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Before starting the development of the general theory of C^*-algebras we consider a few motivating examples.

Example 1.1. Consider the Hilbert spaces $\ell^2(\mathbb{Z})$, $L^2(\mathbb{R})$ and define unitary operators U, V by $(U\xi)(r) = \xi(r-1)$ and $(V\xi)(r) = e^{2\pi i\theta r}\xi(r)$ for $\theta \in \mathbb{R}$. Consider the algebra generated by these two operators. $VU = e^{2\pi i\theta}VU$ so the linear span of U^mV^n is an algebra. Denote its operator norm closure by $\mathcal{A}_\theta$.
The algebras generated by U and V individually are both isomorphic to $C(\mathbb{S}^1)$ and if $\theta \in \mathbb{Z}$, U and V commute so $\mathcal{A}_\theta \cong C(\mathbb{S}^1 \times \mathbb{S}^1)$. Otherwise, we call $\mathcal{A}_\theta$ a noncommutative 2-torus.
Consider the self adjoint operator, called an almost Matthieu operator

$$H = (U + U^*) + \lambda(V + V^*)$$

Q: What is the spectrum of H?

- For rational θ, it is a disjoint union of intervals
- 1976: Hofstader plotted the spectra for rational θ to produce a fractal like pattern – the Hofstader Butterfly. This led to the conjecture that the spectrum of H for irrational θ was a Cantor set.
- 1981: Mark Kat offered 10 martinis for an answer to this conjecture.
- Barry Simon popularized this and it became known as the 10 martini problem
- 2005: Jitomirskaya, Avila gave an affirmative answer

Much of the partial progress on this problem came from the theory of C^*-algebras but the final proof used completely different techniques.

Example 1.2. Let G be a group. A (Hilbert Space) unitary representation of G is a group homomorphism $\pi : G \to \mathcal{U}(H)$. Consider the algebra of operators generated by $\{\pi(x) : x \in G\}$, ie. elements of the form $\pi_f = \sum_{x\in G} f(x)\pi(x)$ for $f : G \to \mathbb{C}$ finnitely supported. We define a product on this set by

$$(\sum f(x)\pi(x))(\sum g(y)\pi(y)) = \sum_{x,y} f(x)g(y)\pi(xy) = \sum_{x,y} f(x)g(x^{-1}y)\pi(y) = \sum_y(\sum_x f(x)g(x^{-1}y))\pi(y)$$

Defining the convolution $*$ by

$$(f * g)(y) = \sum_x f(x)g(x^{-1}y)$$

we have the $\pi_f \cdot \pi_g = \pi_{f*g}$ and this gives an algebra structure on $C_c(G)$ and an algebra homomorphism $\pi : (C_c(G), *) \to B(H)$. The convolution also gives $\ell^1(G)$ an algebra structure since $||f*g||_1 \leq ||f||_1||g||_1$.
For any $f \in C_c(G)$, let $||f||_{C^*(G)} = \sup\{\pi_f : \pi \text{ unitary rep of } G\}$. Observe that

$$||\pi_f|| \leq \sum |f(x)|||\pi(x)|| = \sum |f(x)| = ||f||_1$$

and so $||f||_{C^*(G)} \leq ||f||_1$. Let $C^*(G)$ denote the completion of $C_c(G)$ with respect to this norm. We can represent this algebra as an algebra bounded operators on some Hilbert space.

Note: for any group G we have at least two unitary representations – the trivial representation and the left regular representation on $\ell^2(G)$ by translation.

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Example 2.1. Using the left regular representation: $H = \ell^2(G)$ with $\mathcal{U}_x(\xi)(y) = \underline{\xi(x^{-1}y)}$, we can define a norm on $C_c(G)$ by $||f||_r = ||U_f||$. Note that the involution is given by $f^*(x) = \overline{f(-x)}$. The closure of $C_c(G)$ under this norm is called the reduced C*-algebra for G. Since $||f||_r \leq ||f||_{C^*}$, we only have that $C^*_r(G)$ is a quotient of $C^*(G)$ in general. In fact, the discreprency between the two encodes information about the group G:

Theorem 2.2. $C^*(G) = C^*_r(G)$ if and only if G is amenable.

