MATH 110 Notes

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Contents

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1.1 Overview

Linear Algebra is about:

- Vector spaces (also called linear spaces)
- Linear maps
- Extra structures on spaces/maps (inner products, etc.)

Motivation:

- Physics - we live in a 3D space
- Geometry - even for a curved object, locally it looks like a vector space (tangent space)
- Representation Theory and Group Theory - the set of all matrices has a rich structure and interesting subsets (subgroups)
- Solving Differential Equations - natural tool and solution spaces
- Normal Operators - guaranteed good bases
- Statistics - square matrices, ...
- Applied Math - designing of algorithms, ...

1.2 Ch1 - Vector Spaces

$\mathbb{R}$- set of reals, $\mathbb{R}^2$ - plane, $\mathbb{R}^3$ - 3D space

Key feature: Have addition and scalar multiplication by $\mathbb{R}$

Generalizations: Vector spaces over $\mathbb{R}$ (or a general $\mathbb{F}$)

1.3 1.A: $\mathbb{R}^n$ and $\mathbb{C}^n$

Definition 1.1 ($\mathbb{C}$).
Introduced i such that $i^2 + 1 = 0$
$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$
Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
eg: $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)^2 = 0 + i \cdot i = i$
$\mathbb{R} \subset \mathbb{C}$: view x as $x + 0i$

Theorem 1.2 (Properties of $\mathbb{C}$)**.**
Commutativity: $\alpha+\beta=\beta+\alpha, \quad \alpha\beta=\beta\alpha \quad \forall\alpha,\beta\in\mathbb{C}$
Associativity: $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma), \quad (\alpha\beta)\gamma=\alpha(\beta\gamma) \quad \forall\alpha,\beta,\gamma\in\mathbb{C}$
Additive Identity: $\alpha+0=\alpha \quad \forall\alpha\in\mathbb{C}$
Additive Inverse: $\forall\alpha\in\mathbb{C}$, $\exists!\beta\in\mathbb{C}$ such that $\alpha+\beta=0$
Multiplicative Identity: $\alpha\cdot 1=\alpha \quad \forall\alpha\in\mathbb{C}$
Multiplicative Inverse: $\forall\alpha\neq 0\in\mathbb{C}$ $\exists!\beta\in\mathbb{C}$ such that $\alpha\beta=1$
Distributive Properties: $\lambda(\alpha+\beta)=\lambda\alpha+\lambda+\beta \quad \forall\lambda,\alpha,\beta\in\mathbb{C}$

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2.1 1.A: $\mathbb{R}^n$ and $\mathbb{C}^n$

Example 2.1. Show existence and uniqueness of the multiplicative inverse of $\forall a\neq 0$
Idea: Assume $\alpha=a+bi$ want $(a+bi)(?+?i)=1\rightarrow ?+?i=\frac{1}{a+bi}$ "$=$" $\frac{a-bi}{(a+bi)(a-bi)}=\frac{1-bi}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$

Proof. Assume $\alpha=a+bi$, $a,b\in\mathbb{R}$, not both zero. We see that $\beta=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$ satisfies $(a+bi)(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i)=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=1$. Similarly, $(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i)(a+bi)=1$. $\rightarrow$ existence
Moreover, if there exists $\tilde{\beta}$ such that $\alpha\tilde{\beta}=1$, then $\beta=\beta\alpha\tilde{\beta}=\tilde{\beta}$. $\rightarrow$ uniqueness ∎

Definition 2.2.

- For $\alpha\in\mathbb{C}$, let $-\alpha\in\mathbb{C}$ define the unique element such that $\alpha+(-\alpha)=0$
- For $\alpha\in\mathbb{C}$, let $1/\alpha\in\mathbb{C}$ define the unique element such that $\alpha(1/\alpha)=1$
- Subtraction: $\alpha-\beta=\alpha+(-\beta)$
- Division: $\beta/\alpha=\beta\cdot(1/\alpha)$, $\alpha\neq 0$

$\mathbb{F}$: field(In the book, $\mathbb{R}$ or $\mathbb{C}$)

- In general, generalization of $\mathbb{R}$ or $\mathbb{C}$

Definition 2.3. A set $\mathbb{F}$(with addition "+" and multiplication "$\times$") is a field if:

(i) $\exists 0,1\in\mathbb{F}$, $0\neq 1$

(ii) $+:\mathbb{F}\times\mathbb{F}\rightarrow\mathbb{F}$ by $(a,b)\mapsto a+b$

(iii) $\times:\mathbb{F}\times\mathbb{F}\rightarrow\mathbb{F}$ by $(a,b)\mapsto a\cdot b$

Satisfying:

(a) Commutivity: $a + b = b + a$, $ab = ba$

(b) Associativity: $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$

(c) Inverses: $\forall a$, $\exists -a$ such that $a + (-a) = 0$
$\forall a$, $\exists 1/a$ such that $a \cdot (1/a) = 1$

(d) Distributive: $c(a + b) = ca + cb$

Example 2.4.

1. $\mathbb{R}, \mathbb{C}$
2. $\{0, 1\} \quad +, \times \bmod 2$
3. $\mathbb{F}_p = \{0, \ldots, p - 1\} \quad +, \times \bmod p$, p prime
4. $\mathbb{Q}$: rationals
5. $\{a + b\sqrt{2} : a, b, \in \mathbb{Q}$
6. $\{P(x)/Q(x) : P, Q$ are polynomials over $\mathbb{F}$, $Q \neq 0\}$

We will define $\cdot$ for $\mathbb{F}$. Elements of $\mathbb{F}$ are known as scalars (as opposed to vectors)

Definition 2.5. An n-tuple of elements of $\mathbb{F}$ is $(x_1, \ldots, x_n)$ where each $x_i \in \mathbb{F}$

Definition 2.6. $\mathbb{F}^n = \{$all n-tuples of elements in $\mathbb{F}\}$

Definition 2.7.

- Addition "+": $\mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ by $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto (x_1 + y_1, \ldots, x_n + y_n)$
- Scalar Multiplication "·": $\mathbb{F} \times \mathbb{F}^n$ by $(\lambda, (x_1, \ldots, x_n)) \mapsto (\lambda x_1, \ldots, \lambda x_n)$

Theorem 2.8 (Properties of $\mathbb{F}^n$)**.**

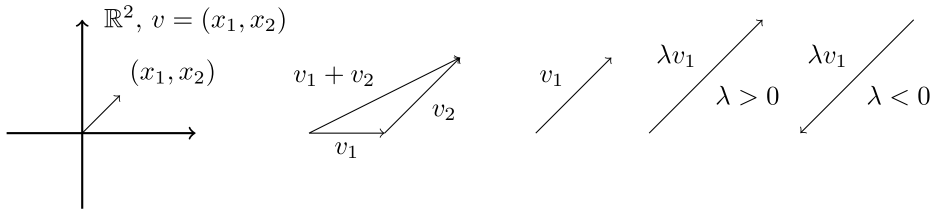
- Addition is commutative: $v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in \mathbb{F}^n$

