MATH 110 Notes

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Contents

1 1/19/2022

1.1 Overview

Linear Algebra is about:

- Vector spaces (also called linear spaces)
- Linear maps
- Extra structures on spaces/maps (inner products, etc.)

Motivation:

- Physics - we live in a 3D space
- Geometry - even for a curved object, locally it looks like a vector space (tangent space)
- Representation Theory and Group Theory - the set of all matrices has a rich structure and interesting subsets (subgroups)
- Solving Differential Equations - natural tool and solution spaces
- Normal Operators - guaranteed good bases
- Statistics - square matrices, ...
- Applied Math - designing of algorithms, ...

1.2 Ch1 - Vector Spaces

$\mathbb{R}$- set of reals, $\mathbb{R}^2$ - plane, $\mathbb{R}^3$ - 3D space

Key feature: Have addition and scalar multiplication by $\mathbb{R}$

Generalizations: Vector spaces over $\mathbb{R}$ (or a general $\mathbb{F}$)

1.3 1.A: $\mathbb{R}^n$ and $\mathbb{C}^n$

Definition 1.1 ($\mathbb{C}$).
Introduced i such that $i^2 + 1 = 0$
$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$
Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
eg: $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)^2 = 0 + i \cdot i = i$
$\mathbb{R} \subset \mathbb{C}$: view x as $x + 0i$

Theorem 1.2 (Properties of $\mathbb{C}$)**.**
Commutativity: $\alpha+\beta=\beta+\alpha, \quad \alpha\beta=\beta\alpha \quad \forall\alpha,\beta\in\mathbb{C}$
Associativity: $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma), \quad (\alpha\beta)\gamma=\alpha(\beta\gamma) \quad \forall\alpha,\beta,\gamma\in\mathbb{C}$
Additive Identity: $\alpha+0=\alpha \quad \forall\alpha\in\mathbb{C}$
Additive Inverse: $\forall\alpha\in\mathbb{C}$, $\exists!\beta\in\mathbb{C}$ such that $\alpha+\beta=0$
Multiplicative Identity: $\alpha\cdot 1=\alpha \quad \forall\alpha\in\mathbb{C}$
Multiplicative Inverse: $\forall\alpha\neq 0\in\mathbb{C}\ \exists!\beta\in\mathbb{C}$ such that $\alpha\beta=1$
Distributive Properties: $\lambda(\alpha+\beta)=\lambda\alpha+\lambda+\beta \quad \forall\lambda,\alpha,\beta\in\mathbb{C}$

2 1/24/2022

2.1 1.A: $\mathbb{R}^n$ and $\mathbb{C}^n$

Example 2.1. Show existence and uniqueness of the multiplicative inverse of $\forall a\neq 0$
Idea: Assume $\alpha=a+bi$ want $(a+bi)(?+?i)=1\to ?+?i=\frac{1}{a+bi}$ "$=$" $\frac{a-bi}{(a+bi)(a-bi)}=\frac{1-bi}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$

Proof. Assume $\alpha=a+bi$, $a,b\in\mathbb{R}$, not both zero. We see that $\beta=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$ satisfies $(a+bi)(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i)=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=1$. Similarly, $(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i)(a+bi)=1$. $\to$ existence
Moreover, if there exists $\tilde{\beta}$ such that $\alpha\tilde{\beta}=1$, then $\beta=\beta\alpha\tilde{\beta}=\tilde{\beta}$. $\to$ uniqueness ☐

Definition 2.2.

- For $\alpha\in\mathbb{C}$, let $-\alpha\in\mathbb{C}$ define the unique element such that $\alpha+(-\alpha)=0$
- For $\alpha\in\mathbb{C}$, let $1/\alpha\in\mathbb{C}$ define the unique element such that $\alpha(1/\alpha)=1$
- Subtraction: $\alpha-\beta=\alpha+(-\beta)$
- Division: $\beta/\alpha=\beta\cdot(1/\alpha)$, $\alpha\neq 0$

$\mathbb{F}$: field(In the book, $\mathbb{R}$ or $\mathbb{C}$)

- In general, generalization of $\mathbb{R}$ or $\mathbb{C}$

Definition 2.3. A set $\mathbb{F}$(with addition "+" and multiplication "$\times$") is a field if:

