Solution to Algebra : Chapter 0 by Paolo Aluffi

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Last updated at February 19, 2020
v0.3.6

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Prologue

Over a few months I want to improve my skills in solving algebra problems. I tried to find a textbook that can serves me good and is good enough to use in self-study.

Eventually, this is what I felt the most "comfortable" book in my opinion. It doesn't contain that much unlike Dummit & Foote, but the writing style, the explanation, and the exercises really served me well.

So here is the solution to Algebra : Chapter 0. There are a few important points to note here:

- The solution is *only* hosted on my GitHub page https://github.com/macyayaya/algebra-chapter-0-solutions. If you find this document outside this page, you might have an outdated version of the solution which might have errors, so please be aware.
- I will update the solution irregularly.
- I'll try to write this beginner-friendly (as I am also a beginner), so the answer might be way too detailed/verbose. Sorry if you find this annoying.
- If you found an error in the solutions, typos, bad grammar or want to give an advise on LaTeX formatting, etc., don't hesitate to open an issue or a pull request on my repo.
- The questions I picked is completely random, so if you want to see some solution of a certain problem (but please not all of them), you can also open an issue to notify me.
- However, I currently do *not* accept any PRs to new solutions; this is more than my note on self-study rather than a complete solution set.

Thanks.

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February 16, 2020

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Chapter II

Groups, first encounter

Unless otherwise specified, in the following G denotes a group, e denotes the identity of G. Some description and hints are omitted for simplicity.

II.1

Problem II.1.8. Let G be a finite abelian group with exactly one element f of order 2. Prove that $\prod_{g\in G} g = f$.

Proof. For all elements that is not of order 2, they have an inverse that is not itself, so they canceled out in the product $\prod_{g\in G} g$, leaving only elements that is of order 2, i.e. f. ■

Problem II.1.10. If the order of g is odd, what can you say about the order of g^2 ?

Solution. The order of g^2 is $|g|$ since the only number that divides $|g|$ and in $\{2, 4, ..., 2|g|\}$ is $2|g|$ if $|g|$ is odd. ■

Problem II.1.11. Prove that for all g, h in a group G, $|gh| = |hg|$.

Proof. Simply observe that $e = (gh)^{|gh|} = g(hg)^{(|gh|-1)}h$, therefore

$$g^{-1}h^{-1} = (hg)^{-1} = (hg)^{|gh|-1}$$

hence $(hg)^{|gh|} = e$. The other case is the same. ■

Problem II.1.13. Give an example showing that $|gh| \neq \mathrm{lcm}(|g|, |h|)$ even if g and h commute.

Solution. In C_4, $|1 + 3| = |0| = 1$ but $\mathrm{lcm}(|1|, |3|) = 4$. Clearly C_4 is abelian. ■

Problem II.1.14. As a counterpoint of II.1.13, prove that if g and h commute and $\gcd(|g|, |h|) = 1$, then $|gh| = |g||h|$.

Proof. One has $|gh|$ divides $\mathrm{lcm}(|g|, |h|) = |g||h|$ by Proposition II.1.14, so it suffices to prove that $|g||h|$ divides $|gh|$.

Let $N = |gh|$. Then one sees that $(gh)^N = g^N h^N$ since g and h commutes. Then

$$(gh)^{N|h|} = e^{|h|} = g^{N|h|}h^{N|h|} = g^{N|h|}$$

so we have $|g|$ divides $N|h|$, which implies $|g|$ divides N since $\gcd(|g|, |h|) = 1$. Similarly $|h|$ divides N, therefore $|g||h|$ divides $N = |gh|$, as desired. ■

Problem II.1.15. Let G be a commutative group, and let $g \in G$ be an element of maximal *finite* order. Prove that if h has finite order in G, then $|h|$ *divides* $|g|$.

Proof. Suppose that $|h|$ does not divide $|g|$, then we can assume that $|g| = p^m r$, $|h| = p^n s$, where p is a prime and r, s relatively prime to p and $m < n$. Then by the previous problem we can calculate the order of $g^{p^m}h^s$, which is $p^n r$. But this element has order bigger than g, contradict to the maximality of g. Hence $|h|$ must divide $|g|$. ■

II.2

Problem II.2.10. Prove that $\mathbb{Z}/n\mathbb{Z}$ consists of precisely n elements.

Proof. Trivial. ■

Problem II.2.14. Show that the multiplication in $\mathbb{Z}/n\mathbb{Z}$ is a well-defined action.

Proof. If $a \equiv a' \mod n$ and $b \equiv b' \mod n$, then $a = a' + kn$, $b = b' + ln$ for $k, l \in \mathbb{Z}$, therefore

$$(ab) - (a'b') = (a' + kn)(b' + ln) - a'b' = a'ln + b'kn + kln^2 \equiv 0 \mod n$$

as desired. ■

Problem II.2.16. Find the last digit of $1238237^{18238456}$.

Solution. $1238237^{18238456} \equiv 7^{18238456} = 49^{9119228} = 2401^{4559614} \equiv 1^{4559614} = 1 \mod 10.$ ■

Problem II.2.17. Show that if $m \equiv m' \mod n$, then $\gcd(m, n) = 1$ if and only if $\gcd(m', n) = 1$.

Proof. We can write $m = nk + m'$ for $n \in \mathbb{Z}$ and use Euclidean Algorithm to conclude. ■

II.3

Problem II.3.3. Show that if G, H are abelian groups, then $G \times H$ satisfies the universal property for coproducts in Ab.

Proof. Let A be an arbitrary abelian group, f_G, f_H be homomorphisms, i_G, i_H be inclusions.

$$\begin{tikzcd}
G \arrow[dr, "i_G"] \arrow[drr, bend left, "f_G"] & & \\
& G \times H \arrow[r, "\exists ! \varphi"] & A \\
H \arrow[ur, "i_H"] \arrow[urr, bend right, "f_H"'] & &
\end{tikzcd}$$

To check the universal property, define $\varphi(g, h) := f_G(g)f_H(h)$. Now φ is a homomorphism since for $g_1, g_2 \in G, h_1, h_2 \in H$,

$$\begin{aligned}
\varphi((g_1, h_1)(g_2, h_2)) = \varphi(g_1g_2, h_1h_2) = f_G(g_1g_2)f_H(h_1h_2) &= f_G(g_1)f_G(g_2)f_H(h_1)f_H(h_2) \\
&\xlongequal{\text{abelian}} f_G(g_1)f_H(h_1)f_G(g_2)f_H(h_2) = \varphi(g_1, h_1)\varphi(g_2, h_2)
\end{aligned}$$

as desired. ■

Problem II.3.6. Consider the product $C_2 \times C_3$, which is a coproduct in Ab. Show that it is *not* a coproduct of C_2 and C_3 in Grp.

Proof. If $C_2 \times C_3$ is a coproduct, then take $A = S_3$. Although there are injective homomorphisms

$$\begin{aligned}
\varphi_1 &: C_2 \to S_3 \text{ by } \varphi_1(1) = (12) \text{ or other two cycle} \\
\varphi_2 &: C_3 \to S_3 \text{ by } \varphi_2(1) = (123) \text{ or other three cycle}
\end{aligned}$$

but there are no homomorphisms $\varphi : C_2 \times C_3 \to S_3$ that satisfies the universal property of coproducts: Observe that any choice of cycles in φ_1 and φ_2 will exhaust all possible element of S_3, hence force φ to be an isomorphism. But the element $\varphi(1, 1)$ must be either a 2(or 3)-cycle, and neither $(1, 1)^2$ nor $(1, 1)^3$ are $(0, 0)$, and φ will map a non-identity element to the identity, a contradiction. ■

II.4

Problem II.4.3. Prove that a group of order n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ if and only if it contains an element of order n.

