}_1 \cap \mathcal{O}_2 = (\bigcup U_\alpha^1) \cap (\bigcup U_\beta^2) = \bigcup_{\alpha,\beta}(U_\alpha^1 \cap U_\beta^2)$.

Definition 1.3.3. Given a topological space $(X, \mathcal{T})$, a base for it is a set of subsets, $\mathcal{B}$, of $\mathcal{T}$, with the property that every element of $\mathcal{T}$ is a (maybe infinite) union of elements of $\mathcal{B}$.

If $\mathcal{S}$ is a subbase for $\mathcal{T}$, then $\mathcal{I}^{\mathcal{S}}$ is a base for $\mathcal{T}$.
Note: definition does not require $\mathcal{B}$ to be closed under finite intersection
(X, d) is a metric space, let $\mathcal{B}$ be the set of open balls. Then $\mathcal{B}$ is a base for the metric topology but usually the intersection of two open balls is not an open ball.
The intersection of finitely many elements of $\mathcal{B}$ is the union of elements of $\mathcal{B}$.

Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces, and $\mathcal{S}$ a subbase of $\mathcal{T}^Y$. Let $f : X \to Y$, then f is continuous if for every $U \in \mathcal{S}$, $f^{-1}(U) \in \mathcal{T}^X$.

Example 1.3.4. For $X = \mathbb{R}$, $\mathcal{S} = \{(-\infty, a), (b, +\infty) \,:\, a, b \in \mathbb{Q}\}$ generates the usual topology.

1.4 August 31

1.4.1 Initial Topologies

Definition 1.4.1. Let X be a set. Suppose we have a collection of topologies $(Y_\alpha, \mathcal{T}_\alpha)$, and for each α a function $f_\alpha : X \to Y_\alpha$. The smallest topology $\mathcal{T}$ such that each f_α is continuous is called the initial topology.

For each α, $U \in \mathcal{T}_\alpha$, must have $f_\alpha^{-1}(U) \in \mathcal{T}$ so a subbase of $\mathcal{T}$ is $\{f_\alpha^{-1}(U) \,:\, \text{for all } \alpha, U \in \mathcal{T}_\alpha\}$

Example 1.4.2. Have (Y, T^Y), let X be a subset of Y. $f : X \hookrightarrow Y$ by $f(x) = x$.
Inital topology on X has subbase $f^{-1}(U) = U \cap X \subseteq X$ for $U \in \mathcal{T}^Y$. Further, $\{U \cap X \,:\, U \in \mathcal{T}^Y\}$ is a topology.
This topology is called the relative topology on X.

Example 1.4.3. $Y = \mathbb{R}$, $X = [0,1]$, relative topology contains $[0, \frac{1}{2})$, not in the original topology

Example 1.4.4. Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces. Form set $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$.
We have projections $p^X : X \times Y \to X$ and $p^Y : X \times Y \to Y$. The initial topology has basis $(p^X)^{-1}(U) = U \times Y$ for $U \in \mathcal{T}^X$, $(p^Y)^{-1}(V) = X \times V$ for $V \in \mathcal{T}^Y$.
Further, $(U \times Y) \cap (X \times V) = U \times V$ (rectangle) so the open sets of the initial topology consist of all arbitrary unions of rectangles $U \times V$ for $U \in \mathcal{T}^X, V \in \mathcal{T}^Y$, called the product topology on $X \times Y$.

Example 1.4.5. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The product topology contains rectangles $(a,b) \times (c,d)$
Gives same topolgy as the euclidean metric

- Given $(X_1, \mathcal{T}^{X_1}), (X_2, \mathcal{T}^{X_2}), \dots, (X_n, \mathcal{T}^{X_n})$ can form $X_1 \times X_2 \times \cdots \times X_n$ with projections $p_1 : X_1 \times X_2 \times \cdots \times X_n \to X_i$. The product topology is generated by "rectangles" $U_1 \times U_2 \times \cdots \times U_n$ with $U_i \in \mathcal{T}^{X_i}$

- Suppose for $n \in \mathbb{N}$ we have $(X_n, \mathcal{T}^n)$, can form ΠX_n with $p_j : \Pi X_n \to X_j$, $\forall j$.
Only needs to contain finite intersections so we have a base of $U_1 \times U_2 \times \cdots \times U_m \times X_{m+1} \times X_{m+2} \times \cdots$ with $U_j \in \mathcal{T}^j$.

