MATH 104 Notes

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Contents

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1.1 Natural Numbers

- $\mathbb{N} = \{0, 1, 2, 3, \ldots, \}$
- successor construction: 2 is the successor of 1, 3 is the successor of 2. So starting from 0 one can reach all rational numbers (for any given natural number, it can be reached from 0 in finitely many steps)
- Peano Axioms for natural Numbers (see Tao 1)
  - Mathematical Induction Property (Axiom 5): let n be a natural number and let $P(n)$ be a statement depending on n, if the following two conditions hold:
    * $P(0)$ is true
    * If $P(k)$ is true, then $P(k+1)$ is true

    then $P(n)$ is true for all $n \in \mathbb{N}$
- operations allowed for $\mathbb{N} : +, \times$
  - if $n, m \in \mathbb{N}$, then $n + m \in \mathbb{N}$ and $n \times m \in \mathbb{N}$
  - $-, /$ are not always defined

1.2 Integers

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- allowed operations: $+, -, \times$ (formally, $\mathbb{Z}$ is a ring)

1.3 Rational Numbers

- $\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$
- We have all four operations $+, -, \cdot, /$
- $\mathbb{Q}$ is now a field

Theorem 1.1 (Field Axioms(Ross 3))**.**
Addition:

- $a + (b + c) = (a + b) + c$ for all a, b, c
- $a + b = b + a$ for all a, b
- $a + 0 = a$ for all a
- For each a, there is an element $-a$ such that $a + (-a) = 0$

Multiplication:

- $a(bc) = (ab) = c$ for all a, b, c
- $ab = ba$ for all a, b
- $a \cdot 1 = a$ for all a
- For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$

Distributive Law:

- $a(b + c) = ab + ac$ for all a, b, c

Theorem 1.2 (Useful Properties of Fields(Ross 3))**.**

- $a + c = b + c$ implies $a = b$
- $(-a)b = -ab$ for all a, b
- $(-a)(-b) = ab$ for all a, b
- $ac = bc$ and $c \neq 0$ imply $a = b$
- $ab = 0$ implies either $a = 0$ or $b = 0$

for $a, b, c \in \mathbb{Q}$

$\mathbb{Q}$ is an ordered field, there is a "relation" $\leq$

Definition 1.3. A relation S is a subset of $\mathbb{Q} \times \mathbb{Q}$, if $(a, b) \in S$ we say "a and b have relation S" or "aSb"

The relation "$\leq$" has 3 properties:

- if $a \leq b$ and $b \leq a$, then $a = b$
- if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity)
- for any $a, b \in \mathbb{Q}$, at least one of the following is true: $a \leq b$ or $b \leq a$

Since $\mathbb{Q}$ is an ordered field, the field structure $(+, -, \cdot, /)$ is compatible with $(\leq)$

- If $a \leq b$, then $a + c \leq b + c$ for all $c \in \mathbb{Q}$
- If $a \geq 0$ and $b \geq 0$, then $ab \geq 0$

Theorem 1.4 (Useful Properties of Ordered Fields(Ross 3))**.**

- If $a \leq b$, then $-b \leq a$
- If $a \leq b$ and $c \geq 0$, then $ac \leq bc$
- If $a \leq b$ and $c \leq 0$, then $bc \leq ac$
- $0 \leq a^2$ for all a
- $0 < 1$
- If $0 < a$, then $0 < a^{-1}$
- If $0 < a < b$, then $0 < b^{-1} < a^{-1}$

for $a, b, c \in \mathbb{Q}$

1.4 What's lacking in $\mathbb{Q}$?

1. There are certain gaps in $\mathbb{Q}$. For example, the equation $x^2 - 2$ cannot be solved in $\mathbb{Q}$

2. For a bounded set in $\mathbb{Q}$, E, it may not have a "most economical" or "sharpest" upper bound in $\mathbb{Q}$
Ex: $E = \{x \in \mathbb{Q} | x^2 < 2\}$ there is no least upper bound(sup) of E in $\mathbb{Q}$ (we want to take $\sqrt{2}$ as $\sup(E)$ but $\sqrt{2}$ is not a rational number)

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2.1 Rational Zeros Theorem

Definition 2.1. An integer coefficient polynomial in x is of the form: $c_n x^2 + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ $c_1, \dots, c_n \in \mathbb{Z}$, $c_n \neq 0$.

1. A $\mathbb{Z}$-coefficient equation is $f(x) = 0$

2. One can ask: when does a $\mathbb{Z}-$coefficient equation have roots in $\mathbb{Q}$

Fact 2.2. A degree n polynomial has n roots in $\mathbb{C}$, ie. $\exists z_1, \dots, z_n \in \mathbb{C}$ such that $f(x) = c_n(x - z_1) \cdots (x - z_n)$

Theorem 2.3. If a rational number r satisfies the equation $x_n x^n + \cdots + c_1 x + c_0 = 0$, with $c_i \in \mathbb{Z}, c_n, c_0 \neq 0$ and $r = \frac{c}{d}$(where c and d are coprime integers). Then c divides c_0 and d divides c_n.

