MATH 214 Hw4

Jad Damaj

Feb 15, 2024

Exercise 1 (Lee 3.1)**.** Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_pN$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof. First, suppose that F is a smooth map such that dF_p is the zero map for each $p \in M$. We first consider the case where M and N and $\mathbb{R}^m$ and $\mathbb{R}^n$. In this case the map dF_p is the matrix $(\partial F^1/\partial x^j(p))_{ij}$ and so we have the all partials of F are 0 at all points p and so F must be constant on each component. In the case where M and N are arbitrary manifolds, consider charts (U, φ) and (V, ψ) for the points p and $F(p)$, respectively. If dF_p is 0 for all p then $d(\psi \circ F \circ \varphi^{-1})_p = 0$. Then, as this is a map between open subsets of Euclidean space the above proof shows that $\psi \circ F \circ \varphi^{-1}$ must be constant on each component and so, since ψ, φ are homeomorphisms F must be also. Hence, we have shown that for each point $p \in M$, F is constant each component in a neighborhood of p and so it follows that this holds for all of M. $\square$

Exercise 2 (Lee 3.3)**.** Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Proof. Suppose M and N are smooth manifolds. Recall that for each $p \in M$ we have an isomorphism of the tangent spaces $T_p(M \times N)$ and $T_pM \times T_pN$ given by $\alpha(v) = (d(\pi_1)_p(v), d(\pi_2)_p(v))$. Define a map $T(M \times N) \to TM \times TN$ by $f(((p, q), v)) = ((p, d(\pi_1)_{(p,q)}(v)), (q, d(\pi_2)_{(p,q)}(v)))$. Since the map α is an isomorphism it is clear that this map is bijective so to show it is a diffeomorphism we show it is smooth with smooth inverse. $\square$

Exercise 3 (Lee 3.4)**.** Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Exercise 4. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) \colon \max\{|x|, |y|\} = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no diffeomorphism with the same property.

Exercise 5 (Lee 3.7)**.** Let M be a smooth manifold with or without boundary and p a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\phi : \mathcal{D}_pM \to T_pM$ by $(\phi v)f - v([f]_p)$. Show that ϕ is an isomorphism.

Exercise 6 (Lee 3.8)**.** Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of curves starting at p under the relation $\gamma_1 \sim \gamma_2$ of $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\psi : \mathcal{V}_pM \to T_pM$ defined by $(\psi[\gamma])(f) = (f \circ \gamma)'(0)$ is well defined and bijective.