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1.1 Motivation

Before starting the development of the general theory of C^*-algebras we consider a few motivating examples.

Example 1.1. Consider the Hilbert spaces $\ell^2(\mathbb{Z})$, $L^2(\mathbb{R})$ and define unitary operators U, V by $(U\xi)(r) = \xi(r-1)$ and $(V\xi)(r) = e^{2\pi i\theta r}\xi(r)$ for $\theta \in \mathbb{R}$. Consider the algebra generated by these two operators. $VU = e^{2\pi i\theta}VU$ so the linear span of U^mV^n is an algebra. Denote its operator norm closure by $\mathcal{A}_\theta$.
The algebras generated by U and V individually are both isomorphic to $C(\mathbb{S}^1)$ and if $\theta \in \mathbb{Z}$, U and V commute so $\mathcal{A}_\theta \cong C(\mathbb{S}^1 \times \mathbb{S}^1)$. Otherwise, we call $\mathcal{A}_\theta$ a noncommutative 2-torus.
Consider the self adjoint operator, called an almost Matthieu operator

$$H = (U + U^*) + \lambda(V + V^*)$$

Q: What is the spectrum of H?

- For rational θ, it is a disjoint union of intervals
- 1976: Hofstader plotted the spectra for rational θ to produce a fractal like pattern – the Hofstader Butterfly. This led to the conjecture that the spectrum of H for irrational θ was a Cantor set.
- 1981: Mark Kat offered 10 martinis for an answer to this conjecture.
- Barry Simon popularized this and it became known as the 10 martini problem

- 2005: Jitomirskaya, Avila gave an affirmative answer

Much of the partial progress on this problem came from the theory of C^*-algebras but the final proof used completely different techniques.

Example 1.2. Let G be a group. A (Hilbert Space) unitary representation of G is a group homomorphism $\pi : G \to \mathcal{U}(H)$. Consider the algebra of operators generated by $\{\pi(x)\colon x \in G\}$, ie. elements of the form $\pi_f = \sum_{x \in G} f(x)\pi(x)$ for $f : G \to \mathbb{C}$ finnitely supported. We define a product on this set by

$$(\sum f(x)\pi(x))(\sum g(y)\pi(y)) = \sum_{x,y} f(x)g(y)\pi(xy) = \sum_{x,y} f(x)g(x^{-1}y)\pi(y) = \sum_y (\sum_x f(x)g(x^{-1}y))\pi(y)$$

Defining the convolution $*$ by

$$(f * g)(y) = \sum_x f(x)g(x^{-1}y)$$

we have the $\pi_f \cdot \pi_g = \pi_{f*g}$ and this gives an algebra structure on $C_c(G)$ and an algebra homomorphism $\pi : (C_c(G), *) \to B(H)$. The convolution also gives $\ell^1(G)$ an algebra structure since $||f*g||_1 \leq ||f||_1||g||_1$.

For any $f \in C_c(G)$, let $||f||_{C^*(G)} = \sup\{\pi_f \colon \pi \text{ unitary rep of } G\}$. Observe that

$$||\pi_f|| \leq \sum |f(x)|||\pi(x)|| = \sum |f(x)| = ||f||_1$$

and so $||f||_{C^*(G)} \leq ||f||_1$. Let $C^*(G)$ denote the completion of $C_c(G)$ with respect to this norm. We can represent this algebra as an algebra bounded operators on some Hilbert space.

Note: for any group G we have at least two unitary representations – the trivial representation and the left regular representation on $\ell^2(G)$ by translation.

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Example 2.1. Using the left regular representation: $H = \ell^2(G)$ with $\mathcal{U}_x(\xi)(y) = \underline{\xi(x^{-1}y)}$, we can define a norm on $C_c(G)$ by $||f||_r = ||U_f||$. Note that the involution is given by $f^*(x) = \overline{f(-x)}$. The closure of $C_c(G)$ under this norm is called the reduced C*-algebra for G. Since $||f||_r \leq ||f||_{C^*}$, we only have that $C_r^*(G)$ is a quotient of $C^*(G)$ in general. In fact, the discreprency between the two encodes information about the group G:

Theorem 2.2. $C^*(G) = C_r^*(G)$ if and only if G is amenable.

Example 2.3. For X a compact Hausdorff space, we can consider a homomorphism $\alpha : G \to \text{Homeo}(x)$ (eg. a dynamical system). This induces a map $\alpha : G \to \text{Aut}(C(X))$. We can often use this map to give insight into the behavior of the dynamical system as passing to the level of algebras gives more structure to leverage.
More generally, for a C*-algebra $\mathcal{A}$, we call a map $\alpha : G \to \text{Aut}(\mathcal{A})$ a "noncommutative dynamical system". We will look at representations of these systems: $U : G \to \mathcal{U}(H)$, $\pi : \mathcal{A} \to B(H)$ satisfying $\pi(\alpha_x(a)) = U_x\pi(a)U_x^{-1}$ (the covariance relation). The algebra generated by $(U_x, \pi(a))$ is called the crossproduct algebra.

Example 2.4. Let $X = \mathbb{S}^1$ and a^θ be rotation by θ.

Ex: $X = \mathbb{S}^1$, a^θ = rotation by θ. Then $\mathbb{Z}$ acts by $(\alpha^\theta)^n$ and the crossproduct algebra is $\mathcal{A}_\theta$.

