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Before starting the development of the general theory of C^*-algebras we consider a few motivating examples.

Example 1.1. Consider the Hilbert spaces $\ell^2(\mathbb{Z})$, $L^2(\mathbb{R})$ and define unitary operators U, V by $(U\xi)(r) = \xi(r-1)$ and $(V\xi)(r) = e^{2\pi i\theta r}\xi(r)$ for $\theta \in \mathbb{R}$. Consider the algebra generated by these two operators. $VU = e^{2\pi i\theta}VU$ so the linear span of $U^m V^n$ is an algebra. Denote its operator norm closure by $\mathcal{A}_\theta$.

The algebras generated by U and V individually are both isomorphic to $C(\mathbb{S}^1)$ and if $\theta \in \mathbb{Z}$, U and V commute so $\mathcal{A}_\theta \cong C(\mathbb{S}^1 \times \mathbb{S}^1)$. Otherwise, we call $\mathcal{A}_\theta$ a noncommutative 2-torus.

Consider the self adjoint operator, called an almost Matthieu operator

$$H = (U + U^*) + \lambda(V + V^*)$$

Q: What is the spectrum of H?

- For rational θ, it is a disjoint union of intervals

- 1976: Hofstader plotted the spectra for rational θ to produce a fractal like pattern – the Hofstader Butterfly. This led to the conjecture that the spectrum of H for irrational θ was a Cantor set.

- 1981: Mark Kat offered 10 martinis for an answer to this conjecture.

- Barry Simon popularized this and it became known as the 10 martini problem

- 2005: Jitomirskaya, Avila gave an affirmative answer

Much of the partial progress on this problem came from the theory of C^*-algebras but the final proof used completely different techniques.

Example 1.2. Let G be a group. A (Hilbert Space) unitary representation of G is a group homomorphism $\pi: G \to \mathcal{U}(H)$. Consider the algebra of operators generated by $\{\pi(x): x \in G\}$, ie. elements of the form $\pi_f = \sum_{x \in G} f(x)\pi(x)$ for $f: G \to \mathbb{C}$ finitely supported. We define a product on this set by

$$\left(\sum f(x)\pi(x)\right)\left(\sum g(y)\pi(y)\right) = \sum_{x,y} f(x)g(y)\pi(xy) = \sum_{x,y} f(x)g(x^{-1}y)\pi(y) = \sum_y \left(\sum_x f(x)g(x^{-1}y)\right)\pi(y)$$

Defining the convolution $*$ by

$$(f * g)(y) = \sum_x f(x)g(x^{-1}y)$$

we have the $\pi_f \cdot \pi_g = \pi_{f*g}$ and this gives an algebra structure on $C_c(G)$ and an algebra homomorphism $\pi : (C_c(G), *) \to B(H)$. The convolution also gives $\ell^1(G)$ an algebra structure since $||f*g||_1 \leq ||f||_1 ||g||_1$.

For any $f \in C_c(G)$, let $||f||_{C^*(G)} = \sup\{\pi_f : \pi \text{ unitary rep of } G\}$. Observe that

$$||\pi_f|| \leq \sum |f(x)|||\pi(x)|| = \sum |f(x)| = ||f||_1$$

and so $||f||_{C^*(G)} \leq ||f||_1$. Let $C^*(G)$ denote the completion of $C_c(G)$ with respect to this norm. We can represent this algebra as an algebra bounded operators on some Hilbert space.

Note: for any group G we have at least two unitary representations – the trivial representation and the left regular representation on $\ell^2(G)$ by translation.

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Example 2.1. Using the left regular representation: $H = \ell^2(G)$ with $\mathcal{U}_x(\xi)(y) = \xi(x^{-1}y)$, we can define a norm on $C_c(G)$ by $||f||_r = ||U_f||$. Note that the involution is given by $f^*(x) = \overline{f(-x)}$. The closure of $C_c(G)$ under this norm is called the reduced C*-algebra for G. Since $||f||_r \leq ||f||_{C^*}$, we only have that $C_r^*(G)$ is a quotient of $C^*(G)$ in general. In fact, the discreprency between the two encodes information about the group G:

Theorem 2.2. $C^*(G) = C_r^*(G)$ if and only if G is amenable.