Proof. Let G be such group.
($\Rightarrow$) Trivial.
($\Leftarrow$) Let g be an element of order n. Then consider a homomorphism $\varphi : G \to \mathbb{Z}/n\mathbb{Z}$ with $\varphi(g) = \bar{1}$. It is a direct check that this is an isomorphism. ■

Problem II.4.8. Let $g \in G$. Prove that the function $\gamma_g : G \to G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism of G. Prove that the function $G \to \mathrm{Aut}(G)$ defined by $g \to \gamma_g$ is a homomorphism, and show that this homomorphism is trivial if and only if G is abelian.

Proof. γ_g is injective since if $gag^{-1} = gbg^{-1}$ then $a = b$; it is surjective since for $k \in G$ we can find $g^{-1}kg$ so that $\gamma_g(g^{-1}kg) = k$; it is a homomorphism since

$$\gamma_g(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \gamma_g(a)\gamma_g(b).$$

If G is abelian then the automorphism is simply $\gamma_g(a) = a$; conversely if $gag^{-1} = a$ then $ga = ag$ for all $a, g \in G$, hence abelian. ■

Problem II.4.9. Prove that if m, n are positive integers such that $\gcd(m, n) = 1$, then $C_{mn} \cong C_m \times C_n$.

Proof.

$$\varphi : C_{mn} \to C_m \times C_n,\ \varphi(a) = (a \bmod m, a \bmod n)$$

is a homomorphism and a bijection. ■

Problem II.4.11. Assuming the fact that the equation $x^d = 1$ can have at most d solutions in $\mathbb{Z}/p\mathbb{Z}$ for a prime p, prove that $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.

Proof. Let g be an element of maximal order, and by 1.15, all elements have degree that divides $|g|$, i.e. $|h|^{|g|} = 1 \forall h \in G$. Using the fact, we have $|G| \leq |d|$, since only at most $|g|$ elements can be the solution to $h^{|g|} = 1$. Clearly we also have $|G| \geq |d|$, so $|G| = |d|$. Thus the proof is complete by II.4.3. ■

Problem II.4.13. Prove that $\mathrm{Aut}_{\mathsf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$.

Proof. To make an automorphism φ, φ must fix $(0, 0)$, leaving 6 possible permutations for elements $(0, 1), (1, 0), (1, 1)$. It suffices to check that all permutations of these elements are homomorphisms(hence isomorphisms). ■

Problem II.4.16. Prove the *Wilson's theorem*: for $p \in \mathbb{N}_{>1}$, p is a prime if and only if

$$(p-1)! \equiv -1 \mod p$$

Proof. ($\Rightarrow$) Assuming that the result of II.1.8 and II.4.11 is true, consider $G = (\mathbb{Z}/n\mathbb{Z})^*$. It is cyclic, and has exactly one element of order 2 since for $0 \leq k \leq p-2$,

$$(p-1-k)^2 \equiv 1 + 2k + k^2 \equiv 1 \mod p \iff k(k+2) \equiv 0 \mod p$$

and such solution can only be $k = 0$ or $p - 2$ since p is a prime, which correspond to $p - 1$ and 1 (identity). Therefore by II.1.8

$$\prod_{g \in G} g = (p-1)! \equiv (p-1) \equiv -1 \mod p$$

as desired.
($\Leftarrow$) If p is not a prime, then there exists $1 < k < p$ such that $k|p$. Since $k < p$ we have $k|(p-1)!$, i.e.

$$(p-1)! \equiv rk \mod p \text{ for some } r \in \mathbb{Z}$$

and clearly no choice of r will make $rk \equiv -1 \mod p$ by the fact that $k|p$. Therefore p must be a prime. ■

II.5

Problem II.5.3. Use the universal property of free groups to prove that the map $j : A \to F(A)$ is injective.

Proof. If there is $a, b \in A$ such that $j(a) = j(b)$ but $a \neq b$, then let f be a set function such that $f(a) \neq f(b)$; in particular, let $G = \mathbb{Z}$ and let $f(a) = 1, f(b) = 2$. Then there are no homomorphisms that will make the diagram commute, therefore j must be injective. ■

Problem II.5.6. Prove that the group $F(\{x, y\})$ is a coproduct $\mathbb{Z} * \mathbb{Z}$ of $\mathbb{Z}$ by itself in the category Grp.

Proof. We are given the universal property of free group: for $j : \{x, y\} \to F(\{x, y\})$, $\exists G, f$ such that the diagram

$$\begin{tikzcd}
F(\{x,y\}) \arrow[r, "\exists!\varphi"] & G \\
\{x,y\} \arrow[u, "j"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. To check that it is a coproduct, consider the coproduct diagram composed with above. Let $i(0) = x$, j be the inclusion, then we have the following diagram:

$$\begin{tikzcd}
& \mathbb{Z} \arrow[dr, "i"] \arrow[drr, bend left, "f"] & & \\
\{x,y\} \arrow[ur, "\gamma"] \arrow[rr, "j"] \arrow[dr, "\gamma"'] \arrow[rrr, bend right=80, "g"'] & & F\{x,y\} \arrow[r, "\exists!\varphi"] & G \\
& \mathbb{Z} \arrow[ur, "i"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

Note that the arrows j, g, φ comes from the free group diagram. From this, we have $f \circ \gamma = \varphi \circ j$. To check the coproduct diagram commutes, it suffices to check $f = \varphi \circ i$. To do this, define $\gamma(x) = 0, \gamma(y) = 1$. Then

$$f \circ \gamma(x) = f(0) = \varphi(x) = \varphi \circ j(x), \quad f \circ \gamma(y) = f(1) = \varphi(y) = \varphi \circ j(y)$$

Since $f(1) = \varphi \circ i(1) = \varphi(y)$, the homomorphisms agree on the generator, hence are the same. ■

II.6

Problem II.6.5. Let G be a *commutative* group, and let $n > 0$ be an integer. Prove that $\{g^n : g \in G\}$ is a subgroup of G. Prove that this is not necessarily the case if G is not commutative.

Proof. For any two elements a, b in the set, they can be represented as g^n and h^n respectively. Now

$$ab^{-1} = g^n h^{-n} = (gh^{-1})^n$$

which shows that ab^{-1} is also in the set, proving the set is a subgroup. A counterexample would be D_6, the dihedral group with 6 elements, with the choice $n = 3$. Let s denote the reflection, r denotes the rotation, we then have

$$\{g^3 : g \in D_3\} = \{1, r^3, r^{2\cdot 3}, s^3, (sr)^3, (sr^2)^3\} = \{1, 1, 1, s, sr, sr^2\}$$

this set is not a subgroup, as $s^{-1}sr = r$ is not an element of this set. ■

Problem II.6.7. Show that inner automorphisms (the collection of γ_g in II.4.8) form a subgroup $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$, and show that $\mathrm{Inn}(G)$ is cyclic if and only if $\mathrm{Inn}(G)$ is trivial if and only if G is abelian. Deduce that if $\mathrm{Aut}(G)$ is cyclic, then G is abelian.

Proof. $\mathrm{Inn}(G)$ is a subgroup: clearly g_e is the identity, inverse exists, and the associative clearly holds.

