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## 1 1-21

### 1.1 Motivation

Before starting the development of the general theory of  $C^*$ -algebras we consider a few motivating examples.

**Example 1.1.** Consider the Hilbert spaces  $\ell^2(\mathbb{Z})$ ,  $L^2(\mathbb{R})$  and define unitary operators  $U, V$  by  $(U\xi)(r) = \xi(r-1)$  and  $(V\xi)(r) = e^{2\pi i\theta r}\xi(r)$  for  $\theta \in \mathbb{R}$ . Consider the algebra generated by these two operators.  $VU = e^{2\pi i\theta}UV$  so the linear span of  $U^m V^n$  is an algebra. Denote its operator norm closure by  $\mathcal{A}_\theta$ . The algebras generated by  $U$  and  $V$  individually are both isomorphic to  $C(\mathbb{S}^1)$  and if  $\theta \in \mathbb{Z}$ ,  $U$  and  $V$  commute so  $\mathcal{A}_\theta \cong C(\mathbb{S}^1 \times \mathbb{S}^1)$ . Otherwise, we call  $\mathcal{A}_\theta$  a noncommutative 2-torus. Consider the self adjoint operator, called an almost Matthieu operator

$$H = (U + U^*) + \lambda(V + V^*)$$

Q: What is the spectrum of  $H$ ?

- For rational  $\theta$ , it is a disjoint union of intervals
- 1976: Hofstadter plotted the spectra for rational  $\theta$  to produce a fractal like pattern – the Hofstadter Butterfly. This led to the conjecture that the spectrum of  $H$  for irrational  $\theta$  was a Cantor set.
- 1981: Mark Kat offered 10 martinis for an answer to this conjecture.
- Barry Simon popularized this and it became known as the 10 martini problem
- 2005: Jitomirskaya, Avila gave an affirmative answer

Much of the partial progress on this problem came from the theory of  $C^*$ -algebras but the final proof used completely different techniques.

**Example 1.2.** Let  $G$  be a group. A (Hilbert Space) unitary representation of  $G$  is a group homomorphism  $\pi : G \rightarrow \mathcal{U}(H)$ . Consider the algebra of operators generated by  $\{\pi(x) : x \in G\}$ , ie. elements of the form  $\pi_f = \sum_{x \in G} f(x)\pi(x)$  for  $f : G \rightarrow \mathbb{C}$  finitely supported. We define a product on this set by

$$\left(\sum f(x)\pi(x)\right)\left(\sum g(y)\pi(y)\right) = \sum_{x,y} f(x)g(y)\pi(xy) = \sum_{x,y} f(x)g(x^{-1}y)\pi(y) = \sum_y \left(\sum_x f(x)g(x^{-1}y)\right)\pi(y)$$

Defining the convolution  $*$  by

$$(f * g)(y) = \sum_x f(x)g(x^{-1}y)$$

we have the  $\pi_f \cdot \pi_g = \pi_{f*g}$  and this gives an algebra structure on  $C_c(G)$  and an algebra homomorphism  $\pi : (C_c(G), *) \rightarrow B(H)$ . The convolution also gives  $\ell^1(G)$  an algebra structure since  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ .

For any  $f \in C_c(G)$ , let  $\|f\|_{C^*(G)} = \sup\{\|\pi_f\| : \pi \text{ unitary rep of } G\}$ . Observe that

$$\|\pi_f\| \leq \sum |f(x)| \|\pi(x)\| = \sum |f(x)| = \|f\|_1$$

and so  $\|f\|_{C^*(G)} \leq \|f\|_1$ . Let  $C^*(G)$  denote the completion of  $C_c(G)$  with respect to this norm. We can represent this algebra as an algebra bounded operators on some Hilbert space.

Note: for any group  $G$  we have at least two unitary representations – the trivial representation and the left regular representation on  $\ell^2(G)$  by translation.

