MATH 202B Hw4

Jad Damaj

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Exercise 1. Let X be a normed vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Show that addition and scalar multiplication are continuous from $X \times X$ and $\mathbb{F} \times X$ to X, respectively. Show that $x \mapsto ||x||$ is continuous from X to $[0, \infty)$ and that $|\,||x|| - ||y||\,| \leq ||y - x||$ for all $x, y \in X$.

Proof. First, to show addition is continuous fix some (x, y) and $\varepsilon > 0$. Then letting $\delta = \varepsilon/2$ we see that if $||(x, y) - (x_0, y_0)|| < \delta$ then

$$||(x + y) - (x_0 + y_0)|| = ||(x - x_0) + (y - y_0)|| \leq ||x - x_0|| + ||y - y + 0|| \leq 2\delta = \varepsilon$$

Similarly, to see that scalar multiplication is continuous fix some (λ, x) and $\varepsilon > 0$. Then letting $\delta = \min\{\varepsilon/(|c| + ||x|| + 1), 1\}$ we compute that

$$\begin{aligned}
||cx - cx_0|| &= ||cx - cx_0 + cx_0 - c_0x_0|| \\
&\leq ||cx - cx_0|| + ||cx_0 - c_0x_0|| \\
&= |c|\,||x - x_0|| + |c - c_0|\,||x_0|| \\
&\leq |c|\,||x - x_0|| + |c - c_0|\,(||x - x_0|| + ||x||) \\
&\leq |c|\delta + \delta^2 + \delta||x|| \\
&\leq \delta(|c| + ||x|| + 1) \\
&\leq \varepsilon
\end{aligned}$$

Next, we show that $|\,||x|| - ||y||\,| \leq ||x - y||$. This follows since $||x|| \leq ||x - y|| + ||y||$ and so $||x|| - ||y|| \leq ||x - y||$. Then, by a symmetric argument we also have $||y|| - ||x|| \leq ||x - y||$ and so the above claim follows.
Finally, to show that $x \mapsto ||x||$ is continuous fix some x and $\varepsilon > 0$. Setting $\delta = \varepsilon$, $||x - y|| < \delta$ then by the above we have that $|\,||x|| - ||y||\,| \leq ||x - y|| < \varepsilon$ as well.

□

Exercise 2. Show that $\mathcal{L}(X, Y)$ is a vector space, and $V \subset X$ is a normed vector space and that the norm $||\cdot||_{\mathcal{L}(X,Y)}$ defined in our text is a norm on this vector space.

Proof. First, to show that $\mathcal{L}(X, Y)$ is a vector space note that there is a natural pointwise addition and scalar multiplication on maps $T : X \to Y$ so it suffices to show that the sum of two bounded operators is bounded and the scalar multiple of a bounded operator is bounded. However, this is immediate since the sum of continuous functions is continuous and multiplication of a continuous function by a constant is also continuous and continuous is equivalent to bounded for linear operators.
Next, we show that $||\cdot||_{\mathcal{L}(X,Y)}$ is a norm. To show it satisfies the triangle inequality, suppose T_1 and T_2 are operators with $||T_1|| = C_1$ and $||T_2|| = C_2$. Then for each x,

$$||(T_1 + T_2)(x)|| \leq ||T_1x|| + ||T_2x|| \leq (C_1 + C_2)||x||$$

and so we must have $||T_1 + T_2|| \leq C_1 + C_2 = ||T_1|| + ||T_2||$.
Similarly, given any $\lambda \neq 0 \in \mathbb{F}$ we see that for any C, $||(\lambda)Tx|| \leq \lambda C$ iff $||Tx|| \leq C$ and so $||\lambda T|| = \inf\{\lambda C \colon \forall\, ||Tx|| \leq C||x||\} = \lambda||T||$. Finally, if $||T|| = 0$ then we have $||Tx|| = 0$ for all x and so $T(x) = 0$ for all x, ie. $T = 0$. □

Exercise 3. Let X be a normed vector space, and $V \subset X$ a subspace. Show that the closure of V is a subspace of X.

Proof. Suppose X is a vector space and V a subspace to show that $\overline{V}$ is a subspace note that each element of $\overline{V}$ can be written as the limit of a (not necesarrily distinct) elements of V and so given $x, y \in \overline{V}$ write $x = \lim_{i\to\infty} x_i$ and $y = \lim_{i\to\infty} y_i$ with $x_i, y_i \in V$. Then we have that $x + y = \lim_{i\to\infty}(x_i + y_i) \in \overline{V}$ since each $x_i + y_i \in V$ since it is a subspace. Similarly, using the fact that $\lambda(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \lambda x_i$ it follows that if $x \in \overline{V}$ and $\lambda \in \mathbb{F}$ then $\lambda x \in \overline{V}$ as well. $\square$

Exercise 4. Let X be a normed vector space and V a proper closed subspace. Denote the elements of the quotient space X/V by $x + V$ with $x \in X$,

(a) Show that the quantity $||x + V|| = \inf_{v\in V} ||x - v||_X$ is a norm on X.

(b) Show that for any $\varepsilon > 0$ there exists $x \in X$ satisfying $||x||_X = 1$ such that $||x + V|| > 1 - \varepsilon$.

(c) Show that the natural projection map $\pi : X/V$ has norm equal to 1.

(d) Show that if X is complete then so is X/V.

Exercise 5. Let X be a Banach space. Show that if X^* is separable, then X is separable.

Exercise 6. Let l^∞ be the normed vector space of all bounded sequences $x = (x_n : n \in \mathbb{N})$ with $x_n \in \mathbb{F}_n$, with the supremum norm. Show that l^∞ is not separable.

Exercise 7. Let V be a closed subspace of X. By definition a supplement for V is a closed subspace W of X such that $V \cap W = \{0\}$ and $V + W = X$. Show that if X is finite dimensional then V has a supplement.