SUPPLEMENT TO "ASYMPTOTIC THEORY OF GENERALIZED ESTIMATING EQUATIONS BASED ON JACK-KNIFE PSEUDO-OBSERVATIONS"

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This supplement contains an overview of the theory of differentiable functionals used in the primary paper, some details on the p-variation setting, a note on the measurability of the influence functions that play an important role in the primary paper, and a detailed proof of Proposition 4.2 of the primary paper.

1. Differentiability of functionals. There are different concepts of differentiability for functionals between Banach spaces. In the primary paper we use the Fréchet differentiability concept, following especially Dudley and Norvaiša (2011). The notation is, however, inspired by van der Vaart (1998). Let $\mathbf{D}$ and $\mathbf{E}$ be Banach spaces and consider a functional $\phi\colon W \to \mathbf{E}$ for an open subset $W \subseteq \mathbf{D}$. It is said to be differentiable at $f \in W$ if a continuous, linear map $\phi'_f\colon \mathbf{D} \to \mathbf{E}$ exists such that for any bounded set $B \subseteq \mathbf{D}$,

$$\Big\|\frac{\phi(f+th)-\phi(f)}{t} - \phi'_f(h)\Big\|_{\mathbf{E}} \to 0, \tag{1.1}$$

for $t \downarrow 0$, uniformly for $h \in B$, or equivalently

$$\big\|\phi(f+h)-\phi(f)-\phi'_f(h)\big\|_{\mathbf{E}} = o(\|h\|_{\mathbf{D}}). \tag{1.2}$$

It can be seen that the derivative is unique, and that differentiability implies continuity of the functional. We usually call $\phi'_f(h)$ the derivative of ϕ at f in the direction of h.

Various other concepts of functional differentiability have been considered in the litterature and Fréchet is one of the stronger conditions. Fréchet differentiability implies continuity and thereby also in that respect generalizes the ordinary differentiability concept.

Example 1.1. For the Banach spaces $\mathbf{D} = \mathbf{E} = \mathbb{R}$ with the Euclidean norm, a functional $\phi\colon \mathbf{D} \to \mathbf{E}$ is Fréchet differentiable if and only if it is

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differentiable in the ordinary sense, and the Fréchet derivative of ϕ at $f \in \mathbb{R}$ in the direction $h \in \mathbb{R}$ is

$$\phi'_f(h) = \phi'(f) \cdot h, \tag{1.3}$$

where $\phi'(f)$ is the ordinary derivative at $f \in \mathbb{R}$.

When ϕ is differentiable at all $f \in W$, the derivative

$$\phi' \colon f \mapsto \phi'_f \tag{1.4}$$

is a map from W into $L^1(\mathbf{D}, \mathbf{E})$, the space of continuous, linear maps from $\mathbf{D}$ to $\mathbf{E}$. A continuous, linear map, $a \colon \mathbf{D} \to \mathbf{E}$, is also a bounded, linear operator, meaning that there exists a constant $K > 0$ such that

$$\|a(f)\|_{\mathbf{E}} \leq K \|f\|_{\mathbf{D}} \tag{1.5}$$

for all $f \in \mathbf{D}$. The operator norm of a is the smallest of such constants,

$$\|a\|_{\mathrm{op}} = \inf\{K > 0 : \|a(f)\|_{\mathbf{E}} \leq K\|f\|_{\mathbf{D}}, \text{ for all } f \in \mathbf{D}\}, \tag{1.6}$$

or equivalently

$$\|a\|_{\mathrm{op}} = \inf\{K > 0 : \|a(f)\|_{\mathbf{E}} \leq K, \text{ for all } f \in \mathbf{D} \text{ with } \|f\|_{\mathbf{D}} = 1\}. \tag{1.7}$$

Endowed with this norm, $L^1(\mathbf{D}, \mathbf{E})$ is itself a Banach space. We thereby immediately have a continuity and differentiability concept for the derivative ϕ'. Thus, we can define continuous differentiability and higher order derivatives in terms of the operator norm.

