MATH 104: Real Analysis

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Contents

Chapter 1

Sequences and Series

1.1 January 18

1.1.1 Natural Numbers

- $\mathbb{N} = \{0, 1, 2, 3, \ldots, \}$
- successor construction: 2 is the successor of 1, 3 is the successor of 2. So starting from 0 one can reach all rational numbers (for any given natural number, it can be reached from 0 in finitely many steps)
- Peano Axioms for natural Numbers (see Tao 1)
  - Mathematical Induction Property (Axiom 5): let n be a natural number and let $P(n)$ be a statement depending on n, if the following two conditions hold:
    * $P(0)$ is true
    * If $P(k)$ is true, then $P(k+1)$ is true

    then $P(n)$ is true for all $n \in \mathbb{N}$
- operations allowed for $\mathbb{N} : +, \times$
  - if $n, m \in \mathbb{N}$, then $n + m \in \mathbb{N}$ and $n \times m \in \mathbb{N}$
  - $-, /$ are not always defined

1.1.2 Integers

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- allowed operations: $+, -, \times$ (formally, $\mathbb{Z}$ is a ring)

1.1.3 Rational Numbers

- $\mathbb{Q} = \{\frac{m}{n} | m, n \in \mathbb{Z}, n \neq 0\}$
- We have all four operations $+, -, \cdot, /$
- $\mathbb{Q}$ is now a field

Theorem 1.1.1 (Field Axioms(Ross 3)).
Addition:

- $a + (b + c) = (a + b) + c$ for all a, b, c
- $a + b = b + a$ for all a, b
- $a + 0 = a$ for all a
- For each a, there is an element $-a$ such that $a + (-a) = 0$

Multiplication:

- $a(bc) = (ab) = c$ for all a, b, c
- $ab = ba$ for all a, b
- $a \cdot 1 = a$ for all a
- For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$

Distributive Law:

- $a(b + c) = ab + ac$ for all a, b, c

Theorem 1.1.2 (Useful Properties of Fields(Ross 3)).

- $a + c = b + c$ implies $a = b$
- $(-a)b = -ab$ for all a, b
- $(-a)(-b) = ab$ for all a, b
- $ac = bc$ and $c \neq 0$ imply $a = b$
- $ab = 0$ implies either $a = 0$ or $b = 0$

for $a, b, c \in \mathbb{Q}$

$\mathbb{Q}$ is an ordered field, there is a "relation" $\leq$

Definition 1.1.3. A relation S is a subset of $\mathbb{Q} \times \mathbb{Q}$, if $(a, b) \in S$ we say "a and b have relation S" or "aSb"

The relation "$\leq$" has 3 properties:

- if $a \leq b$ and $b \leq a$, then $a = b$
- if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity)
- for any $a, b \in \mathbb{Q}$, at least one of the following is true: $a \leq b$ or $b \leq a$

Since $\mathbb{Q}$ is an ordered field, the field structure $(+, -, \cdot, /)$ is compatible with $(\leq)$

- If $a \leq b$, then $a + c \leq b + c$ for all $c \in \mathbb{Q}$
- If $a \geq 0$ and $b \geq 0$, then $ab \geq 0$

Theorem 1.1.4 (Useful Properties of Ordered Fields(Ross 3))**.**

- If $a \leq b$, then $-b \leq a$
- If $a \leq b$ and $c \geq 0$, then $ac \leq bc$
- If $a \leq b$ and $c \leq 0$, then $bc \leq ac$
- $0 \leq a^2$ for all a
- $0 < 1$
- If $0 < a$, then $0 < a^{-1}$
- If $0 < a < b$, then $0 < b^{-1} < a^{-1}$

for $a, b, c \in \mathbb{Q}$

1.1.4 What's lacking in $\mathbb{Q}$?

1. There are certain gaps in $\mathbb{Q}$. For example, the equation $x^2 - 2$ cannot be solved in $\mathbb{Q}$
2. For a bounded set in $\mathbb{Q}$, E, it may not have a "most economical" or "sharpest" upper bound in $\mathbb{Q}$
   Ex: $E = \{x \in \mathbb{Q} | x^2 < 2\}$ there is no least upper bound(sup) of E in $\mathbb{Q}$ (we want to take $\sqrt{2}$ as $\sup(E)$ but $\sqrt{2}$ is not a rational number)

1.2 January 20

1.2.1 Rational Zeros Theorem

Definition 1.2.1. An integer coefficient polynomial in x is of the form: $c_n x^2 + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ $c_1, \ldots, c_n \in \mathbb{Z}$, $c_n \neq 0$.

1. A $\mathbb{Z}$-coefficient equation is $f(x) = 0$
2. One can ask: when does a $\mathbb{Z}-$coefficient equation have roots in $\mathbb{Q}$

Fact 1.2.2. A degree n polynomial has n roots in $\mathbb{C}$, ie. $\exists z_1, \ldots, z_n \in \mathbb{C}$ such that $f(x) = c_n(x - z_1) \cdots (x - z_n)$

Theorem 1.2.3. If a rational number r satisfies the equation $x_n x^n + \cdots + c_1 x + c_0 = 0$, with $c_i \in \mathbb{Z}, c_n, c_0 \neq 0$ and $r = \frac{c}{d}$(where c and d are coprime integers). Then c divides c_0 and d divides c_n.

Proof. Plug in $x = \frac{c}{d}$ into the equation to get $c_n(\frac{c}{d})^n + c_{n-1}(\frac{c}{d})^{n-1} + \cdots + c_1(\frac{c}{d}) + c_n = 0$ multiply both sides by d^n to get $c_n c^n + c_{n-1}c^{n-1}d + \cdots + c_1 c d^{n-1} + c_0 d = 0$
Since $c_n c^n = -d(c_{n-1}c^{n-1} + \cdots + c_1 d^{n-1})$, d divides $c_n c^n$. Since d and c are coprimes, d does not divide c^n so d has to divide c_n
Also, since $c_0 d^n = -c(c_n c^{n-1} + c_{n-1}c^{n-2}d + \cdots + c_1 d^{n-1})$ by similar reasoning $c | c_0$

Using the rational zeros theorem, we can answer questions about rationality

Example 1.2.4. Show $\sqrt[3]{6}$ is irrational.
$\sqrt[3]{6}$ is rational $\leftrightarrow$ $x^3 - 6$ has rational roots. The only possible rational roots such that $r = \frac{c}{d}$ need $c|6, d|1$. Taking $d = 1$, $c = \pm 1, \pm 2, \pm 3, \pm 6$. Once can check all of these do not satisfy the equation so there is no solution in $\mathbb{Q}$

1.2.2 Historical Construction of $\mathbb{R}$ from $\mathbb{Q}$

1. Dedekind Cut: (Q: if $\sqrt{2} \notin \mathbb{Q}$, how can we save the information of $\sqrt{2}$?)
   A: the subset of $\mathbb{Q}$ $C_{\sqrt{2}} = \{r \in \mathbb{Q} | r > x\}$
   For every $x \in \mathbb{R}$, consider $C_x = \{x \in \mathbb{Q} | r < x\}$. We can define addition, multiplication on the subsets C_x

2. Sequences in $\mathbb{Q}$
   ie. Use a sequence of rational numbers to "aproximate" a real number
   eg. $\sqrt{2}$ can be approximated by $1, 1.4, 1.41.1.414, \ldots$
   Problems:

   (a) Given any real number, how do you get such a sequence?

   (b) How do you determine if 2 different sequences approximate the same real number
   (eg. $1 \leftarrow 1.1, 1.01, 1.001, \ldots$ or $1 \leftarrow 0.9, 0.99, 0.999, \ldots$ or $1 \leftarrow 1, 1, 1, \ldots$) all have the same limit

1.2.3 Properties (Axioms) of $\mathbb{R}$

Given the existence of $\mathbb{R}$, we have certain properties (axoims) of $\mathbb{R}$

Definition 1.2.5. A subset of $\mathbb{R}$ is said to be bounded above if $\exists a \in \mathbb{R}$ such that for any $x \in E$, we have $x \leq a$

Theorem 1.2.6 (Completeness Axiom of $\mathbb{R}$)**.** Given a set $E \subset \mathbb{R}$, bounded above, there exists a unique r such that:

1. r is an upper bound of E

2. for any other upper bound of α, we have $r \leq \alpha$

r is called the least upper bound of E, $r = \sup E$
(ie. $\sup E$ is well defined for subsets that are bounded above)

Example 1.2.7. $\sup([0, 1]) = 1$, $\sup((0, 1)) = 1$, $\sup(\{r \in \mathbb{Q} | r^2 < 2\}) = \sqrt{2}$

Theorem 1.2.8 (Archimedean Property)**.** For any $r \in \mathbb{R}$, $r > 0$ $\exists n \in \mathbb{N}$ such that $nr > 1$ or equivalently, $r > \frac{1}{n}$

1.2.4 $+\infty, -\infty$

- With these symbols, we can say $\sup(\mathbb{N}) = +\infty \leftrightarrow \mathbb{N}$ is not bounded above

- $+\infty, -\infty$ are not real numbers. They have part of the defined operations $\mathbb{R}$ has
  ie. $3 \cdot +\infty = +\infty$, $(-3) \cdot +\infty = -\infty$ but $(+\infty) + (-\infty) =$NAN, $0 \cdot (+\infty) =$ undefined.

1.2.5 Sequences and Limits

- A sequence of real numbers is: $a_0, a_1, a_2, \dots$ denoted $(a_n)_{n=0}^{\infty}$ or shortened (a_n)
- We care about the "eventual behavior" of a sequence

Definition 1.2.9. A sequence (a_n) converges to $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > \mathbb{N}$, $|a_n - a| < \varepsilon$.

1.3 January 25

1.3.1 Sequences and Limits

Definition 1.3.1. A sequence (a_n) is bounded if $\exists M > 0$, $|a_n| \leq M$ for all n.

Theorem 1.3.2. Convergent sequences are bounded.

Proof. Let (a_n) be a convergent sequence that converges to a.
Let $\varepsilon = 1$, then by definition of convergence, there exists $N > 0$ such that $\forall n > n$

$$|a_n - a| < 1 \iff a - 1 < a_n < a + 1 \quad \forall n > N.$$

Let $M = \max\{a_1, a_2, \dots, a_N\}$, $M_2 = \max\{|a-1|, |a+1|\}$ and $M = \max\{M_1, M_2\}$. Thus if $n \leq N$ we have $|a_n| \leq M$, and if $n \geq N$ we have $|a_n| \leq M_2$ so

$$\forall n, |a_n| \leq \max\{M_1, M_2\} = M$$

Remark 1.3.3. One can deal with the first few terms of a sequence easily, it is the "tail of the sequence" that matters.

1.3.2 Operations on Convergent Sequences

Theorem 1.3.4. $c \in \mathbb{R}$, $\forall$ convergent sequences $a_n \to a$, we have $c \cdot a_n \to c \cdot a$.

Proof. If $c = 0$, the result is obvious.
If $c \neq 0$, we want to show for all $\varepsilon > 0$, $\exists N$ such that $\forall n > N$

$$|c \cdot a_n - c \cdot a| < \varepsilon \iff |c| \cdot |a_n - a| \leq \varepsilon \iff |a_n - a| \leq \tfrac{\varepsilon}{|c|}.$$

Now let $\varepsilon' = \frac{\varepsilon}{|c|}$. By definition of $a_n \to a$, we have $N > 0$ such that $|a_n - a| \leq \varepsilon' = \frac{\varepsilon}{|c|}$. This gives the desired N.

Theorem 1.3.5. If $a_n \to a$, $b_n \to b$, then $a_n + b_n \to a + b$.

Proof. We want to show $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$

$$|a_n + b_n - (a+b)| \leq \varepsilon \iff |(a_n - a) + (b_n - b)| \leq \varepsilon. \tag{*}$$

$|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ by the triangle inequality so

$$(*) \leftarrow |a_n - a| < \varepsilon \tag{**}$$

$$\leftarrow \begin{cases} |a_n - a| \leq \varepsilon/2 \\ |b_n - b| \leq \varepsilon/2 \end{cases} \tag{***}$$

By the convergence of a_n and b_n, $\exists N_1, N_2$ such that $\forall n > N_1$, $|a_n - a| \leq \frac{\varepsilon}{2}$, and $\forall n > N$, $|b_n - b| \leq \frac{\varepsilon}{2}$. Take $N = \max\{N_1, N_2\}$, then $\forall n > N$ $(***)$ is satisfied hence $(*)$ is satsified.

Corollary 1.3.6. If $a_n \to a$, $b_n \to b$, then $a_n - b_n \to a - b$.

Proof. Let $c_n = (-1) \cdot b_n$. Then $c_n \to -b$ so $a_n + c_n \to a - b$.

Theorem 1.3.7. If $a_n \to a$, $b_n \to b$, then $a_n \cdot b_n \to ab$.

Proof. Want to show: $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$

$$|a_n - ab| \leq \varepsilon. \tag{*}$$

Since a_n is convergent, it is bounded by some $M > 0$ which yields the following inequalities.

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b - b) + a_n b - ab| \\ &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n(b_n - b)| + |(a_n - a)b| \\ &\leq |a_n| \cdot |b_n - b| + |a_n - a| \cdot |b| \\ &\leq M|b_n - b| + |b||a_n - a| \end{aligned}$$

So

$$(*) \leftarrow \begin{cases} M|b_n - b| \leq \varepsilon/2 \\ |b||a_n - a| \leq \varepsilon/2 \end{cases} . \tag{**}$$

Since $a_n \to a$, let $\varepsilon_1 = \frac{\varepsilon}{2|b|}$, then $\exists N$ such that $\forall n > N$,

$$|a_n - a| < \varepsilon_1 \iff |b||a_n - a| \leq \frac{\varepsilon}{2}.$$

Also, since $b_n \to b$, let $\varepsilon_2 = \frac{\varepsilon}{2M}$, then $\exists N$ such that $\forall n > N$,

$$|b_n - b| \leq \varepsilon_2 \iff M|b_n - b| \leq \frac{\varepsilon}{2}.$$

. Let $N = \max\{N_1, N_2\}$, then for $n > N$, $(**)$ holds so $(*)$ holds.

Theorem 1.3.8. If $a_n \to a$, and $a_n \neq 0 \, \forall n$ and $a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$.

Remark 1.3.9. $a_n \neq 0$ does not imply $a \neq 0$. For example consider the sequence $a_n = \frac{1}{n}$

Proof. Want to show $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$,

$$|\frac{1}{a} - \frac{1}{a_n}| \leq \varepsilon. \tag{*}$$

Observe that

$$|\frac{1}{a} - \frac{1}{a_n}| = |\frac{a - a_n}{a \cdot a_n}| = \frac{|a_n - a|}{|a| \cdot |a_n|}.$$

Claim: $\exists c > 0$ such that $|a_n| > c \, \forall n$.

Proof. Let $\varepsilon' = \frac{\varepsilon}{2}$, then $\exists N'$ such that $\forall n \geq N'$

$$\begin{aligned} |a_n - a| \leq \varepsilon' = \frac{\varepsilon}{2} &\iff -|a|/2 < a_n - a < |a|/2 \\ &\iff a - \frac{|a|}{2} < a_n < a + \frac{|a|}{2} \to |a_n| \geq \frac{|a|}{2} \end{aligned}$$

Let $c_1 = \min\{|a_1|, |a_2|, \ldots, |a_{N'}|\} \geq 0$. Let $c = \min\{c_1, |a|/2\}$.

Thus, $\frac{|a_n - a|}{|a| \cdot |a_n|} \leq \frac{|a_n - a|}{|a| \cdot c}$. Hence

$$(*) \leftarrow \frac{|a_n \cdot a|}{|a| \cdot c} \leq \varepsilon \tag{**}$$

and $(**)$ can be satisfied since $a_n \to a$.

Corollary 1.3.10. If $a_n \to a$, $b_n \to b$ and $b_n \neq 0$, $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.

Proof. $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$. Since by Thm 8, $\frac{1}{b_n} \to \frac{1}{b}$, $\; a_n \cdot \frac{a}{b_n} \to a \cdot \frac{1}{b}$ by Thm 7.

Theorem 1.3.11 (Useful Results)**.**

(1) $\lim_{n\to\infty} \frac{1}{n^p} = 0 \; \forall p > 0$.

(2) $\lim_{n\to\infty} a^n = 0 \; \forall |a| < 1$.

(3) $\lim_{n\to\infty} n^{1/n} = 1$.

(4) $\lim_{n\to\infty} a^{1/n} = 1$ for all $n > 0$.

Proof (Proof of (3)). Let $S_n = n^{1/n} - 1$, then $s_n \geq 0$ $\forall n$ positive integers.

$$1 + s_n = n^{1/n} \iff (1 + s_n)^n = n.$$

Using to binomial theorem we see

$$\begin{aligned} &1 + ns_n + \frac{n(n-1)}{2}s_n^2 + \cdots = n \\ &\to \frac{n(n-1)}{2}s_n^2 \leq n \\ &\to s_n^2 \leq \frac{2}{n-1} \end{aligned}$$

Thus, $s_n \to 0$ as $n \to \infty$.

1.4 January 27

1.4.1 Monotone Sequences

Definition 1.4.1 ($\lim s_n = +\infty$). A sequence (s_n) is said to "diverge to $+\infty$", if for every $M \in \mathbb{R}$ there exists N such that $s_n > M \,\forall n > N$.

