A Guide to Finding Expected Value

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Tips and Tricks:

- Recursive probelms will follow the **Conditional Expectation Formula**

1 Locate the Randomness

To make things easier for us later on, we want to *isolate* the randomness.

2 Fit to a Formula

Now that we know what parts are random, we can find the formula that fits best.

Normal Expectation (one of the loops executes some random amount of times):

$$ET(n) = \sum_{q=1}^{n} T(k=q) \cdot \mathbf{Prob}(k=q)$$

```
for i = 1 to n do
    k = Random(i)
    for j = 1 to k² do
        s = s + A[i] · A[j]
    end for
end for
```

Conditional Expectation (we branch to different loops based on our random variable):

$$ET(n) = \sum_{q=1}^{n} E(X \mid A)\, \mathbf{Prob}(A) + E(X \mid \texttt{not } A)\, \mathbf{Prob}(\texttt{not } A)$$

```
k = Random(n)
if k < n/2 then
    for i = 1 to log(n) do ...
else
    for i = 1 to n do ...
endif
```

3 Find the Expected Value

With the appropriate formula selected, we can solve to find the expected value.

Once we determine the expected value, we can us that result for the rest of our calculation because of the **linearity of expectation**.

This will give us a value that we can then use to solve the rest of our summations with.

4 Solving for Expected Value

From the **Condtional Expectation** code①.

$$ET(n) = \sum_{q=1}^{n} E(T(n) \mid k \le \frac{n}{2})\, \mathbf{Prob}(k \le \frac{n}{2}) + E(T(n) \mid k > \frac{n}{2})\, \mathbf{Prob}(k > \frac{n}{2})$$

$$= \frac{1}{2}\, c\log(n) + \frac{1}{2}\, cn$$

After this point, the problem may not be done but the probability is.

5 Breaking out of a loop early

When there's a random element that causes a loop to exit early, we use the **Normal Expectation Formula②** and the following probability:

$$\sum_{i=1}^{n} \mathbf{Prob}(x \le i)$$

Where x is the number of times that we don't exit early.

6 Breaking out of a loop early (continued)

Now we can simplify the probability

$$\mathbf{Prob}(x \le 1) = \mathbf{Prob}(x \le 1)$$
$$\mathbf{Prob}(x \le 2) = \mathbf{Prob}(x \le 1)^2$$
$$\mathbf{Prob}(x \le 3) = \mathbf{Prob}(x \le 1)^3$$
$$\cdots$$
$$\mathbf{Prob}(x \le q) = \mathbf{Prob}(x \le 1)^q$$

After simplifying, we find that the sum is *geometric.*

$$\sum_{q=1}^{n} \mathbf{Prob}(x \le q)^q$$