Example 2.3. For X a compact Hausdorff space, we can consider a homomorphism $\alpha : G \to \text{Homeo}(x)$ (eg. a dynamical system). This induces a map $\alpha : G \to \text{Aut}(C(X))$. We can often use this map to give insight into the behavior of the dynamical system as passing to the level of algebras gives more structure to leverage.
More generally, for a C*-algebra $\mathcal{A}$, we call a map $\alpha : G \to \text{Aut}(\mathcal{A})$ a "noncommutative dynamical system". We will look at representations of these systems: $U : G \to \mathcal{U}(H)$, $\pi : \mathcal{A} \to B(H)$ satisfying $\pi(\alpha_x(a)) = U_x \pi(a) U_x^{-1}$ (the covariance relation). The algebra generated by $(U_x, \pi(a))$ is called the crossproduct algebra.

Example 2.4. Let $X = \mathbb{S}^1$ and a^θ be rotation by θ.

Ex: $X = \mathbb{S}^1$, $a^\theta = $ rotation by θ. Then $\mathbb{Z}$ acts by $(\alpha^\theta)^n$ and the crossproduct algebra is $\mathcal{A}_\theta$.

Remark 2.5. The construction of $C^*(G)$ also works for groupoids

If X, Y are compact Hausdroff spaces, how are $C(X \times Y)$ and $C(X), C(Y)$ related. Consider functions of the form $(f \otimes g)(x, y) = f(x)g(y)$. The algebra generated by these functions is dense in $C(X \times Y)$ by Stone Weierstrass.

Example 2.6. Can from algebra $\mathcal{A} \oplus \mathcal{A} \oplus \cdots$

Consider direct limit of algebras $M_2 \hookrightarrow M_4 \hookrightarrow M_8 \cdots$ with isometric embeddings $T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$.
Called the CAR-algebra (canonical anticommutative relation).
Q: for different n, are the C*-algebras generated by $M_n \to M_{n^2} \cdots$ nonisomorphic?
Yes (the proof uses methods from noncommutative geometry)
However, the Von-Neumann algebras generated by different n are all isomorphic.

Definition 2.7. A concrete C*-algebra is a subalgebra of $B(H)$, for H a Hilbert space, which is norm closed and closed under $*$.
An abstract C*-algebra is a Banach algebra with a conjugate linear involution $*$ satisfying $||A^*A|| = ||A||^2$.

By the Gelfand-Naimark theorem (1943), every abstract C*-algeba is isomorphic to a concrete C*-algebra.
Little Gelfand-Naimark: If $\mathcal{A}$ is a commutatuve C*-algebra then $\mathcal{A} \cong C_0(X)$ for X LCH.

If $\mathcal{A}$ is a commutative unital C*-algebra over $\mathbb{C}$, define $\hat{\mathcal{A}} = \{\varphi : \mathcal{A} \to \mathbb{C}, \text{ unital hom}\}$. For each $a \in \mathcal{A}$, define $\hat{a}$ on $\hat{\mathcal{A}}$ by $\hat{a}(\varphi) = \varphi(a)$. The map $a \mapsto \hat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ to functions on $\hat{\mathcal{A}}$.

Definition 2.8. For $a \in \mathcal{A}$, define the spectrum of a, $\sigma(a) = \{\lambda : a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}$.

If $\mathcal{A}$ is a Banach algebra, then $(1-a)^{-1} = \sum a^n$ if $||a|| < 1$ and so if $\lambda \in \sigma(a)$ then $|\lambda| < ||a||$.

Observe that if $\varphi \in \hat{\mathcal{A}}$ then $\varphi(a) \in \sigma(a)$ since $\varphi(a - \varphi(a)1) = 0$ and so $a - \varphi(a)1$ is not invertible.

For $a \in \mathcal{A}$, $\hat{a} \in C(\hat{\mathcal{A}})$, $a \mapsto \hat{a}$ called the Gelfand transform.

Gelfand spectral radius formula: $\sup\{|\lambda| : \lambda \in \sigma(a)\} = \lim ||a^n||^{1/n}$.

So, if $||a^2|| = ||a||^2$, then Gelfand transform is isometric.

3   1-26

Definition 3.1. An algebra involution $*$ is a conjugate linear map so that $(ab)^* = (a^*)^* = a$.

Given $C(X)$ a unit C*-algebra, how do we realize it as bounded operators on a Hilber space H?

Suppose μ is a positive measure, viewed as a linear functional. Define a pre-inner product by $\langle f, g \rangle = \int f \bar{g} d\mu = \mu(f\bar{g})$. Quotienting out by norm zero elements, get a Hilbert space on which $C(X)$ acts by pointwise multiplication.