  Proof. Assume $v_1 = (x_1, \ldots, x_n)$, $v_2 = (y_1, \ldots, y_n)$ then
  $v_1 + v_2 = (x_1 + y_1, \ldots, x_n + y_n) = (y_1 + x_1, \ldots, y_n + x_n) = v_2 + v_1$ □

- Existence of $0 \in \mathbb{F}^n$: Denote $0 = (0, \ldots, 0)$. Then $v + 0 = v \ \forall v \in \mathbb{F}^n$
- Additive Inverse: $\forall v \in \mathbb{F}^n$, $\exists!(-v) \in \mathbb{F}^n$ such that $v + (-v) = 0$

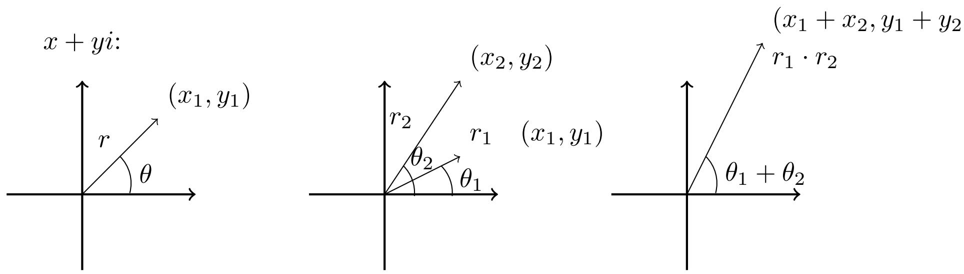
Geometric Meaning for $\mathbb{F} = \mathbb{R}$

Descartes Coordinate System:



Geometric Meaning of Multiplication on $\mathbb{C}$

$x + yi$:



2.2 1B - Vector Spaces

Definition 2.9. Fix a field $\mathbb{F}$. A vector space over $\mathbb{F}$ is a set V with addition "+" and scalar multiplication "$\cdot$ " denoted as $+ : V \times V \to V$ by $(v_1, v_2) \mapsto v_1 + v_2$, $\cdot : \mathbb{F} \times V \to V$ by $(\lambda, v) \mapsto \lambda v$
Satisfies:

(1) $u + v = v + u$, $\forall u, v \in V$

(2) $(u + v) + w = u + (v + w)$, $a(bv) = (ab)v$ $\forall u, v \in \mathbb{V}, a, b \in \mathbb{F}$

(3) $\exists 0 \in \mathbb{V}$ such that $v + 0 = v$, $\forall v \in V$

(4) $\forall v \in V$, $\exists w \in V$ such that $v + w = 0$. (we will show w is unique and denote it as $-v$)

(5) $1 \cdot v = v$, $\forall v \in V$

(6) $a(u + v) = au + av$, $(a + b)v = av + bv$, $\forall a, b \in \mathbb{F}, u, v \in V$

Definition 2.10. Elements in a vector space V are called points or vectors

Definition 2.11. A vector space over $\mathbb{F}(/\mathbb{F})$ is also called an $\mathbb{F}$-vector space

Example 2.12.

(1) $\mathbb{R}^n, \mathbb{C}^n, \mathbb{F}^n$ are all vector spaces

(2) $\mathbb{C}$ is a vector space over $\mathbb{R}$

(3) Let S be a set. Define F^s = the set of all functions from S to $\mathbb{F}$. $\mathbb{F}^S$ is a vector space $/\mathbb{F}$ under the operations $(f+g)(s) = f(s)+g(s)$, $(\lambda f)(s) = \lambda \cdot f(s)$. Each element has additive inverse $(-f)(s) = -f(s)$
$\mathbb{F}^\infty = \mathbb{F}^{\{1,2,3,\dots,\}}$, consists of $(a_1, a_2, a_3, \dots)$ $\forall a_n \in \mathbb{F}$

(4) the set of all sequences of real numbers that converge to 0

(5) the set of all polynomials over $\mathbb{F}$, with deg $\leq n$ in k variables is a vector space $/\mathbb{F}$

Theorem 2.13. A vector space V has a unique additive identity

Proof. Assume 0 and $0'$ are both additive inverses. Then $0 = 0 + 0' = 0'$ □

Theorem 2.14. $\forall v \in V$ has a unique additive inverse.

Proof. If w_1, w_2 are both additive inverses of v, then $w_1 = w_1 + (v + w_2) = (w_1 + v) + w_2 = w_2$ □

Definition 2.15. Let $w - v = w + (-v)$

Notation 2.16. V will be used to denote a vector space over $\mathbb{F}$

Theorem 2.17. $0 \cdot v = 0$, $\forall v \in V$

Proof. $0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v$ so by the existence of additive inverses $0 = 0 \cdot v$ □

Theorem 2.18. $a \cdot 0 = 0$, $\forall a \in \mathbb{F}$

Proof. $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$ so $0 = a \cdot 0$ □

Theorem 2.19. $(-1) \cdot v = -v$, $\forall v \in V$

Proof. $0 = 0v = (1+(-1))v = 1v + (-1)v = v + (-1)v$ so by def $(-1)v = -v$ □

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3.1 1.C - Subspaces

Definition 3.1. Assuming V is a vector space $/\mathbb{F}$. $U \subset V$ is called a subspace of V if U is also a vector space $/\mathbb{F}$ under $+$ and $\cdot$ in V.

Example 3.2. $\{(x_1, x_2, 0) : x_1, x_2, \in \mathbb{F}\}$ is a subspace of $\mathbb{F}^3$

Proposition 3.3. U is a subspace iff

(i) $0 \in V$

(ii) $u_1, u_2 \in U \to u_1 + u_2 \in U$

(iii) $a \in \mathbb{F}$, $u \in U \to a \cdot u \in U$

Proof. →) Suppose conditions hold. Then properties of $+$, $\cdot$ follow from V, U has identity by (i) and additive inverses by (iii). Finally, $+$, $\cdot$ well defined by (ii), (iii) so U is a subspace.
←) Suppose U is a subspace. Then U is nonempty so $0 \cdot u = 0 \in U$ so (i) holds. Also, $+$, $\cdot$ well defined so (ii), (iii) hold. □

Example 3.4.

(a) $\{0\}$ is a subspace

(b) $\{(x_1, x_2, x_3, x_3) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$ is a subspace iff $b = 0$

(c) $C[0,1] = \{\text{continuous real valued functions on } [0,1]\}$ is a subspace of $\mathbb{R}^{[0,1]}$ (over $\mathbb{R}$)

(d) $C^\infty[0,1] = \{\text{smooth real-valued functions on } [0,1]\}$ is a subspace $\mathbb{R}^{[0,1]}$

(e) The real sequences that converge to zero form a subspace of $\mathbb{R}^\infty$

(f) The only subspaces of $\mathbb{F}^1$ are $\{0\}$ and $\mathbb{F}$ (over $\mathbb{F}$)

(g) If U is a subspace of V, W is a subspace of U, then W is a subspace of V

(h) We will show the only subspace of $\mathbb{R}^3$ are $\{0\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$

Definition 3.5. For $U_1, \ldots, U_n$ subspaces of V, define the sum

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

also denoted as $\sum_{j=1}^m U_j$.