Remark 2.5. The construction of $C^*(G)$ also works for groupoids

If X, Y are compact Hausdroff spaces, how are $C(X \times Y)$ and $C(X), C(Y)$ related. Consider functions of the form $(f \otimes g)(x, y) = f(x)g(y)$. The algebra generated by these functions is dense in $C(X \times Y)$ by Stone Weierstrass.

Example 2.6. Can from algebra $\mathcal{A} \oplus \mathcal{A} \oplus \cdots$

Consider direct limit of algebras $M_2 \hookrightarrow M_4 \hookrightarrow M_8 \cdots$ with isometric embeddings $T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$.
Called the CAR-algebra (canonical anticommutative relation).
Q: for different n, are the C*-algebras generated by $M_n \to M_{n^2} \cdots$ nonisomorphic?
Yes (the proof uses methods from noncommutative geometry)
However, the Von-Neumann algebras generated by different n are all isomorphic.

Definition 2.7. A concrete C*-algebra is a subalgebra of $B(H)$, for H a Hilbert space, which is norm closed and closed under $*$.
An abstract C*-algebra is a Banach algebra with a conjugate linear involution $*$ satisfying $||A^*A|| = ||A||^2$.

By the Gelfand-Naimark theorem (1943), every abstract C*-algeba is isomorphic to a concrete C*-algebra.
Little Gelfand-Naimark: If $\mathcal{A}$ is a commutatuve C*-algebra then $\mathcal{A} \cong C_0(X)$ for X LCH.

If $\mathcal{A}$ is a commutative unital C*-algebra over $\mathbb{C}$, define $\hat{\mathcal{A}} = \{\varphi : \mathcal{A} \to \mathbb{C}, \text{ unital hom}\}$. For each $a \in \mathcal{A}$, define $\hat{a}$ on $\hat{\mathcal{A}}$ by $\hat{a}(\varphi) = \varphi(a)$. The map $a \mapsto \hat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ to functions on $\hat{\mathcal{A}}$.

Definition 2.8. For $a \in \mathcal{A}$, define the spectrum of a, $\sigma(a) = \{\lambda \colon a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}$.

If $\mathcal{A}$ is a Banach algebra, then $(1-a)^{-1} = \sum a^n$ if $||a|| < 1$ and so if $\lambda \in \sigma(a)$ then $|\lambda| < ||a||$.
Observe that if $\varphi \in \hat{\mathcal{A}}$ then $\varphi(a) \in \sigma(a)$ since $\varphi(a - \varphi(a)1) = 0$ and so $a - \varphi(a)1$ is not invertible.
For $a \in \mathcal{A}$, $\hat{a} \in C(\hat{\mathcal{A}})$, $a \mapsto \hat{a}$ called the Gelfand transform.
Gelfand spectral radius formula: $\sup\{|\lambda| \colon \lambda \in \sigma(a)\} = \lim ||a^n||^{1/n}$.
So, if $||a^2|| = ||a||^2$, then Gelfand transform is isometric.

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3.1 The GNS Construction

Definition 3.1. An algebra involution $*$ is a conjugate linear map so that $(ab)^* = b^*a*$ and $(a^*)^* = a$.

Given $C(X)$ a unital C*-algebra, how do we realize it as bounded operators on a Hilbert space H?

Suppose μ is a positive measure, viewed as a linear functional. Define a pre-inner product by $\langle f, g\rangle = \int f\overline{g}d\mu = \mu(f\overline{g})$. Quotienting out by norm zero elements, gives a Hilbert space on which $C(X)$ acts by pointwise multiplication.

Example 3.2. Consider $\delta_x(f) = f(x)$. Then $C(X) \subseteq \ell^\infty(X_{\text{disc}})$ acts on $\ell^2(X_{\text{disc}})$. This is called the atomic representation.

We would like to this construction on a general unital *-algebra.

Definition 3.3. Let μ be a linear functional on $\mathcal{A}$ a unital *-algebra. Say that μ is positive if $\mu(a^*a) \geq 0$ for $a \in \mathcal{A}$.

The positive linear functionals are closed under addition and multiplication by nonnegative scalars (ie. they form a cone)

Let μ be a positive linear function on $\mathcal{A}$. Define a pre-inner product on $\mathcal{A}$ by $\langle a, b\rangle_\mu = \mu(b^*a)$. To see that this is a pre-inner product observe that

$$\langle a, a\rangle_\mu = \mu(a^*a) \geq 0 \quad \text{ and } \quad \langle a+b, a+b\rangle_\mu = \langle a, a\rangle_\mu + \langle a, b\rangle_\mu + \langle b, a\rangle_\mu + \langle b, b\rangle_\mu \geq 0$$

so $0 = \mathrm{Im}(\langle a, b\rangle_\mu) + \mathrm{Im}(\langle b, a\rangle_\mu)$ and, replacing b with ib, gives $\mathrm{Re}(\langle a, b\rangle_\mu) = \mathrm{Re}(\langle b, a\rangle_\mu)$.