Consider $\delta_x(f) = f(x)$. $C(X) \subseteq \ell^\infty(X_{\text{disc}})$ acts on $\ell^2(X_{\text{disc}})$. Call this the atomic representation.

Let $\mathcal{A}$ be a unital $*$-algebra. A linear function on $\mathcal{A}$ is positive if $\mu(a^*a) \geq 0$ for $a \in \mathcal{A}$. The positive linera functions are closed under addition and multiplication by nonnegative scalars (ie. they form a cone)

Let μ be a positive linear function on $\mathcal{A}$. Define a pre-inner product on $\mathcal{A}$ by $\langle a, b \rangle_\mu = \mu(b^*a)$.

Check this is a pre-iner product:

Let $N_\mu = \{a \in \mathcal{A} : \langle a, a \rangle_\mu = 0\}$. If $a \in N_\mu$, then $|\langle a, b \rangle_\mu|^2 \leq \langle a, a \rangle_\mu \langle b, b \rangle_\mu = 0$ so $N_\mu = \{a \in A : \langle a, b \rangle_\mu \ \forall b\}$. Hence, N_μ is a subspace of $\mathcal{A}$.

On $\mathcal{A}/N_\mu$ the pre-inner product becomes a product. Call its completion $L^2(\mathcal{A}, \mu)$.

Let $\mathcal{A}$ act on itself on the left (left regular representation): $\pi_a b = ab$. $*$-algebra hom since

$$\langle \pi_a b, c \rangle_\mu = \langle ab, c \rangle_\mu = \mu(c^*ab) = \mu((a^*c)^*b) = \langle b, a^*c \rangle_\mu = \langle b, \pi_{a^*}c \rangle$$

If $a \in \mathcal{A}$, $b \in N_\mu$ then $\langle \pi_a b, \pi_a b \rangle = \langle ab, ab \rangle_\mu = \langle b, a^*ab \rangle_\mu = 0$. So N_μ is a left ideal of $\mathcal{A}$. Hence $\mathcal{A}$ acts on the left on A/N_μ as well.

Example 3.2. Let $\mathcal{A}$ be the algebra of $\mathbb{C}$-valued polynomials on $\mathbb{R}$, and let $\mu(p) = \int_{-\infty}^{\infty} p(t)e^{-t^2} dt$. On $L^2(\mathcal{A}, \mu)$, multiplying by polynomials in an unboudned operator.

Called the GNS (Gelfand-Naimark-Segal) construction

Theorem 3.3. Let $\mathcal{A}$ be a unital normed $*$-algebra, μ a positive linear functional on $\mathcal{A}$.

(1) If $\mathcal{A}$ is complete, then μ is continuous and $||\mu|| = \mu(1)$

(2) If $\mathcal{A}$ is not necessarily complete but μ is continuous

Lemma 3.4. For $\mathcal{A}$ a unital Banach $*$-algebra, if $a \in \mathcal{A}$, $a^* = a$, and $||a|| < 1$ then there is $b \in \mathcal{A}$ so that $b^* = b$, b commutes with a, and $1 - a = b^2 = b^*b$.

4   1-28

Lemma 4.1. Let $\mathcal{A}$ be a unital *-Banach algebra over $\mathbb{C}$. If $a \in \mathcal{A}$, $a^* = a$, and $||a|| < 1$, then there is $b \in \mathcal{A}$ with b^*b and $1 - a = b^2$.

Proof. The function $f(z) = \sqrt{1-z}$, has a power series expansion $f(z) = \sum r_n z^n$, $r_n \in \mathbb{R}$ which converges for $|z| < 1$. Setting $b = \sum r_n a^n$, it follows that $b^2 = (f(a))^2 = (f^2)(a) = 1 - a$. $\square$

Theorem 4.2. If μ is a positive linear function on $\mathcal{A}$, then μ is continuous and $||\mu|| = \mu(1)$.

Theorem 4.3. If μ is a positive linear functional on $\mathcal{A}$, then the left regular representation on $L^2(A, \mu)$ defines a continuous *-representation, ie. each π_a is a bounded linear operator (and $||\pi_a|| \leq ||a||$).

Definition 4.4. Let $\mathcal{A}$ be a unital *-Banach algebra and (H, π) a continuous *-representation. Then $\xi \in H$ is said to be (topologically) cyclic if $\{\pi_a \xi : a \in \mathcal{A}\}$ is dense in H.