Example 3.6. In $\mathbb{F}^3$, what is $\{(x, x, 0)\} + \{(0, y, y)\}$?

Proof. $\{(x, y, z) : y = x + z\}$ □

Theorem 3.7. For subspaces $U_1, \ldots, U_m \subset V$, $\sum_{j=1}^m U_j$ is a subspace. Moreover, it is the smallest subspace containing $U_1, \ldots, U_n$ in the sense that if W contains $U_1, \ldots, U_n$, then $W \supset U$.

Proof. Subspace: (i) $0 \in U_i$ for $i = 1, \ldots, n$ so $0 = 0 + \cdots + 0 \in W$
(ii)/(iii): follow from closedness of each U_j
Containing $U_1, \ldots, U_n$: Consider the sum $0 + \cdots + 0 + u_j + 0 + \cdots + 0$ for $j = 1, \ldots, m$
Smallest Subspace: Suppose W contains $U_1, \ldots, U_m$ then W contains $u_1, \ldots, u_m$ $\forall u_j \in U_j$ so $u_1 + \cdots + u_m \in W$. □

3.2 Direct Sums

Definition 3.8. If $U_1, \dots, U_m$ are subspaces of V then the sum $U_1 + \cdots + U_m$ is a direct sum if each element in $U_1 + \cdots + U_m$ can be written as $u_1 + \cdots + u_m$ in a unique way with $u_j \in U_j$. In this case, we also use $U_1 \oplus \cdots \oplus U_m$ to denote $U_1 + \cdots + U_m$.

Example 3.9.

(1) If $U_1 = \{(x_1, x_2, 0) x_1, x_2 \in \mathbb{F}\}$, $U_2 = \{(0, 0, x_3) x_3 \in \mathbb{F}\}$, then $\mathbb{F}^3 = U_1 \oplus +U_2$.

(2) Let $U = \{(x, x, \dots) \in \mathbb{R}^\infty$, $V = \{(x_1, x_2, \dots) \in \mathbb{R}^\infty : \sum |x_n| < \infty, \sum x_n = 0\}$. Then $U + V$ is a direct sum.
(ex): Prove $U + V \neq \mathbb{R}^\infty$

Theorem 3.10. $U_1 + \cdots + U_m$ is a direct sum iff $\exists!$ way to write 0 as a sum of $u_1 + \cdots + u_m$, $\forall u_j \in U_j$ (which is $0 = 0 + \cdots + 0$).

Proof. $\to$) by def
$\leftarrow$) For $u \in U_1 + \cdots + U_m$, assume $u = u_1 + \cdots + u_m = \tilde{u_1} + \cdots + \tilde{u_n}$, $u_j, \tilde{u_j} \in U_j$. Then $(u_1 - \tilde{u_1}) + (u_2 - \tilde{u_2}) + \cdots + (u_m - \tilde{u_m}) = 0$. Hence $u_1 - \tilde{u_1} = u_2 - \tilde{u_2} = \cdots = 0$. Thus there is only one way to write u as $\sum_{j=1}^m$, $\forall u_j \in U_j$. ☐

Theorem 3.11. For subspaces $U_1, U_2 \in V$, $U_1 + U_2$ is a direct sum iff $U_1 \cap U_2 = \{0\}$.

Proof. $\to$) If $v \in U_1 \cap U_2$, $\underbrace{v}_{\in U_1} + \underbrace{(-v)}_{\in U_2} = 0$ so $v = (-v) = 0$
$\leftarrow$) Take $u \in U_1 + U_2$ assume $u = u_1 + u_2 = \tilde{u_1} + \tilde{u_2}$. Then $\underbrace{u_1 - \tilde{u_1}}_{\in U_1} = \underbrace{-(u_2 - \tilde{u_2}}_{\in U_2}$ so by assumptions, $u_1 = \tilde{u_1}$ and $u_2 = \tilde{u_2}$. ☐

Example 3.12. For subspaces $U_1, \dots, U_m$ of V, TFAE:

(i) $U_1 + \cdots + U_m$ is a direct sum

(ii) $\forall j\ U_j \cap (\sum_{k \neq j} U_k) = \{0\}$

(iii) $\forall j > 1$, $U_j \cap (U_1 + \cdots + U_{j-1}) = \{0\}$

(iv) If $u_1 + \cdots + u_m = 0$, $u_j \in U_j$ then $u_1 = u_2 = \cdots = u_m = 0$

3.3 Chapter 2: Finite Dimensional Vector Spaces

$\mathbb{F}$: field, V : Vector space $/\mathbb{F}$

3.4 2.A: Span and Linear Independence

Motivation: In some V(such as $\mathbb{F}^n$), we can find vectors $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ such that every $v \in V$ can be written as $v = \sum_{j-1}^{n} a_j e_j$ and the choice of a_j is unique.

We will work with such vectors in a general setting.

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4.1 Chapter 2: Finite Dimensional Vector Spaces

- Main motivation: Find "coordinate systems" in a vector space
- Recall in $\mathbb{F}^n$, $(x_1, \ldots, x_n) = x_1(1, 0 \ldots, 0) + x_n(0, \ldots, 0, 1) = x_1 e_1 + \cdots + x_n e_n$.

4.2 2.A: Span and Linear Independence

Definition 4.1. A linear combination of vectors $v_1, \ldots, v_m \in V$ is a vector of the form

$$v = \sum_{j=1}^{m} a_j v_j, \quad a_1, \ldots, a_m \in \mathbb{F}.$$

Example 4.2. $(1, 2, -3) = (1, 0, -1) + 2(0, 1, -1)$

Example 4.3. Is $(1, 2, 3)$ a linear combination of $(1, 0, -1)$ and $(0, 1, 1)$?
No, if $(1, 2, -3) = a_1(1, 0, -1) + a_2(0, 1, 1)$ then $a_1 = 1, a_2 = 2$ but $1(1, 0, -1) + 2(0, 1, 1) = (1, 2, 1) \neq (1, 2, -3)$.

Definition 4.4. The set

$$\{\sum_{j=1}^{m} a_j v_j, a_i \in \mathbb{F}, \forall 1 \leq j \leq m\}$$

is the span of $v_1, \ldots, v_m$, denoted by $\text{span}(v_1, \ldots, v_m)$. Note $\text{span}() = \{0\}$.

Example 4.5. $(1, 2, -3) \in \text{span}((1, 0, -1), (0, 1, -1))$.

Theorem 4.6. $\text{span}(v_1, \ldots, v_m)$ is the smallest subspace of V that contains $v_1, \ldots, v_m$.

