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1.1 Motivation

Before starting the development of the general theory of C^*-algebras we consider a few motivating examples.

Example 1.1. Consider the Hilbert spaces $\ell^2(\mathbb{Z})$, $L^2(\mathbb{R})$ and define unitary operators U, V by $(U\xi)(r) = \xi(r-1)$ and $(V\xi)(r) = e^{2\pi i \theta r}\xi(r)$ for $\theta \in \mathbb{R}$. Consider the algebra generated by these two operators. $VU = e^{2\pi i \theta}VU$ so the linear span of $U^m V^n$ is an algebra. Denote its operator norm closure by $\mathcal{A}_\theta$.

The algebras generated by U and V individually are both isomorphic to $C(\mathbb{S}^1)$ and if $\theta \in \mathbb{Z}$, U and V commute so $\mathcal{A}_\theta \cong C(\mathbb{S}^1 \times \mathbb{S}^1)$. Otherwise, we call $\mathcal{A}_\theta$ a noncommutative 2-torus.

Consider the self adjoint operator, called an almost Matthieu operator

$$H = (U + U^*) + \lambda(V + V^*)$$

Q: What is the spectrum of H?

- For rational θ, it is a disjoint union of intervals

- 1976: Hofstader plotted the spectra for rational θ to produce a fractal like pattern – the Hofstader Butterfly. This led to the conjecture that the spectrum of H for irrational θ was a Cantor set.

- 1981: Mark Kat offered 10 martinis for an answer to this conjecture.

- Barry Simon popularized this and it became known as the 10 martini problem

- 2005: Jitomirskaya, Avila gave an affirmative answer

Much of the partial progress on this problem came from the theory of C^*-algebras but the final proof used completely different techniques.

Example 1.2. Let G be a group. A (Hilbert Space) unitary representation of G is a group homomorphism $\pi : G \to \mathcal{U}(H)$. Consider the algebra of operators generated by $\{\pi(x) \colon x \in G\}$, ie. elements of the form $\pi_f = \sum_{x \in G} f(x)\pi(x)$ for $f : G \to \mathbb{C}$ finnitely supported. We define a product on this set by

$$\left(\sum f(x)\pi(x)\right)\left(\sum g(y)\pi(y)\right) = \sum_{x,y} f(x)g(y)\pi(xy) = \sum_{x,y} f(x)g(x^{-1}y)\pi(y) = \sum_y \left(\sum_x f(x)g(x^{-1}y)\right)\pi(y)$$

Defining the convolution $*$ by

$$(f * g)(y) = \sum_x f(x)g(x^{-1}y)$$

we have the $\pi_f \cdot \pi_g = \pi_{f*g}$ and this gives an algebra structure on $C_c(G)$ and an algebra homomorphism $\pi : (C_c(G), *) \to B(H)$. The convolution also gives $\ell^1(G)$ an algebra structure since $||f*g||_1 \le ||f||_1 ||g||_1$.

For any $f \in C_c(G)$, let $||f||_{C^*(G)} = \sup\{\pi_f \colon \pi \text{ unitary rep of } G\}$. Observe that

$$||\pi_f|| \le \sum |f(x)| ||\pi(x)|| = \sum |f(x)| = ||f||_1$$

and so $||f||_{C^*(G)} \le ||f||_1$. Let $C^*(G)$ denote the completion of $C_c(G)$ with respect to this norm. We can represent this algebra as an algebra bounded operators on some Hilbert space.

Note: for any group G we have at least two unitary representations – the trivial representation and the left regular representation on $\ell^2(G)$ by translation.

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Example 2.1. Using the left regular representation: $H = \ell^2(G)$ with $\mathcal{U}_x(\xi)(y) = \xi(x^{-1}y)$, we can define a norm on $C_c(G)$ by $||f||_r = ||U_f||$. Note that the involution is given by $f^*(x) = \overline{f(-x)}$. The closure of $C_c(G)$ under this norm is called the reduced C*-algebra for G. Since $||f||_r \le ||f||_{C^*}$, we only have that $C_r^*(G)$ is a quotient of $C^*(G)$ in general. In fact, the discreprency between the two encodes information about the group G:

Theorem 2.2. $C^*(G) = C_r^*(G)$ if and only if G is amenable.

Example 2.3. For X a compact Hausdorff space, we can consider a homomorphism $\alpha : G \to \text{Homeo}(x)$ (eg. a dynamical system). This induces a map $\alpha : G \to \text{Aut}(C(X))$. We can often use this map to give insight into the behavior of the dynamical system as passing to the level of algebras gives more structure to leverage.

More generally, for a C*-algebra $\mathcal{A}$, we call a map $\alpha : G \to \text{Aut}(\mathcal{A})$ a "noncommutative dynamical system". We will look at representations of these systems: $U : G \to \mathcal{U}(H)$, $\pi : \mathcal{A} \to B(H)$ satisfying $\pi(\alpha_x(a)) = U_x\pi(a)U_x^{-1}$ (the covariance relation). The algebra generated by $(U_x, \pi(a))$ is called the crossproduct algebra.

Example 2.4. Let $X = \mathbb{S}^1$ and a^θ be rotation by θ.

Ex: $X = \mathbb{S}^1$, $a^\theta = $ rotation by θ. Then $\mathbb{Z}$ acts by $(\alpha^\theta)^n$ and the crossproduct algebra is $\mathcal{A}_\theta$.

Remark 2.5. The construction of $C^*(G)$ also works for groupoids

If X, Y are compact Hausdroff spaces, how are $C(X \times Y)$ and $C(X), C(Y)$ related. Consider functions of the form $(f \otimes g)(x, y) = f(x)g(y)$. The algebra generated by these functions is dense in $C(X \times Y)$ by Stone Weierstrass.

Example 2.6. Can from algebra $\mathcal{A} \oplus \mathcal{A} \oplus \cdots$

Consider direct limit of algebras $M_2 \hookrightarrow M_4 \hookrightarrow M_8 \cdots$ with isometric embeddings $T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$.

Called the CAR-algebra (canonical anticommutative relation).

Q: for different n, are the C*-algebras generated by $M_n \to M_{n^2} \cdots$ nonisomorphic?

Yes (the proof uses methods from noncommutative geometry)

However, the Von-Neumann algebras generated by different n are all isomorphic.

Definition 2.7. A concrete C*-algebra is a subalgebra of $B(H)$, for H a Hilbert space, which is norm closed and closed under $*$.

An abstract C*-algebra is a Banach algebra with a conjugate linear involution $*$ satisfying $||A^*A|| = ||A||^2$.

By the Gelfand-Naimark theorem (1943), every abstract C*-algeba is isomorphic to a concrete C*-algebra.

Little Gelfand-Naimark: If $\mathcal{A}$ is a commutatuve C*-algebra then $\mathcal{A} \cong C_0(X)$ for X LCH.