Definition 1.2. A functional $\phi \colon W \to \mathbf{E}$ is said to be continuously differentiable, or C^1, if it is (Fréchet) differentiable and the derivative $\phi' \colon W \to L^1(\mathbf{D}, \mathbf{E})$ is continuous with respect to the operator norm on $L^1(\mathbf{D}, \mathbf{E})$.

Recursively, for $k > 1$, a functional $\phi \colon W \to \mathbf{E}$ is said to be k times continuously differentiable, or C^k, if it is C^{k-1} and the $(k-1)$th order derivative $\phi^{(k-1)}$ is C^1.

Defining recursively

$$L^k(\mathbf{D}, \mathbf{E}) := L^1(\mathbf{D}, L^{k-1}(\mathbf{D}, \mathbf{E})) \quad \text{(with the operator norm)}, \tag{1.8}$$

the Banach space $L^k(\mathbf{D}, \mathbf{E})$ can be identified with the Banach space of k-linear, continuous operators from $\mathbf{D}$ to $\mathbf{E}$ with the operator norm, cf.

Sections 5.1 and 5.2 of Dudley and Norvaiša (2011), where $\|a\|_{\mathrm{op}}$ for $a \in L^k(\mathbf{D},\mathbf{E})$ is defined as the smallest $K > 0$ such that

$$\|a(f_1,\dots,f_k)\|_{\mathbf{E}} \leq K\|f_1\|_{\mathbf{D}}\cdots\|f_k\|_{\mathbf{D}} \tag{1.9}$$

for all $f_1,\dots,f_k \in \mathbf{D}$. If $\phi\colon W \to \mathbf{E}$ is C^k, the kth order derivative is a continuous map $\phi^{(k)}\colon W \to L^k(\mathbf{D},\mathbf{E})$.

By Theorem 5.27 of Dudley and Norvaiša (2011), the Schwarz Theorem, for each $f \in W$, the derivative $\phi_f^{(k)}$ is not only continuous and k-linear, but also symmetric in its arguments.

A functional $\phi\colon W \to \mathbf{E}$ is said to be analytic if it has a Taylor expansion around each point of W. Analyticity of ϕ on W implies that ϕ is C^k on W for any k, cf. Theorem 5.28 of Dudley and Norvaiša (2011). For a C^k functional $\phi\colon W \to \mathbf{E}$ we have the kth order Taylor expansion with an integral remainder

$$\begin{aligned}\phi(f+g) &= \sum_{j=0}^{k}\frac{1}{j!}\phi_f^{(j)}(g,\dots,g)\\ &\quad+\frac{1}{(k-1)!}\int_0^1(1-s)^{k-1}(\phi_{f+sg}^{(k)}-\phi_f^{(k)})(g,\dots,g)\,\mathrm{d}s,\end{aligned} \tag{1.10}$$

if $f+sg \in W$ for all $s \in [0,1]$. Here $\phi_f^{(0)} = \phi(f)$. This is Theorem 5.42 of Dudley and Norvaiša (2011) or Theorem A.4.1 of Keller (1974).

Let us consider Banach spaces $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F}$, V an open subset of $\mathbf{E}$, and W an open subset of $\mathbf{D}$. We have that compositions of C^k functionals are C^k, cf. Theorem 2.10.0 of Keller (1974).

Proposition 1.3. *If $\phi\colon W \to \mathbf{E}$ is C^k on a neighborhood of $f \in W$, $\phi(f) \in V$, and $\psi\colon V \to \mathbf{F}$ is C^k on a neighborhood of $\phi(f)$, then $\psi\circ\phi$ is C^k on a neighborhood of $f \in W$.*

The first order derivative of $\psi\circ\phi$ at f in direction g is

$$(\psi\circ\phi)'_f(g) = \psi'_{\phi(f)}(\phi'_f(g)) \tag{1.11}$$

under sufficient differentiability assumptions. This is the chain rule.

