

MATH 104 Notes

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1 1/18/2022

1.1 Natural Numbers

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
- successor construction: 2 is the successor of 1, 3 is the successor of 2. So starting from 0 one can reach all natural numbers (for any given natural number, it can be reached from 0 in finitely many steps)
- Peano Axioms for natural Numbers (see Tao 1)
 - Mathematical Induction Property (Axiom 5): let n be a natural number and let $P(n)$ be a statement depending on n , if the following two conditions hold:
 - * $P(0)$ is true
 - * If $P(k)$ is true, then $P(k+1)$ is truethen $P(n)$ is true for all $n \in \mathbb{N}$
- operations allowed for $\mathbb{N} : +, \times$
 - if $n, m \in \mathbb{N}$, then $n + m \in \mathbb{N}$ and $n \times m \in \mathbb{N}$
 - $-, /$ are not always defined

1.2 Integers

- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- allowed operations: $+, -, \times$ (formally, \mathbb{Z} is a ring)

1.3 Rational Numbers

- $\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$
- We have all four operations $+, -, \cdot, /$
- \mathbb{Q} is now a field

Theorem 1.1 (Field Axioms(Ross 3)).

Addition:

- $a + (b + c) = (a + b) + c$ for all a, b, c
- $a + b = b + a$ for all a, b
- $a + 0 = a$ for all a
- For each a , there is an element $-a$ such that $a + (-a) = 0$

Multiplication:

- $a(bc) = (ab)c$ for all a, b, c
- $ab = ba$ for all a, b
- $a \cdot 1 = a$ for all a
- For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$

Distributive Law:

- $a(b + c) = ab + ac$ for all a, b, c

Theorem 1.2 (Useful Properties of Fields(Ross 3)).

- $a + c = b + c$ implies $a = b$
- $(-a)b = -ab$ for all a, b
- $(-a)(-b) = ab$ for all a, b
- $ac = bc$ and $c \neq 0$ imply $a = b$
- $ab = 0$ implies either $a = 0$ or $b = 0$

for $a, b, c \in \mathbb{Q}$

\mathbb{Q} is an ordered field, there is a “relation” \leq

Definition 1.3. A relation S is a subset of $\mathbb{Q} \times \mathbb{Q}$, if $(a, b) \in S$ we say “ a and b have relation S ” or “ aSb ”

The relation “ \leq ” has 3 properties:

- if $a \leq b$ and $b \leq a$, then $a = b$
- if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity)
- for any $a, b \in \mathbb{Q}$, at least one of the following is true: $a \leq b$ or $b \leq a$

Since \mathbb{Q} is an ordered field, the field structure $(+, -, \cdot, /)$ is compatible with (\leq)

- If $a \leq b$, then $a + c \leq b + c$ for all $c \in \mathbb{Q}$
- If $a \geq 0$ and $b \geq 0$, then $ab \geq 0$

Theorem 1.4 (Useful Properties of Ordered Fields(Ross 3)).

- If $a \leq b$, then $-b \leq -a$
- If $a \leq b$ and $c \geq 0$, then $ac \leq bc$
- If $a \leq b$ and $c \leq 0$, then $bc \leq ac$
- $0 \leq a^2$ for all a
- $0 < 1$
- If $0 < a$, then $0 < a^{-1}$
- If $0 < a < b$, then $0 < b^{-1} < a^{-1}$

for $a, b, c \in \mathbb{Q}$

1.4 What's lacking in \mathbb{Q} ?

1. There are certain gaps in \mathbb{Q} . For example, the equation $x^2 - 2$ cannot be solved in \mathbb{Q}
2. For a bounded set in \mathbb{Q} , E , it may not have a “most economical” or “sharpest” upper bound in \mathbb{Q}
Ex: $E = \{x \in \mathbb{Q} | x^2 < 2\}$ there is no least upper bound(sup) of E in \mathbb{Q} (we want to take $\sqrt{2}$ as $\sup(E)$ but $\sqrt{2}$ is not a rational number)

2 1/20/2022

2.1 Rational Zeros Theorem

Definition 2.1. An integer coefficient polynomial in x is of the form: $c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$ $c_1, \dots, c_n \in \mathbb{Z}$, $c_n \neq 0$.