Proof. Subspace: $0 = 0v_1 + \cdots, 0v_n \in \text{span}(v_1, \ldots, v_m)$
Closed under addition: $(a_1 v_1 + \cdots + a_m v_m) + (b_1 v_1 + \cdots + b_m v_m) = (a_1 + b_1)v_1 + \cdots + (a_m + b_m)v_m$.
Closed under multiplication: $\lambda(a_1 v_1 + \cdots + a_m v_m) = (\lambda a_1)v_1 + \cdots + (\lambda a_m)v_m$.
So it is a subspace.
Smallest: If $v_1, \ldots, v_m \in W$ for some subspace W, then $\forall a_1, \ldots, a_n \in \mathbb{F}$, $a_1 v_1, \ldots, a_m v_m \in V$ so $a_1 v_1 + \cdots + a_m v_m \in W$. Thus, $\text{span}(v_1, \ldots, v_m) \subseteq W$. $\square$

Definition 4.7. If $V = \mathrm{span}(v_1, \ldots, v_m)$, then we say the list $v_1, \ldots, v_m$ spans V.

Example 4.8. $e_1, \ldots, e_n$ spans $\mathbb{F}^n$

Definition 4.9. V is called finite dimensional if some (finite) list of vectors spans V.

Example 4.10. $\mathbb{F}^n$ is finite dimensional.

Definition 4.11. A finite expression

$$p(z) = a_0 + a_1 z^1 + \cdots + a_m z^m, \quad a_0, \ldots, a_m \in \mathbb{F}, a_m \neq 0, \tag{*}$$

also written as

$$\sum_{j=0}^{\infty} a_j x_j, a_{n+1} = a_{n+2} = \cdots = 0,$$

is called a polynomial with coefficients in $\mathbb{F}$. (By definition $p = 0$ is a polynomial.)

- Each polynomial over $\mathbb{F}$ gives rise to a function from $\mathbb{F} \to \mathbb{F}$ defined by $p : \mathbb{F} \to \mathbb{F}$ by $z \mapsto p(z)$
- m is the degree of p if p has the form $(*)$. The zero polynomial has degree $-\infty$ by definition.
- $\mathcal{P}(\mathbb{F}) = \{$all polynomials over $\mathbb{F}\}$
- $\mathcal{P}_m(\mathbb{F}) = \{$all polynomials of deg $\leq m$ over $\mathbb{F}\}$

Example 4.12. $\mathcal{P}_m(\mathbb{F})$ and $\mathcal{P}(\mathbb{F})$ are vector spaces over $\mathbb{F}$ (also subspaces of $\mathbb{F}^{\mathbb{F}}$ if viewed as functions.)

Example 4.13.

(a) $\mathcal{P}_m(\mathbb{F})$ is finite dimensional

(b) $\mathcal{P}(\mathbb{F})$ is infinte dimensional

Proof.

(a) $1, z, \ldots, z^m$ spans $\mathcal{P}_m(\mathbb{F})$

(b) For any $p_1, \ldots, p_m \in \mathcal{P}(\mathbb{F})$, assume N is larger than $\deg p_j$ for $1 \leq j \leq m$. Then every $\sum_{j=1}^{m} a_j p_j$ is not equal to z^N.

□

Definition 4.14. $v_1, \ldots, v_m$ is called linearly independent if whenever $0 = \sum_{j=1}^{m} a_j v_j$, $a_1, \ldots, a_m \in \mathbb{F}$, we must have $a_1 = \cdots = a_m = 0$. Otherwise, the list is linearly dependent. The empty list is linearly independent by definition.

Example 4.15.

(a) v is linearly independent iff $v \neq 0$

(b) $e_1, \ldots, e_n$ is linearly independent in $\mathbb{F}^n$

(c) v_1, v_2 is linearly independent iff neither vector is a scalar multiple of the other.

(d) $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}_m(\mathbb{F})$.

(e) $(1, *, *), (0, 1, *), (0, 0, 1)$ where each $*$ is arbitrary is linearly independent in $\mathbb{F}^3$

(f) $(1, 1, \ldots, 1), (a_1, a_2, \ldots, a_n), (a_1^2, a_2^2, \ldots a_n^2), \ldots, (a_1^{n-1}, a_2^{n-1}, \ldots, a_n^{n-1})$ is linearly dependent iff at least two of the a_j's are the same.

(g) Any subset of a linearly independent list is linearly independent.

Example 4.16.

(a) If some vector in a list is a linear combination of the others, then the list is linearly dependent.

(b) Every list containing 0 is linearly dependent.

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5.1 2.A: Span and Linear Independence

Notation 5.1. $\mathcal{P}(\mathbb{F})$ can also be written as $\mathbb{F}[x]$

Lemma 5.2. For $v_1, \ldots, v_n \in V$, TFAE:

(a) $v_1, \ldots, v_n$ is linearly dependent.

(b) $\exists\, 1 \leq j \leq n$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$

(c) $\exists\, 1 \leq j \leq n$ such that $v_j \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ (Note: here $\hat{v}_j$ means v_j is excluded from the list)

(d) $\exists\, 1 \leq j \leq n$ such that $\operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$.

Proof. a $\to$ b) By def, $\exists\, a_1, \ldots, a_n \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \cdots + a_n v_n = 0$. Take the largest j such that $a_j \neq 0$. Then, $a_1 v_1 + \cdots + a_j v_j = 0$. Hence, $v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$ so $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.
b $\to$ c) Notice $\operatorname{span}(v_1, \ldots, v_{j-1}) \subset \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ so $v_j \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$.
c $\to$ d) By assumption $v_j \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$. Also $v_k \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ for $k \neq j$ so $\operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ contains $v_1, \ldots, v_n$. Thus, it contains $\operatorname{span}(v_1, \ldots, v_n)$. Since $\operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n) \subset \operatorname{span}(v_1, \ldots, v_n)$, the two are equal.
d $\to$ a) By assumption, $\exists\, b_k \in \mathbb{F}$, $1 \leq k \leq n$, $k \neq j$ such that $v_j = \sum_{j \neq k} b_k v_k$. So $\sum_{j \neq k} b_k v_k - v_j = 0$ so the set is linearly dependent. $\square$

Theorem 5.3. If $v_1, \ldots, v_m$ spans V, and $u_1, \ldots, u_n \in V$ are linearly independent, then $n \leq m$.