If $\mathcal{A}$ is a commutative unital C*-algebra over $\mathbb{C}$, define $\hat{\mathcal{A}} = \{\varphi : \mathcal{A} \to \mathbb{C}, \text{ unital hom}\}$. For each $a \in \mathcal{A}$, define $\hat{a}$ on $\hat{\mathcal{A}}$ by $\hat{a}(\varphi) = \varphi(a)$. The map $a \mapsto \hat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ to functions on $\hat{\mathcal{A}}$.

Definition 2.8. For $a \in \mathcal{A}$, define the spectrum of a, $\sigma(a) = \{\lambda \colon a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}$.

If $\mathcal{A}$ is a Banach algebra, then $(1-a)^{-1} = \sum a^n$ if $||a|| < 1$ and so if $\lambda \in \sigma(a)$ then $|\lambda| < ||a||$.

Observe that if $\varphi \in \hat{\mathcal{A}}$ then $\varphi(a) \in \sigma(a)$ since $\varphi(a - \varphi(a)1) = 0$ and so $a - \varphi(a)1$ is not invertible.

For $a \in \mathcal{A}$, $\hat{a} \in C(\hat{\mathcal{A}})$, $a \mapsto \hat{a}$ called the Gelfand transform.

Gelfand spectral radius formula: $\sup\{|\lambda| \colon \lambda \in \sigma(a)\} = \lim ||a^n||^{1/n}$.

So, if $||a^2|| = ||a||^2$, then Gelfand transform is isometric.

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3.1   The GNS Construction

Definition 3.1. An algebra involution $*$ is a conjugate linear map so that $(ab)^* = b^*a*$ and $(a^*)^* = a$.

Given $C(X)$ a unital C*-algebra, how do we realize it as bounded operators on a Hilbert space H?

Suppose μ is a positive measure, viewed as a linear functional. Define a pre-inner product by $\langle f, g \rangle = \int f\bar{g}d\mu = \mu(f\bar{g})$. Quotienting out by norm zero elements, gives a Hilbert space on which $C(X)$ acts by pointwise multiplication.

Example 3.2. Consider $\delta_x(f) = f(x)$. Then $C(X) \subseteq \ell^\infty(X_{\text{disc}})$ acts on $\ell^2(X_{\text{disc}})$. This is called the atomic representation.

We would like to this construction on a general unital *-algebra.

Definition 3.3. Let μ be a linear functional on $\mathcal{A}$ a unital *-algebra. Say that μ is positive if $\mu(a^*a) \geq 0$ for $a \in \mathcal{A}$.

The positive linear functionals are closed under addition and multiplication by nonnegative scalars (ie. they form a cone)

Let μ be a positive linear function on $\mathcal{A}$. Define a pre-inner product on $\mathcal{A}$ by $\langle a, b \rangle_\mu = \mu(b^*a)$. To see that this is a pre-inner product observe that

$$\langle a, a \rangle_\mu = \mu(a^*a) \geq 0 \quad \text{and} \quad \langle a+b, a+b \rangle_\mu = \langle a, a \rangle_\mu + \langle a, b \rangle_\mu + \langle b, a \rangle_\mu + \langle b, b \rangle_\mu \geq 0$$

so $0 = \text{Im}(\langle a, b \rangle_\mu) + \text{Im}(\langle b, a \rangle_\mu)$ and, replacing b with ib, gives $\text{Re}(\langle a, b \rangle_\mu) = \text{Re}(\langle b, a \rangle_\mu)$.

Let $N_\mu = \{a \in \mathcal{A} \colon \langle a, a \rangle_\mu = 0\}$. If $a \in N_\mu$, then $|\langle a, b \rangle_\mu|^2 \leq \langle a, a \rangle_\mu \langle b, b \rangle_\mu = 0$ so $N_\mu = \{a \in A \colon \langle a, b \rangle_\mu = 0 \ \forall b\}$. It follows that N_μ is a left ideal of $\mathcal{A}$. On $\mathcal{A}/N_\mu$ the pre-inner product becomes an inner product. Denote its completion by $L^2(\mathcal{A}, \mu)$.

Let $\mathcal{A}$ act on itself on the left via $\pi_a b = ab$. This is a $*$-algebra homomorphism since

$$\langle \pi_a b, c \rangle_\mu = \langle ab, c \rangle_\mu = \mu(c^*ab) = \mu((a^*c)^*b) = \langle b, a^*c \rangle_\mu = \langle b, \pi_{a^*}c \rangle$$

Call this the left regular representation. Since N_μ is a left ideal of $\mathcal{A}$, this gives an action of $\mathcal{A}$ on $\mathcal{A}/N_\mu$ as well. It remains to check that the π_a's are continuous. This may fail without a complete norm as the next example shows.

Example 3.4. Let $\mathcal{A}$ be the algebra of $\mathbb{C}$-valued polynomials on $\mathbb{R}$, and let $\mu(p) = \int_{-\infty}^{\infty} p(t)e^{-t^2} dt$. On $L^2(\mathcal{A}, \mu)$, multiplying by polynomials in an unbounded operator.

The above is called the called the GNS (Gelfand-Naimark-Segal) construction

Theorem 3.5. Let $\mathcal{A}$ be a unital normed $*$-algebra, μ a positive linear functional on $\mathcal{A}$.

(1) If $\mathcal{A}$ is complete, then μ is continuous and $||\mu|| = \mu(1)$

(2) If $\mathcal{A}$ is not necessarily complete but μ is continuous then $||\mu|| = \mu(1)$.

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4.1 Automatic Continuity

Lemma 4.1. Let $\mathcal{A}$ be a unital *-Banach algebra over $\mathbb{C}$. If $a \in \mathcal{A}$, $a^* = a$, and $||a|| < 1$, then there is $b \in \mathcal{A}$ with b^*b and $1 - a = b^2$.

Proof. The function $f(z) = \sqrt{1-z}$, has a power series expansion $f(z) = \sum r_n z^n$, $r_n \in \mathbb{R}$ which converges for $|z| < 1$. Setting $b = \sum r_n a^n$, it follows that $b^2 = (f(a))^2 = (f^2)(a) = 1 - a$. $\square$

Theorem 4.2. If μ is a positive linear function on $\mathcal{A}$, then μ is continuous and $||\mu|| = \mu(1)$.

