LIMITS: INFO SHEETS

Definitions

- **Equals** is designated with an equals sign $=$.
- **Approaching** is designated with an arrow $\rightarrow$.
- A **finite number** is any real number in the set $(-\infty, \infty)$.   Note: ∞'s are not included.
    - Examples of finite numbers include any of the following familiar constants and much much more!

      0, 1,2, -⅓, , 365, 86, -89987665433222, $ln(2)$, $-6log_7(2)$, π, e, $\sqrt{\pi}$, e^{-8}, etc. etc.
- **Infinity** is not a real number! It's something you *approach* but never "get to".
- A **finite nonzero number** is any real number in the set $(-\infty, 0) \cup (0, \infty)$.   Note: zero and ∞'s are not included.
- An **undefined quantity** is nonsense, it gives NO INFORMATION about the value of the limit.
    - Examples of undefined quantities: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, $\infty \cdot 0$
    - If you encounter an undefined quantity, you must proceed by doing valid algebra or other valid math to obtain the answer.

Don't memorize, understand!

The info sheets will help initially,

but you need to learn and understand the material WITHOUT aid of the info sheets!

LIMITS: THE BASICS

Plug x=a into **numerator f(x)** and **denominator g(x)**. **USE ARROWS** to record the results $\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to A}{\to B}$.

- **A,B are both finite nonzero:** $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{A}{B}$
- **A is finite nonzero and ...**
  - **... B is infinite:** $\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to A}{\to\infty}\to 0$
  - **... B is zero:** $\left(\frac{\to A}{\to 0}\to\infty\right)$ ☆

	A positive	A negative
B $\to 0^{+}$	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to A^{(+)}}{\to 0^{(+)}}\to+\infty$	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to A^{(-)}}{\to 0^{(+)}}\to-\infty$
B $\to 0^{-}$	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to A^{(+)}}{\to 0^{(-)}}\to-\infty$	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to A^{(-)}}{\to 0^{(-)}}\to+\infty$

- **B is finite nonzero and ...**
  - **... A is zero:** $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{B}=0$
  - **... A is infinite:** $\left(\frac{\to\infty}{\to B}\to\infty\right)$ ☆

	A $\to +\infty$	A $\to -\infty$
B positive	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to +\infty}{\to B^{(+)}}\to+\infty$	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to -\infty}{\to B^{(+)}}\to-\infty$
B negative	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to +\infty}{\to B^{(-)}}\to-\infty$	$\lim_{x\to a}\frac{f(x)}{g(x)}\frac{\to -\infty}{\to B^{(-)}}\to+\infty$

☆ These cases require an extra step. Mentally test out x-values **VERY VERY** (infinitesimally) close to x=a, to see whether A is positive or negative in the numerator, and to see whether denominator approaches zero with positive or negative numbers. Use your results to add in superscripts indicating your results, then conclude plus/minus infinity according to the charts.

LIMITS: DEALING WITH ZEROS & INFINITIES TOGETHER

Plug x=a into **numerator f(x)** and **denominator g(x)**. **USE ARROWS** to record the results $\lim_{x\to a} \frac{f(x)}{g(x)} \quad \frac{\to A}{\to B}$.

- **A,B are both zero or both infinity:** $\lim_{x\to a} \frac{f(x)}{g(x)} \quad \frac{\to 0}{\to 0}$ OR $\lim_{x\to a} \frac{f(x)}{g(x)} \quad \frac{\to \infty}{\to \infty}$
  - Basic Strategy:
    - Use algebra to simplify your quantity $\frac{f(x)}{g(x)}$.
    - Foil, factor, or multiply by the conjugate -- as is appropriate for your problem.
    - Cancel, simplify, reduce, until eventually the limit evaluates to something ON THE PREVIOUS PAGE.
  - Advanced Strategy:
    - Use L'Hopital's Rule: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
    - Apply L'Hopital's rule *mindfully:* watch out for basic algebra, simplifications, reductions.
    - After L'Hopital & simplifying, eventually the limit will evaluate to something ON THE PREVIOUS PAGE.
- **One of A,B is zero and the other is infinity:** $\lim_{x\to a} \frac{f(x)}{g(x)} \quad \frac{\to 0}{\to \infty}$ OR $\lim_{x\to a} \frac{f(x)}{g(x)} \quad \frac{\to \infty}{\to 0}$
  - Rewrite $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and then use limit laws.
    - **If A is zero, and B is infinity** then you will get...

      $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \cdot \frac{1}{g(x)} = \lim_{x\to a} f(x) \cdot \lim_{x\to a} \frac{1}{g(x)} = 0 \cdot \left(\frac{\to 1}{\to \infty}\right) \to 0 \cdot 0 \to 0 .$$

    - **If A is infinity, and B is zero** then you will GENERALLY get…

      $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \cdot \frac{1}{g(x)} = \lim_{x\to a} f(x) \cdot \lim_{x\to a} \frac{1}{g(x)} = \infty \cdot \left(\frac{\to 1}{\to 0}\right) \to \infty \cdot \infty \to \infty .$$

      ☆ Decide whether the answer is $+\infty$ or $-\infty$ by testing out x-values infinitesimally close to x=a.

      Can you create a chart of the possibilities? Hint: $+\infty \cdot \left(\frac{\to 1}{\to 0^{(-)}}\right) \to \; +\infty \cdot (-\infty) \to \; -\infty$

ADVANCED TOPIC: ZEROS TIMES INFINITY

- Suppose you had a product limit $\lim_{x \to a} f(x) \cdot g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$. Now pick whichever option is easier:
  - Option #1: Rewrite $f(x) \cdot g(x) = \dfrac{f(x)}{\left(\frac{1}{g(x)}\right)}$, then you will get ...

    $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} \dfrac{f(x)}{\left(\frac{1}{g(x)}\right)} \longrightarrow \dfrac{\to 0}{\left(\frac{1}{\to \infty}\right)} \longrightarrow \dfrac{\to 0}{\to 0}$ NOW APPLY L'HOPITAL' RULE, see previous page
  - Option #2: Rewrite $f(x) \cdot g(x) = \dfrac{g(x)}{\left(\frac{1}{f(x)}\right)}$, then you will get …

    $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} \dfrac{g(x)}{\left(\frac{1}{f(x)}\right)} \longrightarrow \dfrac{\to \infty}{\left(\frac{1}{\to 0}\right)} \longrightarrow \dfrac{\to \infty}{\to \infty}$ NOW APPLY L'HOPITAL' RULE, see previous page
  - Having the foresight to determine whether Option #1 or Option #2 is better requires skill.

Practice, practice, practice, in order to get the skills necessary to make the right choices.