If $\mathbf{D}$ and $\mathbf{E}$ are Banach spaces and W is an open subset of $\mathbf{D}$, a map $\phi\colon W \to \mathbf{E}$ is said to be Lipschitz continuous if a constant $K > 0$ exists such that

$$\|\phi(f)-\phi(g)\|_{\mathbf{E}} \leq K\|f-g\|_{\mathbf{D}} \tag{1.12}$$

for all $f, g \in W$, and it is said to be locally Lipschitz continuous if for all $f \in W$ there is a ball, $B_f \subseteq W$, around f for which a constant $K_f > 0$ exists such that

$$\|\phi(g) - \phi(h)\|_{\mathbf{E}} \leq K_f \|g - h\|_{\mathbf{D}} \tag{1.13}$$

for all $g, h \in B_f$.

The following results are likely considered simple facts in functional analysis. We include proofs for completeness.

PROPOSITION 1.4. *A functional, $\phi\colon W \to \mathbf{E}$ that is continuously differentiable (C^1) is also locally Lipschitz continuous.*

PROOF. Let $f \in W$ be given. Due to the continuity of the derivative, there is a ball, $B \subset \mathbf{D}$, around f such that $\|\phi'_g - \phi'_f\|_{\text{op}} \leq 1$ for all $g \in B$. Specifically we have

$$\|\phi'_g\|_{\text{op}} \leq \|\phi'_f\|_{\text{op}} + 1, \quad g \in B. \tag{1.14}$$

The mean value theorem, Theorem 5.3 of Dudley and Norvaiša (2011), ensures

$$\|\phi(g) - \phi(h)\|_{\mathbf{E}} \leq \|g - h\|_{\mathbf{D}} \sup_{\tilde{h} \in B} \|\phi'_{\tilde{h}}\|_{\text{op}} \leq \|g - h\|_{\mathbf{D}} (\|\phi'_f\|_{\text{op}} + 1) \tag{1.15}$$

for $g, h \in B$. □

In the primary paper we are particularly interested in the case where the functional is C^2 with a locally Lipschitz continuous second order derivative, which is somewhere between the C^2 and C^3 properties according to Proposition 1.4. In this case, the following result is nice to have.

PROPOSITION 1.5. *Consider C^2 functionals $\phi\colon W \to \mathbf{E}$ and $\psi\colon V \to \mathbf{F}$. If both ϕ and ψ have locally Lipschitz continuous second order derivatives, then $\psi \circ \phi$ also has a locally Lipschitz continuous second order derivative.*

PROOF. A chain rule result of order 2 gives

$$(\psi \circ \phi)''_f(h, k) = \psi'_{\phi(f)}(\phi''_f(h, k)) + \psi''_{\phi(f)}(\phi'_f(h), \phi'_f(k)). \tag{1.16}$$

Then

$$\begin{aligned} \big((\psi \circ \phi)''_f - (\psi \circ \phi)''_g\big)(h, k) = {} & (\psi'_{\phi(f)} - \psi'_{\phi(g)})(\phi''_f(h, k)) \\ & + \psi'_{\phi(g)}(\phi''_f(h, k) - \phi''_g(h, k)) \\ & + (\psi''_{\phi(f)} - \psi''_{\phi(g)})(\phi'_f(h), \phi'_f(k)) \\ & + \psi''_{\phi(g)}(\phi'_f(h) - \phi'_g(h), \phi'_f(k)) \\ & + \psi''_{\phi(g)}(\phi'_g(h), \phi'_f(k) - \phi'_g(k)). \end{aligned} \tag{1.17}$$

Using the locally Lipschitz continuity for the various functionals by Proposition 1.4, the result can be seen to follow. □

2. *p*-variation and some key functionals. The spaces of bounded p-variation as defined in this section are useful Banach spaces in our applications in the primary paper. We shall see that they are strong enough to make important functionals differentiable and even analytic while still weak enough that we have convergence of the empirical distribution to the true distribution in some sense. According to chain rule these functionals can then be combined and composed in various ways to obtain a wide range of C^k functionals.

