MATH 214 Hw4

Jad Damaj

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Exercise 1 (Lee 3.1)**.** Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $dF_p : T_pM \to T_pN$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof. First, suppose that F is a smooth map such that dF_p is the zero map for each $p \in M$. We first consider the case where M and N and $\mathbb{R}^m$ and $\mathbb{R}^n$. In this case the map dF_p is the matrix $(\partial F^1/\partial x^j(p))_{ij}$ and so we have the all partials of F are 0 at all points p and so F must be constant on each component. In the case where M and N are arbitrary manifolds, consider charts (U, φ) and (V, ψ) for the points p and $F(p)$, respectively. If dF_p is 0 for all p then $d(\psi \circ F \circ \varphi^{-1})_p = 0$. Then, as this is a map between open subsets of Euclidean space the above proof shows that $\psi \circ F \circ \varphi^{-1}$ must be constant on each component and so, since ψ, φ are homeomorphisms F must be also. Hence, we have shown that for each point $p \in M$, F is constant each component in a neighborhood of p and so it follows that this holds for all of M.

Conversely, if F is constant on each component of a manfiold then it follows that the corresponding differential between charts must be 0 for all p since the partial of any constant maps between Euclidean spaces is 0. Hence, it follows that dF_p is 0 as a map between manifolds since it is the composition of a zero map with the maps corresponding do the chart. $\square$

Exercise 2 (Lee 3.3)**.** Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Proof. Suppose M and N are smooth manifolds. Recall that for each $p \in M$ we have an isomorphism of the tangent spaces $T_p(M \times N)$ and $T_pM \times T_pN$ given by $\alpha(v) = (d(\pi_1)_p(v), d(\pi_2)_p(v))$. Define a map $T(M \times N) \to TM \times TN$ by $f(((p,q),v)) = ((p, d(\pi_1)_{(p,q)}(v)), (q, d(\pi_2)_{(p,q)}(v)))$. Since the map α is an isomorphism it is clear that this map is bijective so to show it is a diffeomorphism we show it is smooth with smooth inverse.

First, to check it is smooth at $((p,q),v)$ fix smooth charts (U, φ) and (V, ψ) containing p and q in M and N, respectively. Then, $(\pi^{-1}(U \times V), \widetilde{\varphi \times \psi})$ is a smooth chart containing $((p,q),v)$ and $(\pi^{-1}(U) \times \pi^{-1}(V), \tilde{\varphi} \times \tilde{\psi})$ is a smooth chart containing its inverse. We compute the transition map to be

$$\begin{aligned}
&(\tilde{\varphi} \times \tilde{\psi} \circ F \circ (\widetilde{\varphi \times \psi}))^{-1}(\overline{x}, \overline{y}, \overline{v}, \overline{w}) = (\tilde{\varphi} \times \tilde{\psi}) \circ F \left(\sum v^i \frac{\partial}{\partial x^i} + \sum w^j \frac{\partial}{\partial y^j} \right)|_{(\varphi \times \psi)^{-1}(\overline{x}, \overline{y})} \\
&= (\tilde{\varphi} \times \tilde{\psi}) \left(d(\pi_1) \left(\sum v^i \frac{\partial}{\partial x^i} + \sum w^j \frac{\partial}{\partial y^j} \right)|_{\varphi^{-1}(\overline{x})}, d(\pi_2) \left(\sum v^i \frac{\partial}{\partial x^i} + \sum w^j \frac{\partial}{\partial y^j} \right)|_{\varphi^{-1}(\overline{y})} \right) \\
&= (\tilde{\varphi} \times \tilde{\psi}) \left(\sum v^i \frac{\partial}{\partial x^i}|_{\varphi^{-1}(\overline{x})}, \sum w^j \frac{\partial}{\partial y^j}|_{\varphi^{-1}(\overline{y})} \right) \\
&= ((\overline{x}, \overline{v}), (\overline{y}, \overline{w}))
\end{aligned}$$

which is smooth and has smooth inverse (since the inverse of the transition map is the transition map for F^{-1}) hence it follows that both F and F^{-1} are smooth, as desired. $\square$

Exercise 3 (Lee 3.4)**.** Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Proof. For each $p = e^{i\varphi} \in \mathbb{S}^1$ consider the generating element of $T_p\mathbb{S}^1$ defined by $v_p(f) = (f \circ \gamma)'(0)$ where $\gamma : [-\pi/2, \pi/2] \to \mathbb{S}^1$ is defined by $\gamma(t) = e^{i(\varphi+t)}$. Then each element $w \in T\mathbb{S}^1$ is equal to av_p for some $a \in \mathbb{R}$. Now, define the map $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ by $F((p, av_p)) = (p, a)$. It is clear that F is a bijection. We show that F is smooth. Fix some $(e^{i\varphi}, w) \in T_pM$. We can find some chart (U, θ) of $\mathbb{S}^1$ containing $e^{i\varphi}$ such that θ is an angle function. Now, $(\pi^{-1}(U, \tilde{\theta})$ is a chart containing $(e^{i\varphi}, w)$ and

$(U \times \mathbb{R}, \theta \times \mathrm{id}_{|bR})$ is a chart containing $F((e^{i\varphi}, w))$ and $F(\pi^{-1}(U)) \subseteq U \times \mathbb{R}$ and so to show F is smooth it is enough to show the transition map is smooth. This follows since

$$(\theta \times \mathrm{id}_{\mathbb{R}}) \circ F \circ (\tilde{\theta})^{-1}(x, y) = (\theta \times \mathrm{id}_{\mathbb{R}}) \circ F(e^{ix}, y \cdot v_p) = (\theta \times \mathrm{id}_{\mathbb{R}})(e^{ix}, y) = (x, y)$$

Note that this map has smooth inverse and since its inverse is a transition map for F^{-1} is follows that F^{-1} is smooth as well and so F is a diffeomorphism. $\square$

Exercise 4. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) \colon \max\{|x|, |y|\} = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no diffeomorphism with the same property.

