The Rising Sea: Vakil

Jad Damaj

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Exercises

Chapter 1

Just Enough Category Theory to be Dangerous

1.1 Motivation

1.2 Categories and Functors

Exercise 1.2.1. A category in which each morphism is an isomorphism is called a groupoid.

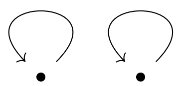
(a) A perverse definition of a group is: is a groupoid with one object. Make sense of this.

(b) Describe a groupoid that is not a group.

Proof.

(a) One can identiy one element groupoids and groups as follows: The elements of the group are the morhpisms of the category and the group law is given by composition. The category has an identiy morphism iff the group has an identity, composition is associative iff the group law is associative, and all morphisms are isomorphisms iff each elememt of the group has an inverse.

(b) The following is a groupoid which is not a group



$\square$

Exercise 1.2.2. If A is an object in the category $\mathcal{C}$, show that the invertible elements of $\mathrm{Mor}(A, A)$ from a group. What are the automorphism groups of a set X and a vector space V.

Proof. Denote the set of invertible elements of $\mathrm{Mor}(A, A)$ by $\mathrm{Aut}(A)$. Observe that $\mathrm{id}_A \in \mathrm{Aut}(A)$ and given $f, g \in \mathrm{Aut}(A)$, $f \circ g \in \mathrm{Aut}(A)$ since $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. Hence, the object A with collection of morphisms $\mathrm{Aut}(A)$ forms a one element subcategory of $\mathcal{C}$. Further, it is a groupoid. Hence, by the previous exercise, $\mathrm{Aut}(A)$ forms a group. When X is a set $\mathrm{Aut}(X)$ is exactly the symmetric group on X: $\mathrm{Sym}(X)$. When V is a vector space, $\mathrm{Aut}(V)$ is the subgroup of $\mathrm{Sym}(V)$ corresponding to permutations which are also linear maps.

Now, given $A, B \in C$ and an isomorphism $f : A \to B$, we can define a map $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$ via $g \mapsto f \circ g \circ f^{-1}$. Since f is invertible this map is a bijection and, further, it is a group homomorphism since

$$g \circ h \mapsto f \circ (g \circ h) f^{-1} = (f \circ g \circ f^{-1}) \circ (f \circ h \circ f^{-1})$$

Hence, A and B have isomorphic automorphism groups. □

Exercise 1.2.3. Let $(\cdot)^{\vee\vee} : f.d.Vec_k \to f.d.Vec_k$ be the double dual functor from the category of finite dimensional vector spaces over k to itself. Show that $(\cdot)^{\vee\vee}$ is naturally isomorphic to the identity functor on $f.d.Vec_k$.

Proof. To show that $(\cdot)^{\vee\vee}$ is naturally isomorphic to the identity, for each $V \in f.d.Vec_k$ we define a map $m_V : V \to V^{\vee\vee}$ by $m_V(v) = \mathrm{ev}_v$, ie. the map which evaluates a linear functional at v. We claim these maps define the desired natural isomorphism. First, observe that each m_V is injective since ev_v is the zero map only when $v = 0$. So, since V and $V^{\vee\vee}$ are finite dimensional vector spaces of the same dimension it follows that each m_V must be an isomorphism. Now, to check that this transformation is natural, we show that for any linear transformation $f : V \to W$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{m_V} & V^{\vee\vee} \\ \Big\downarrow f & & \Big\downarrow f^{\vee\vee} \\ W & \xrightarrow{m_W} & W^{\vee\vee} \end{array}$$

Given $v \in V$ and $\varphi \in W^\vee$, we compute that value of $f^{\vee\vee}(M_V(v))(\varphi)$ and $M_W(f(v))(\varphi)$.

$$f^{\vee\vee}(M_V(v))(\varphi) = (f^{\vee\vee}(\mathrm{ev}_v))(\varphi) = (\mathrm{ev}_v \circ f^\vee)(\varphi) = \mathrm{ev}_v(\varphi \circ f) = \varphi(f(v)) = \mathrm{ev}_{f(v)}(\varphi) = M_W(f(v))(\varphi)$$

□

Exercise 1.2.4. Let $\mathcal{V}$ be the category whose objects are the k-vector spaces k^n for $n \geqslant 0$ and whose morphisms are linear transformations. Show that $\mathcal{V} \to f.d.Vec_k$ gives an equivalence of categories by describing an "inverse" functor.