1. A \mathbb{Z} -coefficient equation is $f(x) = 0$
2. One can ask: when does a \mathbb{Z} -coefficient equation have roots in \mathbb{Q}

Fact 2.2. A degree n polynomial has n roots in \mathbb{C} , ie. $\exists z_1, \dots, z_n \in \mathbb{C}$ such that $f(x) = c_n(x - z_1) \dots (x - z_n)$

Theorem 2.3. If a rational number r satisfies the equation $x^n + \dots + c_1 x + c_0 = 0$, with $c_i \in \mathbb{Z}$, $c_n, c_0 \neq 0$ and $r = \frac{c}{d}$ (where c and d are coprime integers). Then c divides c_0 and d divides c_n .

Proof. Plug in $x = \frac{c}{d}$ into the equation to get $c_n(\frac{c}{d})^n + c_{n-1}(\frac{c}{d})^{n-1} + \dots + c_1(\frac{c}{d}) + c_0 = 0$ multiply both sides by d^n to get $c_n c^n + c_{n-1} c^{n-1} d + \dots + c_1 c d^{n-1} + c_0 d^n = 0$ Since $c_n c^n = -d(c_{n-1} c^{n-1} + \dots + c_1 d^{n-1})$, d divides $c_n c^n$. Since d and c are coprimes, d does not divide c^n so d has to divide c_n
Also, since $c_0 d^n = -c(c_n c^{n-1} + c_{n-1} c^{n-2} d + \dots + c_1 d^{n-1})$ by similar reasoning $c | c_0$ \square

Using the rational zeros theorem, we can answer questions about rationality

Example 2.4. Show $\sqrt[3]{6}$ is irrational.

$\sqrt[3]{6}$ is rational $\leftrightarrow x^3 - 6$ has rational roots. The only possible rational roots such that $r = \frac{c}{d}$ need $c | 6, d | 1$. Taking $d = 1$, $c = \pm 1, \pm 2, \pm 3, \pm 6$. Once can check all of these do not satisfy the equation so there is no solution in \mathbb{Q}

2.2 Historical Construction of \mathbb{R} from \mathbb{Q}

1. Dedekind Cut: (\mathbb{Q} : if $\sqrt{2} \notin \mathbb{Q}$, how can we save the information of $\sqrt{2}$?)
A: the subset of \mathbb{Q} $C_{\sqrt{2}} = \{r \in \mathbb{Q} | r < \sqrt{2}\}$
For every $x \in \mathbb{R}$, consider $C_x = \{x \in \mathbb{Q} | r < x\}$. We can define addition, multiplication on the subsets C_x

2. Sequences in \mathbb{Q}

ie. Use a sequence of rational numbers to “approximate” a real number
eg. $\sqrt{2}$ can be approximated by $1, 1.4, 1.41, 1.414, \dots$

Problems:

- (a) Given any real number, how do you get such a sequence?
- (b) How do you determine if 2 different sequences approximate the same real number
(eg. $1 \leftarrow 1.1, 1.01, 1.001, \dots$ or $1 \leftarrow 0.9, 0.99, 0.999, \dots$ or $1 \leftarrow 1, 1, 1, \dots$) all have the same limit

2.3 Properties (Axioms) of \mathbb{R}

Given the existence of \mathbb{R} , we have certain properties (axioms) of \mathbb{R}

Definition 2.5. A subset of \mathbb{R} is said to be bounded above if $\exists a \in \mathbb{R}$ such that for any $x \in E$, we have $x \leq a$

Theorem 2.6 (Completeness Axiom of \mathbb{R}). Given a set $E \subset \mathbb{R}$, bounded above, there exists a unique r such that:

