

Descriptive Set Theory: Moschovakis

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Part I

Notes

Part II

Exercises

Chapter 1

The Basic Classical Notions

1.1 Perfect Polish Spaces

1.2 The Borel Pointclasses of Finite Order

1.3 Computing with Relations; Closure Properties

1.4 Parameterization and Hierarchy Theorems

1.5 The Projective Sets

1.6 Countable Operations

1.7 Borel Functions and Isomorphisms

Chapter 2

κ -Suslin and λ -Borel

2.1 The Cantor-Bendixson Theorem

2.2 κ -Suslin Sets

2.3 Trees and the Perfect Set Theorem

2.4 Wellfounded Trees

2.5 The Suslin Theorem

2.6 Inductive Analysis of Projective Trees

2.7 The Kunen-Martin Theorem

Exercise 2.7.1. Prove that a binary relation $R(x, y)$ on a set S is wellfounded if and only if there are no infinite $<_R$ -descending chains.

Proof. If $R(x, y)$ is not wellfounded then there must be some nonempty set A such that A has no $<_R$ minimal element. Using A , we construct an infinite $<_R$ -descending chain as follows: Let $x_0 \in A$ arbitrary and given x_i , choose $x_{i+1} \in A$ such that $x_i >_R x_{i+1}$. Such an element will always exist since A has no least element and so $x_0 >_R x_1 > \dots$ is an infinite $<_R$ -descending chain.

Conversely, suppose there is an infinite $<_R$ -descending chain $x_0 >_R x_1 >_R \dots$ and let $A = \{x_i : i \in \omega\}$. A is nonempty set with no $<_R$ minimal element and so $R(x, y)$ is not wellfounded. \square

Exercise 2.7.2. Prove that every Borel wellfounded relation has countable length and every Δ_2^1 wellfounded relation has length less than \aleph_2 .

Proof. Recall that if $R(x, y)$ is a relation we define its strict part to be

$$<_R = \{(x, y) : R(x, y) \& \neg R(y, x)\}$$

So, if R is Δ_n^1 we see that $<_R$ is Δ_n^1 as well. Hence, if R is a Borel relation then $<_R$ must be \aleph_0 -Suslin and so, applying the Kunen-Martin theorem, it must have length less than \aleph_1 . Similarly, if R is Δ_2^1 then $<_R$ is \aleph_1 -Suslin and so has length less than \aleph_2 . \square

Exercise 2.7.3. Let R be a wellfounded relation on S with rank function ρ , and let $f : S \rightarrow \text{Ordinals}$ be an order preserving function, ie.

$$x <_R y \Rightarrow f(x) < f(y)$$

Prove that for every x in S , $\rho(x) \leq f(x)$.

Proof. We show this by induction on the relation. Suppose that for all $y <_R x$, $\rho(y) \leq f(y)$ then, since we have $f(y) < f(x)$ for each $y <_R x$, $f(x) \geq f(y) + 1$ for all $y <_R x$ and so

$$f(x) \geq \sup\{f(y) + 1 : y <_R x\} \geq \sup\{\rho(y) + 1 : y <_R x\} = \rho(x)$$

□

A norm φ on S is called regular if it is onto some ordinal λ .

Given a norm φ on S the associated the binary relation \leq^φ on S is defined by

$$x \leq^\varphi y \Leftrightarrow \varphi(x) \leq \varphi(y)$$

Exercise 2.7.4. Prove that a binary relation \leq on a set S is a prewellordering if and only if there is a norm φ on S such that $\leq = \leq^\varphi$. Moreover if \leq is a prewellordering, then there is a unique regular φ on S such that $\leq = \leq^\varphi$.

Proof.

□