Let $N_\mu = \{a \in \mathcal{A} \colon \langle a, a\rangle_\mu = 0\}$. If $a \in N_\mu$, then $|\langle a, b\rangle_\mu|^2 \leq \langle a, a\rangle_\mu \langle b, b\rangle_\mu = 0$ so $N_\mu = \{a \in A \colon \langle a, b\rangle_\mu = 0 \ \forall b\}$. It follows that N_μ is a left ideal of $\mathcal{A}$. On $\mathcal{A}/N_\mu$ the pre-inner product becomes an inner product. Denote its completion by $L^2(\mathcal{A}, \mu)$.

Let $\mathcal{A}$ act on itself on the left via $\pi_a b = ab$. This is a $*$-algebra homomorphism since

$$\langle \pi_a b, c\rangle_\mu = \langle ab, c\rangle_\mu = \mu(c^*ab) = \mu((a^*c)^*b) = \langle b, a^*c\rangle_\mu = \langle b, \pi_{a^*}c\rangle$$

Call this the left regular representation. Since N_μ is a left ideal of $\mathcal{A}$, this gives an action of $\mathcal{A}$ on $\mathcal{A}/N_\mu$ as well. It remains to check that the π_a's are continuous. This may fail without a complete norm as the next example shows.

Example 3.4. Let $\mathcal{A}$ be the algebra of $\mathbb{C}$-valued polynomials on $\mathbb{R}$, and let $\mu(p) = \int_{-\infty}^{\infty} p(t)e^{-t^2}dt$. On $L^2(\mathcal{A}, \mu)$, multiplying by polynomials in an unbounded operator.

The above is called the called the GNS (Gelfand-Naimark-Segal) construction

Theorem 3.5. Let $\mathcal{A}$ be a unital normed $*$-algebra, μ a positive linear functional on $\mathcal{A}$.

(1) If $\mathcal{A}$ is complete, then μ is continuous and $||\mu|| = \mu(1)$

(2) If $\mathcal{A}$ is not necessarily complete but μ is continuous then $||\mu|| = \mu(1)$.

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4.1 Automatic Continuity

Lemma 4.1. Let $\mathcal{A}$ be a unital *-Banach algebra over $\mathbb{C}$. If $a \in \mathcal{A}$, $a^* = a$, and $||a|| < 1$, then there is $b \in \mathcal{A}$ with b^*b and $1 - a = b^2$.

Proof. The function $f(z) = \sqrt{1-z}$, has a power series expansion $f(z) = \sum r_n z^n$, $r_n \in \mathbb{R}$ which converges for $|z| < 1$. Setting $b = \sum r_n a^n$, it follows that $b^2 = (f(a))^2 = (f^2)(a) = 1 - a$. □

Theorem 4.2. If μ is a positive linear function on $\mathcal{A}$, then μ is continuous and $||\mu|| = \mu(1)$.

Proof. Suppose $a \in \mathcal{A}$ with $a^* = a$ and $||a|| < 1$. Then for b so that $1 - a = b^*b$, we have that

$$\mu(b^*b) = \mu(1) - \mu(a) \geq 0$$

and so $\mu(1) \geq \mu(a)$. It follows that if $a^* = a$, $\mu(a) \leq ||a||\mu(1)$. For general a,

$$|\mu(a)|^2 = |\mu(a1)|^2 = |\langle 1, a\rangle_\mu|^2 \leq \mu(1)\mu(a^*a) \leq \mu(1)^2||a^*a|| \leq \mu(1)^2||a||^2$$

□

If (H, π) is a *-representation of $\mathcal{A}$, then for any $\xi \in H$, set $\mu_\xi(a) = \langle \pi(a)\xi, \xi\rangle$. This defines a positive linear functional with $||\mu_\xi|| = ||\xi||^2$. We also have that

$$||\pi(a)\xi||^2 = \langle \pi(a)\xi, \pi(a)\xi\rangle = \langle \pi(a^*a)\xi, \xi\rangle \leq ||a^*a||\, ||\xi^2|| \leq ||a||^2\, ||\xi||^2$$

and so $||\pi(a)|| \leq ||a||_{\mathcal{A}}$.

Theorem 4.3. If μ is a positive linear functional on $\mathcal{A}$, then the left regular representation on $L^2(A, \mu)$ defines a *-representation, ie. each π_a is a bounded linear operator (and $||\pi_a|| \leq ||a||$).

Proof. For any $a, b \in \mathcal{A}$,

$$||\pi_a(b)||^2 = \langle \pi_a(b), \pi_a(b)\rangle_\mu \langle ab, ab\rangle_\mu = \mu(b^*a^*ab) \leq ||b^*a^*ab|| \leq ||b^*||\, ||a^*a||\, ||b||\mu(1) \leq ||a||^2||b||^2\mu(1)$$

□

4.2 Cyclic Representations

Definition 4.4. Let $\mathcal{A}$ be a unital *-Banach algebra and (H, π) a continuous *-representation. Then $\xi \in H$ is said to be (topologically) cyclic if $\{\pi_a\xi \colon a \in \mathcal{A}\}$ is dense in H.

Proposition 4.5. Let (H, π) be a *-representation of $\mathcal{A}$. If $K \subseteq H$ is a closed π-invariant subspace then $K^\perp$ is also π-invariant

Proof. For any $\eta \in K^\perp$, $\xi \in K$, $\langle \xi, \pi_a\eta\rangle = \langle \pi_a^*\xi, \eta\rangle = 0$. □

Given (H, π) a *-representation, for any nonzero $\xi \in H$, $K = \overline{\{\pi_a \colon a \in \mathcal{A}\}}$ is a closed cyclic subspace and we can write H as the sum of closed π-invariant subspaces $H = K \oplus K^\perp$. Iterating this (possibly a transfinite number of times) shows that any *-representation (H, π) can be decomposed into cylic invariant subspaces.

