

MATH 214 Hw4

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Exercise 1 (Lee 3.1). Suppose M and N are smooth manifolds with or without boundary, and $F : M \rightarrow N$ is a smooth map. Show that $dF_p : T_p M \rightarrow T_p N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M .

Proof. First, suppose that F is a smooth map such that dF_p is the zero map for each $p \in M$. We first consider the case where M and N are \mathbb{R}^m and \mathbb{R}^n . In this case the map dF_p is the matrix $(\partial F^i / \partial x^j(p))_{ij}$ and so we have the all partials of F are 0 at all points p and so F must be constant on each component. In the case where M and N are arbitrary manifolds, consider charts (U, φ) and (V, ψ) for the points p and $F(p)$, respectively. If dF_p is 0 for all p then $d(\psi \circ F \circ \varphi^{-1})_p = 0$. Then, as this is a map between open subsets of Euclidean space the above proof shows that $\psi \circ F \circ \varphi^{-1}$ must be constant on each component and so, since ψ, φ are homeomorphisms F must be also. Hence, we have shown that for each point $p \in M$, F is constant each component in a neighborhood of p and so it follows that this holds for all of M . \square

Exercise 2 (Lee 3.3). Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Proof. Suppose M and N are smooth manifolds. Recall that for each $p \in M$ we have an isomorphism of the tangent spaces $T_p(M \times N)$ and $T_p M \times T_p N$ given by $\alpha(v) = (d(\pi_1)_p(v), d(\pi_2)_p(v))$. Define a map $T(M \times N) \rightarrow TM \times TN$ by $f((p, q), v) = ((p, d(\pi_1)_{(p,q)}(v)), (q, d(\pi_2)_{(p,q)}(v)))$. Since the map α is an isomorphism it is clear that this map is bijective so to show it is a diffeomorphism we show it is smooth with smooth inverse.

First, to check it is smooth at $((p, q), v)$ fix smooth charts (U, φ) and (V, ψ) containing p and q in M and N , respectively. Then, $(\pi^{-1}(U \times V), \widetilde{\varphi \times \psi})$ is a smooth chart containing $((p, q), v)$ and $(\pi^{-1}(U) \times \pi^{-1}(V), \tilde{\varphi} \times \tilde{\psi})$ is a smooth chart containing its inverse. We compute the transition map to be

$$\begin{aligned} (\tilde{\varphi} \times \tilde{\psi} \circ F \circ (\widetilde{\varphi \times \psi}))^{-1}(\bar{x}, \bar{y}, \bar{v}, \bar{w}) &= (\tilde{\varphi} \times \tilde{\psi}) \circ F \left(\sum v^i \frac{\partial}{\partial x^i} + \sum w^j \frac{\partial}{\partial y^j} \right) |_{(\varphi \times \psi)^{-1}(\bar{x}, \bar{y})} \\ &= (\tilde{\varphi} \times \tilde{\psi}) \left(d(\pi_1) \left(\sum v^i \frac{\partial}{\partial x^i} + \sum w^j \frac{\partial}{\partial y^j} \right) |_{\varphi^{-1}(\bar{x})}, d(\pi_2) \left(\sum v^i \frac{\partial}{\partial x^i} + \sum w^j \frac{\partial}{\partial y^j} \right) |_{\varphi^{-1}(\bar{y})} \right) \\ &= (\tilde{\varphi} \times \tilde{\psi}) \left(\sum v^i \frac{\partial}{\partial x^i} |_{\varphi^{-1}(\bar{x})}, \sum w^j \frac{\partial}{\partial y^j} |_{\varphi^{-1}(\bar{y})} \right) \\ &= ((\bar{x}, \bar{v}), (\bar{y}, \bar{w})) \end{aligned}$$

which is smooth and has smooth inverse (since the inverse of the transition map is the transition map for F^{-1}) hence it follows that both F and F^{-1} are smooth, as desired. \square