Definition 1.4.2 (Values of a Sequence). If $(s_n)^\infty)_{n=1}$ is a sequence, then $\{s_n\}_{n=1}^\infty$, the subset of $\mathbb{R}$ consisting of the values of (s_n), is called the value set.

Example 1.4.3.

- $(s_n) = 1, 2, 1, 2, \ldots \quad \{s_n\}_{n=1}^\infty = \{1, 2\}$
- $(s_n) = 1, 1, 2, 2, 1, 1, 2, 2, \ldots \quad \{s_n\}_{n=1}^\infty = \{1, 2\}$
- $(s_n) = 1, 2, 3, 4, \ldots \quad \{s_n\}_{n=1}^\infty = \{1, 2, 3, 4, \ldots\}$

Definition 1.4.4 (Monotone Sequences).

- A sequence (s_n) is monotonically increasing if $a_{n+1} \geq a_n \,\forall n$
- A sequence (s_n) is monotonically increasing if $a_{n+1} \leq a_n \,\forall n$

Example 1.4.5.

- $(a_n) = a$, a constant sequence is monotonically increasing and decreasing
- $(a_n) = 1, 2, 3, \ldots$, is increasing
- $(a_n) = -\frac{1}{n}$, is increasing and bounded above (also below)

Theorem 1.4.6. A bounded monotone sequence is convergent.

Proof. (We will show for increasing, the proof for decreasing is similar.)
Let (a_n) be a bounded monotone increasing sequence and let $\gamma = \sup\{a_n\}_{n=1}^{\infty}$ $(= \sup a_n)$. Then $a_n \leq \gamma\, \forall n$ and for any $\varepsilon > 0$, $\exists a_{n_0}$ such that $a_{n_0} > \gamma - \varepsilon$. Thus for every $\varepsilon > 0$, let $N = n_0$(as defined above), then for every $n > N$, we have $\gamma - \varepsilon < a_{n_0} \leq a_n \leq \gamma$ thus $|a_n - \gamma| < \varepsilon$ then $\lim a_n = \gamma$

Example 1.4.7 (Recursive Definition of Sequences)**.** Let s_n be any positive number and let

$$s_{n+1} = \frac{s_n^2 + 5}{2s_n} \quad \forall n \geq 1. \tag{*}$$

We want to show $\lim s_n$ exists and find it.

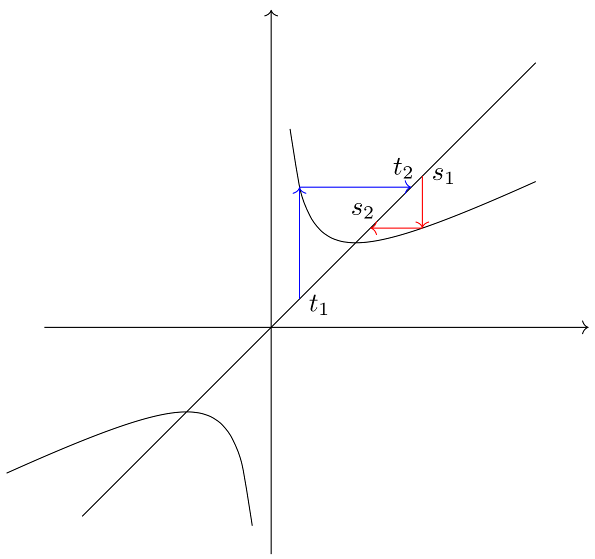
Remark 1.4.8. If we assume $\lim s_n$ exists, call it s, then s satisfies

$$s = \frac{s^2 + 5}{2s} \tag{**}$$

since we can apply $\lim_{n\to\infty}$ to both sides.

$(**) \to 2s^2 = s^2 + 5 \to s = \pm\sqrt{5}$. Since s_n is a positive sequence $\lim s_n$ can only be ≥ 0, thus s can only by $\sqrt{5}$

- To show $\lim s_n$ exists, we can only need to show s_n is bounded and monotone
- Here is a trick: let $f(x) = \frac{x^2+5}{2x}$, then $s_{n+1} = f(s_n)$
  - Consider the graph of f, ie. $y = f(x)$
  - Consider the diagonal, ie. $y = x$



- If $s_1 > \sqrt{5}$, we should try to prove $\sqrt{5} < \cdots s_3 < s_2 < s_1$
- If $0 < s_1 < \sqrt{5}$, then we show that $s_2 > \sqrt{5}$, we can consider $(s_n)_{n=1}^{\infty}$, which reduces to case 1
- If (s_n) is unbounded and increasing, then $\lim s_n = +\infty$
- If (s_n) is unbounded and decreasing, then $\lim s_n = -\infty$

1.4.2 Lim inf and sup of a sequence

Definition 1.4.9 (limsup)**.** Let $(s_n)_{n=1}^{\infty}$ be a sequence,

$$\limsup_{n\to\infty} s_n := \lim_{n\to\infty}(\sup\{s_n\}_{m=1}^{\infty})$$

- $(s_n)_{n=N}^{\infty}$ is called a "tail of the sequence (s_n)" starting at N
- $A_N = \sup\{s_n\}_{n=N}^{\infty} = \sup_{n\geq N} s_n$
- $\limsup s_n = \lim A_n = +\infty$

Example 1.4.10.

(1) $(s_n) = 1, 2, 3, 4, 5, \ldots$
$A_1 = \sup_{n\geq 1} s_n = +\infty$, $A_2 = \sup_{n\geq 2} s_n = +\infty$
$\limsup s_n = \lim A_n = +\infty$

(2) $(s_n) = 1 - \frac{1}{n}$
$A_1 = \sup_{n\geq 1} s_n = 1$, $A_2 = \sup_{n\geq 2} s_n = 1$
$\limsup s_n = \lim A_n = 1$ (for any monotonic increasing sequence $\limsup s_n = \sup s_1 = A_1$)

(3) $s_n = 1 + \frac{1}{n}$ $\quad (s_n) = 2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots$
$A_1 = \sup\{2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots\} = 2$
$A_2 = \sup\{1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, \ldots\} = 1 + \frac{1}{2}$
$A_n = s_n$ so $\limsup s_n = \lim(1 + \frac{1}{n}) = 1$

Lemma 1.4.11. $A_n = \sup_{m\geq n} s_m$ forms a decreasing sequence.

Proof. Since $\{s_n\}_{m=n}^{\infty} \supset \{s_n\}_{m=n+1}^{\infty}$, $\sup\{s_n\}_{m=n}^{\infty} \geq \sup\{s_m\}_{m=n+1}^{\infty}$, ie. $A_n \geq A_{n+1}$

Corollary 1.4.12. $\lim_{n\to\infty} A_n = \inf A_n{}_{n=1}^{\infty}$ $(= \inf_n A_n)$

Example 1.4.13. $s_n = (-1)^n \cdot \frac{1}{n}$ $\quad (s_n) = (-1, \frac{1}{2}, -\frac{1}{3}, \ldots)$
$A_1 = \sup_{n\geq 1} s_n = s_2 = \frac{1}{2}$, $A_2 = \frac{1}{2}$, $A_3 = \frac{1}{4}$, so
$(A_n) = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \ldots$ $\quad \limsup s_n = \lim A_n = 0$
A_n is like the "upper envelope."

1.5 February 1

1.5.1 Cauchy Sequences

Definition 1.5.1 (Cauchy Sequence)**.** A sequence (a_n) is cauchy if $\forall \varepsilon > 0$, $\exists N > 0$, such that $\forall n, m > N$ we have $|a_n - a_m| < \varepsilon$.

Lemma 1.5.2. If (a_n) converges to a, then (a_n) is cauchy.

Proof. Let $\varepsilon_1 = \frac{\varepsilon}{2}$, then since $a_n \to a$, $\exists N_1 > 0$ such that $\forall n, m < N$, $|a_n - a| < \varepsilon_1$ and $|a_m - a| < \varepsilon_1$. Thus,

$$|a_n - a_m| = |(a_n - a) - (a_m - a)| \leq |a_n - a| + |a_m - a| < \varepsilon_1 + \varepsilon_1 = \varepsilon.$$

Remark 1.5.3. This is also for true in $\mathbb{Q}$

Lemma 1.5.4 (Squeze Lemma)**.** Given sequences $(A_n), (B_n), (a_n)$ such that $A_n \geq a_n \geq B_n$ $\forall n$, if $A_n \to a$, $B_n \to a$, then $a_n \to a$.

Proof. $\forall \varepsilon > 0$, we have $N > 0$ such that $\forall n > N$, $|A_n - a| < \varepsilon$ and $|B_n - a| < \varepsilon$. Then $a_n \leq A_n < a + \varepsilon$ and $a_n \geq B_n > a - \varepsilon$ so

$$a - \varepsilon < a_n < a + \varepsilon \leftrightarrow |a_n - a| < \varepsilon.$$

Lemma 1.5.5. Cauchy Sequences are bounded.

Proof. Let $\varepsilon = 1$. Then $\exists N > 0$ such that $\forall n, m > N$, $|s_n - s_m| < \varepsilon$. Consider the term s_{N+1}. Observe that $\forall n < N$, $|s_{N+1} - s_m| < 1$ so $\forall n < N$, $|s_n| < s_{N+1} + 1$. Taking $M = \max\{|s_1|, |s_2|, \ldots, |s_{N+1}|, |s_{N+1}| + 1\}$, we see that $M \geq |s_n|$ for all n.

Theorem 1.5.6. If (a_n) is cauchy in $\mathbb{R}$, then (a_n) is convergent.

Proof. Since (a_n) is cauchy, (a_n) is bounded so $\limsup a_n$ and $\liminf a_n$ exist. Let $A_n = \sup_{m \geq n} a_m$, $B_n = \inf_{m \geq n} a_m$, then $A_n \geq a_n \geq B_n$. Let $A = \lim A_n$ and $B_n = \lim B_n$. By the Squeeze Lemma, we only need to show $A = B$. Since $A_n \geq B_n$, we know $A \geq B$, hence we only have to rule out $A < B$.
Assume $A < B$. Let $\varepsilon = \frac{(A-B)}{3}$. By Cauchy criterion $\exists N > 0$ such that $\forall n, m > N$, $|a_n - a_m| < \varepsilon$. By the previous lemma, since $A = \limsup a_n$ and $B = \liminf a_n$, given ε, N above, we have $n > N$ such that $|a_n - A| < \varepsilon$ and $m > N$ such that $|a_m - B| \leq \varepsilon$. Then

$$|A - B| \leq |A - a_n| + |a_n - a_m| + |a_m - B| < \varepsilon + \varepsilon + \varepsilon = A - B = |A - B|,$$

which is a contradiction.

1.5.2 Subsequences

Let (a_n) be a sequence. If we pick an infinite subset of $\mathbb{N}$, $n_1 < n_2 < n_3 < \cdots$, then we can have a new sequence $b_k = a_{n_k}$, $(b_k) = a_{n_1}, a_{n_2}, a_{n_3}, \ldots$.

Example 1.5.7. For $(a_n) = (-1)^n$, $a_1 = -1, a_2 = +1, \ldots$ does not converge but subsequence consisting of odd terms converges to -1 and subsequence consisting of even terms converges to 1.

Definition 1.5.8. Let (a_n) be a sequence. Then $a \in \mathbb{R}$ is a subsequential limit if there exists (a_{n_k}) such that $\lim_{k\to\infty} a_k = a$.

Theorem 1.5.9. Let (a_n) be a sequence. Then:

(1) a is a subsequential limit of (a_n)

(2) $\leftrightarrow \forall\varepsilon > 0$, $\forall N > 0$, $\exists n > N$ such that $|a_n - a| \leq \varepsilon$

(3) $\leftrightarrow \forall\varepsilon > 0$, the set $A_\varepsilon = \{n||a_n - a| < \varepsilon\}$ is infinite

Proof. $2 \leftrightarrow 3$) follows from definitions.
$1 \to 3$) If $a_{n_k} \to a$, then for a given $\varepsilon > 0$, $\exists K > 0$ such that $|a_{n_k} - a| \leq \varepsilon$. Thus $\{n_k|k > K\} \subset A_\varepsilon$. So A_ε is infinite.
$3 \to 1$) Cantor's Diagonal Trick: Let $A_{\frac{1}{k}} = \{n||a_n - a| \leq \frac{1}{k}\}$.
$A_1 : n_{1,1} < n_{1,2} < n_{1,3} < \cdots$
$A_2 : n_{2,1} < n_{2,2} < n_{2,3} < \cdots$
Observe that $A_{\frac{1}{k+1}} \subset A_{\frac{1}{k}}$, thus $n_{k,i} \leq n_{k+1,i}$.
Claim: $(a_{n_{k,k}}) \to a$.
First observe that this is a valid subsequence since $a_{n_{k,k}} < a_{n_{k,k+1}} \leq a_{n_{k+1,k+1}}$ for all k. Also for $\varepsilon > 0$, $\exists K$ such that $\frac{1}{K} < \varepsilon$ so for all $k > K$, $|a_n - a| < \frac{1}{K} < \varepsilon$ so it converges to a.

1.6 February 3

1.6.1 Subsequences

Proposition 1.6.1. If $s_n \to s$, then all subsequences of s_n converge to s.

Proof. Any tail of a subsequence belongs to a tail of the original sequence to they must converge to the same limit.

Proposition 1.6.2. Any sequence has a monotone subsequence.

Proof. We say that s_n is a dominant term if $s_n > sm$ for all $m > n$.
Case 1: Suppose there are infinitely many dominant terms. Then the subsequence if dominant terms forms a monotone decreasing sequence.
Case 2: There are finitely many dominant terms. Then we can choose $N > 0$ such that for all $n > N$, s_n is not dominant. We can construct an increasing sequence as follows :

- pick $n_1 > N$, and get s_{n_1}
- pick $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. This is posible since otherwise s_{n_1} would be a dominant term.
- continue in this fashion to achieve a sequence such that $s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq \cdots$

Theorem 1.6.3 (Bolzano - Weierstrass)**.** Every bounded sequence has a convergent subsequence.

Proof (Proof 1)**.** Assume WLOG, that the sequence is bounded in $[0,1]$. We may write $[0,1] = [0,\frac{1}{2}]\cup[\frac{1}{2},1]$. Then (s_n) must visit one of the intervals infinitely many times. We can then subdivide that interval and continue in a similar fashion to obtain a decreasing sequence of closed intervals $I_0 = [0,1] \supset I_1 \supset I_2 \supset \cdots$ with $|I_n| = 2^{-n}$. Let $A_n = \{n | n \in I_n\}$. Then $A_k \subset A_{k-1}$. The sequence $(a_{k,k})_k$ is a cauchy sequence since $\forall\varepsilon > 0$, $\exists k_0$ such that $\frac{1}{2^{k_0}} \leq \varepsilon$ for $k_n > k_0$.

Proof (Proof 2)**.** Every sequence contains a monotone sequence so since the sequence is bounded the given monotone sequence converges.

Proposition 1.6.4. Let (s_n) be a sequence, the $\limsup s_n$ is a subsequential limit.

Proof. We know that for $\varepsilon > 0$, $N > 0$, $\exists\, n_0 > N$ such that $|s_{n_0} - \limsup s_n| < \varepsilon$. Thus by the alternative of a subsequential limit, $\limsup s_n$ is a subsequential limit.

Remark 1.6.5. This sequence can be refined to a montone sequence by considering the monotone subsequence of the generated sequence.

Theorem 1.6.6. Let (s_n) be a bounded sequence and let S by the set of subsequential limits of (s_n). Then:

(a) $\sup S = \limsup s_n$, $\inf S = \liminf s_n$ and $\limsup s_n, \liminf s_n \in S$.

(b) $\lim s_n$ exists iff S contains only one element.

(c) S is closed under taking limits. ie. if there is a convergent sequence $t_n \to t$ with $t_n \in S$, we will have $t \in S$.

Proof.

1. For $t \in S$ suppose $s_{n_k} \to t$. Then $\limsup s_{n_k} = \liminf s_{n_k}$. Since $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$, $\liminf s_n \leq \liminf s_{n_k} = \limsup s_{n_k} \leq \limsup s_n$. Thus, $\liminf s_n \leq \inf S \leq \sup S \leq \limsup s_n$. Since by the previous proposition $\limsup s_n, \liminf s_n \in S$, $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.

2. This follows since $s_n \to s$ iff $\limsup s_n = \liminf s_n$.

3. We will show t is a subsequential limit of (s_n). We want to show, $\forall\varepsilon > 0$, $\forall N > 0$, $\exists n_0 > N$ such that $|s_{n_0} - t| \leq \varepsilon$.
   Since $t_n \to t$, $\exists N$ such that $\forall n > N$, $|t_n - t| \leq \frac{\varepsilon}{2}$. For $n_1 < N$, there are infinitely many s_n with $|s_n - t_{n_1}| \leq \frac{\varepsilon}{2}$. Thus, $\exists n_0$ such that $|s_{n_0} - t_{n_1}| \leq \frac{\varepsilon}{2}$. Thus, $|s_{n_0} - t| \leq |s_{n_0} - t_{n_1}| + |t_{n_1} - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

1.7 February 8

1.7.1 liminf and limsup (cont.)