Proof. Suppose $a \in \mathcal{A}$ with $a^* = a$ and $||a|| < 1$. Then for b so that $1 - a = b^*b$, we have that

$$\mu(b^*b) = \mu(1) - \mu(a) \geq 0$$

and so $\mu(1) \geq \mu(a)$. It follows that if $a^* = a$, $\mu(a) \leq ||a||\mu(1)$. For general a,

$$|\mu(a)|^2 = |\mu(a1)|^2 = |\langle 1, a \rangle_\mu|^2 \leq \mu(1)\mu(a^*a) \leq \mu(1)^2||a^*a|| \leq \mu(1)^2||a||^2$$

$\square$

If (H, π) is a *-representation of $\mathcal{A}$, then for any $\xi \in H$, set $\mu_\xi(a) = \langle \pi(a)\xi, \xi \rangle$. This defines a positive linear functional with $||\mu_\xi|| = ||\xi||^2$. We also have that

$$||\pi(a)\xi||^2 = \langle \pi(a)\xi, \pi(a)\xi \rangle = \langle \pi(a^*a)\xi, \xi \rangle \leq ||a^*a|| \, ||\xi^2|| \leq ||a||^2 \, ||\xi||^2$$

and so $||\pi(a)|| \leq ||a||_\mathcal{A}$.

Theorem 4.3. If μ is a positive linear functional on $\mathcal{A}$, then the left regular representation on $L^2(A, \mu)$ defines a *-representation, ie. each π_a is a bounded linear operator (and $||\pi_a|| \leq ||a||$).

Proof. For any $a, b \in \mathcal{A}$,

$$||\pi_a(b)||^2 = \langle \pi_a(b), \pi_a(b) \rangle_\mu \langle ab, ab \rangle_\mu = \mu(b^*a^*ab) \leq ||b^*a^*ab|| \leq ||b^*|| \, ||a^*a|| \, ||b||\mu(1) \leq ||a||^2||b||^2\mu(1)$$

$\square$

4.2 Cyclic Representations

Definition 4.4. Let $\mathcal{A}$ be a unital *-Banach algebra and (H, π) a continuous *-representation. Then $\xi \in H$ is said to be (topologically) cyclic if $\{\pi_a\xi \colon a \in \mathcal{A}\}$ is dense in H.

Proposition 4.5. Let (H, π) be a *-representation of $\mathcal{A}$. If $K \subseteq H$ is a closed π-invariant subspace then $K^\perp$ is also π-invariant

Proof. For any $\eta \in K^\perp$, $\xi \in K$, $\langle \xi, \pi_a\eta \rangle = \langle \pi_a^*\xi, \eta \rangle = 0$. $\square$

Given (H, π) a *-representation, for any nonzero $\xi \in H$, $K = \overline{\{\pi_a \colon a \in \mathcal{A}\}}$ is a closed cyclic subspace and we can write H as the sum of closed π-invariant subspaces $H = K \oplus K^\perp$. Iterating this (possibly a transfinite number of times) shows that any *-representation (H, π) can be decomposed into cylic invariant subspaces.

5   1-30

5.1   Integrated Forms

Let G be a group and U a unitary representation of G. Given $f \in C_c(G)$, define $U_f = \sum f(x)U_x$. Then, $U_f U_g = U_{f*g}$ and $||U_f|| \leq ||f||_1$ so $f \mapsto U_f$ is a continuous *-representation of $\ell^1(G)$. The assignment $f \mapsto U_f$ is called the integrated form corresponding to U. Conversely, if (H, π) is a *-representation of $\ell^1(G)$, then $x \mapsto \pi_{\delta_x}$ gives a unitary representation of G. So, there is a bijection between unitary representations of G and continuous *-representations of $C_c(G)$ or $\ell^1(G)$.

If φ is a continuous positive linear functional on $\ell^1(G)$, then $\varphi \in \ell^\infty(G)$. So to any positive linear functional μ, there is an associated $\mu_f \in \ell^\infty(G)$. We call such a μ_f a function of positive type.

Example 5.1. Let (H, π) be the trivial representation of G on $\mathbb{C}$. Its integrated from is $f \mapsto \sum f(x)$ and so the corresponding function of positive type is $\varphi(x) = 1$ for all x.

Claim: $\delta_e \in \ell^\infty(G)$ is of positive type.

$$\langle f^* * f, \delta_e \rangle = (f^* * f)(e) = \sum \overline{f(y^{-1})}f(y^{-1}e) = \sum |f(y^{-1})|^2 \geq 0$$

What does the GNS construction give?

$$\langle f, g \rangle_{\delta_e} = \sum \overline{g(y)}f(y) = \langle f, g \rangle_{\ell^2(G)}$$

So we recover the left regular representation of G.

5.2   Traces

Definition 5.2. A positive linear functional on a normed *-algebra is tracial (is a trace) if $\mu(ab) = \mu(ba)$ for all $a, b \in \mathcal{A}$.

Claim: δ_e is a trace on $\ell^1(G)$

$$\langle f * g, \delta_e \rangle = \sum f(y)g(y^{-1}) = \sum f(y^{-1})g(y) = \langle g * f, \delta_e \rangle$$

Let $\mathcal{A}$ be a unital Banach *-algebra, μ a tracial positive linear functional. Let $N_\mu = \{a \colon \langle a, b \rangle_\mu = 0 \; \forall b\}$. Then N_μ is a two sided ideal of $\mathcal{A}$ and $\mathcal{A}/N_\mu$ is an A, A-bimodule. So $L^2(A, \mu)$ admits both left and right regular representations.

The representation $L^2(A, \mu)$ is cylic vector with cyclic vector 1. Any vector ξ gives a positive linear functional via $a \mapsto \langle a\xi, \xi \rangle$ and the postive linear function corresponding to 1 gives back μ. So, any positive linear functional on $\mathcal{A}$ comes from a cyclic vector in some *-representation.

Example 5.3. For the regular representations of $\ell^1(G)$ or $\ell^\infty(G)$ on $\ell^2(G)$, the vector δ_e is cyclic for the representation. It defines a trace on the image of $\ell^1(G)$. Let $C_r(G)$ denote the norm closure of the image of $\ell^1(G)$ in $B(\ell^2(G))$. δ_e defines a trace on this sub C*-algebra, $||\delta_e|| = 1$.

It is not possible to extend this to a trace on $B(H)$ in general since iff H is infinite dimensional, then $B(H)$ admits no boudned trace.

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6.1   Direct Sums of Hilbert Spaces

Given a collection of Hilbert spaces $\{H_j\}_{j \in J}$, we can form the algebraic direct sum as

$$\bigoplus_{\text{alg}} H_j = \{\xi \in \prod H_j \colon \xi_j = 0 \text{ for all but finitely many } j\}$$

This is an inner product space with

$$\langle \xi, \eta \rangle = \sum_j \langle x_j, \eta_j \rangle$$

Taking the completion with respect to this inner product gives the direct sum (coproduct) of the H_j's as Hilbert spaces. One can check that

$$\bigoplus H_j = \{\xi \in \prod H_j \colon \sum \|x_i\|_j^2 < \infty\}$$

Given $(T_j)_{j \in J}$ with $T_j \in B(H_j)$, can define $T\xi = (T_j \xi_j)$. If $\{\|T_j\| \colon j \in J\}$ is bounded then $T \in B(H)$ with $\|T\| = \sup_j \|T_j\|$. Similarly, if (H_j, π_j) is a family of unitary representations of G (*-representations of a Banach algebra), we can define a *-representation on $\bigoplus H_j$ by $U_x \xi = (U_x^j, \xi_j)$, $\left(\pi_a \xi = (\pi_a^j \xi_j)\right)$. Note that these are bounded representations since $\|U_x^j\| = 1$ and $\|\pi_a^j\| \leq \|a\|$ for each j.