## 2 1-23

**Example 2.1.** Using the left regular representation:  $H = \ell^2(G)$  with  $\mathcal{U}_x(\xi)(y) = \xi(x^{-1}y)$ , we can define a norm on  $C_c(G)$  by  $\|f\|_r = \|U_f\|$ . Note that the involution is given by  $f^*(x) = f(-x)$ . The closure of  $C_c(G)$  under this norm is called the reduced  $C^*$ -algebra for  $G$ . Since  $\|f\|_r \leq \|f\|_{C^*}$ , we only have that  $C_r^*(G)$  is a quotient of  $C^*(G)$  in general. In fact, the discrepancy between the two encodes information about the group  $G$ :

**Theorem 2.2.**  $C^*(G) = C_r^*(G)$  if and only if  $G$  is amenable.

**Example 2.3.** For  $X$  a compact Hausdorff space, we can consider a homomorphism  $\alpha : G \rightarrow \text{Homeo}(X)$  (eg. a dynamical system). This induces a map  $\alpha : G \rightarrow \text{Aut}(C(X))$ . We can often use this map to give insight into the behavior of the dynamical system as passing to the level of algebras gives more structure to leverage.

More generally, for a  $C^*$ -algebra  $\mathcal{A}$ , we call a map  $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$  a “noncommutative dynamical system”. We will look at representations of these systems:  $U : G \rightarrow \mathcal{U}(H)$ ,  $\pi : \mathcal{A} \rightarrow B(H)$  satisfying  $\pi(\alpha_x(a)) = U_x \pi(a) U_x^{-1}$  (the covariance relation). The algebra generated by  $(U_x, \pi(a))$  is called the crossproduct algebra.

**Example 2.4.** Let  $X = \mathbb{S}^1$  and  $a^\theta$  be rotation by  $\theta$ .

Ex:  $X = \mathbb{S}^1$ ,  $a^\theta$  = rotation by  $\theta$ . Then  $\mathbb{Z}$  acts by  $(a^\theta)^n$  and the crossproduct algebra is  $\mathcal{A}_\theta$ .

**Remark 2.5.** The construction of  $C^*(G)$  also works for groupoids

If  $X, Y$  are compact Hausdorff spaces, how are  $C(X \times Y)$  and  $C(X), C(Y)$  related. Consider functions of the form  $(f \otimes g)(x, y) = f(x)g(y)$ . The algebra generated by these functions is dense in  $C(X \times Y)$  by Stone Weierstrass.

**Example 2.6.** Can form algebra  $\mathcal{A} \oplus \mathcal{A} \oplus \dots$

Consider direct limit of algebras  $M_2 \hookrightarrow M_4 \hookrightarrow M_8 \dots$  with isometric embeddings  $T \mapsto \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ .

Called the CAR-algebra (canonical anticommutative relation).

Q: for different  $n$ , are the  $C^*$ -algebras generated by  $M_n \rightarrow M_{n^2} \dots$  nonisomorphic?

Yes (the proof uses methods from noncommutative geometry)

However, the Von-Neumann algebras generated by different  $n$  are all isomorphic.

**Definition 2.7.** A concrete  $C^*$ -algebra is a subalgebra of  $B(H)$ , for  $H$  a Hilbert space, which is norm closed and closed under  $*$ .

An abstract  $C^*$ -algebra is a Banach algebra with a conjugate linear involution  $*$  satisfying  $\|A^*A\| = \|A\|^2$ .

By the Gelfand-Naimark theorem (1943), every abstract  $C^*$ -algebra is isomorphic to a concrete  $C^*$ -algebra.

Little Gelfand-Naimark: If  $\mathcal{A}$  is a commutative  $C^*$ -algebra then  $\mathcal{A} \cong C_0(X)$  for  $X$  LCH.

If  $\mathcal{A}$  is a commutative unital  $C^*$ -algebra over  $\mathbb{C}$ , define  $\hat{\mathcal{A}} = \{\varphi : \mathcal{A} \rightarrow \mathbb{C}, \text{ unital hom}\}$ . For each  $a \in \mathcal{A}$ , define  $\hat{a}$  on  $\hat{\mathcal{A}}$  by  $\hat{a}(\varphi) = \varphi(a)$ . The map  $a \mapsto \hat{a}$  is a unital algebra homomorphism from  $\mathcal{A}$  to functions on  $\hat{\mathcal{A}}$ .