Proposition 1.7.1. If $A = \limsup a_n$, then $\forall \varepsilon > 0$, $\exists N$ such that $\sup\{a_n : n > N\} \le A + \varepsilon$.

Example 1.7.2. For $a_n = \frac{1}{n}$, $\limsup a_n = 0$ so it is necessary to raise A by ε to have some $a_n \le A + \varepsilon$.

Proposition 1.7.3. Given $a_n \to a$, $a > 0$ and b_n bounded, then $\limsup(a_n b_n) = (\lim a_n) \cdot \limsup b_n$.

Proof. Let $b = \limsup b_n$
$\le$) We plan to show that $a \cdot b$ is a subsequential limit of $a_n \cdot b_n$, then since all subsequential limits $\le$ $\limsup(a_n b_n)$, the result follows.
We know $\exists$ subsequence (b_{n_k}) that converges to b. We also know all subsequences of (a_n) converge to a. Thus, $a_{n_k} \cdot b_{n_k} \to a \cdot b$.
$\ge$) Since $a > 0$, then $\exists N$ such that $a_n > 0$ for all $n > N$. Thus, if we throw away a_n with $n \le N$, we may assume $a_n > 0 \, \forall n$. Then $\lim \frac{1}{a_n} = \frac{1}{a}$. Thus

$$\limsup b_n = \limsup(a_n b_n \tfrac{1}{a_n}) \ge \limsup(a_n b_n) \lim(\frac{1}{a_n}) = \frac{1}{a} \limsup(a_n b_n)$$

so $a \cdot \limsup b_n \ge \limsup(a_n b_n)$

Example 1.7.4. Need $a > 0$. Consider $a_n = -1$, $b_n = 1, 3, 1, 3, \ldots$. Then $\limsup(a_n b_n) = -1$, $\limsup(b_n) = 3$, but $\lim a_n \cdot \limsup a_n b_n = (-1) \cdot 3 = -3$.

Theorem 1.7.5. Let a_n be a sequence of positive real numbers. Then

$$\liminf(\frac{a_{n+1}}{a_n}) \le \liminf a_n^{1/n} \le \limsup a_n^{1/n} \le \limsup(\frac{a_{n+1}}{a_n}).$$

Example 1.7.6.

(1) $a_n = r^n$ for $r > 0$, then $a_n^{1/n} = r$, $\frac{a_{n+1}}{a_n} = r$.

(2) $a_n = C \cdot r^n$ for $C > 0, r > 0$. Then $a_n^{1/n} = C^{1/n} \cdot r$, $\frac{a_{n+1}}{a_n} = r$ and $\lim a_n^{1/n} = r$.

(3) $a_n = \begin{cases} (\frac{1}{2})^n & n \text{ is even} \\ (\frac{1}{3})^n & n \text{ is odd} \end{cases}$, $a_n^{1/n} = \begin{cases} \frac{1}{2} & n \text{ is even} \\ \frac{1}{3} & n \text{ is odd} \end{cases}$.
However, $\lim \frac{a_{n+1}}{a_n}$ has a lot of oscillations.
In general, root test is stronger than ratio test.

Proof. Note $\liminf(\cdots) \le \limsup(\cdots)$ so middle $\le$ is obvious.
We will show $\limsup a_n^{1/n} \le \limsup \frac{a_{n+1}}{a_n}$ (other $\le$ is similar).
Assume $\limsup \frac{a_{n+1}}{a_n} = L < \infty$, then $\forall \varepsilon > 0$, $\exists N > 0$ such that $\sup\{\frac{a_{n+1}}{a_n} : n > N\} \le L + \varepsilon$. We may write $\forall n > N$, $a_n = a_N \cdot \frac{a_{N+1}}{a_N} \cdot \frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_n}{a_{n-1}}$ (N terms). so $a_n \le a_N \cdot (L + \varepsilon)^{n-N} = (\frac{a_n}{(L+\varepsilon)^N})(L + \varepsilon)^n$ so $a_n^{1/n} \le C_N^{1/n}(L + \varepsilon)$ where $C_N = \frac{a-n}{(L+\varepsilon)^N}$. So $\limsup(C_N^{1/n}(L + \varepsilon)) = (\lim C_N^{1/n})(L + \varepsilon) = L + \varepsilon$. So $\limsup a_n^{1/n} \le L + \varepsilon$. Since the holds for any $\varepsilon > 0$, we have $\limsup a_n^{1/n} \le L$. □

1.7.2 Series

- A series is of the form $\sum_{n=1}^{\infty} a_n$

- We denote the partial sum, $S_N = \sum_{n=1}^{N} a_n$ and we say "$\sum_{n=1}^{\infty} = L$ if $\lim S_N = L$. Convergence of a series $\iff$ Convergence of its partial sums.

Definition 1.7.7. $\sum a_n$ is cauchy if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, we have $|a_m + a_{m+1} + \cdots + a_n| \leq \varepsilon$.

Proposition 1.7.8. $\sum a_n$ is convergent $\iff$ $\sum a_n$ is cauchy.

Proposition 1.7.9.

(1) "Sanity Check": if $\sum a_n$ is convergent, then $\lim a_n = 0$.

> **Proof.** Convergence $\to$ Cauchy so if we take $n = m$, then we have $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$, $|a_n| \leq \varepsilon$.

(2) Comparison Test: If a_n is a positive sequence, $0 \leq a_n \leq b_n$ then if $\sum b_n$ is convergent, $\sum a_n$ is convergent.

> **Proof.** $\sum a_n$ is a montonic series since $a_n \geq 0$. Since it is bounded by $\sum b_n$, it converges.

Definition 1.7.10. $\sum a_n$ is "absolutely convergent" if $\sum |a_n|$ is convergent.

Proposition 1.7.11. If $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

> **Proof.** $|a_n + a_{n+1} + \cdots + a_m| \leq |a_n| + |a_{n+1}| + \cdots + |a_m|$ so it follows since $\sum |a_n|$ is cauchy.

Proposition 1.7.12.

- Ratio Test: $\sum a_n$ is absolutely convergent if $\limsup \frac{|a_{n+1}|}{|a_n|} = r < 1$.

- Root Test: $\sum a_n$ is absolutely convergent if $\limsup |a_n|^{1/n} = r < 1$.

> **Proof** (Proof (Root Test))**.** Choose r' such that $r < r' < 1$. $\exists N > 0$ such that $\sup\{|a_n|^{1/n} : n > N\} \leq r'$. ie. $\forall n > N$, $|a_n| \leq (r')^n = \frac{1}{1-r'}$ so $\sum |a_n|$ is convergent.

> **Proof** (Proof (Ratio Test))**.** Follows from root test and theorem 7.5

1.8 February 10

1.8.1 Series

Root Test(extended): Let $R = \limsup |a_n|^{1/n}$

- If $R < 1$, then $\sum a_n$ is absolutely convergent
- If $R > 1$. then $\sum a_n$ is divergent (doesn't satisfy cauchy)
- If $R = 1$, it depends eg. Consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

Integral Test: If $\sum a_n$ has $a_n \geq 0$. If $\exists f(x)$ with graph for $f(x) \geq a_n$ for $x \in [n-1, n]$ and $\int_a^\infty f(x) < \infty$ for some $a > 0$, then $\sum a_n < \infty$.

Example 1.8.1. $\sum \frac{1}{n^2}$ converges since $\int_1^\infty \frac{1}{x^2} dx < \infty$

Alternating Series:

- $\begin{cases} b_1 - b_2 + b_3 - b_4 + \cdots \\ b_n \geq 0 \end{cases}$
- Test: If (b_n) is decreasing, ie. $b_{n+1} \leq b_n$ then $\sum_{n=1}^\infty (-1)^{n+1} b_n$ converges.

  Proof. Define montonic increasing and decreasing seqeunces based on upper and lower bounds of series since each term is absorbed into the following one. Since $b_n \to 0$ the two sequences converge to the same limit. □

Example 1.8.2.

- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots$ is convergent
- $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} - \cdots$ is also convergent

1.8.2 Summation by Parts

Example 1.8.3. Consider $a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$. Let $A_0 = 0$, $A_1 = a_1$, $A_2 = a_1 + a_2$, ... Notice $a_n = A_n - A_{n-1}$.

$$\begin{aligned} a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 &= (A_1 - A_0)b_1 + (A_2 - A_1)b_2 + (A_3 - A_2)b_3 + (A_4 - A_3)b_4 \\ &= A_0b_1 + A_1(b_1 - b_2) + \cdots + A_3(b_3 - b_4) + A_4b_4 \end{aligned}$$

In general, if a_n, b_n are sequences of real numbers, if $A_n = a_1 + \cdots + a_n$, $A_0 = 0$, then for any $p < q$,

$$a_pb_p + \cdots + a_qb_q = -A_{p-1}b_p + \sum_{n=p}^{q-1} A_i(b_i - b_{i+1}) + A_qb_q$$

Theorem 1.8.4. Suppose the partial sum A_n forms a bounded sequence and suppose $b_1 \geq b_2 \geq b_3 \geq \cdots$, $\lim b_n \to 0$. Then $\sum a_nb_n$ is convergent. (if $a_n = (-1)^{n+1}$, gives alternating series).

Proof. Since (A_n) is bounded, $\exists M > 0$ such that $|A_n| < M$ $\forall n$.
WTS $\forall \varepsilon > 0$, $\exists N$ such that $\forall N < p < q$, we have

$$|a_p b_p + \cdots + a_q b_q| < \varepsilon \tag{*}$$

Claim: Since $b_n \to 0$, $\exists N$ such that $\forall n > N$, $b_n < \frac{\varepsilon}{2M}$. This N will satisfy (*).

$$\begin{aligned}
|a_p b_p + \cdots + a_q b_q| &= |-A_{p-1} b_p + \sum_{n=p}^{q-1} A_i (b_i - b_{i+1}) + A_q b_q| \\
&\le M b_p + \sum_{n=p}^{q-1} M(b_i - b_{i+1}) + M b_q \\
&= M[b_p + (b_p + b_{p+1}) + \cdots + (b_{q-1} - b_q) + b_q] \\
&= M \cdot 2 b_p < M \cdot 2 \cdot \frac{\varepsilon}{2M} = \varepsilon
\end{aligned}$$

Example 1.8.5. $\sum_{n=1}^{\infty} \sin(n \cdot 2\pi x) \frac{1}{n}$, where x is irrational, is convergent.
$= \operatorname{Im} \sum_{n=1}^{\infty} e^{i2\pi n x} \frac{1}{n}$.
$A_n = \sum_{n=1}^{N} e^{i2\pi x n} = e^{i2\pi x} \frac{1 - e^{i2\pi x N}}{1 - e^{i2\pi x}}$ so $|A_n| < \frac{2}{|1 - e^{i2\pi x}|}$.

1.8.3 Power Series

- $\sum_{n=0}^{\infty} a_n x^n$, $a_n \in \mathbb{R}$
- If we plug in $x \in \mathbb{R}$, then this becomes a series of numbers. We ask, for which x does $\sum a_n x^n$ converge?

Theorem 1.8.6. Let $\alpha = \limsup |a_n|^{1/n}$, let $R = \frac{1}{\alpha}$ (radius of convergence), then

- if $|x| < R$, $\sum a_n x^n$ is absolutely convergent
- if $|x| > R$, $\sum a_n x^n$ is divergent
- if $|x| = R$, it depends

Proof. $\limsup |a_n x^n|^{1/n} = |x|\alpha$ so follows from root test.

Example 1.8.7.

- $\sum_{n=1}^{\infty} x^n$, $a_n = 1$, $\alpha = 1$, $R = \frac{1}{\alpha} = 1$ so for $|x| < 1$, this is convergent.
- $\sum \frac{x^n}{n!}$, $a_n = \frac{1}{n!}$, $\alpha = \limsup(\frac{1}{n})^{1/n} = 0$, $R = \infty$.

Chapter 2

Topology and Metric Spaces

2.1 February 22

2.1.1 Topology and Metric Spaces

Definition 2.1.1. A metric space is a pair (X, d) such that:

- X is a set
- d is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ (ie. $\forall, x, y \in X$, $d(x, y)$ is nonnegative) satisfying:
  (1) $d(x, y) \geq 0$ and $d(x, y) = 0 \leftrightarrow x = y$
  (2) $d(x, y) = d(y, x)$
  (3) $\forall x, y, x \in X$, $d(x, y) + d(y, z) \geq d(x, z)$

Example 2.1.2.

(1) $X = \mathbb{R}^1$, $d(x, y) = |x - y|$

(2) $X = \mathbb{R}^2 = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$, $d((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ (Euclidean Metric)

(3) $X = \mathbb{R}^2$, $d = d_{\max}$ where $d_{\max} = \max(|x_1 - y_1|, |x_2 - y_2|)$.
$d_{\max}$ satisfies condition 3:

$$\begin{aligned} d(x, y) + d(y, z) &= \max(|x_1 - y_1|, |x_2 - y_2|) + \max(|y_1 - z_1|, |y_2 - z_2|) \\ &\geq \max(|x_1 - y_1| + |y_1 - z_1|, |x_2 - y_2| + |y_2 - z_2|) \\ &\geq \max(|x_1 - z_1|, |x_2 - z_2|) = d(x, z) \end{aligned}$$

(4) "discrete" metric space:
X is a set, $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

(5) Undirected (connected) graph distance:
graph: (vertices, edges)- vertices with labeled with positive distances.
$d(v_1, v_2) = \min(\text{length of paths between } v_1, v_2)$

Terminology (Given (X, d) a metric space):

- Open ball: given $x \in X$, $r > 0$, $B_r(x) = \{y \in X | d(x, y) < r\}$
- Closed ball: Open ball: given $x \in X$, $r > 0$, $\overline{B_r(x)} = \{y \in X | d(x, y) \leq r\}$

Definition 2.1.3. Let (X, d) be a metric space. A subset $U \subset X$ is called an open subset if $\forall x \in U$, $\exists r > 0$ such that $B_r(x) \subset U$.

Example 2.1.4. $(\mathbb{R}^2, d = d_{\text{Euclidean}})$, $U = (0, 1) \times (0, 1) = \{(x_1, x_2) | x_1, x_2 \in (0, 1)\}$. Claim: U is open.

Proof. Let $(x_1, x_2) \in U$, $r = \min(x_1, 1-x_1, x_2, 1-x_2)$. If $y \in B_r(x)$, then $d(x, y) < r$ ie. $\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} < r$ so $|x_1 - y_1| < r$ and $|x_2 - y_2| < r$ so $y_1 \in (x_1 - r, x_1 + r) \subset (0, 1)$ and $y_2 \in (x_2 - r, x_2 + r) \subset (0, 1)$ so $y \in U$. $\square$

Proposition 2.1.5.

(1) $\emptyset$, X are open in X

(2) If $U_1, \ldots, U_n \subset X$ are open then $U_1 \cap U_2 \cap \cdots U_n$ is open.

(3) If $\{U_\alpha\}_{\alpha \in I}$ is an arbitrary collection of open sets then $\bigcup_{\alpha \in I} U_\alpha$ is open.

(4) Every open ball $B_r(x)$ is open.

> **Proof.** WTS, $\forall y \in B_r(x)$, $\exists \varepsilon$ such that $B_\varepsilon(x) \subset B_r(x)$. Let $\varepsilon = r - d(x, y)$. Then $\forall z \in B_\varepsilon(y)$, $d(x, z) \leq d(x, y) + d(y, z) < (r - \varepsilon) + \varepsilon = r$, so $B_\varepsilon(y) \subset B_r(x)$.

2.2 February 24

2.2.1 Metric Spaces

Example 2.2.1.

(1) $\mathbb{R}^n$, $d_p(x, y) = [\sum |x_i - y_i|^p]^{\frac{1}{p}}$

(2) $\mathbb{R}^b$, "$p = \infty$", $d(x, y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$

(3) $\mathbb{R}^n$, $p = 1$, $d(x, y) = \sum |x_1 - y_i|$ "taxi-cab" metric.

Definition 2.2.2. Let (X, d) be a metric space. A sequence in X is denoted $(p_n)_{n=1}^\infty$ or (p_n). We say that $p_n \to p$ for some $p \in X$ if $\forall \varepsilon > 0$, $\exists N > 0$ such that if $n > N$ then $d(p_n, p) < \varepsilon$.

- Cauchy Criterion: $\forall \varepsilon > 0$, $\exists N$ such that $\forall n, m > N$ $d(p_n, p_m) < \varepsilon$.
- Subsequences have an equivalent definition.

Warning: For general metric space, (p_n) convergent $\to$ (p_n) cauchy but the converse is not true, eg. there is no $p \in X$ such that $p_n \to p$

Example 2.2.3.

(1) $\mathbb{Q}$, $d(x,y) = |x-y|$. Let p_n be a sequence that converges to $\sqrt{2}$ (in $\mathbb{R}$). Hence it is cauchy but (p_n) does not converge in $\mathbb{Q}$ (just because "would be" limit is not in X).

(2) $X = (0,1)$, $d(x,y) = |x-y|$, $p_n = \frac{1}{n}$ fails to converge in X ie. there is not $p \in X$ such that $d(p_n, p) \to 0$

Definition 2.2.4. If (X, d_X) is a metric space, $Y \subset X$ a subset. Then restricting d to $Y \times Y \subset X \times X$, makes Y a metric space (Y, d_Y).