If (H, π) is a *-representation, then it decomposes into a direct sum $\bigoplus(H_j, \pi_j)$ with each (H_j, π_j) a cylic representation.

6.2 Intertwining Operators

Definition 6.1. Suppose (H, π), (K, ρ) are *-representations of $\mathcal{A}$ (unitary reps of G). A morphism of representations is an $\mathcal{A}$ module homomorphism $T \colon H \to K$, ie. a bounded linear operator $T \colon H \to K$ so that for each $a \in \mathcal{A}$, $T\pi_a = \rho_a T$. We call such a map an intertwining operator.

Definition 6.2. An isomorphism of representations is an intertwining unitary operator $U \colon H \to K$.

Definition 6.3. A pointed cylic representation (H, π, ξ_*) is a cylic representation with a specified cylic vector ξ_*.

Proposition 6.4. Let (H, π, ξ_*) and (K, ρ, η_*) be pointed cylic representations. If $\mu_{\xi_*} = \mu_{\eta_*}$ there is a unitary intertwining operator U from H to K so that $U\xi_* = \eta_*$.

Proof. Define U on the dense subspace $\{\pi_a \xi_* \colon a \in \mathcal{A}\}$ by $U(\pi_a \xi_*) = \rho_a \eta_*$. U then extends uniquely to a continuous map $H \to K$. It is clear that U is intertwining. Further, for any $a, b \in \mathcal{A}$,

$$\langle U\pi_a \xi_*, U\pi_b \xi_* \rangle = \langle \rho_a \eta_*, \rho_b \eta_* \rangle = \langle \rho_b^* \rho_a \eta_*, \eta_* \rangle = \mu_{\eta_*}(b^* a) = \mu_{\xi_*}(b^* a) = \langle \pi_b^* \pi_a \xi_*, \xi_* \rangle = \langle \pi_a \xi_*, \pi_b \xi_* \rangle$$

So U is unitary as well. $\qquad \square$

It follows that there is a bijection between positive linear functions on $\mathcal{A}$ and isomorphism classes of pointed cylic representations.

Definition 6.5. A *-representation (H, π) irreducible if it has no proper closed π-invariant subspaces.

Given a finite dimensional representation, it can be decomposed into a sum of irreducible representations.

7 2-4

7.1 Schur's Lemma

Definition 7.1. Let H be a Hilbert space and $\mathcal{C} \subseteq B(H)$. The commutant of $\mathcal{C}$ is defined to be

$$\mathcal{C}' = \{T \in B(H) \colon TC = CT \, \forall C \in \mathcal{C}\}$$

The commutant is closed with respect to the strong operator topology and so is a Von Neumman algebra (and hence a C^*-algebra). For any $\mathcal{C}$, $(\mathcal{C}')'$ is the strong closed *-algebra generated by $\mathcal{C}$. The double commutant theorem was proved by Von Neumman in the first paper on operator algebras.

If (H, π) is a representation of $\mathcal{A}$, then $\text{End}_{\mathcal{A}}(H)$ is the commutant of $\{\pi_a\}_{a \in \mathcal{A}}$.

Lemma 7.2 (Schur's Lemma). A representation (H, π) of $\mathcal{A}$ is irreducible if and only if $\text{End}_{\mathcal{A}}(H) = \mathbb{C}$.

Proof. If (H, π) is not irreducible then there is a proper closed invariant subspace $K \subseteq H$. Taking P to be the orthogonal projection onto K, we have that $K \in \text{End}_{\mathcal{A}}(H)$ but K is not a scalar multiple of the identity. Conversely, if $\text{End}_{\mathcal{A}}(H) \neq \mathbb{C}$ then let T be an operator which is not a scalar multiple of the identity. Since either the real or imaginary part of T is also not a scalar multiple of the identity we may assume that T is self adjoint without loss of generality. There are $s \neq t \in \sigma(T)$ and so let f, g be functions so that $f(s) = g(t) = 1$ and $fg = 0$. This $F, G \in C^*(I, T)$ so that $F, G \neq 0$ and $FG = 0$. Now, $\langle FH, GH \rangle = 0$ and $GH \neq 0$ so, since $F \in \text{End}_{\mathcal{A}}(H)$, $\overline{FH}$ is a proper invariant subspace. $\qquad \square$

Corollary 7.3. If $\mathcal{A}$ is a commutative unital normed *-algebra, then every irreducible representation of $\mathcal{A}$ is one dimensional.

Proof. If $\mathcal{A}$ is commutative, then for representation (H, π), $\pi_a \in \text{End}_{\mathcal{A}}(H)$ for $a \in \mathcal{A}$. So if the representation is irreducible it follows that $\pi_a \in \mathbb{C}I$ for each A and so every subspace is invariant. Thus, H must be one dimensional. □

Example 7.4. If X is a compact Hausdorff space, then the irreducible representations of X are precisely evaluations at points in x. More generally, if $\mathcal{A}$ is a commutative *-Banach algebra, then its irreducible representations are exactly the characters on $\mathcal{A}$.

Example 7.5. Consider the algebra $\mathcal{A} = \mathbb{C} \oplus \mathcal{C}$ with $(\alpha, \beta)^* = (\overline{\beta}, \overline{\alpha})$. There are no nonzero *-algebra homomorphisms from $\mathcal{A}$ to $\mathbb{C}$ since $(1, 0)$ generates it as a *-algebra but $(1, 0)(1, 0)^* = 0$ and so $\mu(1, 0) = 0$ for any $\mu : \mathcal{A} \to \mathbb{C}$.

7.2 Radon-Nikodym for Positive Linear Functionals

Definition 7.6. Let $\mathcal{A}$ be a unital *-algebra and let μ, ν be positive linear functionals on $\mathcal{A}$. Say that μ dominates ν, $\mu \geq \nu$ if $\mu - \nu \geq 0$.

How do we get such a ν? Given μ, consider the pointed representation $(H_{\mu}, \pi_{\mu}, \xi_{\mu})$. For any $T \in \text{End}_{\mathcal{A}}(H)$ with $0 \leq T \leq I$, define
$$\nu(a) = \langle \pi_a T \xi_{\mu}, \xi_{\mu} \rangle$$

We have that
$$\nu(a^*a) = \langle T \pi_a \xi_{\mu}, \pi_a \xi_{\mu} \rangle \quad \text{and} \quad (\mu - \nu)(a^*a) = \langle (I - T) \pi_a \xi_{\mu}, \xi_{\mu} \rangle$$

and so $0 \leq \nu \leq \mu$. Here T is the "Radon-Nidokdym derivative" of ν.

