MATH 225A: Metamathmatics

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Fall 2022

Contents

Chapter 1

Structures and Theories

1.1 August 25

1.1.1 Review

Definition 1.1.1. A language $\mathcal{L}$ consists of $\{\mathcal{C}, \mathcal{R}, \mathcal{F}\}$ where $\mathcal{C}$ is the set of constant symbols, $\mathcal{R}$ is the set of relation symbols, $\mathcal{F}$ is the set of function symbols, and and arity function $n : \mathcal{R} \cup \mathcal{F} \to \mathbb{N}$. For $R \in \mathcal{R}$, n_R is the arity of R, for $f \in \mathcal{F}$, n_f is the number of inputs f takes.

Definition 1.1.2. An $\mathcal{L}$-structure consist of

- a set M called the domain
- an element $c^{\mathcal{M}}$ for each $c \in \mathcal{C}$
- a subset $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
- a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$

denoted $\mathcal{M} = (M\ :\ \{c^{\mathcal{M}} : c \in \mathcal{C}\}, \{R^{\mathcal{M}} : R \in \mathcal{R}\}, \{f^{\mathcal{M}} : f \in \mathcal{F}\})$

Definition 1.1.3. An $\mathcal{L}$-embedding $\eta : \mathcal{M} \to \mathcal{N}$ is a one to one function $M \to N$ that preserves interpretation
eg. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$, $\eta(f^{\mathcal{M}})(m_1, \ldots, m_{n_f}) = f^{\mathcal{N}}(\eta(m_1), \ldots, \eta(m_{n_f}))$,
$(m_1, \ldots, m_{n_R}) \in R^{\mathcal{M}} \iff (\eta(m_1), \ldots, \eta(m_n)) \in R^{\mathcal{N}}$

Definition 1.1.4. An $\mathcal{L}$-isomorphim is an $\mathcal{L}$-embedding that is onto.

Definition 1.1.5. $\mathcal{M}$ is a substructure if $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$ if:
$c^{\mathcal{M}} = c^{\mathcal{N}}$, $f^{\mathcal{M}} = f^{\mathcal{N}} \restriction M^{n_f}$, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^{n_R}$

First Order language:

- Use symbols :

  - $\mathcal{L}$
  - Logical symbols: connectives $(\wedge, \vee, \neg)$, quantifiers ($\forall$, $\exists$), equality ($=$), variables $(v_0, v_1, \ldots)$
  - paranthesis and commas

- terms
  - c : constants
  - v_i : variables
  - $f(t_1, \ldots, t_{n_f})$ for terms $t_1, \ldots, t_{n_f}$
- given an $\mathcal{L}$-structure $\mathcal{M}$, a term $t(v_0, \ldots, v_n)$, and $m_0, \ldots, m_n \in M$ we inductively define $t^{\mathcal{M}}(m_0, \ldots, m_n)$
- atomic formulas: $t_1 = t_2$ and $R(t_1, \ldots, t_{n_R})$
- $\mathcal{L}$-formulas: If $\phi and \psi$ are $\mathcal{L}$-formulas, then so are: $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $\exists v\phi$, $\forall v\phi$

Definition 1.1.6. We say a variable v occurs freely in ψ when it is not in a quantifier $\forall v$ or $\exists v$

- an $\mathcal{L}$-sentence is an $\mathcal{L}$-formula with no free variables

Definition 1.1.7. A theory is a set of $\mathcal{L}$-sentences

Definition 1.1.8. Given an $\mathcal{L}$-formla $\psi(v_1, \ldots, v_k)$, $\mathcal{L}$-structure $\mathcal{M}$, $m_1, \ldots, m_k \in M$ we can define $\mathcal{M} \models \phi(m_1, \ldots, m_k)$ inductively. We say $(m_1, \ldots, m_k)$ satisfies ϕ in $\mathcal{M}$ or ϕ is true in $\mathcal{M}, m_1, \ldots, m_k$.

- A theory T is satisfiable if it has a model $\mathcal{M}$, eg. $\mathcal{M}$ such that $\mathcal{M} \models \phi$ for $\phi \in T$

Proposition 1.1.9. If $\mathcal{M} \subseteq \mathcal{N}$, $\phi(\overline{v})$ is quantifier free, $\overline{m} \in M$, then $\mathcal{M} \models \phi(\overline{m}) \leftrightarrow \mathcal{N} \models \phi(\overline{m})$.

Definition 1.1.10. $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$ if for all $\mathcal{L}$-sentences ϕ, $\mathcal{M} \models \phi \leftrightarrow \mathcal{N} \models \phi$, denoted $\mathcal{M} \equiv \mathcal{N}$

- Th($\mathcal{M}$), the full theory of $\mathcal{M}$, is $\{\phi \ \mathcal{L} - \text{sentence } \mid \mathcal{M} \models \phi\}$
- $\mathcal{M} \equiv \mathcal{N} \iff \text{TH}(\mathcal{M}) = \text{Th}(\mathcal{N})$
- A class of $\mathcal{L}$-structures $\mathcal{K}$ is elementary if there is a theory T such that $\mathcal{K}$ is the class of all $\mathcal{M}$ such that $\mathcal{M} \models T$.

Logical implication: $T \models \phi$ if for every $\mathcal{M} \models T$, $\mathcal{M} \models \phi$
Gödels Completeness Theorem: $T \models \phi \leftrightarrow$ there is a formal proof for $T \vdash \phi$

1.1.2 Definable Sets

Definition 1.1.11. $X \subseteq M^n$ is definable if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $b_1, \ldots, b_m \in M$ such that $\forall \overline{a}$, $\overline{a} \in X \leftrightarrow \mathcal{M} \models \phi(\overline{a}, \overline{b})$ (definable over $\overline{b}$)

- Given $A \subseteq M$, X is definable over A, or A-definable, if it is definable over $\overline{b}$ for some $\overline{b} \in A$.

Proposition 1.1.12. Suppose $\mathcal{D} = (D_n : n \in \omega)$ is the smallest collection of subsets $D_n \subseteq \mathcal{P}(M^n)$ such that

- $M^n \in D_n$
- D_n is closed under union, intersection, complement, permutation
- if $X \in D_{n+1}$, then $\pi(X) \in D_n$ where $\pi(m_1, \ldots, m_{n+1}) = (m_1, \ldots, m_n)$
- $\{\overline{b}\} \in D_n$ for $\overline{b} \in M^n$
- $R^{\mathcal{M}} \in D_{n_R}$, graph$(f) \in D_{n_f+1}$
- if $X \in D_n$, $M \times X \in D_{n+1}$
- $\{(m_1, \ldots, m_n) : m_i - m_j\} \in D_n$

Then $X \subseteq \mathcal{M}^n$ is definable $\leftrightarrow X \in D_n$

Chapter 2

Basic Techniques

2.1 August 30

2.1.1 Compactness Theorem

Theorem 2.1.1 (Compactness). If T is finitely satisfiable, then T has a model $\mathcal{M}$. Furthermore, $|\mathcal{M}| \leqslant |\mathcal{L}| + \aleph_0$

- T is finitely satisfiable if every finite subset is satisfiable

Compactness II: if $T \models \phi$, then there is finite $T_0 \subset T$ such that $T_0 \models \phi$
$T \models \phi \leftrightarrow T \cup \{\neg\phi\}$ is not satisfiable

Proposition 1: If T is finitely satisfiable, maximal, and has the witness property, then T has a model $\mathcal{M}$ with $|\mathcal{M}| \leqslant |\mathcal{L}|$

Proposition 2: If T is finitely satisfiable, then there is $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-theory $T^* \supseteq T$ such that T^* is finite;y satisfiable, maximal, and has the witness property. Further, $|\mathcal{L}^*| \leqslant |\mathcal{L}| + \aleph_0$

Definition 2.1.2.

- T is maximal if for any sentence ϕ, either $\phi \in T$ or $\neg\phi \in T$
- T has the witness property if for all $\mathcal{L}$-formulas $\phi(v)$ there is a constant c_ϕ such that $\exists v\phi(v) \to \phi(c_\phi) \in T$

Lemma 1: If T is maximal and finitely satisfiable, if there is finite $\Delta \subseteq T$ such that $\Delta \models \phi$, then $\phi \in T$.

Proof. If $\phi \notin T$, $\neg\phi \in T$. Since $\Delta \models \phi$, $\Delta \cup \{\neg\phi\}$ is not satisfiable, contradicting our assumption.

