

MATH 142: Elementary Algebraic Topology

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Chapter 1

Topology

1.1 August 24

1.1.1 What is Algebraic Topology

Recall Metric Spaces: (X, d) , X is a set, d is a metric on X (ie. $d : X \times X \rightarrow \mathbb{R}$)

1. $d(x, y) = 0$ exactly if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Let V be a vector space, let $\|\cdot\|$ be a norm on V , let $d(v, w) = \|v - w\|$

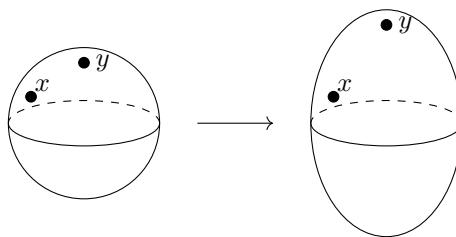
- \mathbb{R}^n : $\|(r_j)\|_2 = (\sum |r_j|^2)^{\frac{1}{2}}$ - Euclidean Norm, $\|(r_j)\|_1 = \sum |r_j|$, $\|(r_j)\|_\infty = \max |r_j|$

If (X, d) is a metric space and if $Y \subseteq X$, let d^Y be the restriction of d to $Y \times Y$. Then (Y, d^Y) is a metric space.

Metric spaces \leftrightarrow geometry: length, area, size of angles.

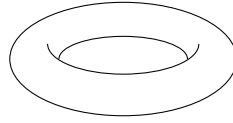
Let X be a balloon on \mathbb{R}^3

- Two natural metrics: inherited metric from \mathbb{R}^3 , path-length metric (eg. length of shortest path on surface between two points)
- Consider a deformation:



the shapes have different Euclidean distances but still have an underlying commonality

- We also observe that the balloon cannot be continuously deformed into the shape below:



We want to be able to prove such things without embedding into a metric space. This is done by attaching algebraic objects to topological spaces such that their isomorphism classes don't change under continuous deformation.

1.1.2 Continuity

Let (X, d^X) and (Y, d^Y) be two metric spaces. Let $f : X \rightarrow Y$ be a function. Let $x_0 \in X$. We say f is continuous at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d^X(x, x_0) < \delta$ then $d^Y(f(x), f(x_0)) < \varepsilon$.

- Let (X, d) be a metric space. By the open ball of radius r about x_0 , we mean $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ (closed ball is $\{x \in X : d(x, x_0) \leq r\}$)
- the above definition can be rephrased as: for any $B(f(x_0), \varepsilon)$ there is an open ball $B(x_0, \delta)$ such that if $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \varepsilon)$.
eg. For every open ball B_1 about $f(x_0)$ there is an open ball B_2 about x_0 such that if $x \in B_2$ then $f(x) \in B_1$

Definition 1.1.1. For (X, d) a metric space, by a neighborhood of a point $x \in X$, we mean any subset of X that contains an open ball about x .

- rephrasing the definition again we get: For any neighborhood $N_{f(x_0)}$ of $f(x_0)$ there is a neighborhood N_{x_0} of x_0 such that if $x \in N_{x_0}$ then $f(x) \in N_{f(x_0)}$

Definition 1.1.2. $f : X \rightarrow Y$ is continuous if it is continuous at each point of X .

1.2 August 26

1.2.1 Continuity

Recall: Given (X, d^X) , (Y, d^Y) and $f : X \rightarrow Y$, f is continuous at x_0 if for any open ball B_1 about $f(x_0)$ there is an open ball B_2 about x_0 such that if $x \in B_2$ then $f(x) \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1)$

Definition 1.2.1. Let (X, d) be a metric space. Let $U \subseteq X$. We say that U is open if for every $x \in U$ there is an open ball B about x such that $B \subseteq U$, ie. U is a neighborhood of each point it contains.

We say $f : X \rightarrow Y$ is continuous if it is continuous at each point of X .