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5.1 Integrated Forms

Let G be a group and U a unitary representation of G. Given $f \in C_c(G)$, define $U_f = \sum f(x)U_x$. Then, $U_fU_g = U_{f*g}$ and $||U_f|| \leq ||f||_1$ so $f \mapsto U_f$ is a continuous *-representation of $\ell^1(G)$. The assignment $f \mapsto U_f$ is called the integrated form corresponding to U. Conversely, if (H, π) is a *-representation of $\ell^1(G)$, then $x \mapsto \pi_{\delta_x}$ gives a unitary representation of G. So, there is a bijection between unitary representations of G and continuous *-representations of $C_c(G)$ or $\ell^1(G)$.

If φ is a continuous positive linear functional on $\ell^1(G)$, then $\varphi \in \ell^\infty(G)$. So to any positive linear functional μ, there is an associated $\mu_f \in \ell^\infty(G)$. We call such a μ_f a function of positive type.

Example 5.1. Let (H, π) be the trivial representation of G on $\mathbb{C}$. Its integrated from is $f \mapsto \sum f(x)$ and so the corresponding function of positive type is $\varphi(x) = 1$ for all x.

Claim: $\delta_e \in \ell^\infty(G)$ is of positive type.

$$\langle f^* * f, \delta_e \rangle = (f^* * f)(e) = \sum \overline{f(y^{-1})} f(y^{-1}e) = \sum |f(y^{-1})|^2 \geq 0$$

What does the GNS construction give?

$$\langle f, g \rangle_{\delta_e} = \sum \overline{g(y)} f(y) = \langle f, g \rangle_{\ell^2(G)}$$

So we recover the left regular representation of G.

5.2 Traces

Definition 5.2. A positive linear functional on a normed *-algebra is tracial (is a trace) if $\mu(ab) = \mu(ba)$ for all $a, b \in \mathcal{A}$.

Claim: δ_e is a trace on $\ell^1(G)$

$$\langle f * g, \delta_e \rangle = \sum f(y) g(y^{-1}) = \sum f(y^{-1}) g(y) = \langle g * f, \delta_e \rangle$$

Let $\mathcal{A}$ be a unital Banach *-algebra, μ a tracial positive linear functional. Let $N_\mu = \{a \colon \langle a, b \rangle_\mu = 0 \ \forall b\}$. Then N_μ is a two sided ideal of $\mathcal{A}$ and $\mathcal{A}/N_\mu$ is an A, A-bimodule. So $L^2(A, \mu)$ admits both left and right regular representations.

The representation $L^2(A, \mu)$ is cylic vector with cyclic vector 1. Any vector ξ gives a positive linear functional via $a \mapsto \langle a\xi, \xi \rangle$ and the postive linear function corresponding to 1 gives back μ. So, any positive linear functional on $\mathcal{A}$ comes from a cyclic vector in some *-representation.

Example 5.3. For the regular representations of $\ell^1(G)$ or $\ell^\infty(G)$ on $\ell^2(G)$, the vector δ_e is cyclic for the representation. It defines a trace on the image of $\ell^1(G)$. Let $C_r(G)$ denote the norm closure of the image of $\ell^1(G)$ in $B(\ell^2(G))$. δ_e defines a trace on this sub C*-algebra, $||\delta_e|| = 1$.

It is not possible to extend this to a trace on $B(H)$ in general since iff H is infinite dimensional, then $B(H)$ admits no boudned trace.

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6.1 Direct Sums of Hilbert Spaces

Given a collection of Hilbert spaces $\{H_j\}_{j \in J}$, we can form the algebraic direct sum as

$$\bigoplus_{\text{alg}} H_j = \{\xi \in \prod H_j \colon \xi_j = 0 \text{ for all but finitely many } j\}$$

This is an inner product space with

$$\langle \xi, \eta \rangle = \sum_j \langle x_j, \eta_j \rangle$$

Taking the completion with respect to this inner product gives the direct sum (coproduct) of the H_j's as Hilbert spaces. One can check that

$$\bigoplus H_j = \{\xi \in \prod H_j \colon \sum ||x_i||_j^2 < \infty\}$$

Given $(T_j)_{j \in J}$ with $T_j \in B(H_j)$, can define $T\xi = (T_j \xi_j)$. If $\{||T_j|| \colon j \in J\}$ is bounded then $T \in B(H)$ with $||T| = \sup_j ||T_j||$. Similarly, if (H_j, π_j) is a family of unitary representations of G (*-representations of a Banach algebra), we can define a *-representation on $\bigoplus H_j$ by $U_x \xi = (U_x^j, \xi_j)$, $\big(\pi_a \xi = (\pi_a^j \xi_j)\big)$. Note that these are bounded representations since $||U_x^j|| = 1$ and $||\pi_a^j|| \leq ||a||$ for each j.

If (H, π) is a *-representation, then it decomposes into a direct sum $\bigoplus (H_j, \pi_j)$ with each (H_j, π_j) a cylic representation.