We will show that all such ν arise in this way.

Theorem 7.7. If $\mu \geq \nu \geq 0$, then $\exists T \in \text{End}_{\mathcal{A}}(H_{\mu})$, $0 \leq T \leq I$ such that $\nu(a) = \langle T \pi_{\mu}(a) \xi_{\mu}, \xi_{\mu} \rangle$.

Fact: Let H be a Hilbert Space, K a dense subspace, and $\langle , \rangle_K$ a pre-inner product on K so that $\langle \eta, \eta \rangle_K \leq \langle \eta, \eta \rangle_H$ for $\eta \in K$. Then, there is an operator $T \in B(H)$ with $0 \leq T \leq I$ and so that $\langle \xi, \eta \rangle_K = \langle T \xi, \eta \rangle$ for $\xi, \eta \in K$.

Proof. Given $\eta \in K$, consider the conjugate-linear functional given by $\varphi_{\eta}(\xi) = \langle \eta, \xi \rangle_K$. φ_{η} extends uniquely to H and so there is some ξ_{η} so that $\varphi_{\eta}(\xi) = \langle \xi_{\eta}, \xi \rangle$ for all $\xi \in K$. Consider the linear map defined on K by $T\eta = \xi_{\eta}$. Note that T is continuous since $||\xi_{\eta}|| = ||\eta||_K \leq ||\eta||_H$ and so extends uniquely to an element in $B(H)$ satisfying the desired conditions. □

Proof of Theorem. Define an inner product on H by $\langle \pi_{\mu}(a) \xi_{\mu}, \pi_{\mu}(b) \xi_{\mu} \rangle_{\nu} = \nu(b^*a)$ then take T to be as above. It remains to check that $T \in \text{End}_{\mathcal{A}}(H)$. For any $a, b \in \mathcal{A}$, we compute that

$$\langle T \pi_{\mu}(b) \pi_{\mu}(a) \xi_{\mu}, \xi_{\mu} \rangle = \langle T \pi_{\mu}(ba) \xi_{\mu}, \xi_{\mu} \rangle = \nu(ba) = \langle T \pi_{\mu}(a) \xi_{\mu}, \pi_{\mu}(b^*) \xi_{\mu} \rangle = \langle \pi_{\mu}(b) T \pi_{\mu}(a) \xi_{\mu}, \xi_{\mu} \rangle$$

□

8 2-6

8.1 Pure States

Definition 8.1. A positive linear functional μ on $\mathcal{A}$ is pure if whenever $\mu \geq \nu \geq 0$, $\mu = r\nu$ for some $r \in [0, 1]$.

Proposition 8.2. A positive linear functional μ is pure if and only if the corresponding representation (H_{μ}, π_{μ}) is irreducible.

Proof. If (H_{μ}, π_{μ}) is irreducible, then by Schur's lemma $\text{End}_{\mathcal{A}}(H) = \mathbb{C}$. So, whenever $0 \leq \nu \leq \mu$, the derivative of ν is equal to rI for some $r \in [0, 1]$ and so $\nu = r\mu$. Conversely, If (H_{μ}, π_{μ}) has a closed proper invariant subspace, let P be the projection onto this subspace. Then $0 \leq P \leq I$ and so, taking ν to be the corresponding positive linear functional, it follows that $0 \leq \nu \leq \mu$ but $\nu \neq r\mu$. □

Definition 8.3. A state on $\mathcal{A}$ is a positive linear functional μ with $\mu(1) = ||1|| = ||\mu||$.

Let $\mathcal{S}(\mathcal{A})$ denote the collection of states on $\mathcal{A}$, called the state space. $\mathcal{S}(\mathcal{A}) \subseteq B_1(\mathcal{A}')$ and is convex. It is also closed for the weak *-topology and so is compact.

Remark 8.4. Physicists call states "mixed states" and use "states" to refer to pure states.

Proposition 8.5. A state $\mu \in \mathcal{S}(A)$ is pure if and only if it is an extreme point of $\mathcal{S}(A)$.

Proof. If μ is not an extreme point then there are $\nu_1, \nu_2 \neq \mu$, $t \in (0,1)$ so that $\mu = t\nu_1 + (1-t)\nu_2$. So $\mu \geq t\nu_1$ but $t\nu_1 \neq r\mu_1$. So μ is not pure. Conversely, if μ is an extreme point and $0 \leq \nu \leq \mu$, then

$$\mu = ||\nu|| \left(\frac{\nu}{||\nu||}\right) + ||\mu - \nu|| \left(\frac{\mu - \nu}{||\mu - \nu||}\right)$$

is a convex combination of states and so $||\nu||\mu = \nu$. Hence, μ is pure. $\square$

Corollary 8.6. For $\mu \in \mathcal{S}(A)$, (H_μ, π_μ) is irreducible if and only if μ is an extreme point of $\mathcal{S}(A)$.

Theorem 8.7 (Krein-Milman)**.** Every compact, convex subset of a Banach space is equal to the closed convex hull of its extreme points.

For any representation (H, π) and $\xi \in H$ with $||\xi|| = 1$ state, we can define a state by $\mu_\xi(a) = \langle \pi(a)\xi, \xi \rangle$. Call such a state a vector state.

Example 8.8. Let G be a discrete group, and consider the left regular representation of $\ell^1(G)$ on $\ell^2(G)$. There is $\xi \in \ell^2(G)$ so that if $x \neq e$, $\lambda_x \xi \neq \xi$. There are lots of vector states in $\mathcal{S}(A)$ so it follows that there are lots of irreducible representations. In particular, for any $x \in G$ there is an irreducible representation (H, U) so that $U_x \neq I_H$ (Gelfand-Raikov Theorem). For $f \in \ell^1(G)$, let $||f|| = \sup\{||\pi_\mu(f)||: \mu \text{ pure state}\}$. Then $||f|| = ||\lambda_f||$ as an operator on $\ell^2(G)$.

9 2-9

9.1 Constructing Positive Linear Functionals

Given a positive linear functional on a unital C*-algebra $\mathcal{A}$, we have seen how to construct a representation. Our next goal is to answer the following question:

Q: If $\mathcal{A}$ is an abstract unital C*-algebra, why does it have any positive linear functionals?

Using the Riesz-representation theorem, we can produce many such functionals for commutative C*-algebras. More generally, suppose $\mathcal{A}$ is a C*-algebra and $\mathcal{B}$ a unital commutative sub C*-algebra. Then $B \cong C(X)$ for some compact Hausdorff space X, and so if μ_0 is a probability measure on X, it defines a linear functional on $C(X)$ with $\mu(1) = 1 = ||\mu_0||$. Using Hahn-Banach, μ_0 can be extended to μ a linear functional on $\mathcal{A}$. However, how do we know that such an extension is a state?