- 1. r is an upper bound of E
- 2. for any other upper bound of α , we have $r \leq \alpha$

r is called the least upper bound of E , $r = \sup E$

(ie. $\sup E$ is well defined for subsets that are bounded above)

Example 2.7. $\sup([0, 1]) = 1$, $\sup((0, 1)) = 1$, $\sup(\{r \in \mathbb{Q} | r^2 < 2\}) = \sqrt{2}$

Theorem 2.8 (Archimedean Property). For any $r \in \mathbb{R}$, $r > 0$ $\exists n \in \mathbb{N}$ such that $nr > 1$ or equivalently, $r > \frac{1}{n}$

2.4 $+\infty, -\infty$

- With these symbols, we can say $\sup(\mathbb{N}) = +\infty \leftrightarrow \mathbb{N}$ is not bounded above
- $+\infty, -\infty$ are not real numbers. They have part of the defined operations \mathbb{R} has
ie. $3 \cdot +\infty = +\infty$, $(-3) \cdot +\infty = -\infty$ but $(+\infty) + (-\infty) = \text{NAN}$, $0 \cdot (+\infty) = \text{undefined}$.

2.5 Sequences and Limits

- A sequence of real numbers is: a_0, a_1, a_2, \dots denoted $(a_n)_{n=0}^\infty$ or shortened (a_n)
- We care about the “eventual behavior” of a sequence

Definition 2.9. A sequence (a_n) converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $|a_n - a| < \varepsilon$.

3 1/25/2022

3.1 Sequences and Limits

Definition 3.1. A sequence (a_n) is bounded if $\exists M > 0, |a_n| \leq M$ for all n .

Theorem 3.2. Convergent sequences are bounded.

Proof. Let (a_n) be a convergent sequence that converges to a .

Let $\varepsilon = 1$, then by definition of convergence, there exists $N > 0$ such that $\forall n > N$

$$|a_n - a| < 1 \iff a - 1 < a_n < a + 1 \quad \forall n > N.$$

Let $M = \max\{|a_1|, |a_2|, \dots, |a_N|\}$, $M_1 = \max\{|a-1|, |a+1|\}$ and $M = \max\{M_1, M_2\}$. Thus if $n \leq N$ we have $|a_n| \leq M$, and if $n \geq N$ we have $|a_n| \leq M_2$ so

$$\forall n, |a_n| \leq \max\{M_1, M_2\} = M$$

□

Remark 3.3. One can deal with the first few terms of a sequence easily, it is the “tail of the sequence” that matters.

3.2 Operations on Convergent Sequences

Theorem 3.4. $c \in \mathbb{R}, \forall$ convergent sequences $a_n \rightarrow a$, we have $c \cdot a_n \rightarrow c \cdot a$.

Proof. If $c = 0$, the result is obvious.

If $c \neq 0$, we want to show for all $\varepsilon > 0, \exists N$ such that $\forall n > N$

$$|c \cdot a_n - c \cdot a| < \varepsilon \iff |c| \cdot |a_n - a| \leq \varepsilon \iff |a_n - a| \leq \frac{\varepsilon}{|c|}.$$

Now let $\varepsilon' = \frac{\varepsilon}{|c|}$. By definition of $a_n \rightarrow a$, we have $N > 0$ such that $|a_n - a| \leq \varepsilon' = \frac{\varepsilon}{|c|}$. This gives the desired N . □

Theorem 3.5. If $a_n \rightarrow a, b_n \rightarrow b$, then $a_n + b_n \rightarrow a + b$.