Proof. Plug in $x = \frac{c}{d}$ into the equation to get $c_n(\frac{c}{d})^n + c_{n-1}(\frac{c}{d})^{n-1} + \cdots + c_1(\frac{c}{d}) + c_n = 0$ multiply both sides by d^n to get $c_n c^n + c_{n-1}c^{n-1}d + \cdots + c_1 c d^{n-1} + c_0 d = 0$
Since $c_n c^n = -d(c_{n-1}c^{n-1} + \cdots + c_1 d^{n-1})$, d divides $c_n c^n$. Since d and c are coprimes, d does not divide c^n so d has to divide c_n
Also, since $c_0 d^n = -c(c_n c^{n-1} + c_{n-1}c^{n-2}d + \cdots + c_1 d^{n-1})$ by similar reasoning $c | c_0$ ☐

Using the rational zeros theorem, we can answer questions about rationality

Example 2.4. Show $\sqrt[3]{6}$ is irrational.
$\sqrt[3]{6}$ is rational $\leftrightarrow$ $x^3 - 6$ has rational roots. The only possible rational roots such that $r = \frac{c}{d}$ need $c|6, d|1$. Taking $d = 1$, $c = \pm 1, \pm 2, \pm 3, \pm 6$. Once can check all of these do not satisfy the equation so there is no solution in $\mathbb{Q}$

2.2 Historical Construction of $\mathbb{R}$ from $\mathbb{Q}$

1. Dedekind Cut: (Q: if $\sqrt{2} \notin \mathbb{Q}$, how can we save the information of $\sqrt{2}$?)
A: the subset of $\mathbb{Q}$ $C_{\sqrt{2}} = \{r \in \mathbb{Q} | r > x\}$
For every $x \in \mathbb{R}$, consider $C_x = \{x \in \mathbb{Q} | r < x\}$. We can define addition, multiplication on the subsets C_x

2. Sequences in $\mathbb{Q}$
   ie. Use a sequence of rational numbers to "aproximate" a real number
   eg. $\sqrt{2}$ can be approximated by $1, 1.4, 1.41.1.414, \ldots$
   Problems:

   (a) Given any real number, how do you get such a sequence?

   (b) How do you determine if 2 different sequences approximate the same real number
   (eg. $1 \leftarrow 1.1, 1.01, 1.001, \ldots$ or $1 \leftarrow 0.9, 0.99, 0.999, \ldots$ or $1 \leftarrow 1, 1, 1, \ldots$) all have the same limit

2.3 Properties (Axioms) of $\mathbb{R}$

Given the existence of $\mathbb{R}$, we have certain properties (axoims) of $\mathbb{R}$

Definition 2.5. A subset of $\mathbb{R}$ is said to be bounded above if $\exists a \in \mathbb{R}$ such that for any $x \in E$, we have $x \leq a$

Theorem 2.6 (Completeness Axiom of $\mathbb{R}$)**.** Given a set $E \subset \mathbb{R}$, bounded above, there exists a unique r such that:

1. r is an upper bound of E

2. for any other upper bound of α, we have $r \leq \alpha$

r is called the least upper bound of E, $r = \sup E$
(ie. $\sup E$ is well defined for subsets that are bounded above)

Example 2.7. $\sup([0,1]) = 1$, $\sup((0,1)) = 1$, $\sup(\{r \in \mathbb{Q} | r^2 < 2\}) = \sqrt{2}$

Theorem 2.8 (Archimedean Property)**.** For any $r \in \mathbb{R}$, $r > 0$ $\exists n \in \mathbb{N}$ such that $nr > 1$ or equivalently, $r > \frac{1}{n}$

2.4 $+\infty, -\infty$

- With these symbols, we can say $\sup(\mathbb{N}) = +\infty \leftrightarrow \mathbb{N}$ is not bounded above

- $+\infty, -\infty$ are not real numbers. They have part of the defined operations $\mathbb{R}$ has
  ie. $3 \cdot +\infty = +\infty$, $(-3) \cdot +\infty = -\infty$ but $(+\infty) + (-\infty) =$NAN, $0 \cdot (+\infty) =$ undefined.

2.5 Sequences and Limits

- A sequence of real numbers is: $a_0, a_1, a_2, \ldots$ denoted $(a_n)_{n=0}^{\infty}$ or shortened (a_n)

- We care about the "eventual behavior" of a sequence

Definition 2.9. A sequence (a_n) converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > \mathbb{N}$, $|a_n - a| < \varepsilon$.

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3.1 Sequences and Limits

Definition 3.1. A sequence (a_n) is bounded if $\exists M > 0$, $|a_n| \leq M$ for all n.

Theorem 3.2. Convergent sequences are bounded.

Proof. Let (a_n) be a convergent sequence that converges to a.
Let $\varepsilon = 1$, then by definition of convergence, there exists $N > 0$ such that $\forall n > n$

$$|a_n - a| < 1 \iff a - 1 < a_n < a + 1 \quad \forall n > N.$$

Let $M = \max\{a_1, a_2, \dots, a_N\}$, $M_2 = \max\{|a-1|, |a+1|\}$ and $M = \max\{M_1, M_2\}$. Thus if $n \leq N$ we have $|a_n| \leq < M$, and if $n \geq N$ we have $|a_n| \leq M_2$ so

$$\forall n, |a_n| \leq \max\{M_1, M_2\} = M$$

□

Remark 3.3. One can deal with the first few terms of a sequence easily, it is the "tail of the sequence" that matters.