2.2.2 Topology

In a metric space (X, d):

- open "ball": $B_r(p) = \{x \in X | d(x,p) < r\}$. $p \in X$ center, $r > 0$ radius.

Definition 2.2.5. A subset $U \subset X$ is open if $\forall p \in U$, $\exists B_r(p) \subset U$.

Proposition 2.2.6.

(0) $\forall p \in X$, $\forall r > 0$ $B_r(p)$ is open.

(1) $\emptyset, X$ is open.

(2) If $U_1, \ldots, U_n$ is open, then $U_1 \cap \cdots \cap U_n$ is open.

(3) If $\{U_\alpha | \alpha \in I\}$ is a collection of open sets, then $\bigcup U_\alpha$ is open.

Proof.

(0) WTS, $\forall x \in B_r(p)$ $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset B_r(p)$. Take $\varepsilon = r - d(x,p)$.

(1) Clear

(2) $\forall p \in U_1 \cap \cdots \cap U_n$ since $p \in U_i$ $\forall i$, and U_i is open then $\exists B_{r_i}(p) \subset U_i$, then $\bigcap B_{r_i}(p) = B_r(p)$ where $r = \min(r_1, \ldots, r_n)$. So $B_r(p) = \bigcap_{i=1}^n B_{r_i}(p) \subset \bigcap_{i=1}^n U_i$.

(3) $If p \in \bigcup_{\alpha \in I} U_\alpha$ then there is a α_0 such that $p \in U_{\alpha_0}$. Since U_{α_0} is open, we have $B_r(p) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in I} U_\alpha$

Definition 2.2.7. If X is a set, $\mathcal{T}$ is a collection of subsets of X such that

(1) $\emptyset, X \in \mathcal{T}$

(2) If $U_1, \ldots, U_n \in \mathcal{T}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$

(3) If $U_\alpha \in \mathcal{T}$ $\forall \alpha \in I$, then $\bigcup U_\alpha \in \mathcal{T}$

Then $\mathcal{T}$ is a topology of X and elements of $\mathcal{T}$ are called open subsets of X.

Example 2.2.8.

(1) $X = \mathbb{R}$, any open interval (a, b) is open. Also, any union of open intervals is open eg. $\bigcup_{n\in\mathbb{Z}}(n, n + \frac{1}{2})$.

(2) Open sets in $\mathbb{R}^2$: open balls are open, open squares are open. Topology on $\mathbb{R}^2$ induced by the metric d_2 equals the topology induced by $d_{\max}$.

Definition 2.2.9 (Closure)**.** If (X, d) is a metric space, $S \subset X$ a subset. $\overline{S} = \{p \in X |$ there is a sequence (p_n) such that $p_n \to p$.

Example 2.2.10. If $S = (0, 1)$, $\overline{S} = [0, 1]$. Also, if $S = (0, 1) \cap \mathbb{Q}$, $\overline{S} = [0, 1]$

Remark 2.2.11. $S \subset \overline{S}$. $\forall p \in S$, take the sequence $p_n = p$, then $p_n \to p$.

Proposition 2.2.12. Let $S \subset X$, then $S = \overline{S} \leftrightarrow S^c (= X \setminus S)$ is open.

Proof. $\to$) To show S^c is open, WTS $\forall p \in S^c$, $\exists B_r(p) \subset S^c$.
Suppose there is no open ball $B_r(p) \subset S^c$, ie $\forall r > 0$ $B_r(p) \not\subset S^c \leftrightarrow B_r(p) \cap S \neq \emptyset$. Then, take $r = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$ and pick $p_n \in B_{\frac{1}{n}}(p) \cap S$. We have $p_n \to p$ so $p \in \overline{S}$ which contradicts $p \in S^c$ and $S = \overline{S}$.
$\leftarrow$) If S^c is open, we need to show $\forall p \in \overline{S}$, we have $p \in S$. Suppose $p \in \overline{S}$ but $p \notin S$. Then $p \in S^c$. Since S^c is open, $\exists B_r(p) \subset S^c$. Since $p \in \overline{S}$, $\exists$ sequence (p_n), $p_n \in S$ $\forall n$, $p_n \to p$. Thus $\exists N$ such that $\forall n > N$, $p_n \in B_r(p)$. This is a contradiction since p_n can't be in $B_r(p)$ and S.

Definition 2.2.13. $S \subset X$ is closed if S^c is open.

Proposition 2.2.14. $\overline{\overline{S}} = \overline{S}$ for any subset $S \subset X$.

Proposition 2.2.15. $\forall S \subset X$, $\overline{S} = \{F \subset X$ closed, $F \supset S\}$

Proposition 2.2.16. For a metric space (X, d):

(0) $\emptyset, X$ are closed

(1) if $F_1, \ldots, F_n$ are closed then $F_1 \cup \cdots \cup F_n$ is closed.

(2) if F_α is closed $\forall \alpha$, $\bigcap F_\alpha$ is closed.

If U is open, then U is the union of open balls.

Proof. $\forall p \in U$, $B_{r(p)}(p) \subset U$ is an open ball so $U \subset \bigcup_{p\in U} B_{r(p)}(p)$, $\bigcup B_{r(p)}(p) \subset U$ hence $U = \bigcup_{p\in U} B_{r(p)}(p)$. $\square$

2.3 March 1

2.3.1 Metric Spaces

Example 2.3.1. X =the set of all pairs of points on $\mathbb{R} = \{\{x_1, x_2\}, x_1 \neq x_2 \in \mathbb{R}\}$. Want to define a reasonable metric on X.
Ideas:

- $\text{dist}(p_1, p_2)$ = distance from smallest point in p_1 to largest point in p_2.
  Fails to satisfy condition since $d(p,p) \neq 0$.

- $\text{dist}(\{x_1, x_2\}, \{y_1, y_2\}) = \min\{d(x_i, y_j) : i = 1, 2\, j = 1, 2\}$
  Fails since $d(\{1,2\}, \{2,3\}) = 0$.

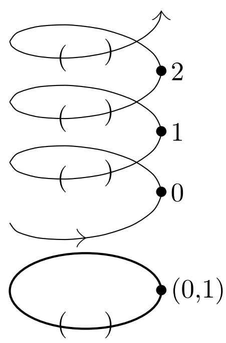
? points in $\mathbb{R}^2$, $\{x_1, x_2\} \mapsto \mathbb{R}^2$. potentially ambiguous lifting but can say $x_1 < x_2$. distance$(\{x_1, x_2\}, \{y_1, y_2\}) = \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2} \quad x_1 < x_2\, y_1 < y_2$

- Alternate Solution: define the distance from a point to a set by $d(p, B) = \inf_{q \in B} d(p, q)$.
  Let $d(A, B) = \sup_{p \in A}(\inf_{q \in B}(p, q)) + \sup_{q \in B}(\inf_{p \in A}(p, q))$.
  For the above example, $\text{dist}(\{x_1, y_1\}, \{x_2, y_2\}) = \max(\min(|x_1 - y_1|, |x_1 - y_2|), \min(|x_2 - y_1|, |x_2 - y_2|)) + \max(\min(|x_1 - y_1|, |x_2 - y_1|), \min(|x_1 - y_2|, |x_2 - y_2|))$.
  This is called the Gromov-Hausdorff distance.

2.3.2 Continuous functions

Definition 2.3.2. Let X, Y be topological spaces, a map of stes $f : X \to Y$ is continous if for any open subset $V \subset Y$, we have $f^{-1}(v)$ open in X.
Here, $f^{-1}(V) = \{x \in A | f(x) \in V\}$.

Example 2.3.3. $f : \mathbb{R} \to S^1$ (circle) $= [0,1]/0 \sim 1$ by $x \mapsto x - \lfloor x \rfloor$.
Continuous as the preimage of an open interval is the union of open intervals, which is open.



Definition 2.3.4 (Inherited Topology)**.** If X is a topological space, $S \subset X$ then a subset $E \subset X$ is said to be open in S if there exists $\tilde{E} \subset X$, open in X such that $\tilde{E} \cap S = E$.

Example 2.3.5 (Inherited or Induced Topology)**.** If $X = \mathbb{R}$, $S = [0,1]$. What are the open sets in S?
$[0, a), (a, b), (b, 1]$ $) < a, b < 1$ are open in S though they may not be open in $\mathbb{R}$.
$[0, a] = (-\varepsilon, a) \cap [0, 1]$. $[0, 1]$ is both closed and open in S.

Example 2.3.6. If we have $f : \mathbb{R} \to [0, 1)$ by $x \mapsto x - \lfloor x \rfloor$
$[0, \frac{1}{2})$ open in $[0, 1)$ but $f^{-1}([0, \frac{1}{2}) = \bigcup_{n\in\mathbb{Z}}[n, n + \frac{1}{2})$ is not open in $\mathbb{R}$ so f is not continuous.
Example 2.3.7. $X = \mathbb{R}$, $S = \mathbb{Q}$. Open sets in $\mathbb{Q}$ come from open sets in $\mathbb{R}$, $\cap\mathbb{Q}$.
eg. $(0, 1) \cap \mathbb{Q}$ is open in $\mathbb{Q}$.
Observe that $[-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is both closed and open in $\mathbb{Q}$.

Definition 2.3.8. Let X, Y be a metric space. $f : X \to Y$ a map of sets. Then f is continuous if $\forall x \in X$, $\forall r_y > 0$, $\exists r_x > 0$ such that $f(B_{r_x}(x)) \subset B_{r_y}(y)$ where $y = f(x)$.

2.4 March 3

2.4.1 Compact Sets

Definition 2.4.1 (Sequential Compactness)**.** In a metric space (X, d), a subset $K \subset X$ is sequentially compact if any sequence in K has a convergent subsequence in K (ie. $\forall (p_n)$ in K, $\exists (p_{n_k})$ such that $\lim_{n\to\infty} p_{n_k} = p \in K$)

Definition 2.4.2 (Open Cover)**.** $A \subset X$, and $\mathcal{U}_\alpha \subset X$ open with $\alpha \in I$ such that $A \subset \bigcup_{\alpha\in I} \mathcal{U}_\alpha$.

- A finite cover means the index set I is finite.
- A subcover of $\{\mathcal{U}_\alpha\}_{\alpha\in I}$, means a subset $I' \subset I$ such that $A \subset \bigcup_{\alpha\in I'} \mathcal{U}_\alpha$

Definition 2.4.3 (Open Cover Compactness)**.** A subset K is (open cover) compact of any open cover of K admits a finite subcover.

Example 2.4.4.

(1) Finite subset $K \subset X$ is both sequentially compact and open cover compact. $K = \{p_1, \ldots, p_n\} \subset X$.
If (x_n) is a sequence in K, there is a p_i that will be visited infinitely many times, take that constant subsequence (it converges to p_i)
If $K \subset \bigcup_{\alpha\in I} \mathcal{U}_\alpha$, then for each $i \in K$, $p_i \in \bigcup_{\alpha\in I} \mathcal{U}_\alpha$ so $\exists \alpha_i \in I$ such that $p_i \in \mathcal{U}_{\alpha_i}$, then $K \subset \mathcal{U}_{\alpha_i} \cup \cdots \cup \mathcal{U}_{\alpha_n}$.

(2) $X = \mathbb{R}, K = \mathbb{R}$.
Claim: K is not sequentially compact: (take sequence $1, 2, 3, 4, \ldots$ then no subsequence converges)
K is not open cover compact: $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}(n - \frac{1}{2}, n + \frac{3}{2})$ but has no finite subcover.

(3) $K = (0, 1) \subset \mathbb{R}$.
Not compact: $\bigcup_{n=1}^{\infty}(0, 1 - (\frac{1}{2})^n) = (0, 1)$ but has no finite subcover.
Also sequence $p_n = 1 - (\frac{1}{2})^n$ is not convergent in K.

(4) $K = [0, 1]$ is sequentially compact and open cover compact.

Proof.

(a) Let (p_n) be a sequence in $[0, 1]$. Since p_n is bounded $\exists\ p_{n_k} \to p$ for $p \in \mathbb{R}$. Since K is closed, the limit of the sequence in also in K. Thus $p \in K$.

(b) Let $\{\mathcal{U}_\alpha\}$ be an open cover of $[0,1]$. Let $a = \sup\{b|[0,b]$ has a finite subcover $\}$. We claim $[0,a]$ also admits a finite subcover. Since there is some open set with $a \in \mathcal{U}_0$, then $\exists \varepsilon > 0$ such that $[a-\varepsilon, a] \subset \mathcal{U}_0$ and $\exists$ b such that $b > a - \varepsilon$ so $[0,b]$ has a finite subcover hence combining this with $\mathcal{U}_0$ so does $[0,a]$.
Now, we will show $a = 1$. If $a < 1$, then the finite subcover of $[0,a]$ also contains $[0, a+\varepsilon]$ for some $\varepsilon > 0$, $0 < a + \varepsilon < 1$ contradicting the maximality of a.

□

Note: If K is open cover compact then:

(1) K is bounded.

(2) K is closed.

Proof.

(1) pick $p \in K$. $K \subset U_{n=1}^{\infty} B_n(p_0)$. By open cover compactness, $K \subset B_{n_0}(p_0)$ for some n_0.

(2) To show K is closed WTS $\forall p \notin K$, $\exists B_r(p) \cap K = \emptyset$.
Lemma: if A_i, B_i disjoint for $i = 1, \ldots, N$. Then $(\bigcup A_i) \cap (\bigcap B_i)) = \emptyset$
$\forall q \in K$ let $B_q = B_{\frac{1}{2}d(p,q)}(q)$. Then $K \subset \bigcup_{q \in K} B_q$ so $K \subset B_{q_1} \cup \cdots \cup B_{q_N}$. Let $r = \min_{1,\ldots,N}(\frac{1}{2}d(p,q))$ then $B_r(p)$ is disjoint from $\bigcup B_q \supset K$.

□

Theorem 2.4.5. Sequential compactness is equivalent to open cover compactness.

Proof. $\leftarrow$) Suppose $K \subset X$ is open cover compact. If $\exists$ (p_n) in K such that there is no convergent subsequence in K then $\forall p \in K$ $\exists r_p > 0$ such that (p_n) visits $B_{r_p} = B_p$ finitely many times, otherwise $\exists p \in K$ such that $\forall r_p > 0$ (p_n) visits $B_{r_p}(p)$ infinitely many times so there is a susbequence that converges to p. Thus, $K \subset \bigcup_{p \in K} B_p$. Since K is compact, $K \subset B_{p_1} \cup \cdots \cup B_{p_n}$ and the sequence has to visit one of the balls infinitely many times, contracting our assumption.

2.5 March 8

2.5.1 More Topology

Example 2.5.1. $(X = \mathbb{R}, d_{\text{std}})$, $Y = \{1,2,3\}$
What is T_Y?
Claim: collection of all subsets of Y: $T_Y = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,23\}\}$
Why is $\{1\}$ open in Y?
$B_{\frac{1}{2}}(1) = \{q \in Y | d(1,q) < \frac{1}{2}\} = \{1\}$. Similarly, $\{2\}$ and $\{3\}$ are open in Y and their unions generate T_Y.
Another Solution: $\{1\} \subset Y$ is open in Y since $(1-\varepsilon, 1+\varepsilon) \subset X = \mathbb{R}$ is open and $(1-\varepsilon, 1+\varepsilon) \cap \{1,2,3\} = \{1\}$.

2.5.2 Completness

Ex of Complete subsets in $\mathbb{R}$?

- $[a.b]$, any closed interval
- Every bounded and closed subset.

Proof (Proof of Thm 12.5 (cont))**.** Sequential Compactness $\rightarrow$ Open cover Compactnes

(1) Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X, we say that $\mathcal{U}$ refines $\mathcal{V}$ if for any $U \in \mathcal{U}$, $\exists V \in \mathcal{V}$ such that $U \subset V$.

Lemma: If $\mathcal{U}$ is a subset of X and $\mathcal{V}$ is a refinement of $\mathcal{U}$, that covers X and $\mathcal{V}$ admits a finite subcover of X, then $\mathcal{U}$ admits a finite subcover of X.

Proof. Since $X = \bigcup_{i=1}^{N} V_i$ and $V_i \subset U_i$, then $X \subset \bigcup_{i=1}^{N} U_i$.

Lemma 1: Assume X is sequentially compact. $\forall r > 0$, the open cover $\{B_r(p) | p \in X\}$ of X admits a finite subcover, ie. $\exists P_1, \ldots, P_n \in X$ such that $X = \bigcup_{i=1}^{N} B_r(P_i)$.

Proof. X cannot contain infinitely many disjoint open balls of radius $r/2$. Pick a "maximally sphere packing" of disjoint $(r/2)-$balls in X to choose $p_1, \ldots, p_n$ such that $\{B_{\frac{r}{2}}(p_i)\}$ disjoint and for any $p \in X$, $B_{\frac{r}{2}}(p) \cap B_{\frac{r}{2}}(p_i) \neq \emptyset$ for some i so $\forall p \in X$ $\exists p_i$ such that $d(p, p_i) < r$. Thus, $X \subset \bigcup_{i=1}^{N} B_r(p_i)$.