Proof. We want to show $\forall \varepsilon > 0, \exists N$ such that $\forall n > N$

$$|a_n + b_n - (a + b)| \leq \varepsilon \iff |(a_n - a) + (b_n - b)| \leq \varepsilon. \quad (*)$$

$|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ by the triangle inequality so

$$(*) \leftarrow |a_n - a| < \varepsilon \quad (**)$$

$$\leftarrow \begin{cases} |a_n - a| \leq \varepsilon/2 \\ |b_n - b| \leq \varepsilon/2 \end{cases} \quad (***)$$

By the convergence of a_n and b_n , $\exists N_1, N_2$ such that $\forall n > N_1, |a_n - a| \leq \frac{\varepsilon}{2}$, and $\forall n > N_2, |b_n - b| \leq \frac{\varepsilon}{2}$. Take $N = \max\{N_1, N_2\}$, then $\forall n > N$ $(***)$ is satisfied hence $(*)$ is satisfied. □

Corollary 3.6. If $a_n \rightarrow a$, $b_n \rightarrow b$, then $a_n - b_n \rightarrow a - b$.

Proof. Let $c_n = (-1) \cdot b_n$. Then $c_n \rightarrow -b$ so $a_n + c_n \rightarrow a - b$. \square

Theorem 3.7. If $a_n \rightarrow a$, $b_n \rightarrow b$, then $a_n \cdot b_n \rightarrow ab$.

Proof. Want to show: $\forall \varepsilon > 0, \exists N$ such that $\forall n > N$

$$|a_n - ab| \leq \varepsilon. \quad (*)$$

Since a_n is convergent, it is bounded by some $M > 0$ which yields the following inequalities.

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b - b) + a_n b - ab| \\ &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n(b_n - b)| + |(a_n - a)b| \\ &\leq |a_n| \cdot |b_n - b| + |a_n - a| \cdot |b| \\ &\leq M|b_n - b| + |b||a_n - a| \end{aligned}$$

So

$$(*) \leftarrow \begin{cases} M|b_n - b| \leq \varepsilon/2 \\ |b||a_n - a| \leq \varepsilon/2 \end{cases}. \quad (**)$$

Since $a_n \rightarrow a$, let $\varepsilon_1 = \frac{\varepsilon}{2|b|}$, then $\exists N$ such that $\forall n > N$,

$$|a_n - a| < \varepsilon_1 \iff |b||a_n - a| \leq \frac{\varepsilon}{2}.$$

Also, since $b_n \rightarrow b$, let $\varepsilon_2 = \frac{\varepsilon}{2M}$, then $\exists N$ such that $\forall n > N$,

$$|b_n - b| \leq \varepsilon_2 \iff M|b_n - b| \leq \frac{\varepsilon}{2}.$$

. Let $N = \max\{N_1, N_2\}$, then for $n > N$, $(**)$ holds so $(*)$ holds. \square

Theorem 3.8. If $a_n \rightarrow a$, and $a_n \neq 0 \forall n$ and $a \neq 0$, then $\frac{1}{a_n} \rightarrow \frac{1}{a}$.

Remark 3.9. $a_n \neq 0$ does not imply $a \neq 0$. For example consider the sequence $a_n = \frac{1}{n}$

Proof. Want to show $\forall \varepsilon > 0, \exists N$ such that $\forall n > N$,

$$\left| \frac{1}{a} - \frac{1}{a_n} \right| \leq \varepsilon. \quad (*)$$

Observe that

$$\left| \frac{1}{a} - \frac{1}{a_n} \right| = \left| \frac{a - a_n}{a \cdot a_n} \right| = \frac{|a_n - a|}{|a| \cdot |a_n|}.$$

Claim: $\exists c > 0$ such that $|a_n| > c \forall n$.