The p-variation of a function $f\colon J \to \mathbb{R}$ on the interval $J \subseteq \mathbb{R}$ is defined by

$$v_p(f;J) = \sup \sum_{i=1}^{m} |f(x_{i-1}) - f(x_i)|^p \tag{2.1}$$

where the supremum is over $m \in \mathbb{N}$ and points $x_0 < x_1 < \cdots < x_m$ in the interval J. The space of functions of bounded p-variation

$$\mathcal{W}_p := \mathcal{W}_p(J) := \{f : J \to \mathbb{R} \mid v_p(f) < \infty\} \tag{2.2}$$

(we will often drop J from the notation when it is clear from the context) is a Banach space when endowed with the p-variation norm,

$$\|f\|_{[p]} = v_p(f;J)^{\frac{1}{p}} + \|f\|_\infty. \tag{2.3}$$

According to Theorem 6.2 of Part I of Dudley and Norvaiša (1999),

$$\|F_n - F\|_{[p]} = O_{\mathrm{P}}(n^{\frac{1-p}{p}} (\log \log n)^{\frac{1}{2}}) \tag{2.4}$$

for $1 \leq p \leq 2$ when F_n is the empirical distribution function of a one-dimensional i.i.d. sample with common distribution function F. In Qian (1998) even stronger results can be found.

In the following we will consider some key functionals for our applications in time-to-event analysis. Many of them are analytic for $1 \leq p < 2$, giving this range a special place in our theory.

Note that for $f, g \in \mathcal{W}_p$ for $p > 1$ ordinary Lebesgue-Stieltjes integrals like $\int f \, \mathrm{d}g$ are not generally well-defined (i.e. when g is not of bounded variation). In this case we let the central Young integral, denoted by $(CY) \int f \, \mathrm{d}g$ in e.g. Dudley and Norvaiša (1999), extend the ordinary Lebesgue-Stieltjes integral without further mention.

The multiplication operator. Based on the inequality

$$\|FG\|_{[p]} \leq \|F\|_{[p]}\|G\|_{[p]}, \quad F, G \in \mathcal{W}_p, \tag{2.5}$$

from Dudley and Norvaiša (1999) (p. 6), cf. also Krabbe (1961), for $p \geq 1$, the bilinear operator $(F, G) \mapsto F \cdot G$ is an analytic map from $\mathcal{W}_p^2$ to $\mathcal{W}_p$ with derivative at (F, G) in direction (f, g) equal to $F \cdot g + f \cdot G$.

The composition operator. For a given function $F\colon \mathbb{R} \to \mathbb{R}$ we call the functional $N_F\colon G \mapsto F \circ G$ the composition operator. It is also known as an autonomous Nemytskii operator in Dudley and Norvaiša (2011). From Theorem 6.74 of Dudley and Norvaiša (2011) it can be seen that for $p \geq 1$ the functional N_F is C^k from $\mathcal{W}_p$ to $\mathcal{W}_p$ if and only if the function F is C^k on $\mathbb{R}$. Also, Corollary 6.80 of Dudley and Norvaiša (2011) states that N_F is analytic if and only if F is analytic on $\mathbb{R}$. We are, however, primarily interested in the functional $G \mapsto \frac{1}{G}$, that is not strictly covered by these results. Since $F\colon s \mapsto \frac{1}{s}$ is analytic on sets that do not include 0, this functional can be seen to be analytic on neighborhoods of functions in $\mathcal{W}_p$ bounded away from 0 based on Corollary 6.79 of Dudley and Norvaiša (2011).

When differentiable, the derivative of $N_F\colon G \mapsto F \circ G$ at G in direction g is the function $s \mapsto F'(G(s))\, g(s)$.

