PHIL 143 Hw4

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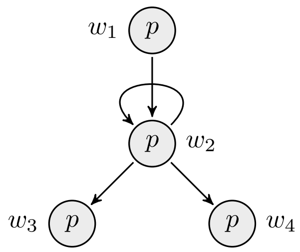
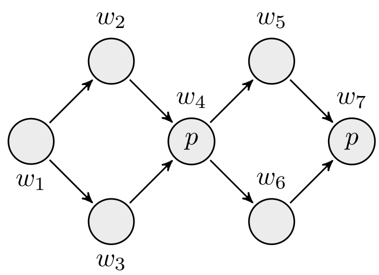
Exercise 1. Draw the filtration of each model through the given formula.

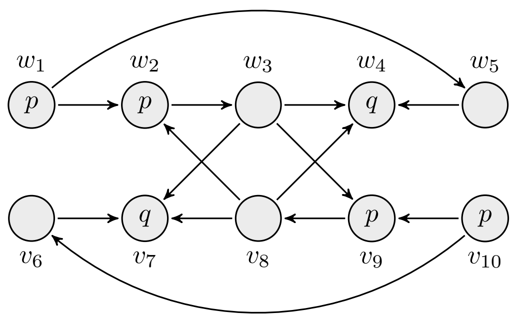
(a) $p \to \Diamond p$



(b) $\Box p \to \Box\Box p$

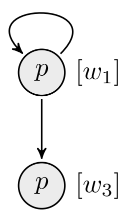


(c) $p \to \Diamond p$

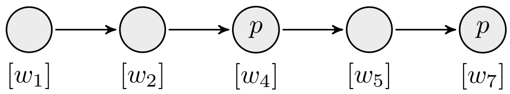


Proof.

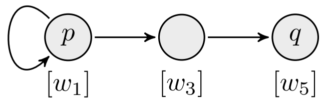
(a)



(b)



(c)



□

Exercise 2.

(a) Prove that if a formula φ contains n connectives, then $|\text{sub}(\varphi)| \leq 2n + 1$.

(b) Show that the bound is sharp. That is, for any n, give an example of formula with n connectives and exactly $2n + 1$ subformulas.

Proof.

(a) We show this by induction on the complexity of the formula.
$\varphi = p$: If φ is atomic then it contains 0 connectives and $|\text{sub}(p)| = 1 = 2 \cdot 0 + 1$.
$\varphi = \neg\psi$: If the claim holds for ψ and ψ has n connectives then φ has $n + 1$ connectives and

$$|\text{sub}(\varphi)| \leq |sub(\psi)| + 1 \leq (2n + 1) + 1 \leq 2(n + 1) + 1$$

$\varphi = \Box\psi$: If the claim holds for ψ and ψ has n connectives then φ has $n + 1$ connectives and

$$|\text{sub}(\varphi)| \leq |sub(\psi)| + 1 \leq (2n + 1) + 1 \leq 2(n + 1) + 1$$

$\varphi = \psi_1 \wedge \psi_2$: If the claim holds for ψ_1 and ψ_1 and they have n_1 and n_2 connectives respectively then φ has $n_1 + n_2 + 1$ connectives and

$$|\text{sub}(\varphi)| \leq |\text{sub}(\psi_1)| + |\text{sub}(\psi_2)| + 1 \leq (2n_1 + 1) + (2n_2 + 1) + 1 = 2(n_1 + n_2 + 1) + 1$$

(b) To see that this bound is sharp inductively define a formulas by $\varphi_0 = p_0$ and $\varphi_{n+1} = (\varphi_n \wedge p_{n+1})$. Then $|\text{sub}(\varphi_0)| = 1$ and $|\text{sub}(\varphi_{n+1})| = 2 + |\text{sub}(\varphi_n)|$ so these witness the upper bound.

□

Exercise 3. Recursively define formulas φ_n as follows:

- $\varphi_0 = p_0$
- $\varphi_{n=1} = (\Diamond\varphi_n \wedge \Box p_{n+1})$

(a) Find a formula for $|\text{sub}(\varphi_n)|$ in terms of n.

(b) Find a formula for $d(\varphi_n)$ in terms of n.

(c) Find a formula for $|\varphi_n|$ in terms of n.

(d) In the limit, which method gives us a better bound on the size of the models satisfying φ_n?

Proof.

(a) $|\text{sub}(\varphi_0)| = 1$ and $|\text{sub}(\varphi_{n+1})| = |\text{sub}(\varphi_n)| + 4$ and $|\text{sub}(\varphi_n)| = 4n + 1$.

(b) $d(\varphi_0) = 0$ and $d(\varphi_{n+1}) = \max\{d(\varphi_n) + 1, 1\}$ so $d(\varphi_n) = n$.

(c) $|\varphi_0| = 1$ and $|\varphi_{n+1}| = |\varphi_n| + 6$ so $|\varphi_n| = 6n + 1$.

(d) Filtration gives a bound of 2^{4n+1} and selection gives a bound of $(6n+1)^n$ so filtration gives a better bound in the limit.

$\square$

Exercise 4. Show that the following are theorems of **K.**

1. $\Box(p \to \Diamond p) \to (\Diamond\Diamond p \vee \neg\Diamond p)$

2. $(\Diamond\Box(p \to q) \wedge \Box\Diamond p) \to \Diamond\Diamond q$

Proof. To show these are theorems of K we that they are valid from which their theoremhood follows by completeness.

(a) Suppose $\mathcal{M}, w$ is a pointed model such that $\mathcal{M}, w \models \Box(p \to \Diamond p)$. If there is some v such that wRv and $\mathcal{M}, v \models p$ we have that $\mathcal{M}, v \models p \to \Diamond p$ and so $\mathcal{M}, v \models \Diamond p$. Hence, it follows that $\mathcal{M}, w \models \Diamond\Diamond p$. Otherwise, no such v exists and we have that $\mathcal{M}, w \models \neg\Diamond p$. Thus, $\mathcal{M}, w \models (\Diamond\Diamond p \vee \neg\Diamond p)$, as desired.

(b) Given some $\mathcal{M}, w$ such that $\mathcal{M}, w \models \Diamond\Box(p \to q) \wedge \Box\Diamond p$ we show that $\mathcal{M}, w \models \Diamond\Diamond p$. By assumption, there is some v such that wRv and $\mathcal{M}, v \models \Box(p \to q)$. Further, we must also have $\mathcal{M}, v \models \Diamond p$ and so there is some v' such that vRv' and $\mathcal{M}, v' \models p$. Then, since $\mathcal{M}, v \models \Box(p \to q)$, $\mathcal{M}, v' \models p \to q$ and so $\mathcal{M}, v' \models q$. Thus, we have that $\mathcal{M}, w \models \Diamond\Diamond q$, as desired.

$\square$