MATH 214 Hw4

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Exercise 1 (Lee 3.1). Suppose M and N are smooth manifolds with or without boundary, and $F: M \to N$ is a smooth map. Show that $dF_p: T_pM \to T_pN$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

Proof. First, suppose that F is a smooth map such that dF_p is the zero map for each $p \in M$. We first consider the case where M and N and \mathbb{R}^m and \mathbb{R}^n . In this case the map dF_p is the matrix $(\partial F^1/\partial x^j(p))_{ij}$ and so we have the all partials of F are 0 at all points p and so F must be constant on each component. In the case where M and N are arbitrary manifolds, consider charts (U,φ) and (V,ψ) for the points p and F(p), respectively. If dF_p is 0 for all p then $d(\psi \circ F \circ \varphi^{-1})_p = 0$. Then, as this is a map between open subsets of Euclidean space the above proof shows that $\psi \circ F \circ \varphi^{-1}$ must be constant on each component and so, since ψ, φ are homeomorphisms F must be also. Hence, we have shown that for each point $p \in M$, F is constant each component in a neighborhood of p and so it follows that this holds for all of M

Exercise 2 (Lee 3.3). Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Proof. Suppose M and N are smooth manifolds. Recall that for each $p \in M$ we have an isomorphism of the tangent spaces $T_p(M \times N)$ and $T_pM \times T_pN$ given by $\alpha(v) = (d(\pi_1)_p(v), d(\pi_2)_p(v))$. Define a map $T(M \times N) \to TM \times TN$ by $f(((p,q),v)) = ((p,d(\pi_1)_{(p,q)}(v)), (q,d(\pi_2)_{(p,q)}(v)))$. Since the map α is an isomorphism it is clear that this map is bijective so to show it is a diffeomorphism we show it is smooth with smooth inverse.

First, to check it is smooth at ((p,q),v) fix smooth charts (U,φ) and (V,ψ) containing p and q in M and N, respectively. Then, $(\pi^{-1}(U\times V),\widetilde{\varphi}\times\psi)$ is a smooth chart containing ((p,q),v) and $(\pi^{-1}(U)\times\pi^{-1}(V),\widetilde{\varphi}\times\widetilde{\psi})$ is a smooth chart containing its inverse. We compute the transition map to be

$$\begin{split} &(\tilde{\varphi}\times\tilde{\psi}\circ F\circ (\widetilde{\varphi\times\psi}))^{-1}(\overline{x},\overline{y},\overline{v},\overline{w}) = (\tilde{\varphi}\times\tilde{\psi})\circ F\left(\sum v^i\frac{\partial}{\partial x^i} + \sum w^j\frac{\partial}{\partial y^j}\right)|_{(\varphi\times\psi)^{-1}(\overline{x},\overline{y})} \\ &= (\tilde{\varphi}\times\tilde{\psi})\left(d(\pi_1)\left(\sum v^i\frac{\partial}{\partial x^i} + \sum w^j\frac{\partial}{\partial y^j}\right)|_{\varphi^{-1}(\overline{x})}, d(\pi_2)\left(\sum v^i\frac{\partial}{\partial x^i} + \sum w^j\frac{\partial}{\partial y^j}\right)|_{\varphi^{-1}(\overline{y})}\right) \\ &= (\tilde{\varphi}\times\tilde{\psi})\left(\sum v^i\frac{\partial}{\partial x^i}|_{\varphi^{-1}(\overline{x})}, \sum w^j\frac{\partial}{\partial y^j}|_{\varphi^{-1}(\overline{y})}\right) \\ &= ((\overline{x},\overline{v}), (\overline{y},\overline{w})) \end{split}$$

which is smooth and has smooth inverse (since the inverse of the transition map is the transition map for F^{-1})hence it follows that both F and F^{-1} are smooth, as desired.

Exercise 3 (Lee 3.4). Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Exercise 4. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x,y) \colon \max\{|x|,|y|\} = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no diffeomorphism with the same property.

Exercise 5 (Lee 3.7). Let M be a smooth manifold with or without boundary and p a point of M. Let $C_p^{\infty}(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_p M$ denote the vector space of derivations of $C_p^{\infty}(M)$. Define a map $\phi: \mathcal{D}_p M \to T_p M$ by $(\phi v)f = v([f]_p)$. Show that ϕ is an isomorphism.

Proof. To show that ϕ is an isomorphism of vector spaces first note that it is linear since

$$(\phi(av + bw))(f) = (av + bw)([f]_p) = av([f]_p) + bw([f]_p) = a\phi(v)(f) + b\phi(w)(f)$$

 ϕ is injective since if $\phi(v) = 0$ then $v([f]_p) = 0$ for all f and so v must have been the 0 map to begin with. Finally, ϕ is surjective since given any $w \in T_pM$ consider $v \in \mathcal{D}_pM$ defined by $v([f]_p) = w(f)$. Note that this is well defined since if $[f]_p = [g]_p$ then f and g agree on some open set containing p so we must have w(f) = w(g). Then it is clear that $\phi(v) = w$ and so ϕ is surjective. Thus it is an isomorphism. \square

Exercise 6 (Lee 3.8). Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$ denote the set of equivalence classes of curves starting at p under the relation $\gamma_1 \sim \gamma_2$ of $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\psi: \mathcal{V}_p M \to T_p M$ defined by $(\psi[\gamma])(f) = (f \circ \gamma)'(0)$ is well defined and bijective.

Proof. First, to see that ψ is well defined suppose that $\gamma_1 \sim \gamma_2$, ie. $[\gamma_1] = [\gamma_2]$. Then

$$(\psi[\gamma_1])(f) = (f \circ \gamma_1)(0) = (f \circ \gamma_2)(0) = (\psi[\gamma_2])(f)$$

for all f so $\psi[\gamma_1] = \psi[\gamma_2]$. By linearity of the derivative it is clear that ψ is linear as well. ψ is injective since if $\psi[\gamma] \equiv 0$ then $(f \circ \gamma)'(0) = 0$ for all f and so $\gamma \sim 0$, ie, $[\gamma] = [0]$. Finally, to see that ψ is surjective fix some chart (U, φ) containing p. It suffices to show its image contains $\frac{\partial}{\partial x^i}|_p$ since these form a basis for T_pM . Now, observe that if $\gamma:[-1,1] \to \mathbb{R}^n$ is defined by $\gamma_i(t) = (p_1, \ldots, p_i + t, \ldots, p_n)$ then $(f \circ \gamma_i)'(0) = \partial f/\partial x^i(p)$ for each $f: \mathbb{R}^n \to \mathbb{R}$. Hence, taking the curve $\varphi^{-1} \circ \gamma_i: [-1,1] \to M$, we see that for each $f: M \to \mathbb{R}$,

$$\frac{\partial}{\partial x^i}|_p(f) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})(p) = (f \circ \varphi^{-1} \circ \gamma_i)'(0) = \psi([\varphi^{-1} \circ \gamma_i])(f)$$