

MATH 110 Notes

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1 1/19/2022

1.1 Overview

Linear Algebra is about:

- Vector spaces (also called linear spaces)
- Linear maps
- Extra structures on spaces/maps (inner products, etc.)

Motivation:

- Physics - we live in a 3D space
- Geometry - even for a curved object, locally it looks like a vector space (tangent space)
- Representation Theory and Group Theory - the set of all matrices has a rich structure and interesting subsets (subgroups)
- Solving Differential Equations - natural tool and solution spaces
- Normal Operators - guaranteed good bases
- Statistics - square matrices, ...
- Applied Math - designing of algorithms, ...

1.2 Ch1 - Vector Spaces

\mathbb{R} - set of reals, \mathbb{R}^2 - plane, \mathbb{R}^3 - 3D space

Key feature: Have addition and scalar multiplication by \mathbb{R}

Generalizations: Vector spaces over \mathbb{R} (or a general \mathbb{F})

1.3 1.A: \mathbb{R}^n and \mathbb{C}^n

Definition 1.1 (\mathbb{C}).

Introduced i such that $i^2 + 1 = 0$

$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

eg: $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)^2 = 0 + i \cdot i = i$

$\mathbb{R} \subset \mathbb{C}$: view x as $x + 0i$

Theorem 1.2 (Properties of \mathbb{C}).

Commutativity: $\alpha + \beta = \beta + \alpha$, $\alpha\beta = \beta\alpha$ $\forall \alpha, \beta \in \mathbb{C}$

Associativity: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ $\forall \alpha, \beta, \gamma \in \mathbb{C}$

Additive Identity: $\alpha + 0 = \alpha$ $\forall \alpha \in \mathbb{C}$

Additive Inverse: $\forall \alpha \in \mathbb{C}$, $\exists! \beta \in \mathbb{C}$ such that $\alpha + \beta = 0$

Multiplicative Identity: $\alpha \cdot 1 = \alpha$ $\forall \alpha \in \mathbb{C}$

Multiplicative Inverse: $\forall \alpha \neq 0 \in \mathbb{C}$ $\exists! \beta \in \mathbb{C}$ such that $\alpha\beta = 1$

Distributive Properties: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ $\forall \lambda, \alpha, \beta \in \mathbb{C}$

2 1/24/2022

2.1 1.A: \mathbb{R}^n and \mathbb{C}^n

Example 2.1. Show existence and uniqueness of the multiplicative inverse of $\forall a \neq 0$

Idea: Assume $\alpha = a + bi$ want $(a + bi)(? + ?i) = 1 \rightarrow ? + ?i = \frac{1}{a + bi}$ “=”
 $\frac{a - bi}{(a + bi)(a - bi)} = \frac{1 - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$

Proof. Assume $\alpha = a + bi$, $a, b \in \mathbb{R}$, not both zero. We see that $\beta = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$ satisfies $(a + bi)(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i) = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$. Similarly, $(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i)(a + bi) = 1$. \rightarrow existence

Moreover, if there exists $\tilde{\beta}$ such that $\alpha\tilde{\beta} = 1$, then $\beta = \beta\alpha\tilde{\beta} = \tilde{\beta}$. \rightarrow uniqueness \square

Definition 2.2.

- For $\alpha \in \mathbb{C}$, let $-\alpha \in \mathbb{C}$ define the unique element such that $\alpha + (-\alpha) = 0$
- For $\alpha \in \mathbb{C}$, let $1/\alpha \in \mathbb{C}$ define the unique element such that $\alpha(1/\alpha) = 1$
- Subtraction: $\alpha - \beta = \alpha + (-\beta)$
- Division: $\beta/\alpha = \beta \cdot (1/\alpha)$, $\alpha \neq 0$

\mathbb{F} : field(In the book, \mathbb{R} or \mathbb{C})

- In general, generalization of \mathbb{R} or \mathbb{C}

Definition 2.3. A set \mathbb{F} (with addition “+” and multiplication “ \times ”) is a field if:

- $\exists 0, 1 \in \mathbb{F}$, $0 \neq 1$
- $+: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ by $(a, b) \mapsto a + b$
- $\times: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ by $(a, b) \mapsto a \cdot b$

Satisfying:

- (a) Commutivity: $a + b = b + a$, $ab = ba$
- (b) Associativity: $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$
- (c) Inverses: $\forall a, \exists -a$ such that $a + (-a) = 0$
 $\forall a, \exists 1/a$ such that $a \cdot (1/a) = 1$
- (d) Distributive: $c(a + b) = ca + cb$

Example 2.4.

