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1 1-21

1.1 Motivation

Before starting the development of the general theory of C^* -algebras we consider a few motivating examples.

Example 1.1. Consider the Hilbert spaces $\ell^2(\mathbb{Z})$, $L^2(\mathbb{R})$ and define unitary operators U, V by $(U\xi)(r) = \xi(r-1)$ and $(V\xi)(r) = e^{2\pi i \theta r} \xi(r)$ for $\theta \in \mathbb{R}$. Consider the algebra generated by these two operators. $VU = e^{2\pi i \theta} UV$ so the linear span of $U^m V^n$ is an algebra. Denote its operator norm closure by \mathcal{A}_θ .

The algebras generated by U and V individually are both isomorphic to $C(\mathbb{S}^1)$ and if $\theta \in \mathbb{Z}$, U and V commute so $\mathcal{A}_\theta \cong C(\mathbb{S}^1 \times \mathbb{S}^1)$. Otherwise, we call \mathcal{A}_θ a noncommutative 2-torus.

Consider the self adjoint operator, called an almost Matthieu operator

$$H = (U + U^*) + \lambda(V + V^*)$$

Q: What is the spectrum of H ?

- For rational θ , it is a disjoint union of intervals
- 1976: Hofstader plotted the spectra for rational θ to produce a fractal like pattern – the Hofstader Butterfly. This led to the conjecture that the spectrum of H for irrational θ was a Cantor set.
- 1981: Mark Kat offered 10 martinis for an answer to this conjecture.
- Barry Simon popularized this and it became known as the 10 martini problem
- 2005: Jitomirskaya, Avila gave an affirmative answer

Much of the partial progress on this problem came from the theory of C^* -algebras but the final proof used completely different techniques.

Example 1.2. Let G be a group. A (Hilbert Space) unitary representation of G is a group homomorphism $\pi : G \rightarrow \mathcal{U}(H)$. Consider the algebra of operators generated by $\{\pi(x) : x \in G\}$, ie. elements of the form $\pi_f = \sum_{x \in G} f(x)\pi(x)$ for $f : G \rightarrow \mathbb{C}$ finnitely supported. We define a product on this set by

$$(\sum f(x)\pi(x))(\sum g(y)\pi(y)) = \sum_{x,y} f(x)g(y)\pi(xy) = \sum_{x,y} f(x)g(x^{-1}y)\pi(y) = \sum_y (\sum_x f(x)g(x^{-1}y))\pi(y)$$

Defining the convolution $*$ by

$$(f * g)(y) = \sum_x f(x)g(x^{-1}y)$$

we have the $\pi_f \cdot \pi_g = \pi_{f*g}$ and this gives an algebra structure on $C_c(G)$ and an algebra homomorphism $\pi : (C_c(G), *) \rightarrow B(H)$. The convolution also gives $\ell^1(G)$ an algebra structure since $\|f*g\|_1 \leq \|f\|_1\|g\|_1$.

For any $f \in C_c(G)$, let $\|f\|_{C^*(G)} = \sup\{\pi_f : \pi \text{ unitary rep of } G\}$. Observe that

$$\|\pi_f\| \leq \sum |f(x)|\|\pi(x)\| = \sum |f(x)| = \|f\|_1$$

and so $\|f\|_{C^*(G)} \leq \|f\|_1$. Let $C^*(G)$ denote the completion of $C_c(G)$ with respect to this norm. We can represent this algebra as an algebra bounded operators on some Hilbert space.

Note: for any group G we have at least two unitary representations – the trivial representation and the left regular representation on $\ell^2(G)$ by translation.

2 1-23

Example 2.1. Using the left regular representation: $H = \ell^2(G)$ with $\mathcal{U}_x(\xi)(y) = \xi(x^{-1}y)$, we can define a norm on $C_c(G)$ by $\|f\|_r = \|U_f\|$. Note that the involution is given by $f^*(x) = f(-x)$. The closure of $C_c(G)$ under this norm is called the reduced C^* -algebra for G . Since $\|f\|_r \leq \|f\|_{C^*}$, we only have that $C_r^*(G)$ is a quotient of $C^*(G)$ in general. In fact, the discrepancy between the two encodes information about the group G :

Theorem 2.2. $C^*(G) = C_r^*(G)$ if and only if G is amenable.