Proof. We define functors F and G to give this equivalence. Let $F : \mathcal{V} \to f.d.Vec_k$ be the inclusion functor. We define a functor $G : f.d.Vec_k \to \mathcal{V}$ as follows: For each vector space V, fix some basis $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ and let $M_V : V \to k^n$ denote the map defined by $v_i \mapsto e_i$, the ith element of the standard basis of k^n. Note that each M_V is an isomorphism. WLOG we may assume that that these bases are chosen such that if $V = k^n$ for some n, then $\mathcal{B}_V$ is the standard basis.
Now, G sends a vector space V to k^n where n is the dimension of V and sends a map $f : V \to W$ to the map $M_W \circ f \circ M_V^{-1} : k^n \to k^m$ where n, m are the dimensions of V, W respectively. By our choice of bases, we have the $G \circ F = \mathrm{id}_{\mathcal{V}}$ and so it remains to show that $F \circ G$ is naturally isomorphic to $\mathrm{id}_{f.d.Vec_k}$. To check this we define the maps $V \to F \circ G(V) = k^n$ to be the previously defined isomorphisms M_V. Further, given any $f : V \to W$, we have that $G(f) \circ M_V = M_W \circ f$ by definition of G and so the diagram below commutes and this transformation is natural.

$$\begin{array}{ccc} V & \xrightarrow{M_V} & k^n \\ \Big\downarrow f & & \Big\downarrow G(f) = M_W \circ f \circ M_W^{-1} \\ W & \xrightarrow{M_W} & k^m \end{array}$$

□

1.3 Universal Properties Determine an Object up to Isomorphism

Exercise 1.3.1. Show that any two inital objects are uniquely isomorphic. Show that any two final objects are uniquely isomorphic.

Exercise 1.3.2. What are the initial objects in *Sets*, *Rings*, and *Top*? How about in the category of subsets of a set or the category of open subsets of a topological space?

Exercise 1.3.3. Show that $A \to S^{-1}A$ is injective if and only if S contains no zerodivisors.

Exercise 1.3.4. Show that $A \to S^{-1}A$ satisfies the following universal property: $S^{-1}A$ is inital among A-algebras B where every element of S is sent to an invertible element in B.

Exercise 1.3.5. Show that $\phi : M \to S^{-1}M$ exists, by constructing something satisfying the universal property.

Exercise 1.3.6.

(a) Show that localization commutes with finite products, or equivalently, finite direct sums.

(b) Show that localization commutes with arbitrary direct sums.

(c) Show that "localization does not necessarily commute with infinite products": the obvious map $S^{-1}(\prod_i M_i) \to \prod_i S^{-1}M_i$ induced by the universal property is not always an isomorphism.

Exercise 1.3.7. Show that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$

Exercise 1.3.8. Show that $(\cdot)\otimes_A N$ gives a covariant functor $Mod_A \to Mod_A$. Show that $(\cdot)\otimes_A N$ is right-exact.

Exercise 1.3.9. Show that $(T, t : M \times N \to T)$ is unique up to isomorphism.

Exercise 1.3.10. Show that the construction of 1.3.5 satisfies the universal property of the tensor product.

Exercise 1.3.11.

(a) if M is an A-module and $A \to B$ is a morphism of rings, give $B \otimes_A M$ the structure of a B-module. Show that this describes a functor $Mod_A \to Mod_B$

(b) If further $A \to C$ is another morphism of rings, show that $B \otimes_A C$ has a natural structure of a ring.

Exercise 1.3.12. If S is a multiplicative subset of A and M is an A-module, describe a natural isomorphism $(S^{-1}A) \otimes_A M \to S^{-1}M$.

Exercise 1.3.13. Show that tensor products commute with arbitrary direct sums: If M and $\{N_i\}_{i\in I}$ are A-modules, describe an isomorphism

$$M \otimes (\oplus_{i\in I} N) \longrightarrow \otimes_{i\in I}(M \otimes N_i)$$

Exercise 1.3.14. Show that in *Sets*,

$$X \times_Z Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$$

Exercise 1.3.15. If X is a topological space, show that fibered products always exist in the category of open sets of X, by describing what a fibered product is.

Exercise 1.3.16. If Z is a final object in a category $\mathcal{C}$, and $X, Y \in \mathcal{C}$, show that "$X \times_Z Y = X \times Y$": the fibered product over Z is uniquely isomorphic to the product.

Exercise 1.3.17. If the two squares in the following commutative diagram are Cartesian diagrams, show that the outside rectangle is also a Cartesian diagram.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Exercise 1.3.18. Given morphisms $X_1 \to Y$, $X_2 \to Y$, show that there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$, assuming that both fibered products exist.