- 1. \mathbb{R}, \mathbb{C}
- 2. $\{0, 1\}$ $+, \times \text{ mod } 2$
- 3. $\mathbb{F}_p = \{0, \dots, p-1\}$ $+, \times \text{ mod } p$, p prime
- 4. \mathbb{Q} : rationals
- 5. $\{a + b\sqrt{2} : a, b, \in \mathbb{Q}\}$
- 6. $\{P(x)/Q(x) : P, Q \text{ are polynomials over } \mathbb{F}, Q \neq 0\}$

We will define \cdot for \mathbb{F} . Elements of \mathbb{F} are known as scalars (as opposed to vectors)

Definition 2.5. An n -tuple of elements of \mathbb{F} is (x_1, \dots, x_n) where each $x_i \in \mathbb{F}$

Definition 2.6. $\mathbb{F}^n = \{\text{all } n\text{-tuples of elements in } \mathbb{F}\}$

Definition 2.7.

- Addition “+”: $\mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}^n$ by $((x_1, \dots, x_n), (y_1, \dots, y_n)) \mapsto (x_1 + y_1, \dots, x_n + y_n)$
- Scalar Multiplication “ \cdot ”: $\mathbb{F} \times \mathbb{F}^n$ by $(\lambda, (x_1, \dots, x_n)) \mapsto (\lambda x_1, \dots, \lambda x_n)$

Theorem 2.8 (Properties of \mathbb{F}^n).

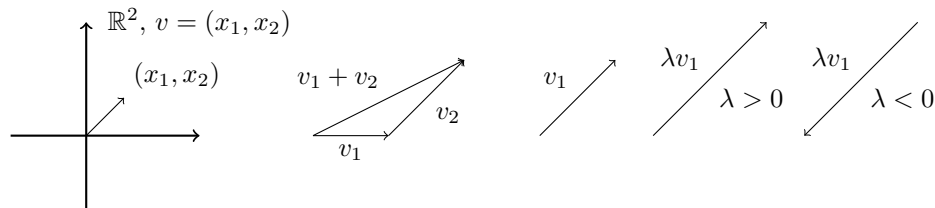
- Addition is commutative: $v_1 + v_2 = v_2 + v_1 \quad \forall v_1, v_2 \in \mathbb{F}^n$

Proof. Assume $v_1 = (x_1, \dots, x_n)$, $v_2 = (y_1, \dots, y_n)$ then
 $v_1 + v_2 = (x_1 + y_1, \dots, x_n + y_n) = (y_1 + x_1, \dots, y_n + x_n) = v_2 + v_1 \quad \square$

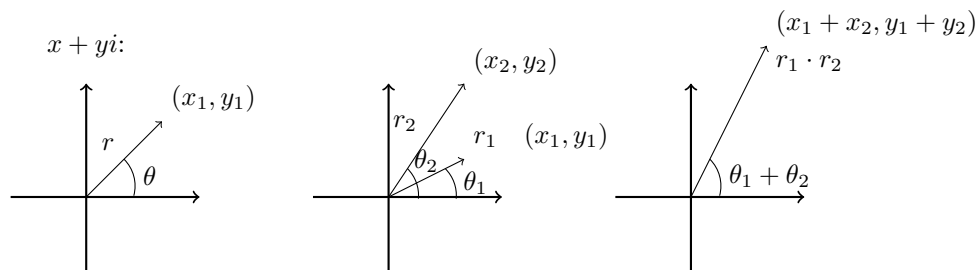
- Existence of $0 \in \mathbb{F}^n$: Denote $0 = (0, \dots, 0)$. Then $v + 0 = v \quad \forall v \in \mathbb{F}^n$
- Additive Inverse: $\forall v \in \mathbb{F}^n, \exists!(-v) \in \mathbb{F}^n$ such that $v + (-v) = 0$