If $\mathrm{Inn}(G)$ is cyclic, then let $\gamma_g(a) = gag^{-1}$ be a generator of order n. Then $\forall b \in G$ we have $\gamma_b = \gamma_g^n$, i.e. $gbg^{-1} = b^n b b^{-n}$. This gives $gb = bg \ \forall b \in G$, so γ_g is in fact trivial, and hence G is abelian by II.4.8. The last statement follows from Proposition II.6.11 that every subgroup of cyclic group is cyclic. ■

Problem II.6.9. Prove that an *abelian* group G is finitely generated if and only if there is a surjective homomorphism

$$\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}} \twoheadrightarrow G$$

for some n.

Proof.
($\Rightarrow$) As the group is abelian, for $G = \langle a_1, \cdots a_n \rangle$, we can represen an element g uniquely as

$$g = a_1^{p_1} \cdots a_n^{p_n}$$

where $p_i \in \mathbb{Z}, \ i = 1, \cdots n$. Therefore we can explictly write down the surjective homomorphism

$$\varphi : \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}} \twoheadrightarrow G \quad \text{by} \quad \varphi(p_1, \cdots, p_n) = a_1^{p_1} \cdots a_n^{p_n} = g$$

as desired.
($\Leftarrow$) By the universal property of $\mathbb{Z}^{\oplus n}$ we have the following diagram that commutes:

$$\begin{tikzcd} \mathbb{Z}^{\oplus n} \arrow[r, two heads, "\exists!\varphi"] & G \\ \{1, \cdots, n\} \arrow[u, "j"] \arrow[ur, "f"'] & \end{tikzcd} \tag{*}$$

To prove, it suffices to "replace" the set $\{1, \cdots, n\}$ by a subset of G.

$$\begin{tikzcd}
& \mathbb{Z}^{\oplus n} \arrow[r, two heads, "\exists!\varphi"] & G \\
\{1, \cdots, n\} \arrow[ur, "j"] \arrow[r, "f"] & A \arrow[u, "\tilde{j}"] \arrow[ur, "i"] &
\end{tikzcd}$$

By the diagram (*), we have $i \circ f = \varphi \circ j$. It is a fast check that the diagram formed by $\tilde{j}, i$ and φ commutes. Finally since A is a finite set and $\operatorname{im} \varphi = G$, it follows by definition that G is finitely generated. ∎

Problem II.6.14. Let ϕ be the Euler's ϕ-function. Prove that for $n \in \mathbb{N}$,

$$\sum_{m>0, m|n} \phi(m) = n.$$

Proof. Let $\langle x \rangle = C_n$. We have the trivial equation

$$\sum_{g \in C_n} 1 = n$$

Now note that every element in C_n generates a cyclic subgroup. To establish the result, we show that for every $d > 0$ that is a division of n, the subgroup of order d is *unique*, i.e. the unique subgroup is given by

$$\langle x^{n/d} \rangle = \{g \in G : g^d = 1\}$$

Indeed, if $g = x^{kn/d}$ for some positive integer k, then $g^d = x^{kn} = 1$. Conversely, if $g^d = 1$, then we have $g = x^m$ for some m since x is a generator. But this means that $x^{md} = 1$, and this implies $n|md$. Hence we have

$$g = x^m = x^{n/d \cdot dm/n} = x^{n/d} \in \langle x^{n/d} \rangle$$

as desired.

Now we count the generators of each subgroup of C_n, which is $\phi(d)$ for every d that is a divisor of n. Since every element in C_n generates a cyclic subgroup C_d, the sum of generator along each subgroup is exactly n, namely

$$\sum_{g \in C_n} 1 = \sum_{m : m|n} \phi(m) = n$$

which proved the assertion. ∎

Problem II.6.15. Prove that if $\varphi : G \to G'$ has a left inverse, then φ is a monomorphism.

Proof. If $a, b \in G$ are distinct elements that satisfies $\varphi(a) = \varphi(b)$, then having left inverse means there exists a homomorphism ψ such that $\psi \circ \varphi = id_G$. Then we would have $\psi \circ \varphi(a) = \psi \circ \varphi(b)$, which means $a = b$, a contradiction. ∎

II.7

Problem II.7.7. Let n be a positive integer. Let $H \subset G$ be the subgroup generated by all elements of order n in G. Prove that H is normal.

Proof. For $a \in H, g \in G$, since $a^n = e$,

$$(gag^{-1})^n = ga^n g^{-1} = e$$

we have $gag^{-1} \in H$, hence normal. ∎

Problem II.7.11. Prove that the commutator subgroup $[G, G]$ is normal, and the quotient $G/[G, G]$ is commutative.

Proof. Observe

$$gaba^{-1}b^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) = xyx^{-1}y^{-1} \in [G, G]$$

for $x = gag^{-1}, y = gbg^{-1}$. The quotient is commutative since $aba^{-1}b^{-1}[G, G] = [G, G]$ implies $ab[G, G] = ba[G, G]$. ■

II.8

Problem II.8.7. Let $(A|\mathscr{R}), (A'|\mathscr{R}')$, be the presentation for groups G, G', respectively, and assume that A and A' are disjoint. Prove that

$$G * G' := (A \cup A' \mid \mathscr{R} \cup \mathscr{R}')$$

satisfies the universal property for the coproduct of G and G' in Grp.

Proof. Write $H = \mathscr{R} \cup \mathscr{R}'$. Let us construct a homomorphism from G to $G * G'$. As $G = F(A)/R$, by the universal property of quotient we have a commutative diagram

$$\begin{tikzcd} F(A) \arrow[rr, "f"] \arrow[dr, "\pi"'] & & G * G \\ & F(A)/\mathscr{R} \arrow[ur, "\exists! \varphi_1"'] & \end{tikzcd}$$

In particular, we let f be an quotient map, i.e. $f(w) = wH$. Then naturally we have $\varphi_1(w\mathscr{R}) = wH$. Similarly, for G' we have another homomorphism $\varphi_2(v\mathscr{R}') = vH$.

Now it suffices to check the universal property. For every homomorphism that maps G and G' to a group K, which we call them f_1 and f_2, we can define $\phi : G * G' \to K$ by

$$\phi(wH) = \prod_{i=1}^{|w|} \left(f_1(w_i\mathscr{R})\chi_{F(A)}(w_i) + f_2(w_i\mathscr{R}')\chi_{F(A')}(w_i) \right)$$

where $w = w_1 \cdots w_n$, χ is the indicator function. The commutative of the coproduct diagram is clear, and ϕ is clearly a homomorphism since we can clearly combine two finite product to one. ■

Problem II.8.13. Let G be a finite group, and assume $|G|$ is odd. Prove that every element of G is a square.

Proof. Let $|G| = 2n - 1$, $n \in \mathbb{N}$. For every $g \in G$, we have

$$g = g \cdot g^{2n-1} = g^{2n} = (g^n)^2$$

which implies that every element in G is a square. ■

Problem II.8.13. Generalize the result of II.8.13: if G is a group of order n and k is an integer relatively prime to n, then the function $G \to G, g \to g^k$ is surjective.

Proof. By the prime condition, we can apply Bezout's identity, namely there exists integers a, b such that $an + bk = 1$. Then for every $g \in G$, we have

$$g = g \cdot g^{-an} = g^{1-an} = g^{bk} = (g^b)^k$$

which implies that every element in G is a k-power of some element in G. ■

Problem II.8.17. Assume that G is a finite abelian group, and let p be a prime divisor of $|G|$. Prove that there exists an element in G of order p.