(i) $\exists 0,1\in\mathbb{F}$, $0\neq 1$

(ii) $+:\mathbb{F}\times\mathbb{F}\to\mathbb{F}$ by $(a,b)\mapsto a+b$

(iii) $\times:\mathbb{F}\times\mathbb{F}\to\mathbb{F}$ by $(a,b)\mapsto a\cdot b$

Satisfying:

(a) Commutivity: $a + b = b + a$, $ab = ba$

(b) Associativity: $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$

(c) Inverses: $\forall a$, $\exists -a$ such that $a + (-a) = 0$
$\forall a$, $\exists 1/a$ such that $a \cdot (1/a) = 1$

(d) Distributive: $c(a + b) = ca + cb$

Example 2.4.

1. $\mathbb{R}, \mathbb{C}$
2. $\{0, 1\} \quad +, \times \bmod 2$
3. $\mathbb{F}_p = \{0, \ldots, p-1\} \quad +, \times \bmod p$, p prime
4. $\mathbb{Q}$: rationals
5. $\{a + b\sqrt{2} : a, b, \in \mathbb{Q}$
6. $\{P(x)/Q(x) : P, Q$ are polynomials over $\mathbb{F}$, $Q \neq 0\}$

We will define $\cdot$ for $\mathbb{F}$. Elements of $\mathbb{F}$ are known as scalars (as opposed to vectors)

Definition 2.5. An n-tuple of elements of $\mathbb{F}$ is $(x_1, \ldots, x_n)$ where each $x_i \in \mathbb{F}$

Definition 2.6. $\mathbb{F}^n = \{\text{all } n\text{-tuples of elements in } \mathbb{F}\}$

Definition 2.7.

- Addition "+": $\mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ by $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto (x_1 + y_1, \ldots, x_n + y_n)$
- Scalar Multiplication "$\cdot$": $\mathbb{F} \times \mathbb{F}^n$ by $(\lambda, (x_1, \ldots, x_n)) \mapsto (\lambda x_1, \ldots, \lambda x_n)$

Theorem 2.8 (Properties of $\mathbb{F}^n$)**.**

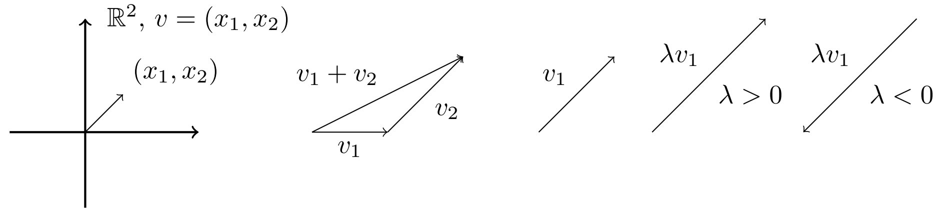
- Addition is commutative: $v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in \mathbb{F}^n$

  Proof. Assume $v_1 = (x_1, \ldots, x_n)$, $v_2 = (y_1, \ldots, y_n)$ then
  $v_1 + v_2 = (x_1 + y_1, \ldots, x_n + y_n) = (y_1 + x_1, \ldots, y_n + x_n) = v_2 + v_1$ □

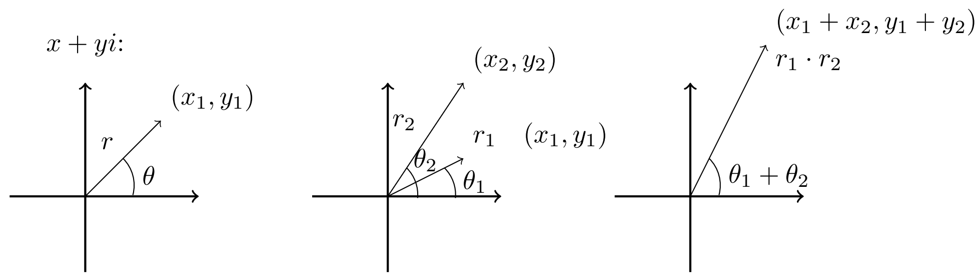
- Existence of $0 \in \mathbb{F}^n$: Denote $0 = (0, \ldots, 0)$. Then $v + 0 = v \ \forall v \in \mathbb{F}^n$
- Additive Inverse: $\forall v \in \mathbb{F}^n$, $\exists!(-v) \in \mathbb{F}^n$ such that $v + (-v) = 0$

