Descriptive Set Theory: Moschovakis

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Exercise 2.7.1. Prove that a binary relation $R(x,y)$ on a set S is wellfounded if an only if there are no infinite $<_R$-descending chains.

Proof. If $R(x,y)$ is not wellfounded then there must be some nonempty set A such that A has no $<_R$ minimal element. Using A, we construct an infinite $<_R$-descending chain as follows: Let $x_0 \in A$ arbitrary and given x_i, choose $x_{i+1} \in A$ such that $x_i >_R x_{i+1}$. Such an element will always exist since A has no least element and so $x_0 >_R x_1 > \cdots$ is an infinite $<_R$-descending chain.
Conversely, suppose there is an infinite $<_R$-descending chain $x_0 >_R x_1 >_R \cdots$ and let $A = \{x_i \colon in \in \omega\}$. A is nonempty set with no $<_R$ minimal element and so $R(x,y)$ is not wellfounded. $\square$

Exercise 2.7.2. Prove that every Borel wellfounded relation has countable length and every $\utilde{\Delta}^1_2$ wellfounded relation has length less than $\aleph_2$.

Proof. Recall that if $R(x,y)$ is a relation we define its strict part to be

$$<_R = \{(x,y) \colon R(x,y) \& \neg R(y,x)\}$$

So, if R is $\utilde{\Delta}^1_n$ we see that $<_R$ is $\utilde{\Delta}^1_n$ as well. Hence, if R is a Borel relation then $<_R$ must be $\aleph_0$-Suslin and so, applying the Kunen-Martin theorem, it must have length less than $\aleph_1$. Similarly, if R is $\utilde{\Delta}^1_2$ then $<_R$ is $\aleph_1$-Suslin and so has length less than $\aleph_2$. $\square$

Exercise 2.7.3. Let R be a wellfounded relation on S with rank function ρ, and let $f : S \to$ Ordinals be an order preserving function, ie.

$$x <_R y \Rightarrow f(x) < f(y)$$

Prove that for every x in S, $\rho(x) \leqslant f(x)$.

Proof. We show this by induction on the relation. Suppose that for all $y <_R x$, $\rho(y) \leqslant f(y)$ then, since we have $f(y) < f(x)$ for each $y <_R x$, $f(x) \geqslant f(y) + 1$ for all $y <_R x$ and so

$$f(x) \geqslant \sup\{f(y) + 1 \colon y <_R x\} \geqslant \sup\{\rho(y) + 1 \colon y <_R x\} = \rho(x)$$

□

A norm φ on S is called regular if it is onto some ordinal λ.
Given a norm φ on S the associated the binary relation $\leqslant^{\varphi}$ on S is defined by

$$x \leqslant^{\varphi} y \Leftrightarrow \varphi(x) \leqslant \varphi(y)$$

Exercise 2.7.4. Prove that a binary relation $\preceq$ on a set S is a prewellordering if and only if there is a norm φ on S such that $\preceq = \leqslant^{\varphi}$. Moreover if $\preceq$ is a prewellordering, then there is a unique regular φ on S such that $\preceq = \leqslant^{\varphi}$.

Proof. If $\preceq$ is a prewellordering, we claim that its rank function ρ is a norm on S such that $\preceq = \leqslant^{\rho}$, ie. for all $x, y \in S$

$$x \preceq y \Leftrightarrow \rho(x) \leqslant \rho(y)$$

This follows from the fact that if $x \preceq y$ then $\rho(x) \leqslant \rho(x)$ since all elemnts of S strictly below x will also be strictly below y. Similarly, if $\neg x \preceq y$, then y is strictly below x so $\rho(y) < \rho(x)$.
Conversely, if there is some norm φ on S such that $\preceq = \leqslant^{\varphi}$, $\preceq$ must be prewellordering since $\leqslant$ is a wellorder and, under the pullback, the only condition it may fail to satisfy is antisymmetry.
To show that there is a unique regular norm satisfying this property note that the rank function ρ is a regular norm since it is onto $\lambda = |\preceq|$ and so to show uniqueness it is enough to show any regular norm is equal to ρ. We do this by induction on the ordering: Suppose that φ is another regular norm and for all y strictly below x, $\varphi(y) = \rho(y)$. We must have

$$\varphi(x) \geqslant \sup\{\varphi(y) + 1 \colon y \prec x\} = \sup\{\rho(y) + 1 \colon y \prec x\} = \rho(x)$$

and if this inequality is strict, since φ is surjective, there must be some z such that $\varphi(z) = \rho(x)$. We cannot have $z \prec x$ (otherwise $\varphi(z) = \rho(z) < \rho(x)$) and so $x \preceq z$. However, this implies $\varphi(x) \leqslant \varphi(z)$ contradicting our choice of z. Hence, we must have $\varphi(x) = \rho(x)$ as well. □

2.8 Category and Measure

Exercise 2.8.1. Prove that in a complete metric space no open ball in meager.

