Fiber bundles and cocycles.

Principal bundles.

Connections

Gauge groups

Clifford algebras and spinor bundles

Representations → Edson de Faria

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① Gauge theory: Theory of principal bundles and associated vector bundles.

② Yang-Mills theory = Theory of principal bundles alone.

Fibre bundles can be thought of as twisted, non-trivial products between a base manifold and a fibre manifold (much like TM)

Principal and vector bundles are fibre bundles whose fibres are (resp.) Lie groups and vector spaces. In these cases the bundle admits a special type of bundle atlas, preserving some of the additional structure of the fibres.

③ Fundamental object of a gauge theory: A principal bundle over spacetime with structure group given by the gauge group.

The fibres of a principal bundle are sometimes thought of as an internal space at every space-time point (as is the case with the tangent bundle, of course), not belonging to space-time itself (like the concept of virtual displacements in mech.)

Connections on principal bundles will describe gauge fields, whose particle excitations in QFT are the gauge bosons that transmit interactions.

Matter fields in the standard model (quarks, leptons), or scalar fields (Higgs, for instance) are sections of vector bundles associated to the principal bundle (and twisted by spinor bundles in the case of fermions).

The underlying reason for the interaction between matter and gauge and matter fields is that they are both related to the same principal bundle

Gauge fields $\longrightarrow$ Connections on principal bundle

Matter fields, scalar fields $\longrightarrow$ Sections on a vector bundle assoc. to
Fermion fields $\qquad$ the principal bundle.

General fibre bundles

Let E, M be smooth manifolds and $\pi : E \longrightarrow M$ a surjective differentiable map between them.

Definition.

① Let $x \in M$ be any point. The set (nonempty, for π is onto)

$$E_x := \pi^{-1}\{x\} \subset E$$

is called the fibre of π over x.

② For a subset $U \subset M$ we set

$$E_U := \pi^{-1}(U) \subset E,$$

the part of E "above" U.

③ A differentiable map $s: M \longrightarrow E$ such that
$$\pi \circ s = id_M$$
is called a (global) section of π. A diff. map $s: U \subset M \longrightarrow E$ defined on some open subset $U \subset M$ of M, such that
$$\pi \circ s = id_U ,$$
is called a local section of π.

Note that $s: U \subset M \longrightarrow E$ is a (local) section of $\pi: E \longrightarrow U$ iff
$$s(x) \in E_x , \quad \forall x \in U,$$
this is obvious.

One observes that for a general surjective map, the fibres E_x and E_y over points $x \neq y \in M$ can be very complicated and different, and in part. they may not be submanifolds of E, and even if they are, it may be the case that $E_x \neq E_y$. The simplest example where
$$E_x \text{ is submanifold of } E, \ \forall x \in M, \text{ and}$$
$$E_x \simeq E_y . \ \forall x, y \in M,$$
is for $E := M \times F$ and $\pi : M \times F \longrightarrow M$ the projection onto the first factor, M.

Fiber bundles are an important generalisation of products $E := M \times F$ and can be viewed as twisted products. The fibres E_x for $x \in M$ of a fibre bundle are still submanifolds of E and are all diffeomorphic. However, the fibration E is only locally trivial, i.e., locally a product.

Definition. Let E, F, M be manifolds and $\pi : E \longrightarrow M$ a surjective differentiable map. Then $(E, \pi, M; F)$ is called a fibre bundle if the following holds:

$\forall x \in M$, there is an open neighbourhood $U \in \mathcal{U}_M(x)$ such that $\pi | E_U$ can be trivialized in the following sense:

there exists a diffeomorphism

$$\varphi_U : E_U \longrightarrow U \times F,$$

such that

$$pr_1 \circ \varphi_U = \pi .$$

Hence π factors through φ_U and the canonical projection:

$$\begin{tikzcd} E_U \arrow[r, "\varphi_U"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "pr_1"] \\ & U \end{tikzcd}$$

We also write

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

to denote a fibre bundle. We call:

E the total space

M the base manifold

F the general fibre

π the projection

(U, φ_U) a local trivialisation or bundle chart

Using a local trivialisation (U, φ_U), we see that the fibre

$$E_x := \pi^{-1}\{x\}$$

is an embeded submanifold of the total space E, $\forall x \in M$.

Also, the map

$$\varphi_{U_x} := pr_2 \circ \varphi_U | E_x : E_x \longrightarrow F$$

$$\boxed{\begin{tikzcd} E_x \arrow[r, "\varphi_U|E_x"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "pr_1"] \\ & \{x\} \end{tikzcd}}$$

is always a diffeomorphism between the fibre over $x \in U$, E_x, and the general fibre, F.

Note that in a local trivialisation the map

$$\varphi_U : E_U \longrightarrow U \times F$$

is a diffeomorphism and

$$pr_1 : U \times F \longrightarrow U$$

is a submersion (its differential is everywhere surjective). This implies that the projection $\pi : E \longrightarrow M$ of a fibre bundle is always a submersion.