Proof. Let $\varepsilon' = \frac{\varepsilon}{2}$, then $\exists N'$ such that $\forall n \geq N'$

$$\begin{aligned} |a_n - a| \leq \varepsilon' = \frac{\varepsilon}{2} &\iff -|a|/2 < a_n - a < |a|/2 \\ &\iff a + \frac{|a|}{2} < a_n < a + \frac{|a|}{2} \rightarrow |a_n| \geq \frac{|a|}{2} \end{aligned}$$

Let $c_1 = \min\{|a_1|, |a_2|, \dots, |a_{N'}|\} \geq 0$. Let $c = \min\{c_1, |a|/2\}$. \square

Thus, $\frac{|a_n - a|}{|a| \cdot |a_n|} \leq \frac{|a_n - a|}{|a| \cdot c}$. Hence

$$(*) \leftarrow \frac{|a_n \cdot a|}{|a| \cdot c} \leq \varepsilon \quad (**)$$

and $(**)$ can be satisfied since $a_n \rightarrow a$. \square

Corollary 3.10. If $a_n \rightarrow a$, $b_n \rightarrow b$ and $b_n \neq 0$, $b \neq 0$, then $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$.

Proof. $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$. Since by Thm 8, $\frac{1}{b_n} \rightarrow \frac{1}{b}$, $a_n \cdot \frac{a}{b_n} \rightarrow a \cdot \frac{1}{b}$ by Thm 7. \square

Theorem 3.11 (Useful Results).

- (1) $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \ \forall p > 0$.
- (2) $\lim_{n \rightarrow \infty} a^n = 0 \ \forall |a| < 1$.
- (3) $\lim_{n \rightarrow \infty} n^{1/n} = 1$.
- (4) $\lim_{n \rightarrow \infty} a^{1/n} = 1$ for all $n > 0$.

Proof of (3). Let $S_n = n^{1/n} - 1$, then $s_n \geq 0 \ \forall n$ positive integers.

$$1 + s_n = n^{1/n} \iff (1 + s_n)^n = n.$$

Using to binomial theorem we see

$$\begin{aligned} 1 + ns_n + \frac{n(n-1)}{2} s_n^2 + \dots &= n \\ \rightarrow \frac{n(n-1)}{2} s_n^2 &\leq n \\ \rightarrow s_n^2 &\leq \frac{2}{n-1} \end{aligned}$$

Thus, $s_n \rightarrow 0$ as $n \rightarrow \infty$. \square

4 1/27/2022

4.1 Monotone Sequences

Definition 4.1 ($\lim s_n = +\infty$). A sequence (s_n) is said to “diverge to $+\infty$ ”, if for every $M \in \mathbb{R}$ there exists N such that $s_n > M \forall n > N$.

Definition 4.2 (Values of a Sequence). If $(s_n)_{n=1}^{\infty}$ is a sequence, then $\{s_n\}_{n=1}^{\infty}$, the subset of \mathbb{R} consisting of the values of (s_n) , is called the value set.

Example 4.3.

- $(s_n) = 1, 2, 1, 2, \dots \quad \{s_n\}_{n=1}^{\infty} = \{1, 2\}$
- $(s_n) = 1, 1, 2, 2, 1, 1, 2, 2, \dots \quad \{s_n\}_{n=1}^{\infty} = \{1, 2\}$
- $(s_n) = 1, 2, 3, 4, \dots \quad \{s_n\}_{n=1}^{\infty} = \{1, 2, 3, 4, \dots\}$

Definition 4.4 (Monotone Sequences).

- A sequence (s_n) is monotonically increasing if $a_{n+1} \geq a_n \forall n$
- A sequence (s_n) is monotonically decreasing if $a_{n+1} \leq a_n \forall n$

Example 4.5.

- $(a_n) = a$, a constant sequence is monotonically increasing and decreasing
- $(a_n) = 1, 2, 3, \dots$, is increasing
- $(a_n) = -\frac{1}{n}$, is increasing and bounded above (also below)

Theorem 4.6. A bounded monotone sequence is convergent.