Geometric Meaning for $\mathbb{F} = \mathbb{R}$

Descartes Coordinate System:



Geometric Meaning of Multiplication on $\mathbb{C}$



2.2 1B - Vector Spaces

Definition 2.9. Fix a field $\mathbb{F}$. A vector space over $\mathbb{F}$ is a set V with addition "+" and scalar multiplication "$\cdot$ " denoted as $+ : V \times V \to V$ by $(v_1, v_2) \mapsto v_1 + v_2$, $\cdot : \mathbb{F} \times V \to V$ by $(\lambda, v) \mapsto \lambda v$
Satisfies:

(1) $u + v = v + u$, $\forall u, v \in V$

(2) $(u + v) + w = u + (v + w)$, $a(bv) = (ab)v \ \forall u, v \in \mathbb{V}, a, b \in \mathbb{F}$

(3) $\exists 0 \in \mathbb{V}$ such that $v + 0 = v$, $\forall v \in V$

(4) $\forall v \in V$, $\exists w \in V$ such that $v + w = 0$. (we will show w is unique and denote it as $-v$)

(5) $1 \cdot v = v$, $\forall v \in V$

(6) $a(u + v) = au + av$, $(a + b)v = av + bv$, $\forall a, b \in \mathbb{F}, u, v \in V$

Definition 2.10. Elements in a vector space V are called points or vectors

Definition 2.11. A vector space over $\mathbb{F}(/\mathbb{F})$ is also called an $\mathbb{F}$-vector space

Example 2.12.

(1) $\mathbb{R}^n, \mathbb{C}^n, \mathbb{F}^n$ are all vector spaces

(2) $\mathbb{C}$ is a vector space over $\mathbb{R}$

(3) Let S be a set. Define F^s = the set of all functions from S to $\mathbb{F}$. $\mathbb{F}^S$ is a vector space $/\mathbb{F}$ under the operations $(f+g)(s) = f(s)+g(s)$, $(\lambda f)(s) = \lambda \cdot f(s)$. Each element has additive inverse $(-f)(s) = -f(s)$
$\mathbb{F}^\infty = \mathbb{F}^{\{1,2,3,\ldots,\}}$, consists of $(a_1, a_2, a_3, \ldots)$ $\forall a_n \in \mathbb{F}$

(4) the set of all sequences of real numbers that converge to 0

(5) the set of all polynomials over $\mathbb{F}$, with deg $\leq n$ in k variables is a vector space $/\mathbb{F}$

Theorem 2.13. A vector space V has a unique additive identity

Proof. Assume 0 and $0'$ are both additive inverses. Then $0 = 0 + 0' = 0'$ $\square$

Theorem 2.14. $\forall v \in V$ has a unique additive inverse.

Proof. If w_1, w_2 are both additive inverses of v, then $w_1 = w_1 + (v + w_2) = (w_1 + v) + w_2 = w_2$ $\square$

Definition 2.15. Let $w - v = w + (-v)$

Notation 2.16. V will be used to denote a vector space over $\mathbb{F}$

Theorem 2.17. $0 \cdot v = 0$, $\forall v \in V$

Proof. $0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v$ so by the existence of additive inverses $0 = 0 \cdot v$ $\square$

Theorem 2.18. $a \cdot 0 = 0$, $\forall a \in \mathbb{F}$

Proof. $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$ so $0 = a \cdot 0$ $\square$

Theorem 2.19. $(-1) \cdot v = -v$, $\forall v \in V$

Proof. $0 = 0v = (1+(-1))v = 1v + (-1)v = v + (-1)v$ so by def $(-1)v = -v$ $\square$

3 1/26/2022

3.1 1.C - Subspaces

Definition 3.1. Assuming V is a vector space $/\mathbb{F}$. $U \subset V$ is called a subspace of V if U is also a vector space $/\mathbb{F}$ under $+$ and $\cdot$ in V.