Exercise 1.3.19. Suppose that we are given morphisms $X_1, X_2 \to Y$ and $Y \to Z$. Show that the following diagram is a Cartesian square.

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

Exercise 1.3.20. Show that the coproduct for *Sets* is disjoint union.

Exercise 1.3.21. Suppose $A \to B$ and $A \to C$ are two ring morphisms, so in particular B and C are A-modules. Recall that $B \otimes_A C$ has a ring structure. Show that there is a natural morphism $B \to B \otimes_A C$ given by $b \mapsto b \otimes 1$. Similarly, there is a natural morphism $C \to B \otimes_A C$. Show that this gives a fibered coproduct on rings, ie. that

$$\begin{tikzcd} B \otimes_A C & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Exercise 1.3.22. Show that the composition of two monomorphisms is a monomorphism.

Exercise 1.3.23. Prove that a morphism $\pi : Y \to Z$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced diagonal morphism $\delta_\pi : X \to X \times_Y X$ is an isomorphism.

Exercise 1.3.24. Show that if $Y \to Z$ is a monomorphism, then the natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ is an isomorphism.

1.4 Limits and Colimits

1.5 Adjoints

1.6 An Introduction to Abelian Categories

Chapter 2

Sheaves

2.1 Motivating Example: The Sheaf of Smooth Functions

Exercise 2.1.1. Let $\mathcal{O}_p$ be the ring of germs at some point p. Show that m_p, the ideal of germs vanishing at p, is the only maximal ideal of $\mathcal{O}_p$.

Proof. Note that this ideal is maximal since $\mathcal{O}_p/m_p \cong \mathbb{R}$. To show that it is the only maximal ideal it is enough to show that every element of $\mathcal{O}_p \backslash m_p$ is invertible. Let (f, U) be a germ not vanishing at p, we can find some open set $p \in V \subset U$ such that f does not vanish on V. We will have $(f, U) = (f, V)$ as germs. Now, the function $1/f$ will be continuous on V and so the germ $(1/f, V)$ is a germ and satisfies $(1/f, V) \cdot (f, V) = (1, V)$ and so (f, U) is invertible, as desired. $\square$

2.2 Definition of Sheaf and Presheaf

Exercise 2.2.1. Given a topological space X, verify that the data of a presheaf is precisely the data of a contravariant functor of open sets of X to the category of sets.

Proof. A contravariant functor $\mathcal{F} : \mathcal{O}_X \to Sets$ consist of the following data: For each $U \in \mathcal{O}_X$ a set $\mathcal{F}(U)$ and for each morphism $i : U \to V$ in $\mathcal{O}_X$, a morphism $F(i) : F(V) \to F(U)$ in sets. This is exactly the data given in the definition of a presheaf as $\mathcal{F}(U)$ is just the set of sections of $\mathcal{F}$ over U and $F(i)$ is the map $\mathrm{res}_{V,U}$.
Further, the data of the functor must satisfy the following: For each U, $F(\mathrm{id}_U) = \mathrm{id}_{F(U)}$ and $F(i \circ j) = F(j) \circ (i)$. Again, these are exactly the two conditions the data of a presheaf must satisfy since $F(\mathrm{id}_U)$ is just $\mathrm{res}_{U,U}$ and given $j : U \to V$, $i : V \to W$, the second condition can be rewritten as $\mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}$. $\square$

Exercise 2.2.2. Show that the following are presheaves on $\mathbb{C}$ but not sheaves: (a) bounded functions, (b) holomorphic functions admitting a holomorphic square root.

Proof. The restriction maps in both cases are the usual restriction maps for functions and so it is clear that they satisfy the presheaf axioms. However, neither of these are sheaves as they both fail to satisfy the gluability condition:

(a) Let $\{U_i\}$ be an open cover of $\mathbb{C}$ where each U_i is bounded. Consider the collection of holomorphic function $\{f_i\}$ where each f_i is the identity on U_i. These satisfy the hypothesis for the gluability condition but there is no bounded holomorphic function restricting to f_i on each U_i as any such function would be unbounded.

(b) Consider the open cover $\{U_1, U_2\}$ of $\mathbb{C}$ where $U_1 = \mathbb{C}\backslash\{0\} \times [0, \infty)$ and $U_2 = \mathbb{C}\backslash\{0\} \times (-\infty, 0]$. Consider the collection $\{f_1, f_2\}$ where f_i is the identity on U_i. Using some branch of the logarithm we can define a holomorphic square root of f_i on U_i however, since the identity has no holomorphic square root on all of $\mathbb{C}$, these fail to satisfy the gluability condition.