Proposition 4.5. Let (H, π) be a *-representation of $\mathcal{A}$. If $K \subseteq H$ is a closed π-invariant subspace then $K^\perp$ is also π-invariant

Proof. For any $\eta \in K^\perp$, $\xi \in K$, $\langle \xi, \pi_a \eta \rangle = \langle \pi_a^* \xi, \eta \rangle = 0$. $\square$

Given (H, π) a *-representation, for any nonzero $\xi \in H$, $K = \overline{\{\pi_a : a \in \mathcal{A}\}}$ is a closed cyclic subspace and we can write H as the sum of closed π-invariant subspaces $H = K \oplus K^\perp$. Iterating this (possibly a transfinite number of times) shows that any *-representation (H, π) can be decomposed into cylic invariant subspaces.

5   1-30

5.1   Integrated Forms

Let G be a group and U a unitary representation of G. Given $f \in C_c(G)$, define $U_f = \sum f(x) U_x$. Then, $U_f U_g = U_{f*g}$ and $||U_f|| \leq ||f||_1$ so $f \mapsto U_f$ is a continuous *-representation of $\ell^1(G)$. The assignment $f \mapsto U_f$ is called the integrated form corresponding to U. Conversely, if (H, π) is a *-representation of $\ell^1(G)$, then $x \mapsto \pi_{\delta_x}$ gives a unitary representation of G. So, there is a bijection between unitary representations of G and continuous *-representations of $C_c(G)$ or $\ell^1(G)$.

If φ is a continuous positive linear functional on $\ell^1(G)$, then $\varphi \in \ell^\infty(G)$. So to any positive linear functional μ, there is an associated $\mu_f \in \ell^\infty(G)$. We call such a μ_f a function of positive type.

Example 5.1. Let (H, π) be the trivial representation of G on $\mathbb{C}$. Its integrated from is $f \mapsto \sum f(x)$ and so the corresponding function of positive type is $\varphi(x) = 1$ for all x.

Claim: $\delta_e \in \ell^\infty(G)$ is of positive type.

$$\langle f^* * f, \delta_e \rangle = (f^* * f)(e) = \sum \overline{f(y^{-1})} f(y^{-1} e) = \sum |f(y^{-1})|^2 \geq 0$$

What does the GNS construction give?

$$\langle f, g \rangle_{\delta_e} = \sum \overline{g(y)} f(y) = \langle f, g \rangle_{\ell^2(G)}$$

So we recover the left regular representation of G.

5.2   Traces

Definition 5.2. A positive linear functional on a normed *-algebra is tracial (is a trace) if $\mu(ab) = \mu(ba)$ for all $a, b \in \mathcal{A}$.

Claim: δ_e is a trace on $\ell^1(G)$

$$\langle f * g, \delta_e \rangle = \sum f(y) g(y^{-1}) = \sum f(y^{-1}) g(y) = \langle g * f, \delta_e \rangle$$

Let $\mathcal{A}$ be a unital Banach *-algebra, μ a tracial positive linear functional. Let $N_\mu = \{a : \langle a, b \rangle_\mu = 0 \ \forall b\}$. Then N_μ is a two sided ideal of $\mathcal{A}$ and $\mathcal{A}/N_\mu$ is an A, A-bimodule. So $L^2(A, \mu)$ admits both left and right regular representations.

The representation $L^2(A, \mu)$ is cyclic vector with cyclic vector 1. Any vector ξ gives a positive linear functional via $a \mapsto \langle a\xi, \xi \rangle$ and the postive linear function corresponding to 1 gives back μ. So, any positive linear functional on $\mathcal{A}$ comes from a cyclic vector in some *-representation.

Example 5.3. For the regular representations of $\ell^1(G)$ or $\ell^\infty(G)$ on $\ell^2(G)$, the vector δ_e is cyclic for the representation. It defines a trace on the image of $\ell^1(G)$. Let $C_r(G)$ denote the norm closure of the image of $\ell^1(G)$ in $B(\ell^2(G))$. δ_e defines a trace on this sub C*-algebra, $||\delta_e|| = 1$.

It is not possible to extend this to a trace on $B(H)$ in general since iff H is infinite dimensional, then $B(H)$ admits no boudned trace.