Example 2.3. For X a compact Hausdorff space, we can consider a homomorphism $\alpha : G \rightarrow \text{Homeo}(x)$ (eg. a dynamical system). This induces a map $\alpha : G \rightarrow \text{Aut}(C(X))$. We can often use this map to give insight into the behavior of the dynamical system as passing to the level of algebras gives more structure to leverage.

More generally, for a C^* -algebra \mathcal{A} , we call a map $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ a “noncommutative dynamical system”. We will look at representations of these systems: $U : G \rightarrow \mathcal{U}(H)$, $\pi : \mathcal{A} \rightarrow B(H)$ satisfying $\pi(\alpha_x(a)) = U_x\pi(a)U_x^{-1}$ (the covariance relation). The algebra generated by $(U_x, \pi(a))$ is called the crossproduct algebra.

Example 2.4. Let $X = \mathbb{S}^1$ and a^θ be rotation by θ .

Ex: $X = \mathbb{S}^1$, a^θ = rotation by θ . Then \mathbb{Z} acts by $(a^\theta)^n$ and the crossproduct algebra is \mathcal{A}_θ .

Remark 2.5. The construction of $C^*(G)$ also works for groupoids

If X, Y are compact Hausdorff spaces, how are $C(X \times Y)$ and $C(X), C(Y)$ related. Consider functions of the form $(f \otimes g)(x, y) = f(x)g(y)$. The algebra generated by these functions is dense in $C(X \times Y)$ by Stone Weierstrass.

Example 2.6. Can from algebra $\mathcal{A} \oplus \mathcal{A} \oplus \dots$

Consider direct limit of algebras $M_2 \hookrightarrow M_4 \hookrightarrow M_8 \dots$ with isometric embeddings $T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$.

Called the CAR-algebra (canonical anticommutative relation).

Q: for different n , are the C^* -algebras generated by $M_n \rightarrow M_{n^2} \dots$ nonisomorphic?

Yes (the proof uses methods from noncommutative geometry)

However, the Von-Neumann algebras generated by different n are all isomorphic.

Definition 2.7. A concrete C^* -algebra is a subalgebra of $B(H)$, for H a Hilbert space, which is norm closed and closed under $*$.

An abstract C^* -algebra is a Banach algebra with a conjugate linear involution $*$ satisfying $\|A^*A\| = \|A\|^2$.

By the Gelfand-Naimark theorem (1943), every abstract C^* -algebra is isomorphic to a concrete C^* -algebra.

Little Gelfand-Naimark: If \mathcal{A} is a commutative C^* -algebra then $\mathcal{A} \cong C_0(X)$ for X LCH.

If \mathcal{A} is a commutative unital C^* -algebra over \mathbb{C} , define $\hat{\mathcal{A}} = \{\varphi : \mathcal{A} \rightarrow \mathbb{C}, \text{ unital hom}\}$. For each $a \in \mathcal{A}$, define \hat{a} on $\hat{\mathcal{A}}$ by $\hat{a}(\varphi) = \varphi(a)$. The map $a \mapsto \hat{a}$ is a unital algebra homomorphism from \mathcal{A} to functions on $\hat{\mathcal{A}}$.

Definition 2.8. For $a \in \mathcal{A}$, define the spectrum of a , $\sigma(a) = \{\lambda : a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}$.

If \mathcal{A} is a Banach algebra, then $(1 - a)^{-1} = \sum a^n$ if $\|a\| < 1$ and so if $\lambda \in \sigma(a)$ then $|\lambda| < \|a\|$. Observe that if $\varphi \in \hat{\mathcal{A}}$ then $\varphi(a) \in \sigma(a)$ since $\varphi(a - \varphi(a)1) = 0$ and so $a - \varphi(a)1$ is not invertible. For $a \in \mathcal{A}$, $\hat{a} \in C(\hat{\mathcal{A}})$, $a \mapsto \hat{a}$ called the Gelfand transform. Gelfand spectral radius formula: $\sup\{|\lambda| : \lambda \in \sigma(a)\} = \lim \|a^n\|^{1/n}$. So, if $\|a^2\| = \|a\|^2$, then Gelfand transform is isometric.