The integration operator. The integration operator is the functional $\mathcal{I} : \mathcal{W}_p \times \mathcal{W}_p \to \mathcal{W}_p$ given by $\mathcal{I}(f, g) = \int f \,\mathrm{d}g$. For $1 \leq p < 2$ it is analytic, cf. Corollary 4.6 of Part I of Dudley and Norvaiša (1999), with its Taylor expansion at (F, G) given by

$$\int (F + f)\,\mathrm{d}(G + g) = \int F \,\mathrm{d}G + \int F \,\mathrm{d}g + \int f \,\mathrm{d}G + \int f \,\mathrm{d}g. \tag{2.6}$$

The first order derivative of $\mathcal{I}$ at (F, G) in direction (f, g) is $\int F \,\mathrm{d}g + \int f \,\mathrm{d}G$, the second order derivative of $\mathcal{I}$ at (F, G) in directions (f_1, g_1) and (f_2, g_2) is $\int f_2 \,\mathrm{d}g_1 + \int f_1 \,\mathrm{d}g_2$ and higher order derivatives are 0.

The product integral operator. The product integral operator is the functional $\mathcal{P} : \mathcal{W}_p([a, b]) \to \mathcal{W}_p([a, b])$ given by $\alpha \mapsto \prod_a^{(\cdot)}(1 + \mathrm{d}\alpha)$. This operator is described in detail in Dudley and Norvaiša (1999, 2011) and Gill and Johansen (1990), and $\mu = \mathcal{P}(\alpha)$ is the unique solution to the equation

$$\mu = 1 + \int_a^{(\cdot)} \mu(s-)\,\mathrm{d}\alpha(s), \tag{2.7}$$

and thus coincides with $s \mapsto e^{\alpha(s)}$ when α is continuous. For $1 \leq p < 2$, the operator $\mathcal{P}$ is analytic according to Section 9.10 of Dudley and Norvaiša

(2011). Let $\mu(J) = \prod_J(1 + \mathrm{d}\alpha(s))$ for any interval $J \subseteq [a, b]$. Then the first order derivative at α in direction β can be written as

$$\int_a^{(\cdot)} \mu([a,s))\mu((s,(\cdot)])\,\mathrm{d}\beta(s) \tag{2.8}$$

in our commutative case. When α is continuous this reduces to $\mathcal{P}(\alpha)\beta$, similar to the derivative of $\alpha \mapsto e^{\alpha}$.

2.1. *A proof of equation (3.21) of the primary paper.* In the primary paper, using $\delta_x = (Y_x, N_{x,0}, \ldots, N_{x,d})^T$, we consider $F_n = \frac{1}{n}\sum \delta_{X_i}$ a member of the Banach space $\mathcal{W}_p^{d+2}([0,t])$ with a norm $\|\cdot\|_{[p]}$ obtained by summation of the entrywise p-variation norms. With $F = (H, H_0, \ldots, H_d)^T$, we claim that the following result holds.

LEMMA 2.1. *Consider $p \in [1,2]$. If $\|\cdot\|_{[p]}$ is the norm on $\mathcal{W}_p^{d+2}$ given by the sum of the p-variation norms on each entry, we have*

$$\|F_n - F\|_{[p]} = O_{\mathrm{P}}(n^{\frac{1-p}{p}}(\log\log n)^{\frac{1}{2}}), \tag{2.9}$$

in similarity to (2.4).

PROOF. We need the stated bound for each entry. We let

$$F_n = (H_n, H_{n,0}, \ldots, H_{n,d}) \tag{2.10}$$

denote the empirical counterpart of $F = (H, H_0, \ldots, H_d)$. The first entry is of the right size according to (2.4). For $i = 0, \ldots, d$ let

$$n_i = \#\{k : \tilde{T}_k \le t, \tilde{\Delta}_k = i\}. \tag{2.11}$$

Almost surely $H_{n,i}(t) = 0$ if $H_i(t) = 0$, so assume $H_i(t) > 0$. Then $n_i > 0$ with high probability when n is large. We see that