6 2-2

6.1 Direct Sums of Hilbert Spaces

Given a collection of Hilbert spaces $\{H_j\}_{j \in J}$, we can form the algebraic direct sum as

$$\bigoplus_{\text{alg}} H_j = \{\xi \in \prod H_j \colon \xi_j = 0 \text{ for all but finitely many } j\}$$

This is an inner product space with

$$\langle \xi, \eta \rangle = \sum_j \langle x_j, \eta_j \rangle$$

Taking the completion with respect to this inner product gives the direct sum (coproduct) of the H_j's as Hilbert spaces. One can check that

$$\bigoplus H_j = \{\xi \in \prod H_j \colon \sum \|x_i\|_j^2 < \infty\}$$

Given $(T_j)_{j \in J}$ with $T_j \in B(H_j)$, can define $T\xi = (T_j \xi_j)$. If $\{\|T_j\| \colon j \in J\}$ is bounded then $T \in B(H)$ with $\|T\| = \sup_j \|T_j\|$. Similarly, if (H_j, π_j) is a family of unitary representations of G (*-representations of a Banach algebra), we can define a *-representation on $\bigoplus H_j$ by $U_x \xi = (U_x^j \xi_j)$, $\left(\pi_a \xi = (\pi_a^j \xi_j)\right)$. Note that these are bounded representations since $\|U_x^j\| = 1$ and $\|\pi_a^j\| \le \|a\|$ for each j.

If (H, π) is a *-representation, then it decomposes into a direct sum $\bigoplus (H_j, \pi_j)$ with each (H_j, π_j) a cylic representation.

6.2 Intertwining Operators

Definition 6.1. Suppose (H, π), (K, ρ) are *-representations of $\mathcal{A}$ (unitary reps of G). A morphism of representations is an $\mathcal{A}$ module homomorphism $T \colon H \to K$, ie. a bounded linear operator $T \colon H \to K$ so that for each $a \in \mathcal{A}$, $T\pi_a = \rho_a T$. We call such a map an intertwining operator.

Definition 6.2. An isomorphism of representations is an intertwining unitary operator $U \colon H \to K$.

Definition 6.3. A pointed cylic representation (H, π, ξ_*) is a cyclic representation with a specified cyclic vector ξ_*.

Proposition 6.4. Let (H, π, ξ_*) and (K, ρ, η_*) be pointed cylic representations. If $\mu_{\xi_*} = \mu_{\eta_*}$ there is a unitary intertwining operator U from H to K so that $U\xi_* = \eta_*$.

Proof. Define U on the dense subspace $\{\pi_a \xi_* \colon a \in \mathcal{A}\}$ by $U(\pi_a \xi_*) = \rho_a \eta_*$. U then extends uniquely to a continuous map $H \to K$. It is clear that U is intertwining. Further, for any $a, b \in \mathcal{A}$,

$$\langle U\pi_a \xi_*, U\pi_b \xi_* \rangle = \langle \rho_a \eta_*, \rho_b \eta_* \rangle = \langle \rho_b^* \rho_a \eta_*, \eta_* \rangle = \mu_{\eta_*}(b^* a) = \mu_{\xi_*}(b^* a) = \langle \pi_b^* \pi_a \xi_*, \xi_* \rangle = \langle \pi_a \xi_*, \pi_b \xi_* \rangle$$

So U is unitary as well. $\qquad \square$

It follows that there is a bijection between positive linear functions on $\mathcal{A}$ and isomorphism classes of pointed cylic representations.

Definition 6.5. A *-representation (H, π) irreducible if it has no proper closed π-invariant subspaces.

Given a finite dimensional representation, it can be decomposed into a sum of irreducible representations.

7 2-4

7.1 Schur's Lemma

Definition 7.1. Let H be a Hilbert space and $\mathcal{C} \subseteq B(H)$. The commutant of $\mathcal{C}$ is defined to be

$$\mathcal{C}' = \{T \in B(H) \colon TC = CT \; \forall C \in \mathcal{C}\}$$

The commutant is closed with respect to the strong operator topology and so is a Von Neumman algebra (and hence a C^*-algebra). For any $\mathcal{C}$, $(\mathcal{C}')'$ is the strong closed *-algebra generated by $\mathcal{C}$. The double commutant theorem was proved by Von Neumman in the first paper on operator algebras.

If (H, π) is a representation of $\mathcal{A}$, then $\text{End}_{\mathcal{A}}(H)$ is the commutant of $\{\pi_a\}_{a \in \mathcal{A}}$.

Lemma 7.2 (Schur's Lemma)**.** A representation (H, π) of $\mathcal{A}$ is irreducible if and only if $\text{End}_{\mathcal{A}}(H) = \mathbb{C}$.