3 1-26

3.1 The GNS Construction

Definition 3.1. An algebra involution $*$ is a conjugate linear map so that $(ab)^* = b^*a^*$ and $(a^*)^* = a$.

Given $C(X)$ a unital C^* -algebra, how do we realize it as bounded operators on a Hilbert space H ?

Suppose μ is a positive measure, viewed as a linear functional. Define a pre-inner product by $\langle f, g \rangle = \int f\bar{g}d\mu = \mu(f\bar{g})$. Quotienting out by norm zero elements, gives a Hilbert space on which $C(X)$ acts by pointwise multiplication.

Example 3.2. Consider $\delta_x(f) = f(x)$. Then $C(X) \subseteq \ell^\infty(X_{\text{disc}})$ acts on $\ell^2(X_{\text{disc}})$. This is called the atomic representation.

We would like to this construction on a general unital $*$ -algebra.

Definition 3.3. Let μ be a linear functional on \mathcal{A} a unital $*$ -algebra. Say that μ is positive if $\mu(a^*a) \geq 0$ for $a \in \mathcal{A}$.

The positive linear functionals are closed under addition and multiplication by nonnegative scalars (ie. they form a cone)

Let μ be a positive linear function on \mathcal{A} . Define a pre-inner product on \mathcal{A} by $\langle a, b \rangle_\mu = \mu(b^*a)$. To see that this is a pre-inner product observe that

$$\langle a, a \rangle_\mu = \mu(a^*a) \geq 0 \quad \text{and} \quad \langle a + b, a + b \rangle_\mu = \langle a, a \rangle_\mu + \langle a, b \rangle_\mu + \langle b, a \rangle_\mu + \langle b, b \rangle_\mu \geq 0$$

so $0 = \text{Im}(\langle a, b \rangle_\mu) + \text{Im}(\langle b, a \rangle_\mu)$ and, replacing b with ib , gives $\text{Re}(\langle a, b \rangle_\mu) = \text{Re}(\langle b, a \rangle_\mu)$.

Let $N_\mu = \{a \in \mathcal{A} : \langle a, a \rangle_\mu = 0\}$. If $a \in N_\mu$, then $|\langle a, b \rangle_\mu|^2 \leq \langle a, a \rangle_\mu \langle b, b \rangle_\mu = 0$ so $N_\mu = \{a \in \mathcal{A} : \langle a, b \rangle_\mu = 0 \ \forall b\}$. It follows that N_μ is a left ideal of \mathcal{A} . On \mathcal{A}/N_μ the pre-inner product becomes an inner product. Denote its completion by $L^2(\mathcal{A}, \mu)$.

Let \mathcal{A} act on itself on the left via $\pi_a b = ab$. This is a $*$ -algebra homomorphism since

$$\langle \pi_a b, c \rangle_\mu = \langle ab, c \rangle_\mu = \mu(c^*ab) = \mu((a^*c)^*b) = \langle b, a^*c \rangle_\mu = \langle b, \pi_{a^*}c \rangle$$

Call this the left regular representation. Since N_μ is a left ideal of \mathcal{A} , this gives an action of \mathcal{A} on \mathcal{A}/N_μ as well. It remains to check that the π_a 's are continuous. This may fail without a complete norm as the next example shows.

Example 3.4. Let \mathcal{A} be the algebra of \mathbb{C} -valued polynomials on \mathbb{R} , and let $\mu(p) = \int_{-\infty}^{\infty} p(t)e^{-t^2} dt$. On $L^2(\mathcal{A}, \mu)$, multiplying by polynomials in an unbounded operator.

The above is called the GNS (Gelfand-Naimark-Segal) construction

Theorem 3.5. Let \mathcal{A} be a unital normed $*$ -algebra, μ a positive linear functional on \mathcal{A} .

- (1) If \mathcal{A} is complete, then μ is continuous and $\|\mu\| = \mu(1)$
- (2) If \mathcal{A} is not necessarily complete but μ is continuous then $\|\mu\| = \mu(1)$.