Let U be an open set in Y , $x \in X$ with $f(x) \in U$. For each ball B_1 in U about $f(x)$, there is an open ball about x $B_2 \subseteq X$ such that if $x' \in B_2$ then $f(x') \in B_1$, ie. $B_2 \subseteq f^{-1}(B_1) \subseteq f^{-1}(U)$
ie. if $x \in f^{-1}(U)$ then there is an open ball B_2 about x with $B_2 \subseteq f^{-1}(U)$

ie. $f^{-1}(U)$ is open

Conversely, if the preimage $f^{-1}(U)$ of every open set U in Y is open, then f is continuous. This is because if $x_0 \in X$, B_1 an open ball about $f(x_0)$, then $f^{-1}(B_1)$ is open in X . $f(x_0) \in B_1$ so we have an open ball $B_2 \subseteq X$ about x_0 such that $B_2 \subseteq f^{-1}(B_1)$ so f is continuous at x_0 .

Thus, $f : X \rightarrow Y$ is continuous exactly if for any open U in Y , $f^{-1}(U)$ is open in X .

1.2.2 Topology

Let (X, d) be a metric space. Let J be the collection of open subsets in X of d . J has the following properties:

1. $X \in J$, $\emptyset \in J$
2. an arbitrary, maybe infinite, union of open sets is open
3. a finite intersection of open sets is open.

Proof (of (3)). If U_1, \dots, U_n are open sets and $x \in U_1 \cap \dots \cap U_n$ then there are $r_1, \dots, r_n \in \mathbb{R}$ such that $B(x, r_j) \subseteq U_j$ for $j = 1, \dots, n$. Let $r = \min\{r_1, \dots, r_n\}$, then $B(x, r) \subseteq U_j$ for each j so $B(x, r) \subseteq U_1 \cap \dots \cap U_n$. Thus, $U_1 \cap \dots \cap U_n$ is open.

Note: This does not hold for infinite intersections, consider $\bigcap_{i \in \mathbb{N}} B(x, \frac{1}{n}) = \{x\}$ in the plane.

This motivates the following definition:

Definition 1.2.2. Let X be a set. By a topology on X we mean a collection, \mathcal{T} , of subsets of X (called the open sets of the topology) satisfying **1**, **2**, and **3** above.

Definition 1.2.3. If (X, \mathcal{T}^X) , (Y, \mathcal{T}^Y) are topological spaces, $f : X \rightarrow Y$ is continuous if for every $U \in \mathcal{T}^Y$, $f^{-1}(U) \in \mathcal{T}^X$.

Example 1.2.4. Given X , let \mathcal{T}_X be all subsets of X . This is called the discrete topology on X .

- This topology can also be given by the metric $d(x, y) = 1$ if $x \neq y$

Definition 1.2.5. If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on X , we say \mathcal{T}_1 is bigger, or finer, than \mathcal{T}_2 if $\mathcal{T}_1 \supseteq \mathcal{T}_2$.

- the discrete topology is the biggest topology on X .

Example 1.2.6. $\mathcal{T} = \{X, \emptyset\}$, called the indiscrete topology on X .

Note: this topology can not be given by a metric if X has 2 or more points.

1.3 August 29

1.3.1 Bases and Subbases

Let (X, \mathcal{T}) be a topological space.

Definition 1.3.1. A subset A of X is said to be closed if $A' (X - A)$ is open.

Let \mathcal{C} be the collection of closed subsets

1. $X, \emptyset \in \mathcal{C}$
2. any (maybe infinite) intersection of closed sets is closed
3. A finite union of closed sets is closed

Let X be a set, any (maybe infinite) intersection of topologies on X is a topology on X .

Thus, for any \mathcal{S} , a subset of X , there is a smallest topology that contains \mathcal{S} , namely the intersection of all topologies that contain \mathcal{S} . We say that \mathcal{S} generates this topology.

Definition 1.3.2. If \mathcal{S} has the property that $\bigcup(U \in \mathcal{S}) = X$, then \mathcal{S} is called a subbasis of the topology it generates.

Let $\mathcal{I}^{\mathcal{S}}$ be the collection of all finite intersection of elements of \mathcal{S} , then the intersection of a finite number of elements of $\mathcal{I}^{\mathcal{S}}$ is in $\mathcal{I}^{\mathcal{S}}$.