3.2 Operations on Convergent Sequences

Theorem 3.4. $c \in \mathbb{R}$, $\forall$ convergent sequences $a_n \to a$, we have $c \cdot a_n \to c \cdot a$.

Proof. If $c = 0$, the result is obvious.
If $c \neq 0$, we want to show for all $\varepsilon > 0$, $\exists N$ such that $\forall n > N$

$$|c \cdot a_n - c \cdot a| < \varepsilon \iff |c| \cdot |a_n - a| \leq \varepsilon \iff |a_n - a| \leq \tfrac{\varepsilon}{|c|}.$$

Now let $\varepsilon' = \frac{\varepsilon}{|c|}$. By definition of $a_n \to a$, we have $N > 0$ such that $|a_n - a| \leq \varepsilon' = \frac{\varepsilon}{|c|}$. This gives the desired N. □

Theorem 3.5. If $a_n \to a$, $b_n \to b$, then $a_n + b_n \to a + b$.

Proof. We want to show $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$

$$|a_n + b_n - (a + b)| \leq \varepsilon \iff |(a_n - a) + (b_n - b)| \leq \varepsilon. \tag{*}$$

$|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ by the triangle inequality so

$$(*) \leftarrow |a_n - a| < \varepsilon \tag{**}$$

$$\leftarrow \begin{cases} |a_n - a| \leq \varepsilon/2 \\ |b_n - b| \leq \varepsilon/2 \end{cases} \tag{***}$$

By the convergence of a_n and b_n, $\exists N_1, N_2$ such that $\forall n > N_1$, $|a_n - a| \leq \frac{\varepsilon}{2}$, and $\forall n > N$, $|b_n - b| \leq \frac{\varepsilon}{2}$. Take $N = \max\{N_1, N_2\}$, then $\forall n > N$ $(***)$ is satisfied hence $(*)$ is satsified. □

Corollary 3.6. If $a_n \to a$, $b_n \to b$, then $a_n - b_n \to a - b$.

Proof. Let $c_n = (-1) \cdot b_n$. Then $c_n \to -b$ so $a_n + c_n \to a - b$. ☐

Theorem 3.7. If $a_n \to a$, $b_n \to b$, then $a_n \cdot b_n \to ab$.

Proof. Want to show: $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$

$$|a_n - ab| \leq \varepsilon. \tag{*}$$

Since a_n is convergent, it is bounded by some $M > 0$ which yields the following inequalities.

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b - b) + a_n b - ab| \\ &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n(b_n - b)| + |(a_n - a)b| \\ &\leq |a_n| \cdot |b_n - b| + |a_n - a| \cdot |b| \\ &\leq M|b_n - b| + |b||a_n - a| \end{aligned}$$

So

$$(*) \leftarrow \begin{cases} M|b_n - b| \leq \varepsilon/2 \\ |b||a_n - a| \leq \varepsilon/2 \end{cases}. \tag{**}$$

Since $a_n \to a$, let $\varepsilon_1 = \frac{\varepsilon}{2|b|}$, then $\exists N$ such that $\forall n > N$,

$$|a_n - a| < \varepsilon_1 \iff |b||a_n - a| \leq \frac{\varepsilon}{2}.$$

Also, since $b_n \to b$, let $\varepsilon_2 = \frac{\varepsilon}{2M}$, then $\exists N$ such that $\forall n > N$,

$$|b_n - b| \leq \varepsilon_2 \iff M|b_n - b| \leq \frac{\varepsilon}{2}.$$

. Let $N = \max\{N_1, N_2\}$, then for $n > N$, $(**)$ holds so $(*)$ holds. ☐

Theorem 3.8. If $a_n \to a$, and $a_n \neq 0 \, \forall n$ and $a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$.

Remark 3.9. $a_n \neq 0$ does not imply $a \neq 0$. For example consider the sequence $a_n = \frac{1}{n}$

Proof. Want to show $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$,

$$|\frac{1}{a} - \frac{1}{a_n}| \leq \varepsilon. \tag{*}$$

Observe that

$$|\frac{1}{a} - \frac{1}{a_n}| = |\frac{a - a_n}{a \cdot a_n}| = \frac{|a_n - a|}{|a| \cdot |a_n|}.$$

Claim: $\exists c > 0$ such that $|a_n| > c \, \forall n$.

Proof. Let $\varepsilon' = \frac{\varepsilon}{2}$, then $\exists N'$ such that $\forall n \geq N'$

$$
\begin{aligned}
|a_n - a| \leq \varepsilon' = \frac{\varepsilon}{2} &\iff -|a|/2 < a_n - a < |a|/2 \\
&\iff a + \frac{|a|}{2} < a_n < a + \frac{|a|}{2} \to |a_n| \geq \frac{|a|}{2}
\end{aligned}
$$

Let $c_1 = \min\{|a_1|, |a_2|, \ldots, |a_{N'}|\} \geq 0$. Let $c = \min\{c_1, |a|/2\}$. □

Thus, $\frac{|a_n - a|}{|a| \cdot |a_n|} \leq \frac{|a_n - a|}{|a| \cdot c}$. Hence

$$
(*) \leftarrow \frac{|a_n \cdot a|}{|a| \cdot c} \leq \varepsilon \qquad (**)
$$

and $(**)$ can be satisfied since $a_n \to a$. □

Corollary 3.10. If $a_n \to a$, $b_n \to b$ and $b_n \neq 0$, $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.