Example 1.4.6. $X_j = \{0,1\}$ with discrete topology. $\Pi_{j=1}^{\infty} X_j$ not discrete, also compact.

Example 1.4.7. $C([0,1])$, set of continuous functions on $[0,1]$, $||f||_\infty = \sup\{f(t) : t \in [0,1]\} \to$ metric $d(f,g) = ||f-g||_\infty$
Given an normed vector space $(V, || \, ||)$, let V'= all continuous linear functionsal on V.
eg. for $g \in C([0,1])$ we have $\varphi_g(f) = \int_0^1 f(t)g(t)dt$
For $C([0,1]) \xrightarrow{\varphi_g} \mathbb{R}$, given topology not the smallest that makes each φ_g continuous.

1.5 September 2

1.5.1 Quotient Topologies

Definition 1.5.1. Let Y be a set. Let $(X_\alpha, \mathcal{T}_\alpha)$ be a topology with, for each α, a function $f_\alpha : Y_\alpha \to Y$. The final topology is the largest topology that makes each f_α is continuous.

So for $A \subset Y$, in order for A to be in $\mathcal{T}$ need $f_\alpha^{-1}(A) \in \mathcal{T}_\alpha$ for all α.
For fixed α, we want $\{A \in Y : f_\alpha^{-1}(A) \in \mathcal{T}_\alpha\}$. This is a topology, denote it $\mathcal{T}_\alpha^Y$. It follows that $T = \bigcap_\alpha \mathcal{T}_\alpha^Y$
Let Y be a set $(X, \mathcal{T}^X)$, $f : X \to Y$, we require f is onto Y. Then $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}^X\}$ is the smallest topology that makes f continuous. It is called the quotient topology.

Other view: Let X, Y be sets, $f : X \to Y$ onto. Then f defines an equivalence relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$.
If we have an equivalence relation on a set, it defines are partition of the set.
If you have a partition, P, of a set X, then a set P is a set where the elements are nonempty subsets of X. Then define $f : X \to P$, where $f(x)$ is the element, A, of P such that $x \in A$. Then $f : X \to P$ onto.

Definition 1.5.2. $(X, \mathcal{T}^X)$ and $(Y, \mathcal{T}^Y)$ are homeomorphic if their $f : X \to Y$, one to one, onto such that f and f^{-1} are continuous.

Example 1.5.3. $\mathbb{R}_d = (\mathbb{R}, \mathcal{T})$ with discrete topology.
Consider $\mathbb{R}_d \xrightarrow{f} \mathbb{R}$ by $f(t) = t$. f is one to one, onto, and continuous but f^{-1} not continuous so it is not a homeomorphism.

Example 1.5.4. Let $X = [0,1]$, define an equivalence relation $0 \sim 1$ and $r \nsim s$ of $r \neq s$ and $0 < r < 1$. $[0,1]/\sim$ homeomorphic to the circle. Let $f(t) = e^{2\pi i t}$, we see $f(0) = f(1)$, f is a homeomorphism.
(Insert Figure)

Example 1.5.5. $X = [0,1] \times [0,2]$
(Insert Figure) equivalence relation defined by $(0,r) \sim (2,r)$ for $0 \leqslant r \leqslant 1$
Quotient space is homeomorphic to a cylinder.
Suppose we define $(0,1) \sim (2,1-r)$ $0 \leqslant r \leqslant 1$
(Insert Figure) Quotient space homeomorphic to the mobius strip.

Example 1.5.6. Let X be the unit sphere $\mathbb{R}^3 = \{v \in \mathbb{R} \mid ||v|| = 1\}$.
Put an equivalence relation: for $v \in X$, $v \sim -v$
$X/\sim$ is called a projective space.

1.6 September 7

1.6.1 Group Actions on Topological Spaces

For a topologial spaees $(X, \mathcal{T})$ the set of homeomorphisms of X to X forms a group under composition, auto-homeomorphisms, $\mathrm{Aut}((X,\mathcal{T}))$
Then if G is a group, then of an action of G on a topological space is a group homomorphism α, $\alpha : G \to \mathrm{Aut}((X,\mathcal{T}))$, so for each $g \in G$, α_g is a homeomorphism if $(X,\mathcal{T})$
$\alpha_{g_1} \circ \alpha_{g_2} = \alpha_{g_1 g_2}$, $\alpha_{g_1^{-1}} = (\alpha_{g_1})^{-1}$