Lemma 2: Let (X, d) be sequentially compact. Let $\mathcal{U}$ be an open cover of X. Then $\exists r > 0$ such that the open cover $\{B_r(p) | p \in X\}$ refines $\mathcal{U}$, ie.$\forall p \in X$, $\exists U \in \mathcal{U}$ such that $B_r(p) \subset U$.

Proof. Suppose not. then $\forall r > 0$, $\exists p \in X$ such that $B_r(p)$ is not contained in $U \in \mathcal{U}$. Then for $r = \frac{1}{n}$, $n = 1, 2, \ldots$ pick p_n such that $B_{\frac{1}{n}}(p_n)$ not in $U \in \mathcal{U}$. Then (p_n) subconverges to $p \in X$, but $p \in X$ so $\exists U_0 \subset\in \mathcal{U}$ such that $p \in U_0$ so $\exists B_{r_0}(p) \subset U_0$. So $\exists N > 0$ such that $d(p_N, p) < \frac{r_0}{2}$, and $\frac{1}{N} < \frac{r}{2}$ so $B_{\frac{1}{N}}(p_N) \subset B_{r_0}(p)$. Thus $B_{\frac{1}{N}}(p_N) \subset U_0$ contradicting the construction of p_N.

For any open cover, the theorem follows by taking the refinement of $r > 0$ balls guaranteed by Lemma 2 and finding a finite subcover using Lemma 1.

Remark 2.5.2. Such an r is called a Lebesgue number of the open cover $\mathcal{U}$.

Theorem 2.5.3. $[0, 1]^d \subset \mathbb{R}^d$ is compact $\forall d = 1, 2, \ldots$

Proof. Prove the sequential compactness definition. We need to show $\forall (p_n)$ in $[0, 1]^d$ there is a subsequence that converges to $P \in \mathbb{R}^d$.
Lemma: The distances $d_{\max}, d_1, d_2$ are "equivalent" (d, d are equivalent if $\exists c_1, c_2 > 0$ such that $\forall x, y \in X$ $d(x, y) \leq c_1 d'(x, y)$ and $d'(x, y) \leq c_2 d(x, y)$.

- $d_1 = \sum |x_i - y_i| \quad d_2 = |\sum |x_i - y_i||^{\frac{1}{2}} \quad d_{\max} = \max(|x_i - y_i|)$

A sequence converges in $\mathbb{R}^d$ if it converges in all its coordinates.
For $d = 2$, $(x_{1,1}, x_{1,2}), (x_{2,1}, x_{2,2}), \ldots \to (x_1, x_2) \in \mathbb{R}^2$ iff $\lim x_{n,1} = x_1$, $\lim x_{n,2} = x_2$.
Given p_n for each coordinate we can then refine it to a convergent series iteratively.

Theorem 2.5.4 (Heine Borel)**.** A $K \subset \mathbb{R}^n$ is compact iff it is closed and bounded.

Proof.

- K closed in $\mathbb{R}^n \to K$ is sequentially compact in $\mathbb{R}^n$ (works only for $\mathbb{R}^n$)
- K is compact $\to K$ is closed and bounded (true for all metric spaces)

2.6 March 10

2.6.1 Connectedness

Example 2.6.1. $X = \{1, 2, 3, \ldots, \}$ with a funny topology. Open sets:

- $\emptyset$, X
- $\{1, 2, \ldots, n\}$ for some n integer ≥ 1.

Is X connected?

Definition 2.6.2. Let X be a topological space. X is connected if X cannot be written as the disjoint union of two nonempty open subsets.

Example 2.6.3.

- $X = \{1, 2\}$ with usual topology (ie. discrete) is not connected since $X = \{1\} \sqcup \{2\}$ and $\{1\}, \{2\}$ are open in X.
- $X = [0, 1]$ (under induced topology) is connected.

Example 2.6.4. $\mathbb{Q}$ is disconnected.
$\mathbb{Q} = [(-\infty, \sqrt{2}) \cap \mathbb{Q}] \sqcup [(\sqrt{2}, -\infty) \cap \mathbb{Q}]$

Remark 2.6.5. If $X = G \sqcup H$, G, H open in X then G, H are closed in X since $G = X \setminus H$, and complement of an open set is closed.

Theorem 2.6.6. Let $E \subset \mathbb{R}$, then E is connected iff $\forall x, y \in E$ and $x < y$ we have $[x, y] \subset E$.

Proof. $\to$) Suppose E is connected and suppose $\exists x, y \in E$ with $z \in (x, y)$ but $z \notin E$. Then let $E_1 = (-\infty, z) \cap E$, $E_2 = (z, +\infty) \cap E$ then

- E_1, E_2 are nonempty, $x \in E_1$, $y \in E_2$
- E_1, E_2 are open in E

So $E = E_1 \sqcup E_2$ is not connected, contradicting our assumption.
$\leftarrow$) If E satisfies the condition above and if E is not connected. $A = A \sqcup B$, A, B nonempty subsets of E. Pick $x \in A$, $y \in B$ and assume WLOG $x < y$. Then let $A' = [x, y] \cap A$, $B' = [x, y] \cap B$. Since $x, y \in E$, by assumption $[x, y] \subset E$.
$[x, y] = [x, y] \cap E = ([x, y] \cap A) \sqcup ([x, y] \cap B) = A' \sqcup B'$.
Let $z = \sup A'$ and consider the following cases:

(a) $z = x$, then $A' = \{x\}$ not open in $[x, y]$

(b) $x < z < y$. If $z \in A'$ then A' is not open ($B_\varepsilon(z)$ will not be in A'). Similarly if $z \in B'$ is not open.

(c) If $z = y$, then $z \in B'$ so B' is not open.

In all cases there is a contradiction, thus E must be connected.

Remark 2.6.7.

- Being connected is an intrinsic property of a topological space
- If X is a topological space, $E \subset X$, then if we ask "Is E connected" we treat E with respect to the induced topology.

Definition 2.6.8 (Separated - Rudin)**.** Let X be a topological space. $G, H \subset X$ we say that G, H are separated if $\overline{G} \cap H = \emptyset$, $G \cap \overline{H} = \emptyset$.

Definition 2.6.9. $X = \mathbb{R}$, $G = (0, 1)$, $H = (1, 2)$
$\overline{G} \cap H = [0, 1] \cap (1, 2) = \emptyset \quad G \cap \overline{H} = (0, 1) \cap [1, 2] = \emptyset$ so G, H separated.

Example 2.6.10. $G = (0, 1), H = [1, 2] \quad G, H$ not separated.

Proposition 2.6.11. Let X be a topological space, $E \subset X$, then E is connected iff E cannot be written as $G \sqcup H$ with G, H separated (in X)

Proof. $\rightarrow$) Suppose E is connected and $E = G \sqcup H$, G, H separated. We want to show that G, H are open in E, or equivalently G, H are closed in E.
Since $\overline{G} \cap H = \emptyset$, $\overline{G} = \overline{G} \cap E = \overline{G} \cap (G \cup H) = \overline{G} \cap G = G$ so G is closed in E. Similarly, H is closed in E so E is not connected.

Let $f : X \to Y$ be a continuous map between topological spaces. Then

(1) If $A \subset X$ is compact, then $f(A)$ is compact

(2) If $A \subset X$ is connected, then $f(A)$ is connected.

(3) If $X = \mathbb{R}, Y = \mathbb{R}$, $A = [a, b]$, then $f(A) = [c, d]$ for some c, d.

2.7 March 15

2.7.1 Completeness and Compactness are Preserved by Continuous Maps

Proposition 2.7.1. Let $f : X \to Y$ be a continuous map, if X is compact then $f(X)$ is compact.

Proof. (use open cover compactness) Let $\{V_\alpha\}$ be a collection of open sets in Y covering $f(X)$. Then $f(x) \subset \bigcup_\alpha V_\alpha$ so $X \subset \bigcup_\alpha f^{-1}(V_\alpha)$. By continuity of f, $f^{-1}(V_\alpha)$ is open so by the compactness of X there is a finite subcover $X \subset \bigcup_{i=1}^N f^{-1}(V_{\alpha_i})$ so $f(X) \subset \bigcup_{i=1}^N f(f^{-1}(V_{\alpha_i})) \subset \bigcup_{i=1}^N .V_{\alpha_i}$. Thus we have a finite subcover of $f(X)$.

Corollary 2.7.2. If $f : X \to Y$ continuous, and $K \subset X$ is compact, then $f(K)$ is compact.

Proof. Let $g = f|_K : K \to Y$, still continuous. Follows from previous thm.

Remark 2.7.3. *Proof.* (Using sequential compactness). Given a sequence (y_n) in $f(X)$ we can choose x_n in X such that $f(x_n) = y$. Then (x_n) is a sequence in X. By sequential compactness $\exists(x_{n_k})$ converging to x_0, thus $y_{n_k} = f(x_{n_k})$ converges to $f(x_0)$. □

Lemma 2.7.4.

(a) If $f : X \to Y$ continuous, $E \subset X$ any subset, then the restriction $f|_E : E \to Y$ is continuous.

(b) If $f : X \to Y$ is continuous, then $g : X \to f(X)$.

Proof.

(a) For any open $V \subset Y$, $(f|_E)^{-1}(V) = F^{-1}(V) \cap E$ is open in E so $f|_E$ is continuous.

(b) For any $F \subset f(X)$ open, $\exists \tilde{F} \subset Y$ open such that $F = \tilde{F} \cap f(X)$, then $g^{-1}(F) = f^{-1}(\tilde{F})$, hence is open in X.

Proposition 2.7.5. If $f : X \to Y$ is continuous and X is connected, $f(X)$ is connected.

Proof. let $g : X \to f(X)$ be the restriction of f, then g is continuous. If $f(X) = U \sqcup V$ of 2 nonzero open sets in $f(X)$, then $X = g^{-1}(U) \sqcup g^{-1}(V)$, nonempty and open. Hence X is not connected, contradicting our premise. Thus, $f(X)$ is connected.

Theorem 2.7.6 (Intermediate Value Theorem)**.** Suppose $f : [a, b] \to \mathbb{R}$ continuous. if $f(a) = \alpha$, $f(b) = \beta$ and $\gamma \in (\alpha, \beta)$ then $\exists x \in (a, b)$ such that $f(x) = \gamma$.

Proof. Since $[a, b]$ connected, then $f([a, b])$ connected. Since $\alpha, \beta \in f([a, b])$ then $[\alpha, \beta] \subset f([\alpha, \beta])$ so $\gamma \in f([\alpha, \beta])$ so $\exists x \in (a, b)$ such that $f(x) = \gamma$.

If f continuous

- f does not preserve openness. $f : \{0\} \to \mathbb{R}$, $\{0\}$ open in X but not in $\mathbb{R}$.

- f does not preserve boundedness. $f : (0,1) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. (If X is compact, then $f(X)$ is bounded)

2.7.2 Uniformly Continuous Maps Between Metric Spaces

Definition 2.7.7. $f : X \to Y$ is a uniform continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \varepsilon$.

Example 2.7.8.

(1) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is not uniformly continuous.

Proof. Suppose that for all $\varepsilon > 0$, $\exists \delta > 0$ such that $|x_1 - x_2| < \delta \to |x_1^2 - x_2^2| < \varepsilon$. Then let $x_1 = n$, $x_2 = n + \frac{\delta}{2}$, we have

$$|n^2 - (n + \frac{\delta}{2})^2| \geq |n\delta + (\frac{\delta}{2})^2| > n\delta > \varepsilon$$

for large enough n. □

(2) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$ is uniformly continuous.

(3) $f : [0,1] \to \mathbb{R}$ by $x \mapsto \sqrt{x}$ is uniformly continuous even though the slope is unbounded at $x = 0$.

Theorem 2.7.9. If $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given, we need to find $\delta > 0$ such that $\forall x_1, x_2 \in X$, $d(x_1, x_2) < \delta$, we have $d(f(x_1), f(x)_2) < \varepsilon$. Since f is continuous $X \to Y$, $\forall x \in X$, $\forall r_y > 0$, $\exists r_x > 0$ such that if $x_1, x_2 \in B_{r_x}(x)$, then $d(f(x_1), f(x_2)) < 2r_y$. $\forall x \in X$, choose $r_x > 0$ such that $f(B_{2r_x}(x)) \subset B_{\varepsilon/2}(f(x))$. Then $X \subset \bigcup_{x \in X} B_{r_x}(X)$. By compactness of X, pick a finite open cover such that $X = \bigcup_{i=1}^N B_{r_i}(x_i)$, where $r_i = x_i$. Let $\delta = \min\{r_1, \ldots, r_N\}$. $\forall p_1, p_2 \in X$, $p_1 \in B_{r_i}(x_i)$ for some i. Since $d(p_2, p_1) < \delta < r_i$, $d(p_2, x_i) \leq d(p_2, p_1) + d(p_1, x_i) < r_i + r_i = 2r_i$. Since $f(p_1), f(p_2) \in f(B_{2r_i}(x_i)) \subset B_{\varepsilon/2}(f(x_i))$, we have $d(f(p_1), f(p_2)) < \varepsilon$.

2.7.3 Discontinuity

Definition 2.7.10 (Limit of a Function at a Point)**.** Let $E \subset X$ and $f : E \to Y$ be a map. Let $p \in \overline{E}$, then we say $\lim_{x \to p} f(x) = y \in Y$, if for all sequences of points $x_n \to p$, $x_n \in E$, we have $\lim_{n \to \infty} f(x_n) = y$.

- For $f : (a,b) \to \mathbb{R}$, $\forall x \in (a,b)$ we let $f(x-)$ and $f(x+)$ denote the "left" and "right" limits. $\lim(x-) = \lim_{\substack{t \to x \\ t \in (a,x)}} f(x) = \lim_{t \to x^-} f(x)$ and $\lim(x+) = \lim_{\substack{t \to x \\ t \in (x,b)}} f(x) = \lim_{t \to x^+} f(x)$. (They need not exist)
- f is continuous at $f \leftrightarrow f(x) = f(x-) = f(x+)$
- Discontinuity of the first kind: $f(x+)$ and $f(x-)$ exists but f is discontinuous at x.
- else discontinuity of the second kind.

Example 2.7.11.

(1) $f(x) = \begin{cases} x & x \leq 0 \\ \sin(\frac{1}{x}) & x > 0 \end{cases}$ has a discontinuity of the second kind at 0.

(2) $f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{Q} \cup \{0\} \\ \frac{1}{q} & x \in \mathbb{Q} \setminus \{0\}, x = \frac{p}{q}\ p, q \text{ coprime} \end{cases}$
Claim: $f(x)$ is continuous on all $\mathbb{R} \setminus \mathbb{Q}$ and 0.

(3) $f(x) = \begin{cases} 1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$ is discontinuous at all points in $\mathbb{R}$.

Theorem 2.7.12. If $f(x)$ is a monotonic increasing function on (a, b) (if $x_1 < x_2$, $f(x_1) \leq f(x_2)$), then $f(x)$ can have at most countably many discontinuities, all of the first kind.

2.8 March 17

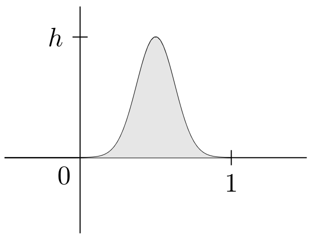
2.8.1 Sequences and Series of Functions

Sequence: $f_1(x), f_2(x), f_3(x), \ldots$
Series: $\sum_{n=1}^{\infty} f_n(x)$

Example 2.8.1. $f_n(x) = \frac{x^2}{(1+x^2)^n}$, $\quad f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{1+x^2}{(1+x^2)^n}$
fix an x, forms a geometric series: $x^2 \sum (\frac{1}{1+x^2})^n = x^2 \frac{1}{1-\frac{1}{x^2+1}} = x^2 \frac{1+x^2}{x^2} = 1 + x^2$.
so $f(x) = \begin{cases} 1 + x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$

Example 2.8.2. $f_m(x) = \lim_{n\to\infty} [\cos(m!\pi x)]^{2n}$, $\quad f(x) = \lim_{m\to\infty} f_m(x)$.
if $m!\pi x = n\pi x$, $m!x$ is an integer then $\cos(m!\pi x) = \pm 1$. This happens if x is a rational number, $x = \frac{p}{q}$ and $q|m!$. $f_m(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q},\ m!x \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$ so
$f(x) = \lim_{m\to\infty} f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

Example 2.8.3. Suppose there is f such that $\int_0^1 f(x)dx = 1$.



$f_n(x) = nf(nx)$ so $\int_{\mathbb{R}} nf(nx)dx = \int f(u)du = 1$.
for any $x \in \mathbb{R}$, $\lim f_n(x) = \begin{cases} 0 & x \notin (0,1) \\ 0 & x \in (0,1) \end{cases}$
so $\int (\lim f_n(x))dx = 0 \neq \lim_{n\to\infty} \int f_n(x)dx = 1$

2.8.2 Uniform Convergence

Definition 2.8.4. Let $f_n : (a,b) \to \mathbb{R}$ be a sequence of functions and $f : (a,b) \to \mathbb{R}$. We say $f_n \to f$ uniformly if for any $\varepsilon > 0$ there exists $N > 0$ such that

$$\forall n > N, \forall x \in (a,v) \quad |f_n(x) - f(x)| \leq \varepsilon$$

Remark 2.8.5. Uniform convergence means N does not depend on x.