Proof. If (H, π) is not irreducible then there is a proper closed invariant subspace $K \subseteq H$. Taking P to be the orthogonal projection onto K, we have that $K \in \text{End}_{\mathcal{A}}(H)$ but K is not a scalar multiple of the identity. Conversely, if $\text{End}_{\mathcal{A}}(H) \neq \mathbb{C}$ then let T be an operator which is not a scalar multiple of the identity. Since either the real or imaginary part of T is also not a scalar multiple of the identity we may assume that T is self adjoint without loss of generality. There are $s \neq t \in \sigma(T)$ and so let f, g be functions so that $f(s) = g(t) = 1$ and $fg = 0$. This $F, G \in C^*(I, T)$ so that $F, G \neq 0$ and $FG = 0$. Now, $\langle FH, GH \rangle = 0$ and $GH \neq 0$ so, since $F \in \text{End}_{\mathcal{A}}(H)$, $\overline{FH}$ is a proper invariant subspace. $\square$

Corollary 7.3. If $\mathcal{A}$ is a commutative unital normed *-algebra, then every irreducible representation of $\mathcal{A}$ is one dimensional.

Proof. If $\mathcal{A}$ is commutative, then for representation (H, π), $\pi_a \in \text{End}_{\mathcal{A}}(H)$ for $a \in \mathcal{A}$. So if the representation is irreducible it follows that $\pi_a \in \mathbb{C}I$ for each A and so every subspace is invariant. Thus, H must be one dimensional. $\square$

Example 7.4. If X is a compact Hausdorff space, then the irreducible representations of X are precisely evaluations at points in x. More generally, if $\mathcal{A}$ is a commutative *-Banach algebra, then its irreducible representations are exactly the characters on $\mathcal{A}$.

Example 7.5. Consider the algebra $\mathcal{A} = \mathbb{C} \oplus \mathcal{C}$ with $(\alpha, \beta)^* = (\overline{\beta}, \overline{\alpha})$. There are no nonzero *-algebra homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ since $(1, 0)$ generates it as a *-algebra but $(1,0)(1,0)^* = 0$ and so $\mu(1, 0) = 0$ for any $\mu : \mathcal{A} \to \mathbb{C}$.

7.2 Radon-Nikodym for Positive Linear Functionals

Definition 7.6. Let $\mathcal{A}$ be a unital *-algebra and let μ, ν be positive linear functionals on $\mathcal{A}$. Say that μ dominates ν, $\mu \geq \nu$ if $\mu - \nu \geq 0$.

How do we get such a ν? Given μ, consider the pointed representation $(H_\mu, \pi_\mu, \xi_\mu)$. For any $T \in \text{End}_{\mathcal{A}}(H)$ with $0 \leq T \leq I$, define
$$\nu(a) = \langle \pi_a T \xi_\mu, \xi_\mu \rangle$$

We have that
$$\nu(a^* a) = \langle T \pi_a \xi_\mu, \pi_a \xi_\mu \rangle \quad \text{and} \quad (\mu - \nu)(a^* a) = \langle (I - T) \pi_a \xi_\mu, \xi_\mu \rangle$$

and so $0 \leq \nu \leq \mu$. Here T is the "Radon-Nidokdym derivative" of ν.

We will show that all such ν arise in this way.

Theorem 7.7. If $\mu \geq \nu \geq 0$, then $\exists T \in \text{End}_{\mathcal{A}}(H_\mu)$, $0 \leq T \leq I$ such that $\nu(a) = \langle \pi_a T \xi_\mu, \xi_\mu \rangle$.

Fact: Let H be a Hilbert Space, K a dense subspace, and $\langle \, , \rangle_K$ a pre-inner product on K so that $\langle \eta, \eta \rangle_K \leq \langle \eta, \eta \rangle_H$ for $\eta \in K$. Then, there is an operator $T \in B(H)$ with $0 \leq T \leq I$ and so that $\langle \eta, \eta \rangle = \langle T\eta, \eta \rangle$ for $\eta \in K$.

Proof. Given $\eta \in K$, consider the conjugate-linear functional given by $\varphi_\eta(\xi) = \langle \eta, \xi \rangle_K$. φ_η extends uniquely to H and so there is some ξ_η so that $\varphi_\eta(\xi) = \langle \xi_\eta, \xi \rangle$ for all $\xi \in K$. Consider the linear map defined on K by $T\eta = \xi_\eta$. Note that T is continuous since $||\xi_\eta|| = ||\eta||_K \leq ||\eta||_H$ and so extends uniquely to an element in $B(H)$ satisfying the desired conditions. $\square$