**Definition 2.8.** For  $a \in \mathcal{A}$ , define the spectrum of  $a$ ,  $\sigma(a) = \{\lambda: a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}$ .

If  $\mathcal{A}$  is a Banach algebra, then  $(1 - a)^{-1} = \sum a^n$  if  $\|a\| < 1$  and so if  $\lambda \in \sigma(a)$  then  $|\lambda| < \|a\|$ .  
Observe that if  $\varphi \in \hat{\mathcal{A}}$  then  $\varphi(a) \in \sigma(a)$  since  $\varphi(a - \varphi(a)1) = 0$  and so  $a - \varphi(a)1$  is not invertible.  
For  $a \in \mathcal{A}$ ,  $\hat{a} \in C(\hat{\mathcal{A}})$ ,  $a \mapsto \hat{a}$  called the Gelfand transform.  
Gelfand spectral radius formula:  $\sup\{|\lambda|: \lambda \in \sigma(a)\} = \lim \|a^n\|^{1/n}$ .  
So, if  $\|a^2\| = \|a\|^2$ , then Gelfand transform is isometric.

### 3 1-26

#### 3.1 The GNS Construction

**Definition 3.1.** An algebra involution  $*$  is a conjugate linear map so that  $(ab)^* = b^*a^*$  and  $(a^*)^* = a$ .

Given  $C(X)$  a unital  $C^*$ -algebra, how do we realize it as bounded operators on a Hilbert space  $H$ ?

Suppose  $\mu$  is a positive measure, viewed as a linear functional. Define a pre-inner product by  $\langle f, g \rangle = \int f\bar{g}d\mu = \mu(f\bar{g})$ . Quotienting out by norm zero elements, gives a Hilbert space on which  $C(X)$  acts by pointwise multiplication.

**Example 3.2.** Consider  $\delta_x(f) = f(x)$ . Then  $C(X) \subseteq \ell^\infty(X_{\text{disc}})$  acts on  $\ell^2(X_{\text{disc}})$ . This is called the atomic representation.

We would like to this construction on a general unital  $*$ -algebra.

**Definition 3.3.** Let  $\mu$  be a linear functional on  $\mathcal{A}$  a unital  $*$ -algebra. Say that  $\mu$  is positive if  $\mu(a^*a) \geq 0$  for  $a \in \mathcal{A}$ .

The positive linear functionals are closed under addition and multiplication by nonnegative scalars (ie. they form a cone)

Let  $\mu$  be a positive linear function on  $\mathcal{A}$ . Define a pre-inner product on  $\mathcal{A}$  by  $\langle a, b \rangle_\mu = \mu(b^*a)$ . To see that this is a pre-inner product observe that

$$\langle a, a \rangle_\mu = \mu(a^*a) \geq 0 \quad \text{and} \quad \langle a + b, a + b \rangle_\mu = \langle a, a \rangle_\mu + \langle a, b \rangle_\mu + \langle b, a \rangle_\mu + \langle b, b \rangle_\mu \geq 0$$

so  $0 = \text{Im}(\langle a, b \rangle_\mu) + \text{Im}(\langle b, a \rangle_\mu)$  and, replacing  $b$  with  $ib$ , gives  $\text{Re}(\langle a, b \rangle_\mu) = \text{Re}(\langle b, a \rangle_\mu)$ .

Let  $N_\mu = \{a \in \mathcal{A}: \langle a, a \rangle_\mu = 0\}$ . If  $a \in N_\mu$ , then  $|\langle a, b \rangle_\mu|^2 \leq \langle a, a \rangle_\mu \langle b, b \rangle_\mu = 0$  so  $N_\mu = \{a \in \mathcal{A}: \langle a, b \rangle_\mu = 0 \forall b\}$ . It follows that  $N_\mu$  is a left ideal of  $\mathcal{A}$ . On  $\mathcal{A}/N_\mu$  the pre-inner product becomes an inner product. Denote its completion by  $L^2(\mathcal{A}, \mu)$ .