$$\begin{aligned} H_{n,i} - H_i = {}& \frac{n_i}{n}\left(\frac{n}{n_i}H_{n,i} - \mathrm{P}(\tilde{T} \le (\cdot) \mid \tilde{T} \le t, \tilde{\Delta} = i)\right) \\ &+ \left(\frac{n_i}{n} - \mathrm{P}(\tilde{T} \le t, \tilde{\Delta} = 1)\right)\left(\mathrm{P}(\tilde{T} \le (\cdot) \mid \tilde{T} \le t, \tilde{\Delta} = i)\right) \end{aligned} \tag{2.12}$$

such that $\|H_{n,i} - H_i\|_{[p]}$ is $O_{\mathrm{P}}(n^{\frac{1-p}{p}}(\log\log n)^{\frac{1}{2}})$ since applying (2.4) yields $\|\frac{n}{n_i}H_{n,i} - \mathrm{P}(\tilde{T} \le (\cdot) \mid \tilde{T} \le t, \tilde{\Delta} = i)\|_{[p]} = O_{\mathrm{P}}(n_i^{\frac{1-p}{p}}(\log\log n_i)^{\frac{1}{2}})$ and since $\left|\frac{n_i}{n} - \mathrm{P}(\tilde{T} \le t, \tilde{\Delta} = 1)\right| = O_{\mathrm{P}}(n^{-\frac{1}{2}})$. Since $n_i = O_{\mathrm{P}}(n)$, the last $d+1$ entries also have the right size giving the desired result. □

3. Measurability. Consider an estimator $\hat{\theta}(\cdot)$ such that the estimate $\hat{\theta}_n$, based on an i.i.d. sample $X_1, \ldots, X_n$, can be considered on the form $\hat{\theta}_n = \phi(F_n)$ for a C^2 functional ϕ from (a subset of) a Banach space into the parameter space, $\mathbb{R}$, where $F_n = \frac{1}{n}\sum \delta_{X_i} \in \mathbf{D}$ is some summary statistic derived from the empirical distribution of the X_is. If $\hat{\theta}(\cdot)$ is a useful estimator, it seems a reasonable assumption that the estimator is a measurable map from the set of observations to the estimate, i.e. we assume

$$\mathcal{X}^n \ni (x_1, \ldots, x_n) \mapsto \phi(\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}) \in \mathbb{R} \tag{3.1}$$

is measurable (with respect to the product σ-algebra on $\mathcal{X}^n$). We want to argue that under this assumption we also have measurability of

$$\mathcal{X} \ni x \mapsto \dot{\phi}(x) \in \mathbb{R}, \tag{3.2}$$

the first order influence function, and

$$\mathcal{X}^2 \ni (x_1, x_2) \mapsto \ddot{\phi}(x_1, x_2) \in \mathbb{R}, \tag{3.3}$$

the second order influence function. Our argument will, however, rely on the additional assumption that $F_n \to F$ in $\mathbf{D}$ in probability, or rather that F can be approximated (in $\mathbf{D}$) by a sequence of the type $(\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i})_n$, which is also an assumption that will always be met.

So, let (x_k) be a sequence of elements in $\mathcal{X}$ such that $f_m = \frac{1}{m}\sum_{i=1}^{m} \delta_{x_i} \to F$ for $m \to \infty$ and let $(x_k)_{k=1}^{m} \in \mathcal{X}^m$ denote the first m elements. For convenience, we introduce the maps

$$G_n : \mathcal{X}^n \ni (x_1, \ldots, x_n) \mapsto \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i} \in \mathbf{D}. \tag{3.4}$$