First, we may assume that the extension is a *-algebra homomorphism by replacing μ with

$$\tilde{\mu}(a) = \frac{\mu(a) + \mu(a^*)}{2} + i \frac{\mu(a) - \mu(a^*)}{2}$$

Then $\tilde{\mu}(a^*) = \overline{\tilde{\mu}(a)}$, $\tilde{\mu} \restriction \mathcal{B} = \mu_0 \restriction \mathcal{B}$, and $||\tilde{\mu}|| = \tilde{\mu}(1) = 1$. So it remains to check that μ is positive.

If $\mathcal{C}$ is a commutative unital C*-algebra, $\mathcal{C} \cong C(Y)$ and so $\mu \restriction_{\mathcal{C}}$ arises from a Radon measure on Y. Taking the Jordan decomposition gives $\mu \restriction_{\mathcal{C}} = (\mu \restriction_{\mathcal{C}})^+ - (\mu \restriction_{\mathcal{C}})^-$. Since

$$1 = \mu(1) = ||\mu \restriction_{\mathcal{C}}|| = ||(\mu \restriction_{\mathcal{C}})^+|| + ||(\mu \restriction_{\mathcal{C}})^-||$$

it follows that $(\mu \restriction_{\mathcal{C}})^- = 0$. So, if $\mathcal{A}$ is a unital C*-algebra, it has lots of linear functionals whose restriction to every commutative unital subalgebra is a state of that subalgebra.

In particular, for normal elements, we have that $\mu(a^*a) \geq 0$ but why is this true for general a?

Definition 9.1. For any C*-algebra $\mathcal{A}$ and $a \in \mathcal{A}$, we say that a is positive if $a^* = a$ and $\sigma(a) \subseteq \mathbb{R}_{\geq 0}$.

If $a \geq 0$ then, restricting to the commutative C*-algebra generated by a, it follows that $\mu(a) \geq 0$. So to show that $\mu(a^*a) \geq 0$ for any a, it suffices to show that a^*a is positive.

Proposition 9.2. If $a \in \mathcal{A}$ a unital C*-algebra, then $a^*a \geq 0$.

Proof. We have $C^*(a^*a, 1) \cong C(\sigma(a^*a))$. Note that we are taking for granted that the spectrum of an element is invariant when passing to a sub C^*-algebra, but this will be shown later.

Suppose towards a contradiction that $\sigma(a^*a) \not\subseteq \mathbb{R}_{\geq 0}$, then there is some $c \in C(\sigma(a^*a))$ with $c \geq 0$ and so that $\sigma(ca^*a) \subseteq \mathbb{R}_{\leq 0}$. Let $d = (ca)$ so that $\sigma(d^*d) \subseteq \mathbb{R}_{\leq 0}$.

Write $d = h + ik$ with h, k self adjoint. Then

$$d^*d + dd^* = (h - ik)(h + ik) + (h + ik)(h - ik) = 2(h^2 + k^2)$$

so that $dd^* = 2(h^2 + k^2) - d^*d$.

To complete the proof, we will need two facts: that the sum of two positive elements is positive, and that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$ for any a, b. Since both $2(h^2 + k^2)$ and $-d^*d$ are positive, this would imply dd^* is positive. However $\sigma(dd^*) \cup \{0\} = \sigma(d^*d) \cup \{0\} \subseteq \mathbb{R}_{\leq 0}$, a contradiction. $\square$

Proposition 9.3. If $\mathcal{A}$ is a unital C^*-algebra and if $a, b \in \mathcal{A}$ are so that $a, b \geq 0$, then $a + b \geq 0$.

For any C^*-algebra, its norm is completely determined by its *-algebraic structure:

$$||a||^2 = ||a^*a|| = r(a^*a)$$

The next claim shows that the order structure on $\mathcal{A}^{sa}$ is completely determined by norm.

Claim: For $a^* = a$, $a \geq 0$ if and only if $\exists t \geq ||a||$ with $||a - t1|| \leq t$ if and only if $\forall t \geq ||a||$ $||a - t1|| \leq t$.

Proof. It suffices to show this for $\mathcal{A} = C(X)$. Given $f \in C_{\mathbb{R}}(X)$ and $t \geq ||f||$, $||f - t1|| \leq t$ iff $|f(x) - t| \leq t$ for all x iff $f(x) \geq 0$ for all x. $\square$

Proof of Proposition (Kelley-Vaught, 1953).

Suppose $a, b \geq 0$ and let $s = ||a||$, $t = ||b||$. Then $||a - s1|| \leq s$ and $||b - t1|| \leq t$, so

$$||(a + b) - (t + s)1|| \leq ||a - s1|| + ||b - t1|| \leq s + t$$

Since, $||a + b|| \leq s + t$, it follows that $a + b \geq 0$. $\square$

10   2-11

10.1   The Gelfand-Naimark Theorem

Theorem 10.1. Let $\mathcal{A}$ be any unital algebra over any field. Then for $a, b \in \mathcal{A}$, $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

Proof. For $\lambda \neq 0$, $(ab - \lambda 1) = \lambda(\lambda^{-1}ab - 1)$ so it suffices to show that $1 - ab$ is invertible whenever $1 - ba$ is. Writing

$$(1 - ab)^{-1} = \sum_{n=0}^{\infty}(ab)^n = 1 + a\left(\sum_{n=0}^{\infty}(ba)^n\right)b = 1 + a(1 - ba^{-1})b$$

Hence, if $1 - ba$ is invertible, it should be that $(1 - ab)^{-1} = 1 + a(1 - ba)^{-1}b$. An easy computation shows that this is true. $\square$

Remark 10.2. We give a brief historical account of connections to physics:

- Heisenberg, Schrodinger (1926): Gave the first way to model quantum physics

- Heisenberg: Observables should be modeled by self adjoint operators on a Hilbert Space.

  Let P model momentum and Q model position. Then

  $$PQ - QP = ihI_H$$

  where h is Planck's constant.

- Schrodinger: Let $H = L^2(\mathbb{R})$ and define

  $$P(\xi)(t) = t\xi(t) \qquad Q(\xi)(t) = ih\frac{d}{dt}\xi(t)$$

  so that we again have $PQ - QP = ihI_H$.

  Note that the operators P and Q above are not bounded so we must restrict their domain.

If $\mathcal{A}$ is a unital Banach algebra, then there are no $a, b \in \mathcal{A}$ so that $ab - ba = \lambda 1$ as this would imply $\sigma(ba + \lambda I) = \sigma(ba) + \lambda$ while $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$, which is impossible. This shows that using unbounded operators is necessary to model the above situation.