Example 2.3. For X a compact Hausdorff space, we can consider a homomorphism $\alpha : G \to \mathrm{Homeo}(x)$ (eg. a dynamical system). This induces a map $\alpha : G \to \mathrm{Aut}(C(X))$. We can often use this map to give insight into the behavior of the dynamical system as passing to the level of algebras gives more structure to leverage.
More generally, for a C*-algebra $\mathcal{A}$, we call a map $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ a "noncommutative dynamical system". We will look at representations of these systems: $U : G \to \mathcal{U}(H)$, $\pi : \mathcal{A} \to B(H)$ satisfying $\pi(\alpha_x(a)) = U_x \pi(a) U_x^{-1}$ (the covariance relation). The algebra generated by $(U_x, \pi(a))$ is called the crossproduct algebra.

Example 2.4. Let $X = \mathbb{S}^1$ and a^θ be rotation by θ.

Ex: $X = \mathbb{S}^1$, a^θ = rotation by θ. Then $\mathbb{Z}$ acts by $(\alpha^\theta)^n$ and the crossproduct algebra is $\mathcal{A}_\theta$.

Remark 2.5. The construction of $C^*(G)$ also works for groupoids

If X, Y are compact Hausdroff spaces, how are $C(X \times Y)$ and $C(X), C(Y)$ related. Consider functions of the form $(f \otimes g)(x, y) = f(x)g(y)$. The algebra generated by these functions is dense in $C(X \times Y)$ by Stone Weierstrass.

Example 2.6. Can from algebra $\mathcal{A} \oplus \mathcal{A} \oplus \cdots$

Consider direct limit of algebras $M_2 \hookrightarrow M_4 \hookrightarrow M_8 \cdots$ with isometric embeddings $T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$.
Called the CAR-algebra (canonical anticommutative relation).
Q: for different n, are the C*-algebras generated by $M_n \to M_{n^2} \cdots$ nonisomorphic?
Yes (the proof uses methods from noncommutative geometry)
However, the Von-Neumann algebras generated by different n are all isomorphic.

Definition 2.7. A concrete C*-algebra is a subalgebra of $B(H)$, for H a Hilbert space, which is norm closed and closed under $*$.
An abstract C*-algebra is a Banach algebra with a conjugate linear involution $*$ satisfying $||A^*A|| = ||A||^2$.

By the Gelfand-Naimark theorem (1943), every abstract C*-algeba is isomorphic to a concrete C*-algebra.
Little Gelfand-Naimark: If $\mathcal{A}$ is a commutatuve C*-algebra then $\mathcal{A} \cong C_0(X)$ for X LCH.
If $\mathcal{A}$ is a commutative unital C*-algebra over $\mathbb{C}$, define $\hat{\mathcal{A}} = \{\varphi : \mathcal{A} \to \mathbb{C}, \text{ unital hom}\}$. For each $a \in \mathcal{A}$, define $\hat{a}$ on $\hat{\mathcal{A}}$ by $\hat{a}(\varphi) = \varphi(a)$. The map $a \mapsto \hat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ to functions on $\hat{\mathcal{A}}$.

Definition 2.8. For $a \in \mathcal{A}$, define the spectrum of a, $\sigma(a) = \{\lambda \colon a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}$.

If $\mathcal{A}$ is a Banach algebra, then $(1-a)^{-1} = \sum a^n$ if $||a|| < 1$ and so if $\lambda \in \sigma(a)$ then $|\lambda| < ||a||$.
Observe that if $\varphi \in \hat{\mathcal{A}}$ then $\varphi(a) \in \sigma(a)$ since $\varphi(a - \varphi(a)1) = 0$ and so $a - \varphi(a)1$ is not invertible.
For $a \in \mathcal{A}$, $\hat{a} \in C(\hat{\mathcal{A}})$, $a \mapsto \hat{a}$ called the Gelfand transform.
Gelfand spectral radius formula: $\sup\{|\lambda| \colon \lambda \in \sigma(a)\} = \lim ||a^n||^{1/n}$.
So, if $||a^2|| = ||a||^2$, then Gelfand transform is isometric.