Proof. We proceed by induction. Clearly if $|G| = 1$ then the statement is true. Now suppose for all abelian group with order less than n, we can find a element whose order is a prime and a divisor of G. Then for any group G that has order n, consider an element $g \in G$, and consider the subgroup generated by g, $H = \langle g \rangle$.

Clearly H is cyclic, so we can find a element $g^{|g|/q}$ of order q where q is a prime since

$$1 = g^{|g|} = (g^{|g|/q})^q$$

provided that $q \mid |g|$. Now if $q = p$, then we are done; otherwise, we replace G with $G/\langle h \rangle$, where $h = g^{|g|/q}$ (note that all subgroups are normal since G is abelian). Now this quotient has order less than n, and by induction, we can find an element of order p in it, which we call it $m\langle h \rangle$. Finally the element mh^q has order p, since

$$(mh^q)^p = m^p g^{p|g|} = 1$$

Note that the commutative is used here. ■

Problem II.8.20. Assume that G is a finite abelian group, and let d be a divisor of $|G|$. Prove that there exists a *subgroup* $H \subseteq G$ of order d.

Proof. We proceed by induction. Clearly if $|G| = 1$ then the statement is true. Now suppose for all abelian group with order less than n, we can find a subgroup whose order is a divisor of $|G|$. Then if $|G| = n$, then by II.8.18, we have an element in G that is of order p, where p is a prime and a divisor of d. If $p = d$, then we are done. Otherwise, we consider the quotient $G/\langle p \rangle$. This group has order $|G|/p$, and by induction hypothesis, we can find a subgroup H in the quotient that is of order d/p. Now we claim that the set

$$H' = \{gp^n : n \in \{0, \cdots, p-1\}, g\langle p \rangle \in H\}$$

is a subgroup of order d. It is indeed a subgroup since for $g, h \in H'$,

$$gh^{-1} = ap^k b^{-1} p^{-l} = ab^{-1} p^{k-l} \in H'$$

for some a, b that is a coset representative ($ab^{-1}\langle p \rangle \in H$ since H is a subgroup). As the cosets are disjoint, there are precisely $p \cdot d/p = d$ elements in H', proving the assertion. ■

Problem II.8.22. Let $\varphi : G \to G'$ be a group homomorphism, and let N be the smallest normal subgroup containing $\operatorname{im} \varphi$. Prove that G'/N satisfies the universal property of $\operatorname{coker} \varphi$ in Grp.

Proof. By universal property of quotient, for every homomorphism $\alpha : G' \to L$, the homomorphism $\bar{\alpha} : G'/N \to L$ exists and is unique. Now it suffices to check the universal property of cokernel. For any $\alpha : G' \to L$ such that $\alpha \circ \varphi = 0$, define $\bar{\alpha}(gN) = \alpha(g)$. We need to check that this is well defined. If $\bar{\alpha}(gN) = \bar{\alpha}(hN)$ but $\alpha(g) \neq \alpha(h)$, then $gh^{-1} \notin \ker \alpha$. However since $\alpha \circ \varphi = 0$, $\operatorname{im} \varphi \subseteq \ker \alpha$. By noting that N is normal and minimal, we have

$$\ker \alpha \supseteq N \ni gh^{-1}$$

since $gN = hN$. This is a contradiction, therefore $\alpha(g) = \alpha(h)$, showing the well-definedness of $\bar{\alpha}$. Then

$$\bar{\alpha}(\pi(\varphi(g)) = \bar{\alpha}(N) = \alpha(e) = e_L$$

for all $g \in G$. This shows $\bar{\alpha} \circ \pi \circ \varphi = 0$, and the assertion is proved. ■

Problem II.8.24. Show that epimorphisms in Grp do not necessarily have right-inverses.

Proof. Let

$$\varphi : \mathbb{Z} \to \mathbb{Z}_2, \quad \varphi(x) = x \mod 2$$

this map has no right inverses as any homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}$ can only be the identity map. ■

II.9

Problem II.9.7. Prove that stabilizers are indeed subgroups.

Proof. Assume G acts on A, and pick $a \in A$. For $g, h \in \operatorname{Stab}_G(a)$, we have

$$gh^{-1}a = g(h(h^{-1}a)) = ga = a$$

as required. ■

Problem II.9.11. Let G be a finite group, and let H be a subgroup of index p, where p is the *smallest prime dividing* $|G|$. Prove that H is normal in G.

Proof. We consider the left-multiplication action of G on the left cosets of H, which is $g \cdot hH = ghH$. This induces a homomorphism $\varphi : G \to S_p$, whose kernel includes H since

$$\text{if } g \in \ker \varphi, \text{ then } aH = gaH \ \forall a \in G \Rightarrow g = gH \Rightarrow g \in H.$$

Then $G/\ker\varphi \cong \operatorname{im}\varphi$, so $G/\ker\varphi$ is a subgroup of S_p, therefore it has order dividing $p!$. However by Lagrange, such order also divides $|G|$, and hence must be divisible by p, so $|G/\ker\varphi| = p$. Finally

$$p = [G : H] = [G : \ker\varphi][\ker\varphi : H] = p[\ker\varphi : H]$$

which leads to $[\ker\varphi : H] = 1$. Since $\ker\varphi \subseteq H$, $\ker\varphi = H$ by index consideration, proving the assertion. ■

Problem II.9.13. Prove 'by hand' that that for all subgroups H of a group G and $\forall g \in G$, G/H and $G/(gHg^{-1})$ (endowed with the action of G by left-multiplication) are isomorphic in G-Set.

Proof. We want to find a *bijection* function $\varphi : G/H \to G/gHg^{-1}$ such that the diagram

$$\begin{array}{ccc} G \times G/H & \xrightarrow{id_G \times \varphi} & G \times G/gHg^{-1} \\ \downarrow \rho & & \downarrow \rho' \\ G/H & \xrightarrow{\varphi} & G/gHg^{-1} \end{array}$$

commutes. Indeed the most natural map would be $\varphi(xH) = (gxg^{-1})gHg^{-1}$. We check that this is well-defined; if $aH = bH$, then $gaHg^{-1} = gbHg^{-1}$ clearly. We now check that this is a bijection, by explicitly give the inverse

$$\phi : G/gHg^{-1} \to G/H, \quad \phi(xgHg^{-1}) = (g^{-1}xg)H$$

so $\varphi \circ \phi = id$. Therefore G/H and $G/(gHg^{-1})$ are isomorphic in G-Set. Note that if we assume $\varphi(xH) = xgHg^{-1}$, then H would need to be normal in order to be well-defined. ■

Problem II.9.17. Consider G as a G-set, by acting with left-multiplication. Prove that $\operatorname{Aut}_{G-\mathsf{Set}(G)} \cong G$.

Proof. The set of automorphisms on $G - \mathsf{Set}(G)$ are bijections that satisfies $g\varphi(h) = \varphi(gh)$. In particular we can define

$$\varphi_g(h) = g^{-1}h$$

this is clearly a bijection and forms a group structure by $\varphi_g\varphi_h = \varphi_{gh}$. We now consider the map $\psi : \mathrm{Aut}_{G-\mathsf{Set}(G)} \to G$ by $\psi(\varphi_g) = g$. We claim that this is an isomorphism. Indeed, its kernel is precisely φ_e, which is the identity of $\mathrm{Aut}_{G-\mathsf{Set}(G)}$. The map is clearly surjective, and it is an homomorphism by construction. Therefore $\mathrm{Aut}_{G-\mathsf{Set}(G)} \cong G$. ■

Chapter III

Rings and modules

Unless otherwise specified, in the following $R = (R, +, \cdot)$ denotes an arbitrary ring *with identity*, $0, 1$ denotes the additive and multiplicative identity of R, respectively. In the case of possible confusion, I will use $0_R, 1_R$ instead.