Proof. (We will show for increasing, the proof for decreasing is similar.)
Let (a_n) be a bounded monotone increasing sequence and let $\gamma = \sup\{a_n\}_{n=1}^{\infty}$ ($= \sup a_n$). Then $a_n \leq \gamma \forall n$ and for any $\varepsilon > 0$, $\exists a_{n_0}$ such that $a_{n_0} > \gamma - \varepsilon$. Thus for every $\varepsilon > 0$, let $N = n_0$ (as defined above), then for every $n > N$, we have $\gamma - \varepsilon < a_{n_0} \leq a_n \leq \gamma$ thus $|a_n - \gamma| < \varepsilon$ then $\lim a_n = \gamma$ \square

Example 4.7 (Recursive Definition of Sequences). Let s_n be any positive number and let

$$s_{n+1} = \frac{s_n^2 + 5}{2s_n} \quad \forall n \geq 1. \quad (*)$$

We want to show $\lim s_n$ exists and find it.

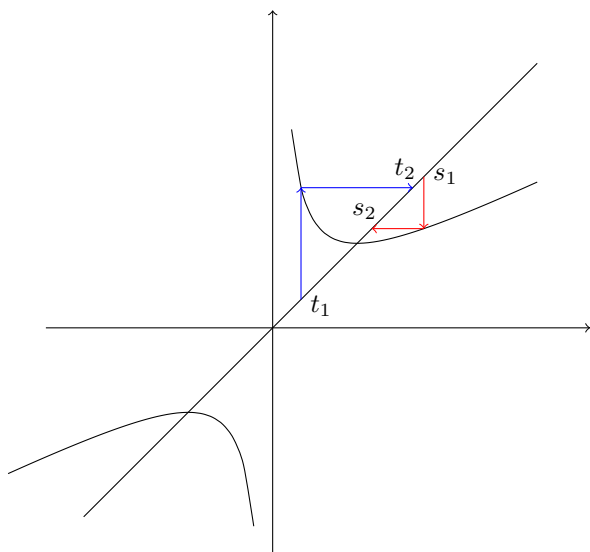
Remark 4.8. If we assume $\lim s_n$ exists, call it s , then s satisfies

$$s = \frac{s^2 + 5}{2s} \quad (**)$$

since we can apply $\lim_{n \rightarrow \infty}$ to both sides.

(**) $\rightarrow 2s^2 = s^2 + 5 \rightarrow s = \pm\sqrt{5}$. Since s_n is a positive sequence $\lim s_n$ can only be ≥ 0 , thus s can only be $\sqrt{5}$

- To show $\lim s_n$ exists, we can only need to show s_n is bounded and monotone
- Here is a trick: let $f(x) = \frac{x^2+5}{2x}$, then $s_{n+1} = f(s_n)$
 - Consider the graph of f , ie. $y = f(x)$
 - Consider the diagonal, ie. $y = x$



- If $s_1 > \sqrt{5}$, we should try to prove $\sqrt{5} < \dots s_3 < s_2 < s_1$
- If $0 < s_1 < \sqrt{5}$, then we show that $s_2 > \sqrt{5}$, we can consider $(s_n)_{n=1}^{\infty}$, which reduces to case 1
- If (s_n) is unbounded and increasing, then $\lim s_n = +\infty$
- If (s_n) is unbounded and decreasing, then $\lim s_n = -\infty$

4.2 Lim inf and sup of a sequence

Definition 4.9 (limsup). Let $(s_n)_{n=1}^{\infty}$ be a sequence,

$$\limsup_{n \rightarrow \infty} s_n := \lim_{n \rightarrow \infty} (\sup\{s_n\}_{m=1}^{\infty})$$

- $(s_n)_{n=N}^{\infty}$ is called a “tail of the sequence (s_n) ” starting at N
- $A_N = \sup\{s_n\}_{n=N}^{\infty} = \sup_{n \geq N} s_n$

- $\limsup s_n = \lim A_n = +\infty$

Example 4.10.