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Definition 3.1. An algebra involution $*$ is a conjugate linear map so that $(ab)^* = (a^*)^* = a$.

Given $C(X)$ a unit C*-algebra, how do we realize it as bounded operators on a Hilber space H?
Suppose μ is a positive measure, viewed as a linear functional. Define a pre-inner product by $\langle f, g \rangle = \int f\overline{g} d\mu = \mu(f\overline{g})$. Quotienting out by norm zero elements, get a Hilbert space on which $C(X)$ acts by pointwise multiplication.
Consider $\delta_x(f) = f(x)$. $C(X) \subseteq \ell^\infty(X_{\text{disc}})$ acts on $\ell^2(X_{\text{disc}})$. Call this the atomic representation.
Let $\mathcal{A}$ be a unital $*$-algebra. A linear function on $\mathcal{A}$ is positive if $\mu(a^*a) \geq 0$ for $a \in \mathcal{A}$. The positive linera functions are closed under addition and multiplication by nonnegative scalars (ie. they form a cone)
Let μ be a positive linear function on $\mathcal{A}$. Define a pre-inner product on $\mathcal{A}$ by $\langle a, b \rangle_\mu = \mu(b^*a)$.
Check this is a pre-iner product:

Let $N_\mu = \{a \in \mathcal{A} \colon \langle a, a\rangle_\mu = 0\}$. If $a \in N_\mu$, then $|\langle a, b\rangle_\mu|^2 \leq \langle a, a\rangle_\mu \langle b, b\rangle_\mu = 0$ so $N_\mu = \{a \in A \colon \langle a, b\rangle_\mu \quad \forall b\}$. Hence, N_μ is a subspace of $\mathcal{A}$.

On $\mathcal{A}/N_\mu$ the pre-inner product becomes a product. Call its completion $L^2(\mathcal{A}, \mu)$.

Let $\mathcal{A}$ act on itself on the left (left regular representation): $\pi_a b = ab$. $*$-algebra hom since

$$\langle \pi_a b, c\rangle_\mu = \langle ab, c\rangle_\mu = \mu(c^* ab) = \mu((a^* c)^* b) = \langle b, a^* c\rangle_\mu = \langle b, \pi_{a^*} c\rangle$$

If $a \in \mathcal{A}$, $b \in N_\mu$ then $\langle \pi_a b, \pi_a b\rangle = \langle ab, ab\rangle_\mu = \langle b, a^* ab\rangle_\mu = 0$. So N_μ is a left ideal of $\mathcal{A}$. Hence $\mathcal{A}$ acts on the left on A/N_μ as well.

Example 3.2. Let $\mathcal{A}$ be the algebra of $\mathbb{C}$-valued polynomials on $\mathbb{R}$, and let $\mu(p) = \int_{-\infty}^{\infty} p(t) e^{-t^2} dt$. On $L^2(\mathcal{A}, \mu)$, multiplying by polynomials in an unboudned operator.

Called the GNS (Gelfand-Naimark-Segal) construction

Theorem 3.3. Let $\mathcal{A}$ be a unital normed $*$-algebra, μ a positive linear functional on $\mathcal{A}$.

(1) If $\mathcal{A}$ is complete, then μ is continuous and $||\mu|| = \mu(1)$

(2) If $\mathcal{A}$ is not necessarily complete but μ is continuous

Lemma 3.4. For $\mathcal{A}$ a unital Banach $*$-algebra, if $a \in \mathcal{A}$, $a^* = a$, and $||a|| < 1$ then there is $b \in \mathcal{A}$ so that $b^* = b$, b commutes with a, and $1 - a = b^2 = b^* b$.