Example 10.3. Consider $\ell^2(\mathbb{N})$. Define S via $(S\xi)_n = x_{n+1}$. Then $(S^*\xi)_n = \xi_{n-1}$ for $n > 0$ and 0 otherwise. We have that $S^*S = I$ but $SS^* = I - \pi_0$ and so $\sigma(S^*S) \neq \sigma(SS^*)$.

A unital C*-algebra $\mathcal{A}$ is called finite if there is no a so that $a^*a = 1$ and $aa^* \neq 1$.

We are finally ready to complete the proof of the Gelfand-Naimark theorem.

Theorem 10.4. Let $\mathcal{A}$ be an abstract unital C*-algebra. Then $\mathcal{A}$ is isometrically isomorphic to a C*-subalgebra of bounded operators on some Hilbert space H.

Proof. For each $\mu \in \mathcal{S}(\mathcal{A})$, form (H_μ, π_μ) and let $H = \bigoplus_{\mu \in \mathcal{S}(\mathcal{A})}(H_\mu, \pi_\mu)$. We show this representation of $\mathcal{A}$ is isometric. For each a, $||\pi(a)|| = \sup_\mu ||\pi_\mu(a)|| \leq ||a||$. This supremum is achieved since for any $a \in \mathcal{A}$, there is a state μ_0 on $C^*(a^*a, 1)$ so that $\mu_0(a^*a) = ||a^*a||$. We can then extend μ_0 to a state μ on $\mathcal{A}$ and for this μ we will have

$$||\pi_\mu(a)||^2 = ||\pi_\mu(a^*a)|| = \mu(a^*a) = ||a^*a|| = ||a||^2$$

$\square$

11   2-13

11.1   Invariance of Spectra

Theorem 11.1. Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{B}$ a unital sub C*-algebra. For any $b \in \mathcal{B}$, $\sigma_{\mathcal{A}}(b) = \sigma_{\mathcal{B}}(b)$

Example 11.2. This may fail for general Banach algebras. Consider $\ell^1(\mathbb{N}) \subseteq \ell^1(\mathbb{Z})$. The element δ_1 is invertible in $\ell^1(\mathbb{Z})$ but not $\ell^1(\mathbb{N})$.

To prove the above theorem, it suffices to show that if $b \in \mathcal{B}$ has an inverse in $\mathcal{A}$, then it has an inverse in $\mathcal{B}$. We draw the motivation for this proof from algebra: Let R be a unital ring, S a unital subring. If b is a zerodivisor in S, then it has no inverse in R, ie. b cannot be made invertible in some ring containing S. This condition is equivalent to being a zero divisor since if $b \in S$ is not a zerodivisor then S embeds into S_b, in which b is invertible.

For normed algebras, we first consider this in the commutative case. An element $f \in C(X)$ is not invertible if and only if it is an "approximate zero divisor", ie. there exists a sequence $\{g_n\} \in C(X)$ and K so that $K \leq ||g_n||$ with $||g_n f|| \to 0$. If such a sequence exists we may assume $||g_n|| \leq 1$ for all n.

Proof of Theorem. Suppose b is not invertible in $\mathcal{B}$. Then b^*b is also not invertible and so is an approximate zero divisor in $\mathcal{B}$. Such a sequence also witnesses that b^*b (and hence b) is not invertible in $\mathcal{A}$. $\square$

Note that if b_n is a sequence winessing that a^*a is a right approximate zero divisor then either $b_n a^*$ is bounded below so a is also a right approximate zero divisor or b_n witnesses that a is a left approximate zero divisor. So $a \in \mathcal{A}$ is invertible if and only if it is not a (left or right) approximate zero divisor.

We conclude by showing that for elements of $B(H)$, the notion of positivity for operators and elements of a C*-algebra coincide.

Proposition 11.3. For $T \in B(H)$, $T^* = T$, $\sigma(T) \subseteq \mathbb{R}_{\geq 0}$ if and only if $\langle T\xi, \xi \rangle \geq 0$ for all ξ.

Proof. If $\sigma(T) \subseteq \mathbb{R}_{\geq 0}$, then there is some S so that $T = S^*S$ and so $\langle T\xi, \xi \rangle = \langle S\xi, S\xi \rangle \geq 0$ for all ξ. Conversely, suppose $\langle T\xi, xi \rangle \geq 0$ for all ξ. If $\lambda \in R$ is so that $(T - \lambda I)$ is not invertible, λ is an approximate eigenvalue so there are $\{\xi_n\}$ with $||\xi_n|| = 1$ and $(T - \lambda I)\xi_n \to 0$. Then $\langle (T - \lambda I)\xi_n, \xi_n \rangle = \langle T\xi_n, \xi_n \rangle - \lambda \to 0$ and so $\lambda \geq 0$. $\square$

11.2 Approximate Identities

Our next topic will be ideals in C*-algbras. To study these we will first need to understand with C*-algebras without identities. Many C*-algebras we may work with do not have identity elements, for instance $C_\infty(\mathbb{R})$. Despite this the identity element can be approximated in this algebra: let e_n be a sequence of compactly supported positive fuctions so that $e_n \equiv 1$ on $[-n, n]$. Then for any $f \in C_\infty(\mathbb{R})$, $fe_n \to f$.

Definition 11.4. Let $\mathcal{A}$ be a normed algebra. A left approximate identity is a net $\{e_\lambda\}_{\lambda \in \Lambda}$ so that for any $a \in \mathcal{A}$, $||e_\lambda a - a|| \to 0$. We can also define right, bounded, self adjoint, and positive approximate identities.

Example 11.5. If X is locally compact, then $\Lambda = f \in C_c(X)$ with $0 \le f \le 1$ is an approximate identity.

Proposition 11.6. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{L}$ a left ideal of $\mathcal{A}$ (not necessarily closed). There is a right approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{L}$ (positive of norm 1). If $\mathcal{L}$ is separable, this can be taken to be a sequence.

Proof. Let $\mathcal{C}$ be a dense subset of $\mathcal{L}$ and let Γ be the collection of finite subsets of $\mathcal{C}$ ordered by inclusion. For $\lambda \in \Lambda$, set $b_\lambda = \sum_{a \in \lambda} a^* a$. Note that $b_\lambda \ge 0$ and so $(b_\lambda + \frac{1}{|\lambda|})^{-1}$ exists and $e_\lambda = (b_\lambda + 1)^{-1} b_\lambda \in \mathcal{L}$. We claim that e_λ is a right approximate identity for $\mathcal{L}$.

It suffices to show that for $a \in \mathcal{C}$. Assume $a \in \lambda$ so that $a^* a \le b_\lambda$. Then

$$(a - ae_\lambda)^*(a - ae_\lambda) = (1 - e_\lambda)a^* a(1 - e_\lambda) \le (1 - e_\lambda)b_\lambda(1 - e_\lambda)$$

Which, writing $t = b_\lambda$ is the function

$$f(t) = (1 - (t + \tfrac{1}{|\lambda|})^{-1}t)^2 t$$

As $|\lambda| \to \infty$, $||f(t)|| \to 0$ and so we have that $ae_\lambda \to a$, as desired. $\qquad\square$

12   2-18

12.1   Ideals in C*-algebras

Proposition 12.1. If I is a closed right ideal with right self adjoint approximate identity, then $d^* \in I$ whenever $d \in I$ and I is a two sided ideal.