Definition 1.6.1. For an action α, of G on some set X, given $x_0 \in X$, the orbit of x_0 for the action α is $\{\alpha_g(x_0) : g \in G\}$. The orbits from a partition of X. (orbits of $\alpha_g(x_0)$ same as x_0, $\alpha_{g_1}^{-1}(\alpha_g(x_0)) = x_0$)

Let X/α be the set of orbits. Have "quotient map" $X \to X/\alpha$ by $x \mapsto$ orbit of x.
If X has a topology and α acts by homeomorphism, puts quotient topology on X/α

Example 1.6.2. Symmetry of letters:
X=A given $Z_2 = \mathbb{Z}/2\mathbb{Z}$ act by reflection. X/α= (Insert Figure)
$X = H$, $Z_2 \times Z_2$, X/α = (Insert Figure)

Example 1.6.3. Let $G = \mathbb{Z}$, let $X = \mathbb{R}$, let α be an action of $\mathbb{Z}$ on $\mathbb{R}$ by translation, $\alpha_n(t) = t + n$ each of $\{\ldots, t_0 - 1, t_0, t_0 + 1, \ldots\}$. What is $\mathbb{R}/\alpha$

Example 1.6.4. A fundamental domain for α is a subset of X that contains exactly one element of each orbit.

- For the above example, fundamnetal domain $[0,1)$ with open subsets "wrapped around edges" so $\mathbb{R}/\alpha$ is homeorphic to the circle. Homoemorphism given by $t = e^{2\pi i t}$, constant on equivalence classes.

Example 1.6.5. The antipodal relation on the unit sphere with $v \sim -v$ acted on by $Z_2 = (0,1)$ by $\alpha_1(v) = -v$
Let Y be set of all lines in $\mathbb{R}^3$ through 0. Let $\mathbb{R} - \{0\}$, have an action on $\mathbb{R}^3$ by $\alpha_t(r,s,v) = (tr, ts, tv)$
Orbits in $\mathbb{R}^3 - \{0\}$, set of all lines through 0, (with 0 removed). Each line intersects the unit spehr in 2 antipodal points. Quotient topology gives a topology on the set of lines.

1.6.2 Connectedness

Definition 1.6.6. A topological space $(X, \mathcal{T})$ is connect if it does have two, nonempty, disjoint open sets A, B with $A \cup B = X$

- If this is the acse, A, B also closed - called "clopen"

Theorem 1.6.7. If $(X, \mathcal{T})$ is connected, $f : X \to Y$ is continuous, $f(X)$ =range(f) with the inherited topology is connected.

1.7 September 9

1.7.1 Connectedness

$(X, \mathcal{T})$ is connected if the only clopen sets are $X, \varnothing$

Proposition 1.7.1. If $(X, \mathcal{T})$, $A \subseteq X$, give A the relative topology, then if A is connected then so is $\overline{A}$

Proof. Suppose that C is a clopen subset of $\overline{A}$, then $C \cap A$ is a clopen subset of A so either $C \cap A = A$ or $C \cap A = \varnothing$. If $C \cap A = \varnothing$, $C \cap \overline{A} = \varnothing$ since C is open. If $C \cap A$, $C \supseteq A$ so $C \supseteq \overline{A}$ since C is closed. So $C = \varnothing$ or $\overline{A}$

Proposition 1.7.2. Given $(X, \mathcal{T})$ a collection of $\{F_\alpha\}$ of subsets of X, let $Y = \bigcup_\alpha F_\alpha$. Suppose that each F_α is connected. If $\exists p \in \bigcap F_\alpha$ then Y is connected.

Proof. Let C bea aclopen subset of Y. We can asssume that $p \in C$, then for each α, $C \cap F_\alpha \neq \varnothing$, $C \cap F_\alpha$ is clopen so $C \cap F_\alpha = F_\alpha$ so $F_\alpha \subseteq C$. Thus $C \supseteq \bigcup F_\alpha = Y$, so $C = Y$.

Proposition 1.7.3. Let $(X, \mathcal{T}^X)$, $(Y, \mathcal{T}^Y)$ be topological spaces and suppose that each is connected. THen $X \times Y$ with the product topology is connected.