Geometric Meaning for $\mathbb{F} = \mathbb{R}$

Descartes Coordinate System:



Geometric Meaning of Multiplication on \mathbb{C}



2.2 1B - Vector Spaces

Definition 2.9. Fix a field \mathbb{F} . A vector space over \mathbb{F} is a set V with addition “+” and scalar multiplication “ \cdot ” denoted as $+$: $V \times V \rightarrow V$ by $(v_1, v_2) \mapsto v_1 + v_2$, \cdot : $\mathbb{F} \times V \rightarrow V$ by $(\lambda, v) \mapsto \lambda v$

Satisfies:

- (1) $u + v = v + u, \forall u, v \in V$
- (2) $(u + v) + w = u + (v + w), a(bv) = (ab)v \forall u, v \in V, a, b \in \mathbb{F}$
- (3) $\exists 0 \in V$ such that $v + 0 = v, \forall v \in V$
- (4) $\forall v \in V, \exists w \in V$ such that $v + w = 0$. (we will show w is unique and denote it as $-v$)
- (5) $1 \cdot v = v, \forall v \in V$
- (6) $a(u + v) = au + av, (a + b)v = av + bv, \forall a, b \in \mathbb{F}, u, v \in V$

Definition 2.10. Elements in a vector space V are called points or vectors

Definition 2.11. A vector space over $\mathbb{F}(\mathbb{F})$ is also called an \mathbb{F} -vector space

Example 2.12.

- (1) $\mathbb{R}^n, \mathbb{C}^n, \mathbb{F}^n$ are all vector spaces
- (2) \mathbb{C} is a vector space over \mathbb{R}

- (3) Let S be a set. Define F^S = the set of all functions from S to \mathbb{F} . \mathbb{F}^S is a vector space $/\mathbb{F}$ under the operations $(f + g)(s) = f(s) + g(s)$, $(\lambda f)(s) = \lambda \cdot f(s)$. Each element has additive inverse $(-f)(s) = -f(s)$
 $\mathbb{F}^\infty = \mathbb{F}^{\{1,2,3,\dots\}}$, consists of $(a_1, a_2, a_3, \dots) \forall a_n \in \mathbb{F}$
- (4) the set of all sequences of real numbers that converge to 0
- (5) the set of all polynomials over \mathbb{F} , with $\deg \leq n$ in k variables is a vector space $/\mathbb{F}$

Theorem 2.13. A vector space V has a unique additive identity

Proof. Assume 0 and $0'$ are both additive inverses. Then $0 = 0 + 0' = 0'$ \square

Theorem 2.14. $\forall v \in V$ has a unique additive inverse.

Proof. If w_1, w_2 are both additive inverses of v , then $w_1 = w_1 + (v + w_2) = (w_1 + v) + w_2 = w_2$ \square

Definition 2.15. Let $w - v = w + (-v)$

Notation 2.16. V will be used to denote a vector space over \mathbb{F}

Theorem 2.17. $0 \cdot v = 0, \forall v \in V$

Proof. $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v$ so by the existence of additive inverses $0 = 0 \cdot v$ \square

Theorem 2.18. $a \cdot 0 = 0, \forall a \in \mathbb{F}$

Proof. $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$ so $0 = a \cdot 0$ \square

Theorem 2.19. $(-1) \cdot v = -v, \forall v \in V$

Proof. $0 = 0v = (1 + (-1))v = 1v + (-1)v = v + (-1)v$ so by def $(-1)v = -v$ \square

3 1/26/2022

3.1 1.C - Subspaces

Definition 3.1. Assuming V is a vector space $/\mathbb{F}$. $U \subset V$ is called a subspace of V if U is also a vector space $/\mathbb{F}$ under $+$ and \cdot in V .

Example 3.2. $\{(x_1, x_2, 0) : x_1, x_2, \in \mathbb{F}\}$ is a subspace of \mathbb{F}^3

Proposition 3.3. U is a subspace iff

- (i) $0 \in U$
- (ii) $u_1, u_2 \in U \rightarrow u_1 + u_2 \in U$
- (iii) $a \in \mathbb{F}, u \in U \rightarrow a \cdot u \in U$

Proof. \rightarrow) Suppose conditions hold. Then properties of $+$, \cdot follow from V , U has identity by (i) and additive inverses by (iii). Finally, $+$, \cdot well defined by (ii), (iii) so U is a subspace.