Let \mathcal{I} be a collection of subsets of X (union of elements of \mathcal{I} is X) with the property that the intersection of a finite number of elements of \mathcal{I} is in \mathcal{I} . Then the collection, \mathcal{T} , of arbitrary unions of elements of \mathcal{I} is a topology (the smallest topology containing \mathcal{I})

Why is a finite intersection of elements of \mathcal{T} in \mathcal{T} ?

Suppose $\mathcal{O}_1 = \bigcup_{\alpha} U_{\alpha}^1$, $\mathcal{O}_2 = \bigcup_{\beta} U_{\beta}^2$ with $U_{\alpha}^1, U_{\beta}^2 \in \mathcal{I}$, then $\mathcal{O}_1 \cap \mathcal{O}_2 = (\bigcup U_{\alpha}^1) \cap (\bigcup U_{\beta}^2) = \bigcup_{\alpha, \beta} (U_{\alpha}^1 \cap U_{\beta}^2)$.

Definition 1.3.3. Given a topological space (X, \mathcal{T}) , a base for it is a set of subsets, \mathcal{B} , of \mathcal{T} , with the property that every element of \mathcal{T} is a (maybe infinite) union of elements of \mathcal{B} .

If \mathcal{S} is a subbase for \mathcal{T} , then $\mathcal{I}^{\mathcal{S}}$ is a base for \mathcal{T} .

Note: definition does not require \mathcal{B} to be closed under finite intersection

(X, d) is a metric space, let \mathcal{B} be the set of open balls. Then \mathcal{B} is a base for the metric topology but usually the intersection of two open balls is not an open ball.

The intersection of finitely many elements of \mathcal{B} is the union of elements of \mathcal{B} .

Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces, and \mathcal{S} a subbase of \mathcal{T}^Y . Let $f : X \rightarrow Y$, then f is continuous if for every $U \in \mathcal{S}$, $f^{-1}(U) \in \mathcal{T}^X$.

Example 1.3.4. For $X = \mathbb{R}$, $\mathcal{S} = \{(-\infty, a), (b, +\infty) : a, b \in \mathbb{Q}\}$ generates the usual topology.

1.4 August 31

1.4.1 Initial Topologies

Definition 1.4.1. Let X be a set. Suppose we have a collection of topologies $(Y_{\alpha}, \mathcal{T}_{\alpha})$, and for each α a function $f_{\alpha} : X \rightarrow Y_{\alpha}$. The smallest topology \mathcal{T} such that each f_{α} is continuous is called the initial topology.

For each α , $U \in \mathcal{T}_{\alpha}$, must have $f_{\alpha}^{-1}(U) \in \mathcal{T}$ so a subbase of \mathcal{T} is $\{f_{\alpha}^{-1}(U) : \text{for all } \alpha, U \in \mathcal{T}_{\alpha}\}$

Example 1.4.2. Have (Y, \mathcal{T}^Y) , let X be a subset of Y . $f : X \hookrightarrow Y$ by $f(x) = x$.

Initial topology on X has subbase $f^{-1}(U) = U \cap X \subseteq X$ for $U \in \mathcal{T}^Y$. Further, $\{U \cap X : U \in \mathcal{T}^Y\}$ is a topology. This topology is called the relative topology on X .

Example 1.4.3. $Y = \mathbb{R}$, $X = [0, 1]$, relative topology contains $[0, \frac{1}{2})$, not in the original topology

Example 1.4.4. Let $(X, \mathcal{T}^X), (Y, \mathcal{T}^Y)$ be topological spaces. Form set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

We have projections $p^X : X \times Y \rightarrow X$ and $p^Y : X \times Y \rightarrow Y$. The initial topology has basis $(p^X)^{-1}(U) = U \times Y$ for $U \in \mathcal{T}^X$, $(p^Y)^{-1}(V) = X \times V$ for $V \in \mathcal{T}^Y$.

Further, $(U \times Y) \cap (X \times V) = U \times V$ (rectangle) so the open sets of the initial topology consist of all arbitrary unions of rectangles $U \times V$ for $U \in \mathcal{T}^X, V \in \mathcal{T}^Y$, called the product topology on $X \times Y$.