Our main assumption can be restated as measurability of $\phi \circ G_n$ for any n. For any $x \in \mathcal{X}$ and any $t \in [0, 1]$, let $s_n(x) = (\{x\}^{\lceil nt \rceil}, (x_k)_{k=1}^{n-\lceil nt \rceil}) \in \mathcal{X}^n$ where $\lceil \cdot \rceil$ denotes the ceiling function. The map $x \mapsto s_n(x)$ is obviously measurable for any n. This implies that

$$\mathcal{X} \ni x \mapsto \phi \circ G_n(s_n(x)) = \phi\big(\frac{\lceil tn \rceil}{n}\delta_x + \frac{n - \lceil nt \rceil}{n} f_{n-\lceil nt \rceil}\big) \in \mathbb{R} \tag{3.5}$$

is measurable for any n. The limit for $n \to \infty$, which is

$$\mathcal{X} \ni x \mapsto \phi(t\delta_x + (1-t)F) \in \mathbb{R} \tag{3.6}$$

due to the continuity of ϕ, is then also a measurable map. Since $t \in [0,1]$ was arbitrary, we can consider a sequence (t_n) such that $t_n \downarrow 0$, and obtain the result that

$$x \mapsto \dot{\phi}(x) = \phi'_F(\delta_x - F) = \lim_{n\to\infty} t_n^{-1}(\phi(F + t_n(\delta_x - F)) - \phi(F)) \tag{3.7}$$

is measurable. A similar argument gives the measurability of $(x_1, x_2) \mapsto \ddot{\phi}(x_1, x_2)$.

4. Proof of Proposition 4.2 of the primary paper. A proof of Proposition 4.2 of the primary paper was essentially given in Graw, Gerds and Schumacher (2009) in the proof of their Lemma 2. Here we give a detailed martingale-based proof, using the notation of Section 4 of the primary paper. Recall that the claim is that we have

$$\mathrm{E}\left(\gamma'_F(s; \delta_X - F) \mid Z\right) = F_{Z,1}(s) - F_1(s), \quad s \in [0,t], \tag{4.1}$$

under the assumption of completely independent censorings. The expression

$$\begin{aligned} \gamma'_F(s; \delta_x - F) = & \int_0^s \frac{1}{G(u)} \mathrm{d}N_{x,1}(u) - F_1(s) \\ & + \int_0^s \frac{1}{G(u)} \int_0^{u-} \frac{1}{H(v)} \mathrm{d}M_{x,0}(v) \mathrm{d}H_1(u). \end{aligned} \tag{4.2}$$

can be established as for the influence function $\dot{\phi}(x) = \gamma'_F(t; \delta_x - F)$ in the primary paper. Since the $F_1(s)$-term fits, the proposition is proven by showing that

$$\mathrm{E}\left(\int_0^s \frac{1}{G(u)} \mathrm{d}N_{X,1}(u) \mid Z\right) = F_{Z,1}(s) \tag{4.3}$$

and

$$\mathrm{E}\left(\int_0^s \frac{1}{G(u)} \int_0^{u-} \frac{1}{H(v)} \mathrm{d}M_{X,0}(v) \mathrm{d}H_1(u) \mid Z\right) = 0. \tag{4.4}$$

Note that if $N_C(u) := 1(C \le u)$ defines the counting process for the censoring alone, then

$$N_{X,0}(u) = \int_0^u Y_X(v) \mathrm{d}N_C(v). \tag{4.5}$$

Thus, with $M_C(u) := \int_0^u 1(C \ge v) \mathrm{d}(N_C - \Lambda_0)(v)$, we have

$$M_{X,0}(u) = \int_0^u Y_X(v) \mathrm{d}M_C(v). \tag{4.6}$$

Since, under the assumption of completely independent censorings, M_C is a martingale with respect to the filter given by $\tilde{\mathcal{F}}_u^Z = \sigma(\sigma(Z) \cup \tilde{\mathcal{F}}_u)$, where $\tilde{\mathcal{F}}_u = \sigma(N_C(v), N_{X,0}(v), \ldots, N_{X,d}(v) \mid v \leq u)$, we see that $M_{X,0}$ is a martingale with respect to $(\tilde{\mathcal{F}}_u^Z)$. Since it depends on N_C only through the other components, it is also a martingale with respect to the smaller filter $\mathcal{F}_u^Z = \sigma(\sigma(Z) \cup \mathcal{F}_u)$, where $\mathcal{F}_u = \sigma(N_{X,0}(v), \ldots, N_{X,d}(v) \mid v \leq u)$. This implies that

$$u \mapsto \int_0^u \frac{1}{H(v)} \, \mathrm{d}M_{X,0}(v) \tag{4.7}$$

also defines an $(\mathcal{F}_u^Z)$-martingale, which in turn implies that

$$\mathrm{E}\left(\int_0^{u-} \frac{1}{H(v)} \, \mathrm{d}M_{X,0}(v) \mid Z \right) = 0, \quad u \leq s. \tag{4.8}$$

So, we see that (4.4) holds.