Let  $\mathcal{A}$  act on itself on the left via  $\pi_a b = ab$ . This is a  $*$ -algebra homomorphism since

$$\langle \pi_a b, c \rangle_\mu = \langle ab, c \rangle_\mu = \mu(c^*ab) = \mu((a^*c)^*b) = \langle b, a^*c \rangle_\mu = \langle b, \pi_{a^*}c \rangle$$

Call this the left regular representation. Since  $N_\mu$  is a left ideal of  $\mathcal{A}$ , this gives an action of  $\mathcal{A}$  on  $\mathcal{A}/N_\mu$  as well. It remains to check that the  $\pi_a$ 's are continuous. This may fail without a complete norm as the next example shows.

**Example 3.4.** Let  $\mathcal{A}$  be the algebra of  $\mathbb{C}$ -valued polynomials on  $\mathbb{R}$ , and let  $\mu(p) = \int_{-\infty}^{\infty} p(t)e^{-t^2}dt$ . On  $L^2(\mathcal{A}, \mu)$ , multiplying by polynomials in an unbounded operator.

The above is called the GNS (Gelfand-Naimark-Segal) construction

**Theorem 3.5.** Let  $\mathcal{A}$  be a unital normed  $*$ -algebra,  $\mu$  a positive linear functional on  $\mathcal{A}$ .

- (1) If  $\mathcal{A}$  is complete, then  $\mu$  is continuous and  $\|\mu\| = \mu(1)$
- (2) If  $\mathcal{A}$  is not necessarily complete but  $\mu$  is continuous then  $\|\mu\| = \mu(1)$ .

## 4 1-28

### 4.1 Automatic Continuity

**Lemma 4.1.** Let  $\mathcal{A}$  be a unital  $*$ -Banach algebra over  $\mathbb{C}$ . If  $a \in \mathcal{A}$ ,  $a^* = a$ , and  $\|a\| < 1$ , then there is  $b \in \mathcal{A}$  with  $b^*b$  and  $1 - a = b^2$ .

*Proof.* The function  $f(z) = \sqrt{1-z}$ , has a power series expansion  $f(z) = \sum r_n z^n$ ,  $r_n \in \mathbb{R}$  which converges for  $|z| < 1$ . Setting  $b = \sum r_n a^n$ , it follows that  $b^2 = (f(a))^2 = (f^2)(a) = 1 - a$ .  $\square$

**Theorem 4.2.** If  $\mu$  is a positive linear function on  $\mathcal{A}$ , then  $\mu$  is continuous and  $\|\mu\| = \mu(1)$ .

*Proof.* Suppose  $a \in \mathcal{A}$  with  $a^* = a$  and  $\|a\| < 1$ . Then for  $b$  so that  $1 - a = b^*b$ , we have that

$$\mu(b^*b) = \mu(1) - \mu(a) \geq 0$$

and so  $\mu(1) \geq \mu(a)$ . It follows that if  $a^* = a$ ,  $\mu(a) \leq \|a\|\mu(1)$ . For general  $a$ ,

$$|\mu(a)|^2 = |\mu(a1)|^2 = |\langle 1, a \rangle_\mu|^2 \leq \mu(1)\mu(a^*a) \leq \mu(1)^2\|a^*a\| \leq \mu(1)^2\|a\|^2$$

$\square$

If  $(H, \pi)$  is a  $*$ -representation of  $\mathcal{A}$ , then for any  $\xi \in H$ , set  $\mu_\xi(a) = \langle \pi(a)\xi, \xi \rangle$ . This defines a positive linear functional with  $\|\mu_\xi\| = \|\xi\|^2$ . We also have that

$$\|\pi(a)\xi\|^2 = \langle \pi(a)\xi, \pi(a)\xi \rangle = \langle \pi(a^*a)\xi, \xi \rangle \leq \|a^*a\| \|\xi\|^2 \leq \|a\|^2 \|\xi\|^2$$

and so  $\|\pi(a)\| \leq \|a\|_{\mathcal{A}}$ .

**Theorem 4.3.** If  $\mu$  is a positive linear functional on  $\mathcal{A}$ , then the left regular representation on  $L^2(\mathcal{A}, \mu)$  defines a  $*$ -representation, ie. each  $\pi_a$  is a bounded linear operator (and  $\|\pi_a\| \leq \|a\|$ ).