Some description and hints are omitted for simplicity.

III.1

Problem III.1.1. Prove that if $0 = 1$ in a ring R, then R is a zero ring.

Proof. If r is any nonzero element in R, then

$$r = r \cdot 1 = r \cdot 0 = 0$$

showing that $R = 0$. ■

Problem III.1.6. Prove that if a and b are nilpotent in R and $ab = ba$, then so is $a + b$.

Proof. If $a^n = 0, b^m = 0$, then

$$(a+b)^{n+m} = a^{n+m} + \binom{n+m-1}{1} a^{n+m-1}b + ... + b^{n+m}$$

and all terms are zeros since every term either have a^n or b^m. If we do not assume that $ab = ba$, then the statement would be false, for example, in $M_n(\mathbb{Z})$,

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

are nilpotent of degree 3, but $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is not nilpotent. ■

Problem III.1.7. Prove that $[m]$ is nilpotent in $\mathbb{Z}/n\mathbb{Z}$ if and only if m is divisible by all prime factors of n.

Proof.
($\Rightarrow$) If $[m]^k = [0]$ for some integer k, then this implies $m^k = dn$ for some integer d. Now we write $n = p_1^{a_1} \cdots p_n^{a_n}$, where p_i are primes, and a_i are positive integers. Then

$$m^k = dp_1^{a_1} \cdots p_n^{a_n}$$

and it is clear to see that m must contain each p_i at least once.
($\Leftarrow$) If $n = p_1^{a_1} \cdots p_n^{a_n}$ where p_i are primes, and a_i are positive integers, then we can write

$$m = p_1^{b_1} \cdots p_n^{b_n} d$$

where b_i, d are positive integers, and $p_i \nmid d$ for all i. Define

$$f = \text{floor}\left(\max\left\{\frac{a_1}{b_1}, \cdots \frac{a_n}{b_n}\right\}\right)$$

then let $r = m^f/n$, which is an integer larger than 0 by the choice of f. Finally

$$m^f = nr = 0 \mod n$$

showing that m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$. ■

Problem III.1.9. Prove Proposition 1.12, that is:

- The inverse of a two-sided unit is unique;
- two-sided units form a group under multiplication.

Proof. For a two-sided unit v, we have $uv = 1$ and $vw = 1$ for some $u, w \in R$. Then

$$w = 1 \cdot w = uvw = u \cdot 1 = u$$

showing that $w = u$, so the inverse can be uniquely defined as $v^{-1} = u$. Now as the inverse is unique, we can properly define a group structure, using the multiplication from the ring R. ■

Problem III.1.15. Prove that $R[x]$ is a domain if and only if R is a domain.

Proof.
($\Rightarrow$) Trivial since $R \subset R[x]$.
($\Leftarrow$) Assume the contrary that $R[x]$ is not a domain. Then we can find $f = \sum_{i=0}^{n} a_i x^i$, $g = \sum_{j=0}^{m} b_j x^j$, $f \neq 0, g \neq 0$ such that $fg = 0$. Then we would have $a_n b_m = 0$, and since R is a domain, either a_n or b_m is zero. Without loss of generality, we can reduce the case to $f = a_0 \neq 0$. Then by the same argument, we would arrive at $a_0 b_0 = 0$, since all higher terms must be zero. But this contradict to the assumption that R is a domain, since $f = a_0$ and $g = b_0$ are nonzero. Hence $R[x]$ must be a domain. ■

III.2

Problem III.2.1. Prove that if there is a homomorphism from a zero ring to a ring R, then R is a zero ring.

Proof. If 1_R is the multiplicative identity of R, then for any homomorphism $\varphi : 0 \to R$,

$$0_R = \varphi(0) = \varphi(1) = 1_R$$

and by III.1.1, R is a zero-ring. ■

Problem III.2.6. Verify the 'extension property' of polynomial ring:
Let $\alpha : R \to S$ be a fixed ring homomorphism, and let $s \in S$ be an element commuting with $\alpha(r)$ for all $r \in R$. Then there is a unique ring homomorphism $\bar{\alpha} : R[x] \to S$ extending α and sending x to s.

Proof. Indeed, for $\sum_{i\geq 0} a_i x^i \in R[x]$, we have no choice but to define

$$\bar{\alpha}\left(\sum_{i\geq 0} a_i x^i\right) = \sum_{i\geq 0} \alpha(a_i) s^i \tag{1}$$

so that $\bar{\alpha}(r) = \alpha(r)$ and x sends to s in this map. It is clearly a homomorphism (note that the commutativity of s is used in the proof of $\bar{\alpha}(fg) = \bar{\alpha}(f)\bar{\alpha}(g)$), so it suffices to check that $\bar{\alpha}$ is unique. But it is clear by the fact that any map that extends α and send x to s must have the same value evaluated as in (1). ■

Problem III.2.9. Prove that the center of R is a subring. Moreover, prove that the center of a division ring is a field.

Proof. A subset of a ring S is a subring if it is a subgroup of $(R, +)$, closed under multiplication, and 1 is in it. So we check that:

- it is a subgroup of $(R, +)$: for $a, b \in C$, for all $r \in R$,

  $$(a-b)r = ar - br = ra - rb = r(a-b)$$

  showing that $a - b \in C$, hence a subgroup;
- closed under multiplication: for $a, b \in C$, for all $r \in R$,

  $$abr = a(br) = a(rb) = (ar)b = (ra)b = rab$$

  showing that $ab \in C$;
- finally, 1 is in C since $1r = r1$ for all $r \in R$.

Clearly the center forms a commutative ring since for $a, b \in C$, $ab = ba$. Then it follows by definition that a commutative division ring is a field. ■

Problem III.2.10. Prove that the centralizer of a is a subring for every $a \in R$. Prove that the center is the intersection of all its centralizers, and prove that every centralizer of a division ring is a division ring.

Proof. We use the same test as above. Let C_x denotes the centralizer of x.

- It is a subgroup of $(R, +)$: for $a, b \in C_x$,

  $$(a-b)x = ax - bx = xa - xb = x(a-b)$$

  showing that $a - b \in C_x$, hence a subgroup;
- closed under multiplication: for $a, b \in C_x$,

  $$abx = a(bx) = a(xb) = (ax)b = (xa)b = xab$$

  showing that $ab \in C_x$;
- finally, 1 is in C_x since $1x = x1$.

It is easy that the center is the intersection of all its centralizers, since such elemet in the intersection must commute with the whole ring R. Finally, if R is a division ring, then for every element $a \in C_x$, then we show that $a^{-1} \in C_x$:

$$ax = xa \Rightarrow axa^{-1} = x \Rightarrow xa^{-1} = a^{-1}x$$

as desired. ■

Problem III.2.11. Prove that a division ring R which consists of p^2 elements where p is a prime, is commutative.

Proof. Suppose the contrary that R is not commutative. Then the center C must be a proper subring, which can only consist of p elements by Lagrange. Now let $r \in R \backslash C$. Then the centralizer of r will contain at least r and C by III.2.10, therefore the centralizer of r must be R itself (again by Lagrange), for every $r \in R \backslash C$. But then the intersection of all centralizer are now R (element of center has centralizer R clearly), which is a contradiction to that C is proper. Therefore R must be commutative, i.e. a field. ∎

Problem III.2.12. Consider the inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Describe the cokernel of ι in Ab and its cokernel in Ring.