Henekin Construction:
We want to define $\mathcal{M} = (M, c^{\mathcal{M}}, R^{\mathcal{M}}, f^{\mathcal{M}})$

- Let $M = \mathcal{C}/\sim$ where $\mathcal{C}$ is the set of constant symbols and $\sim$ is the equivalence relation defined by $c_1 \sim c_2 \leftrightarrow c_1 = c_2 \in T$
- $R^{\mathcal{M}} \subseteq M^{n_R}$ by $(c_1^*, \ldots, c_{n_R}^*) \in R^{\mathcal{M}} \leftrightarrow R(c_1, \ldots, c_n) \in T$ where c^* equivalence class of c
  This is well defined since if we have $c_1' \sim c_1, \ldots, c_n' \sim c_n, R(c_1, \ldots, c_n) \in T$ then $R(c_1', \ldots, c_n') \in T$

- $f^{\mathcal{M}}$ by $f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \leftrightarrow f(c_1, \ldots, c_n) = d \in T$. SUch a d^* exists since T has the witness property: $\exists v f(c_1, \ldots, c_n) = v \to f(c_1, \ldots, c_n) \in T$
- $c^{\mathcal{M}} := c^*$

Claim: For every formula $\phi(v_1, \ldots, v_k)$ and constant symbols $c_1, \ldots, c_k$, $\mathcal{M} \models \phi(c_1^*, \ldots, c_n^*) \leftrightarrow \phi(c_1, \ldots, c_n) \in T$
This implies $\mathcal{M} \models T$

Proof. By induction on formulas $\phi(v_1, \ldots, v_l)$

- atomic formulas: $\phi(v_1, \ldots, v_k)$ is $t_1(v_1, \ldots, v_k) = t_2(v_1, \ldots, v_k)$
  Subclaim: $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = c^* \leftrightarrow t(c_1 \ldots, c_n) = c \in T$
  Proved by induction on terms
- $\phi(v_1, \ldots, v_k)$ is $R(v_1, \ldots, v_k)$. Follows by deifnition of $R^{\mathcal{M}}$
- Suppose $\phi(\overline{v})$ is $\psi_1(\overline{v}) \wedge \psi_2(\overline{v})$, then
  $\mathcal{M} \models \psi_1 \wedge \psi_2(\overline{v}) \leftrightarrow \mathcal{M} \models \psi_1(\overline{v})$ and $\mathcal{M} \models \psi_2(\overline{v}) \overset{\text{IH}}{\leftrightarrow} \psi_1(\overline{c}) \in T$ and $\psi_2(\overline{c}) \in T \overset{\text{lemma}}{\leftrightarrow} \psi_1 \wedge \psi_2(\overline{c}) \in T$
- Suppose $\phi(\overline{v})$ is $\neg\psi(\overline{v})$, then
  $\mathcal{M} \models \neg\psi(\overline{c}^*) \leftrightarrow \mathcal{M} \not\models \psi(\overline{c}^*) \overset{\text{IH}}{\leftrightarrow} \varphi(\overline{c}) \notin T \overset{\text{maximality}}{\leftrightarrow} \neg\psi(\bar{(c)}) \in T$
- Suppose $phi(\overline{v})$ is $\exists w \varphi(\overline{v}, w)$, then
  $\mathcal{M} \models \exists w \varphi(\overline{c}^*, w) \leftrightarrow \exists d \in M$ such that $\mathcal{M} \models \phi(\overline{c}^*, d) \leftrightarrow \exists d \in M$ such that $\varphi(\overline{c}, d) \in T \overset{\text{witness principle}}{\leftrightarrow}$ $\exists w \varphi(\overline{c} w) \in T$

2.2 September 1

2.2.1 Compactness

Proof of Compactness continued:
We now prove proposition 2
Lemma 1: If T is finitely satisfiable then there is $\mathcal{L}^* \supset \mathcal{L}$, $T^* \supset T$ such that T^* has the witness property and is finitely satisfiable

Proof. For each $\mathcal{L}$-formula define a new constant symbol c_ϕ. Let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v)\, \mathcal{L} - \text{formula}\}$, $T_1 = T \cup \{\exists v \phi(v) \to \phi(c_\phi) : \phi(v)\, \mathcal{L} - \text{formula}\}$.
Claim: T_1 is finitely satisfiable.
Take $\Delta \subseteq T_1$ finite. $\Delta = T' \cup \{\exists v \phi_i(v) \to c_{\phi_i} : i = 1, \ldots, k\}$ for finite T' in T. We make an $\mathcal{L}_1$-structure $\mathcal{M}_1$ that satisfies Δ. Take $\mathcal{M} \models T'$, $\mathcal{M}$ $\mathcal{L}$-structure. Make $\mathcal{M}$ an $\mathcal{L}_1$-structure by defining $c_\phi^{\mathcal{M}_1}$ for each c_ϕ. If $\mathcal{M} \models \exists v \phi(v)$ let $c^{\mathcal{M}_1}$ be such a v otherwise let $c^{\mathcal{M}_1}$ be anything.
We repeat this process, defining $\mathcal{L}_{n+1}$ from $\mathcal{L}_n$ similarly.
We have $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \cdots$, $T \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq \cdots$ such that each T_i is finitely satisfiable and for $\phi(v)$ an $\mathcal{L}_{i-1}$-formula, there is c_ϕ in $\mathcal{L}_i$ such that $\exists v \phi(v) \to \phi(c_\phi) \in T_i$.
Let $\mathcal{L}^* = \bigcup_{n \in \omega} \mathcal{L}_n$, $T^* = \bigcup_{n \in \omega} T_n$. We see T^* has the witness property.
Sub-claim: If $T_0 \subset T_1 \subset T_2 \subset \cdots$ all finitely satisfiable, then $U_{n \in \omega} T_n$ is finitely satisfiable.

Lemma 2: If T is finitely satisfiable and ϕ a sentence, one of $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable.

Proof. Assume that both $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are not finitely satisfiable. Then there are $T_0, T_1 \subseteq T$ such that $T_0 \cup \{\phi\}$ and $T_1 \cup \{\neg\phi\}$ are not satisfiable. Let $\mathcal{M} \models T_0 \cup T_1$, then $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$ so $T_0 \cup \{\phi\}$ or $T_1 \cup \{\neg\phi\}$ is satisfiable, contradicting our assumption.

Zorn's Lemma: Let $\mathcal{A}$ be a collection of sets such that for any chain $\mathcal{C} \in \mathcal{A}$. $\bigcup \mathcal{C} \in \mathcal{A}$ where $\mathcal{C}$ is a chain if for $A, B \in \mathcal{C}$ either $A \subseteq B$ or $B \subseteq A$, then $\mathcal{A}$ has a maximal element, eg. $A \in \mathcal{A}$ such that there is not $B \in \mathcal{A}$ with $A \subsetneq B$.

Lemma: For every T, finitely satisfiable, there is $T' \supseteq T$ that is maximal and finitely satisfiable.

Proof. Let $\mathcal{A} = \{S\ \mathcal{L}\text{-theory} \mid S \supseteq T,\ S \text{ finitely satisfiable }\}$. Can apply zorns lemma since for any $\mathcal{C} \subseteq A$, $\bigcup \mathcal{C} \in \mathcal{A}$ so we have a maximal S.

Example 2.2.1. Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. In a group G, $g \in G$, ord$g$ = least n such that $\overbrace{g \cdots g}^{n \text{ times}} = e$, if it exists.
Observation: If T is an $\mathcal{L}$-theory extending the axioms of groups, $\phi(v)$ such that for every n there is $G_n \models T$, $g_n \in G_n$ of order greater than n such that $G_n \models \phi(g_n)$. Then there is $G \models T$ and $g \in G$, $\text{ord}(g) = \infty$ such that $G \models \phi(g)$.

Proof. Let $\mathcal{L}' = \{\cdot, e, c\}$. Let $T^* = T \cup \phi(c) \cup \{\psi_n\}$ where ψ_n is $\underbrace{c \cdot c}_{n \text{ times}} \neq e$. T^* finitely satisfiable so follows by compactness.

This tells us that there is not sentence that axiomatizes when an element is torsion.

Lemma 2.2.2. Let κ be a carindal $\kappa \geqslant |\mathcal{L}|$. Let T be a satisfiable theory such that $\forall n \in \mathbb{N}$, there is $\mathcal{M} \models T$ such that $|\mathcal{M}| > n$. Then T has a model of size κ.

Proof. Extend the language by adding κ may new constant symbols c_i for $i \in \kappa$. $T^* = T \cup \{c_i \neq c_j \mid i \neq j\}$. If $\mathcal{M} \models T^*$, $|\mathcal{M}| \geqslant \kappa$. T^* is finitely satisfiable so by compactness T^* has a model $\mathcal{M}$, $|\mathcal{M}| \leqslant |\mathcal{L}^*| + \aleph_0 = \kappa$. Thus, $|M| = \kappa$.

2.3 September 6

2.3.1 Complete Theories

Definition 2.3.1. Let κ be an infinite cardinal. A theory T is κ-categorical if all models of T of size κ are isomorphic (and there is at least one).