□

Exercise 2.2.3. The identity and gluability axioms may be interpreted as saying that $\mathcal{F}(\bigcup_{i\in I} U_i)$ is a certain limit. What is the limit?

Proof. We claim that these conditions are equivalent to $\mathcal{F}(\bigcup_{i\in I} U_i)$ being the limit of the following diagram

$$\prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{r_2} \prod_{(i,j)\in I\times I} F(U_i \cap U_j)$$

where r_1 is given by $r_1((f_i)_{i\in I}) = (f_i|_{U_i\cap U_j})_{(i,j)\in I\times I}$ and $r_2((f_j)_{j\in I}) = (f_j|_{U_i\cap U_j})_{(i,j)\in ItimesI}$. Note that an element $(f_i)_{i\in I}$ having the same image under r_1 and r_2 is equivalent to the collection $f_{i\, i\in I}$ satisfying the hypothesis for the gluability condition.
First, suppose that $\mathcal{F}$ satisfies the gluability and identity conditions and let X be a set along with a function $h : X \to \prod_{i\in I} U_i$ making the diagram commute. Then for all $x \in X$, $h(x) \in \prod_{i\in I} U_i$ satisfies the hypothesis for gluability condition and so there is $f \in \mathcal{F}(\cup_{i\in I} U_i)$ mapping to $h(x)$. So, lifting each element in $h(X)$ to $\mathcal{F}(\cup_{i\in I} U_i)$ gives a map $X \to \mathcal{F}(\cup_{i\in I} U_i)$ which h factors through. Further, the identity condition ensures this map is unique as for each $h(x)$ there is a unique element of $\mathcal{F}(\cup_{i\in I} U_i)$ mapping to it and so the desired commutivity conditions fully determine the map. Conversely, suppose that $\mathcal{F}(\bigcup_{i\in I} U_i)$ is the limit of this diagram. To see that $\mathcal{F}$ satisfies the identity axiom, suppose that f_1 and f_2 are two elements of $\mathcal{F}(\cup_{i\in I} U_i)$ mapping to the same element of $\prod_{i\in I} \mathcal{F}(U_i)$ and consider the map from the singleton to this element. Since this element is the image of an element in $\mathcal{F}(\cup_{i\in I} U_i)$ its compositions with r_1 and r_2 are equal and so their it factors uniquely through $\mathcal{F}(\cup_{i\in I} U_i)$. However, the map from the singleton to f_1 and f_2 both satisfy the condition for the limit we see that they must be equal, ie. $f_1 = f_2$. Next, to see that $\mathcal{F}$ satisfies gluability suppose that $(f_i)_{i\in I}$ is some collection of elements satisfying the hypothesis for the gluability condition. Then the map from the singleton to this element comutes with the diagram and so we have a map from the singleton to $\mathcal{F}(\cup_{i\in I} U_i)$ mapping to this element. This gives the desired function. □

Exercise 2.2.4. Show that the real-valued continuous functions on a topological space form a sheaf.

Proof. Since the restriction maps are the usual restriction maps for functions, this forms a presheaf. Further, this collection satisfies the identity axiom since if $\{U_i\}$ is an open cover of U and f_1 and f_2 agree when restricted to each U_i we see that $f_1 = f_2$ since for any $u \in U$ there is i such that $u \in U_i$ and so, since $f_1|_{U_i} = f_2|_{U_i}$, we have $f_1(u) = f_2(u)$. Finally, to check gluability let $\{f_i\}_{i\in I}$ be a collection of functions satisfying the hypothesis for the gluability condition. They can be glued to form some function f which is equal to f_i when restricted to each U_i we just need to check that this resulting f is continuous. Suppose $V \subseteq \mathbb{R}$ is open, notice that $f^{-1}(V) = \bigcup_{i\in I} f_i^{-1}(V)$ is open and so f is continuous, as desired. □

Exercise 2.2.5. Let $\mathcal{F}(U)$ be the maps to S are locally constant. Show that this is a sheaf. We denote this sheaf $\underline{S}$

Proof. Since the restriction maps are the usual restriction maps for functions, this forms a presheaf. We can also check that the identity axiom holds pointwise, as above. Hence, to check this is a sheaf we need to ensure that when a collection of locally constnat maps are glued together, the resulting function is also locally constant. However, this is immediate since if $\{U_i\}$ is a cover of U and $p \in U$, we can find some i such that $u \in U_i$ and so there is an neighborhood of u in U_i on which f_i constant. Then, f will still be constant on this neighborhood of u. □

Exercise 2.2.6. Suppose Y is a topological space. Show that "continuous maps to Y" from a sheaf of sets on X.