Idea. If $m = 2$, why can't $n = 3$?
Consider the functions:

$$\begin{aligned} u_1 &= a_{1,1}v_1 + a_{1,2}v_2 \\ u_2 &= a_{2,1}v_1 + a_{2,2}v_2 \\ u_3 &= a_{3,1}v_1 + a_{3,2}v_2 \end{aligned}$$

We must rearrange u_1, u_2, u_3 to show they are linearly dependent (3 equations in 2 variables.) □

Proof. We will proceed by induction on m.
Note that for $m = 0$, span$() = \{0\}$ so this is trivially true.
Basis: If $m = 1$, $n \geq 2$. Let v_1 span V and let $u_1, u_2 \in V$ be arbitrary. Then $u_1 = \lambda_1 v_1$ and $u_2 = \lambda_2 v_2$. If $\lambda_1 = 0$, then $u_1 = 0$ and the set is linearly dependent so assume $\lambda_1 \neq 0$. Then $\lambda_2 u_1 - \lambda_1 u_2 = 0$ so the list is linearly dependent.
Inductive Step: Assume that the theorem holds for $m = k$. It suffices to show the $m = k+1$ case. Let $v_1, \ldots, v_{k+1}$ be a spanning list of V. If $n \geq k+2$, let

$$u_i = \sum_{j=1}^{k+1} a_{i,j} v_j, \quad 1 \leq i \leq k+2, \quad a_{i,j} \in \mathbb{F},$$

be a list of $k+2$ vectors.
If all $a_{i,k+1} = 0$, then the list of vectors can be represented using only the vectors $v_1, \ldots, v_k$ so they would be linearly independent by the IH.
Otherwise, WLOG, assume $a_{k+2,k+1} \neq 0$ (if not we can relabel the vectors to achieve this). Then

$$u_i - \frac{a_{i,k+1}}{a_{k+2,k+1}} u_{k+2} \in \text{span}(v_1, \ldots, v_k)$$

for $1 \leq i \leq k+1$.
By IH, $\exists\, b_1, \ldots, b_{k+1} \in \mathbb{F}$, not all 0, such that

$$b_1(u_1 - \frac{a_{1,k+1}}{a_{k+2,k+1}} u_{k+2}) + \cdots + b_{k+1}(u_{k+1} - \frac{a_{k+1,k+1}}{a_{k+2,k+1}} u_{k+1} u_{k+2}) = 0$$

so

$$b_1 u_1 + \cdots + b_{k+1} u_{k+1} - (b_1 \frac{a_{1,k+1}}{a_{k+2,k+2}} + \cdots + b_{k+1} \frac{a_{k+1,k+1}}{a_{k+1,k+2}}) u_{k+2} = 0$$

so the list $u_1, \ldots, u_{k+2}$ is linearly dependent. □

Example 5.4. $e_1, \ldots, e_n$ spans $\mathbb{F}^n$ and is linearly independent so:

- $(1,2,3), (4,5,8), (4,6,7), (-3,2,8)$ are linearly dependent in $\mathbb{F}^3$

- $(1,2,3,-5),(4,5,8,-3),(4,6,7,-1)$ does not span $\mathbb{F}^4$

Proposition 5.5. Every subspace of a finite dimensional vector space is finite dimensional.

Proof. Assume V is spanned by $v_1,\dots,v_m$, and U is a subspace of V.
Start from the empty list () in U and add vectors 1 by 1 so that no vector is in the span if the previous vectors in the list. This produces a linearly independent list in U.
By the thm, this process must terminate since the length of a list of linearly independent vectors in V cannot be greater than m.
Assume we have $u_1,\dots,u_n$. Then each $u \in U$ is a linear combination of $u_1,\dots,u_n$, otherwise we could add it to the list and it would produce a longer linearly independent list. Thus, $u_1,\dots,u_n$ spans U. □

5.2 2.B - Bases

Definition 5.6. A basis of V is a list of vectors in V that is linearly independent and spans V.

Theorem 5.7. Every finitely dimensional vector space has a basis.

Proof. Take $U = V$ in the proof of proposition 5.5. Then we can generate a linearly independent list in V that spans V. Thus V has a basis. □

Example 5.8.

(a) $e_1,\dots,e_n$ forms a basis of $\mathbb{F}^n$ (standard basis)

(b) $(1,2,3),(3,4,6),(0,0,1)$ is a basis of $\mathbb{F}^3$ unless char $\mathbb{F} = 3$

(c) $(1,-1,0),(0,1,-1)$ is a basis of $\{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$

(d) $1,z,\dots,z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$

(e) $f_0,f_1,\dots,f_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$ if $\deg f_j = j$, $0 \le j \le m$

Proposition 5.9. $v_1,\dots,v_m$ forms a basis of V iff $\forall v \in V$ can be uniquely represented as $v = \sum_{j=1}^n a_j v_j$, $a_j \in \mathbb{F}$.

Proof. If $v_1,\dots,v_n$ forms a basis of V, then they span V so all vectors can be represented in the desired form. Suppose $\exists\, a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{F}$ such that $a_1v_1+\cdots+a_nv_n = v = b_1v_1+\cdots+b_nv_n$, then $(a_1-b_1)v_1+\cdots+(a_n-b_n)v_n = 0$. Since the set is linearly independent, $a_1-b_1 = \cdots = a_n-b_n = 0$ so $a_i = b_i$ for all i, thus the representation is unique.
If the stated conditions hold, then the list spans v. Also, 0 has a unique representation so the list is linearly independent and hence a basis. □

Proposition 5.10. Every spanning list in a finite dimensional vector space contains a basis.

Proof 1. Starting from (), we can create a linearly independent list consisting of vectors in the spanning list by the same procedure as proposition 5.5. This process must terminate since the spanning list is finite. So we produce a linearly independent list that spans V, eg. a basis. $\square$

Proof 2. We can also start with the spanning list $v_1, \ldots, v_m$ and at each step, if the list is linearly dependent, we can choose v_j such that $\text{span}(v_1, \ldots, v_n) = \text{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$. This process must terminate in a linearly independent list since the spanning list is finite. So we produce a linearly independent list that spans V, eg. a basis. $\square$

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6.1 2.B - Bases

Proposition 6.1. Every linearly independent list of vectors in a finite dimensional vector space can be extended to a basis.

Proof. Starting with a linear independent list we can continue to add vectors that are not in the span of the previous vectors of the list to product a linearly independent list. This will eventually terminate since a linearly independent list cannot be longer than the spanning list. Thus, it will produce a linearly independent spanning list, eg. a basis. $\square$

Proposition 6.2. If V is finite dimensional and U is a subspace of V, then there exists a subspace $W \subset V$ such that $V = U \oplus W$.

Proof. U is finite dimensional so take a basis $u_1, \ldots, u_n$ of U. Extend this to a basis $u_1, \ldots, u_m, u_{m+1}, \ldots, u_n$ of V. We will show $W = \text{span}(u_{m+1}, \ldots, u_n)$ suffices.
Since $u_1, \ldots, u_n$ is a basis of V, every $v \in V$ can be written as $\underbrace{a_1 u_1 + \cdots + a_m v_m}_{\in U} + \underbrace{a_{m+1} u_{m+1} + \cdots + a_n u_n}_{\in W}$ so $U + W = V$.
Moreover, if $w \in U \cap W$, then $w = \sum_{j=1}^{m} b_j v_j$ and $w = \sum_{j=m+1}^{n} b_j v_j$ for $b_1, \ldots, b_n \in \mathbb{F}$. Hence, since $\sum_{j=1}^{m} b_j v_j - \sum_{j=m+1}^{n} b_j v_j = 0$, all $b_j = 0$ so $w = 0$. $\square$

6.2 2C - Dimension

Theorem 6.3. Any two bases of a finite dimensional vector space have the same length.