Proof. Choose a point $b \in X$ (a "basepoint"). Then $E = \{b\} \times Y$ is homoemorphic to Y and so is connected. For each $y \in Y$, let $H_y = X \times \{y\}$. Homoemorphic to X so connected. For each $y \in Y$, let $T_y = H_y \cup E$, connected since (y, b) is in both. Choose a basepoint $c \in Y$ so $(b, c) \in E$ and (b, c) is in each T_y so $X \times Y = \bigcup_{y \in Y} T_y$ is connected.

Follows that if $X_1, \ldots, X_n$ are topological spaces and each is connected then $X_1 \times \cdot \times X_n$ is connnected.

Any open interval (a', b') in $\mathbb{R}$ is connected. (False for (a, b) in $\mathbb{Q}$)
Suppose $C \subseteq (a', b')$ is clopen and $\neq \varnothing$ and suppose we have $a \in C$, $b \in C'$, $a < b$. Consider $A = \{r \in C : r < b\}$. $a \in A$ and b is an upper bound. Let c be its least upper bound then $c \in A$ since if $c \in C$ then there is an open ball about c contained in C (since C is open), but $c \notin C'$ for a similar reason.

1.7.2 Connected Components

Given $(X, \mathcal{T})$ define an equivalence realtion on X by $x \sim y$ if there is a connected subset that contains both of them.
Reflexivity, symmetry clear. If $x \sim y$, $y \sim z$, then $x, y \in C$, $y, z \in D$ so $y \in C \cap D$ so $C \cup D$ is connected.

1.8 September 12

1.8.1 Connected Components

$(X, \mathcal{T})$ a topological space. Define an equivalence relation on X by $x \sim y$ if there is a connected subset of X containing both x and y.
Transitivity: If $x \sim y$ and $y \sim z$, there is connected A with $x, y \in A$ and connected B with $y, z \in B$ then $A \cup B$ is conected since $y \in A, y \in B$, $x, z \in A \cup B$.
The equivalence classes foe this equivalence relation are called the connected components of X. Given $x \in X$, the equivalence class of x is the union of all connected sets containing x. So the equivalence class is the largest connected set containing x.
Since the closure of a connected set is conected, the equivalence classes are closed subsets of X.

Example 1.8.1. $X = \mathbb{Q}$, the connected components we get are the one point subsets.
($\mathbb{Q}$ is totally disconnected, as is $\prod_{m=1}^{\infty}\{0, 1\}$, "0 dimensional")

Definition 1.8.2. By a parametrized path in X we mean a continuous function, f, from some interval $[a.b] \subseteq \mathbb{R}$. This path connects $f(a)$ to $f(b)$.

Define an equivalence relation on $(X, \mathcal{T})$ by $x \sim y$ if there is a path in X connecting x to y.
Reflexive: Assume $f : [0, 1] \to X$, $f(0) = x$, $f(1) = y$ set $g(t) = f(1 - t)$, then $g(0) = y$, $g(1 = x)$
Transitive: If $f : [a, b] \to X$, $f(a) = x, f(b) = y$ and $g : [c, d] \to X$, $g(c) = y, g(d) = z$ change interval such that $g : [b, c]$ with $g(b) = y, g(e) = z$. $[a, e] = [a, b] \cup [b, e]$ so define $h : [a, c] \to X$ by $h(t) = \begin{cases} f(t) & t \in [a, b] \\ g(t) & t \in [b, e] \end{cases}$
The equivalence classes are called path components of $(X, \mathcal{T})$
Note: path connected $\to$ connected.

Example 1.8.3. Let $f : (0, 1], f(t) = (t, \sin(\frac{1}{t}))$, graph of $\sin(\frac{1}{t})$.
Subset is path connected but not closed. Closure is graph $\cup\{0\} \times [0, 1]$. Closure consists of 2 path connected components but only 1 connected component. In closure, 1 path connected component is not closed, while the other is closed but not open.

Definition 1.8.4. $(X, \mathcal{T})$ is locally connected if $\forall x \in X$ $\forall$ open $\mathcal{O}$ if $x \in \mathcal{O}$ there is an open U, $x \in U \subseteq \mathcal{O}$ with U connected.

- If $(X, \mathcal{T})$ is locally connected then all conected components are open, and hence clopen.

Definition 1.8.5. $(X, \mathcal{T})$ is locally path connected if $\forall x \in X$ $\forall$ open $\mathcal{O}$ if $x \in \mathcal{O}$ there is an open U, $x \in U \subseteq \mathcal{O}$ with U path connected.