Proof. Suppose towards a contradiction that B is open ball such that $B = \bigcup_n A_n$ where each A_n is nowhere dense. Then we have $B \subset \bigcup \overline{A_n}$. Since A_1 is not dense in B we can find some $B_1 \subset B$ such that $\overline{B_1} \subset B \backslash \overline{A_1}$ and such that radius$(B_1) <$. Similarly, given B_i we can find B_{i+1} such that $\overline{B_i} \subset B_i \backslash \overline{A_{i+1}}$ and radius$(B_{i+1}) < 2^{-(i+1)}$. Now, there exists a unique $b \in \bigcap_n \overline{B_n}$ and by construction $b \notin \overline{A_i}$ for all i and so $b \in B \backslash \bigcup_n A_n$, contradicting our assumption. Hence, B cannot be meager. □

A pointset P has the property of Baire if there is some open poinset P^* such that $P \Delta P^*$ is meager.

Exercise 2.8.2. Prove that every Borel poinset has the property of Baire.

Proof. We show that the class of poinsets having the property of Baire contains all the open and closed sets, and is closed under negation and countable union. If P is an open pointset, then $P\Delta P = \varnothing$ is meager so P has the property of Baire. If P is a closed pointset, consider $P^* = P^\circ$. $P\Delta P^* = P\backslash P^*$ is meager since it is closed and has empty interior since $(P\backslash P*)^\circ \subseteq P^\circ$ and $(P\backslash P^*) \cap P^\circ = \varnothing$. Next, suppose P has the property of Baire, we show $\neg P$ does as well. If P^* is an open set such that $P\Delta P^*$ is meager then, since $\neg P\Delta\neg P^* = P\Delta P^*$, $\neg P\Delta\neg P^*$ must be meager as well. Now, it follows that $\neg P\Delta(\neg P^*)^\circ \subseteq (\neg P\Delta\neg P^*) \cup (\neg P^*)\backslash(\neg P^*)^\circ$ is meager and so $\neg P$ has the property of Baire as well. Finally, we show that if $\{P_n\}_{n\in\omega}$ are such that each P_i has the property of Baire $\bigcup_i P_i$ doees as well. Suppose we have open P_i^* such that $P_i\Delta P_i^*$ is meager. Then, $\bigcup_i P_i\Delta\bigcup_i P_i^* \subseteq \bigcup_i(P_i\Delta P_i^*)$ is meager and so $\bigcup_i P_i$ has the property of Baire as well. □

Exercise 2.8.3. Prove that for every poinset $P \subseteq \mathcal{X}$, there is an F_σ set $\tilde{P} \supseteq P$ such that if $A \subseteq P^*\backslash P$ is Borel, then A is meager.

Proof. Suppose P is an arbitary pointset. Consider the set

$$D = \{x\colon \text{ for all neighboroods of} N \text{ of } x\, N \cap P \text{ is not meager}\}$$

D is open since if $x \in \neg D$, then there is some $B_r(x)$ such that $B_r(x) \cap P$ is meager. Now, for all $y \in B_{r/2}(x)$, $B_{r/2}(y) \subseteq B_r(x)$ and so $B_{r/2}(y) \cap P$ is meager. Hence, $B_{r/2}(x) \subseteq \neg D$.
Next, observe that $P\backslash D$ must be meager since each point $x \in P\backslash D$ must be contained in some basic open neighborhood, N_x such that $N_x \cap P$ is meager and so, since there are only countable many such neighborhoods, the union $\bigcup_{x\in P\backslash D}(N_x \cap P)$ is a countable union of meager sets and so must be meager. Further, this set must be contained in some F_σ meager set W since if $P\backslash D = \bigcup A_n$ where each A_n is no where dense, then $P\backslash D \subset \bigcup \overline{A_n}$ as well.
We define $P^* = D \subseteq W$ and claim that for any Borel set $A \subseteq P^* \subseteq P$, A must be meager. To see why this is true suppose A is such a Borel set. Then, there is some open set A^* such that $A\Delta A^*$ is meager. $A^* \cap P \subseteq (A^*\backslash A)$ is meager and so $A^* \cap D = \varnothing$. However, this implies $A^* \subseteq W \cup (A^*\backslash A)$ which is meager and so we must have $A^* = \varnothing$. Hence, A must be meager. □

Exercise 2.8.4. Prove that the collection of pointsets with the property of Baire is closed under the operation $\mathcal{A}$; in particular $\underset{\sim}{\Sigma}^1_1$ sets have the property of Baire.

Proof. Let $\mathcal{C}$ be the σ-algebra of open sets and let J be the σ-ideal consisting of the meager sets. The previous exercise showed that J is regular from above relative to $\mathcal{C}$ and so, by the Approximation theorem, the collection of all pointsets that are in $\mathcal{C}$ modulo J, ie. the sets with the property of Baire, is closed under $\mathcal{A}$. In particular, since this class contains all Borel sets, it must contain all $\underset{\sim}{\Sigma}^1_1$ sets as well. □