4 1-28

4.1 Automatic Continuity

Lemma 4.1. Let \mathcal{A} be a unital *-Banach algebra over \mathbb{C} . If $a \in \mathcal{A}$, $a^* = a$, and $\|a\| < 1$, then there is $b \in \mathcal{A}$ with b^*b and $1 - a = b^2$.

Proof. The function $f(z) = \sqrt{1-z}$, has a power series expansion $f(z) = \sum r_n z^n$, $r_n \in \mathbb{R}$ which converges for $|z| < 1$. Setting $b = \sum r_n a^n$, it follows that $b^2 = (f(a))^2 = (f^2)(a) = 1 - a$. \square

Theorem 4.2. If μ is a positive linear function on \mathcal{A} , then μ is continuous and $\|\mu\| = \mu(1)$.

Proof. Suppose $a \in \mathcal{A}$ with $a^* = a$ and $\|a\| < 1$. Then for b so that $1 - a = b^*b$, we have that

$$\mu(b^*b) = \mu(1) - \mu(a) \geq 0$$

and so $\mu(1) \geq \mu(a)$. It follows that if $a^* = a$, $\mu(a) \leq \|a\|\mu(1)$. For general a ,

$$|\mu(a)|^2 = |\mu(a1)|^2 = |\langle 1, a \rangle_\mu|^2 \leq \mu(1)\mu(a^*a) \leq \mu(1)^2\|a^*a\| \leq \mu(1)^2\|a\|^2$$

\square

If (H, π) is a *-representation of \mathcal{A} , then for any $\xi \in H$, set $\mu_\xi(a) = \langle \pi(a)\xi, \xi \rangle$. This defines a positive linear functional with $\|\mu_\xi\| = \|\xi\|^2$. We also have that

$$\|\pi(a)\xi\|^2 = \langle \pi(a)\xi, \pi(a)\xi \rangle = \langle \pi(a^*a)\xi, \xi \rangle \leq \|a^*a\| \|\xi\|^2 \leq \|a\|^2 \|\xi\|^2$$

and so $\|\pi(a)\| \leq \|a\|_{\mathcal{A}}$.

Theorem 4.3. If μ is a positive linear functional on \mathcal{A} , then the left regular representation on $L^2(\mathcal{A}, \mu)$ defines a *-representation, ie. each π_a is a bounded linear operator (and $\|\pi_a\| \leq \|a\|$).

Proof. For any $a, b \in \mathcal{A}$,

$$\|\pi_a(b)\|^2 = \langle \pi_a(b), \pi_a(b) \rangle_\mu \langle ab, ab \rangle_\mu = \mu(b^*a^*ab) \leq \|b^*a^*ab\| \leq \|b^*\| \|a^*a\| \|b\| \mu(1) \leq \|a\|^2 \|b\|^2 \mu(1)$$

\square

4.2 Cyclic Representations

Definition 4.4. Let \mathcal{A} be a unital *-Banach algebra and (H, π) a continuous *-representation. Then $\xi \in H$ is said to be (topologically) cyclic if $\{\pi_a\xi : a \in \mathcal{A}\}$ is dense in H .

Proposition 4.5. Let (H, π) be a *-representation of \mathcal{A} . If $K \subseteq H$ is a closed π -invariant subspace then K^\perp is also π -invariant

Proof. For any $\eta \in K^\perp$, $\xi \in K$, $\langle \xi, \pi_a\eta \rangle = \langle \pi_a^*\xi, \eta \rangle = 0$. \square

Given (H, π) a *-representation, for any nonzero $\xi \in H$, $K = \overline{\{\pi_a : a \in \mathcal{A}\}}$ is a closed cyclic subspace and we can write H as the sum of closed π -invariant subspaces $H = K \oplus K^\perp$. Iterating this (possibly a transfinite number of times) shows that any *-representation (H, π) can be decomposed into cyclic invariant subspaces.