Now, we need to prove (4.3). As the compensator of $N_{X,1}$ with respect to $(\mathcal{F}_u^Z)$ is

$$u \mapsto \int_0^u Y_X(v) \, \mathrm{d}\Lambda_{Z,1}(v) \tag{4.9}$$

with

$$\Lambda_{Z,1}(u) = \int_0^u \frac{1}{S_Z(v)} dF_{Z,1}(v), \tag{4.10}$$

an $(\mathcal{F}_u^Z)$-martingale is defined by

$$M_{X,Z,1}(u) = N_{X,1}(u) - \int_0^u Y_X(v) \, \mathrm{d}\Lambda_{Z,1}(v). \tag{4.11}$$

This implies that

$$u \mapsto \int_0^u \frac{1}{G(v)} \, \mathrm{d}M_{X,Z,1}(v) \tag{4.12}$$

also is an $(\mathcal{F}_u^Z)$-martingale, and, letting

$$H_Z(u) = \mathrm{P}\left(\tilde{T} > u \mid Z\right) = G(u) S_Z(u), \tag{4.13}$$

we can conclude

$$\begin{aligned} \mathrm{E}\left(\int_0^s \frac{1}{G(u)} \, \mathrm{d}N_{X,1}(u) \mid Z \right) &= \mathrm{E}\left(\int_0^s \frac{Y_X(u)}{G(u)} \, \mathrm{d}\Lambda_{Z,1}(u) \mid Z \right) \\ &= \int_0^s \frac{H_Z(u)}{G(u)} \, \mathrm{d}\Lambda_{Z,1}(u) \\ &= \int_0^s S_Z(u) \, \mathrm{d}\Lambda_{Z,1}(u) = F_{Z,1}(s), \end{aligned} \tag{4.14}$$

which gives us the result.

REFERENCES

DUDLEY, R. M. and NORVAIŠA, R. (1999). *Differentiability of six operators on nonsmooth functions and p-variation. Lecture Notes in Mathematics* **1703**. Springer-Verlag, Berlin With the collaboration of Jinghua Qian. MR1705318 (2000e:46053)

DUDLEY, R. M. and NORVAIŠA, R. (2011). *Concrete functional calculus. Springer Monographs in Mathematics*. Springer, New York. MR2732563 (2012e:47191)

GILL, R. D. and JOHANSEN, S. (1990). A survey of product-integration with a view toward application in survival analysis. *Ann. Statist.* **18** 1501–1555. MR1074422 (92f:60125)

GRAW, F., GERDS, T. A. and SCHUMACHER, M. (2009). On pseudo-values for regression analysis in competing risks models. *Lifetime Data Anal.* **15** 241–255. MR2510866 (2010e:62237)

KELLER, H. H. (1974). *Differential calculus in locally convex spaces. Lecture Notes in Mathematics, Vol. 417*. Springer-Verlag, Berlin-New York. MR0440592 (55 ##13466)

KRABBE, G. L. (1961). Integration with respect to operator-valued functions. *Acta Sci. Math. (Szeged)* **22** 301–319. MR0132814 (24 ##A2650)

QIAN, J. (1998). The p-variation of partial sum processes and the empirical process. *Ann. Probab.* **26** 1370–1383. MR1640349 (99i:60052)

VAN DER VAART, A. W. (1998). *Asymptotic statistics. Cambridge Series in Statistical and Probabilistic Mathematics* **3**. Cambridge University Press, Cambridge. MR1652247 (2000c:62003)

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