Example 2.3.2. The theory of torsion free abelian division groups (TFADG) is κ categorical for all uncountable κ.

Language = $\{\cdot, e\}$, TFADG = group axioms, commutativity, torsion free - $\forall a \neq e\ \overbrace{a \cdot a \cdots a}^{n} \neq e$ for $n \in \omega$,

divisible - $\forall a \exists b\, \overbrace{b + b + \cdots + b}^{n}$ for each $n \in \omega$
Observation: TFADG are essentialy $\mathbb{Q}$-vector spaces
For $a \in G$, $n \in \mathbb{N}$ $a \cdot n = \overbrace{a + \cdots + a}^{n \text{ times}}$ $\frac{a}{n}$ is b such that $b \cdot n = a$. Such a b exists since the group is division and is uniquely defined since if $b \cdot n = a = b' \cdot n$, $(b - b') \cdot n = 0$ so since the group is torsion free, $b - b' = 0$. For $a \in G$, $\frac{p}{q} \in \mathbb{Q}$ we define $a \cdot \frac{p}{q} = \frac{a}{q} \cdot p$
Two vector $\mathbb{Q}$-vector spaces are isomorphic $\leftrightarrow$ they have the same dimension. A $\mathbb{Q}$ vector space of size κ must have dimension κ so two $\mathbb{Q}$ vector spaces of size κ must be isomorphic.

Let ACF_p be the theory of algebraicly closed fields of characteristic p.
Language $= \{0, 1, +, \times\}$. ACF_P: field axioms, char p - $\underbrace{1 + \cdots + 1}_{p} = 0$, char 0 - $\underbrace{1 + \cdots + 1}_{n} \neq 0$ for $n \in \omega$, algebraicly closed - every non-constant polynomial has at least one root: for degree n, $\forall z_0, z_1, \ldots, z_n\, z_n \neq 0\, \exists x(z_n x^n + z_{n-1} x^{n-1} + \cdots + z_0 = 0)$. For each $n \in \omega$

Proposition 2.3.3. ACF is κ categorical for all uncountable κ.

Facts and Definitions

- Every fielf F has a prime subfield $P = \{\frac{\overbrace{1 + \cdots + 1}^{p}}{\underbrace{1 + \cdots + 1}_{q}} : p \in \mathbb{Z}, q \in \mathbb{N}\}$
  - if F has char $p > 0$, then the prime subfield is $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$
  - If F has char $o = 0$, then the prime subfield in $\mathbb{Q}$
- An element $a \in F$ is algebraic if there is a polynomial $p(x) \in P[x]$ such that $p(x) = 0$. (Can think of as a polynomial in $\mathbb{Z}[x]$)
- Otherwise a is transcendental
- A tuple $\overline{a}$ is algebraicly independent if there is no nontrivial polynomial $p(\overline{x}) \in P[x]$ such that $p(\overline{x}) = 0$.
- the transcendence degree of a field F is the size fo a maximal algebraicly independent set.
- Algebraicly closed fields are isomorphic $\leftrightarrow$ they have the same transcendence degree.

Observation: an ACF_p of size κ must have transcendence degree κ
If $M \subset F$ is a maximal algebraicly independent set, $\forall a \in F$ there is a polynomial $p(\overline{x}, y) \in P[\overline{x}, y]$ and $\overline{m} \in M$ such that $p(\overline{m}, a) = 0$.

Definition 2.3.4. A theory T is complete if for all $\mathcal{L}$-sentences, ϕ either $T \models \phi$ or $T \models \neg\phi$

Theorem 2.3.5 (Vaught's Test)**.** If T is satisfiable and has no finite models and is κ-categorical for $\kappa > |\mathcal{L}|$, then T is complete.

Corollary 2.3.6. ALL ACF_p satisfy the same sentences.

Proof. Suppose not. There is ϕ such taht $T \not\models \phi$, $T \not\models \neg\phi$ so $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$ are satisfiable. Both have models of size κ, contradicting κ-categoricity.

Definition 2.3.7. T is decidable if there is an algorithm to decdide $T \models \phi$ given ϕ

Observation: If T is computably enumerable and complete then T is decidable

Corollary 2.3.8. $\mathrm{Th}(\mathbb{C}; 0, 1, +, \times)$ is decidable.

2.4 September 8

2.4.1 Complete Theories

Observation: Let f be a function $: k \to k$. If f is one to one then f is onto, provided k is finite.

Theorem 2.4.1. Every injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective.
(A polynomial map consists of n polynomials $p_1[x_1, \ldots, x_n], \ldots, p_n[x_1, \ldots, x_n] \in \mathbb{C}[x]$)

Lemma 2.4.2. Let *phi* be a senctence in the language $\{0, 1, +, \times\}$. TFAE

1. $C \models \phi$
2. ϕ is true in any algebraically closed field of characteristic 0.
3. ϕ is true in some algebraically closed field of characteristic 0.
4. There are arbitrarily large primes p such that ϕ is true in some $F \models \mathrm{ACF}_p$
5. There is an $m \in \mathbb{N}$ such that for all $p \geqslant n$ and all $F \models \mathrm{ACF}_p$, $F \models \phi$

Proof. (1), (2), (3) equivalent since ACF_0 is complete. (4) $\to$ (5) clear.
(2) $\to$ (5) $\mathrm{ACF}_0 \models \phi$. There is finite $\Delta \subseteq \mathrm{ACF}_0$ such that $\Delta \models \phi$. If $p \geqslant n$ for an all n such that "$\overbrace{1 + \cdot + 1}$ $\neq 0$" shows up in Δ, then if $F \models \mathrm{ACF}_p$, $F \models \Delta$ so $f \models \phi$
(4) $\to$ (3) If (3) was false, $\mathrm{ACF}_0 \models\neq \phi$ and for some n, all $p > n$, if $F \models \mathrm{ACF}_p$ then $F \models \neg\phi$ so (4) is false.

Claim: Every injective polynomial function $f : (\mathbb{F}_p^{\mathrm{alg}})^n \to (\mathbb{F}_p^{\mathrm{alg}})^n$ is onto
where $\mathbb{F}_p^{\mathrm{alg}}$ is the algebraic closure of $\mathbb{F}_p : \mathbb{Z}/p\mathbb{Z}$. $\mathbb{F}_p^{\mathrm{alg}} = \bigcup_{n \in \mathbb{N}} \mathbb{F}_{p^n}$ where $\mathbb{F}_{p^n}$ is the unique field of size p^n.

For every polynomial $p(\overline{x}) \in F$ there is an atomic $t(\overline{x}, \overline{z})$ and parameters $\overline{c} \in F$ such that $p(\overline{x}) = t(\overline{x}, \overline{c})$ so $t_1(\overline{x}, \overline{c}), \ldots, t_n(\overline{x}, \overline{c})$ for $\overline{c} \in \mathbb{F}_p^{\mathrm{alg}}$, $\overline{x} = x_1, \ldots, x_n$
Claim states $\forall \overline{z} (\forall \overline{x} \forall \overline{y} \bigwedge_{i=1}^n t_i(x_i, z) = t_i(y_1, z) \to \overline{x} = \overline{y}) \to (\forall \overline{w} \exists \overline{x} \bigwedge_{i=1}^n t_i(\overline{x}, z) = w_i)$

Proof (Pf of Claim)**.** Take $\overline{b} \in (\mathbb{F}_p^{\text{alg}})^n$, want to show $\overline{b}$ is in the range of f
Let k be the finite subfield of $\mathbb{F}_p^{\text{alg}}$ generated by $\overline{c}$ and $\overline{b}$. $\mathbb{F}_p(\overline{c}, \overline{d})$
Restricting f to k^n, we get a one to one function from k^n to k^n so $f \restriction k^n$ is onto so $\overline{b}$ is in the range of f

2.4.2 Up and Down

Definition 2.4.3. A map $j : \mathcal{M} \to \mathcal{N}$ is an elementary embedding if for all formulas $\phi(\overline{x})$, all $m \in M$

$$\mathcal{M} \models \phi(\overline{m}) \leftrightarrow \mathcal{N} \models \phi(j(\overline{m}))$$

Definition 2.4.4. If for $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M}$ is an elementary subset of $\mathcal{N}$ if $i : M \hookrightarrow N$ is elementary ($\mathcal{M} \preceq \mathcal{N}$)

- $\mathbb{Q}^{\text{alg}} \preceq \mathbb{C}$, $(\mathbb{Q}, 0, +) \preceq (\mathbb{R}, 0, +)$

Definition 2.4.5. Given $\mathcal{M}$, let $\mathcal{L}_M = \mathcal{L} \cup \{c_m \mid m \in M\}$. $\mathcal{M}$ can be made into an $\mathcal{L}_m$-structure $\mathcal{M}^*$ by letting $c_m^{\mathcal{M}^*} = m$