6.2 Intertwining Operators

Definition 6.1. Suppose (H, π), (K, ρ) are *-representations of $\mathcal{A}$ (unitary reps of G). A morphism of representations is an $\mathcal{A}$ module homomorphism $T : H \to K$, ie. a bounded linear operator $T : H \to K$ so that for each $a \in \mathcal{A}$, $T\pi_a = \rho_a T$. We call such a map an intertwining operator.

Definition 6.2. An isomorphism of representations is an intertwining unitary operator $U : H \to K$.

Definition 6.3. A pointed cylic representation (H, π, ξ_*) is a cylic representation with a specified cylic vector ξ_*.

Proposition 6.4. Let (H, π, ξ_*) and (K, ρ, η_*) be pointed cylic representations. If $\mu_{\xi_*} = \mu_{\eta_*}$ there is a unitary intertwining operator U from H to K so that $U\xi_* = \eta_*$.

Proof. Define U on the dense subspace $\{\pi_a\xi_* : a \in \mathcal{A}\}$ by $U(\pi_a\xi_*) = \rho_a\eta_*$. U then extends uniquely to a continuous map $H \to K$. It is clear that U is intertwining. Further, for any $a, b \in \mathcal{A}$,

$$\langle U\pi_a\xi_*, U\pi_b\xi_*\rangle = \langle \rho_a\eta_*, \rho_b\eta_*\rangle = \langle \rho_b^*\rho_a\eta_*, \eta_*\rangle = \mu_{\eta_*}(b^*a) = \mu_{\xi_*}(b^*a) = \langle \pi_b^*\pi_a\xi_*, \xi_*\rangle = \langle \pi_a\xi_*, \pi_b\xi_*\rangle$$

So U is unitary as well. □

It follows that there is a bijection between positive linear functions on $\mathcal{A}$ and isomorphism classes of pointed cylic representations.

Definition 6.5. A *-representation (H, π) irreducible if it has no proper closed π-invariant subspaces.

Given a finite dimensional representation, it can be decomposed into a sum of irreducible representations.

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7.1 Schur's Lemma

Definition 7.1. Let H be a Hilbert space and $\mathcal{C} \subseteq B(H)$. The commutant of $\mathcal{C}$ is defined to be

$$\mathcal{C}' = \{T \in B(H) : TC = CT\, \forall C \in \mathcal{C}\}$$

The commutant is closed with respect to the strong operator topology and so is a Von Neumman algebra (and hence a C^*-algebra). For any $\mathcal{C}$, $(\mathcal{C}')'$ is the strong closed *-algebra generated by $\mathcal{C}$. The double commutant theorem was proved by Von Neumman in the first paper on operator algebras.

If (H, π) is a representation of $\mathcal{A}$, then $\mathrm{End}_{\mathcal{A}}(H)$ is the commutant of $\{\pi_a\}_{a\in\mathcal{A}}$.

Lemma 7.2 (Schur's Lemma)**.** A representation (H, π) of $\mathcal{A}$ is irreducible if and only if $\mathrm{End}_{\mathcal{A}}(H) = \mathbb{C}$.

Proof. If (H, π) is not irreducible then there is a proper closed invariant subspace $K \subseteq H$. Taking P to be the orthogonal projection onto K, we have that $K \in \mathrm{End}_{\mathcal{A}}(H)$ but K is not a scalar multiple of the identity. Conversely, if $\mathrm{End}_{\mathcal{A}}(H) \neq \mathbb{C}$ then let T be an operator which is not a scalar multiple of the identity. Since either the real or imaginary part of T is also not a scalar multiple of the identity we may assume that T is self adjoint without loss of generality. There are $s \neq t \in \sigma(T)$ and so let f, g be functions so that $f(s) = g(t) = 1$ and $fg = 0$. This $F, G \in C^*(I, T)$ so that $F, G \neq 0$ and $FG = 0$. Now, $\langle FH, GH\rangle = 0$ and $GH \neq 0$ so, since $F \in \mathrm{End}_{\mathcal{A}}(H)$, $\overline{FH}$ is a proper invariant subspace. □

Corollary 7.3. If $\mathcal{A}$ is a commutative unital normed *-algebra, then every irreducible representation of $\mathcal{A}$ is one dimensional.

Proof. If $\mathcal{A}$ is commutative, then for representation (H, π), $\pi_a \in \mathrm{End}_{\mathcal{A}}(H)$ for $a \in \mathcal{A}$. So if the representation is irreducible it follows that $\pi_a \in \mathbb{C}I$ for each A and so every subspace is invariant. Thus, H must be one dimensional. □

Example 7.4. If X is a compact Hausdorff space, then the irreducible representations of X are precisely evaluations at points in x. More generally, if $\mathcal{A}$ is a commutative *-Banach algebra, then its irreducible representations are exactly the characters on $\mathcal{A}$.

Example 7.5. Consider the algebra $\mathcal{A} = \mathbb{C} \oplus \mathcal{C}$ with $(\alpha, \beta)^* = (\overline{\beta}, \overline{\alpha})$. There are no nonzero $*$-algebra homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ since $(1, 0)$ generates it as a $*$-algebra but $(1, 0)(1, 0)^* = 0$ and so $\mu(1, 0) = 0$ for any $\mu : \mathcal{A} \to \mathbb{C}$.