- (1) $(s_n) = 1, 2, 3, 4, 5, \dots$
 $A_1 = \sup_{n \geq 1} s_n = +\infty$, $A_2 = \sup_{n \geq 2} s_n = +\infty$
 $\limsup s_n = \lim A_n = +\infty$
- (2) $(s_n) = 1 - \frac{1}{n}$
 $A_1 = \sup_{n \geq 1} s_n = 1$, $A_2 = \sup_{n \geq 2} s_n = 1$
 $\limsup s_n = \lim A_n = 1$ (for any monotonic increasing sequence $\limsup s_n = \sup s_1 = A_1$)
- (3) $s_n = 1 + \frac{1}{n}$ $(s_n) = 2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \dots$
 $A_1 = \sup\{2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \dots\} = 2$
 $A_2 = \sup\{1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \dots\} = 1 + \frac{1}{2}$
 $A_n = s_n$ so $\limsup s_n = \lim(1 + \frac{1}{n}) = 1$

Lemma 4.11. $A_n = \sup_{m \geq n} s_m$ forms a decreasing sequence.

Proof. Since $\{s_n\}_{m=n}^\infty \supset \{s_n\}_{m=n+1}^\infty$, $\sup\{s_n\}_{m=n}^\infty \geq \sup\{s_m\}_{m=n+1}^\infty$, ie. $A_n \geq A_{n+1}$ \square

Corollary 4.12. $\lim_{n \rightarrow \infty} A_n = \inf_{n \geq 1} A_n (= \inf_n A_n)$

Example 4.13. $s_n = (-1)^n \cdot \frac{1}{n}$ $(s_n) = (-1, \frac{1}{2}, -\frac{1}{3}, \dots)$
 $A_1 = \sup_{n \geq 1} s_n = s_2 = \frac{1}{2}$, $A_2 = \frac{1}{2}$, $A_3 = \frac{1}{4}$, so
 $(A_n) = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \dots$ $\limsup s_n = \lim A_n = 0$
 A_n is like the “upper envelope.”

5 2/1/2022

5.1 Cauchy Sequences

Definition 5.1 (Cauchy Sequence). A sequence (a_n) is cauchy if $\forall \varepsilon > 0$, $\exists N > 0$, such that $\forall n, m > N$ we have $|a_n - a_m| < \varepsilon$.

Lemma 5.2. If (a_n) converges to a , then (a_n) is cauchy.

Proof. Let $\varepsilon_1 = \frac{\varepsilon}{2}$, then since $a_n \rightarrow a$, $\exists N_1 > 0$ such that $\forall n, m < N$, $|a_n - a| < \varepsilon_1$ and $|a_m - a| < \varepsilon_1$. Thus,

$$|a_n - a_m| = |(a_n - a) - (a_m - a)| \leq |a_n - a| + |a_m - a| < \varepsilon_1 + \varepsilon_1 = \varepsilon.$$

\square

Remark 5.3. This is also for true in \mathbb{Q}

Lemma 5.4 (Squeeze Lemma). Given sequences $(A_n), (B_n), (a_n)$ such that $A_n \geq a_n \geq B_n \forall n$, if $A_n \rightarrow a$, $B_n \rightarrow a$, then $a_n \rightarrow a$.

Proof. $\forall \varepsilon > 0$, we have $N > 0$ such that $\forall n > N$, $|A_n - a| < \varepsilon$ and $|B_n - a| < \varepsilon$. Then $a_n \leq A_n < a + \varepsilon$ and $a_n \geq B_n > a - \varepsilon$ so

$$a - \varepsilon < a_n < a + \varepsilon \leftrightarrow |a_n - a| < \varepsilon.$$

□

Lemma 5.5. Cauchy Sequences are bounded.

Proof. Let $\varepsilon = 1$. Then $\exists N > 0$ such that $\forall n, m > N$, $|s_n - s_m| < \varepsilon$. Consider the term s_{N+1} . Observe that $\forall n < N$, $|s_{N+1} - s_n| < 1$ so $\forall n < N$, $|s_n| < s_{N+1} + 1$. Taking $M = \max\{|s_1|, |s_2|, \dots, |s_{N+1}|, |s_{N+1}| + 1\}$, we see that $M \geq |s_n|$ for all n . □

Theorem 5.6. If (a_n) is cauchy in \mathbb{R} , then (a_n) is convergent.