Solution. In Ab, this is easy: it is just $\mathbb{Q}/\operatorname{im}\iota = \mathbb{Q}/\mathbb{Z}$. However in Ring, we notice that for any map $\alpha : \mathbb{Q} \to F$ that satisfy $\alpha \circ \iota = 0$, we have

$$0_F = \alpha(1) = \alpha \circ \iota(1) = \alpha(1) = 1_F$$

which shows that F must be the zero ring by III.1.1. Now the unique homomorphism $\bar{\alpha} : \operatorname{coker} \iota \to F$ must also be the zero map, and by the requirement $\bar{\alpha} \circ \pi \circ \iota = 0$, we finally have $\pi \circ \iota = 0$, and by the same argument as above, we have that the codomain of π is the zero ring, i.e. $\operatorname{coker} \iota = 0$. ∎

III.3

Problem III.3.2. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of S. Prove that $\varphi^{-1}(J)$ is an ideal.

Proof. The ideal is clearly nonempty, so it suffices to check that $\varphi^{-1}(J)$ is a additive subgroup and satisfies the absorption property. For $x, y \in \varphi^{-1}(J)$, we have $\varphi(x), \varphi(y) \in J$, so $\varphi(x) - \varphi(y) = \varphi(x - y) \in J$, therefore $x - y \in \varphi^{-1}(J)$, showing that it is a subgroup of $(R, +)$.

Now for any $r \in R, a \in \varphi^{-1}(J)$, we have $\varphi(a) \in J$, so $\varphi(r)\varphi(a) = \varphi(ra) \in J$, and hence $ra \in \varphi^{-1}(J)$, showing the left-absorption property. The right case is the same. ∎

Problem III.3.3. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of R.

- Show that $\varphi(J)$ need not be an ideal of S.
- Assume that φ is surjective; then prove that $\varphi(J)$ *is* an ideal of S.
- Assume that φ is surjective, and let $I = \ker \varphi$. Let $\bar{J} = \varphi(J)$. Prove that

$$\frac{R/I}{\bar{J}} \cong \frac{R}{I + J}.$$

Proof. Let $\varphi : \mathbb{Z} \hookrightarrow \mathbb{R}$ be inclusion (and clearly a homomorphism). Then every ideal of $\mathbb{Z}$ will be directly transformed into $\mathbb{R}$. But since $\mathbb{R}$ is a field, by III.3.8 (which will be proved later) the possible ideal of $\mathbb{R}$ are only $\{0\}$ and $\mathbb{R}$ itself, so the image of a homomorphism need not to be an ideal.

However, If φ is surjective, Then $\varphi(J)$ is indeed an ideal: if $\varphi(x), \varphi(y) \in \varphi(J)$, then so is $\varphi(x) - \varphi(y) = \varphi(x - y) \in \varphi(J)$. The absorption property is also true since $\varphi(r)\varphi(x) = \varphi(rx) \in \varphi(J)$.

Finally, we consider the homomorphism

$$\phi : R/I \to R/(I + J), \quad \phi(a + I) = a + I + J$$

ϕ is clearly a surjective homomorphism, and by first isomorphism theorem

$$\frac{R/I}{\ker \phi} \cong \frac{R}{I+J}$$

so it remains to solve $\ker \phi$, which is

$$\begin{aligned}\ker \phi &= \{a + I : a + I + J = I + J\} \\ &= \{a + b + I : a \in I, b \in J\} \\ &= \{b + I : b \in J\} \\ &= \{\varphi(b) \in S : b \in J\} \quad \text{(regarding } R/I \text{ as } S) \\ &= \varphi(J) = \bar{J}\end{aligned}$$

therefore

$$\frac{R/I}{\bar{J}} \cong \frac{R}{I+J}$$

as required. ■

Problem III.3.7. Let R be a ring, and let $a \in R$. Prove that Ra is a left-ideal of R and aR is a right-ideal of R. Prove that a is a left-, resp. right-, unit if and only if $R = aR$, resp. $R = Ra$.

Proof. We prove only the left-ideal case since the same argument holds for right-ideal case. Ra is a subgroup of $(R, +)$ since for $ra, sa \in Ra$, $ra - sa = (r - s)a \in Ra$. The absorption property follows easily since $rsa = (rs)a \in Ra$.

If a is a right unit, then there exists u such that $ua = 1$. Then 1 is contained in Ra, and since for all $r \in R$, $r \cdot 1 \in Ra$, we conclude that $R = Ra$. ■

Problem III.3.8. Prove that R is a division ring if and only if its only left-ideals and right-ideals are $\{0\}$ and R.

In particular, a commutative ring R is a field if and only if the only ideals of R are $\{0\}$ and R.

Proof.
($\Rightarrow$) If a nonzero element a is in the left-ideal I, then so is 1 since

$$1 = a^{-1}a \in I \text{ by definition}$$

Therefore any nonzero left-ideals are automatically R itself. The right-ideal case is the same.
($\Leftarrow$) If a nonzero element a does not have a left inverse, then aR would be a proper right-ideal by III.3.7. Therefore all elements must have left(and hence right) inverse. ■

Problem III.3.10. Let $\varphi : k \to R$ be a ring homomorphism, where k is a field and R is a nonzero ring. Prove that φ is *injective*.

Proof. φ is injective if and only if $\ker \varphi = \{0\}$ by Proposition III.2.4. Also, the ideals of k are only $\{0\}$ and k by III.3.8. If $\ker \varphi = \{0\}$ then there is nothing to prove, so let $\ker \varphi = k$. But this means that $\varphi = 0$, so we have

$$1_R = \varphi(1) = 0 = \varphi(0) = 0_R$$

and by III.1.1, R is a zero ring, a contradiction to the hypothesis. Therefore $\ker \varphi = \{0\}$, showing that φ is injective. ■

Problem III.3.12. Let R be a *commutative* ring. Prove that the set of nilpotent elements forms an ideal of R. This ideal is called the *nilradical* of R.

Proof. From III.1.6 we already know that it forms a subgroup of $(R,+)$ by relpacing b with $-b$, so it remains to check that it is an ideal. Let I be such ideal. If $a \in R, r \in I$ and $r^n = 0$, then since

$$(ar)^n \stackrel{!}{=} a^n r^n = 0$$

in which ! is where commutative is used. Therefore $ar \in I$, proving the absorption property.

For an counter-example where R is not commutative, simply consider the example of III.1.6: it is not even a subgroup of $(R,+)$. ■

Problem III.3.13. Let R be a commutative ring, and let N be its nilradical. Prove that R/N contains no nonzero nilpotent elements. Such a ring is said to be *reduced.*

Proof. Pick an element $a \in R \backslash N$. Then for every integer $n > 0$,

$$(a+N)^n = a^n + \binom{n}{1} a^{n-1} N + \cdots + N^n = a^n + N$$

Since a is not nilpotent, $a^n \neq 0$ for every n, showing that $a + N$ is not nilpotent for $a \in R \backslash N$. ■

III.4

Problem III.4.1. Let R be a ring, and let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals of R. We let

$$\sum_{\alpha \in A} I_\alpha := \left\{ \sum_{\alpha \in A} r_\alpha \text{ such that } r_\alpha \in I_\alpha \text{ and } r_\alpha = 0 \text{ for all but finitely many } \alpha \right\}.$$

Prove that $\{I_\alpha\}_{\alpha \in A}$ is an ideal of R and that it is the smallest ideal containing all of the ideals I_α.