7.2 Radon-Nikodym for Positive Linear Functionals

Definition 7.6. Let $\mathcal{A}$ be a unital *-algebra and let μ, ν be positive linear functionals on $\mathcal{A}$. Say that μ dominates ν, $\mu \geq \nu$ if $\mu - \nu \geq 0$.

How do we get such a ν? Given μ, consider the pointed representation $(H_\mu, \pi_\mu, \xi_\mu)$. For any $T \in \mathrm{End}_{\mathcal{A}}(H)$ with $0 \leq T \leq I$, define

$$\nu(a) = \langle \pi_a T \xi_\mu, \xi_\mu \rangle$$

We have that

$$\nu(a^*a) = \langle T\pi_a\xi_\mu, \pi_a\xi_\mu \rangle \quad \text{and} \quad (\mu - \nu)(a^*a) = \langle (I - T)\pi_a\xi_\mu, \xi_\mu \rangle$$

and so $0 \leq \nu \leq \mu$. Here T is the "Radon-Nidokdym derivative" of ν.

We will show that all such ν arise in this way.

Theorem 7.7. If $\mu \geq \nu \geq 0$, then $\exists T \in \mathrm{End}_{\mathcal{A}}(H_\mu)$, $0 \leq T \leq I$ such that $\nu(a) = \langle T\pi_\mu(a)\xi_\mu, \xi_\mu \rangle$.

Fact: Let H be a Hilbert Space, K a dense subspace, and $\langle \, , \rangle_K$ a pre-inner product on K so that $\langle \eta, \eta \rangle_K \leq \langle \eta, \eta \rangle_H$ for $\eta \in K$. Then, there is an operator $T \in B(H)$ with $0 \leq T \leq I$ and so that $\langle \xi, \eta \rangle_K = \langle T\xi, \eta \rangle$ for $\xi, \eta \in K$.

Proof. Given $\eta \in K$, consider the conjugate-linear functional given by $\varphi_\eta(\xi) = \langle \eta, \xi \rangle_K$. φ_η extends uniquely to H and so there is some ξ_η so that $\varphi_\eta(\xi) = \langle \xi_\eta, \xi \rangle$ for all $\xi \in K$. Consider the linear map defined on K by $T\eta = \xi_\eta$. Note that T is continuous since $||\xi_\eta|| = ||\eta||_K \leq ||\eta||_H$ and so extends uniquely to an element in $B(H)$ satisfying the desired conditions. □

Proof of Theorem. Define an inner product on H by $\langle \pi_\mu(a)\xi_\mu, \pi_\mu(b)\xi_\mu \rangle_\nu = \nu(b^*a)$ then take T to be as above. It remains to check that $T \in \mathrm{End}_{\mathcal{A}}(H)$. For any $a, b \in \mathcal{A}$, we compute that

$$\langle T\pi_\mu(b)\pi_\mu(a)\xi_\mu, \xi_\mu \rangle = \langle T\pi_\mu(ba)\xi_\mu, \xi_\mu \rangle = \nu(ba) = \langle T\pi_\mu(a)\xi_\mu, \pi_\mu(b^*)\xi_\mu \rangle = \langle \pi_\mu(b)T\pi_\mu(a)\xi_\mu, \xi_\mu \rangle$$

□

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8.1 Pure States

Definition 8.1. A positive linear functional μ on $\mathcal{A}$ is pure if whenever $\mu \geq \nu \geq 0$, $\mu = r\nu$ for some $r \in [0, 1]$.

Proposition 8.2. A positive linear functional μ is pure if and only if the corresponding representation (H_μ, π_μ) is irreducible.

Proof. If (H_μ, π_μ) is irreducible, then by Schur's lemma $\mathrm{End}_{\mathcal{A}}(H) = \mathbb{C}$. So, whenever $0 \leq \nu \leq \mu$, the derivative of ν is equal to rI for some $r \in [0, 1]$ and so $\nu = r\mu$. Conversely, If (H_μ, π_μ) has a closed proper invariant subspace, let P be the projection onto this subspace. Then $0 \leq P \leq I$ and so, taking ν to be the corresponding positive linear functional, it follows that $0 \leq \nu \leq \mu$ but $\nu \neq r\mu$. □

Definition 8.3. A state on $\mathcal{A}$ is a positive linear functional μ with $\mu(1) = ||1|| = ||\mu||$.

Let $\mathcal{S}(\mathcal{A})$ denote the collection of states on $\mathcal{A}$, called the state space. $\mathcal{S}(\mathcal{A}) \subseteq B_1(\mathcal{A}')$ and is convex. It is also closed for the weak *-topology and so is compact.

Remark 8.4. Physicists call states "mixed states" and use "states" to refer to pure states.

Proposition 8.5. A state $\mu \in \mathcal{S}(A)$ is pure if and only if it is an extreme point of $\mathcal{S}(A)$.