Alternatively, we define distances between 2 functions $f, g : X \to Y$, X, Y metric spaces by $d_\infty(f,g) = \sup_{x\in X} d_Y(f(x), g(x))$. We say $f_n \to f$ uniformly if $\lim_{n\to\infty} d_\infty(f_n, f) = 0$.

Example 2.8.6. With f as in Ex 3, $d_\infty(f,0) = \sup|f(x)-0| = h$, and $d_\infty(f_n,0) = n\cdot h$ so f_n does not converge uniformly.

$g, f : \mathbb{R} \to \mathbb{R}$, $d_2(f,g) = [\int |f(x) - g(x)|^2 dx]^{\frac{1}{2}}$
(Warning: only makes sense for "nice enough" f, g)
Define $d_1(f,g), d_\infty(f,g)$ similarly.

Theorem 2.8.7. Let $f_n : X \to Y$ be a sequence of continuous functions between 2 metric spaces. If $f_n \to f$ uniformly, then f is continuous.

Proof. To show f is continuous, WTS $\forall x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $d(x', x) < \delta$, then $d_Y(f(x'), f(x)) < \varepsilon$. Fix $x_0 \in X$, we will show f is continuous at x_0.

- By uniform convergence of $f_n \to f$, we know $\exists N$ such that $\forall n \geq N$, $\forall x \in X$ $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$. Fix $n_0 = N$.
- Since $f_{n_0}(x)$ continuous at x_0, we know $\exists \delta > 0$ such that $d(x, x_0) < \delta \to d_Y(f_{n_0}(x_0), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Thus, if $d(x, x_0) < \delta$,

$$\begin{aligned} d(f(x), f(x_0)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(x_0)) + d(f_{n_0}(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Definition 2.8.8. A sequence of functions f_n if uniformly Cauchy if $\forall \varepsilon > 0$, $\exists N > 0$ such that $\forall n, m > N$, $d_\infty(f_n, f_m) < \varepsilon$, ie. $\forall x \in \mathbb{R}$, $|f_n(x) - f_m(x)| < \varepsilon$.

Proposition 2.8.9. If $f_n : \mathbb{R} \to \mathbb{R}$ satisfies the uniform Cauchy condition then f_n is uniformly convergent to some $f : \mathbb{R} \to \mathbb{R}$.

Proof. For each $x \in \mathbb{R}$, $f_n(x)$ from a sequence of numbers in $\mathbb{R}$ and is Cauchy in $\mathbb{R}$, hence it is convergent. Let $f(x) := \lim_{n\to\infty} f_n(x)$. WTS $f_n \to f$ uniformly.
To show $f_n \to f$ uniformly, fix $\varepsilon > 0$, WTS $\exists N > 0$ such that $\forall x \in \mathbb{R}$, $\forall n > N$, $|f_n(x) - f(x)| < \varepsilon$. Choose N large enough such that $\forall n, m > N$, $|f_m(x) - f_m(x)| < \varepsilon$. Fix n, let $m \to \infty$, then $\lim m \to \infty$, then $\lim f_m(x) = f(x)$, $|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| < \varepsilon$.

Chapter 3

Differentiation and Integration

3.1 March 29

3.1.1 Differentiation

Given a nice function, $f'(p) =$ the slope of the tangent line of p.

Definition 3.1.1. A function $f : [a, b] \to \mathbb{R}$ is differentiable at a point $p \in [a, b]$ if the limit $\lim_{t\to p} \frac{f(t)-f(p)}{t-p}$ exists. If so, we call it $f'(p)$.

Proposition 3.1.2. If $f(x)$ is differentiable at p, then $f(x)$ is continuous at p, ie. $\lim_{x\to p} f(x) = f(p)$.

Proof. $f(x)-f(p) = \frac{f(x)-f(p)}{x-p}\cdot(x-p)$ so $\lim_{x\to p}[f(x)-f(p)] = \lim_{x\to p}[\frac{f(x)-f(p)}{x-p}\cdot(x-p)] = \lim_{x\to p}(\frac{f(x)-f(p)}{x-p)})\cdot$ $\lim_{x\to p}(x-p) = f'(0)\cdot 0 = 0$.

Example 3.1.3. $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ -x^2 & x \notin \mathbb{Q} \end{cases}$. Claim: $f'(0) = 0$.

Proof. $f'(0) = \lim_{x\to p} \frac{f(x)-f(0)}{x-0}$. $\lim_{x\to 0} |\frac{f(x)-f(0)}{x-0}| - \lim_{z\to 0} \frac{|\pm x^2|}{|x|} = \lim x \to 0|x| = 0$. ☐

Theorem 3.1.4. If $f, g : [a.b] \to \mathbb{R}$, differentiable at a point $x_0 \in [a, b]$.

(1) $\forall c$, $(c\cdot f)'(x_0) = c\cdot(f'(x_0))$

(2) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

(3) $(fg)'(x_0) = f'(x_0)g(x_0) + g'(x_0)f(x_0)$

Theorem 3.1.5 (Chain Rule)**.** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, ie. $f(x_0) = y_0$, $f'(x_0)$ exists and if $g : \mathbb{R} \to \mathbb{R}$, is differentiable at y_0, ie. $g(y_0) = z_0$, $g'(y_0)$ exists. The composition $h = g \circ f$, ie $h(x) = g(f(x))$

is differentiable at x_0, $h'(x_0) = g'(y_0) \cdot f'(x_0)$.

Proof. Use "baby taylor expansion".
$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + (x - x_0) \cdot r_f(x) \quad \lim_{x \to x_0} r_f(x) = 0$
$g(x) = g(x_0) + g'(x_0) \cdot (x - x_0) + (x - x_0) \cdot r_g(x) \quad \lim_{x \to x_0} r_g(x) = 0$
Then

$$\begin{aligned} h(x) - h(0) &= g(f(x)) - g(f(x_0)) \\ &= (f(x) - f(x_0))(g'(f(x_0)) + r_g(f(x))) \\ &= (x - x_0)(f'(x_0) + r_f(x))(g'(f(x_0)) + r_g(f(x))) \end{aligned}$$

Dividing both sides by $(x - x_0)$ and taking the limit as $x \to x_0$ but $x \neq x_0$, we see that $h'(x_0) = f'(x_0)g'(f(x_0))$, as desired.

Example 3.1.6. $h(x) = \sin^2 x$
$f(x) = x^2$, $f'(x) = 2x \quad g(x) = \sin x$, $g'(x) = \cos x$
$h'(x) = f'(x)g'(f(x)) = 2x \cos(x^2)$

Definition 3.1.7. $f : [a, b] \to \mathbb{R}$, we say $p \in [a, b]$ is a local maximum if $\exists \delta > 0$ such that $\forall x \in [a, b] \cap (p - \delta, p + \delta)$, $f(p) \geq f(x)$.

Proposition 3.1.8. If p is a local maximum of f and $f'(p)$ exists, then $f'(p) = 0$.

Proof. If $f'(p)$ exists, $\lim_{x \to p^+} \frac{f(x)-f(p)}{x-p} = \lim_{x \to p^-} \frac{f(x)-f(p)}{x-p}$. For $x > p$, $\frac{f(x)-f(p)}{x-p} \geq 0$, for $x < p$, $\frac{f(x)-f(p)}{x-p} \leq 0$ so we must have $\lim_{x \to p} \frac{f(x)-f(p)}{x-p} = 0$.

Theorem 3.1.9 (Rolle)**.** If $f : [a, b] \to \mathbb{R}$ is continuous and if f is differentiable on (a, b), if $f(a) = f(b)$, then $\exists c \in (a, b)$ with $f'(c) = 0$.

Proof. Suffices to find a local max or local min of f on (a, b). If constant then $f'(x) = 0$ for all $x \in (a, b)$ otherwise must either increase so must have local max or min.

3.2 March 31

3.2.1 Differentiation

Theorem 3.2.1 (Generalized Mean Value Theorem)**.** Let $f, g : [a, b] \to \mathbb{R}$ be differentiable on (a, b) and continuous on $[a, b]$ then $\exists c \in (a, b)$, $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$ ie. $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ if $g(a) - g(b), g(c) \neq 0$.

- For simple case take $g(x) = x$.

Proof. Define $h(x) = [f(b) - f(a)][g(x) - g(a)] - [f(x) - f(a)][g(b) - g(a)]$. Then $h(a) = 0$, $h(b) = 0$, so by Rolle's Theorem $\exists c$ such that $h'(c) = 0 = [f(b) - f(a)]g'(c) - f'(c)[g(b) - g(a)]$.

Remark 3.2.2. If $f(b) - f(a) = g(b) - g(a) = 1$, then $\exists c$ such that $f'(c) = g'(c)$.

Corollary 3.2.3. Suppose $f : \mathbb{R} \to \mathbb{R}$ differentiable $\forall x \in R$, $|f'(x)| \leq M$ for some constant M, then f is uniformly continuous.

Proof. To show f is uniformly continuous we need to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(x)| < \varepsilon$. Hence we can take $\delta = \frac{\varepsilon}{M}$, then by MVT, $f(x) - f(y) = f'(c)(x - y)$ for some $c \in (x, y)$. Thus $|f(x) - f(y)| = |f'(c)| \cdot |x - y| < M \cdot \delta < \varepsilon$.

Corollary 3.2.4. If $f'(x) \geq 0$ $\forall x \in [a, b]$ then $y > x \to f(y) \geq f(x)$. (monotonic increasing)

Proof. $f(y) - f(x) = f'(c) \cdot (y - x) \geq 0$.

Theorem 3.2.5 (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be differentiable, $f(a) \leq f(b)$. For μ such that $f'(a) < \mu < f'(b)$, $\exists c \in (a, b)$ such that $f'(c) = \mu$.

Remark 3.2.6. Since $f'(x)$ as a function on $[a, b]$ may not be continuous so cannot use mean value theorem for $f'(x)$.

Proof. Let $h(x) = f(x) - \mu \cdot x$, $h'(x) = f'(x) - \mu$ then $h'(a) < 0 < h'(b)$. Consider $h : [a, b] \to \mathbb{R}$, let $c \in [a, b]$ such that $h(c) = \min h(x)$, $x \in [a, b]$. Want to show $c \neq a$, $c \neq b$. By definition of $h'(a)$, we know $\frac{h(t)-h(a)}{t-a} < 0$ then for t close enough to a, $t > a$, $h(t) < h(a)$. Thus $h(a) \neq \min h(b)$. Similarly, $h(b) \neq \min(h)$.

3.2.2 L'Hopital's Rule

Example 3.2.7.

(1) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1$.

(2) $\lim_{x\to 0} \frac{\log x}{x} = \lim_{x\to 0} \frac{1/x}{x} = \lim_{x\to 0} \frac{1}{x} = 0$.

Theorem 3.2.8 (L'Hopital's Rule)**.** Assume $f, g : (a, b) \to \mathbb{R}$ differentiable, $g(x) > 0$ over (a, b). If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = A \in \mathbb{R} \cup \{+\infty, -\infty\}$ and one of the following are true:

(1) $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$

(2) $\lim_{x\to a} g(x) = \infty$.

Then, $\lim_{x\to a} \frac{f(x)}{g(x)} = A$.

Proof. Assume for simplicity, $A \in \mathbb{R}$. The cases where $A = \pm\infty$ are similar.
Case 1: $\lim_{x\to a} g(x) = 0$, $\lim_{x\to a} f(x) = 0$.
Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in (a, a+\delta)$, then $|\frac{f'(x)}{g'(x)} - A| < \varepsilon$. Then for α, β such that $a < \alpha < \beta < a+\delta$, $\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\gamma)}{g'(\gamma)} \in (A-\varepsilon, A+\varepsilon)$ for some $\gamma \in (\alpha, \beta)$. Take the limit $\alpha \to a$, then $f(\alpha), g(\alpha) \to 0$ so $\frac{f(\beta)}{g(\beta)} = \lim_{\alpha\to a}(\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} \in [A-\varepsilon, A+\varepsilon]$. Then $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $\forall\beta \in (\alpha, \alpha+\delta)$, $\frac{f(\beta)}{g(\beta)} \in [A - ve, A+\varepsilon]$. Thus $\lim \frac{f(\beta)}{f(\alpha)} = A$.
Case 2: $\lim g(x) = \infty$
Consider $a < \alpha < \beta < b$, $\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}$ as above. Then $(A-\varepsilon)(\frac{g(a\alpha)-g(\beta)}{g(\alpha)} < \frac{f(\alpha)-f(\beta)}{g(\alpha)} \cdot \frac{g(\alpha)-g(\beta)}{g(\alpha)} < (A+\varepsilon)(\frac{g(\alpha)-g(\beta)}{g(\alpha)})$. Then as $\alpha \to a$, $A - \varepsilon \leq \liminf_{\alpha\to a} \frac{f(\alpha)-f(\beta)}{g(\alpha)} = \liminf_{\alpha\to a} \frac{f(\alpha)}{g(\alpha)} \leq \limsup_{\alpha\to a} \frac{f(a)}{g(a)} = \limsup_{\alpha\to a} \frac{f(a)-f(\beta)}{g(\alpha)} \leq (A+\varepsilon)$. Since $\varepsilon > 0$ was arbitrary $\lim \frac{f(\alpha)}{g(\alpha)} = A$.

3.3 April 7

3.3.1 Higher Derivatives

- If $f : \mathbb{R} \to \mathbb{R}$ continuous, if $f'(x)$ exists fro all $x \in \mathbb{R}$ and $f'(x)$ is continuous, we say $f \in C^1(\mathbb{R})$
- If $f'(x)$ is also differentiable, $(f')'(x) = \lim_{\varepsilon\to 0} \frac{f'(x+\varepsilon)-f'(x)}{\varepsilon}$, and if $f''(x) = f^{(2)}(x)$ exists for all x and is continuous, then $f \in C^2(\mathbb{R})$.
- If $f^{(k)}(x)$ exists and is continuous, $f \in C^k(\mathbb{R})$
- If $f \in C^k(\mathbb{R})$ $\forall k = 1, 2, 3, \dots$ then $f \in C^\infty(\mathbb{R})$ is called a smooth function.

Example 3.3.1.

1. if $f(x) = a_0 + \frac{a_1}{1}x + \frac{a_2}{1\cdot 2}x^2 + \frac{a_3}{1\cdot 2\cdot 3}x^3 + \cdots + \frac{a_n}{n!}x^n$, then $f'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-1} + a_{n-2}(n-2)x^{n-2} + \cdots + a_1$.
   $f^{(k)}(x)$ exists and is a polynomial. Thus, $f \in C^\infty(\mathbb{R})$.
2. $f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0 \end{cases}$, $f \in C^1(\mathbb{R})$ but $f''(x) = \begin{cases} 0 & x < 0 \\ \text{DNE} & x = 0 \\ x^2 & x > 0 \end{cases}$

3.3.2 Taylor Approximation of Smooth Functions

Remark 3.3.2. $P(x) = a_0 + \frac{a_1}{1}x = \frac{a_2}{1\cdot 2}x^2 + \frac{a_3}{1\cdot 2\cdot 3}x^3 + \cdots + \frac{a_n}{n!}x^n$
$P'(x) = a_1 + a_2 x + \frac{a_3}{1\cdot 2}x^2 + \cdots + \frac{a_n}{(n-1)!}x^{n-1}$
$P'(0) = a_1, P''(0) = a_2, \cdots, P^{(k)}(0) = a_k$
There exists a nice function such that its value at the kth derivative $(k = 1, \dots, n)$ can be specified.
$P_{x_0}(x) = P(x - x_0) = a_0 + a_1(x - x_0) + \frac{a_2}{2!}(x - x_0)^2 + \cdots + \frac{a_n}{n!}(x - x_0)^n$. Then, $P_{x_0}(x_0) = P(0) = a_1$, $P'_{x_0}(x_0) = a_1, \dots,$
nth Taylor Expansion Centered at a point:

- Assume $f : \mathbb{R} \to \mathbb{R}$ is a C^k functions. Then we can use $f(x_0), f'(x_0), \ldots, f^{(k)}(x_0)$ to cook up a polynomial.
  $P_{x_0}(x) = f(x_0) + f'(x_0)\frac{x-x_0}{1} + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^n(x_0)\frac{(x-x_0)^n}{n!}$.
  Note: $P_{x_0}^{(k)}(x_0) = f^{(k)}(x_0)$

Theorem 3.3.3 (Taylor's Theorem)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is $C^n(\mathbb{R})$ and $f^{(n+1)}$ exists (may not be continuous)

- Let $P(x)$ be the nth order taylor approximation of f at x_0.
  $P(x) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!}$
- Then $\forall x \in \mathbb{R}$, $\exists \theta \in [0,1]$ such that if $x_\theta = x_0(1-\theta) + x\theta$
  $f(x) - P_{x_0}(x) = f^{(n+1)}(x_\theta)\frac{(x-x_0)^{n+1}}{n!}$.