Proof. We only consider the case when $A = \{1,2\}$: Any other A follows the same exact argument.

Let $I = I_1 + I_2$. I is a subgroup of $(R,+)$: the two elements in I can be represented as $r_1 + r_2$ and $r_1' + r_2'$, and clearly $(r_1 - r_1') + (r_2 - r_2')$ is in I. The absorption property is also clear, since $r(r_1 + r_2) = (rr_1 + rr_2) \in I$.

Now it suffice to show that I is minimal. For every ideal that contains I_1 and I_2, they must also contain $r_1 + r_2$ for $r_1 \in I_1$ and $r_2 \in I_2$, since ideal is a subgroup of $(R,+)$. Therefore every such ideal must also contain I, proving the minimality of I. ■

Problem III.4.2. Prove that the homomorphic image of a Noetherian ring is Noetherian.

Proof. Let R be Noetherian, S be any ring, $\varphi : R \to S$ be a surjective ring homomorphism. Let J be an ideal of S. By III.3.2, the preimage is an ideal, which we call $I = \langle a_1, ...a_n \rangle$. We claim that $J = \langle \varphi(a_1), ...\varphi(a_n) \rangle$, so every finitely generated ideal will map to a finitely generated ideal, proving that S is Noetherian.

Indeed, since $a_i \in \varphi^{-1}(J)$, $\varphi(a_i) \in J$ for $i = 1, ..., n$, so $\langle \varphi(a_1), ...\varphi(a_n) \rangle \subseteq J$. On the other hand, for an element $j \in J$, there exists $i \in R$ such that $\varphi(i) = j$ by surjectivity, therefore $i \in I$, so i is generated by elements $a_1, ..., a_n$, i.e. $i = r_1 a_1 + ... + r_n a_n$. Then since φ is a homomorphism,

$$\varphi(i) = j = \varphi(r_1 a_1 + ... + r_n a_n) = s_1 \varphi(a_1) + ... + s_n \varphi(a_n)$$

so $J \subseteq \langle \varphi(a_1), ...\varphi(a_n) \rangle$, and the claim is proved. ■

Problem III.4.4. Prove that if k is a field, then $k[x]$ is a PID.

Proof. Let I be any ideal of $k[x]$. If $I = (0)$, then there is nothing to prove. Otherwise, there is some polynomial $f \in I$ that has minimal degree in I and is monic (since you can do scalar division). We claim that $I = (f)$. Indeed, for $g \in I$, we can use division algorithm to write

$$g(x) = f(x)q(x) + r(x)$$

where $\deg r(x) < \deg f(x)$. Since $k[x]$ is a subgroup, $r = g - fq \in I$, and by the minimality of f, $r(x) = 0$, so every element of I can be written as $g(x)f(x)$ for some $g \in k[x]$, showing that $k[x]$ is a PID. ■

Problem III.4.5. Let I, J be ideals in a commutative ring R, such that $I + J = (1)$. Prove that $IJ = I \cap J$.

Proof. If $x \in IJ$, then it can be represented as ij for some $i \in I, j \in J$, and by the property of ideal, $ji \in I, ij \in J$, so $ij \in I \cap J$. Conversely, we have

$$I \cap J = (I \cap J)(1) = (I \cap J)(I + J) = (I \cap J)I + (I \cap J)J \subseteq IJ + IJ = IJ$$

showing the identity. ■

Problem III.4.7. Let $R = k$ be a field. Prove that every nonzero (principle) ideal in $k[x]$ is generated by a unique *monic* polynomial.

Proof. From III.4.4 we already know that every ideal is generated by a single polynomial f. Since k is a field, we can do division, so there is a monic polynomial $f(x)/a$ where a is the coefficient of the largest degree in f. Then it's trivial that $(f) = (f/a)$. ■

Problem III.4.11. Let R be a commutative ring, $a \in R$, and $f_1(x), \dots, f_r(x) \in R[x]$.

- Prove the equality of ideals

$$(f_1(x), \dots, f_r(x), x - a) = (f_1(a), \dots, f_r(a), x - a).$$

- Prove the useful substitution trick

$$\frac{R[x]}{(f_1(x), \dots, f_r(x), x - a)} \cong \frac{R}{(f_1(a), \dots, f_r(a))}$$

Proof. We consider only the case $k = 1$; the other cases are just extending the same argument. We are required to prove that

$$(f(x), x - a) = (f(a), x - a)$$

For $f(x)$, we can apply division algorithm to get

$$f(x) = q(x)(x - a) + r$$

where $q(x) \in R[x], r \in R$. By plug in $x = a$, we obtain $r = f(a)$. Therefore $f(x)$ is generated by $f(a)$ and $(x - a)$, showing $f(x) \in (f(a), x - a)$. On the other hand, note the division algorithm also implies

$$f(a) = f(x) - q(x)(x - a) \in (f(x), x - a)$$

therefore $f(a) \in (f(x), x - a)$, so $(f(x), x - a) = (f(a), x - a)$. Now since $R[x]/(x - a) \cong R$, by III.3.3

$$\frac{R}{\varphi(J)} \cong \frac{R[x]}{\ker \varphi + J}$$

for an ideal $J \in R[x]$, $\varphi : R[x] \to R$ a surjective homomorphism. It is clear that how should we choose these: by taking

$$J = (f_1(x), \ldots, f_r(x)), \quad \varphi(f(x)) = f(a)$$

we have

$$\frac{R}{(f_1(a), \ldots, f_r(a))} \cong \frac{R[x]}{(f_1(x), \ldots, f_r(x), x - a)}$$

as desired (note that φ is surjective). ■

Problem III.4.13. Let R be an integral domain. For all $k = 1, \ldots, n$, prove that $(x_1, \ldots, x_k)$ is prime in $R[x_1, \ldots, x_n]$.

Proof. We proceed by induction. For the case $k = 1$, we have

$$\frac{R[x]}{(x)} \cong R \quad \text{(p.p.151)}$$

and since R is a domain, it follows by definition that (x) is a prime ideal. Suppose that for $k < n$, the argument holds. Then for $k = n$, choose

$$J = (x_1, \ldots, x_{n-1}), \quad \varphi : R[x_1, \ldots, x_n] \hookrightarrow R[x_1, \ldots, x_{n-1}]$$

where φ is the inclusion map and $\ker \varphi = (x_n)$. Then by III.3.3

$$\frac{R[x_1, \ldots, x_n]/(x_n)}{(x_1, \ldots, x_{n-1})} \cong \frac{R[x_1, \ldots, x_n]}{(x_1, \ldots, x_{n-1}) + (x_n)}$$

which simplifies to

$$\frac{R[x_1, \ldots, x_{n-1}]}{(x_1, \ldots, x_{n-1})} \cong \frac{R[x_1, \ldots, x_n]}{(x_1, \ldots, x_n)}$$

By induction hypothesis, the quotient on the left is a domain since $(x_1, \ldots, x_{n-1})$ is a prime ideal, therefore by definition, $(x_1, \ldots, x_n)$ is a prime ideal. ■

Problem III.4.16. Let R be a commutative ring, and let P be a prime ideal of R. Suppose 0 is the only zero-divisor of R contained in P. Prove that R is an integral domain.