Example 3.2. $\{(x_1, x_2, 0) : x_1, x_2, \in \mathbb{F}\}$ is a subspace of $\mathbb{F}^3$

Proposition 3.3. U is a subspace iff

(i) $0 \in V$

(ii) $u_1, u_2 \in U \to u_1 + u_2 \in U$

(iii) $a \in \mathbb{F}$, $u \in U \to a \cdot u \in U$

Proof. $\to$) Suppose conditions hold. Then properties of $+$, $\cdot$ follow from V, U has identity by (i) and additive inverses by (iii). Finally, $+$, $\cdot$ well defined by (ii), (iii) so U is a subspace.
$\leftarrow$) Suppose U is a subspace. Then U is nonempty so $0 \cdot u = 0 \in U$ so (i) holds. Also, $+$, $\cdot$ well defined so (ii), (iii) hold. $\square$

Example 3.4.

(a) $\{0\}$ is a subspace

(b) $\{(x_1, x_2, x_3, x_3) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$ is a subspace iff $b = 0$

(c) $C[0,1] = \{$continuous real valued functions on $[0,1]\}$ is a subspace of $\mathbb{R}^{[0,1]}$ (over $\mathbb{R}$)

(d) $C^\infty[0,1] = \{$smooth real-valued functions on $[0,1]\}$ is a subspace $\mathbb{R}^{[0,1]}$

(e) The real sequences that converge to zero form a subspace of $\mathbb{R}^\infty$

(f) The only subspaces of $\mathbb{F}^1$ are $\{0\}$ and $\mathbb{F}$ (over $\mathbb{F}$)

(g) If U is a subspace of V, W is a subspace of U, then W is a subspace of V

(h) We will show the only subspace of $\mathbb{R}^3$ are $\{0\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$

Definition 3.5. For $U_1, \ldots, U_n$ subspaces of V, define the sum

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

also denoted as $\sum_{j=1}^m U_j$.

Example 3.6. In $\mathbb{F}^3$, what is $\{(x,x,0)\} + \{(0,y,y)\}$?

Proof. $\{(x,y,z) : y = x + z\}$ $\square$

Theorem 3.7. For subspaces $U_1, \ldots, U_m \subset V$, $\sum_{j=1}^m$ is a subspace. Moreover, it is the smallest subspace containing $U_1, \ldots, U_n$ in the sense that if W contains $U_1, \ldots, U_n$, then $W \supset U$.

Proof. Subspace: (i) $0 \in U_i$ for $i = 1, \ldots, n$ so $0 = 0 + \cdots + 0 \in W$
(ii)/(iii): follow from closedness of each U_j
Containing $U_1, \ldots, U_n$: Consider the sum $0 + \cdots + 0 + u_j + 0 + \cdots + 0$ for $j = 1, \ldots, m$
Smallest Subspace: Suppose W contains $U_1, \ldots, U_m$ then W contains $u_1, \ldots, u_m$ $\forall u_j \in U_j$ so $u_1 + \cdots + u_m \in W$. $\square$

3.2 Direct Sums

Definition 3.8. If $U_1, \dots, U_m$ are subspaces of V then the sum $U_1 + \cdots + U_m$ is a direct sum if each element in $U_1 + \cdots + U_m$ can be written as $u_1 + \cdots + u_m$ in a unique way with $u_j \in U_j$. In this case, we also use $U_1 \oplus \cdots \oplus U_m$ to denote $U_1 + \cdots + U_m$.

Example 3.9.

(1) If $U_1 = \{(x_1, x_2, 0) x_1, x_2 \in \mathbb{F}\}$, $U_2 = \{(0, 0, x_3) x_3 \in \mathbb{F}\}$, then $\mathbb{F}^3 = U_1 \oplus +U_2$.

(2) Let $U = \{(x, x, \dots) \in \mathbb{R}^\infty$, $V = \{(x_1, x_2, \dots) \in \mathbb{R}^\infty : \sum |x_n| < \infty, \sum x_n = 0\}$. Then $U + V$ is a direct sum.
(ex): Prove $U + V \neq \mathbb{R}^\infty$

Theorem 3.10. $U_1 + \cdots + U_m$ is a direct sum iff $\exists!$ way to write 0 as a sum of $u_1 + \cdots + u_m$, $\forall u_j \in U_j$ (which is $0 = 0 + \cdots + 0$).