Exercise 3 (Lee 3.4). Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Proof. For each $p = e^{i\varphi} \in \mathbb{S}^1$ consider the generating element of $T_p \mathbb{S}^1$ defined by $v_p(f) = (f \circ \gamma)'(0)$ where $\gamma : [-\pi/2, \pi/2] \rightarrow \mathbb{S}^1$ is defined by $\gamma(t) = e^{i(\varphi+t)}$. Then each element $w \in T\mathbb{S}^1$ is equal to av_p for some $a \in \mathbb{R}$. Now, define the map $F : T\mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{R}$ by $F((p, av_p)) = (p, a)$. It is clear that F is a bijection. We show that F is smooth. Fix some $(e^{i\varphi}, w) \in T_p M$. We can find some chart (U, θ) of \mathbb{S}^1 containing $e^{i\varphi}$ such that θ is an angle function. Now, $(\pi^{-1}(U, \tilde{\theta}))$ is a chart containing $(e^{i\varphi}, w)$ and $(U \times \mathbb{R}, \theta \times \text{id}_{\mathbb{R}})$ is a chart containing $F((e^{i\varphi}, w))$ and $F(\pi^{-1}(U)) \subseteq U \times \mathbb{R}$ and so to show F is smooth it is enough to show the transition map is smooth. This follows since

$$(\theta \times \text{id}_{\mathbb{R}}) \circ F \circ (\tilde{\theta})^{-1}(x, y) = (\theta \times \text{id}_{\mathbb{R}}) \circ F(e^{ix}, y \cdot v_p) = (\theta \times \text{id}_{\mathbb{R}})(e^{ix}, y) = (x, y)$$

Note that this map has smooth inverse and since its inverse is a transition map for F^{-1} it follows that F^{-1} is smooth as well and so F is a diffeomorphism. \square

Exercise 4. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max\{|x|, |y|\} = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no diffeomorphism with the same property.

Exercise 5 (Lee 3.7). Let M be a smooth manifold with or without boundary and p a point of M . Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p , and let $\mathcal{D}_p M$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\phi : \mathcal{D}_p M \rightarrow T_p M$ by $(\phi v)f = v([f]_p)$. Show that ϕ is an isomorphism.

Proof. To show that ϕ is an isomorphism of vector spaces first note that it is linear since

$$(\phi(av + bw))(f) = (av + bw)([f]_p) = av([f]_p) + bw([f]_p) = a\phi(v)(f) + b\phi(w)(f)$$

ϕ is injective since if $\phi(v) = 0$ then $v([f]_p) = 0$ for all f and so v must have been the 0 map to begin with. Finally, ϕ is surjective since given any $w \in T_p M$ consider $v \in \mathcal{D}_p M$ defined by $v([f]_p) = w(f)$. Note that this is well defined since if $[f]_p = [g]_p$ then f and g agree on some open set containing p so we must have $w(f) = w(g)$. Then it is clear that $\phi(v) = w$ and so ϕ is surjective. Thus it is an isomorphism. \square

Exercise 6 (Lee 3.8). Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$ denote the set of equivalence classes of curves starting at p under the relation $\gamma_1 \sim \gamma_2$ of $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p . Show that the map $\psi : \mathcal{V}_p M \rightarrow T_p M$ defined by $(\psi[\gamma])(f) = (f \circ \gamma)'(0)$ is well defined and bijective.

Proof. First, to see that ψ is well defined suppose that $\gamma_1 \sim \gamma_2$, ie. $[\gamma_1] = [\gamma_2]$. Then

$$(\psi[\gamma_1])(f) = (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) = (\psi[\gamma_2])(f)$$

for all f so $\psi[\gamma_1] = \psi[\gamma_2]$. By linearity of the derivative it is clear that ψ is linear as well.

ψ is injective since if $\psi[\gamma] \equiv 0$ then $(f \circ \gamma)'(0) = 0$ for all f and so $\gamma \sim 0$, ie. $[\gamma] = [0]$.

Finally, to see that ψ is surjective fix some chart (U, φ) containing p . It suffices to show its image contains $\frac{\partial}{\partial x^i}|_p$ since these form a basis for $T_p M$. Now, observe that if $\gamma : [-1, 1] \rightarrow \mathbb{R}^n$ is defined by $\gamma_i(t) = (p_1, \dots, p_i + t, \dots, p_n)$ then $(f \circ \gamma_i)'(0) = \partial f / \partial x^i(p)$ for each $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Hence, taking the curve $\varphi^{-1} \circ \gamma_i : [-1, 1] \rightarrow M$, we see that for each $f : M \rightarrow \mathbb{R}$,

$$\frac{\partial}{\partial x^i}|_p(f) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})(p) = (f \circ \varphi^{-1} \circ \gamma_i)'(0) = \psi([\varphi^{-1} \circ \gamma_i])(f)$$

\square