Proof. First, to show that such a homeomorphism exists we define a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ using polar coordinates, setting $F(0) = 0$ and

$$F(r, \theta) = \begin{cases} (\frac{4r}{\pi}(t + \pi/4), r) & \theta \in [0, \pi/2] \\ (-r, -\frac{4r}{\pi}(t - \pi/4)) & \theta \in [\pi/2, \pi] \\ (\frac{4r}{\pi}(t - \pi/4), -r) & \theta \in [\pi, 3\pi/2] \\ (r, \frac{4r}{\pi}(t - \pi/4)) & \theta in [3\pi/2, 2\pi] \end{cases}$$

then F is a homoemorphism that sends each quadrant of the circle of radius r to a side of the square of side length $2r$ and so $F(\mathbb{S}^1) = K$.
To show that no diffeomorphism satisfying this condition exists, let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ denote the path on the circle given by $\varphi \mapsto (\cos\varphi, \sin\varphi)$. γ is a smooth map so if $F : \mathbb{R}^2 \to \mathbb{R}^2$ was a diffeomorphism with $F(\mathbb{S}^1) = K$ then $F \circ \gamma$ would be a path around the square K. However, no such path can be smooth if $t \in [0, 2\pi]$ is such that $F(t) = (1, 1)$ then $\frac{\partial F}{\partial t} = (\partial/\partial t(F^1 \circ \cos t), \partial/\partial t(F^2 \circ \sin t))$ and so the derivative at $(1, 1)$ must be nonzero in some component. However, it is clear that this is not the case since as t approaches the corner from below the x-component of the derivative is 0 and as it approaches $(1, 1)$ from the right its y-component is 0. Hence, no such F can exist. $\square$

Exercise 5 (Lee 3.7)**.** Let M be a smooth manifold with or without boundary and p a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\phi : \mathcal{D}_pM \to T_pM$ by $(\phi v)f = v([f]_p)$. Show that ϕ is an isomorphism.

Proof. To show that ϕ is an isomorphism of vector spaces first note that it is linear since

$$(\phi(av + bw))(f) = (av + bw)([f]_p) = av([f]_p) + bw([f]_p) = a\phi(v)(f) + b\phi(w)(f)$$

ϕ is injective since if $\phi(v) = 0$ then $v([f]_p) = 0$ for all f and so v must have been the 0 map to begin with. Finally, ϕ is surjective since given any $w \in T_pM$ consider $v \in \mathcal{D}_pM$ defined by $v([f]_p) = w(f)$. Note that this is well defined since if $[f]_p = [g]_p$ then f and g agree on some open set containing p so we must have $w(f) = w(g)$. Then it is clear that $\phi(v) = w$ and so ϕ is surjective. Thus it is an isomorphism. $\square$

Exercise 6 (Lee 3.8)**.** Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of curves starting at p under the relation $\gamma_1 \sim \gamma_2$ of $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\psi : \mathcal{V}_pM \to T_pM$ defined by $(\psi[\gamma])(f) = (f \circ \gamma)'(0)$ is well defined and bijective.

Proof. First, to see that ψ is well defined suppose that $\gamma_1 \sim \gamma_2$, ie. $[\gamma_1] = [\gamma_2]$. Then

$$(\psi[\gamma_1])(f) = (f \circ \gamma_1)(0) = (f \circ \gamma_2)(0) = (\psi[\gamma_2])(f)$$

for all f so $\psi[\gamma_1] = \psi[\gamma_2]$. By linearity of the deriviative it is clear that ψ is linear as well.
ψ is injective since if $\psi[\gamma] \equiv 0$ then $(f \circ \gamma)'(0) = 0$ for all f and so $\gamma \sim 0$, ie, $[\gamma] = [0]$.
Finally, to see that ψ is surjective fix some chart (U, φ) containing p. It suffices to show its image contains $\frac{\partial}{\partial x^i}|_p$ since these form a basis for T_pM. Now, observe that if $\gamma : [-1, 1] \to \mathbb{R}^n$ is defined by $\gamma_i(t) = (p_1, \ldots, p_i + t, \ldots, p_n)$ then $(f \circ \gamma_i)'(0) = \partial f/\partial x^i(p)$ for each $f : \mathbb{R}^n \to \mathbb{R}$. Hence, taking the curve $\varphi^{-1} \circ \gamma_i : [-1, 1] \to M$, we see that for each $f : M \to \mathbb{R}$,

$$\frac{\partial}{\partial x^i}|_p(f) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})(p) = (f \circ \varphi^{-1} \circ \gamma_i)'(0) = \psi([\varphi^{-1} \circ \gamma_i])(f)$$

$\square$