*Proof.* For any  $a, b \in \mathcal{A}$ ,

$$\|\pi_a(b)\|^2 = \langle \pi_a(b), \pi_a(b) \rangle_\mu = \langle ab, ab \rangle_\mu = \mu(b^*a^*ab) \leq \|b^*a^*ab\| \leq \|b^*\| \|a^*a\| \|b\| \mu(1) \leq \|a\|^2 \|b\|^2 \mu(1)$$

$\square$

### 4.2 Cyclic Representations

**Definition 4.4.** Let  $\mathcal{A}$  be a unital  $*$ -Banach algebra and  $(H, \pi)$  a continuous  $*$ -representation. Then  $\xi \in H$  is said to be (topologically) cyclic if  $\{\pi_a\xi : a \in \mathcal{A}\}$  is dense in  $H$ .

**Proposition 4.5.** Let  $(H, \pi)$  be a  $*$ -representation of  $\mathcal{A}$ . If  $K \subseteq H$  is a closed  $\pi$ -invariant subspace then  $K^\perp$  is also  $\pi$ -invariant

*Proof.* For any  $\eta \in K^\perp$ ,  $\xi \in K$ ,  $\langle \xi, \pi_a\eta \rangle = \langle \pi_a^*\xi, \eta \rangle = 0$ .  $\square$

Given  $(H, \pi)$  a  $*$ -representation, for any nonzero  $\xi \in H$ ,  $K = \overline{\{\pi_a : a \in \mathcal{A}\}\xi}$  is a closed cyclic subspace and we can write  $H$  as the sum of closed  $\pi$ -invariant subspaces  $H = K \oplus K^\perp$ . Iterating this (possibly a transfinite number of times) shows that any  $*$ -representation  $(H, \pi)$  can be decomposed into cyclic invariant subspaces.

## 5 1-30

### 5.1 Integrated Forms

Let  $G$  be a group and  $U$  a unitary representation of  $G$ . Given  $f \in C_c(G)$ , define  $U_f = \sum f(x)U_x$ . Then,  $U_f U_g = U_{f * g}$  and  $\|U_f\| \leq \|f\|_1$  so  $f \mapsto U_f$  is a continuous  $*$ -representation of  $\ell^1(G)$ . The assignment  $f \mapsto U_f$  is called the integrated form corresponding to  $U$ . Conversely, if  $(H, \pi)$  is a  $*$ -representation of  $\ell^1(G)$ , then  $x \mapsto \pi_{\delta_x}$  gives a unitary representation of  $G$ . So, there is a bijection between unitary representations of  $G$  and continuous  $*$ -representations of  $C_c(G)$  or  $\ell^1(G)$ .

If  $\varphi$  is a continuous positive linear functional on  $\ell^1(G)$ , then  $\varphi \in \ell^\infty(G)$ . So to any positive linear functional  $\mu$ , there is an associated  $\mu_f \in \ell^\infty(G)$ . We call such a  $\mu_f$  a function of positive type.

**Example 5.1.** Let  $(H, \pi)$  be the trivial representation of  $G$  on  $\mathbb{C}$ . Its integrated form is  $f \mapsto \sum f(x)$  and so the corresponding function of positive type is  $\varphi(x) = 1$  for all  $x$ .

**Claim:**  $\delta_e \in \ell^\infty(G)$  is of positive type.

$$\langle f^* * f, \delta_e \rangle = (f^* * f)(e) = \sum \overline{f(y^{-1})} f(y^{-1}e) = \sum |f(y^{-1})|^2 \geq 0$$

What does the GNS construction give?

$$\langle f, g \rangle_{\delta_e} = \sum \overline{g(y)} f(y) = \langle f, g \rangle_{\ell^2(G)}$$

So we recover the left regular representation of  $G$ .

## 5.2 Traces

**Definition 5.2.** A positive linear functional on a normed  $*$ -algebra is tracial (is a trace) if  $\mu(ab) = \mu(ba)$  for all  $a, b \in \mathcal{A}$ .