Definition 2.4.6. Diag($\mathcal{M}$) the atomic diagram of $\mathcal{M} = \{\phi \mid \phi$ atomic $\mathcal{L}_M$ sentence such that $\mathcal{M} \models \phi\} \cup \{\neg\phi \mid \phi$ atomic $\mathcal{L}_M$ sentence such that $\mathcal{M} \models \neg\phi\}$
This is equivalent to $\{\phi \mid \phi$ is an $\mathcal{L}$-formula $\mathcal{M} \models \phi\}$
$\text{Diag}_{\text{el}}(\mathcal{M})$, the elementary diagram of $\mathcal{M}$ is $\{\phi \mid \phi$ is an $\mathcal{L}$formula $\mathcal{M} \models \phi\}$

Lemma 2.4.7. (i) if $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$ then there is an $\mathcal{L}$-embedding $\mathcal{M} \to \mathcal{N}$ (where $\mathcal{N}$ the restriction of $\mathcal{N}^*$ to $\mathcal{L}$)

Proof. Suppose $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$. If $\phi(\overline{x})$ is an $\mathcal{L}$ formula and $\overline{c_m}$ new constants, we can give an embedding by $m \mapsto c_m^{\mathcal{M}^*}$
$\mathcal{M} \models \phi(\overline{m}) \leftrightarrow \mathcal{M}^* \models \phi(\overline{c_m}) \leftrightarrow \mathcal{N}^* \models \phi(\overline{c_m}) \leftrightarrow \mathcal{N} \models \phi(\overline{m})$

Example 2.4.8. $\mathcal{M} = (\mathbb{Z}, +)$, $\mathcal{L} = \{*\}$, $\mathcal{L}_M = \{*, c_0, c_1, c_2, \ldots, c_{-1}, c_{-2}, \ldots\}$, in $\mathcal{M}^*$, $c_n^{\mathcal{M}^*} = n$
$\mathcal{N} = (\mathbb{R}, \times)$, define $\mathcal{N}^*$ by $c_n^{\mathcal{N}^*} = 2^n$. $\mathcal{N}^* = (\mathbb{R}, \times, c_n \mapsto 2^n)$
$\mathcal{N}^* \models \text{Diag}(\mathcal{M})$ size $(\mathbb{Z}, +) \to (\mathbb{R}, \times)$ by $n \mapsto 2^n$ is an embedding.
If $j : \mathcal{M} \to \mathcal{N}$ is an embedding, let $c_m^{\mathcal{M}^*} = j(m)$. Then $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$

2.5 September 13

2.5.1 Up and Down

Definition 2.5.1. $\mathcal{L}^- \subseteq \mathcal{L}$, $\mathcal{M}$ is an $\mathcal{L}$-stucture, then $\mathcal{L}^-$ reduct of $\mathcal{M}$ is the $\mathcal{L}^-$ stucture with the same domain and $\mathcal{L}^-$ interpretations of $\mathcal{M}$. We say that $\mathcal{M}^-$ is a reduction of $\mathcal{M}$, $\mathcal{M}$ is an expansion of $\mathcal{M}^-$

Lemma 2.5.2. Consider $\mathcal{L}$ structures $\mathcal{M}, \mathcal{N}$

1. there is an embedding $\mathcal{M} \to \mathcal{N} \leftrightarrow$ there is an $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ such that $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$
2. there is an elementary embedding $\mathcal{M} \to \mathcal{N} \leftrightarrow$ there is an $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ such that $\mathcal{N}^* \models \text{Diag}_{\text{el}}(\mathcal{M})$

Here $\mathcal{N}^* = (\mathcal{N}, c_m^{\mathcal{N}} \in N \text{ for } m \in M)$

Proof. $\to$) Suppose $f : \mathcal{M} \to \mathcal{N}$ is an embeeding. We need to find a $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ by defining $c_m^{\mathcal{N}}$ for $m \in M$ such that for all $\mathcal{L}$-formulas $\varphi(\overline{x})$, all $\overline{m} \in M$, if $\varphi(\overline{c_m}) \in \text{Diag}(M) \to \mathcal{N}^* \models \varphi(\overline{c_m})$. Let $c_m^{\mathcal{N}} = f(m)$ so $\varphi(\overline{c_m}) \in \text{Diag}(\mathcal{M}) \leftrightarrow \mathcal{M} \models \varphi(\overline{m}) \leftrightarrow \mathcal{N} \models \varphi(f(\overline{m})) \leftrightarrow \mathcal{N}^* \models \varphi(c_m^{\mathcal{N}})$
$\leftarrow$) Given the $\mathcal{L}_M$ expansion $\mathcal{N}^*$ of $\mathcal{N}$ such that $\mathcal{N}^* \models \text{Diag}(\mathcal{M})$. Let $f : \mathcal{M} \to \mathcal{N}$ by $f(m) = c_m^{\mathcal{N}^*}$

Theorem 2.5.3 (Upwards Lowenheim-Skolem). Let $\mathcal{M}$ be an infinite $\mathcal{L}$-strucutre. For every $\kappa \geqslant |M| + |\mathcal{L}|$ there is an $\mathcal{L}$-strucure $\mathcal{N}$ such that $|\mathcal{N}| = \kappa$ and $\mathcal{M} \preceq \mathcal{N}$.

Proof. It suffices to show there is an elementary embedding $j : \mathcal{M} \to \mathcal{N}$ as $\mathcal{M}$ can be identified with its image. Lt $\mathcal{N}^*$ be a model of $\text{Diag}(\mathcal{M})$ of size κ. Let $\mathcal{N}$ be the $\mathcal{L}$-reduct of $\mathcal{N}^*$

Example 2.5.4. $(\mathbb{Q}, 0, +) \preceq (\mathbb{R}, 0,_{)}$ κ-categorical so there is only structure of size $2^{\aleph_0}$ up to isomorphism

Example 2.5.5. $(\mathbb{Q}^{\text{deg}}, 0, 1, +, \times) \preceq (\mathbb{C}, 0, 1, +, \times)$

Theorem 2.5.6 (Downward Lowenheim-Skolem). Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. For all $X \subseteq M$, there is an $\mathcal{L}$ structure $\mathcal{N} \subseteq \mathcal{M}$, $|\mathcal{N}| = |X| + |\mathcal{L}| + \aleph_0$ and $\mathcal{N} \preceq \mathcal{M}$

Proposition 2.5.7 (Tarski-Vaught Test). Suppose $\mathcal{N} \subseteq \mathcal{M}$. Then $\mathcal{N} \preceq \mathcal{M} \leftrightarrow$ formulas $\phi(\overline{v}, w)$ and all $\overline{n} \in N$ if $\mathcal{M} \models \exists \phi(\overline{n}, w)$ then there is $c \in N$ such that $\mathcal{M} \models \phi(\overline{n}, c)$.

Proof. $\to$) Assume $\mathcal{N} \preceq \mathcal{M}$, $\mathcal{M} \models \exists w \phi(\overline{n}, w)$ then $\mathcal{N} \models \exists w \phi(\overline{n}, w)$ so there is $c \in N$ such taht $\mathcal{N} \models \phi(\overline{n}, c)$ so $\mathcal{M} \models \phi(\overline{n}, c)$
$\leftarrow$) We use induction on $\mathcal{L}$-formulas to show that for all formulas $\psi(\overline{x})$ and all $\overline{n}$, $\mathcal{N} \models \psi(\overline{n}) \leftrightarrow \mathcal{M} \models \psi(\overline{n})$

- For ψ atomic, this follows since $\mathcal{N} \subseteq \mathcal{M}$
- For $\psi = \psi_1 \wedge \psi_2$, $\neg\psi_1$ clear by applying IH
- For $\psi(\overline{x})$ of the form $\exists \phi(\overline{x}, w)$, pick $\overline{n} \in N$, $\mathcal{M} \models \psi(\overline{n}) \leftrightarrow \mathcal{M} \models \exists w \phi(\overline{n}, w) \leftrightarrow$ there is $c \in N$ such that $\mathcal{M} \models \phi(\overline{n}, c) \overset{\text{IH}}{\leftrightarrow}$ there is $c \in N$ such that $\mathcal{N} \models \phi(\overline{n}, c) \leftrightarrow \mathcal{N} \models \exists w \phi(\overline{n}, w) \leftrightarrow \mathcal{N} \models \psi(\overline{n})$.