Proof. Bases are spanning lists and linearly independent lists so for two bases B_1, B_2, $\text{len}B_1 \leq \text{len}B_2$ and $\text{len}B_2 \leq \text{len}B_1$ so $\text{len}B_1 = \text{len}B_2$. $\square$

Definition 6.4. The dimension of a finite dimensional vector space is the length of every basis, denoted $\dim V$

Example 6.5.

(a) $\dim \mathbb{F}^n = n$

(b) $\mathbb{C}$ as a vector space over $\mathbb{R}$ has dimension 2. eg. $\dim_{\mathbb{R}} \mathbb{C} = 2$

(c) $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$

(d) $\dim\{(x_1, \ldots, x_n) \in \mathbb{F}^n : x_1 + \cdots + x_n = 0\} = n - 1$.
A basis is $(1, -1, 0, \ldots, 0), (0, 1, -1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1, -1)$.

(e) Every subspace $U \subset V$ such that $U \neq V$ has $\dim U < \dim V$.

Proof. Take a basis of U and extend to a basis of V. We must add ≥ 1 element, otherwise $U = V$. □

(f) Every vector space $\neq \{0\}$ has $\dim \geq 1$.

Proof. Take a nonzero element (linearly independent) and extend to a basis. Thus $\dim \geq 1$. □

Theorem 6.6. If V is fin dim with $\dim V = n$, then if a list of n vectors is linearly independent it is a basis.

Proof. Extend the list to a basis. Since the basis has length n no vectors were added so the list is already a basis. □

Theorem 6.7. If V is finite dimensional with $\dim V = n$, then if a list of n vectors spans V, it must be a basis.

Proof. Refine the list to a basis. The basis has n vectors so no vectors were removed. Thus, the list is already a basis. □

Example 6.8. $U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\}$, [for $p(x) = \sum_{j=0}^{\infty} a_j x_j$, define $p'(x) = \sum_{j=1}^{\infty} j a_j x^{j-1}$], has $\dim \leq 3$. $1, (x-5)^2, (x-5)^3$ are linearly independent so $\dim U \geq 3$. Thus, $\dim U = 3$.

Theorem 6.9. If U_1, U_2 both subspaces of V, $\dim V < \infty$. Then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof. Find a basis $u_1, \ldots, u_n$ of $U_1 \cap U_2$. Extend to a basis $u_1, \ldots, u_n, v_1, \ldots, v_m$ of U_1 and a basis $u_1, \ldots, u_n, w_1, \ldots, w_k$ of U_2. We claim $u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_k$ is a basis of $U_1 + U_2$.
First $\forall v \in U_1 + U_2$, $v = u_1 + u_2$ for $u_1 \in U_1$, $u_2 \in U_2$. Consider $u_1 = \sum_{j=1}^{n} a_j u_j + \sum_{j=1}^{m} b_j v_j$, $u_2 = \sum_{j=1}^{n} c_j u_j + \sum_{j=1}^{k} d_j w_j$. Then, $v = u_1 = u_2 + u_1 = \sum_{j=1}^{n} (a_j +$

$c_j)u_j + \sum_{j=1}^m b_j v_j + \sum_{j=1}^k d_j w_j$. Hence $u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_k$ spans $U_1 + U_2$.
Moreover, if $\sum_j \alpha_j u_j + \sum_j \beta_j v_j + \sum_j \gamma_j w_j = 0$ for $\alpha_j, \beta_j, \gamma_j \in \mathbb{F}$, then

$$\underbrace{(\sum_j \alpha_j u_j + \sum_j \beta_j v_j)}_{\in U_1} = \underbrace{-\sum_j \gamma_j w_j}_{\in U_2}$$

so both in $U_1 \cap U_2$. So $\sum_j \gamma_j w_j = \sum_j \delta_j u_j$ for $\delta_1, \ldots, \delta_n \in \mathbb{F}$ so $\gamma_1 = \cdots = \gamma_n = 0$. Hence $\sum_j \alpha_j u_j + \sum_j \beta_j v_j = 0$ so all $\alpha_j, \beta_j = 0$. Hence, $u_1, \ldots, u_n, v_1, \ldots, v_m, w_1, \ldots, w_k$ is linearly dependent and the claim holds.
Now, $\dim(U_1+U_2) = n+m+k$, $\dim U_1 = n+m$, $\dim U_2 = n+k$, $\dim(U_1 \cap U_2) = n$ so theorem follows by a direct computation. □

6.3 Ch3 - Linear Maps

Notation 6.10. U, V, W will represent subspaces.

6.4 3.A - Linear Maps as a Vector Space

Definition 6.11. $T : V \to W$ is called a linear map if $\begin{cases} T(u+v) = Tu + Tv & \forall u, v \in V \\ T(\lambda v) = \lambda T v & \forall \lambda \in \mathbb{F}, v \in V \end{cases}$.

Note: V is called the domain of T.

Definition 6.12. $\{$linear maps from V to $W\}$ is denoted by $\mathrm{Hom}(V, W)$ ($\mathcal{L}(V, W)$).
$\mathrm{Hom}(V, V) = \mathrm{End}(V)$.

Example 6.13.

(1) Zero map: $0 \in \mathrm{Hom}(V, W)$ $\quad 0 : V \to W$ by $v \mapsto 0$

(2) Identity: $I \in \mathrm{End}(V)$ $\quad I : V \to W$ by $v \mapsto v$

(3) Inclusion: "i''. If $V \subseteq W$, $i : V \to W$ by $v \mapsto v$

(4) Differentiation: $D : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ by $\sum_{j=0}^\infty a_j x^j \mapsto \sum_{j=1}^\infty j a_j x^{j-1}$. $D \in \mathrm{End}(\mathcal{P}(\mathbb{F}))$

(5) Integration from 0 to $1 \in \mathrm{End}(\mathcal{P}(\mathbb{R}))$

(6) "Multiplication by f": Fix $f \in \mathcal{P}(\mathbb{F})$. Let $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ by $g \mapsto g \cdot f$. $[(\sum_j a_j x^j)(\sum_j b_j x^j) = \sum_{k=0}^\infty (\sum_{j_1+j_2=k} a_{j_1} b_{j_2}) x^k]$. $T \in \mathrm{End}(\mathcal{P}(\mathbb{F}))$.

(7) For a matrix
$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \cdots & \\ a_{m,1} & \cdots & a_{m_n} \end{pmatrix}$$
$T : \mathbb{F}^m \to \mathbb{F}^m$ by $(x_1, \ldots, x_n) \mapsto (a_{1,1}x_1 + \cdots + a_{1,n}x_n, \ldots, a_{1,m}x_1 + \cdots + a_{n,m}x_n)$. $T \in \mathrm{Hom}(\mathbb{F}^n, \mathbb{F}^m)$.

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7.1 3.A- Linear Maps a Vector Space

Theorem 7.1. $\mathrm{Hom}(V, W)$ is a vector space with respect to:
$+ : (T_1 + T_2)v = T_1 v + T_2 v$
$\cdot : (\lambda T_1)v = \lambda \cdot T_1 v$

Theorem 7.2. If $T \in \mathrm{Hom}(V, W)$, then $T0 = 0$.