Example. (Trivial bundle) Let M and F be arbitrary smooth manifolds and $E = M \times F$; then $\pi := pr_1$ defines the trivial bundle:

$$\begin{tikzcd} F \arrow[r] & M \times F \arrow[d, "pr_1"] \\ & M \end{tikzcd} \qquad (\text{trivial, for } \varphi_U = id_U).$$

Bundle maps. Let $F \longrightarrow E \xrightarrow{\pi} M$ and $F' \longrightarrow E' \xrightarrow{\pi'} M$ be fibre bundles over the same manifold M.

A bundle morphism of these bundles is a smooth map

$$H : E \longrightarrow E'$$

such that

$$\pi' \circ H = \pi,$$

i.e., such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "H"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

If H is a diffeomorphism, then it is called a bundle isomorphism, and we write $E \simeq E'$.

Remark. Note that a morphism $H : E \longrightarrow E'$ maps a point in the fibre E_x of E over $x \in M$ to a point in the fibre E'_x of E' over $x \in M$; indeed,

$$p \in E_x \iff \pi(p) = x; \text{ and so } \pi' \circ H(p) = \pi(p) = x \iff H(p) \in E'_x.$$

A bundle morphism $H : E \longrightarrow E'$ therefore covers the identity of M. Further, it is clear that a bundle isomorphism induces a diffeomorphism between the fibres of E and E' over any $x \in M$.

Definition. Fibre bundles isomorphic to a trivial bundle are also called trivial.

Bundle atlases

Definition A <u>bundle atlas</u> for a fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

is an open cover $\{O_\alpha\}_{\alpha \in \Omega}$ of M together with bundle charts

$$\varphi_i : E_{O_i} \longrightarrow O_i \times F .$$

Definition Let $\{(O_i, \varphi_i)\}_{i \in \Omega}$ be a bundle atlas for a fibre bundle $F \longrightarrow E \xrightarrow{\pi} M$. If $O_i \cap O_j \neq \emptyset$, we define the <u>transition functions</u> by the <u>diffeomorphisms</u>:

$$\varphi_j \circ \varphi_i^{-1} |(O_i \cap O_j) \times F : (O_i \cap O_j) \times F \longrightarrow (O_i \cap O_j) \times F . \quad (*)$$

These maps have a special structure, because <u>they preserve fibres</u>: For every $x \in O_i \cap O_j$ we get a diffeomorphism

$$\varphi_{jx} \circ \varphi_{ix}^{-1} : F \longrightarrow F,$$

given by the restriction of the map $(*)$ to $\{x\} \times F$.

The maps

$$\varphi_{ij} : O_i \cap O_j \longrightarrow \mathrm{Diff}(F); \; x \longmapsto \varphi_{jx} \circ \varphi_{ix}^{-1}$$

from $O_i \cap O_j$ into the group of diffeomorphisms of F are also called <u>transition functions</u>

Lemma (Cocycle condition) The transition functions $\{\varphi_{ij}\}_{ij\in\Omega}$ satisfy the following equations:

$$\varphi_{ii}(x) = id_F,$$

$$\varphi_{ij}(x) \circ \varphi_{ji}(x) = id_F,$$

$$\varphi_{ij}(x) \circ \varphi_{jk}(x) \circ \varphi_{ki}(x) = id_F.$$

Proof: Immediate from the definition of φ_{ij}.

Sections of bundles

We want to study sections of fibre bundles, which are remarkably simple in the case of trivial bundles.

Definition. Let $F \longrightarrow E \xrightarrow{\pi} M$ be a fibre bundle. We denote the set of all smooth global sections $s: M \longrightarrow E$ by $\Gamma(E)$ (smooth right inverses of the projection $\pi: E \to M$).

Likewise, the set of all smooth local sections $s: U \subset M \longrightarrow E$, for $U \subset M$ open, is denoted by $\Gamma(U, E)$.

Remark. Note that if $E := M \times F$ is a trivial bundle, then for each smooth map $\psi: M \longrightarrow F$, we have exactly one map $\overline{\psi}: M \longrightarrow M \times F =: E$ which is right inverse to $pr_1: E = M \times F \longrightarrow M$, und zwar:

$$\overline{\psi}(a) = (a, \psi(a)), \quad \forall a \in M.$$

COROLLARY: (Existence of (local) sections)

① Every trivial fibre bundle has smooth global sections (for example, under the shown 1-1 correspondence, one could take constant maps $\psi: M \longrightarrow F$.)

② Every fibre bundle has smooth local sections, since by def. every fibre bundle is locally trivial.

Principal fibre bundles:

We talk about fibre bundles that also have a Lie group action, so that both structures are compatible in a certain way.

Principal bundles, along with connections play an important role in gauge theory. In general terms, principal bundles are the primary place where Lie groups appear in gauge theories.

Definition (Principal bundle). Let

$$\begin{tikzcd} G \arrow[r] & P \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

be a fibre bundle with a Lie group G as a general fibre and a smooth action $P \times G \longrightarrow P$ on the right. Then P is called a principal G-bundle if the following conditions hold: