

# Descriptive Set Theory: Moschovakis

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# Part I

## Notes

# Part II

## Exercises

# Chapter 1

## The Basic Classical Notions

1.1 Perfect Polish Spaces

1.2 The Borel Pointclasses of Finite Order

1.3 Computing with Relations; Closure Properties

1.4 Parameterization and Hierarchy Theorems

1.5 The Projective Sets

1.6 Countable Operations

1.7 Borel Functions and Isomorphisms

## Chapter 2

# $\kappa$ -Suslin and $\lambda$ -Borel

### 2.1 The Cantor-Bendixson Theorem

### 2.2 $\kappa$ -Suslin Sets

### 2.3 Trees and the Perfect Set Theorem

### 2.4 Wellfounded Trees

### 2.5 The Suslin Theorem

### 2.6 Inductive Analysis of Projective Trees

### 2.7 The Kunen-Martin Theorem

**Exercise 2.7.1.** Prove that a binary relation  $R(x, y)$  on a set  $S$  is wellfounded if and only if there are no infinite  $<_R$ -descending chains.

*Proof.* If  $R(x, y)$  is not wellfounded then there must be some nonempty set  $A$  such that  $A$  has no  $<_R$  minimal element. Using  $A$ , we construct an infinite  $<_R$ -descending chain as follows: Let  $x_0 \in A$  arbitrary and given  $x_i$ , choose  $x_{i+1} \in A$  such that  $x_i >_R x_{i+1}$ . Such an element will always exist since  $A$  has no least element and so  $x_0 >_R x_1 > \dots$  is an infinite  $<_R$ -descending chain.

Conversely, suppose there is an infinite  $<_R$ -descending chain  $x_0 >_R x_1 >_R \dots$  and let  $A = \{x_i : i \in \omega\}$ .  $A$  is nonempty set with no  $<_R$  minimal element and so  $R(x, y)$  is not wellfounded.  $\square$

**Exercise 2.7.2.** Prove that every Borel wellfounded relation has countable length and every  $\Delta_2^1$  wellfounded relation has length less than  $\aleph_2$ .

*Proof.* Recall that if  $R(x, y)$  is a relation we define its strict part to be

$$<_R = \{(x, y) : R(x, y) \& \neg R(y, x)\}$$

So, if  $R$  is  $\Delta_n^1$  we see that  $<_R$  is  $\Delta_n^1$  as well. Hence, if  $R$  is a Borel relation then  $<_R$  must be  $\aleph_0$ -Suslin and so, applying the Kunen-Martin theorem, it must have length less than  $\aleph_1$ . Similarly, if  $R$  is  $\Delta_2^1$  then  $<_R$  is  $\aleph_1$ -Suslin and so has length less than  $\aleph_2$ .  $\square$

**Exercise 2.7.3.** Let  $R$  be a wellfounded relation on  $S$  with rank function  $\rho$ , and let  $f : S \rightarrow \text{Ordinals}$  be an order preserving function, ie.

$$x <_R y \Rightarrow f(x) < f(y)$$

Prove that for every  $x$  in  $S$ ,  $\rho(x) \leq f(x)$ .

*Proof.* We show this by induction on the relation. Suppose that for all  $y <_R x$ ,  $\rho(y) \leq f(y)$  then, since we have  $f(y) < f(x)$  for each  $y <_R x$ ,  $f(x) \geq f(y) + 1$  for all  $y <_R x$  and so

$$f(x) \geq \sup\{f(y) + 1 : y <_R x\} \geq \sup\{\rho(y) + 1 : y <_R x\} = \rho(x)$$

□

A norm  $\varphi$  on  $S$  is called regular if it is onto some ordinal  $\lambda$ .

Given a norm  $\varphi$  on  $S$  the associated the binary relation  $\leq^\varphi$  on  $S$  is defined by

$$x \leq^\varphi y \Leftrightarrow \varphi(x) \leq \varphi(y)$$

**Exercise 2.7.4.** Prove that a binary relation  $\leq$  on a set  $S$  is a prewellordering if and only if there is a norm  $\varphi$  on  $S$  such that  $\leq = \leq^\varphi$ . Moreover if  $\leq$  is a prewellordering, then there is a unique regular  $\varphi$  on  $S$  such that  $\leq = \leq^\varphi$ .

*Proof.* If  $\leq$  is a prewellordering, we claim that its rank function  $\rho$  is a norm on  $S$  such that  $\leq = \leq^\rho$ , ie. for all  $x, y \in S$

$$x \leq y \Leftrightarrow \rho(x) \leq \rho(y)$$

This follows from the fact that if  $x \leq y$  then  $\rho(x) \leq \rho(y)$  since all elements of  $S$  strictly below  $x$  will also be strictly below  $y$ . Similarly, if  $\neg x \leq y$ , then  $y$  is strictly below  $x$  so  $\rho(y) < \rho(x)$ .