Proof. $T0 = T(0 + 0) = T0 + T0$ so $0 = T0$. $\square$

Product of linear maps defined by composition

Definition 7.3. If $T \in \mathrm{Hom}(U, V)$, $S \in \mathrm{Hom}(V, W)$. Then the product (defined by composition) $ST \in \mathrm{Hom}(U, W)$ is defined as $ST : U \to W$ by $v \mapsto S(Tv)$

Proof that ST is linear.
$(ST)(v_1 + v_2) = S(T(v_1 + v_2)) = S(Tv_1 + Tv_2) = S(Tv_1) + S(Tv_2) = (ST)v_1 + (ST)v_2$
$(ST)(\lambda v) = S(T(\lambda v)) = S(\lambda Tv) = \lambda S(Tv) = \lambda(ST)v$ $\square$

Proposition 7.4.

(1) $(T_1T_2)T_3 = T_1(T_2T_3)$ as long as everything is defined

(2) $TI = IT = T$

(3) $(S_1 + S_2)T = S_1T + S_2T$, $S(T_1 + T_2) = ST_1 + ST_2$ as long as everything is defined.

- Assuming $S : U_1 \to U_2$, $T : V_1 \to V_2$ where ST makes sense (ie. $V_2 = U_1$). TS may not make sense
- Even if TS also makes sense (ie. $U_2 = V_1, V_2 = U_1$), $TS : U_1 \to U_1$ but $ST : U_2 \to U_2$
- Even if $U_1 = U_2 = V_1 = V_2$, TS might not equal ST.
  eg. $U_1 = U_2 = V_1 = V_2 = \mathcal{P}(\mathbb{R})$, S : Differentiation, T: multiply by x.
  Then $(ST)(p) = S(T(p)) = S(xp) = p + xp'$ but $(TS)(p) = T(S(p))' = T(p') = xp'$.

Theorem 7.5. If $v_1, \dots, v_m$ is a basis of V and $w_1, \dots, w_m \in W$ then $\exists!$ linear map $T : V \to W$ such that $Tv_j = w_j$, $1 \le j \le n$.

Proof.
Existence: $\forall a_1, \dots, a_n \in \mathbb{F}$ define $T(\sum a_j v_j) = \sum a_j w_j$
Well defined: only one way to write $\forall v \in V$ as some $\sum a_j v_j$
Linear: For $\lambda \in \mathbb{F}$, $u_1, u_2 \in V$ write $u_1 = \sum_{j=1}^n b_j v_j$, $u_2 = \sum_{j=1}^n c_j v_j$, $b_j, c_j \in \mathbb{F}$. Then $T(u_1 + u_2) = T(\sum_j (b_j + c_j)v_j) = \sum_j (b_j + c_j)w_j = \sum_j b_j w_j + \sum_j c_j w_j = T(\sum b_j v_j) + T(\sum c_j v_j) = Tu_1 + Tu_2$.

$T(\lambda v_1) = T(\sum_j (\lambda b_j) w_j = \lambda(\sum_j b_j w_j)\lambda T u_1$
Uniqueness: If $T_1 v_j = T_2 v_g = w_j$, $\forall 1 \leq j \leq n$, then $\forall v \in V$, write $v = \sum_{j=1}^n d_j v_j$, $d_j \in \mathbb{F}$, $1 \leq j \leq n$ so $T_1 v = T(\sum d_j v_j) = \sum (T d_j v_j) = \sum d_j T_1(v_j) = \sum d_j w_j$ and $T_2 v = \sum d_j v_j$ for the same reason so $T_1 v = T_2 v$. □

7.2 3.B - Kernels and Images

Definition 7.6. For $T \in \mathrm{Hom}(V, W)$, the kernel (or null space) of T is $\ker T = \{v \in V : Tv = 0\}$.

Example 7.7.

(1) $0 : V \to W \quad \ker 0 = V$

(2) If $V \subset W$, $i : V \to W \quad \ker i = \{0\}$

(3) $D : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$, char $F = 0 \quad \ker D = \{\text{constants}\}$

Proposition 7.8. $\forall T \in \mathrm{Hom}(V, W)$, $\ker T$ is a subspace

Definition 7.9. A map $f : S_1 \to S_2$ is called injective if $f(x_1) = f(x_2) \to x_1 = x_2$.

Proposition 7.10. If $T \in \mathrm{Hom}(V, W)$, then T is injective iff $\ker T = \{0\}$

Proof. $\to$) $0 \in \ker T$. By injectivity, nothing else is mapped to 0.
$\leftarrow$) If $Tv_1 = Tv_2$, then $T(v_1 - v_2) = 0$. Thus with $\ker T = \{0\}$ implies that $v_1 - v_2 = 0$ so $v_1 = v_2$ □

Definition 7.11. If $T \in \mathrm{Hom}(V, W)$, then image (or range) of T is defined as $\mathrm{im}T = \{w \in W : \exists v \in V \text{ such that } w = Tv\}$

Example 7.12.

(1) $\mathrm{im}0 = \{0\}$

(2) $V \subset W$, $i : V \to W$ has image V

(3) $D : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$, char $\mathbb{F} = 0 \quad \mathrm{im}D = \mathcal{P}(\mathbb{F})$

Proposition 7.13. $\forall T \in \mathrm{Hom}(V, W)$, $\mathrm{im}T$ is a subspace.

Proof. $\forall w_1, w_2 \in \mathrm{im}T$, find $v_1, v_2 \in V$ such that $Tv_1 = w_1$, $Tv_2 = w_2$. Then $T(v_1 + v_2) = w_1 + w_2$, $T(\lambda v_1) = \lambda w_1$. □

Definition 7.14. A map $f : S_1 \to S_2$ is surjective if $\{f(s) : s \in S_1\} = S_2$.

Observation: $\forall T \in \mathrm{Hom}(V, W)$, T is sujective iff $\mathrm{im}T = W$

Theorem 7.15 (Fundamental Theorem of Linear Maps)**.** Assume V is finite dimensional and $T \in \mathrm{Hom}(V, W)$, then $\dim V = \dim(\mathrm{im}T) + \dim(\ker T)$

Proof. If $v_1, \dots, v_n$ is a basis of $\ker T$, extend it to a basis $v_1, \dots, v_n, v_{n+1}, \dots, v_m$ of V. We claim: $Tv_{n+1}, \dots, Tv_m$ iss a basis of im$T$.
Spans: $\forall w \in \text{im}T$, $\exists v \in V$ such that $Tv = w$. Write $v = \sum_{j=1}^m a_j v_j$. Then $Tv = \sum_{j=1}^m a_j Tv_j = \sum_{n<j\leq m} a_j Tv_j$. Hence $Tv_{n+1}, \dots, Tv_m$ spans im$T$.
Lin. Indpendent: If $b_{n+1}, \dots, b_m \in \mathbb{F}$ such that $b_{n+1}Tv_{n+1} + \cdots + b_m Tv_n$. Then $T(\sum_{n<j\leq m} b_j v_j) = 0$ so $\sum_{n<j\leq m} b_j v_j \in \ker T$. So $\exists a_1, \dots, a_n$ such that $\sum_{n<j\leq m} = \sum_{j=1}^n c_j v_j$ so all $b_j = 0$.
Hence the claim is verified. Thus, $\dim V = m, \dim(\ker V) = n, \dim(\text{im}V) = m - n$. ☐

8 2/14/2022

8.1 3.B - Kernels and Images

Corollary 8.1. If $\dim V > \dim W$, then no $T \in \text{Hom}(V, W)$ is injective.