\leftarrow) Suppose U is a subspace. Then U is nonempty so $0 \cdot u = 0 \in U$ so (i) holds. Also, $+$, \cdot well defined so (ii), (iii) hold. \square

Example 3.4.

- (a) $\{0\}$ is a subspace
- (b) $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$ is a subspace iff $b = 0$
- (c) $C[0, 1] = \{\text{continuous real valued functions on } [0, 1]\}$ is a subspace of $\mathbb{R}^{[0,1]}$ (over \mathbb{R})
- (d) $C^\infty[0, 1] = \{\text{smooth real-valued functions on } [0, 1]\}$ is a subspace $\mathbb{R}^{[0,1]}$
- (e) The real sequences that converge to zero form a subspace of \mathbb{R}^∞
- (f) The only subspaces of \mathbb{F}^1 are $\{0\}$ and \mathbb{F} (over \mathbb{F})
- (g) If U is a subspace of V , W is a subspace of U , then W is a subspace of V
- (h) We will show the only subspace of \mathbb{R}^3 are $\{0\}$, lines through the origin, planes through the origin, and \mathbb{R}^3

Definition 3.5. For U_1, \dots, U_n subspaces of V , define the sum

$$U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$$

also denoted as $\sum_{j=1}^m U_j$.

Example 3.6. In \mathbb{F}^3 , what is $\{(x, x, 0)\} + \{(0, y, y)\}$?

Proof. $\{(x, y, z) : y = x + z\}$ \square

Theorem 3.7. For subspaces $U_1, \dots, U_m \subset V$, $\sum_{j=1}^m U_j$ is a subspace. Moreover, it is the smallest subspace containing U_1, \dots, U_m in the sense that if W contains U_1, \dots, U_m , then $W \supset \sum_{j=1}^m U_j$.

Proof. Subspace: (i) $0 \in U_i$ for $i = 1, \dots, m$ so $0 = 0 + \dots + 0 \in \sum_{j=1}^m U_j$

(ii)/(iii): follow from closedness of each U_j

Containing U_1, \dots, U_m : Consider the sum $0 + \dots + 0 + u_j + 0 + \dots + 0$ for $j = 1, \dots, m$

Smallest Subspace: Suppose W contains U_1, \dots, U_m then W contains u_1, \dots, u_m $\forall u_j \in U_j$ so $u_1 + \dots + u_m \in W$. \square

3.2 Direct Sums

Definition 3.8. If U_1, \dots, U_m are subspaces of V then the sum $U_1 + \dots + U_m$ is a direct sum if each element in $U_1 + \dots + U_m$ can be written as $u_1 + \dots + u_m$ in a unique way with $u_j \in U_j$. In this case, we also use $U_1 \oplus \dots \oplus U_m$ to denote $U_1 + \dots + U_m$.

Example 3.9.

- (1) If $U_1 = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{F}\}$, $U_2 = \{(0, 0, x_3) \mid x_3 \in \mathbb{F}\}$, then $\mathbb{F}^3 = U_1 \oplus U_2$.
- (2) Let $U = \{(x, x, \dots) \in \mathbb{R}^\infty\}$, $V = \{(x_1, x_2, \dots) \in \mathbb{R}^\infty : \sum |x_n| < \infty, \sum x_n = 0\}$. Then $U + V$ is a direct sum.
(ex): Prove $U + V \neq \mathbb{R}^\infty$

Theorem 3.10. $U_1 + \dots + U_m$ is a direct sum iff $\exists!$ way to write 0 as a sum of $u_1 + \dots + u_m$, $\forall u_j \in U_j$ (which is $0 = 0 + \dots + 0$).