5 1-30

5.1 Integrated Forms

Let G be a group and U a unitary representation of G . Given $f \in C_c(G)$, define $U_f = \sum f(x)U_x$. Then, $U_f U_g = U_{f*g}$ and $\|U_f\| \leq \|f\|_1$ so $f \mapsto U_f$ is a continuous *-representation of $\ell^1(G)$. The assignment $f \mapsto U_f$ is called the integrated form corresponding to U . Conversely, if (H, π) is a *-representation of $\ell^1(G)$, then $x \mapsto \pi_{\delta_x}$ gives a unitary representation of G . So, there is a bijection between unitary representations of G and continuous *-representations of $C_c(G)$ or $\ell^1(G)$.

If φ is a continuous positive linear functional on $\ell^1(G)$, then $\varphi \in \ell^\infty(G)$. So to any positive linear functional μ , there is an associated $\mu_f \in \ell^\infty(G)$. We call such a μ_f a function of positive type.

Example 5.1. Let (H, π) be the trivial representation of G on \mathbb{C} . Its integrated from is $f \mapsto \sum f(x)$ and so the corresponding function of positive type is $\varphi(x) = 1$ for all x .

Claim: $\delta_e \in \ell^\infty(G)$ is of positive type.

$$\langle f^* * f, \delta_e \rangle = (f^* * f)(e) = \sum \overline{f(y^{-1})} f(y^{-1}e) = \sum |f(y^{-1})|^2 \geq 0$$

What does the GNS construction give?

$$\langle f, g \rangle_{\delta_e} = \sum \overline{g(y)} f(y) = \langle f, g \rangle_{\ell^2(G)}$$

So we recover the left regular representation of G .

5.2 Traces

Definition 5.2. A positive linear functional on a normed $*$ -algebra is tracial (is a trace) if $\mu(ab) = \mu(ba)$ for all $a, b \in \mathcal{A}$.

Claim: δ_e is a trace on $\ell^1(G)$

$$\langle f * g, \delta_e \rangle = \sum f(y) g(y^{-1}) = \sum f(y^{-1}) g(y) = \langle g * f, \delta_e \rangle$$

Let \mathcal{A} be a unital Banach $*$ -algebra, μ a tracial positive linear functional. Let $N_\mu = \{a: \langle a, b \rangle_\mu = 0 \ \forall b\}$. Then N_μ is a two sided ideal of \mathcal{A} and \mathcal{A}/N_μ is an A, A -bimodule. So $L^2(A, \mu)$ admits both left and right regular representations.

The representation $L^2(A, \mu)$ is cyclic vector with cyclic vector 1. Any vector ξ gives a positive linear functional via $a \mapsto \langle a\xi, \xi \rangle$ and the positive linear function corresponding to 1 gives back μ . So, any positive linear functional on \mathcal{A} comes from a cyclic vector in some $*$ -representation.

Example 5.3. For the regular representations of $\ell^1(G)$ or $\ell^\infty(G)$ on $\ell^2(G)$, the vector δ_e is cyclic for the representation. It defines a trace on the image of $\ell^1(G)$. Let $C_r(G)$ denote the norm closure of the image of $\ell^1(G)$ in $B(\ell^2(G))$. δ_e defines a trace on this sub C*-algebra, $\|\delta_e\| = 1$.

It is not possible to extend this to a trace on $B(H)$ in general since iff H is infinite dimensional, then $B(H)$ admits no bounded trace.

6 2-2

6.1 Direct Sums of Hilbert Spaces

Given a collection of Hilbert spaces $\{H_j\}_{j \in J}$, we can form the algebraic direct sum as

$$\bigoplus_{\text{alg}} H_j = \{\xi \in \prod H_j : \xi_j = 0 \text{ for all but finitely many } j\}$$

This is an inner product space with

$$\langle \xi, \eta \rangle = \sum_j \langle x_j, \eta_j \rangle$$

Taking the completion with respect to this inner product gives the direct sum (coproduct) of the H_j 's as Hilbert spaces. One can check that

$$\bigoplus H_j = \{\xi \in \prod H_j : \sum \|x_i\|_j^2 < \infty\}$$

Given $(T_j)_{j \in J}$ with $T_j \in B(H_j)$, can define $T\xi = (T_j \xi_j)$. If $\{\|T_j\| : j \in J\}$ is bounded then $T \in B(H)$ with $\|T\| = \sup_j \|T_j\|$. Similarly, if (H_j, π_j) is a family of unitary representations of G ($*$ -representations of a Banach algebra), we can define a $*$ -representation on $\bigoplus H_j$ by $U_x \xi = (U_x^j, \xi_j)$, $(\pi_a \xi = (\pi_a^j \xi_j))$. Note that these are bounded representations since $\|U_x^j\| = 1$ and $\|\pi_a^j\| \leq \|a\|$ for each j .