Corollary 8.2. If $\dim V < \dim W$, then no $T \in \text{Hom}(V, W)$ is surjective.

Corollary 8.3. $D : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ is not surjective

Theorem 8.4. A homogeneous system of linear equations

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \cdots \\ f_m(x_1, \dots, x_n) = 0 \end{cases} \qquad \text{where } f_j(x_1, \dots, x_n) = \sum_{j=1}^n A_{j,k} x_k$$

with more variables than equations has a nonzero solution.

Proof. Construct a linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ by $(x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$. Then, $\dim \ker T = \dim \mathbb{F}^n - \dim \text{im}T \geq n - m \geq 1$. Take a nonzero element in the kernel and that is a nonzero solution. ☐

Theorem 8.5. An inhomogenous system of linear equations

$$\begin{cases} f_1(x_1, \dots, x_n) = a_1 \\ \cdots \\ f_m(x_1, \dots, x_n) = a_m \end{cases} \qquad \text{where } f_j(x_1, \dots, x_n) = \sum_{j=1}^n A_{j,k} x_k$$

with more equations than variables has no solutions for some choice of constant terms.

Proof. Define T as in the proof above. Then T is not going to be surjective so there exists $(a_1, \dots, a_n)$ not in the image of T so take that vector as the choice of constants. ☐

8.2 3.C - Matrices

A linear map can be represented by a matrix.

Definition 8.6. An $m \times n$ matrix is an array of scalars in the from

$$A = \left.\underbrace{\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ A_{2,1} & \cdots & A_{2,n} \\ \cdots & \cdots & \cdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}}_{n \text{ columns}}\right\} m \text{ rows}$$

Also written as $(A_{i,j})_{m\times n}$. $\mathbb{F}^{m,n} = \{\text{all } m \times n \text{ matrices}\}$.

Definition 8.7 (Matrix of a Linear Map)**.** If $T \in \mathrm{Hom}(V, W)$, $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n$ is a basis of W. Assume $Tv_k = \sum_{j=1}^{m} A_{j,k} v_j$. Then $(A_{j,k})_{m\times n}$ is called the matrix of T with respect to the bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$, denoted by $\mathcal{M}(T)$.

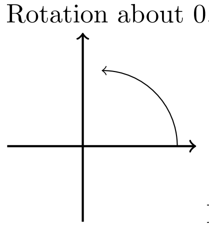
Digest:

$$\begin{array}{cc} & \begin{array}{ccc} \mathrm{v_1} & \cdots & \mathrm{v_n} \end{array} \\ \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array} & \begin{pmatrix} A_{1,1} & \cdots & \\ \vdots & & \vdots \\ A_{1,n} & \cdots & \end{pmatrix} \end{array} \qquad \begin{array}{rcl} \text{columns} & \leftrightarrow & \text{element in basis of domain} \\ \text{rows} & \leftrightarrow & \text{element in basis of target space} \end{array}$$

Motivation: Matrix Multiplication

Example 8.8. In $\mathbb{R}^2$

(a) Rotation about 0.

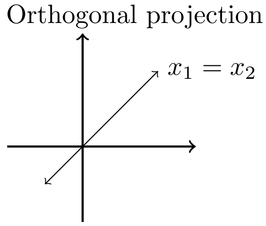


Rotate by $\frac{\pi}{2}$ counterclockwise.

Matrix with respect to (e_1, e_2) is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

More generally, rotation by θ with respect to (e_1, e_2) is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(b) Orthogonal projection to L but then included into $\mathbb{R}^2$.



Matrix with respect to (e_1, e_2): $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

Matrix with respect to $((1,1),(1,-1))$: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(c) $i : V \to W$ (assume $V \subset W$) with respect to $(v_1, \dots, v_n)$, $(v_1, \dots, v_n, v_{n+1}, \dots, v_m)$.

$$\mathcal{M}(i) = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow n\text{th row} \\ \\ \\ \end{array}$$

Definition 8.9. If $A, B \in \mathbb{F}^{m,n}$, $\lambda \in \mathbb{F}$, $A + B$, λA are defined as entrywise addition and scalar multiplication.

Proposition 8.10. If $T_1, T_2 \in \operatorname{Hom}(V, W)$. Fix a basis of V and a basis of W. Then $\mathcal{M}(T_1 + T_2) = \mathcal{M}(T_1) + \mathcal{M}(T_2)$, $\mathcal{M}(\lambda T_1) = \lambda \mathcal{M}(T_1)$.

Proposition 8.11. $\mathbb{F}^{m,n}$ is a vector space with dimension mn.

Proof. The list of all possible $m \times n$ matrices with 0 in all entries except one (where the entry is 1) form a basis. $\square$

8.3 Matrix Multiplication

- Motivated by looking for matrix of ST.

Definition 8.12. For $A \in \mathbb{F}^{m,n}$, $B \in \mathbb{F}^{n,p}$, define $AB \in \mathbb{F}^{m,p}$ such that $(AB)_{i,k} = \sum_{j=1}^{n} A_{i,j} B_{j,k}$.

Proposition 8.13. If $T \in \operatorname{Hom}(V, W)$, $S \in \operatorname{Hom}(V, W)$, $u_1, \dots, u_p$ is a basis of U, $v_1, \dots, v_n$ is a basis of V, and $w_1, \dots, w_m$ is a basis of W, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Proof. Assume $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = B$. $\forall k \in \{1, \dots, p\}$

$$\begin{aligned} (ST)u_k &= S(Tu_k) \\ &= S(\sum_{j=1}^{n} B_{j,k} v_j) \\ &= \sum_{j=1}^{n} B_{j,k} (S v_j) \\ &= \sum_{j=1}^{n} B_{j,k} (\sum_{i=1}^{m} A_{i,j} w_i) \\ &= \sum_{i=1}^{m} (\sum_{j=1}^{n} A_{i,j} B_{j,k} w_i \end{aligned}$$

Hence $(\mathcal{M}(S(T))_{i,k} = \sum_{j=1}^{m} A_{i,j} B_{j,k} = (AB)_{j,k}$. □

Example 8.14. $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 26 & 31 \end{pmatrix}$

Proposition 8.15. $(AB)_{i,j}$ =(ith row of A) $\cdot$ (jth column of B), here "$\cdot$" is the dot product.

Proposition 8.16. The jth column of $AB = A$(jth column of B).

Proposition 8.17. If $A \in \mathbb{F}^{m,n}$, $c \in \mathbb{F}^{n,1} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, then Ac is a linear combination of the columns of A: $Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$.