Proof. If μ is not an extreme point then there are $\nu_1, \nu_2 \neq \mu$, $t \in (0, 1)$ so that $\mu = t\nu_1 + (1 - t)\nu_2$. So $\mu \geq t\nu_1$ but $t\nu_1 \neq r\mu_1$. So μ is not pure. Conversely, if μ is an extreme point and $0 \leq \nu \leq \mu$, then

$$\mu = ||\nu|| \left(\frac{\nu}{||\nu||} \right) + ||\mu - \nu|| \left(\frac{\mu - \nu}{||\mu - \nu||} \right)$$

is a convex combination of states and so $||\nu||\mu = \nu$. Hence, μ is pure. □

Corollary 8.6. For $\mu \in \mathcal{S}(A)$, (H_μ, π_μ) is irreducible if and only if μ is an extreme point of $\mathcal{S}(A)$.

Theorem 8.7 (Krein-Milman)**.** Every compact, convex subset of a Banach space is equal to the closed convex hull of its extreme points.

For any representation (H, π) and $\xi \in H$ with $||\xi|| = 1$ state, we can define a state by $\mu_\xi(a) = \langle \pi(a)\xi, \xi\rangle$. Call such a state a vector state.

Example 8.8. Let G be a discrete group, and consider the left regular representation of $\ell^1(G)$ on $\ell^2(G)$. There is $\xi \in \ell^2(G)$ so that if $x \neq e$, $\lambda_x\xi \neq \xi$. There are lots of vector states in $\mathcal{S}(A)$ so it follows that there are lots of irreducible representations. In particular, for any $x \in G$ there is an irreducible representation (H, U) so that $U_x \neq I_H$ (Gelfand-Raikov Theorem). For $f \in \ell^1(G)$, let $||f|| = \sup\{||\pi_\mu(f)||\colon \mu \text{ pure state}\}$. Then $||f|| = ||\lambda_f||$ as an operator on $\ell^2(G)$.

9 2-9

9.1 Constructing Positive Linear Functionals

Given a positive linear functional on a unital C*-algebra $\mathcal{A}$, we have seen how to construct a representation. Our next goal is to answer the following question:
Q: If $\mathcal{A}$ is an abstract unital C*-algebra, why does it have any positive linear functionals?

Using the Riesz-representation theorem, we can produce many such functionals for commutative C*-algebras. More generally, suppose $\mathcal{A}$ is a C*-algebra and $\mathcal{B}$ a unital commutative sub C*-algebra. Then $B \cong C(X)$ for some compact Hausdorff space X, and so if μ_0 is a probability measure on X, it defines a linear functional on $C(X)$ with $\mu(1) = 1 = ||\mu_0||$. Using Hahn-Banach, μ_0 can be extended to μ a linear functional on $\mathcal{A}$. However, how do we know that such an extension is a state?

First, we may assume that the extension is a *-algebra homomorphism by replacing μ with

$$\tilde{\mu}(a) = \frac{\mu(a) + \mu(a^*)}{2} + i\,\frac{\mu(a) - \mu(a^*)}{2}$$

Then $\tilde{\mu}(a^*) = \overline{\tilde{\mu}(a)}$, $\tilde{\mu} \restriction \mathcal{B} = \mu_0 \restriction \mathcal{B}$, and $||\tilde{\mu}|| = \tilde{\mu}(1) = 1$. So it remains to check that μ is positive.

If $\mathcal{C}$ is a commutative unital C*-algebra, $\mathcal{C} \cong C(Y)$ and so $\mu \restriction_{\mathcal{C}}$ arises from a Radon measure on Y. Taking the Jordan decomposition gives $\mu \restriction_{\mathcal{C}} = (\mu \restriction_{\mathcal{C}})^+ - (\mu \restriction_{\mathcal{C}})^-$. Since

$$1 = \mu(1) = ||\mu \restriction_{\mathcal{C}}|| = ||(\mu \restriction_{\mathcal{C}})^+|| + ||(\mu \restriction_{\mathcal{C}})^-||$$

it follows that $(\mu \restriction_{\mathcal{C}})^- = 0$. So, if $\mathcal{A}$ is a unital C*-algebra, it has lots of linear functionals whose restriction to every commutative unital subalgebra is a state of that subalgebra.

In particular, for normal elements, we have that $\mu(a^*a) \geq 0$ but why is this true for general a?

Definition 9.1. For any C*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$, we say that a is positive if $a^* = a$ and $\sigma(a) \subseteq \mathbb{R}_{\geq 0}$.

If $a \geq 0$ then, restricting to the commutative C*-algebra generated by a, it follows that $\mu(a) \geq 0$. So to show that $\mu(a^*a) \geq 0$ for any a, it suffices to show that a^*a is positive.

Proposition 9.2. If $a \in \mathcal{A}$ a unital C*-algebra, then $a^*a \geq 0$.

Proof. We have $C^*(a^*a, 1) \cong C(\sigma(a^*a))$. Note that we are taking for granted that the spectrum of an element is invariant when passing to a sub C*-algebra, but this will be shown later.
Suppose towards a contradiction that $\sigma(a^*a) \not\subseteq \mathbb{R}_{\geq 0}$, then there is some $c \in C(\sigma(a^*a))$ with $c \geq 0$ and so that $\sigma(ca^*a) \subseteq \mathbb{R}_{\leq 0}$. Let $d = (ca)$ so that $\sigma(d^*d) \subseteq \mathbb{R}_{\leq 0}$.

Write $d = h + ik$ with h, k self adjoint. Then

$$d^*d + dd^* = (h - ik)(h + ik) + (h + ik)(h - ik) = 2(h^2 + k^2)$$

so that $dd^* = 2(h^2 + k^2) - d^*d$.