- If $(X, \mathcal{T})$ is locallt path connected, then all path connected components are clopen. path components = connected components.

Definition 1.8.6. A topological manifold of dimension n is a topological space $(X, \mathcal{T})$ with the property that every $x \in X$ has an open set $\mathcal{O}$ such that $x \in \mathcal{O}$ with $\mathcal{O}$ homoemorphic to an open set in $\mathbb{R}^n$ (open ball in $\mathbb{R}^n$, all of $\mathbb{R}^n$).

1.9 September 14

1.9.1 Compactness

Definition 1.9.1. Let $(X, \mathcal{T})$ be a topological space. By an open cover of X we mean a subset $\mathcal{C}$ of $|cT$, ie. a family of open sets such that $\bigcup\{\mathcal{O} \in C\} = X$. By a subcover of $\mathcal{C}$ we mean a subset $\mathcal{D}$ of $\mathcal{C}$ such that $\mathcal{D}$ is a cover of X.

Definition 1.9.2. $(X, \mathcal{T})$ is said ot be compact if every open cover of X has a finite subcover.

- $[0, 1] \subseteq \mathbb{R}$ is compact
- Heine - Borel Property: any bounded closed subset of $\mathbb{R}^n$ is compact.

Let $(X, \mathcal{T})$ be a topological space. Let A be a subset of X, give A the relative topology. Then A is compact iff for any $\mathcal{C} \subseteq \mathcal{T}$ such that $\bigcup\{\mathcal{O} \in \mathcal{C}\} = A$ there is a finite subcover $\mathcal{D}$ of $\mathcal{C}$ such that $\bigcup\{\mathcal{O} \in \mathcal{D}\} \supseteq A$

Proposition 1.9.3. Let $(X, \mathcal{T})$ be compact. If $A \subseteq X$ is closed, then A is compact.

Proof. If $\mathcal{C} \subseteq T$ such that $\bigcup\{\mathcal{O} \in \mathcal{C}\} \supseteq A$, since A closed, A' open so $\mathcal{C} \cup \{A'\}$ is an open cover of X. Since X is compact, there is a finite subcover, $\mathcal{D}$. If we remove A' from $\mathcal{D}$ (if $A' \in \mathcal{D}$) we get a finite subcover $\mathcal{C}$ covering A.

Any set with the indiscrete topology is compact band any subset of it is compact bu tnot necesarily closed.

Proposition 1.9.4. Given $(X, \mathcal{T})$ and $A \subseteq X$ compact. If $(X, \mathcal{T})$ is Hausdorff then for any $x \in X$, $x \notin A$ there are disjoint open sets U, V with $A \subseteq U$, $x \in V$

Proof. Fror any $a \in A$,by Hausdorff, ther are open sets U_a, V_a disjoint with $a \in U_a, x \in V_a$. The collection of sets $\{U_a : a \in A\}$ covers A. Since A is compact ther is a finite subcover $U_{a_1}, \ldots, U_{a_n}$. Let $U = U_{a_1} \cup \cdots \cup U_{a_n} \supseteq A$, let $V = V_{a_1} \cap \cdots \cap V_{a_n}$ so we get $x \in V$, U, V disjoint.

Corollary 1.9.5. Given $(X, \mathcal{T})$ Hausdorff, $A \subseteq X compact$, then A is closed.

Proof. A' open since for $x \in A'$ can find open set containing x, disjoint from A.

Theorem 1.9.6. Given $(X, \mathcal{T})$ compact, and $f : X \to Y$ continuous, then $f(X)$ is compact.

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$. Since for $\mathcal{O} \in \mathcal{T}^Y$, $f^{-1}(\mathcal{O}) \in \mathcal{T}^X$, then $\{f^{-1}(\mathcal{O}) : \mathcal{O} \in \mathcal{C}\}$ is an open cover of X. Since X is compact, there is a finite subcover $f^{-1}(\mathcal{O}_1), \ldots, f^{-1}(\mathcal{O}_n)$. Then $\mathcal{O}_1, \ldots, \mathcal{O}_n$ is an open cover of $f(X)$

Example 1.9.7. Given $f : [0,1] \to \mathbb{R}$ cotinuous, $f([0,1])$ is connectd, compact so is some $[a,b]$. So f attains its suprenum $= \text{lub}\{f(t) : t \in [a,b]\}$

Theorem 1.9.8. Given $(X,\mathcal{T})$, $(Y,\mathcal{T})$, $f : X \to Y$ continuous, assume f is continuous, one to one, onto, X is compact, Y is Hausdorff. Then f is homoemorphism.