Proof. The proof is identical to the previous exercises: the presheaf conditions and identity follow immediately for collections of functions and the when continuous functions are glued together they remain continuous. □

Exercise 2.2.7. Suppose we are given a continuous map $\mu : Y \to X$. Show that the "sections of μ" form a sheaf. More precisely, to each open set U of X, associate the set of continuous maps $s : U \to Y$ such that $\mu \circ s = \mathrm{id}_U$ Show that this forms a sheaf.

Proof. Again, the presheaf conditions and identity are immediate and gluability holds as when sections are glued together they remain a section (since the property is local). □

Exercise 2.2.8. Suppose $\pi : X \to Y$ is a continuous map, and $\mathcal{F}$ a presheaf on X. Then define $\pi_*\mathcal{F}$ by $\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$, where V is an open subset of Y. Show that $\pi_*\mathcal{F}$ is a presheaf on Y, and is a sheaf if $\mathcal{F}$ is.

Proof. A continuous map $\pi : X \to Y$ induces a covariant functor $\pi^{-1} : \mathcal{O}_Y \to \mathcal{O}_X$, sending an open set in Y to its inverse image. So, viewing a presheaf on X as a contravariant functor $\mathcal{F} : \mathcal{O}_X \to \mathit{Sets}$, we see that the pushforward of $\mathcal{F}$ is exactly the contravariant mapping $\mathcal{F} \circ \pi^{-1} : \mathcal{O}_Y \to \mathit{Sets}$ which is a functor (ie. a presheaf) since the composition of two functors is a functor. Further, suppose that $\mathcal{F}$ is a sheaf. Exercise C showed that the sheaf conditions are equivalent to

$$\mathcal{F}(\mathrm{colim} U_i) = \lim(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j))$$

To see that this also holds for $\mathcal{F} \circ \pi^{-1}$, note that π^{-1} has a right adjoint and so it commutes with colimits. Thus, we have that

$$\begin{aligned}\mathcal{F} \circ \pi^{-1}(\mathrm{colim}\, U_i) &= \mathcal{F}(\mathrm{colim}\, \pi^{-1}(U_i)) \\ &= \lim(\prod \mathcal{F}(\pi^{-1}(U_i)) \rightrightarrows \prod \mathcal{F}(\pi^{-1}(U_i) \cap \pi^{-1}(U_j))) \\ &= \lim(\prod (\mathcal{F} \circ \pi^{-1})(U_i) \rightrightarrows \prod (\mathcal{F} \circ \pi^{-1})(U_i \cap U_j))\end{aligned}$$

□

Exercise 2.2.9. Suppose $\pi X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf of sets on X. If $\pi(p) = q$, describe a natural morphism of stalks $(\pi_*\mathcal{F})_q \to \mathcal{F}_p$.

Proof. Since each stalk is the colimit of open sets containing a point, to produce a map $(\pi_*\mathcal{F})_q \to \mathcal{F}_p$ it is enough to give a maps $\pi_*\mathcal{F}(V) \to \mathcal{F}_p$ for each open V contianing p, commuting with the restriction maps. Given such a V, define the map m_V by $m_V(f) = (f, \pi^{-1}(V))$. Note that the m_V's commute with the restriction maps since given $V' \subseteq V$, $(\mathrm{res}_{V,V'}(f), \pi^{-1}(V')) = (f, \pi^{-1}(V))$ as elements of $\mathcal{F}_p$ since they are equal when restricted to $\pi^{-1}(V')$ (since $\mathrm{res}_{V,V'}$ and $\mathrm{res}_{\pi^{-1}(V),\pi^{-1}(V)}$ are the same map). □

Exercise 2.2.10. If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, describe how for each $p \in X$, $\mathcal{F}_p$ is an $\mathcal{O}_{X,p}$-module.

Proof. We define an $\mathcal{O}_{X,p}$ action on $\mathcal{F}_p$ by $[(f, U)] \cdot [(g, V)] = [(f|_{U \cap V} \cdot_{U \cap V} g|_{U \cap V}, U \cap V)]$, ie. we take the $\mathcal{O}_X$-module action of f on g viewed as elements of the sections above $U \cap V$. Note that this action is well defined since given two representatives of the same germ, restricting them to an open set on which they agree, we see they induce the same action since the restriction maps commute the original $\mathcal{O}_X$ module action. □