Proof. $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$. Since by Thm 8, $\frac{1}{b_n} \to \frac{1}{b}$, $a_n \cdot \frac{a}{b_n} \to a \cdot \frac{1}{b}$ by Thm 7. □

Theorem 3.11 (Useful Results)**.**

(1) $\lim_{n\to\infty} \frac{1}{n^p} = 0 \ \forall p > 0$.

(2) $\lim_{n\to\infty} a^n = 0 \ \forall |a| < 1$.

(3) $\lim_{n\to\infty} n^{1/n} = 1$.

(4) $\lim_{n\to\infty} a^{1/n} = 1$ for all $n > 0$.

Proof of (3). Let $S_n = n^{1/n} - 1$, then $s_n \geq 0 \ \forall n$ positive integers.

$$
1 + s_n = n^{1/n} \iff (1 + s_n)^n = n.
$$

Using to binomial theorem we see

$$
\begin{aligned}
&1 + ns_n + \frac{n(n-1)}{2} s_n^2 + \cdots = n \\
&\to \frac{n(n-1)}{2} s_n^2 \leq n \\
&\to s_n^2 \leq \frac{2}{n-1}
\end{aligned}
$$

Thus, $s_n \to 0$ as $n \to \infty$. □

4 1/27/2022

4.1 Monotone Sequences

Definition 4.1 ($\lim s_n = +\infty$)**.** A sequence (s_n) is said to "diverge to $+\infty$", if for every $M \in \mathbb{R}$ there exists N such that $s_n > M \, \forall n > N$.

Definition 4.2 (Values of a Sequence)**.** If $(s_n)^\infty)_{n=1}$ is a sequence, then $\{s_n\}_{n=1}^\infty$, the subset of $\mathbb{R}$ consisting of the values of (s_n), is called the value set.

Example 4.3.

- $(s_n) = 1, 2, 1, 2, \ldots \quad \{s_n\}_{n=1}^\infty = \{1, 2\}$
- $(s_n) = 1, 1, 2, 2, 1, 1, 2, 2, \ldots \quad \{s_n\}_{n=1}^\infty = \{1, 2\}$
- $(s_n) = 1, 2, 3, 4, \ldots \quad \{s_n\}_{n=1}^\infty = \{1, 2, 3, 4, \ldots\}$

Definition 4.4 (Monotone Sequences)**.**

- A sequence (s_n) is monotonically increasing if $a_{n+1} \geq a_n \, \forall n$
- A sequence (s_n) is monotonically increasing if $a_{n+1} \leq a_n \, \forall n$

Example 4.5.

- $(a_n) = a$, a constant sequence is monotonically increasing and decreasing
- $(a_n) = 1, 2, 3, \ldots$, is increasing
- $(a_n) = -\frac{1}{n}$, is increasing and bounded above (also below)

Theorem 4.6. A bounded monotone sequence is convergent.

Proof. (We will show for increasing, the proof for decreasing is similar.)
Let (a_n) be a bounded monotone increasing sequence and let $\gamma = \sup\{a_n\}_{n=1}^\infty$ $(= \sup a_n)$. Then $a_n \leq \gamma \, \forall n$ and for any $\varepsilon > 0$, $\exists a_{n_0}$ such that $a_{n_0} > \gamma - \varepsilon$. Thus for every $\varepsilon > 0$, let $N = n_0$(as defined above), then for every $n > N$, we have $\gamma - \varepsilon < a_{n_0} \leq a_n \leq \gamma$ thus $|a_n - \gamma| < \varepsilon$ then $\lim a_n = \gamma$ □

Example 4.7 (Recursive Definition of Sequences)**.** Let s_n be any positive number and let

$$s_{n+1} = \frac{s_n^2 + 5}{2s_n} \quad \forall n \geq 1. \tag{*}$$

We want to show $\lim s_n$ exists and find it.

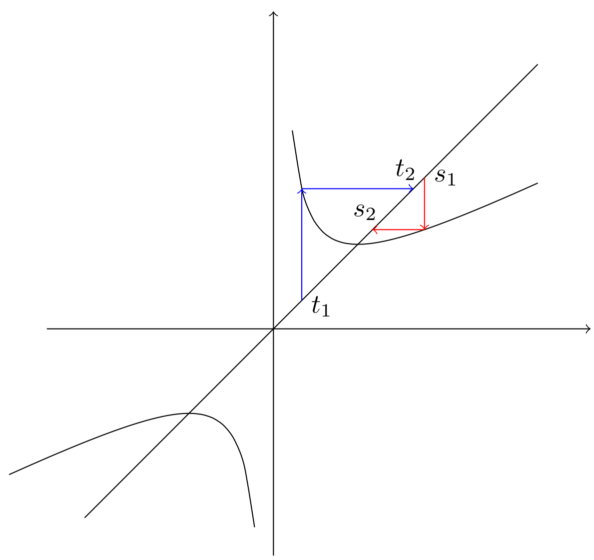
Remark 4.8. If we assume $\lim s_n$ exists, call it s, then s satisfies

$$s = \frac{s^2 + 5}{2s} \tag{**}$$

since we can apply $\lim_{n \to \infty}$ to both sides.

