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Chapter 1

Just Enough Category Theory to be Dangerous

1.1 Motivation

1.2 Categories and Functors

Exercise 1.2.1. A category in which each morphism is an isomorphism is called a groupoid.

(a) A perverse definition of a group is: is a groupoid with one object. Make sense of this.

(b) Describe a groupoid that is not a group.

Proof.

(a) One can identiy one element groupoids and groups as follows: The elements of the group are the morhpisms of the category and the group law is given by composition. The category has an identiy morphism iff the group has an identity, composition is associative iff the group law is associative, and all morphisms are isomorphisms iff each elememt of the group has an inverse.

(b) The following is a groupoid which is not a group

$$\begin{tikzcd} \bullet \arrow[loop above] & \bullet \arrow[loop above] \end{tikzcd}$$

□

Exercise 1.2.2. If A is an object in the category $\mathcal{C}$, show that the invertible elements of $\mathrm{Mor}(A, A)$ from a group. What are the automorphism groups of a set X and a vector space V.

Proof. Denote the set of invertible elements of $\mathrm{Mor}(A, A)$ by $\mathrm{Aut}(A)$. Observe that $\mathrm{id}_A \in \mathrm{Aut}(A)$ and given $f, g \in \mathrm{Aut}(A)$, $f \circ g \in \mathrm{Aut}(A)$ since $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. Hence, the object A with collection of morphisms $\mathrm{Aut}(A)$ forms a one element subcategory of $\mathcal{C}$. Further, it is a groupoid. Hence, by the previous exercise, $\mathrm{Aut}(A)$ forms a group. When X is a set $\mathrm{Aut}(X)$ is exactly the symmetric group on X: $\mathrm{Sym}(X)$. When V is a vector space, $\mathrm{Aut}(V)$ is the subgroup of $\mathrm{Sym}(V)$ corresponding to permutations which are also linear maps.

Now, given $A, B \in C$ and an isomorphism $f : A \to B$, we can define a map $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$ via $g \mapsto f \circ g \circ f^{-1}$. Since f is invertible this map is a bijection and, further, it is a group homomorphism since

$$g \circ h \mapsto f \circ (g \circ h) f^{-1} = (f \circ g \circ f^{-1}) \circ (f \circ h f^{-1})$$

Hence, A and B have isomorphic automorphism groups. $\square$

Exercise 1.2.3. Let $(\cdot)^{\vee\vee} : f.d.Vec_k \to f.d.Vec_k$ be the double dual functor from the category of finite dimensional vector spaces over k to itself. Show that $(\cdot)^{\vee\vee}$ is naturally isomorphic to the identity functor on $f.d.Vec_k$.

Proof. To show that $(\cdot)^{\vee\vee}$ is naturally isomorphic to the identity, for each $V \in f.d.Vec_k$ we define a map $m_V : V \to V^{\vee\vee}$ by $m_V(v) = \mathrm{ev}_v$, ie. the map which evaluates a linear functional at v. We claim these maps define the desired natural isomorphism. First, observe that each m_V is injective since ev_v is the zero map only when $v = 0$. So, since V and $V^{\vee\vee}$ are finite dimensional vector spaces of the same dimension it follows that each m_V must be an isomorphism. Now, to check that this transformation is natural, we show that for any linear transformation $f : V \to W$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{m_V} & V^{\vee\vee} \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f^{\vee\vee}} \\ W & \xrightarrow{m_W} & W^{\vee\vee} \end{array}$$

Given $v \in V$ and $\varphi \in W^{\vee}$, we compute that value of $f^{\vee\vee}(M_V(v))(\varphi)$ and $M_W(f(v))(\varphi)$.

$$f^{\vee\vee}(M_V(v))(\varphi) = (f^{\vee\vee}(\mathrm{ev}_v))(\varphi) = (\mathrm{ev}_v \circ f^{\vee})(\varphi) = \mathrm{ev}_v(\varphi \circ f) = \varphi(f(v)) = \mathrm{ev}_{f(v)}(\varphi) = M_W(f(v))(\varphi)$$

$\square$

Exercise 1.2.4. Let $\mathcal{V}$ be the category whose objects are the k-vector spaces k^n for $n \geqslant 0$ and whose morphisms are linear transformations. Show that $\mathcal{V} \to f.d.Vec_k$ gives an equivalence of categories by describing an "inverse" functor.