Example 1.4.5. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The product topology contains rectangles $(a, b) \times (c, d)$

Gives same topology as the euclidean metric

- Given $(X_1, \mathcal{T}^{X_1}), (X_2, \mathcal{T}^{X_2}), \dots, (X_n, \mathcal{T}^{X_n})$ can form $X_1 \times X_2 \times \dots \times X_n$ with projections $p_i : X_1 \times X_2 \times \dots \times X_n \rightarrow X_i$. The product topology is generated by “rectangles” $U_1 \times U_2 \times \dots \times U_n$ with $U_i \in \mathcal{T}^{X_i}$
- Suppose for $n \in \mathbb{N}$ we have (X_n, \mathcal{T}^n) , can form $\prod X_n$ with $p_j : \prod X_n \rightarrow X_j, \forall j$.
Only needs to contain finite intersections so we have a base of $U_1 \times U_2 \times \dots \times U_m \times X_{m+1} \times X_{m+2} \times \dots$ with $U_j \in \mathcal{T}^j$.

Example 1.4.6. $X_j = \{0, 1\}$ with discrete topology. $\prod_{j=1}^{\infty} X_j$ not discrete, also compact.

Example 1.4.7. $C([0, 1])$, set of continuous functions on $[0, 1]$, $\|f\|_{\infty} = \sup\{f(t) : t \in [0, 1]\} \rightarrow$ metric $d(f, g) = \|f - g\|_{\infty}$

Given an normed vector space $(V, \|\cdot\|)$, let $V' =$ all continuous linear functionals on V .

eg. for $g \in C([0, 1])$ we have $\varphi_g(f) = \int_0^1 f(t)g(t)dt$

For $C([0, 1]) \xrightarrow{\varphi_g} \mathbb{R}$, given topology not the smallest that makes each φ_g continuous.

1.5 September 2

1.5.1 Quotient Topologies

Definition 1.5.1. Let Y be a set. Let $(X_{\alpha}, \mathcal{T}_{\alpha})$ be a topology with, for each α , a function $f_{\alpha} : Y_{\alpha} \rightarrow Y$. The final topology is the largest topology that makes each f_{α} continuous.

So for $A \subset Y$, in order for A to be in \mathcal{T} need $f_{\alpha}^{-1}(A) \in \mathcal{T}_{\alpha}$ for all α .

For fixed α , we want $\{A \in Y : f_{\alpha}^{-1}(A) \in \mathcal{T}_{\alpha}\}$. This is a topology, denote it \mathcal{T}_{α}^Y . It follows that $T = \bigcap_{\alpha} \mathcal{T}_{\alpha}^Y$

Let Y be a set (X, \mathcal{T}^X) , $f : X \rightarrow Y$, we require f is onto Y . Then $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}^X\}$ is the smallest topology that makes f continuous. It is called the quotient topology.

Other view: Let X, Y be sets, $f : X \rightarrow Y$ onto. Then f defines an equivalence relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$.

If we have an equivalence relation on a set, it defines a partition of the set.

If you have a partition, P , of a set X , then a set P is a set where the elements are nonempty subsets of X . Then define $f : X \rightarrow P$, where $f(x)$ is the element, A , of P such that $x \in A$. Then $f : X \rightarrow P$ onto.

Definition 1.5.2. (X, \mathcal{T}^X) and (Y, \mathcal{T}^Y) are homeomorphic if their $f : X \rightarrow Y$, one to one, onto such that f and f^{-1} are continuous.

Example 1.5.3. $\mathbb{R}_d = (\mathbb{R}, \mathcal{T})$ with discrete topology.

Consider $\mathbb{R}_d \xrightarrow{f} \mathbb{R}$ by $f(t) = t$. f is one to one, onto, and continuous but f^{-1} not continuous so it is not a homeomorphism.