Proof. $\rightarrow$) by def
$\leftarrow$) For $u \in U_1 + \cdots + U_m$, assume $u = u_1 + \cdots + u_m = \tilde{u_1} + \cdots + \tilde{u_n}$, $u_j, \tilde{u_j} \in U_j$. Then $(u_1 - \tilde{u_1}) + (u_2 - \tilde{u_2}) + \cdots + (u_m - \tilde{u_m}) = 0$. Hence $u_1 - \tilde{u_1} = u_2 - \tilde{u_2} = \cdots = 0$. Thus there is only one way to write u as $\sum_{j=1}^m$, $\forall u_j \in U_j$. $\square$

Theorem 3.11. For subspaces $U_1, U_2 \in V$, $U_1 + U_2$ is a direct sum iff $U_1 \cap U_2 = \{0\}$.

Proof. $\rightarrow$) If $v \in U_1 \cap U_2$, $\underbrace{v}_{\in U_1} + \underbrace{(-v)}_{\in U_2} = 0$ so $v = (-v) = 0$
$\leftarrow$) Take $u \in U_1 + U_2$ assume $u = u_1 + u_2 = \tilde{u_1} + \tilde{u_2}$. Then $\underbrace{u_1 - \tilde{u_1}}_{\in U_1} = \underbrace{-(u_2 - \tilde{u_2}}_{\in U_2}$ so by assumptions, $u_1 = \tilde{u_1}$ and $u_2 = \tilde{u_2}$. $\square$

Example 3.12. For subspaces $U_1, \dots, U_m$ of V, TFAE:

(i) $U_1 + \cdots + U_m$ is a direct sum

(ii) $\forall j$ $U_j \cap (\sum_{k \neq j} U_k) = \{0\}$

(iii) $\forall j > 1$, $U_j \cap (U_1 + \cdots + U_{j-1}) = \{0\}$

(iv) If $u_1 + \cdots + u_m = 0$, $u_j \in U_j$ then $u_1 = u_2 = \cdots = u_m = 0$

3.3 Chapter 2: Finite Dimensional Vector Spaces

$\mathbb{F}$: field, V : Vector space $/\mathbb{F}$

3.4 2.A: Span and Linear Independence

Motivation: In some V(such as $\mathbb{F}^n$), we can find vectors $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ such that every $v \in V$ can be written as $v = \sum_{j-1}^{n} a_j e_j$ and the choice of a_j is unique.

We will work with such vectors in a general setting.

4 1/31/2022

4.1 Chapter 2: Finite Dimensional Vector Spaces

- Main motivation: Find "coordinate systems" in a vector space
- Recall in $\mathbb{F}^n$, $(x_1, \ldots, x_n) = x_1(1, 0 \ldots, 0) + x_n(0, \ldots, 0, 1) = x_1 e_1 + \cdots + x_n e_n$.

4.2 2.A: Span and Linear Independence

Definition 4.1. A linear combination of vectors $v_1, \ldots, v_m \in V$ is a vector of the form

$$v = \sum_{j=1}^{m} a_j v_j, \quad a_1, \ldots, a_m \in \mathbb{F}.$$

Example 4.2. $(1, 2, -3) = (1, 0, -1) + 2(0, 1, -1)$

Example 4.3. Is $(1, 2, 3)$ a linear combination of $(1, 0, -1)$ and $(0, 1, 1)$?
No, if $(1, 2, -3) = a_1(1, 0, -1) + a_2(0, 1, 1)$ then $a_1 = 1, a_2 = 2$ but $1(1, 0, -1) + 2(0, 1, 1) = (1, 2, 1) \neq (1, 2, -3)$.

Definition 4.4. The set

$$\{\sum_{j=1}^{m} a_j v_j, a_i \in \mathbb{F}, \forall 1 \leq j \leq m\}$$

is the span of $v_1, \ldots, v_m$, denoted by $\operatorname{span}(v_1, \ldots, v_m)$. Note $\operatorname{span}() = \{0\}$.

Example 4.5. $(1, 2, -3) \in \operatorname{span}((1, 0, -1), (0, 1, -1))$.

Theorem 4.6. $\operatorname{span}(v_1, \ldots, v_m)$ is the smallest subspace of V that contains $v_1, \ldots, v_m$.

