# Descriptive Set Theory: Moschovakis

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**Exercise 2.7.1.** Prove that a binary relation $R(x, y)$ on a set $S$ is wellfounded if an only if there are no infinite $<_R$-descending chains.

*Proof.* If $R(x, y)$ is not wellfounded then there must be some nonempty set $A$ such that $A$ has no $<_R$ minimal element. Using $A$, we construct an infinite $<_R$-descending chain as follows: Let $x_0 \in A$ arbitrary and given $x_i$, choose $x_{i+1} \in A$ such that $x_i >_R x_{i+1}$. Such an element will always exist since $A$ has no least element and so $x_0 >_R x_1 > \cdots$ is an infinite $<_R$-descending chain.
Conversely, suppose there is an infinite $<_R$-descending chain $x_0 >_R x_1 >_R \cdots$ and let $A = \{x_i \colon in \in \omega\}$. $A$ is nonempty set with no $<_R$ minimal element and so $R(x, y)$ is not wellfounded. □

**Exercise 2.7.2.** Prove that every Borel wellfounded relation has countable length and every $\utilde{\Delta}^1_2$ wellfounded relation has length less than $\aleph_2$.

*Proof.* Recall that if $R(x, y)$ is a relation we define its strict part to be

$$<_R = \{(x, y) \colon R(x, y) \& \neg R(y, x)\}$$

So, if $R$ is $\utilde{\Delta}^1_n$ we see that $<_R$ is $\utilde{\Delta}^1_n$ as well. Hence, if $R$ is a Borel relation then $<_R$ must be $\aleph_0$-Suslin and so, applying the Kunen-Martin theorem, it must have length less than $\aleph_1$. Similarly, if $R$ is $\utilde{\Delta}^1_2$ then $<_R$ is $\aleph_1$-Suslin and so has length less thatn $\aleph_2$. □

**Exercise 2.7.3.** Let $R$ be a wellfounded relation on $S$ with rank function $\rho$, and let $f : S \to$ Ordinals be an order preserving function, ie.

$$x <_R y \Rightarrow f(x) < f(y)$$

Prove that for every $x$ in $S$, $\rho(x) \leqslant f(x)$.

*Proof.* We show this by induction on the relation. Suppose that for all $y <_R x$, $\rho(y) \leqslant f(y)$ then, since we have $f(y) < f(x)$ for each $y <_R x$, $f(x) \geqslant f(y) + 1$ for all $y <_R x$ and so

$$f(x) \geqslant \sup\{f(y) + 1 \colon y <_R x\} \geqslant \sup\{\rho(y) + 1 \colon y <_R x\} = \rho(x)$$

□

A norm $\varphi$ on $S$ is called regular if it is onto some ordinal $\lambda$.
Given a norm $\varphi$ on $S$ the associated the binary relation $\leqslant^\varphi$ on $S$ is defined by

$$x \leqslant^\varphi y \Leftrightarrow \varphi(x) \leqslant \varphi(y)$$

**Exercise 2.7.4.** Prove that a binary relation $\preceq$ on a set $S$ is a prewellordering if and only if there is a norm $\varphi$ on $S$ such that $\preceq = \leqslant^\varphi$. Moreover if $\preceq$ is a prewellordering, then there is a unique regular $\varphi$ on $S$ such that $\preceq = \leqslant^\varphi$.

*Proof.* □