$(**) \to 2s^2 = s^2 + 5 \to s = \pm\sqrt{5}$. Since s_n is a positive sequence $\lim s_n$ can only be ≥ 0, thus s can only by $\sqrt{5}$

- To show $\lim s_n$ exists, we can only need to show s_n is bounded and monotone
- Here is a trick: let $f(x) = \frac{x^2+5}{2x}$, then $s_{n+1} = f(s_n)$
  - Consider the graph of f, ie. $y = f(x)$
  - Consider the diagonal, ie. $y = x$



- If $s_1 > \sqrt{5}$, we should try to prove $\sqrt{5} < \cdots s_3 < s_2 < s_1$
- If $0 < s_1 < \sqrt{5}$, then we show that $s_2 > \sqrt{5}$, we can consider $(s_n)_{n=1}^{\infty}$, which reduces to case 1
- If (s_n) is unbounded and increasing, then $\lim s_n = +\infty$
- If (s_n) is unbounded and decreasing, then $\lim s_n = -\infty$

4.2 Lim inf and sup of a sequence

Definition 4.9 (limsup)**.** Let $(s_n)_{n=1}^{\infty}$ be a sequence,

$$\limsup_{n\to\infty} s_n := \lim_{n\to\infty} (\sup\{s_n\}_{m=1}^{\infty})$$

- $(s_n)_{n=N}^{\infty}$ is called a "tail of the sequence (s_n)" starting at N
- $A_N = \sup\{s_n\}_{n=N}^{\infty} = \sup_{n\geq N} s_n$

- $\limsup s_n = \lim A_n = +\infty$

Example 4.10.

(1) $(s_n) = 1, 2, 3, 4, 5, \ldots$
$A_1 = \sup_{n \geq 1} s_n = +\infty$, $A_2 = \sup_{n \geq 2} s_n = +\infty$
$\limsup s_n = \lim A_n = +\infty$

(2) $(s_n) = 1 - \frac{1}{n}$
$A_1 = \sup_{n \geq 1} s_n = 1$, $A_2 = \sup_{n \geq 2} s_n = 1$
$\limsup s_n = \lim A_n = 1$ (for any monotonic increasing sequence $\limsup s_n = \sup s_1 = A_1$)

(3) $s_n = 1 + \frac{1}{n} \quad (s_n) = 2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots$
$A_1 = \sup\{2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots\} = 2$
$A_2 = \sup\{1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \ldots\} = 1 + \frac{1}{2}$
$A_n = s_n$ so $\limsup s_n = \lim(1 + \frac{1}{n}) = 1$

Lemma 4.11. $A_n = \sup_{m \geq n} s_m$ forms a decreasing sequence.

Proof. Since $\{s_n\}_{m=n}^{\infty} \supset \{s_n\}_{m=n+1}^{\infty}$, $\sup\{s_n\}_{m=n}^{\infty} \geq \sup\{s_m\}_{m=n+1}^{\infty}$, ie. $A_n \geq A_{n+1}$ $\square$

Corollary 4.12. $\lim_{n \to \infty} A_n = \inf A_{n}{}_{n=1}^{\infty}$ $(= \inf_n A_n)$

Example 4.13. $s_n = (-1)^n \cdot \frac{1}{n} \quad (s_n) = (-1, \frac{1}{2}, -\frac{1}{3}, \ldots)$
$A_1 = \sup_{n \geq 1} s_n = s_2 = \frac{1}{2}$, $A_2 = \frac{1}{2}$, $A_3 = \frac{1}{4}$, so
$(A_n) = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \ldots \quad \limsup s_n = \lim A_n = 0$
A_n is like the "upper envelope."

5 2/1/2022

5.1 Cauchy Sequences

Definition 5.1 (Cauchy Sequence)**.** A sequence (a_n) is cauchy if $\forall \varepsilon > 0$, $\exists N > 0$, such that $\forall n, m > N$ we have $|a_n - a_m| < \varepsilon$.

Lemma 5.2. If (a_n) converges to a, then (a_n) is cauchy.

Proof. Let $\varepsilon_1 = \frac{\varepsilon}{2}$, then since $a_n \to a$, $\exists N_1 > 0$ such that $\forall n, m < N$, $|a_n - a| < \varepsilon_1$ and $|a_m - a| < \varepsilon_1$. Thus,

$$|a_n - a_m| = |(a_n - a) - (a_m - a)| \leq |a_n - a| + |a_m - a| < \varepsilon_1 + \varepsilon_1 = \varepsilon.$$

$\square$

Remark 5.3. This is also for true in $\mathbb{Q}$

Lemma 5.4 (Squeze Lemma)**.** Given sequences $(A_n), (B_n), (a_n)$ such that $A_n \geq a_n \geq B_n$ $\forall n$, if $A_n \to a$, $B_n \to a$, then $a_n \to a$.