If (H, π) is a $*$ -representation, then it decomposes into a direct sum $\bigoplus (H_j, \pi_j)$ with each (H_j, π_j) a cyclic representation.

6.2 Intertwining Operators

Definition 6.1. Suppose (H, π) , (K, ρ) are $*$ -representations of \mathcal{A} (unitary reps of G). A morphism of representations is an \mathcal{A} module homomorphism $T : H \rightarrow K$, ie. a bounded linear operator $T : H \rightarrow K$ so that for each $a \in \mathcal{A}$, $T\pi_a = \rho_a T$. We call such a map an intertwining operator.

Definition 6.2. An isomorphism of representations is an intertwining unitary operator $U : H \rightarrow K$.

Definition 6.3. A pointed cyclic representation (H, π, ξ_*) is a cyclic representation with a specified cyclic vector ξ_* .

Proposition 6.4. Let (H, π, ξ_*) and (K, ρ, η_*) be pointed cyclic representations. If $\mu_{\xi_*} = \mu_{\eta_*}$ there is a unitary intertwining operator U from H to K so that $U\xi_* = \eta_*$.

Proof. Define U on the dense subspace $\{\pi_a \xi_* : a \in \mathcal{A}\}$ by $U(\pi_a \xi_*) = \rho_a \eta_*$. U then extends uniquely to a continuous map $H \rightarrow K$. It is clear that U is intertwining. Further, for any $a, b \in \mathcal{A}$,

$$\langle U\pi_a \xi_*, U\pi_b \xi_* \rangle = \langle \rho_a \eta_*, \rho_b \eta_* \rangle = \langle \rho_b^* \rho_a \eta_*, \eta_* \rangle = \mu_{\eta_*}(b^* a) = \mu_{\xi_*}(b^* a) = \langle \pi_b^* \pi_a \xi_*, \xi_* \rangle = \langle \pi_a \xi_*, \pi_b \xi_* \rangle$$

So U is unitary as well. \square

It follows that there is a bijection between positive linear functions on \mathcal{A} and isomorphism classes of pointed cyclic representations.

Definition 6.5. A $*$ -representation (H, π) irreducible if it has no proper closed π -invariant subspaces.

Given a finite dimensional representation, it can be decomposed into a sum of irreducible representations.

7 2-4

7.1 Schur's Lemma

Definition 7.1. Let H be a Hilbert space and $\mathcal{C} \subseteq B(H)$. The commutant of \mathcal{C} is defined to be

$$\mathcal{C}' = \{T \in B(H) : TC = CT \forall C \in \mathcal{C}\}$$

The commutant is closed with respect to the strong operator topology and so is a Von Neumann algebra (and hence a C^* -algebra). For any \mathcal{C} , $(\mathcal{C}')'$ is the strong closed $*$ -algebra generated by \mathcal{C} . The double commutant theorem was proved by Von Neumann in the first paper on operator algebras.

If (H, π) is a representation of \mathcal{A} , then $\text{End}_{\mathcal{A}}(H)$ is the commutant of $\{\pi_a\}_{a \in \mathcal{A}}$.

Lemma 7.2 (Schur's Lemma). A representation (H, π) of \mathcal{A} is irreducible if and only if $\text{End}_{\mathcal{A}}(H) = \mathbb{C}$.