Proof. Subspace: $0 = 0v_1 + \cdots, 0v_n \in \operatorname{span}(v_1, \ldots, v_m)$
Closed under addition: $(a_1v_1 + \cdots + a_mv_m) + (b_1v_1 + \cdots + b_mv_m) = (a_1 + b_1)v_1 + \cdots + (a_m + b_m)v_m$.
Closed under multiplication: $\lambda(a_1v_1 + \cdots + a_mv_m) = (\lambda a_1)v_1 + \cdots + (\lambda a_m)v_m$.
So it is a subspace.
Smallest: If $v_1, \ldots, v_m \in W$ for some subspace W, then $\forall a_1, \ldots, a_n \in \mathbb{F}$, $a_1v_1, \ldots, a_mv_m \in V$ so $a_1v_1 + \cdots + a_mv_m \in W$. Thus, $\operatorname{span}(v_1, \ldots, v_m) \subseteq W$. $\square$

Definition 4.7. If $V = \text{span}(v_1, \dots, v_m)$, then we say the list $v_1, \dots, v_m$ spans V.

Example 4.8. $e_1, \dots, e_n$ spans $\mathbb{F}^n$

Definition 4.9. V is called finite dimensional if some (finite) list of vectors spans V.

Example 4.10. $\mathbb{F}^n$ is finite dimensional.

Definition 4.11. A finite expression

$$p(z) = a_0 + a_1 z^1 + \cdots + a_m z^m, \quad a_0, \dots, a_m \in \mathbb{F}, a_m \neq 0,$$

also written as

$$\sum_{j=0}^{\infty} a_j x_j, a_{n+1} = a_{n+2} = \cdots = 0,$$

is called a polynomial with coefficients in $\mathbb{F}$. (By definition $p = 0$ is a polynomial.)

- Each polynomial over $\mathbb{F}$ gives rise to a function from $\mathbb{F} \to \mathbb{F}$ defined by $p : \mathbb{F} \to \mathbb{F}$ by $z \mapsto p(z)$
- m is the degree of p if p has the form $(*)$. The zero polynomial has degree $-\infty$ by definition.
- $\mathcal{P}(\mathbb{F}) = \{\text{all polynomials over } \mathbb{F}\}$
- $\mathcal{P}_m(\mathbb{F}) = \{\text{all polynomials of deg} \leq m \text{ over } \mathbb{F}\}$

Example 4.12. $\mathcal{P}_m(\mathbb{F})$ and $\mathcal{P}(\mathbb{F})$ are vector spaces over $\mathbb{F}$ (also subspaces of $\mathbb{F}^{\mathbb{F}}$ if viewed as functions.)

Example 4.13.

(a) $\mathcal{P}_m(\mathbb{F})$ is finite dimensional

(b) $\mathcal{P}(\mathbb{F})$ is infinte dimensional

Proof.

(a) $1, z, \dots, z^m$ spans $\mathcal{P}_m(\mathbb{F})$

(b) For any $p_1, \dots, p_m \in \mathcal{P}(\mathbb{F})$, assume N is larger than $\deg p_j$ for $1 \leq j \leq m$. Then every $\sum_{j=1}^{m} a_j p_j$ is not equal to z^N.

$\square$

Definition 4.14. $v_1, \dots, v_m$ is called linearly independent if whenever $0 \sum_{j=1}^{m} a_j v_j$, $a_1, \dots, a_m \in \mathbb{F}$, we must have $a_1 = \cdots = a_m = 0$. Otherwise, the list is linearly dependent. The empty list is linearly independent by definition.

Example 4.15.

(a) v is linearly independent iff $v \neq 0$

(b) $e_1, \ldots, e_n$ is linearly independent in $\mathbb{F}^n$

(c) v_1, v_2 is linearly independent iff neither vector is a scalar multiple of the other.

(d) $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}_m(\mathbb{F})$.

(e) $(1, *, *), (0, 1, *), (0, 0, 1)$ where each $*$ is arbitrary is linearly independent in $\mathbb{F}^3$

(f) $(1, 1, \ldots, 1), (a_1, a_2, \ldots, a_n), (a_1^2, a_2^2, \ldots a_n^2), \ldots, (a_1^{n-1}, a_2^{n-1}, \ldots, a_n^{n-1})$ is linearly dependent iff at least two of the a_j's are the same.

(g) Any subset of a linearly independent list is linearly independent.

Example 4.16.

(a) If some vector in a list is a linear combination of the others, then the list is linearly dependent.