Proof. $\forall \varepsilon > 0$, we have $N > 0$ such that $\forall n > N$, $|A_n - a| < \varepsilon$ and $|B_n - a| < \varepsilon$. Then $a_n \leq A_n < a + \varepsilon$ and $a_n \geq B_n > a - \varepsilon$ so

$$a - \varepsilon < a_n < a + \varepsilon \leftrightarrow |a_n - a| < \varepsilon.$$

□

Lemma 5.5. Cauchy Sequences are bounded.

Proof. Let $\varepsilon = 1$. Then $\exists N > 0$ such that $\forall n, m > N$, $|s_n - s_m| < \varepsilon$. Consider the term s_{N+1}. Observe that $\forall n < N$, $|s_{N+1} - s_m| < 1$ so $\forall n < N$, $|s_n| < s_{N+1} + 1$. Taking $M = \max\{|s_1|, |s_2|, \ldots, |s_{N+1}|, |s_{N+1}| + 1\}$, we see that $M \geq |s_n|$ for all n. □

Theorem 5.6. If (a_n) is cauchy in $\mathbb{R}$, then (a_n) is convergent.

Proof. Since (a_n) is cauchy, (a_n) is bounded so $\limsup a_n$ and $\liminf a_n$ exist. Let $A_n = \sup_{m \geq n} a_m$, $B_n = \inf_{m \geq n} a_m$, then $A_n \geq a_n \geq B_n$. Let $A = \lim A_n$ and $B_n \lim B_n$. By the Squeeze Lemma, we only need to show $A = B$. Since $A_n \geq B_n$, we know $A \geq B$, hence we only have to rule out $A < B$.
Assume $A < B$. Let $\varepsilon = \frac{(A-B)}{3}$. By Cauchy criterion $\exists N > 0$ such that $\forall n, m > N$, $|a_n - a_m| < \varepsilon$. By the previous lemma, since $A = \limsup a_n$ and $B = \liminf a_n$, given ε, N above, we have $n > N$ such that $|a_n - A| < \varepsilon$ and $m > N$ such that $|a_m - B| \leq \varepsilon$. Then

$$|A - B| \leq |A - a_n| + |a_n - a_m| + |a_m - B| < \varepsilon + \varepsilon + \varepsilon = A - B = |A - B|,$$

which is a contradiction. □

5.2 Subsequences

Let (a_n) be a sequence. If we pick an infinite subset of $\mathbb{N}$, $n_1 < n_2 < n_3 < \cdots$, then we can have a new sequence $b_k = a_{n_k}$, $(b_k) = a_{n_1}, a_{n_2}, a_{n_3}, \ldots$.

Example 5.7. For $(a_n) = (-1)^n$, $a_1 = -1, a_2 = +1, \ldots$ does not converge but subsequence consisting of odd terms converges to -1 and subsequence consisting of even terms converges to 1.

Definition 5.8. Let (a_n) be a sequence. Then $a \in \mathbb{R}$ is a subsequential limit if there exists (a_{n_k}) such that $\lim_{k \to \infty} a_k = a$.

Theorem 5.9. Let (a_n) be a sequence. Then:

(1) $a$is a subsequential limit of (a_n)

(2) $\leftrightarrow \forall \varepsilon > 0$, $\forall N > 0$, $\exists n > N$ such that $|a_n - a| \leq \varepsilon$

(3) $\leftrightarrow \forall \varepsilon > 0$, the set $A_\varepsilon = \{n || a_n - a| < \varepsilon\}$ is infinite

Proof. $2 \leftrightarrow 3$) follows from definitions.
$1 \to 3$) If $a_{n_k} \to a$, then for a given $\varepsilon > 0$, $\exists K > 0$ such that $|a_{n_k} - a| \le \varepsilon$. Thus $\{n_k | k > K\} \subset A_\varepsilon$. So A_ε is infinite.
$3 \to 1$) Cantor's Diagonal Trick: Let $A_{\frac{1}{k}} = \{n || a_n - a| \le \frac{1}{k}\}$.
$A_1 : n_{1,1} < n_{1,2} < n_{1,3} < \cdots$
$A_2 : n_{2,1} < n_{2,2} < n_{2,3} < \cdots$
Observe that $A_{\frac{1}{k+1}} \subset A_{\frac{1}{k}}$, thus $n_{k,i} \le n_{k+1,i}$.
Claim: $(a_{n_{k,k}}) \to a$.
First observe that this is a valid subsequence since $a_{n_{k,k}} < a_{n_{k,k+1}} \le a_{n_{k+1,k+1}}$ for all k. Also for $\varepsilon > 0$, $\exists K$ such that $\frac{1}{K} < \varepsilon$ so for all $k > K$, $|a_n - a| < \frac{1}{K} < \varepsilon$ so it converges to a. □

6 2/3/2022

6.1 Subsequences

Proposition 6.1. If $s_n \to s$, then all subsequences of s_n converge to s.

Proof. Any tail of a subsequence belongs to a tail of the original sequence to they must converge to the same limit. □

Proposition 6.2. Any sequence has a monotone subsequence.