Proof (Proof of Lowenheim Skolem). Let $X = X_0$. For any $\overline{n} \in X$ and $\varphi(\overline{v}, w)$ if $\mathcal{M} \models \exists w \varphi(\overline{n}, w)$. let $c_{\overline{n},\varphi} \in m$ such that $\mathcal{M} \models \phi(\overline{n}, c_{\overline{n},\varphi})$. Let $X_1 = \{c_{\overline{n},\varphi} | \varphi\, \mathcal{L} \text{ forumula }, \overline{n} \in X_0, \mathcal{M} \models \exists w \varphi(\overline{n}, w)\} \cup X_0$
We can define X_{n+1} from X_n similiarly and let $N = \bigcup_{i \in \omega} X_i$

$|X_1| = (\#\ \mathcal{L}$ forumas$) \times (\#$ terms $X_0) = (|\mathcal{L}| + \aleph_0) \times (|X_0|)$
Since $|\mathcal{N}| \leqslant |\mathcal{L}| + |\aleph_0| + |X_0|$, then $|X| \leqslant |\mathcal{N}| \leqslant |X| + |\mathcal{L}| + \aleph_0$.
We define $\mathcal{N}$ with domain N by restricting functions, relations, and constants from $\mathcal{M}$. If $\varphi(\overline{x}, w)$ is the formula $f(\overline{x}) = w$ and $\overline{n} \in X$, $\mathcal{M} \models \exists w f(\overline{m}) = w$ in X_{i+1} so $c_{\varphi,n}$ satisfies $f(\overline{n}) = c_{\varphi.n}$

2.6 September 15

2.6.1 Universal Axiomatizations

Example 2.6.1. Consider $\mathcal{M} = (\mathbb{Z}, 0, +)$, $\mathcal{N} = (2\mathbb{Z}, 0, +)$, $\mathcal{N} \subset \mathcal{M}$, $\mathcal{N} \equiv \mathcal{M}$ but $\mathcal{N} \npreceq \mathcal{M}$. Consider $\varphi(x) = \exists y(y + y = z)$. $\mathcal{M} \models \varphi(2)$, $\mathcal{N} \models \neg\varphi(2)$.
We have $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$ but $\mathcal{N} \not\models \mathrm{Diag}_{\mathrm{el}}(\mathcal{N})$.

Definition 2.6.2. A universal formula is of the form $\forall x_1, \dots, x_n \varphi(x_1, \dots, x_n, \overbrace{\ y\ })$ where ψ is quantifier free.

Observation: If $\mathcal{M} \subseteq \mathcal{N}$ and $\varphi(\overline{x})$ is a univeral formulas, $\overline{m} \in M$, if $\mathcal{N} \models \varphi(\overline{m})$, then $\mathcal{M} \models \varphi(\overline{n})$

Definition 2.6.3. T has a univeral axiomatization if there is a set of universal sentences Γ such that $T \models \Gamma$ and $\Gamma \models T$

Observation: If T has a universalthen if $\mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \models T$

Example 2.6.4. Group axioms, if $\mathcal{L} = \{\cdot, e\}$, not universal, $(\mathbb{N}, 0, +) \subseteq (\mathbb{Z}, 0, +)$ but is not a group.
If we consider $\mathcal{L} = \{\cdot, e, (\cdot)^{-1}\}$, universal, $\forall x(x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e)$

Theorem 2.6.5. If T is such that $\forall \mathcal{M} \subseteq \mathcal{N}(\mathcal{N} \models T \to \mathcal{M} \models T)$, then T has a universal axiomatization.

Proof. Let $\Gamma\{\varphi$ universal $|\, T \models \varphi\}$. Clearly $T \models \Gamma$, want to show $\Gamma \models T$. Suppose $\mathcal{M} \models \Gamma$, we want to show $\mathcal{M} \models T$, We want $\mathcal{N} \supseteq \mathcal{M}$ such that $\mathcal{N} \models T$. $\mathcal{N} \supseteq \mathcal{M} \leftrightarrow \mathcal{N} \models \mathrm{Diag}(\mathcal{M})$ so want $\mathrm{Diag}(\mathcal{M}) \cup T$ is satisfiable.
Claim: $T \cup \mathrm{Diag}(\mathcal{M})$ is satisfiable.
Let $\Delta \subseteq T \cup \mathrm{Diag}(\mathcal{M})$ be finite. $\Delta = T_0 \cup \{\phi_1(\overline{c_m}), \dots, \phi_k(\overline{c_m})$. Can assume only one formula ϕ (can take the conjugation) so ϕ is quantifier free such that $\mathcal{M} \models \phi(\overline{c_m})$. $\mathcal{M} \models \phi(\overline{m}) \to \mathcal{M} \not\models \forall \overline{v} \neg\phi(\overline{v}) \to T \not\models \forall \overline{v} \neg\phi(\overline{v})$ so $T \cup \{\exists \overline{v} \phi(\overline{v})\}$ is satisfiable. Thus, $T \cup \{\phi(\overline{c_m})\}$ is satisfiable since if $\mathcal{A} \models \exists v \phi(v)$, for some $\overline{a} \in A$, $\mathcal{A} \models \phi(\overline{a})$ so let $\overline{c_m} = \overline{a}$. $(\mathcal{A}, \overline{c_m} \mapsto \overline{a}) \models \phi(\overline{c_m})$

- If $\overline{c}$ does not occur in T, ϕ, then $T \cup \{\exists \overline{v} \phi(\overline{v})$ is satisfiable $\to T \cup \phi(\overline{c})$ is satisfiable. Equivalently, $T \models \psi(\overline{c}) \to T \models \forall \overline{v} \psi(\overline{v})$

Suppose $(I, <)$ is a linear order. For each $i \in I$, $\mathcal{M}_i$ is an $\mathcal{L}$-structure, $\forall i < j\ \mathcal{M}_i \subseteq \mathcal{M}_j$ is called a chain (elementary chain if $\mathcal{M}_i \preceq \mathcal{M}_j$). Let $\mathcal{M} = \bigcup_{i \in} I \mathcal{M}_i$, $M = \bigcup_{i \in I} M_i$, $f^{\mathcal{M}} = \bigcup_{i \in I} f^{\mathcal{M}_i}$

Proposition 2.6.6. If $(\mathcal{M}_i : i \in I)$ is an elementary chain, $\forall i\, \mathcal{M}_i \preceq \mathcal{M}$

Proof. Use induction on formulas $\phi(\overline{v})$ to show that $\forall i$, $\forall m \in \mathcal{M}_i$, $\mathcal{M}_i \models \phi(\overline{m}) \leftrightarrow \mathcal{M} \models \phi(\overline{m})$

- ϕ quantifier free true since substructure
- ϕ is $\neg\psi, \psi_1 \wedge \psi_2$ clear by induction
- $\phi(\overline{x})$ is $\exists v\psi(\overline{x}, v)$ $\mathcal{M} \models \exists v\psi(\overline{x}, v) \leftrightarrow \exists n \in \mathcal{M}_j$ for some $j \in I$ such that $\mathcal{M} \models \psi(\overline{m}, n)$ $\overset{\text{IH}}{\leftrightarrow} \mathcal{M}_j \models \psi(\overline{x}, n) \leftrightarrow \mathcal{M}_j \models \exists v\phi(\overline{x}, v) \overset{M_i \preceq M_j}{\leftrightarrow} \mathcal{M}_i \models \exists v\phi(\overline{m}, v)$

2.7 September 20

2.7.1 Ultrafilters

Definition 2.7.1. A filter on I is a subset $\mathcal{D} \subseteq \mathcal{P}(I)$ such that

(i) $\varnothing \notin \mathcal{D}$, $I \in \mathcal{D}$

(ii) If $A \in cD$, $B \supseteq A \rightarrow B \in \mathcal{D}$

(iii) if $A, B \in \mathcal{D}$, then $A \cap B \in \mathcal{D}$

Example 2.7.2. (a) $I = \mathbb{R}$, $\mathcal{D} = \{X \subseteq \mathbb{R} \,|\, X \text{ has full measure }\}$ eg. $\lambda(\mathbb{R} \backslash X) = 0$

(b) $I = \mathbb{R}$, $\mathcal{D} = \{X \subseteq \,|\, X \text{ is meager }\}$

(c) For $\kappa \leqslant |I|$, $\mathcal{D} = \{X \subseteq I \,|\, |I \backslash X| < \kappa\}$
For $\kappa = \aleph_0$, $\mathcal{D}$ is called the Frechet filter or the cofinite filter

(d) For $x \in I$, $\mathcal{D} = \{X \subseteq I \,|\, x \in X\}$ called principle filter

(e) For $I = \mathbb{N}$, $\{X \subseteq N \ \lim_{n\to\infty} \frac{|X \cap n|}{n} = 1\}$

Definition 2.7.3. $\mathcal{D}$ is an ultrafilter if it is a filter and for all $X \subseteq I$, either $X \in \mathcal{D}$ or $X^C \in \mathcal{D}$