Proof. \rightarrow) by def

\leftarrow) For $u \in U_1 + \dots + U_m$, assume $u = u_1 + \dots + u_m = \tilde{u}_1 + \dots + \tilde{u}_m$, $u_j, \tilde{u}_j \in U_j$. Then $(u_1 - \tilde{u}_1) + (u_2 - \tilde{u}_2) + \dots + (u_m - \tilde{u}_m) = 0$. Hence $u_1 - \tilde{u}_1 = u_2 - \tilde{u}_2 = \dots = 0$. Thus there is only one way to write u as $\sum_{j=1}^m u_j$, $\forall u_j \in U_j$. \square

Theorem 3.11. For subspaces $U_1, U_2 \in V$, $U_1 + U_2$ is a direct sum iff $U_1 \cap U_2 = \{0\}$.

Proof. \rightarrow) If $v \in U_1 \cap U_2$, $\underbrace{v}_{\in U_1} + \underbrace{(-v)}_{\in U_2} = 0$ so $v = (-v) = 0$

\leftarrow) Take $u \in U_1 + U_2$ assume $u = u_1 + u_2 = \tilde{u}_1 + \tilde{u}_2$. Then $\underbrace{u_1 - \tilde{u}_1}_{\in U_1} = -\underbrace{(u_2 - \tilde{u}_2)}_{\in U_2}$
so by assumptions, $u_1 = \tilde{u}_1$ and $u_2 = \tilde{u}_2$. \square

Example 3.12. For subspaces U_1, \dots, U_m of V , TFAE:

- (i) $U_1 + \dots + U_m$ is a direct sum
- (ii) $\forall j \ U_j \cap (\sum_{k \neq j} U_k) = \{0\}$
- (iii) $\forall j > 1, U_j \cap (U_1 + \dots + U_{j-1}) = \{0\}$
- (iv) If $u_1 + \dots + u_m = 0$, $u_j \in U_j$ then $u_1 = u_2 = \dots = u_m = 0$

3.3 Chapter 2: Finite Dimensional Vector Spaces

\mathbb{F} : field, V : Vector space / \mathbb{F}

3.4 2.A: Span and Linear Independence

Motivation: In some V (such as \mathbb{F}^n), we can find vectors $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ such that every $v \in V$ can be written as $v = \sum_{j=1}^n a_j e_j$ and the choice of a_j is unique.

We will work with such vectors in a general setting.

4 1/31/2022

4.1 Chapter 2: Finite Dimensional Vector Spaces

- Main motivation: Find “coordinate systems” in a vector space
- Recall in \mathbb{F}^n , $(x_1, \dots, x_n) = x_1(1, 0, \dots, 0) + x_n(0, \dots, 0, 1) = x_1 e_1 + \dots + x_n e_n$.

4.2 2.A: Span and Linear Independence

Definition 4.1. A linear combination of vectors $v_1, \dots, v_m \in V$ is a vector of the form

$$v = \sum_{j=1}^m a_j v_j, \quad a_1, \dots, a_m \in \mathbb{F}.$$

Example 4.2. $(1, 2, -3) = (1, 0, -1) + 2(0, 1, -1)$

Example 4.3. Is $(1, 2, 3)$ a linear combination of $(1, 0, -1)$ and $(0, 1, 1)$?

No, if $(1, 2, -3) = a_1(1, 0, -1) + a_2(0, 1, 1)$ then $a_1 = 1, a_2 = 2$ but $1(1, 0, -1) + 2(0, 1, 1) = (1, 2, 1) \neq (1, 2, -3)$.

Definition 4.4. The set

$$\left\{ \sum_{j=1}^m a_j v_j, a_i \in \mathbb{F}, \forall 1 \leq j \leq m \right\}$$

is the span of v_1, \dots, v_m , denoted by $\text{span}(v_1, \dots, v_m)$. Note $\text{span}() = \{0\}$.

Example 4.5. $(1, 2, -3) \in \text{span}((1, 0, -1), (0, 1, -1))$.

Theorem 4.6. $\text{span}(v_1, \dots, v_m)$ is the smallest subspace of V that contains v_1, \dots, v_m .

Proof. Subspace: $0 = 0v_1 + \dots, 0v_n \in \text{span}(v_1, \dots, v_m)$

Closed under addition: $(a_1 v_1 + \dots + a_m v_m) + (b_1 v_1 + \dots + b_m v_m) = (a_1 + b_1)v_1 + \dots + (a_m + b_m)v_m$.