Proof. Need to show f^{-1} contniuous, so need $f(\mathcal{O}) \in \mathcal{T}^Y$ for $\mathcal{O} \in \mathcal{T}^Y$, equivalently, if A is closed in X, then $f(A)$ is closed in Y. If A closed, A compact so $f(A)$ is compact, but Y is Hausdorff so $f(A)$ is closed. □

1.10 September 16

1.10.1 Compactness

Proposition 1.10.1 (The Tube Lemma)**.** Given $(X,\mathcal{T}^X), (Y,\mathcal{T}^Y)$ and assume Y is compact. Given $x_0 \in X$ and some $\mathcal{O}$ open set in $X \times Y$ such that $\{x_0\} \times Y$ is contained in $\mathcal{O}$. Then there is an open neighborhood, U, of x_0 such that such that $U \times Y \subseteq \mathcal{O}$, called the tube about $\{x_0\} \times Y$

Proof. Note that $\{x_0\} \times Y$ is homeomorphic to Y so $\{x_0\} \times Y$ is compact. For $y \in Y$, $(x_0, y) \in \mathcal{O}$ si there is some $U_y \subseteq X, V_y \subseteq Y$ such that $(x_0, y) \in U_y \times V_y$. The V_y's cover Y so since Y is compact there is a finnite subcover, $V_{y_1}, V_{y_2}, \ldots, V_{y-n}$. Then, let $U = \bigcap_{i=1}^n U_{y_i}$, U is open and we claim $U \times Y \subseteq \mathcal{O}$. Given $(x,y) \in U \times Y$, $\exists j$ such that $y \in V_j$ and $U_j \times V_j \subseteq \mathcal{O}$ so $U \times V_j \subseteq \mathcal{O}$ so $U \times Y \subseteq \mathcal{O}$.

Theorem 1.10.2. If $(X,\mathcal{T}^X)$ and $(Y,\mathcal{T}^Y)$ are both compact then $X \times Y$ is compact.

Proof. If $\mathcal{C}$ is an open cover of $X \times Y$, for each x, $\mathcal{C}$ covers $\{x\} \times Y$ so there is a finite cover $\mathcal{C}_x$, take the union to get an open set $\mathcal{O}_x$ containing $\{x\} \times Y$, so there is an open neighborhood $U_x \times Y$ such that $U_x \times Y \subseteq \mathcal{O}$. The U_x's form an open cover of X, since X is compact there is a finite subcover $U_{x_1}, \ldots, U_{x_n}$. The $(U_{x_j} \times Y)$ cover $X \times Y$. $\mathcal{C}_{x_j}$ is a cover of $(U_{x_j} \times Y)$ so $\bigcup_{j=1}^n \{\mathcal{O} \in C_{x_j}\}$ cover $X \times Y$.

By induction, given $(X_1,\mathcal{T}_1), (X_2,\mathcal{T}_2), \ldots, (X_n,\mathcal{T}_n)$ all compact, then $X_1 \times X_2 \times \cdots \times X_n$ is compact.
Let $\mathcal{F}$ is an infinite collection of topologies such that $(X_\alpha, \mathcal{T}_\alpha)$ ceach compact, then is $\prod X_\alpha$ compact?

1.11 September 19

1.11.1 Compactness

If X is any set, and if $\mathcal{C}$ is a collection of closed subets of X, then $\bigcup\{A \in \mathcal{C}\} = X$ iff $\bigcap\{A' : A \in \mathcal{C}\} = \varnothing$. So $(X,\mathcal{T})$ is compact if whenever $\mathcal{C}$ is a colelction of subsets such that $\bigcap\{C \in \mathcal{C}\} = \varnothing$ then there is a finite subset $\mathcal{F} \subseteq \mathcal{C}$ such taht $\bigcap\{A \in \mathcal{F}\} \neq \varnothing$

Definition 1.11.1. A collection $\mathcal{C}$ of subsets of a set X has the finite intersection property (FIP), if for any finite finite $\mathcal{F} \subseteq \mathcal{C}$ we ahve $\bigcap\{A \in \mathcal{F}\} \neq \varnothing$

Then $(X, \mathcal{T})$ is compact if for any collection $\mathcal{C}$ of closed subsets with FIP, $\bigcap\{A \in \mathcal{C}\} \neq \varnothing$

Definition 1.11.2. $(X, \mathcal{T})$ is locally compact of each point $x \in X$ has a compact neighborhood, ie. $\mathcal{O}, x \in \mathcal{O}$ and $\overline{\mathcal{O}}$ copmact.