Proof. Let $d \in I$. Then $||d^* - e_\lambda d^*|| = ||d - de_\lambda|| \to 0$ and so, since $e_\lambda d^* \in I$ for each λ and I is closed, $d^* \in I$ as well. Now, I is two sided since for any $a \in \mathcal{A}$, $b \in I$, $(ab)^* = b^* a^* \in I$ and so $ab \in I$. $\qquad\square$

Corollary 12.2. Any closed two-sided ideal of $\mathcal{A}$ is a C*-algebra.

Proposition 12.3. If I is a closed two-sided ideal of $\mathcal{A}$ and J is a closed two-sided ideal of I, then J is a two-sided ideal of $\mathcal{A}$.

Proof. Let e_λ be an approximate identity of I. For any $a \in \mathcal{A}$ and $b \in J$, $ab = \lim_\lambda ae_\lambda b \in J$ since each $ae_\lambda \in I$ and J is closed. $\qquad\square$

Proposition 12.4. If I, J are closed two-sided ideals then $IJ = I \cap J$.

Proof. We have that $IJ \subseteq I \cap J$ in general. Since a $\qquad\square$

Corollary 12.5. If I is a closed two-sided ideal then $I^2 = I$.

Q: How do we handle the case where $\mathcal{A}$ has no identity element?

We can formally adjoin an identity. Let $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}1$. If $\mathcal{A}$ is a normed algebra, then defining $||A + \lambda 1|| = ||a|| + |\lambda|$ gives $\tilde{\mathcal{A}}$ the structure of a normed algebra. It will be a Banach algebra if $\mathcal{A}$ is.

Example 12.6. $\mathcal{A} = C_\infty(\mathbb{R})$, $||f + \lambda 1|| = ||f|| + |\lambda|$. If $-1 \le f < 0$, then $||f - 1|| \le ||f||$ and so $||f - 1|| \ne \sup_{x \in \mathbb{R}} |f(x) - 1|$ and so this norm is not a C*-algebra norm.

Given $\mathcal{A}$ normed algebra, consider the left regular representation on itself with

$$||L_a|| = \sup\{||ab||\colon ||b|| \leq 1\}$$

If $\mathcal{A}$ has an approximate identity of norm 1, then the representation of $\tilde{\mathcal{A}}$ on $\mathcal{A}$ is isometric on $\mathcal{A}$.

For any ring (with unit), consider R^n viewed as a right R-module. This is a free right R-module. If $1 \in R$, then $\text{End}_R(R^n) = M_n(R)$, with the usual left action.

Definition 12.7. A projective right R-module is a module that this isomorphic to a direct summand of a free module ie. V is projective if there is a right module W wih $V \oplus W \cong R^n$ for some n.

Theorem 12.8 (Swan's Theorem). Let X be a compact Hausdorff space and $\mathcal{A} = C(X)$. There is a bijection between isomorphism classes of projective R-modules and isomorphism classes of vector bundles over X.

The study of projective modules will lead into K-theory

13   2-20

13.1   Adjoining Units

For $T \in \text{End}_{\mathcal{A}}(\mathcal{A})$, can define T^* in the usual way. If $\mathcal{A}$ has a unit, $\text{End}_{\mathcal{A}}(\mathcal{A}) = M_n()$. It may not exist in general however, and so we restrict our attention to operators with adjoints.

For $a \in \mathcal{A}$, $||L_a e_\lambda|| = ||a e_\lambda|| \to ||a||$ so $||L_a|| = ||a||$

We can still carry out the GNS construction in this setting: If μ is a positive continuous linear functional on $\mathcal{A}$, it extends to $\tilde{\mu}$ on $\tilde{\mathcal{A}}$ by setting $\mu(1) = ||\mu||$.

Definition 13.1. For $\mathcal{A}$ a $*$-normed algebra, a $*$-representation of $\mathcal{A}$ on H is said to be nondegenerate if $\{a\xi\colon a \in \mathcal{A}, \, \xi \in H\}$ is dense in H.

If $\mathcal{A}$ has a bounded approximate identity $\{e_\lambda\}$, the above is equivalent to $\pi(e_\lambda)\xi \to \xi$ for all $\xi \in H$.

Claim: For a GNS representation of $\tilde{\mathcal{A}}$ on $(H_{\tilde{\mu}}, \pi_{\tilde{\mu}}, \xi_{\tilde{\mu}})$, we can restrict this representation to $\mathcal{A}$. Then ξ_μ is still cylic for $\pi_{\tilde{\mu}} \upharpoonright_{\mathcal{A}}$ and so the representation restricted to $\mathcal{A}$ is nondegenerate.

Proof of Claim. Let $\{a_j\} \subseteq \mathcal{A}$ be a sequence with $||a_j|| \leq 1$ and so that $\mu(a_j) \nearrow ||\mu||$. Then

$$||\pi_\mu(a_j)\xi_\mu - \xi_\mu||^2 = ||a_j - 1||^2 = ||(a_j-1)^*(a_j-1)|| = \mu(a_j^* a_j) - \overline{\mu(a_j)} - \mu(a_j) + ||\mu|| \leq 2(||\mu|| - \mu(a_j)) \to 0$$

Hence, $\xi_\mu \in \overline{\{\pi(a)\xi_\mu\colon a \in \mathcal{A}\}}$ and so, since this set is $\tilde{\mathcal{A}}$-invariant, it must be equal to H. $\square$

Theorem 13.2 (Segal, 1947). Let $\mathcal{A}$ be a C*-algebra and let I be a closed two-sided ideal. Then $\mathcal{A}/I$ is a C*-algebra.

Example 13.3. Let $B_0(H)$ be the two-sided ideal of compact operators on H. Then $B(H)/B_0(H)$ is called the Calkin algebra.

For a normed space V and a closed subspace W, put a norm on V/W by

$$||\dot{v}|| = \inf\{||v - w||\colon w \in W\} = d(v, W)$$

If $\mathcal{A}$ is a normed algebra and I a closed two-sided ideal, then the quotient norm on $\mathcal{A}/I$ is still an algebra norm. Given a, b we show that $||\dot{a}\dot{b}|| \leq ||\dot{a}|| \, ||\dot{b}||$. For $c, d \in I$, $(a-c)(b-d) = ab - (cb + ad - cd)$. So, we have

$$||\dot{a}\dot{b}|| \leq ||(a-c)(b-d)|| \leq ||a - c|| \, ||b - d||$$

Taking the infimum over $c, d \in I$ gives the dsired result.