- principle filters are ultrafilters

Observataion: If $\mathcal{U}$ is an ultra filter, $A \cup B \in \mathcal{U} \leftrightarrow A \in \mathcal{U}$ or $B \in \mathcal{U}$
If $A, B \notin \mathcal{U}$, $A^C, B^C \in \mathcal{U}$ so $A^C \cap B^C \in \mathcal{U}$ so $(A^C \cap B^C)^C = A \cup B \notin \mathcal{U}$
Similarly, $C \cap D \notin \mathcal{U} \leftrightarrow C \notin \mathcal{U}$ and $D \notin \mathcal{U}$

Theorem 2.7.4. Every filter $\mathcal{D}$ on I can be extended to an ultrafilter

Proof. Let $\mathcal{A} = \{\mathcal{F} \subseteq \mathcal{P}(I) \,|\, \mathcal{F} \text{ filter and } \mathcal{D} \subseteq \mathcal{F}\}$. To apply Zorn's lemma to get a maximal $\mathcal{U}$ in $\mathcal{A}$, we need to show if $\mathcal{C} \subseteq \mathcal{A}$ is a chain then $\bigcup\mathcal{C} \in \mathcal{A}$. Clear that $\varnothing \notin \bigcup\mathcal{C}, I \in \bigcup\mathcal{C}$, and closed upwards. For $A, B \in \bigcup\mathcal{C}$, $\exists\mathcal{F}, \mathcal{F}' \in \mathcal{C}$ such that $A \in \mathcal{F}, B \in \mathcal{F}'$. WLOG assume $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}' \subseteq \bigcup\mathcal{C}$
Suppose $\mathcal{U}$ is a maximal filter. We show that it is an ultrafilter. Take $X \subseteq I$. We show $X \in \mathcal{U}$ or $X^C \in \mathcal{U}$. Suppose not. Let $\mathcal{D}_0$ = filter generated by $X, \mathcal{U} = \{Y \,|\, \exists V \in \mathcal{U}\, Y \supseteq V \cap X\}$, $\mathcal{D}_1 = \{Z \,|\, \exists W \in \mathcal{U}\, Z \supseteq W \cap X^C\}$. $\mathcal{D}_0, \mathcal{D}_1$ satsify all conditions except we dont know if they contain $\varnothing$. If $\varnothing \in \mathcal{D}_0, \mathcal{D}_1$, there is

$V \in \mathcal{U}$ $V \cap X = \varnothing$, $W \in \mathcal{U}$ $W \cap X^C = \varnothing$ so $V \subseteq X^C, W \subseteq X$ so $V \cap W = \varnothing$, contradicting $V, W \in \mathcal{U}$

To get a nonprinciple ultrafilter take $\mathcal{D} = \{X \subseteq I \mid I \backslash X \text{ finte }\}$ and extend to ultrafilter $\supseteq \mathcal{D}$

Observataion: $\forall x \in I$, $I \backslash \{x\} \in \mathcal{D} \subseteq \mathcal{U}$ so $\{x\} \notin \mathcal{U}$

An ultrafilter is not principle $\leftrightarrow \mathcal{U} \supseteq$ Frechet filter

Observataion: If $\mathcal{U}$ is an ultrafilter and contains a finite set $\mathcal{A} = \{a_0, \ldots, a_n\}$ then $\mathcal{U}$ is principle since $\mathcal{A} = \{a_0\} \cup \{a_1\} \cup \cdots \cup \{a_n\}$

Definition 2.7.5 (Ultraproduct)**.** I an infinte set, $\mathcal{U}$ an ultafilter of I, $\{\mathcal{M}_i : i \in I\}$ a collection of cL structures. Define $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ as follows:

- Given $g, h \in \prod_{i \in I} M_I$, $g \sim h$ iff $\{i \in I \mid g(i) = h(i)\} \in \mathcal{U}$. $M = \prod_{i \in I} M_i / \sim$
- $c^{\mathcal{M}} = [i \mapsto c^{\mathcal{M}_i}]$
- $f^{\mathcal{M}}(g_1, \ldots, g_n) = [i \mapsto f^{\mathcal{M}_i}(g_1(i), \ldots, g_n(i))]$
- $(g_1, \ldots, g_n) \in R^{\mathcal{M}} \iff \{i \in I \mid (g_1(i), \ldots, g_n(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U}$

Claims:

1. $\sim$ is an equivalence relation on $\prod_{i \in I} M_i$
   Reflexivity, symmetry clear. $g \sim h, h \sim f \to g \sim f$ since $\{i \mid g(i) = f(i)\} \supseteq \{i \mid g(i) = h(i)\} \cap \{i \mid h(i) = f(i)\}$
2. $f^{\mathcal{M}}$ is well defined.
   $g_1 \sim g_1', \ldots, g_n \sim g_n' \to f^{\mathcal{M}}(g_1, \ldots, g_n) = f^{\mathcal{M}}(g_1', \ldots, g_n')$ since $\{i \mid f^{\mathcal{M}_i}(g_1(i), \ldots, g_n(i)) = f^{\mathcal{M}_i}(g_1'(i), \ldots, g_n'(i))\} \supseteq \bigcap_{j=1}^n \{i \mid g_j(i) = g_j'(i)\}$
3. $R^{\mathcal{M}}$ well defined for a similar reason.

Definition 2.7.6. The $\mathcal{U}$ ultrapower of $\mathcal{M}$ is $\prod \mathcal{M}/\mathcal{U}$

- $\mathcal{M} \preceq \prod \mathcal{M}/\mathcal{U}$

2.8 September 22

2.8.1 Ultrafilters

Theorem 2.8.1 (Los' Theorem)**.** For every forumla $\varphi(v_1, \ldots, v_k)$ and $g_1, \ldots, g_k \in \prod_{i \in I} M_i$, $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$, $\mathcal{M} \models \phi([g_1], \ldots, [g_n]) \leftrightarrow \{i \in I \mid \mathcal{M}_i \models \phi(g_1(i), \ldots, g_k(i))\} \in \mathcal{U}$

Corollary 2.8.2. $\mathcal{M} \preceq \mathcal{M}^I / \mathcal{U}$ by $m \mapsto g_m$ where $g_m(i) = i$ $\forall i \in I$

Proof. By induction on formulas φ

- φ atomic. $([g_1], \ldots, [g_n]) \in R^{\mathcal{M}} \iff \{i \in I \mid (g_1(i), \ldots, g_n(i)) \in R^{\mathcal{M}_i}\} \in \mathcal{U}$ by definition.
  Similar for $=$

- φ is $\psi_1 \wedge \psi_2$. $\mathcal{M} \models \varphi([\overline{g}]) \leftrightarrow \mathcal{M} \models \psi_1[\overline{g}]$ and $\mathcal{M} \models \psi_2[\overline{g}] \overset{\text{IH}}{\leftrightarrow} \{i \,|\, \mathcal{M}_i \models \psi_1(\overline{g(i)})\} \in \mathcal{U}$ and $\{i \,|\, \mathcal{M}_i \models \psi_2(\overline{g(i)})\} \in \mathcal{U} \leftrightarrow \{i \,|\, \mathcal{M}_i \models \psi_1(\overline{g(i)})\} \cap \{i \,|\, \mathcal{M}_i \models \psi_2(\overline{g(i)})\} \in \mathcal{U} \leftrightarrow \{i \,|\, \mathcal{M}_i \models \psi_1(\overline{g(i)}) \wedge \psi_2(\overline{g(i)})\} \in \mathcal{U}$
  φ is $\psi_1 \vee \psi_2$ is similar
- φ is $\neg\psi$. $\mathcal{M} \models \varphi \leftrightarrow \mathcal{M} \not\models \psi \leftrightarrow \{i \,\mathcal{M}_i \models \psi\} \notin \mathcal{U} \leftrightarrow \{i \,|\mathcal{M}_i \models \varphi\} \in \mathcal{U}$
- $\varphi(\overline{v})$ is $\exists \overline{x} \psi(x, \overline{v})$. $\mathcal{M} \models \varphi[\overline{g}] \leftrightarrow$ there is $h \in M$ such that $\mathcal{M} \models \psi([h], [\overline{g}]) \overset{\text{IH}}{\leftrightarrow} \{i \,|\, \mathcal{M}_i \models \psi(h(i), \overline{g(i)})\} \in \mathcal{U}$ for some $h \leftrightarrow \{i \,|\, \mathcal{M}_i \models \exists x \psi(x, \overline{g(i)})\} \in \mathcal{U}$

Proof (Proof of Compactness)**.** Let T be finitely satisfiable. For every $\Delta \subseteq T$ finite, there is $\mathcal{M}_\Delta \models \Delta$. Let $I = \{\Delta \subseteq I | \Delta \text{ finite }\}$. For $\Sigma \in I$, let $X_\Sigma = \{\Delta \subseteq I \,|\, \Sigma \subseteq \Delta\} \subseteq I$. Let $\mathcal{D} = \{Y \subseteq I \,|$ for some Σ, $Y \supseteq X_\Sigma\}$ (filter generated by $X_\Sigma' s$). Claim $\mathcal{D}$ is a filter, $\varnothing \notin \mathcal{D}, I \in \mathcal{D}$, closed upwards. $X_\Sigma \cap X_{\Sigma'} = X_{\Sigma \cup \Sigma'}$ so closed under intersection. Let $\mathcal{U} \supseteq \mathcal{D}$ be an ultrafilter. Let $\mathcal{M} = \prod_{\Delta \in I} \mathcal{M}_\Delta \;/cU$. For $\varphi \in T$, $X_{\{\varphi\}} \in \mathcal{U}$ and for all $\Delta \in X_{\{\varphi\}}$, $\mathcal{M}_\Delta \models \varphi$ so $\{\Delta \in I \,|\, \mathcal{M}_\Delta \models \varphi\} \supseteq X_{\{\varphi\}} \in \mathcal{U}$ so $\mathcal{M} \models \varphi$ by Los' thm.