- $\mathbb{R}, \mathbb{R}^n$ locally compact

Proposition 1.11.3. Let $(X, \mathcal{T})$ be locally compact and Hausdorff. Then for any $x \in X$ and $\mathcal{O} \in \mathcal{T}$ with $x \in \mathcal{O}$ there is $U \in \mathcal{T}$, $x \in U$, $\overline{U} \subseteq \mathcal{O}$ is compact.

Proof. By local compactness, there is open V, $x \in V$, $\overline{V}$ compact. Then $V \cap \mathcal{O}$ is open, $x \in V \cap \mathcal{O}$ so we can replace $\mathcal{O}$ with $V \cap \mathcal{O}$, $\overline{V \cap \mathcal{O}}$ is compact. Thus we can assume that $\overline{\mathcal{O}}$ is compact. Let $C = \overline{\mathcal{O}} \backslash \mathcal{O}$, closed, compact, $x \notin C$. By Hausdorff, $\exists U, V \in \mathcal{T}$ disjoint $x \in U$, $C \subseteq V, U \subseteq \mathcal{O}$, $C' \supseteq V'$, $U \subseteq V'$ closed so $\overline{U} \subseteq V'$ so $\overline{U} \cap V = \varnothing$ so $\overline{U} \cap C = \varnothing$ so $\overline{U} \subseteq \mathcal{O}$

Chapter 2

Algebraic Topology

2.1 September 19

2.1.1 Homotopy

Definition 2.1.1. Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$. Let $f_0, f_1 : X \to Y$ continuous, then f_0 and f_1 are homotopic if $F : X \times [0,1] \to Y$ continuous such that $F(x,0) = f_0(x), F(x,1) = f_1(x)$. F is called a homotopy from f_0 to f_1.

Proposition 2.1.2. Homotopy is an equivalence relation on the set of continuous functions from X to Y

Proof.

1. $f \sim f$ by constant homotopy
2. If $f_0 \sim f_1$, set $F'(x,t) = F(x, 1-t)$, $f_1 \sim f_0$
3. $f \sim g$ and $g \sim h$ with homotopies F, G. Define $H : X \times [0,2] \to Y$. $H(x,t) = \begin{cases} F(x,t) & t \in [0,1] \\ G(x,t-1) & t \in [1,2] \end{cases}$.

   If $t = 1$, $F(x,1) = g(x)$, $G(x, 1-1) = g(x)$

Lemma 2.1.3 (Pasting Lemma)**.** If $(X, \mathcal{T})$, $X = A \cup B$, A, B closed and if $\varphi : X \to Y$ and if $\varphi|_A$ is continuous and if $\varphi|_B$ is continous then φ on X is continuous.

Proof. If $C \subseteq X$ closed $\varphi^{-1}(C) = (\varphi|_A)^{-1}(C) \cup (\varphi|_B)^{-1}(C)$. $(\varphi|_B)^{-1}(C)$ closed in B so closed in X. Similarly, for $(\varphi|_A)^{-1}(C)$ so $\varphi^{-1}(C)$ is closed.

2.2 September 21

2.2.1 Path Homotopy

Definition 2.2.1. (X, T) (usually path connected). Two paths $f, g : [0,1] \to X$ are path homotopiv if $f(0) = g(0), f(1) = g(1)$ and if they are homotopic via a homotopy $F : [0,1] \times [0,1] \to X$ with $F(t,0) = f(0), F(t,1) = f(1)$ for all t.

Path homotopy is an equivalence relation.