Proof. We say that s_n is a dominant term if $s_n > sm$ for all $m > n$.
Case 1: Suppose there are infinitely many dominant terms. Then the subsequence if dominant terms forms a monotone decreasing sequence.
Case 2: There are finitely many dominant terms. Then we can choose $N > 0$ such that for all $n > N$, s_n is not dominant. We can construct an increasing sequence as follows :

- pick $n_1 > N$, and get s_{n_1}
- pick $n_2 > n_1$ such that $s_{n_2} \ge s_{n_1}$. This is posible since otherwise s_{n_1} would be a dominant term.
- continue in this fashion to achieve a sequence such that $s_{n_1} \le s_{n_2} \le s_{n_3} \le \dots$

□

Theorem 6.3 (Bolzano - Weierstrass)**.** Every bounded sequence has a convergent subsequence.

Proof 1. Assume WLOG, that the sequence is bounded in $[0, 1]$. We may write $[0, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$. Then (s_n) must visit one of the intervals infinitely many times. We can then subdivide that interval and continue in a similar fashion to obtain a decreasing sequence of closed intervals $I_0 = [0, 1] \supset I_1 \supset I_2 \supset \cdots$ with $|I_n| = 2^{-n}$. Let $A_n = \{n | n \in I_n\}$. Then $A_k \subset A_{k-1}$. The sequence $(a_{k,k})_k$ is a cauchy sequence since $\forall \varepsilon > 0$, $\exists k_0$ such that $\frac{1}{2^{k_0}} \le \varepsilon$ for $k_n > k_0$. □

Proof 2. Every sequence contains a monotone sequence so since the sequence is bounded the given monotone sequence converges. □

Proposition 6.4. Let (s_n) be a sequence, the $\limsup s_n$ is a subsequential limit.

Proof. We know that for $\varepsilon > 0$, $N > 0$, $\exists\, n_0 > N$ such that $|s_{n_0} - \limsup s_n| < \varepsilon$. Thus by the alternative of a subsequential limit, $\limsup s_n$ is a subsequential limit.

Remark 6.5. This sequence can be refined to a montone sequence by considering the monotone subsequence of the generated sequence.

□

Theorem 6.6. Let (s_n) be a bounded sequence and let S by the set of subsequential limits of (s_n). Then:

(a) $\sup S = \limsup s_n$, $\inf S = \liminf s_n$ and $\limsup s_n, \liminf s_n \in S$.

(b) $\lim s_n$ exists iff S contains only one element.

(c) S is closed under taking limits. ie. if there is a convergent sequence $t_n \to t$ with $t_n \in S$, we will have $t \in S$.

Proof.

1. For $t \in S$ suppose $s_{n_k} \to t$. Then $\limsup s_{n_k} = \liminf s_{n_k}$. Since $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$, $\liminf s_n \le \liminf s_{n_k} = \limsup s_{n_k} \le \limsup s_n$. Thus, $\liminf s_n \le \inf S \le \sup S \le \limsup s_n$. Since by the previous proposition $\limsup s_n, \liminf s_n \in S$, $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.

2. This follows since $s_n \to s$ iff $\limsup s_n = \liminf s_n$.

3. We will show t is a subsequential limit of (s_n). We want to show, $\forall \varepsilon > 0$, $\forall N > 0$, $\exists n_0 > N$ such that $|s_{n_0} - t| \le \varepsilon$.
Since $t_n \to t$, $\exists N$ such that $\forall n > N$, $|t_n - t| \le \frac{\varepsilon}{2}$. For $n_1 < N$, there are infinitely many s_n with $|s_n - t_{n_1}| \le \frac{\varepsilon}{2}$. Thus, $\exists n_0$ such that $|s_{n_0} - t_{n_1}| \le \frac{\varepsilon}{2}$. Thus, $|s_{n_0} - t| \le |s_{n_0} - t_{n_1}| + |t_{n_1} - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

□

7 2/8/2022

7.1 liminf and limsup (cont.)

Proposition 7.1. If $A = \limsup a_n$, then $\forall \varepsilon > 0$, $\exists N$ such that $\sup\{a_n : n > N\} \le A + \varepsilon$.

Example 7.2. For $a_n = \frac{1}{n}$, $\limsup a_n = 0$ so it is necessary to raise A by ε to have some $a_n \leq A + \varepsilon$.

Proposition 7.3. Given $a_n \to a$, $a > 0$ and b_n bounded, then $\limsup(a_n b_n) = (\lim a_n) \cdot \limsup b_n$.

Proof. Let $b = \limsup b_n$
$\leq$) We plan to show that $a \cdot b$ is a subsequential limit of $a_n \cdot b_n$, then since all subsequential limits $\leq \limsup(a_n b_n)$, the result follows.
We know $\exists$ subsequence (b_{n_k}) that converges to b. We also know all subsequences of (a_n) converge to a. Thus, $a_{n_k} \cdot b_{n_k} \to a \cdot b$.
$\geq$) Since $a > 0$, then $\exists N$ such that $a_n \geq 0$ for all N. Thus, if we throw away a_n with $n \leq N$, we may assume $a_n > 0 \, \forall n$. Then $\lim \frac{1}{a_n} = a$. Thus

$$\limsup b_n = \limsup(a_n b_n) \cdot \frac{1}{a_n} \geq \limsup(a_n b_n) \lim(\frac{1}{a_n}) = \frac{1}{a} \limsup(b_n)$$

so $a \cdot \limsup b_n \geq \limsup(a_n b_n)$ $\square$

Example 7.4. Need $a > 0$. Consider $a_n = -1$, $b_n = 1, 3, 1, 3, \ldots$. Then $\limsup(a_n b_n) = -1$, $\limsup(b_n) = 3$, but $\lim a_n \cdot \limsup a_n b_n = (-1) \cdot 3 = -3$.