2.8.2 Back and Forth Proofs

Example 2.8.3. DLO - dense linear orders without endpoints, $\mathcal{L} = \{\leqslant\}$
$(\mathbb{Q}, \leqslant), (\mathbb{R}, \leqslant), (\mathbb{R}^2, \text{lex}), (2^{<\omega})$ orderd by binary tree with ends removed.

Theorem 2.8.4 (Cantor)**.** DLO is $\aleph_0$ categorical, complete, and decidable. If $A, B \models$ DLO, countable then $A \equiv B$

Proof. Given $A = \{a_0, a_1, a_2, \ldots\}, B = \{b_0, b_1, b_2, \ldots\}$ we specify an isomorphism as follows.
Choose where to send a_0 arbitrarily, choose an element in A, not already chosen, to map to b_0 such that it respects order. At each step continue ensuring a_i is in the domain, b_i is in the range while preserving order. This is possible the ordering is dense and has no endpoints.

Chapter 3

Quantifier Elimination

3.1 September 29

3.1.1 Quantifier Elimination

Sometimes formulas with quantifiers are equivalent to ones without quantifiers $\mathbb{R} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (v^2 - 4ac \geqslant 0 \wedge u \neq 0) \vee (u = 0 \wedge (w = 0 \vee v \neq 0))$
$\mathbb{C} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (u \neq 0 \vee v \neq 0 \vee w = 0)$

No always possible though. In $\mathbb{N}$, there are polynomials $p(x, \overline{v})$ such taht $\{n \mid \mathbb{N} \models \exists \overline{v} p(n, \overline{v})\}$ is not computable.

Definition 3.1.1. T has quantifier elimination if for every formula $\phi(\overline{v})$ ther is a quantifier free $\varphi(\overline{v})$ such that $T \models \forall \overline{v}(\phi(\overline{v}) \leftrightarrow \varphi(\overline{v}))$

Note: we allow quantifier free formulas for true and false.

Example 3.1.2. Theories with quantifier elimination

- DLO - dense linear orders
- Th(Rado Graph)
- DAG - torsion free divisible abelian groups
- ODAG - ordered divisble abelian groups
- Presburger Arithmetic - $\mathrm{Th}(\mathbb{Z}, +, -, 0, 1, <)$ adding preducates $P_n(x) \leftrightarrow \exists y(n \cdot y = x)$
- Algebraically closed fields - $\mathrm{Th}(\mathbb{C}, +, \times)$
- Real closed fields $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$

Theorem 3.1.3. DLO has quantifier elimination

Proof. Let $\phi(\overline{v})$ be a formula, $\mathcal{L} = \{\leqslant\}$

3.1.2 Quantifier Elimination

Sometimes formulas with quantifiers are equivalent to ones without quantifiers $\mathbb{R} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (v^2 - 4ac \geqslant 0 \wedge u \neq 0) \vee (u = 0 \wedge (w = 0 \vee v \neq 0))$
$\mathbb{C} \models \exists x(ux^2 + vx + w = 0) \leftrightarrow (u \neq 0 \vee v \neq 0 \vee w = 0)$

No always possible though. In $\mathbb{N}$, there are polynomials $p(x, \overline{v})$ such taht $\{n \mid \mathbb{N} \models \exists \overline{v} p(n, \overline{v})\}$ is not computable.

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- Algebraically closed fields - $\mathrm{Th}(\mathbb{C}, +, \times)$
- Real closed fields $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$

Theorem 3.1.6. DLO has quantifier elimination

Proof. Let $\phi(\overline{v})$ be a formula, $\mathcal{L} = \{\leqslant\}$
If ϕ is a sentence, then since DLO is complete either DLO $\models \phi \leftrightarrow \top$ or DLO $\models \phi \iff \bot$
Suppose ϕ not a sentence, $\overline{v} = v_1, \ldots, v_n$. Given $\sigma : \{(i,j) : i < j < n\} \to \{0,1,2\}$ define $\chi_\sigma(\overline{v}) = \left(\bigwedge_{\substack{1<j<n \\ \sigma(i,j)=0}} v_i = v_j\right) \wedge \left(\bigwedge_{\substack{1<j<n \\ \sigma(i,j)=1}} v_i < v_j\right) \wedge \left(\bigwedge_{\substack{1<j<n \\ \sigma(i,j)=2}} v_i > v_j\right)$
Observation: If $\overline{a}, \overline{b} \in \mathbb{Q}$, σ, and $\mathbb{Q} \models \chi_\sigma(\overline{a}), \chi_\sigma(\overline{b}) \to (\mathbb{Q}, \overline{a}) \cong (\mathbb{Q}, \overline{b}) \to \mathbb{Q}(a) \leftrightarrow \mathbb{Q}(b)$. Let $\Lambda_\phi = \{\sigma \mid \forall x(\chi_\sigma(\overline{x}) \to \phi(\overline{x}))\}$. Let $\psi(\overline{v}) = \bigvee_{\sigma \in \Lambda_\phi} \chi_\sigma(\overline{v})$. Claim: DLO $\models \forall v \phi(\overline{v}) \leftrightarrow \psi(v)$

Lemma 3.1.7. Suppose that for every quantifier free formula $\phi(\overline{v}, w)$ there is a quantifier free formula $\psi(\overline{v})$ such that $T \models \forall \overline{v}(\exists w \phi(\overline{v}, w) \leftrightarrow \psi(\overline{v}))$ then T has quantifier elimination.

Proof. Idea: We can construc the quantifier free formula by working from the inside out, eliminating one quantifier at a time.

Theorem 3.1.8. For T, an $\mathcal{L}$ theory, $\phi(\overline{v})$ a formula the following are equivalent

(i) There is a quantifier free formula $\varphi(\overline{v})$ such that $T \models \forall \overline{v}(\phi(\overline{v}) \leftrightarrow \varphi(\overline{v}))$

(ii) For every $\mathcal{M}, \mathcal{N}, \mathcal{A}, \overline{a} \in A$ with $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{M}, \mathcal{N}$ We have $\mathcal{M} \models \phi(\overline{a}) \leftrightarrow \mathcal{N} \models \phi(\overline{a})$

Proof. (i) → (ii): $\mathcal{M} \models \phi(\overline{a}) \leftrightarrow \mathcal{M} \models \psi(\overline{a}) \leftrightarrow \mathcal{A} \models \psi(\overline{a}) \iff \mathcal{N} \models \psi(\overline{a}) \leftrightarrow \mathcal{N} \models \phi(\overline{a})$