Sanity Check: for the $n = 0$ case, $P_{x_0}(x) = f(x_0)$ then $\exists\theta$ such that
$f(x) - f(x_0) = f'(x_\theta)\left(\frac{x-x_0}{1}\right)$, ie. $f'(x_\theta) = \frac{f(x)-f(x_0)}{x-x_0}$ (mean value theorem)

Proof. Fix x_0 and $x_1 \in \mathbb{R}$, WTS there is x_θ such that $f(x_1) - P_{x_0}(x_1) = f^{(n+1)}(x_\theta) \cdot \frac{(x-x_0)^{n+1}}{(n+1)!}$

- Define $M \in \mathbb{R}$ such that $f(x_1) - P_{x_0}(x_1) = (x_1 - x_0)^{n+1} \cdot M$
- Let $g(x) := f(x) - P_{x_0}(x) = M(x-x_0)^{n+1}$.

Then $g(x_0) = f(x_0) - P_{x_0}(x_0) - 0 = 0$ and
$g(x_1) = f(x_1) - P_{x_0}(x_1) - M(x_1 - x_0)^{n+1} = 0$
Moreover, $g^{(k)}(x_0) = f^{(k)}(x_0) - P_{x_0}^{(k)}(x_0) - 0 = 0 \; 0 \le k \le n$
Step 1: Use $g(x_0) = 0$, $g(x_1) = 0 \to a_1 \in (x_0, x_1)$ such that $g'(a_1) = 0$
Step 2: Use $g'(x_0) = 0$, $g'(a_1) = 0 \to a_2 \in (x_0, a_1)$ such that $g''(a_2) = 0$
$\vdots$
Step k: Use $g^{(n)}(x) = 0$, $g^{(n)}(a_n) = 0 \to a_{n+1} \in (x_0, a_n)$ such that $g^{(n+1)}(a_{n+1}) = 0$
$0 = g^{(n+1)}(a_{n+1}) = f^{(n+1)}(a_{n+1}) - 0 - M(n+1)!$
Thus, $f(x_1) - P_{x_0}(x_1) = (x_1 - x_0)^{n+1}\frac{f^{(n+1)}(a_{n+1}}{(n+1)!}$

3.4 April 12

3.4.1 Taylor Expansions/Power Series

- Taylor expansion: Let $f : \mathbb{R} \to \mathbb{R}$, C^∞ (smooth) functions. Let $x_0 \in \mathbb{R}$, let N be a positive integer. The Nth order taylor expansion of f centered at x_0 is the polynomial $P(x)$, such that
  $\begin{cases} P^{(k)}(x_0) - f^{(k)}(x_0) & \forall k = 0, 1, \ldots, N \\ \text{and } \deg p \le N \end{cases}$
  Concretely: $P_{x_0,N}(x) = \sum_{k=0}^{N} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!}$
  Remainder: $f(x) - P(x) = R_{x_0,N}(x)$ has the property that $R_{x_0,N}^{(k)}(x_0) = 0$ for $k = 0, 1, \ldots, N$.

Definition 3.4.1 (Analytic Function)**.** We say a smooth function us analytic at a point x_0 if $\exists R > 0$ such that $f(x) = \sum_{k=0}^{\infty} a_n(x - x_0)^n$ for all $|x - x_0| < R$. If f is analytic at x_0, then $a_n = \frac{f^{(n)}(x_0)}{n!}$.

Remark 3.4.2. There exists a smooth function such that $f(0) = 0, f'(0) = 0, \dots, f^{(n)}(0) = 0, \dots$ but $f(x)$ is not identically 0. $f(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases}$

Lemma 3.4.3.

$$\lim_{x \to 0^+} \frac{e^{-x}}{x^n} = 0 \tag{*}$$

Proof. Let $u = \frac{1}{x}$, then $(*)$ equivalent to $\lim_{n\to\infty} \frac{e^{-u}}{(1/u)^n} = \lim_{n\to\infty} \frac{u^n}{e^u} = \lim_{n\to\infty} \frac{n!}{e^u} = 0$ by L'Hopitals.

Thus f is smooth but not analytic at $x = 0$

Example 3.4.4. For $f(x) = \frac{1}{1+x}$, if f analytic?
We need to study $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x - 0)^n$.
$f'(x) = (-1)\frac{1}{(1+x)^2}$, $f'(x) = (-1)(-2)\frac{1}{(1+x)^3}$, $f^{(n)}(x) = \frac{(-1)\cdots(-n)}{(1+x)^{n+1}}$
$f^{(n)}(0) = (-1)^n n!$, $\sum_{n=1}^{\infty}(-1)^n x^n$, a sufficient and necessary condition to converge is $|x| < 1$.
We know:

(1) $\forall 0 < r < 1$, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

(2) If $\sum |a_n| < \infty$, $\sum a_n$ converges

$\frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{1}{1+x}$ when $|x| < 1$

Theorem 3.4.5. Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ be a power series centered at x_0, then let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$, $R = \frac{1}{\alpha}$, then if $|x - x_0| < R$, the series converges. If $|x - x_0| > R$, the series diverges. If $|x - x_0| = R$, it depends. (if $\alpha = 0$, $R = \infty$ so the series is always convergent)

Example 3.4.6. $\sum \frac{1}{n^2} \cdot x^n$, $\alpha = \limsup(\frac{1}{n^2})^{1/n}$, $R = 1$
If $|x - x_0| < R = 1$, it converges
If $|x - x_0| > R = 1$, it diverges
If $|x - x_0| = r$ it still converges. (Not always true, consider $\sum \frac{1}{n} \cdot x^n$)

Remark 3.4.7. Taylor Expression is just one way to approximate a function

- If only cares about 1 point
- Suppose you wanted a polynomial $p(x)$ such that $P(x_i) = f(x_i)$ for $x_1, \dots, x_n \in \mathbb{R}$. We can use interpolation.

3.4.2 Integration

What is Integration?

- Can be thought of as signed area bounded between a graph and the x-axis

- Want to know when our method of approximating area converges (eg. when the integral is defined)

- Let $f : [a, b] \to \mathbb{R}$ be a bounded function (may not be continuous)
- Let $P = \{a = x_0 \leq x_1 \leq \cdots \leq x_N = b\}$ be a partion. Let $\Delta x_i = x_i - x_{i-1}$: the i-th segment.
- $M_i = \sup_{[x_{i-1}:x_i]} f(x)$, $m_i = \inf_{[x_{i-1},x_i]} f(x)$. For a partition P, $U(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$, $L(P, f) = \sum_{i=1}^{N} m_i \Delta x_i$
- We say a partition Q refines P if $Q \supset P$ as a set of "cut" points.

Lemma 3.4.8. If Q refines P, then $L(Q, f) \geq L(P, f)$ and $U(Q, f) \leq U(P, f)$.

Definition 3.4.9. $L(f)(= \underline{\int_a^b} f dx) := \sup L(P, f)$ over all partitions.
$U(f)(= \overline{\int_a^b} f dx) := \inf U(P, f)$ over all partitions.

- We say that f is Riemann integrable if $\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$ and denote the common value by $\int_a^b f dx$.

Example 3.4.10 (Non-Integrable)**.** $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \cap [0, 1] \\ 1 & x \in \mathbb{Q} \cap [0, 1] \end{cases}$
$\underline{\int_a^b} f dx = 0$, $\overline{\int_a^b} f dx = 1$

Theorem 3.4.11. If $f : [a, b] \to \mathbb{R}$ is a continuous (hence bounded, and uniformly continuous) then f is Reimann Integrable.

Proof. WTS, $\forall \varepsilon > 0$, $\exists P$ partition such that $\overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \varepsilon$.
Let $\tilde{\varepsilon} = \frac{\varepsilon}{b-a}$, by uniform continuity $\exists \delta$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \tilde{\varepsilon}$. Choose a partition P such $\Delta x_i < \delta$ (eg. take $N = \lceil \frac{b-a}{\delta} \rceil$) then even partition works. Then $M_i = \sup_{[} x_{i-1}, x_i] f(x) = f(s_i)$ for some $s_i \in [x_{i-1}, x_i]$, $m_i = \inf_{[} x_{i-1}, x_i] f(x) = f(t_i)$ for some $t_i \in [x_{i-1}, x_i]$ so $|m_i - m_i| = |f(s_i) - f(t_i)| < \tilde{\varepsilon}$. Thus, $U(P, f) - L(P, f) = \sum (M_i - m_i) \Delta x_i \leq \sum \tilde{\varepsilon} \Delta x_i = \tilde{\varepsilon}(b - a) = \varepsilon$.

Corollary 3.4.12. If $f(x)$ is piecewise continuous on $[a, b]$ ie. discontinuous on finitely many points, then f is integrable.

3.5 April 14

3.5.1 Integration

Example 3.5.1. $f : [0, 1] \to \mathbb{R}$, bounded by $f(x) = \begin{cases} 0 & x = 0 \\ \sin(\frac{1}{x}) & x \in (0, 1] \end{cases}$

Proof. Yes, $\forall \varepsilon > 0$, consider the partition of the form $[0, \frac{\varepsilon}{4}]$, some partition of $[\frac{\varepsilon}{4}, 1]$. Let P' be a partition such that $U(P', f, [\frac{\varepsilon}{4}, 1]) - L(P', f, [\frac{\varepsilon}{4}, 1]) < \frac{\varepsilon}{2}$. Let $P = [0, \frac{\varepsilon}{4}] \cup P'$. Then, $U(P, f, [0,1]) - L(P, f, [0,1]) \leq (1-(-1))(\frac{\varepsilon}{4}) + [U(P', f) - L(p', f)] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ □

Proposition 3.5.2. If $f : [a,b] \to \mathbb{R}$ is a bounded function with finitely many discontinuities, then f is integrable.

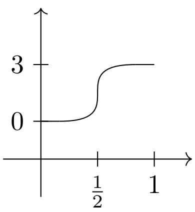
Theorem 3.5.3. If f is a monotonic function over $[a,b]$, then f is integrable.

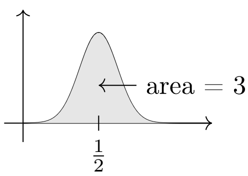
Proof. WLOG f is increasing. Let $\varepsilon > 0$, $\forall m > 0$ integer, consider the partition P_n with n segments such that each segment has length $\frac{b-a}{n} = \delta$ then $U(P_n, f) - L(P_n, f) = \sum_{i=1}^{n} (M_i - m_i)\delta = \delta \sum f(x_i) - f(x_{i-1}) = \delta(f(b) - f(a))$. By making n large enough, $\frac{(f(b)-f(a))(b-a)}{n} < \varepsilon$.

3.5.2 Reimann - Stieltjes Integral (density included)

- Want to assign a "density" function $\rho(x)$ that assigned a different weight to different parts of a function - mass of a small segment
- One general way is to replace $\rho(x)dx$ by $d(\alpha(x))$, $\alpha(x)$ called the "cumulative mass" function. $\alpha(x)$= mass of the interval $[a, x] = \int_a^x \rho(x)dx$.
- Want: $\alpha(x)$ to be monotone increasing.

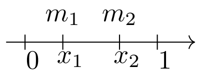
Example 3.5.4. $[a, b] = [0, 1]$

- $\alpha(x) = x$, then $d(\alpha(x)) = dx$ so $\rho(x) = 1$
- $\alpha(x) =$ 

  $d(\alpha(x))$"="$\rho(x) =$ 

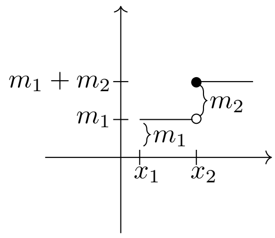
- $\alpha(x) = \begin{cases} 0 & x \leq \frac{1}{2} \\ 3 & \frac{1}{2} < x \leq 1 \end{cases}$ then $d(\alpha(x)) = 3\delta(x - \frac{1}{2})dx$
  (Here δ is the function with infinite value at 1 but area of 1)

Example 3.5.5. Suppose we ant to compute the center of mass of two points.



$c_m = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} = \frac{\int_0^1 x d(\alpha(x))}{\int_0^1 d(\alpha(x)}$)

$\alpha(x) =$



Definition 3.5.6 (Reimann - Stieltjes Integral)**.**

- Let $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing.
  Let $f : [a, b] \to \mathbb{R}$ be bounded.
  Let P be a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$.
  Let $I_i = (x_{i-1}, i]$, $\Delta\alpha(I_i) = \alpha(x_i) - \alpha(x_{i-1})$, $M_i = \sup_{I_i} f$, $m_i = \inf_{I_i} f$
- $U(P, f, \alpha) = \sum m_i \Delta\alpha(I_i)$, $L(P, f, \alpha) = \sum m_i \Delta\alpha(I_i)$
  $U(f, \alpha) = \inf_P U(P, f, \alpha)$, $L(f, \alpha) = \sup_P L(P, f, \alpha)$
- if $U(f, \alpha) = L(f, \alpha)$, we say f is "Reimann integrable with respect to α" denoted as $f \in R(a)$.

Theorem 3.5.7. Suppose f is continuous, then $\int_a^b f d\alpha$ exists.

Theorem 3.5.8. Suppose f is montonic, α is continuous and monotonic then $\int_a^b f d\alpha$ exists.

Remark 3.5.9. If $f = \begin{cases} 0 & x \in [0, 1/2) \\ 1 & x \in [1/2, 1] \end{cases}$ then $U(P, f, \alpha) - L(P, f, \alpha) = 1$.

- If P has a segment containing $1/2$ in the interior, $U - L = (1 - 0) \cdot 1 = 1$
- If $P = (0, \frac{1}{2}]$, $(\frac{1}{2}, 1]$, then $U - L = (1 - 0) \cdot 1 + (1 - 1) \cdot 0 = 1$

Proof. Since α is continuous and monotonic on $[a, b]$

- For each n integer, let $y_0, y_1, \ldots, y_n$ be an even partition of $[\alpha(a), \alpha(b)]$. Let x_i be chosen such that $d(x_i) = y_i$.

- then $\alpha(x_i) - \alpha(x_{i-1}) = y_i - y_{i-1} = \frac{\alpha(a)-\alpha(b)}{n} = \delta$

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1}+)] \cdot \delta \\ &\leq \sum_{i=1}^{m} [f(x_i) - f(x_{i-1})] \cdot \delta \\ &= (f(b) - f(a)) \cdot \delta \end{aligned}$$

3.6 April 19

3.6.1 Reimann Steiltjes Itegral

Example 3.6.1.

(1) If $\alpha(x)$ is smooth, $[a, b] = [0, 1]$, $\alpha(x) = 2x + 3$, $f(x) = 1$.
$\int_0^1 f(x)\alpha(x) = \lim_{P \text{ partition}} \sum f(x_i)\alpha(\Delta x_i) = \alpha(1) - \alpha(0) = 3$

- If α is a smooth function (or at least differentiable), say $\alpha(x) = \rho(x)$, $d(\alpha(x)) = \rho(x)dx$.
- Applying this to above integral, $\int_0^1 1d(2 + 3x) = \int_0^1 1 \cdot 3dx = 3$

(2) If f has finitely many jumps $\alpha(x) = \begin{cases} x & x \in [0, 1] \\ x + 1 & x \in (1, 2] \\ x + 2 & x \in (2, 3] \end{cases}$, then

$$\begin{aligned} \int_0^3 1d(\alpha(x)) &= \int_{0+}^{1-} 1 \cdot d(\alpha(x)) + \int_{1+}^{2-} 1 \cdot d(\alpha(x)) + \int_{2+}^{3-} 1 \cdot d(\alpha(x)) + \sum_{p\text{:jumps of } \alpha} (\alpha(p+) - \alpha(p-)) \\ &= 1 + 1 + 1 + 1 + 1 = 5 \end{aligned}$$

Theorem 3.6.2. If f continuous on $[a, b]$, α monotonically increasing, then $\int_a^b fd(\alpha(x))$ exists.

Theorem 3.6.3. If f is monotonic on $[a, b]$ and α is continuous and monotonically increasing on $[a, b]$ then $\int_a^b fd\alpha$ exists.

Theorem 3.6.4. If $f : [a, b] \to \mathbb{R}$ is bounded and has finitely many discontinuities. If α is continuous when f is discontinuous, then $\int_a^b fd\alpha$ exists.

Remark 3.6.5. If $alpha(x) = x$, the usual Reimann Integral, we show this by contstructing partitions areound the jump points of f.

Proof. Let $\varepsilon > 0$ be given, let $M = \sup |f(x)|$. Let $E = \{c_1, \dots, c_n\}$ be the points where f is discontinuous.
Step 1: Chosse a small enough interval $[u_j, v_j]$ containing c_j such that $\sum_{\alpha(v_j)-\alpha(u_j)} < \varepsilon$ and the intervals are disjoint. By continuity of α at c_j we can have this.

Step 2: Let $K = [a,b] \setminus \bigcup_{i=1}^m (u_j, v_j)$, still compact.
Choose a partition P of K fine enough such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$. Then $\tilde{P} = P \cup \bigcup_{i=1}^m [u_j, v_j]$,

$$U(\tilde{P}) - L(\tilde{P}) < \varepsilon + \sum_{i=1}^m (M - (-M)) \cdot \Delta\alpha_i < \varepsilon + 2M\varepsilon = (1+2M)\varepsilon$$

By making α small enough, we can make the differnce small.

Theorem 3.6.6.