Theorem 7.5. Let a_n be a sequence of positive real numbers. Then

$$\liminf(\frac{a_{n+1}}{a_n} \leq \liminf a_n^{1/n} \leq \limsup a_n^{1/n} \leq \limsup(\frac{a_{n+1} a_n}{\cdot}$$

Example 7.6.

(1) $a_n = r^n$ for $r > 0$, then $a_n^{1/n} = r$, $\frac{a_{n+1}}{a_n} = r$.

(2) $a_n = C \cdot r^n$ for $C > 0, r > 0$. Then $a_n^{1/n} = C^{1/n} \cdot r$, $\frac{a_{n+1}}{a_n} = r$ and $\lim a_n^{1/n} = r$.

(3) $a_n = \begin{cases} (\frac{1}{2})^n & n \text{ is even} \\ (\frac{1}{3})^n & n \text{ is odd} \end{cases}$, $a_n^{1/n} = \begin{cases} \frac{1}{2} & n \text{ is even} \\ \frac{1}{3} & n \text{ is odd} \end{cases}$.
However, $\lim \frac{a_{n+1}}{a_n}$ has a lot of oscillations.
In general, root test is stronger than ratio test.

Proof. Note $\liminf(\cdots) \leq \limsup(\cdots)$ so middle $\leq$ is obvious.
We will show $\limsup a_n^{1/n} \leq \limsup \frac{a_{n+1}}{a_n}$ (other $\leq$ is similar).
Assume $\limsup \frac{a_{n+1}}{a_n} = L < \infty$, then $\forall \varepsilon > 0$, $\exists N > 0$ such that $\sup\{\frac{a_{n+1}}{a_n} : n > N\} \leq L + \varepsilon$. We may write $\forall n > N$, $a_n = a_N \cdot \frac{a_{N+1}}{a_N} \cdot \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_n}{a_{n-1}}$ (N terms). so $an \leq a_N \cdot (L + \varepsilon)^{n-N} = (\frac{a_n}{(L+\varepsilon)^N})(L + \varepsilon)^n$ so $a_n^{1/n} \leq C_N^{1/n}(L + \varepsilon)$ where $C_N = \frac{a-n}{(L+\varepsilon)^N}$. So $\limsup(C_N^{1/n}(L + \varepsilon)) = (\lim C_N^{1/n})(L + \varepsilon) = L + \varepsilon$. So $\limsup a_n^{1/n} \leq L+\varepsilon$. Since the holds for any $\varepsilon > 0$, we have $\limsup a_n^{1/n} \leq L$. $\square$

7.2 Series

- A series is of the form $\sum_{n=1}^{\infty} a_n$
- We denote the partial sum, $S_N = \sum_{n=1}^{N} a_n$ and we say "$\sum_{n=1}^{\infty} = L$ if $\lim S_N = L$. Convergence of a series $\iff$ Convergence of its partial sums.

Definition 7.7. $\sum a_n$ is cauchy if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, we have $|a_m + a_{m+1} + \cdots + a_n| \leq \varepsilon$.

Proposition 7.8. $\sum a_n$ is convergent $\iff$ $\sum a_n$ is cauchy.

Proposition 7.9.

(1) "Sanity Check": if $\sum a_n$ is convergent, then $\lim a_n = 0$.

Proof. Convergence $\to$ Cauchy so if we take $n = m$, then we have $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, $|a_n| \leq \varepsilon$. $\square$

(2) Comparison Test: If a_n is a positive sequence, $0 \leq a_n \leq b_n$ then if $\sum b_n$ is convergent, $\sum a_n$ is convergent.

Proof. $\sum a_n$ is a montonic series since $a_n \geq 0$. Since it is bounded by $\sum b_n$, it converges. $\square$

Definition 7.10. $\sum a_n$ is "absolutely convergent" if $\sum |a_n|$ is convergent.

Proposition 7.11. If $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

Proof. $|a_n + a_{n+1} + \cdots + a_m| \leq |a_n| + |a_{n+1}| + \cdots + |a_m|$ so it follows since $\sum |a_n|$ is cauchy. $\square$

Proposition 7.12.

- Ratio Test: $\sum a_n$ is absolutely convergent if $\limsup \frac{|a_{n+1}|}{|a_n|} = r < 1$.
- Root Test: $\sum a_n$ is absolutely convergent if $\limsup |a_n|^{1/n} = r < 1$.

Proof (Root Test). Choose r' such that $r < r' < 1$. $\exists N > 0$ such that $\sup\{|a_n|^{1/n} : n > N\} \leq r'$. ie. $\forall n > N$, $|a)n| \leq (r')^n = \frac{1}{1-r'}$ so $\sum |a_n|$ is convergent. $\square$

Proof (Ratio Test). Follows from root test and theorem 7.5 $\square$