Closed under multiplication: $\lambda(a_1 v_1 + \dots + a_m v_m) = (\lambda a_1)v_1 + \dots + (\lambda a_m)v_m$. So it is a subspace.

Smallest: If $v_1, \dots, v_m \in W$ for some subspace W , then $\forall a_1, \dots, a_n \in \mathbb{F}$, $a_1 v_1, \dots, a_m v_m \in W$ so $a_1 v_1 + \dots + a_m v_m \in W$. Thus, $\text{span}(v_1, \dots, v_m) \subseteq W$. \square

Definition 4.7. If $V = \text{span}(v_1, \dots, v_m)$, then we say the list v_1, \dots, v_m spans V .

Example 4.8. e_1, \dots, e_n spans \mathbb{F}^n

Definition 4.9. V is called finite dimensional if some (finite) list of vectors spans V .

Example 4.10. \mathbb{F}^n is finite dimensional.

Definition 4.11. A finite expression

$$p(z) = a_0 + a_1 z^1 + \dots + a_m z^m, \quad a_0, \dots, a_m \in \mathbb{F}, a_m \neq 0,$$

also written as

$$\sum_{j=0}^{\infty} a_j x_j, a_{n+1} = a_{n+2} = \dots = 0,$$

is called a polynomial with coefficients in \mathbb{F} . (By definition $p = 0$ is a polynomial.)

- Each polynomial over \mathbb{F} gives rise to a function from $\mathbb{F} \rightarrow \mathbb{F}$ defined by $p : \mathbb{F} \rightarrow \mathbb{F}$ by $z \mapsto p(z)$
- m is the degree of p if p has the form $(*)$. The zero polynomial has degree $-\infty$ by definition.
- $\mathcal{P}(\mathbb{F}) = \{\text{all polynomials over } \mathbb{F}\}$
- $\mathcal{P}_m(\mathbb{F}) = \{\text{all polynomials of } \deg \leq m \text{ over } \mathbb{F}\}$

Example 4.12. $\mathcal{P}_m(\mathbb{F})$ and $\mathcal{P}(\mathbb{F})$ are vector spaces over \mathbb{F} (also subspaces of $\mathbb{F}^{\mathbb{F}}$ if viewed as functions.)

Example 4.13.

- (a) $\mathcal{P}_m(\mathbb{F})$ is finite dimensional
- (b) $\mathcal{P}(\mathbb{F})$ is infinite dimensional

Proof.

- (a) $1, z, \dots, z^m$ spans $\mathcal{P}_m(\mathbb{F})$
- (b) For any $p_1, \dots, p_m \in \mathcal{P}(\mathbb{F})$, assume N is larger than $\deg p_j$ for $1 \leq j \leq m$. Then every $\sum_{j=1}^m a_j p_j$ is not equal to z^N .

□

Definition 4.14. v_1, \dots, v_m is called linearly independent if whenever $0 \sum_{j=1}^m a_j v_j$, $a_1, \dots, a_m \in \mathbb{F}$, we must have $a_1 = \dots = a_m = 0$. Otherwise, the list is linearly dependent. The empty list is linearly independent by definition.

Example 4.15.

- (a) v is linearly independent iff $v \neq 0$
- (b) e_1, \dots, e_n is linearly independent in \mathbb{F}^n
- (c) v_1, v_2 is linearly independent iff neither vector is a scalar multiple of the other.
- (d) $1, z, \dots, z^m$ is linearly independent in $\mathcal{P}_m(\mathbb{F})$.
- (e) $(1, *, *), (0, 1, *), (0, 0, 1)$ where each $*$ is arbitrary is linearly independent in \mathbb{F}^3
- (f) $(1, 1, \dots, 1), (a_1, a_2, \dots, a_n), (a_1^2, a_2^2, \dots, a_n^2), \dots, (a_1^{n-1}, a_2^{n-1}, \dots, a_n^{n-1})$ is linearly dependent iff at least two of the a_j 's are the same.
- (g) Any subset of a linearly independent list is linearly independent.

Example 4.16.

- (a) If some vector in a list is a linear combination of the others, then the list is linearly dependent.
- (b) Every list containing 0 is linearly dependent.