**Claim:**  $\delta_e$  is a trace on  $\ell^1(G)$

$$\langle f * g, \delta_e \rangle = \sum f(y)g(y^{-1}) = \sum f(y^{-1})g(y) = \langle g * f, \delta_e \rangle$$

Let  $\mathcal{A}$  be a unital Banach  $*$ -algebra,  $\mu$  a tracial positive linear functional. Let  $N_\mu = \{a : \langle a, b \rangle_\mu = 0 \ \forall b\}$ . Then  $N_\mu$  is a two sided ideal of  $\mathcal{A}$  and  $\mathcal{A}/N_\mu$  is an  $A, A$ -bimodule. So  $L^2(A, \mu)$  admits both left and right regular representations.

The representation  $L^2(A, \mu)$  is cyclic vector with cyclic vector 1. Any vector  $\xi$  gives a positive linear functional via  $a \mapsto \langle a\xi, \xi \rangle$  and the positive linear function corresponding to 1 gives back  $\mu$ . So, any positive linear functional on  $\mathcal{A}$  comes from a cyclic vector in some  $*$ -representation.

**Example 5.3.** For the regular representations of  $\ell^1(G)$  or  $\ell^\infty(G)$  on  $\ell^2(G)$ , the vector  $\delta_e$  is cyclic for the representation. It defines a trace on the image of  $\ell^1(G)$ . Let  $C_r(G)$  denote the norm closure of the image of  $\ell^1(G)$  in  $B(\ell^2(G))$ .  $\delta_e$  defines a trace on this sub  $C^*$ -algebra,  $\|\delta_e\| = 1$ .

It is not possible to extend this to a trace on  $B(H)$  in general since iff  $H$  is infinite dimensional, then  $B(H)$  admits no bounded trace.

## 6 2-2

### 6.1 Direct Sums of Hilbert Spaces

Given a collection of Hilbert spaces  $\{H_j\}_{j \in J}$ , we can form the algebraic direct sum as

$$\bigoplus_{\text{alg}} H_j = \{\xi \in \prod H_j : \xi_j = 0 \text{ for all but finitely many } j\}$$

This is an inner product space with

$$\langle \xi, \eta \rangle = \sum_j \langle x_j, \eta_j \rangle$$

Taking the completion with respect to this inner product gives the direct sum (coproduct) of the  $H_j$ 's as Hilbert spaces. One can check that

$$\bigoplus H_j = \{\xi \in \prod H_j : \sum \|x_i\|_j^2 < \infty\}$$

Given  $(T_j)_{j \in J}$  with  $T_j \in B(H_j)$ , can define  $T\xi = (T_j\xi_j)$ . If  $\{\|T_j\| : j \in J\}$  is bounded then  $T \in B(H)$  with  $\|T\| = \sup_j \|T_j\|$ . Similarly, if  $(H_j, \pi_j)$  is a family of unitary representations of  $G$  ( $*$ -representations of a Banach algebra), we can define a  $*$ -representation on  $\bigoplus H_j$  by  $U_x\xi = (U_x^j\xi_j)$ ,  $(\pi_a\xi = (\pi_a^j\xi_j))$ . Note that these are bounded representations since  $\|U_x^j\| = 1$  and  $\|\pi_a^j\| \leq \|a\|$  for each  $j$ .

If  $(H, \pi)$  is a  $*$ -representation, then it decomposes into a direct sum  $\bigoplus (H_j, \pi_j)$  with each  $(H_j, \pi_j)$  a cyclic representation.

## 6.2 Intertwining Operators

**Definition 6.1.** Suppose  $(H, \pi)$ ,  $(K, \rho)$  are  $*$ -representations of  $\mathcal{A}$  (unitary reps of  $G$ ). A morphism of representations is an  $\mathcal{A}$  module homomorphism  $T : H \rightarrow K$ , ie. a bounded linear operator  $T : H \rightarrow K$  so that for each  $a \in \mathcal{A}$ ,  $T\pi_a = \rho_a T$ . We call such a map an intertwining operator.

**Definition 6.2.** An isomorphism of representations is an intertwining unitary operator  $U : H \rightarrow K$ .

**Definition 6.3.** A pointed cyclic representation  $(H, \pi, \xi_*)$  is a cyclic representation with a specified cyclic vector  $\xi_*$ .