Conversely, if there is some norm  $\varphi$  on  $S$  such that  $\leq = \leq^\varphi$ ,  $\leq$  must be prewellordering since  $\leq$  is a wellorder and, under the pullback, the only condition it may fail to satisfy is antisymmetry.

To show that there is a unique regular norm satisfying this property note that the rank function  $\rho$  is a regular norm since it is onto  $\lambda = |\leq|$  and so to show uniqueness it is enough to show any regular norm is equal to  $\rho$ . We do this by induction on the ordering: Suppose that  $\varphi$  is another regular norm and for all  $y$  strictly below  $x$ ,  $\varphi(y) = \rho(y)$ . We must have

$$\varphi(x) \geq \sup\{\varphi(y) + 1 : y < x\} = \sup\{\rho(y) + 1 : y < x\} = \rho(x)$$

and if this inequality is strict, since  $\varphi$  is surjective, there must be some  $z$  such that  $\varphi(z) = \rho(x)$ . We cannot have  $z < x$  (otherwise  $\varphi(z) = \rho(z) < \rho(x)$ ) and so  $x \leq z$ . However, this implies  $\varphi(x) \leq \varphi(z)$  contradicting our choice of  $z$ . Hence, we must have  $\varphi(x) = \rho(x)$  as well. □

## 2.8 Category and Measure

**Exercise 2.8.1.** Prove that in a complete metric space no open ball is meager.

*Proof.* Suppose towards a contradiction that  $B$  is open ball such that  $B = \bigcup_n A_n$  where each  $A_n$  is nowhere dense. Then we have  $B \subset \bigcup_n \overline{A_n}$ . Since  $A_1$  is not dense in  $B$  we can find some  $B_1 \subset B$  such that  $\overline{B_1} \subset B \setminus \overline{A_1}$  and such that  $\text{radius}(B_1) < \text{radius}(B)$ . Similarly, given  $B_i$  we can find  $B_{i+1}$  such that  $\overline{B_{i+1}} \subset B_i \setminus \overline{A_{i+1}}$  and  $\text{radius}(B_{i+1}) < 2^{-(i+1)}$ . Now, there exists a unique  $b \in \bigcap_n \overline{B_n}$  and by construction  $b \notin \overline{A_i}$  for all  $i$  and so  $b \in B \setminus \bigcup_n A_n$ , contradicting our assumption. Hence,  $B$  cannot be meager. □

A poinset  $P$  has the property of Baire if there is some open poinset  $P^*$  such that  $P \Delta P^*$  is meager.

**Exercise 2.8.2.** Prove that every Borel poinset has the property of Baire.

*Proof.* We show that the class of poinsets having the property of Baire contains all the open and closed sets, and is closed under negation and countable union. If  $P$  is an open poinset, then  $P\Delta P = \emptyset$  is meager so  $P$  has the property of Baire. If  $P$  is a closed poinset, consider  $P^* = P^\circ$ .  $P\Delta P^* = P \setminus P^*$  is meager since it is closed and has empty interior since  $(P \setminus P^*)^\circ \subseteq P^\circ$  and  $(P \setminus P^*) \cap P^\circ = \emptyset$ . Next, suppose  $P$  has the property of Baire, we show  $\neg P$  does as well. If  $P^*$  is an open set such that  $P\Delta P^*$  is meager then, since  $\neg P\Delta\neg P^* = P\Delta P^*$ ,  $\neg P\Delta\neg P^*$  must be meager as well. Now, it follows that  $\neg P\Delta(\neg P^*)^\circ \subseteq (\neg P\Delta\neg P^*) \cup (\neg P^*) \setminus (\neg P^*)^\circ$  is meager and so  $\neg P$  has the property of Baire as well. Finally, we show that if  $\{P_n\}_{n \in \omega}$  are such that each  $P_i$  has the property of Baire  $\bigcup_i P_i$  does as well. Suppose we have open  $P_i^*$  such that  $P_i\Delta P_i^*$  is meager. Then,  $\bigcup_i P_i\Delta \bigcup_i P_i^* \subseteq \bigcup_i (P_i\Delta P_i^*)$  is meager and so  $\bigcup_i P_i$  has the property of Baire as well.  $\square$

**Exercise 2.8.3.** Prove that for every poinset  $P \subseteq \mathcal{X}$ , there is an  $F_\sigma$  set  $\tilde{P} \supseteq P$  such that if  $A \subseteq P^* \setminus P$  is Borel, then  $A$  is meager.

**Exercise 2.8.4.** Prove that the collection of pointsets with the property of Baire is closed under the operation  $\mathcal{A}$ ; in particular  $\Sigma_1^1$  sets have the property of Baire.