- Can compose equivalence classes. If f and g are paths, $f(1) = g(0)$ can compose them viewing g as a path on $[1,2]$ (instead of $[0,1]$). Define $(f * g)$ on $[0,2]$ by $(f * g)(t) = \begin{cases} f(t) & t \in [0,1] \\ g(t) & t \in [1,2] \end{cases}$

Proposition 2.2.2. If $f \sim f'$, $g \sim g'$ then $f * g \sim f' * g'$

Proof. Show first that $f * g \sim f' * g$. If F is a homotopy from f to f', let $\tilde{F}(r,t) = \begin{cases} F(r,t) & t \in [0,1] \\ g(r) & t \in [1,2] \end{cases}$.
Similarly, $f' * g \sim f' * g'$

- let $\mathcal{G}$ be the collection of path-homotopy classes of X. Then $*$ is a partially defined product. It is associative (when it makes sense). So for path-homotopic equivalence classes it is associative.
- Each $x \in X$ provides an equivalence class $e_x : [0,1] \to X$ by $e_x(t) = x$. If F is a path from x to y then $e_x * f \sim f$, $f * e_y \sim f$ so have an identity element for $x \in X$
- Each element has an inverse. Given f from x to y, let $f^{-1}(t) = f(1-t)$, $f^{-1}(0) = f(1)$, $f^{-1}(1) = f(0)$, $f * f^{-1} \sim e_x$, $f^{-1} * f \sim e_y$. So equivalence classes in $\mathcal{G}$ has inverses.
- This is an example of a groupoid. $\mathcal{G}$ path groupoid for X. In fact, $\mathcal{G}$ is a topological groupoid.

Given $x_0 \in X$, consider all paths rom x_0 to x_0. Path homotopic equivalence classes form a group $\pi_1(X, x_0)$. This is the fundamental group of X for the basepoint x_0.
If we change base point from x_0 to x_0', f a path from x_0 to x_0', from a loop α based at x_0' $f * \alpha * f^{-1}$ is a loop based at x_0. This gives an isomorphism from $\pi_1(X, x_0')$ to $\pi_1(X, x_0)$. Isomorphism depends on f up to homotopy.

2.3 September 23

2.3.1 The Fundamental Group

By a pointed set (or space) we mean a set together with a selectes special point.
(X, x_0) path connected $x_0 \in X$, can attach to (X, x_0) the group $\pi_1(X, x_0)$ (= the set of homotopy classes of loops on X based at x_0)
Given (X, x_0) (Y, y_0), $\varphi : X \to Y$ continuous. Let f be a path in X, then $f \circ \varphi$ is a path in Y. If $\varphi(x_0) = y_0$, we map loops based on x_0 to loops based at y_0
Let F be a homotopy betwen a path f and a path g on X, then $\varphi \circ F$ is a homotopy from $\varphi \circ f$ to $\varphi \circ g$.
So $\varphi : X \to Y$, $\varphi(x_0) = y_0$ gives a function $\tilde{\varphi}$ from homotopy classes of loops based at x_0 to homotopy classes of loops based at y_0. $\tilde{\varphi} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

Theorem 2.3.1. $\tilde{\varphi}$ is a group homomorphism.

Proof. Let f and g be paths in X. $f * g$, view f as defined on $[0,1]$, g as defined on $[1,2]$. $(f*g)(r) = \begin{cases} f(r) & r \in [0,1] \\ q(r-1) & r \in [1,2] \end{cases}$, then $(\varphi \circ f) * (\varphi \circ g)(r) = \begin{cases} vp \circ f(r) & r \in [0,1] \\ vp \circ q(r-1) & r \in [1,2] \end{cases} = \varphi(f * g)$. Passes to homotopy classes.

Theorem 2.3.2. $(X, x_0), (Y, y_0), (Z, z_0)$, $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$, $\pi_1(X) \xrightarrow{\hat{\varphi}} \pi_1(Y) \xrightarrow{\hat{\psi}} \pi_1(Z)$, we have $\hat{\psi} \circ \hat{\varphi} = \widehat{\psi \circ \varphi}$

Proof. If f path on X, $(\hat{\psi} \circ \hat{\varphi})(f) = \hat{\psi}(\varphi \circ f) = \psi \circ (\varphi \circ f) = (\psi \circ \varphi) \circ f = (\widehat{\psi \circ \varphi})(f)$

$(X, x_0), (Y, y_0)$, $\varphi : X \to Y$. Assume φ is a homoemorphism. $\varphi^{-1} \circ \varphi = \mathrm{id}_X$, $\varphi \circ \varphi^{-1} = \mathrm{id}_Y$. Then $\pi_1(\varphi^{-1})\pi_1(\varphi) = \pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X)}$, $\pi_1(\varphi)\pi_1(\varphi^{-1}) = \pi_1(\mathrm{id}_Y) = \mathrm{id}_{\pi_1(Y)}$ ie, $\pi_1(\varphi)$ is a group isomorphims of $\pi_1(X)$ and $\pi_1(Y)$.