**Proposition 6.4.** Let  $(H, \pi, \xi_*)$  and  $(K, \rho, \eta_*)$  be pointed cyclic representations. If  $\mu_{\xi_*} = \mu_{\eta_*}$  there is a unitary intertwining operator  $U$  from  $H$  to  $K$  so that  $U\xi_* = \eta_*$ .

*Proof.* Define  $U$  on the dense subspace  $\{\pi_a \xi_* : a \in \mathcal{A}\}$  by  $U(\pi_a \xi_*) = \rho_a \eta_*$ .  $U$  then extends uniquely to a continuous map  $H \rightarrow K$ . It is clear that  $U$  is intertwining. Further, for any  $a, b \in \mathcal{A}$ ,

$$\langle U\pi_a \xi_*, U\pi_b \xi_* \rangle = \langle \rho_a \eta_*, \rho_b \eta_* \rangle = \langle \rho_b^* \rho_a \eta_*, \eta_* \rangle = \mu_{\eta_*}(b^* a) = \mu_{\xi_*}(b^* a) = \langle \pi_b^* \pi_a \xi_*, \xi_* \rangle = \langle \pi_a \xi_*, \pi_b \xi_* \rangle$$

So  $U$  is unitary as well.  $\square$

It follows that there is a bijection between positive linear functions on  $\mathcal{A}$  and isomorphism classes of pointed cyclic representations.

**Definition 6.5.** A  $*$ -representation  $(H, \pi)$  irreducible if it has no proper closed  $\pi$ -invariant subspaces.

Given a finite dimensional representation, it can be decomposed into a sum of irreducible representations.

## 7 2-4

### 7.1 Schur's Lemma

**Definition 7.1.** Let  $H$  be a Hilbert space and  $\mathcal{C} \subseteq B(H)$ . The commutant of  $\mathcal{C}$  is defined to be

$$\mathcal{C}' = \{T \in B(H) : TC = CT \forall C \in \mathcal{C}\}$$

The commutant is closed with respect to the strong operator topology and so is a Von Neumann algebra (and hence a  $C^*$ -algebra). For any  $\mathcal{C}$ ,  $(\mathcal{C}')'$  is the strong closed  $*$ -algebra generated by  $\mathcal{C}$ . The double commutant theorem was proved by Von Neumann in the first paper on operator algebras.

If  $(H, \pi)$  is a representation of  $\mathcal{A}$ , then  $\text{End}_{\mathcal{A}}(H)$  is the commutant of  $\{\pi_a\}_{a \in \mathcal{A}}$ .

**Lemma 7.2** (Schur's Lemma). A representation  $(H, \pi)$  of  $\mathcal{A}$  is irreducible if and only if  $\text{End}_{\mathcal{A}}(H) = \mathbb{C}$ .

*Proof.* If  $(H, \pi)$  is not irreducible then there is a proper closed invariant subspace  $K \subseteq H$ . Taking  $P$  to be the orthogonal projection onto  $K$ , we have that  $P \in \text{End}_{\mathcal{A}}(H)$  but  $P$  is not a scalar multiple of the identity. Conversely, if  $\text{End}_{\mathcal{A}}(H) \neq \mathbb{C}$  then let  $T$  be an operator which is not a scalar multiple of the identity. Since either the real or imaginary part of  $T$  is also not a scalar multiple of the identity we may assume that  $T$  is self adjoint without loss of generality. There are  $s \neq t \in \sigma(T)$  and so let  $f, g$  be functions so that  $f(s) = g(t) = 1$  and  $fg = 0$ . This  $F, G \in C^*(I, T)$  so that  $F, G \neq 0$  and  $FG = 0$ . Now,  $\langle FH, GH \rangle = 0$  and  $GH \neq 0$  so, since  $F \in \text{End}_{\mathcal{A}}(H)$ ,  $\overline{F}H$  is a proper invariant subspace.  $\square$

**Corollary 7.3.** If  $\mathcal{A}$  is a commutative unital normed  $*$ -algebra, then every irreducible representation of  $\mathcal{A}$  is one dimensional.