3.2 October 4

3.2.1 Quantifier Elimintation

Proof (Proof (Cont))**.** (ii) → (i): Define $\Gamma(\overline{v}) = \{\psi(\overline{v}) :$ quantifier free $T \models \forall \overline{v}(\phi(\overline{v}) \to \psi(\overline{v}))\}$. Let $\overline{d}$ be a tuple of new constants. Note $T \cup \phi(\overline{d}) \models \Gamma(\overline{d})$ by definition.
Claim: $T \cup \Gamma(\overline{d}) \models \phi(\overline{d})$
This would be enough to because by the compactness theorem there are $\psi_0, \ldots, \psi_k \in \Gamma$ such that $T \cup \{\psi_0(\overline{d}), \ldots, \psi_k(\overline{d})\} \models \phi(\overline{d})$ so $T \models \psi_0(\overline{d}) \wedge \ldots \wedge \psi_k(\overline{d}) \leftrightarrow \phi(\overline{d})$
Proof of claim: Suppose not, Let $\mathcal{M} \models T \cup \Gamma(\overline{d}) \cup \{\neg\phi(\overline{d})\}$, let $\mathcal{A} = \langle \overline{d} \rangle$ be the substructure of $\mathcal{M}$ generated by $\overline{d}$ (For every $a \in A$, there is a term $t(\overline{x})$ such that $a = t(\overline{d})$). We want to find $\mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{N}$ and $\mathcal{N} \models \phi(\overline{d})$ we need to show $\Sigma = T \cup \mathrm{Diag}(\mathcal{A}) \cup \phi(\overline{d})$ is satisfiable. If not there is a formula $\psi(a) \in \mathrm{Diag}(\mathcal{A})$ sich that $T \cup \psi(\overline{a}) \models \neg\phi(\overline{d})$. Let $\tilde{\psi}(\overline{d}) = \psi(\overline{t}(\overline{a}))$. $T \cup \tilde{\psi} \models \neg\phi(\overline{d})$ so $T \models \tilde{\psi}(\overline{d}) \to \neg\phi(\overline{d})$ so $T \models \phi(d) \to \neg\tilde{\psi}(d)$ so $T \models \forall \overline{v}(\phi(\overline{v}) \to \neg\tilde{\psi}(\overline{v}))$ so $\neg\tilde{\psi}(\overline{v}) \in \Gamma(\overline{v})$ so $\mathcal{M} \models \neg\psi(\overline{d})$ so $\mathcal{A} \models \neg\tilde{\psi}(\overline{d})$ contradicting $\tilde{\psi}(\overline{d}) \in \mathrm{Diag}(\mathcal{A})$
Now, let $\mathcal{N} \models \Sigma$, $\mathcal{A} \subseteq \mathcal{N}$, $\mathcal{N} \models \phi(\overline{d}), \mathcal{N} \models T$

Corollary 3.2.1. Let T be an $\mathcal{L}$ theory, suppose that for all quantifier free $\theta(\overline{v}, w)$, for all $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{N}, \mathcal{N}$, $\overline{a} \in A$, $\exists m \in M$ $\mathcal{M} \models \theta(\overline{a}, m) \leftrightarrow \exists n \in N$ $\mathcal{N} \models \theta(\overline{a}, n)$, then T has quantifier elimination.

DAG is the theory of torsion free divisble abelain groups. $\mathcal{L} = \{0, +\cdot\}$

Lemma 3.2.2. Let, $G, H \models$ DAG, nontrivial $G \subseteq H$. Let $\psi(\overline{v}, w)$ bea a quantifier free formula, $\overline{a} \in G$. If $H \models \exists w \psi(\overline{a}, w)$, there is $c \in G$ $H \models \psi(\overline{a}, c)$

Proof. $\psi(\overline{v}, w) = \bigvee_{i=1}^{r} \bigwedge_{j=1}^{n} \theta_{ij}(\overline{v}, w)$ where each θ_{ij} is atomic or the negation of an atomic. $H \models \exists w \psi(\overline{a}, w)$ so for some i, $H \models \exists w \psi_i(\overline{a}, w)$ so $\exists c \in G$, $H \models \psi_i(\overline{a}, c)$ where $\psi_i(v, w)$ is $\bigcap_{j=1}^{n} \theta_j(\overline{v}, w)$. θ_j is of the form $t_1(\overline{v}, w) = t_2(\overline{v}, w)$ or $t_1(\overline{v}, w) \neq t_2(\overline{v}, w)$, ie. $t_1(\overline{v}, w) - t_2(\overline{v}, w) = 0$ or $t_1(\overline{v}, w) - t_2(\overline{v}, w) \neq 0$. These can be rewritten as $t(\overline{v}, w) = 0$ or $t(\overline{v}, w) \neq 0$. These have the form $\sum n_i v_i + mw = 0$ or $\sum n_i v_i + mw \neq 0$ for $n_i, m \in \mathbb{Z}$. If it is of the first form $w = \frac{-\sum n_i a_i}{m} \in G$, given $m \neq 0$. If all formulas are of the second form, $w \neq \frac{-\sum n_i a_i}{m}$, then there is a witness since G is infinite and there are finitely many inequalities.

Definition 3.2.3. T is strongly minimal or o-minimal if for every $\mathcal{M} \models T$, every definable subset of $\mathcal{M}$ is either finite or cofinite.

We showed if $H \models$ DAG, that every set that is definable by quantifier free formulas are finite or cofinite. After proving DAG has quantifier elimination, we get that DAG is strongly minimal.

3.3 October 6

3.3.1 Quantifier Elimination

Lemma 3.3.1. Let G be a TFAG. There is DAG $H \supseteq G$ (called the divisble hull of G) such that for all DAG $H' \supseteq G$ there is an embedding $h : H \to H'$ with $h \restriction G = \text{id}$.

Proof. Let $H = \{(g,n) : g \in G, n \in \mathbb{N}\}/\sim$ where $(g,n) \sim (g',n') \leftrightarrow \underbrace{g + \cdots + g}_{n'} = \underbrace{g' + \cdots + g'}_{n}$ then $[(g,n)] +_H [(h,m)] = [(m \cdot g + n \cdot h, mn)]$. We need to show if $(g,n) \sim (g',n')$ $(h,m) \sim (h',m')$ then $(m \cdot g + n \cdot h, mn) \sim (m' \cdot g' + n' \cdot h', n'm')$
Let $i : G \to H$ by $i(g) = (g/1)$
Given DAG $H' \supseteq G$, we define $h : H \to H'$ by $[(g,n)] = \frac{g}{n}$ in H'

Theorem 3.3.2. DAG has quantifier elimination.

Proof. Let $\theta(\overline{v}, w)$ be quantifier free. $G \subseteq G_1, G_2, \overline{a} \in G$. We want to show $G_1 \models \exists\theta(\overline{a}, w) \leftrightarrow G_2 \models \exists\theta(\overline{a}, w)$
Observation: G is a TFAG
Let H be the hull of G. $G_1 \models \exists w\theta(\overline{a}, w) \leftrightarrow H \models \exists w\theta(\overline{a}, w) \leftrightarrow G_2 \models exw\theta(\overline{a}, w)$

Definition 3.3.3. Given T, let $T_\forall\{\varphi\forall\text{-sentence} : T \models \varphi\}$

Claim: $\text{DAG}_\forall$, $\text{ACF}_\forall$ = integral domains, $\text{RCF}_\forall$ = ordered integral domains

Lemma 3.3.4. $\mathcal{A} \models T_\forall \leftrightarrow \exists\mathcal{M} \supseteq \mathcal{A}, \mathcal{M} \models T$

Proof. $\to$) $\mathcal{M} \models T$, $\mathcal{A} \supseteq \mathcal{M}$ and $\varphi \in T_\forall$, $T \models \varphi \to \mathcal{M} \models \varphi \to \mathcal{A} \models \varphi$
$\leftarrow$) Given $\mathcal{A} \models T_\forall$. We want to show $T \cup \text{Diag}(\mathcal{A})$ is satisfiable. If $T \cup \text{Diag}(\mathcal{A})$ is not satisfiable is not satisfiable then there are $\psi_1(\overline{a}), \ldots, \psi_k(\overline{a})$ such that $T \models \neg(\psi_1(\overline{a}) \wedge \cdots \wedge \psi_k(\overline{a}))$ so $T \models \neg\psi(\overline{a})$ for $\psi(\overline{a}) \in \text{Diag}(\mathcal{A})$. Then $T \models \forall\overline{x}\neg\psi(\overline{x})$ so $\forall x\neg\psi(\overline{x}) \in T_\forall$ so $\mathcal{A} \models \forall\overline{x}\neg\psi(\overline{x})$ contradicting $\mathcal{A} \models \psi(\overline{a})$

Claim follows since $G \models$ TFAG $\leftrightarrow$ there is $H \supseteq G$, $H \models$ DAG $\leftrightarrow G \models \text{DAG}_\forall$

Definition 3.3.5. T has algebraically prime models if for every $\mathcal{A} \models T_\forall$ there is an $\mathcal{M} \supseteq \mathcal{A}, \mathcal{M} \models T$ such that if $\mathcal{M}' \supseteq \mathcal{A}$, $\mathcal{M}' \models T$ then there is an embedding $h : \mathcal{M} \to \mathcal{M}'$ with $h \restriction \mathcal{A} = \text{id}_\mathcal{A}$.

DAG has prime models. Also true for ACF, RCF.

Definition 3.3.6. $\mathcal{M} \supseteq \mathcal{N}$, we say that $\mathcal{M}$ is simply closed $\mathcal{N}$ ($\mathcal{M} \preceq_S \mathcal{N}$) if for every quantifier free formula $\theta(\overline{v}, w)$ and $\overline{a} \in M$ if $\mathcal{N} \models \exists w \theta(\overline{a}, w)$ then $\exists m \in M$, $\mathcal{N} \models \theta(\overline{a}, m)$

Theorem 3.3.7. For T an $\mathcal{L}$-theory if

(i) T has algebraically prime models

(ii) For all $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \subseteq \mathcal{N} \to \mathcal{M} \preceq_S \mathcal{N}$

Then T has quantifier elimination.