Example 1.5.4. Let $X = [0, 1]$, define an equivalence relation $0 \sim 1$ and $r \not\sim s$ if $r \neq s$ and $0 < r < 1$. $[0, 1]/\sim$ is homeomorphic to the circle. Let $f(t) = e^{2\pi it}$, we see $f(0) = f(1)$, f is a homeomorphism.
(Insert Figure)

Example 1.5.5. $X = [0, 1] \times [0, 2]$

(Insert Figure) equivalence relation defined by $(0, r) \sim (2, r)$ for $0 \leq r \leq 1$

Quotient space is homeomorphic to a cylinder.

Suppose we define $(0, 1) \sim (2, 1 - r)$ $0 \leq r \leq 1$

(Insert Figure) Quotient space homeomorphic to the mobius strip.

Example 1.5.6. Let X be the unit sphere $\mathbb{R}^3 = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$.

Put an equivalence relation: for $v \in X$, $v \sim -v$

X/\sim is called a projective space.

1.6 September 7

1.6.1 Group Actions on Topological Spaces

For a topological spaces (X, \mathcal{T}) the set of homeomorphisms of X to X forms a group under composition, auto-homeomorphisms, $\text{Aut}((X, \mathcal{T}))$

Then if G is a group, then of an action of G on a topological space is a group homomorphism $\alpha, \alpha : G \rightarrow \text{Aut}((X, \mathcal{T}))$, so for each $g \in G$, α_g is a homeomorphism if (X, \mathcal{T})

$$\alpha_{g_1} \circ \alpha_{g_2} = \alpha_{g_1 g_2}, \alpha_{g_1^{-1}} = (\alpha_{g_1})^{-1}$$

Definition 1.6.1. For an action α , of G on some set X , given $x_0 \in X$, the orbit of x_0 for the action α is $\{\alpha_g(x_0) : g \in G\}$. The orbits form a partition of X . (orbits of $\alpha_g(x_0)$ same as x_0 , $\alpha_{g_1^{-1}}(\alpha_g(x_0)) = x_0$)

Let X/α be the set of orbits. Have “quotient map” $X \rightarrow X/\alpha$ by $x \mapsto \text{orbit of } x$.

If X has a topology and α acts by homeomorphism, puts quotient topology on X/α

Example 1.6.2. Symmetry of letters:

$X = A$ given $Z_2 = \mathbb{Z}/2\mathbb{Z}$ act by reflection. $X/\alpha =$ (Insert Figure)

$X = H$, $Z_2 \times Z_2$, $X/\alpha =$ (Insert Figure)

Example 1.6.3. Let $G = \mathbb{Z}$, let $X = \mathbb{R}$, let α be an action of \mathbb{Z} on \mathbb{R} by translation, $\alpha_n(t) = t + n$

each of $\{\dots, t_0 - 1, t_0, t_0 + 1, \dots\}$. What is \mathbb{R}/α

Example 1.6.4. A fundamental domain for α is a subset of X that contains exactly one element of each orbit.

- For the above example, fundamental domain $[0, 1)$ with open subsets “wrapped around edges” so \mathbb{R}/α is homeomorphic to the circle. Homeomorphism given by $t = e^{2\pi it}$, constant on equivalence classes.

Example 1.6.5. The antipodal relation on the unit sphere with $v \sim -v$ acted on by $Z_2 = (0, 1)$ by $\alpha_1(v) = -v$

Let Y be set of all lines in \mathbb{R}^3 through 0. Let $\mathbb{R} - \{0\}$, have an action on \mathbb{R}^3 by $\alpha_t(r, s, v) = (tr, ts, tv)$

Orbits in $\mathbb{R}^3 - \{0\}$, set of all lines through 0, (with 0 removed). Each line intersects the unit sphere in 2 antipodal points. Quotient topology gives a topology on the set of lines.

1.6.2 Connectedness

Definition 1.6.6. A topological space (X, \mathcal{T}) is connected if it does not have two, nonempty, disjoint open sets A, B with $A \cup B = X$

- If this is the case, A, B are also closed - called “clopen”

Theorem 1.6.7. If (X, \mathcal{T}) is connected, $f : X \rightarrow Y$ is continuous, $f(X) = \text{range}(f)$ with the inherited topology is connected.