(b) Every list containing 0 is linearly dependent.

5 2/2/2022

5.1 2.A: Span and Linear Independence

Notation 5.1. $\mathcal{P}(\mathbb{F})$ can also be written as $\mathbb{F}[x]$

Lemma 5.2. For $v_1, \ldots, v_n \in V$, TFAE:

(a) $v_1, \ldots, v_n$ is linearly dependent.

(b) $\exists\, 1 \leq j \leq n$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1}$

(c) $\exists\, 1 \leq j \leq n$ such that $v_j \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ (Note: here $\hat{v}_j$ means v_j is excluded from the list)

(d) $\exists\, 1 \leq j \leq n$ such that $\operatorname{span}(v_1, \ldots, v_n) = \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$.

Proof. a $\to$ b) By def, $\exists\, a_1, \ldots, a_n \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \cdots + a_n v_n = 0$. Take the largest j such that $a_j \neq 0$. Then, $a_1 v_1 + \cdots + a_j v_j = 0$. Hence, $v_j = -\frac{a_1}{a_j} v_1 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$ so $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.
b $\to$ c) Notice $\operatorname{span}(v_1, \ldots, v_{j-1}) \subset \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ so $v_j \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$.
c $\to$ d) By assumption $v_j \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$. Also $v_k \in \operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ for $k \neq j$ so $\operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$ contains $v_1, \ldots, v_n$. Thus, it contains $\operatorname{span}(v_1, \ldots, v_n)$. Since $\operatorname{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n) \subset \operatorname{span}(v_1, \ldots, v_n)$, the two are equal.
d $\to$ a) By assumption, $\exists\, b_k \in \mathbb{F}$, $1 \leq k \leq n$, $k \neq j$ such that $v_j = \sum_{j \neq k} b_k v_k$. So $\sum_{j \neq k} b_k v_k - v_j = 0$ so the set is linearly dependent. $\square$

Theorem 5.3. If $v_1, \ldots, v_m$ spans V, and $u_1, \ldots, u_n \in V$ are linearly independent, then $n \leq m$.

Idea. If $m = 2$, why can't $n = 3$?
Consider the functions:

$$\begin{aligned} u_1 &= a_{1,1}v_1 + a_{1,2}v_2 \\ u_2 &= a_{2,1}v_1 + a_{2,2}v_2 \\ u_3 &= a_{3,1}v_1 + a_{3,2}v_2 \end{aligned}$$

We must rearrange u_1, u_2, u_3 to show they are linearly dependent (3 equations in 2 variables.) □

Proof. We will proceed by induction on m.
Note that for $m = 0$, span() $= \{0\}$ so this is trivially true.
Basis: If $m = 1$, $n \geq 2$. Let v_1 span V and let $u_1, u_2 \in V$ be arbitrary. Then $u_1 = \lambda_1 v_1$ and $u_2 = \lambda_2 v_2$. If $\lambda_1 = 0$, then $u_1 = 0$ and the set is linearly dependent so assume $\lambda_1 \neq 0$. Then $\lambda_2 u_1 - \lambda_1 u_2 = 0$ so the list is linearly dependent.
Inductive Step: Assume that the theorem holds for $m = k$. It suffices to show the $m = k + 1$ case. Let $v_1, \ldots, v_{k+1}$ be a spanning list of V. If $n \geq K + 2$, let

$$u_i = \sum_{j=1}^{k+1} a_{i,j} v_j, \quad 1 \leq i \leq k+2, \quad a_{i,j} \in \mathbb{F},$$

be a list of $k + 2$ vectors.
If all $a_{i,k+1} = 0$, then the list of vectors can be represented using only the vectors $v_1, \ldots, v_k$ so they would be linearly independent by the IH.
Otherwise, WLOG, assume $a_{k+2,k+1} \neq 0$ (if not we can relabel the vectors to achieve this). Then

$$u_i - \frac{a_{i,k+1}}{a_{k+2,k+1}} u_{k+2} \in \operatorname{span}(v_1, \ldots, v_k)$$

for $1 \leq i \leq k + 1$.
By IH, $\exists\, b_1, \ldots, b_{k+1} \in \mathbb{F}$, not all 0, such that