- Let $f : [a,b] \to \mathbb{R}$ be integrable with respect to $\alpha(x)$, assume $f([a,b]) \subset [m, M]$
- if $\varphi : [m, M] \to \mathbb{R}$ continuous, then $h(x) = \varphi(f(x))$ is integrable with respect to $\alpha(x)$

Example 3.6.7. $\alpha(x) = x$, $f(x)$ =some monotonic function. $[a,b] \xrightarrow{f} [M,m] \overset{\varphi=\exp|\,|}{\to} \mathbb{R}$ if $\int_a^b f(x)dx$ exist, then $\int e^{|f(x)|}dx$ exists.

Proof. Fix an $\varepsilon > 0$ since φ is continuous on $[M,m]$, it is uniformly continuous. Then $\exists \delta > 0$ such that if $|y_1 - y_2| < \delta$, then $|\varphi(y_1) - \varphi(y_2)| < \varepsilon$

- Since f is integrable, $\exists$ a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$
- For interval $I_i = [x_{i-1}, x_i]$, let $M_i = \sup_{I_i} f$, $m_i = \inf_{I_i} f$. Let $M_i^* \sup_{x \in I_i} \varphi(f(x))$, $m_i^* = \inf_{x \in I_i} f(x)$
- We say I_i is of the "short" type, $i \in A$, if $M_i - m_i < \delta$. Then $M_i^* - m_i^* < \varepsilon$.
  Note: $M_i^* - m_i^* = \sup_{x_1, x_2 \in I} |h(x_1) - h(x_2)|$ since if $x_1, x_2 \in I$, then $f(x_1), f(x_2) \in [m_i, M_i] < \delta$, thus by uniform continuity of φ, $|\varphi(f(x_1)) - \varphi(f(x_2))| < \varepsilon$
- Otherwise, say I_i is of the "long" type, $i \in B$, $M_i^* - m_i^* \leq \sup |\varphi(x)| = 2K$.

Also,

$$\delta \cdot \sum_{i \in B} \delta\alpha_i \leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i \leq U(P,f,\alpha) - L(P,f,\alpha) \leq \delta^2 \text{ so } \sum_{i \in B} \Delta\alpha_i < \delta.$$

Thus,

$$\begin{aligned}
U(P,h,\alpha) - L(P,h,\alpha) &= \sum_{i=1}^m (M_i^* - m_i^*)\Delta\alpha_i \\
&= \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i \\
&\leq \sum_{i \in A} \varepsilon \cdot \Delta\alpha_i + \sum_{i \in B} 2K \cdot \Delta\alpha_i \\
&\leq \varepsilon \cdot (\sum_i^n \Delta\alpha_i) + 2K \cdot \sum \Delta\alpha_i \\
&= \varepsilon[\alpha(b) - \alpha(a)] + 2K \cdot \delta \\
&\leq \varepsilon(\alpha(b) - \alpha(a) + 2K) \qquad \text{(since we can assume WLOG } \delta < \varepsilon\text{)}
\end{aligned}$$

Theorem 3.6.8. "$\int$ is linear in both f and α"

(1) If f, g are integrable with respect to α, then

- $\int cf d\alpha = c \int f d\alpha$ exists $\forall c \in \mathbb{R}$
- $\int f + g d\alpha = \int f d\alpha + \int g d\alpha$ exists

(2) If f is integrable with respect to α_1 and α_2, then

- f is integrable with respect to $c \cdot \alpha_1$) ($c \geq 0$) and $\int f d(c\alpha_1) = c \int f d\alpha$)
- f is integrable with respect to $a_1 + a_2$ then $\int f d(\alpha_1 + \alpha_2) = \int f d\alpha_1 + \int f d\alpha_2$

Theorem 3.6.9.

1. if f and g are integrable with respect to α then $\int fg d\alpha$ is integrable with respect to α.
2. If f is integrable, then $|f|$ is integrable.
   (follows by taking $\varphi(x) = |x|$, continuous then $\varphi(f(x)) = |f(x)|$)

3.7 April 21

3.7.1 Properties of Integrals

Lemma 3.7.1 (Sampling Lemma)**.** Given a partition $a = x_0 < x_1 < \ldots < x_n = b$, $f : [a, b] \to \mathbb{R}$ bounded, $\alpha : [a, b] \to \mathbb{R}$ monotone increasing.

- $\forall i = 1, \ldots, n$ pick $s_i \in I_i$. Then $L(P, f, \alpha) \leq \sum f(s_i)\Delta\alpha)i \leq U(P, f, \alpha)$
- If $U - L < \varepsilon$, then for any $s_i, t_i \in I_i$,

$$\sum_i |f(s_i) - f(t_i)|\Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = U - L < \varepsilon$$

Theorem 3.7.2. if f is bounded, α increasing, if α' exists and is integrable. Then

(1) $f \in R(\alpha) \leftrightarrow f\alpha' \in R$

(2) If $f \in R(\alpha)$, then $\int_1^b f d\alpha = \int_b^a f\alpha' dx$.

Proof. Need to prove $\overline{\int_a^b} f d\alpha = \overline{\int_a^b} f\alpha' dx$ and $\underline{\int_a^b} f d\alpha = \underline{\int_a^b} f\alpha' dx$.
We are going to show $\forall \varepsilon > 0$, $\exists P$ partition such that $|U(P, f, \alpha) - U(P, f\alpha')| < \varepsilon$. α' is integrable so $\exists$ partition P, $a = x_0 < \ldots < x_n = b$ such that $U(P, \alpha') - L(P, \alpha') < \varepsilon$.

- By Mean Value theorem, $\exists t_i \in (x_{i-1}, x_i)$ such that $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)\Delta x_i$.
- By Sampling lemma, $\forall s_i \in [x_{i-1}, x_i]$, $\sum |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \varepsilon$.

Thus, $\forall s_i \in I$, $\sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i| = |\sum f(s_i)\alpha'(t_i)\Delta x_i - f(s_i)\alpha'(s_i)\Delta x_i|$.
Let $M = \sup_{[a,b]} |f|$, then above sum

$$\begin{aligned}
&\leq \sum |f(s_i)||\alpha'(t_i) - \alpha'(s_i)|\Delta x_i \\
&\leq M \cdot \sum |\alpha'(f_i) - \alpha'(s_i)|\Delta x_i \\
&\leq M\varepsilon
\end{aligned}$$

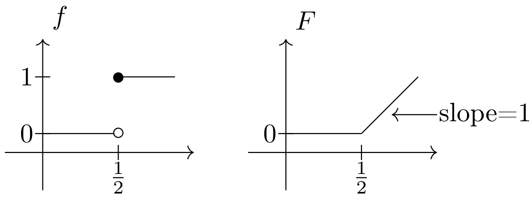
Thus, $\forall s_i \in I$, $|\sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i| < M\varepsilon$ so $\sum f(s_i)\Delta\alpha_i \leq \sum f(s_i)\alpha'(s_i)\Delta x_i + M\varepsilon \leq U(P, f, \alpha') + M\varepsilon$ so taking the sup over all partitions, we get $U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon$. Similarly, $\sum f(s_i)\Delta\alpha_i \leq \sum f(s_i)\alpha'(s_i)\Delta x_i + M\varepsilon$ so $U(P, f\alpha') \leq U(P, f, \alpha) + M\varepsilon$. Thus, $|U(P, f\alpha') - U(P, f, \alpha) < \varepsilon$. For any refinement Q of P all previous statements still hold. Thus, $\lim_P |U(P, f\alpha') - U(P, f, \alpha)| \leq \varepsilon \cdot M$ hence $|U(P, f\alpha') - U(P, f, \alpha)| = 0$. Similarly, $|L(P, f\alpha') - L(P, f, \alpha)| = 0$. Thus, $(1), (2)$ hold.

Theorem 3.7.3 (Change of Variable)**.** Let α be increasing on $[a, b]$, $f \in R(\alpha)$, elet $\varphi : [A, B] \to [a, b]$ be a strictly increasing function. Define $g : [A, B] \to \mathbb{R}$, $g(y) = f(\varphi(y))$. Define $\beta : [A, B] \to \mathbb{R}$ by $\beta(y) = \alpha(\varphi(y))$. Then $\int_A^B f d\alpha = \int_A^B g d\beta$.

Theorem 3.7.4. Let $f \in R$ on $[a, b]$. For any $a \leq x \leq b$, define $F(x) = \int_a^x f(t)dt$. Then

1. $F(x)$ is a continuous function.
2. if $f(x)$ is continuous at a point $x_0 \in [a, b]$, then $F(x)$ is differentiable at x_0, $F'(x_0) = f(x_0)$

Example 3.7.5. Consider $f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}] \\ 1 & x \in (\frac{1}{2}, 1] \end{cases}$



Proof.

(1) Let $M = \sup_{[a,b]} |f(x)|$, for any $a \leq x \leq y \leq b$, we have

$$|F(x) - F(y)| = |\int_a^x f(t)dt - \int_a^y f(t)dt| = |\int_x^y f(t)dt| \leq \int_x^y |f(t)|dt \leq \int_x^y M dt = M \cdot |y - x|$$

Thus, F is lipschitz continous with constant M.

(2) If f is continuous at x_0, then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. Then, for any $s, t \in [a, b]$ such that $x_0 - \delta < s < x_0 < t < x_0 + \delta$, $\frac{F(t)-F(s)}{t-s} = \frac{1}{t-s}\int_s^t f(u)du$. Also, $f(x_0) = \frac{1}{t-s}\int_s^t f(x_0)du$. So

$$\begin{aligned}\Big|\frac{F(t)-F(s)}{t-s} - f(x_0)\Big| &= \Big|\frac{1}{t-s}\int_s^t (f(u) - f(x_0))du\Big| \\ &\leq \frac{1}{t-s}\int_s^t |f(u) - f(x_0)|du \\ &\leq \frac{1}{t-s}\int_s^t \varepsilon \cdot du = \frac{1}{t-s}(t-s)\cdot\varepsilon = \varepsilon\end{aligned}$$

This implies $\lim_{h\to 0}\frac{F(x_0+h)-F(x_0)}{h} = f(x_0)$

Theorem 3.7.6. Let F be a differentiable function on $[a, b]$, $F'(x) = f(x)$. If $f(x)$ is integrable, then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof. Fix $\varepsilon > 0$,

1. $\exists P$ partition of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.
2. $F(b) - F(a) - \sum_{i=1}^n F(x_i) - F(x_{i-1}) = \sum_{i=1}^n F'(s_i)\Delta x_i$ for $s_i \in (x_{i-1}, x_i) = \sum_{i=1}^n f(s_i)\Delta x_i \in [L(P, f), U(P, f)]$. Also, $\int_a^b f dx \in [L(P, f), U(P, f)]$. Thus, $|F(b) - F(a) - \int_a^b f dx| < U - L < \varepsilon$. Since LHS is independent of ε, and $\varepsilon > 0$ is arbitrary, LHS=0.

Theorem 3.7.7. Suppose f and g are differentiable and their derivatives are integrable, then

$$\int_a^b fg'dx = \int_a^b fdg = \int_a^b d(fg) - gdf = fg|_a^b - \int_a^b gdf$$

Proof. Let $h = f \cdot g$, then h is differentiable, $h' = f'g + fg'$ so

$$\int_a^b f'gdx + \int_a^b fg'dx = \int_a^b h'dx = h(b) - h(a)$$

3.8 April 26

3.8.1 Uniform Convergence with Integration

Recall:

- A sequence of functions $f_n : [a, b] \to \mathbb{R}$ is uniformly convergent to $f : [a, b] \to \mathbb{R}$ if for any $\varepsilon > 0$, $\exists N > 0$ such that $\forall n > N$, $\forall x \in [a, b]$ $|f_n(x) - f(x)| < \varepsilon$.
- Equivalently, define $d_\infty(f_n, f) = \sup_{x\in[a,b]} |f_n(x)-f(x)|$, then $f_n \to f$ uniformly iff $\lim_{n\to\infty} d_\infty(f_n, f) = 0$.

- If $f_n \to f$ uniformly, and $\{f_n\}$ is continuous, then f is cotinuous.

Q:

(1) If $f_n \to f$ uniformly and $f_n(x)$ are integrable (with resepct to some weight function $\alpha(x)$) is $f(x) \in R(\alpha)$? (yes)

(2) If $f_n \to f$ uniformly and $f'n$ exists and is continuous, does f' exist? (No)

Theorem 3.8.1. If $f_n \to f$ uniformly, $f_n \in R(\alpha)$, then $f \in R(\alpha)$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$.

Proof. For any given $\varepsilon > 0$, $\exists N$ such that $\sup |f_n(x) - f(x)| < \varepsilon$, $\forall n > N$. Thus $\forall n > N$, $f_n - \varepsilon < f < f_n + \varepsilon$. Thus $\forall P$ parition, we have

$$L(P, f_n, \alpha) - \varepsilon(\alpha(b) - \alpha(b)) = L(P, f_n - \varepsilon, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P, f_n + \varepsilon, \alpha) = U(P, f_n, \alpha) + \varepsilon(\alpha(b) - \alpha(a))$$

Fix an $n > N$, we can choose a partition P such that $U(P, f_n, \alpha) - L(P, f_n, \alpha) < \varepsilon \cdot (a\alpha(b) - \alpha(a))$. Thus,

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &\leq U(P, f, \alpha) - L(P, f, \alpha) + 2\varepsilon(\alpha(b) - \alpha(a)) \\ &= 3 \cdot \varepsilon(\alpha(b) - \alpha(a)) \end{aligned}$$

Thus, $\forall \varepsilon > 0$, $\exists$ parition P that makes $U(P, f, \alpha) - L(P, f, \alpha)$ small enough. Hence, $f \in R(a)$.

Corollary 3.8.2. Let $f_n(x) \in R(\alpha)$, over $[a, b]$, assume $F(x) = \sum_{n=1}^{\infty} f_n(x)$ is a uniformly convergent series, then

$$\int_a^b F(x)dx = \sum)n = 1^\infty \int_a^b f_n(x)dx$$

Proof. Define $F_N(x) = \sum_{n=1}^{N} f_n(x)$. This is a finite sum of $R(\alpha)$ functions, hence $F_N(x) \in R(\alpha)$. By previous theorem, since $F_N \to F$ uniformly, and f_n is integrable, $F(x) \in R(\alpha)$ and

$$\begin{aligned} \int_a^b F(x) &= \lim_{N\to\infty} F_n(x)dx \\ &= \lim_{N\to\infty} \int_a^b \Big(\sum_{n=1}^{N} f_n(x)dx) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_a^b f_n(x)dx \\ &= \sum_{n=1}^{\infty} \int_a^b f_n(x)dx \end{aligned}$$

3.8.2 Uniform Convergence with Differentiation

Example 3.8.3. $f_n \to 0$ uniformly , f_n' exists and is continuous but $f_n'(x) \not\to 0$
$f_n(x) = \frac{1}{n}\sin(n^2 x)$, $f_n'(x) = n \cdot \cos(n^2 x)$

Despite this, we still have a theorem

Theorem 3.8.4. If $f_n(x)$ is a sequene of differentiable functions on $[a, b]$ such that

(a) $f_n'(x) \to g(x)$ uniformly on $[a, b]$

(b) $\exists x_0 \in [a, b]$ such that $f_n(x_0) \to c$

Then we have

(1) $\exists f$ such that $f_n \to f$, uniformly

(2) f is differentiable and $f'(x) = g(x) = \lim f_n'(x)$

Remark 3.8.5. (b) is necessary otherwise we can have $f_n(x) = n$, $f_n'(x) = 0$ but f is not uniformly continous.

Proof.

(1) $\forall \varepsilon > 0$, choose N large enough such that

(1) $|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \quad \forall n, m > N$
(2) $d_\infty(f_n', f_m') < \frac{\varepsilon}{2} \cdot \frac{1}{b-a} \quad \forall n, m > N$

Apply MVT to $f_n - f_m$ over the interval $[x, t]$,

$$\begin{aligned} |f_n(x) - f_m(x) - (f_n(t) - f_m(t))| &= |f_n'(s) - f_m'(s)| \cdot |x - t| \\ &< \frac{\varepsilon}{2} \cdot \frac{1}{b-a}(b-a) \\ &= \frac{\varepsilon}{2} \end{aligned}$$

Thus, $\forall x \in [a, b]$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0))| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, f_n is uniformly Cauchy, and henec uniformly convergent. Thus, $\exists$ function f such that $f_n \to f$ uniformly.

(2) To prove $f(x)$ is differentiable in $[a, b]$, we fix $x \in [a, b]$

- Define $\phi(t) = \frac{f(t)-f(x)}{t-x}$ Goal: Show $\lim_{t\to x} \phi(t) = g(x)$
- Define $\phi_n(t) = \frac{f_n(t)-f_n(x)}{t-x}$

Sine $\begin{cases} \lim_{n\to\infty} \phi_n(t) = \phi(t) \\ \lim_{t\to x} \phi_n(t) = f_n'(x) \end{cases}$, it suffieces to show

$$\lim_{t\to x} \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) \tag{*}$$

Since by above we have, $\forall \varepsilon > 0$, $\exists N > 0$ such that $\forall n, m > N$

$$|f_n(x) - f_m(x) - (f_n(t) - f_m(t))| < \frac{\varepsilon}{2(b-a)} \cdot |t - x|$$

so dividing both sides by $|t - x|$, we see $\phi_n(t) - \phi_m(t)| < \frac{\varepsilon}{2(b-a)}$ so ϕ uniformly convergent over $[a, b] \setminus \{x\}$. Thus $(*)$ holds.