To complete the proof, we will need two facts: that the sum of two positive elements is positive, and that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$ for any a, b. Since both $2(h^2 + k^2)$ and $-d^*d$ are positive, this would imply dd^* is positive. However $\sigma(dd^*) \cup \{0\} = \sigma(d^*d) \cup \{0\} \subseteq \mathbb{R}_{\leq 0}$, a contradiction. ☐

Proposition 9.3. If $\mathcal{A}$ is a unital C*-algebra and if $a, b \in \mathcal{A}$ are so that $a, b \geq 0$, then $a + b \geq 0$.

For any C*-algebra, its norm is completely determined by its *-algebraic structure:

$$||a||^2 = ||a^*a|| = r(a^*a)$$

The next claim shows that the order structure on $\mathcal{A}^{sa}$ is completely determined by norm.
Claim: For $a^* = a$, $a \geq 0$ if and only if $\exists t \geq ||a||$ with $||a - t1|| \leq t$ if and only if $\forall t \geq ||a|| \quad ||a - t1|| \leq t$.

Proof. It suffices to show this for $\mathcal{A} = C(X)$. Given $f \in C_{\mathbb{R}}(X)$ and $t \geq ||f||$, $||f - t1|| \leq t$ iff $|f(x) - t| \leq t$ for all x iff $f(x) \geq 0$ for all x. □

Proof of Proposition (Kelley-Vaught, 1953).
Suppose $a, b \geq 0$ and let $s = ||a||$, $t = ||b||$. Then $||a - s1|| \leq s$ and $||b - t1|| \leq t$, so

$$||(a + b) - (t + s)1|| \leq ||a - s1|| + ||b - t1|| \leq s + t$$

Since, $||a + b|| \leq s + t$, it follows that $a + b \geq 0$. □

10 2-11

10.1 The Gelfand-Naimark Theorem

Theorem 10.1. Let $\mathcal{A}$ be any unital algebra over any field. Then for $a, b \in \mathcal{A}$, $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

Proof. For $\lambda \neq 0$, $(ab - \lambda 1) = \lambda(\lambda^{-1}ab - 1)$ so it suffices to show that $1 - ab$ is invertible whenever $1 - ba$ is. Writing

$$(1 - ab)^{-1} = \sum_{n=0}^{\infty}(ab)^n = 1 + a\left(\sum_{n=0}^{\infty}(ba)^n\right)b = 1 + a(1 - ba^{-1})b$$

Hence, if $1 - ba$ is invertible, it should be that $(1 - ab)^{-1} = 1 + a(1 - ba)^{-1}b$. An easy computation shows that this is true. □

Remark 10.2. We give a brief historical account of connections to physics:

- Heisenberg, Schrodinger (1926): Gave the first way to model quantum physics
- Heisenberg: Observables should be modeled by self adjoint operators on a Hilbert Space.

  Let P model momentum and Q model position. Then

  $$PQ - QP = ihI_H$$

  where h is Planck's constant.
- Schrodinger: Let $H = L^2(\mathbb{R})$ and define

  $$P(\xi)(t) = t\xi(t) \qquad Q(\xi)(t) = ih\frac{d}{dt}\xi(t)$$

  so that we again have $PQ - QP = ihI_H$.

  Note that the operators P and Q above are not bounded so we must restrict their domain.

If $\mathcal{A}$ is a unital Banach algebra, then there are no $a, b \in \mathcal{A}$ so that $ab - ba = \lambda 1$ as this would imply $\sigma(ba + \lambda I) = \sigma(ba) + \lambda$ while $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$, which is impossible. This shows that using unbounded operators is necessary to model the above situation.

Example 10.3. Consider $\ell^2(\mathbb{N})$. Define S via $(S\xi)_n = x_{n+1}$. Then $(S^*\xi)_n = \xi_{n-1}$ for $n > 0$ and 0 otherwise. We have that $S^*S = I$ but $SS^* = I - \pi_0$ and so $\sigma(S^*S) \neq \sigma(SS^*)$.

A unital C*-algebra $\mathcal{A}$ is called finite if there is no a so that $a^*a = 1$ and $aa^* \neq 1$.

We are finally ready to complete the proof of the Gelfand-Naimark theorem.

Theorem 10.4. Let $\mathcal{A}$ be an abstract unital C*-algebra. Then $\mathcal{A}$ is isometrically isomorphic to a C*-subalgebra of bounded operators on some Hilbert space H.

Proof. For each $\mu \in \mathcal{S}(\mathcal{A})$, form (H_μ, π_μ) and let $H = \bigoplus_{\mu \in \mathcal{S}(\mathcal{A})}(H_\mu, \pi_\mu)$. We show this representation of $\mathcal{A}$ is isometric. For each a, $||\pi(a)|| = \sup_\mu ||\pi_\mu(a)|| \leq ||a||$. This supremum is achieved since for any $a \in \mathcal{A}$, there is a state μ_0 on $C^*(a^*a, 1)$ so that $\mu_0(a^*a) = ||a^*a||$. We can then extend μ_0 to a state μ on $\mathcal{A}$ and for this μ we will have

$$||\pi_\mu(a)||^2 = ||\pi_\mu(a^*a)|| = \mu(a^*a) = ||a^*a|| = ||a||^2$$

□