Proof. If (H, π) is not irreducible then there is a proper closed invariant subspace $K \subseteq H$. Taking P to be the orthogonal projection onto K , we have that $K \in \text{End}_{\mathcal{A}}(H)$ but K is not a scalar multiple of the identity. Conversely, if $\text{End}_{\mathcal{A}}(H) \neq \mathbb{C}$ then let T be an operator which is not a scalar multiple of the identity. Since either the real or imaginary part of T is also not a scalar multiple of the identity we may assume that T is self adjoint without loss of generality. There are $s \neq t \in \sigma(T)$ and so let f, g be functions so that $f(s) = g(t) = 1$ and $fg = 0$. This $F, G \in C^*(I, T)$ so that $F, G \neq 0$ and $FG = 0$. Now, $\langle FH, GH \rangle = 0$ and $GH \neq 0$ so, since $F \in \text{End}_{\mathcal{A}}(H)$, \overline{FH} is a proper invariant subspace. \square

Corollary 7.3. If \mathcal{A} is a commutative unital normed $*$ -algebra, then every irreducible representation of \mathcal{A} is one dimensional.

Proof. If \mathcal{A} is commutative, then for representation (H, π) , $\pi_a \in \text{End}_{\mathcal{A}}(H)$ for $a \in \mathcal{A}$. So if the representation is irreducible it follows that $\pi_a \in \mathbb{C}I$ for each A and so every subspace is invariant. Thus, H must be one dimensional. \square

Example 7.4. If X is a compact Hausdorff space, then the irreducible representations of X are precisely evaluations at points in x . More generally, if \mathcal{A} is a commutative $*$ -Banach algebra, then its irreducible representations are exactly the characters on \mathcal{A} .

Example 7.5. Consider the algebra $\mathcal{A} = \mathbb{C} \oplus \mathcal{C}$ with $(\alpha, \beta)^* = (\bar{\beta}, \bar{\alpha})$. There are no nonzero $*$ -algebra homomorphisms from \mathcal{A} to \mathbb{C} since $(1, 0)$ generates it as a $*$ -algebra but $(1, 0)(1, 0)^* = 0$ and so $\mu(1, 0) = 0$ for any $\mu : \mathcal{A} \rightarrow \mathbb{C}$.

7.2 Radon-Nikodym for Positive Linear Functionals

Definition 7.6. Let \mathcal{A} be a unital *-algebra and let μ, ν be positive linear functionals on \mathcal{A} . Say that μ dominates ν , $\mu \geq \nu$ if $\mu - \nu \geq 0$.

How do we get such a ν ? Given μ , consider the pointed representation $(H_\mu, \pi_\mu, \xi_\mu)$. For any $T \in \text{End}_{\mathcal{A}}(H)$ with $0 \leq T \leq I$, define

$$\nu(a) = \langle \pi_a T \xi_\mu, \xi_\mu \rangle$$

We have that

$$\nu(a^*a) = \langle T \pi_a \xi_\mu, \pi_a \xi_\mu \rangle \quad \text{and} \quad (\mu - \nu)(a^*a) = \langle (I - T) \pi_a \xi_\mu, \xi_\mu \rangle$$

and so $0 \leq \nu \leq \mu$. Here T is the “Radon-Nikodym derivative” of ν .

We will show that all such ν arise in this way.

Theorem 7.7. If $\mu \geq \nu \geq 0$, then $\exists T \in \text{End}_{\mathcal{A}}(H_\mu)$, $0 \leq T \leq I$ such that $\nu(a) = \langle \pi_a T \xi_\mu, \xi_\mu \rangle$.

Fact: Let H be a Hilbert Space, K a dense subspace, and $\langle \cdot, \cdot \rangle_K$ a pre-inner product on K so that $\langle \eta, \eta \rangle_K \leq \langle \eta, \eta \rangle_H$ for $\eta \in K$. Then, there is an operator $T \in B(H)$ with $0 \leq T \leq I$ and so that $\langle \eta, \eta \rangle = \langle T\eta, \eta \rangle$ for $\eta \in K$.

Proof. Given $\eta \in K$, consider the conjugate-linear functional given by $\varphi_\eta(\xi) = \langle \eta, \xi \rangle_K$. φ_η extends uniquely to H and so there is some ξ_η so that $\varphi_\eta(\xi) = \langle \xi_\eta, \xi \rangle$ for all $\xi \in K$. Consider the linear map defined on K by $T\eta = \xi_\eta$. Note that T is continuous since $\|\xi_\eta\| = \|\eta\|_K \leq \|\eta\|_H$ and so extends uniquely to an element in $B(H)$ satisfying the desired conditions. \square