$$b_1(u_1 - \frac{a_{1,k+1}}{a_{k+2,k+1}} u_{k+2}) + \cdots + b_{k+1}(u_{k+1} - \frac{a_{k+1,k+1}}{a_{k+2,k+1}} u_{k+1} u_{k+2}) = 0$$

so

$$b_1 u_1 + \cdots + b_{k+1} u_{k+1} - (b_1 \frac{a_{1,k+1}}{a_{k+2,k+2}} + \cdots + b_{k+1} \frac{a_{k+1,k+1}}{a_{k+1,k+2}} u_{k+2}) = 0$$

so the list $u_1, \ldots, u_{k+2}$ is linearly dependent. □

Example 5.4. $e_1, \ldots, e_n$ spans $\mathbb{F}^n$ and is linearly independent so:

- $(1, 2, 3), (4, 5, 8), (4, 6, 7), (-3, 2, 8)$ are linearly dependent in $\mathbb{F}^3$

- $(1, 2, 3, -5), (4, 5, 8, -3), (4, 6, 7, -1)$ does not span $\mathbb{F}^4$

Proposition 5.5. Every subspace of a finite dimensional vector space is finite dimensional.

Proof. Assume V is spanned by $v_1, \ldots, v_m$, and U is a subspace of V.
Start from the empty list () in U and add vectors 1 by 1 so that no vector is in the span if the previous vectors in the list. This produces a linearly independent list in U.
By the thm, this process must terminate since the length of a list of linearly independent vectors in V cannot be greater than m.
Assume we have $u_1, \ldots, u_n$. Then each $u \in U$ is a linear combination of $u_1, \ldots, u_n$, otherwise we could add it to the list and it would produce a longer linearly independent list. Thus, $u_1, \ldots, u_n$ spans U. $\square$

5.2 2.B - Bases

Definition 5.6. A basis of V is a list of vectors in V that is linearly independent and spans V.

Theorem 5.7. Every finitely dimensional vector space has a basis.

Proof. Take $U = V$ in the proof of proposition 5.5. Then we can generate a linearly independent list in V that spans V. Thus V has a basis. $\square$

Example 5.8.

(a) $e_1, \ldots, e_n$ forms a basis of $\mathbb{F}^n$ (standard basis)

(b) $(1, 2, 3), (3, 4, 6), (0, 0, 1)$ is a basis of $\mathbb{F}^3$ unless char $\mathbb{F} = 3$

(c) $(1, -1, 0), (0, 1, -1)$ is a basis of $\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$

(d) $1, z, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbb{F})$

(e) $f_0, f_1, \ldots, f_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$ if $\deg f_j = j$, $0 \le j \le m$

Proposition 5.9. $v_1, \ldots, v_m$ forms a basis of V iff $\forall v \in V$ can be uniquely represented as $v = \sum_{j=1}^{n} a_j v_j$, $a_j \in \mathbb{F}$.

Proof. If $v_1, \ldots, v_n$ forms a basis of V, then they span V so all vectors can be represented in the desired form. Suppose $\exists\, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}$ such that $a_1v_1 + \cdots + a_nv_n = v = b_1v_1 + \cdots + b_nv_n$, then $(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n = 0$. Since the set is linearly independent, $a_1 - b_1 = \cdots = a_n - b_n = 0$ so $a_i = b_i$ for all i, thus the representation is unique.
If the stated conditions hold, then the list spans v. Also, 0 has a unique representation so the list is linearly independent and hence a basis. $\square$

Proposition 5.10. Every spanning list in a finite dimensional vector space contains a basis.

Proof 1. Starting from (), we can create a linearly independent list consisting of vectors in the spanning list by the same procedure as proposition 5.5. This process must terminate since the spanning list is finite. So we produce a linearly independent list that spans V, eg. a basis. $\square$

Proof 2. We can also start with the spanning list $v_1, \ldots, v_m$ and at each step, if the list is linearly dependent, we can choose v_j such that $\mathrm{span}(v_1, \ldots, v_n) = \mathrm{span}(v_1, \ldots, \hat{v}_j, \ldots, v_n)$. This process must terminate in a linearly independent list since the spanning list is finite. So we produce a linearly independent list that spans V, eg. a basis. $\square$

Proposition 5.11. Every linearly independent list of vectors in a finite dimensional vector space can be extended to a basis.

Proof. Next Class. $\square$