*Proof.* If  $\mathcal{A}$  is commutative, then for representation  $(H, \pi)$ ,  $\pi_a \in \text{End}_{\mathcal{A}}(H)$  for  $a \in \mathcal{A}$ . So if the representation is irreducible it follows that  $\pi_a \in \mathbb{C}I$  for each  $A$  and so every subspace is invariant. Thus,  $H$  must be one dimensional.  $\square$

**Example 7.4.** If  $X$  is a compact Hausdorff space, then the irreducible representations of  $X$  are precisely evaluations at points in  $x$ . More generally, if  $\mathcal{A}$  is a commutative  $*$ -Banach algebra, then its irreducible representations are exactly the characters on  $\mathcal{A}$ .

**Example 7.5.** Consider the algebra  $\mathcal{A} = \mathbb{C} \oplus \mathcal{C}$  with  $(\alpha, \beta)^* = (\overline{\beta}, \overline{\alpha})$ . There are no nonzero  $*$ -algebra homomorphisms from  $\mathcal{A}$  to  $\mathbb{C}$  since  $(1, 0)$  generates it as a  $*$ -algebra but  $(1, 0)(1, 0)^* = 0$  and so  $\mu(1, 0) = 0$  for any  $\mu : \mathcal{A} \rightarrow \mathbb{C}$ .

## 7.2 Radon-Nikodym for Positive Linear Functionals

**Definition 7.6.** Let  $\mathcal{A}$  be a unital  $*$ -algebra and let  $\mu, \nu$  be positive linear functionals on  $\mathcal{A}$ . Say that  $\mu$  dominates  $\nu$ ,  $\mu \geq \nu$  if  $\mu - \nu \geq 0$ .

How do we get such a  $\nu$ ? Given  $\mu$ , consider the pointed representation  $(H_\mu, \pi_\mu, \xi_\mu)$ . For any  $T \in \text{End}_{\mathcal{A}}(H)$  with  $0 \leq T \leq I$ , define

$$\nu(a) = \langle \pi_a T \xi_\mu, \xi_\mu \rangle$$

We have that

$$\nu(a^*a) = \langle T \pi_a \xi_\mu, \pi_a \xi_\mu \rangle \quad \text{and} \quad (\mu - \nu)(a^*a) = \langle (I - T) \pi_a \xi_\mu, \xi_\mu \rangle$$

and so  $0 \leq \nu \leq \mu$ . Here  $T$  is the “Radon-Nikodym derivative” of  $\nu$ .

We will show that all such  $\nu$  arise in this way.

**Theorem 7.7.** If  $\mu \geq \nu \geq 0$ , then  $\exists T \in \text{End}_{\mathcal{A}}(H_\mu)$ ,  $0 \leq T \leq I$  such that  $\nu(a) = \langle \pi_a T \xi_\mu, \xi_\mu \rangle$ .

**Fact:** Let  $H$  be a Hilbert Space,  $K$  a dense subspace, and  $\langle \cdot, \cdot \rangle_K$  a pre-inner product on  $K$  so that  $\langle \eta, \eta \rangle_K \leq \langle \eta, \eta \rangle_H$  for  $\eta \in K$ . Then, there is an operator  $T \in B(H)$  with  $0 \leq T \leq I$  and so that  $\langle \eta, \eta \rangle = \langle T\eta, \eta \rangle$  for  $\eta \in K$ .

*Proof.* Given  $\eta \in K$ , consider the conjugate-linear functional given by  $\varphi_\eta(\xi) = \langle \eta, \xi \rangle_K$ .  $\varphi_\eta$  extends uniquely to  $H$  and so there is some  $\xi_\eta$  so that  $\varphi_\eta(\xi) = \langle \xi_\eta, \xi \rangle$  for all  $\xi \in K$ . Consider the linear map defined on  $K$  by  $T\eta = \xi_\eta$ . Note that  $T$  is continuous since  $\|\xi_\eta\| = \|\eta\|_K \leq \|\eta\|_H$  and so extends uniquely to an element in  $B(H)$  satisfying the desired conditions.  $\square$