Proof. We define functors F and G to give this equivalence. Let $F : \mathcal{V} \to f.d.Vec_k$ be the inclusion functor. We define a functor $G : f.d.Vec_k \to \mathcal{V}$ as follows: For each vector space V, fix some basis $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ and let $M_V : V \to k^n$ denote the map defined by $v_i \mapsto e_i$, the ith element of the standard basis of k^n. Note that each M_V is an isomorphism. WLOG we may assume that that these bases are chosen such that if $V = k^n$ for some n, then $\mathcal{B}_V$ is the standard basis.
Now, G sends a vector space V to k^n where n is the dimension of V and sends a map $f : V \to W$ to the map $M_W \circ f \circ M_V^{-1} : k^n \to k^m$ where n, m are the dimensions of V, W respectively. By our choice of bases, we have the $G \circ F = \mathrm{id}_{\mathcal{V}}$ and so it remains to show that $F \circ G$ is naturally isomorphic to $\mathrm{id}_{f.d.Vec_k}$. To check this we define the maps $V \to F \circ G(V) = k^n$ to be the previously defined isomorphisms M_V. Further, given any $f : V \to W$, we have that $G(f) \circ M_V = M_W \circ f$ by definition of G and so the diagram below commutes and this transformation is natural.

$$\begin{array}{ccc} V & \xrightarrow{M_V} & k^n \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle G(f) = M_W \circ f \circ M_W^{-1}} \\ W & \xrightarrow{M_W} & k^m \end{array}$$

$\square$

1.3 Universal Properties Determine an Object up to Isomorphism

Exercise 1.3.1. Show that any two inital objects are uniquely isomorphic. Show that any two final objects are uniquely isomorphic.

Exercise 1.3.2. What are the initial objects in *Sets*, *Rings*, and *Top*? How about in the category of subsets of a set or the category of open subsets of a topological space?

Exercise 1.3.3. Show that $A \to S^{-1}A$ is injective if and only if S contains no zerodivisors.

Exercise 1.3.4. Show that $A \to S^{-1}A$ satisfies the following universal property: $S^{-1}A$ is inital among A-algebras B where every element of S is sent to an invertible element in B.

Exercise 1.3.5. Show that $\phi : M \to S^{-1}M$ exists, by constructing something satisfying the universal property.

Exercise 1.3.6.

(a) Show that localization commutes with finite products, or equivalently, finite direct sums.

(b) Show that localization commutes with arbitrary direct sums.

(c) Show that "localization does not necessarily commute with infinite products": the obvious map $S^{-1}(\prod_i M_i) \to \prod_i S^{-1}M_i$ induced by the universal property is not always an isomorphism.

Exercise 1.3.7. Show that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$

Exercise 1.3.8. Show that $(\cdot)\otimes_A N$ gives a covariant functor $Mod_A \to Mod_A$. Show that $(\cdot)\otimes_A N$ is right-exact.

Exercise 1.3.9. Show that $(T, t : M \times N \to T)$ is unique up to isomorphism.

Exercise 1.3.10. Show that the construction of 1.3.5 satisfies the universal property of the tensor product.

Exercise 1.3.11.

(a) if M is an A-module and $A \to B$ is a morphism of rings, give $B \otimes_A M$ the structure of a B-module. Show that this describes a functor $Mod_A \to Mod_B$

(b) If further $A \to C$ is another morphism of rings, show that $B \otimes_A C$ has a natural structure of a ring.

Exercise 1.3.12. If S is a multiplicative subset of A and M is an A-module, describe a natural isomorphism $(S^{-1}A) \otimes_A M \to S^{-1}M$.

Exercise 1.3.13. Show that tensor products commute with arbitrary direct sums: If M and $\{N_i\}_{i\in I}$ are A-modules, describe an isomorphism

$$M \otimes (\oplus_{i\in I} N) \longrightarrow \otimes_{i\in I}(M \otimes N_i)$$

Exercise 1.3.14. Show that in *Sets*,

$$X \times_Z Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$$

Exercise 1.3.15. If X is a topological space, show that fibered products always exist in the category of open sets of X, by describing what a fibered product is.

Exercise 1.3.16. If Z is a final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, show that "$X \times_Z Y = X \times Y$": the fibered product over Z is uniquely isomorphic to the product.

Exercise 1.3.17. If the two squares in the following commutative diagram are Cartesian diagrams, show that the outside rectangle is also a Cartesian diagram.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Exercise 1.3.18. Given morphisms $X_1 \to Y$, $X_2 \to Y$, show that there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$, assuming that both fibered products exist.

Exercise 1.3.19. Suppose that we are given morphisms $X_1, X_2 \to Y$ and $Y \to Z$. Show that the following diagram is a Cartesian square.

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

Exercise 1.3.20. Show that the coproduct for *Sets* is disjoint union.

Exercise 1.3.21. Suppose $A \to B$ and $A \to C$ are two ring morphisms, so in particular B and C are A-modules. Recall that $B \otimes_A C$ has a ring structure. Show that there is a natural morphism $B \to B \otimes_A C$ given by $b \mapsto b \otimes 1$. Similarly, there is a natural morphism $C \to B \otimes_A C$. Show that this gives a fibered coproduct on rings, ie. that

$$\begin{tikzcd} B \otimes_A C & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Exercise 1.3.22. Show that the composition of two monomorphisms is a monomorphism.

Exercise 1.3.23. Prove that a morphism $\pi : Y \to Z$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced diagonal morphism $\delta_\pi : X \to X \times_Y X$ is an isomorphism.

Exercise 1.3.24. Show that if $Y \to Z$ is a monomorphism, then the natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ is an isomorphism.

1.4 Limits and Colimits

1.5 Adjoints

1.6 An Introduction to Abelian Categories