Proof. Since (a_n) is cauchy, (a_n) is bounded so $\limsup a_n$ and $\liminf a_n$ exist. Let $A_n = \sup_{m \geq n} a_m$, $B_n = \inf_{m \geq n} a_m$, then $A_n \geq a_n \geq B_n$. Let $A = \lim A_n$ and $B = \lim B_n$. By the Squeeze Lemma, we only need to show $A = B$. Since $A_n \geq B_n$, we know $A \geq B$, hence we only have to rule out $A < B$.

Assume $A < B$. Let $\varepsilon = \frac{(A-B)}{3}$. By Cauchy criterion $\exists N > 0$ such that $\forall n, m > N$, $|a_n - a_m| < \varepsilon$. By the previous lemma, since $A = \limsup a_n$ and $B = \liminf a_n$, given ε, N above, we have $n > N$ such that $|a_n - A| < \varepsilon$ and $m > N$ such that $|a_m - B| \leq \varepsilon$. Then

$$|A - B| \leq |A - a_n| + |a_n - a_m| + |a_m - B| < \varepsilon + \varepsilon + \varepsilon = A - B = |A - B|,$$

which is a contradiction. □

5.2 Subsequences

Let (a_n) be a sequence. If we pick an infinite subset of \mathbb{N} , $n_1 < n_2 < n_3 < \dots$, then we can have a new sequence $b_k = a_{n_k}$, $(b_k) = a_{n_1}, a_{n_2}, a_{n_3}, \dots$

Example 5.7. For $(a_n) = (-1)^n$, $a_1 = -1, a_2 = +1, \dots$ does not converge but subsequence consisting of odd terms converges to -1 and subsequence consisting of even terms converges to 1 .

Definition 5.8. Let (a_n) be a sequence. Then $a \in \mathbb{R}$ is a subsequential limit if there exists (a_{n_k}) such that $\lim_{k \rightarrow \infty} a_k = a$.

Theorem 5.9. Let (a_n) be a sequence. Then:

- (1) a is a subsequential limit of (a_n)
- (2) $\leftrightarrow \forall \varepsilon > 0, \forall N > 0, \exists n > N$ such that $|a_n - a| \leq \varepsilon$
- (3) $\leftrightarrow \forall \varepsilon > 0$, the set $A_\varepsilon = \{n \mid |a_n - a| < \varepsilon\}$ is infinite

Proof. $2 \leftrightarrow 3$) follows from definitions.

$1 \rightarrow 3$) If $a_{n_k} \rightarrow a$, then for a given $\varepsilon > 0$, $\exists K > 0$ such that $|a_{n_k} - a| \leq \varepsilon$.

Thus $\{n_k | k > K\} \subset A_\varepsilon$. So A_ε is infinite.

$3 \rightarrow 1$) Cantor's Diagonal Trick: Let $A_{\frac{1}{k}} = \{n | |a_n - a| \leq \frac{1}{k}\}$.

$A_1 : n_{1,1} < n_{1,2} < n_{1,3} < \dots$

$A_2 : n_{2,1} < n_{2,2} < n_{2,3} < \dots$

Observe that $A_{\frac{1}{k+1}} \subset A_{\frac{1}{k}}$, thus $n_{k,i} \leq n_{k+1,i}$.

Claim: $(a_{n_{k,k}}) \rightarrow a$.

First observe that this is a valid subsequence since $a_{n_{k,k}} < a_{n_{k,k+1}} \leq a_{n_{k+1,k+1}}$ for all k . Also for $\varepsilon > 0$, $\exists K$ such that $\frac{1}{K} < \varepsilon$ so for all $k > K$, $|a_n - a| < \frac{1}{K} < \varepsilon$ so it converges to a . \square