Proof. Let $a, b \in R$ such that $ab = 0$. Then since $0 \in P$, $ab \in P$, so either $a \in P$ or $b \in P$. Without loss of generality, let $a \in P$. If $a = 0$, then we are done; otherwise, $a \neq 0$, and since $ab = 0$, we must have $b = 0$ as a is not a zero divisor (0 is the only zero-divisor in P). In both cases, we show that $ab = 0$ implies $a = 0$ or $b = 0$, showing that R is a domain. ■

Problem III.4.18. Let R be a commutative ring, and let N be its nilradical (III.3.12). Prove that N is contained in every prime ideal of R.

Proof. Let $x^n = 0$ for some positive integer n, and P a prime ideal. Then since $0 \in P$, we have

$$P \ni 0 = x^n = x \cdot x^{n-1}$$

By the property of prime ideal, either $x \in P$ or x^{n-1} in P. If the former case is true, then we are done; else, we can reduce to the case where either $x \in P$ or $x^{n-2} \in P$. By continuing this process, we finally arrived at either $x \in P$ or $x \in P$, showing that in any cases, $x \in P$. Therefore all nilpotent elements are in P, proving the statement. ■

Problem III.4.21. Let k be an algebraic closed field, and let $I \subseteq k[x]$ be an ideal. Prove that I is maximal if and only if $I = (x - c)$ for some $c \in k$.

Proof.
($\Leftarrow$) We have

$$\frac{k[x]}{(x-c)} \cong k \quad \text{(p.p.151)}$$

and since k is a field, it follows by definition that $(x-c)$ is maximal.
($\Rightarrow$) Let J be a maximal ideal. By III.4.4, $k[x]$ is a PID, hence every ideal is being generated by a single *monic* polynomial $f(x) \in k[x]$ (III.4.7). Since k is algebraic closed, we can write $f(x) = q(x)(x-c)$ for some $q(x) \in k[x]$, $c \in k$. Then

$$J = (f(x)) = (q(x)(x-c)) \subseteq (x-c)$$

and by Proposition III.4.11, either $J = (x-c)$ or $J = k[x]$. The latter case could not happen since the maximal can not be $k[x]$ itself, therefore $J = (x-c)$, as desired. ■

In the following, let M be a (left-)module over R.

III.5

Problem III.5.2. Prove claim 5.1.

Proof. Let $\sigma : R \to \mathrm{End}_{\mathsf{Ab}}(M)$ be a ring homomorphism and $\rho : R \times M \to M$ a function. We verify the following properties:

- $\rho(r, m+n) = \rho(r,m) + \rho(r,n)$.
  Note that $\sigma(r)$ is a endomorphism on M. Then

$$\rho(r, m+n) = \sigma(r)(m+n) = \sigma(r)(m) + \sigma(r)(n) = \rho(r,m) + \rho(r,n)$$

- $\rho(r+s, m) = \rho(r,m) + \rho(s,m)$.

$$\rho(r+s, m) = \sigma(r+s)(m) = \sigma(r)(m) + \sigma(s)(m) = \rho(r,m) + \rho(s,m)$$

- $\rho(rs, m) = \rho(r, \rho(s,m))$.

$$\rho(rs, m) = \sigma(rs)(m) = \sigma(r)\sigma(s)(m) = \sigma(r)\rho(s,m) = \rho(r, \rho(s,m))$$

- $\rho(1, m) = m$.

$$\rho(1, m) = \sigma(1)(m) = 1(m) = m$$

■

Problem III.5.3. Prove that $0 \cdot m = 0$ and that $(-1) \cdot m = -m$ for all $m \in M$.

Proof. Since $0m = (0+0)m = 0m + 0m, 0m = 0$. Since $0 = 0m = (-1+1)m = (-1)m + m, (-1)m = -m$. ■

Problem III.5.11. Let R be commutative. Prove that there is a natural bijection between the set of $R[x]$-module structures on M and $\mathrm{End}_{R-\mathsf{Mod}}(M)$.

Proof. If f is a R-endomorphism $f : M \to M$, then we have to show that there are some suitable maps

$$R[x] \times M \to M$$
$$(g(x),\ m) \to ?$$

that makes M into a module. We consider $(g(x), m) \to g(f)(m)$, where if $g(x) = \sum_i a_i x^i$, then

$$g(f)(m) = \sum_i a_i f^i(m) \text{ where } f^i = \underbrace{f \circ \cdots \circ f}_{i \text{ times}}$$

We can easily check by definition that M satisfies the property of $R[x]$-module, so this gives the injectivity of $R[x]$-modules to $\mathrm{End}_{R-\mathsf{Mod}}(M)$. To prove surjectivity, if M is a $R[x]$-module, then define $f(m) = xm$. Then M is indeed an endomorphism, proving the statement. ■

Problem III.5.12. Let M, N be R-modules, and let $\varphi : M \to N$ be a homomorphism of R-modules which has a inverse (therefore a bijection). Prove that φ^{-1} is also a homomorphism of R-modules. Conclude that a bijective R-module homomorphism is a R-module isomorphism.

Proof. Since

$$\varphi(\varphi^{-1}(m) + \varphi^{-1}(n)) = m + n = \varphi(\varphi^{-1}(m + n))$$

we have $\varphi^{-1}(m) + \varphi^{-1}(n) = \varphi^{-1}(m + n)$. And

$$\varphi(r\varphi^{-1}(m)) = r\varphi(\varphi^{-1}(m)) = rm = \varphi(\varphi^{-1}(rm))$$

so $r\varphi^{-1}(m) = \varphi^{-1}(rm)$ indeed. ■

Problem III.5.14. Prove Proposition 5.18, that is:

Let N, P be submodules of an R-module M. Then

- *$N + P$ is a submodule of M;*
- *$N \cap P$ is a submodule of P, and*

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}.$$

Proof. Every element of $N + P$ can be written as $n + p$ where $n \in N, p \in P$. Then it is clear that $r(n + p) = rn + rp \in N + P$ for $r \in M$. For the intersection $N \cap P$, it is also clear that for $p \in P, n \in N \cap P$, $pr \in N$ since $r \in N$, and $pr \in P$ since $p \in P$.

The proof for the second isomorphism theorem follows exactly the same as in groups (Proposition II.8.11). Consider the homomorphism

$$\varphi : P \to \frac{N + P}{N}, \quad \varphi(p) = pN$$

it is surjective since for every $(n + p)N$, there is a corresponding p. Then

$$\ker \varphi = \{p \in P : p \in N\} = P \cap N$$

then it follows by first isomorphism theorem that

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}.$$

■

III.6

Problem III.6.1. Prove Claim 6.3, that is, $F^R(A) \cong R^{\oplus A}$.

Proof. Observe that every element in $R^{\oplus A}$ can be uniquely written as

$$\sum_{a \in A} r_a \chi(a)$$

where $\chi(a) = \chi_a(x)$, the indicator function of a, and $r_a \in R$ for $a \in A$. Then it suffices to check the universal property of free modules: given a function $f : A \to M$ where M is a module, we show that the following diagram

$$\begin{tikzcd} R^{\oplus A} \arrow[r, "\exists! \varphi"] & M \\ A \arrow[u, "\chi"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. Indeed, we define

$$\varphi\left(\sum_{a \in A} r_a \chi(a)\right) = \sum_{a \in A} r_a f(a)$$

then the diagram clearly commutes (and is unique). Finally, φ is a $R-\mathsf{Mod}$ homomorphism since

$$\begin{aligned} \varphi\left(\sum_{a \in A} r_a \chi(a)\right) + \varphi\left(\sum_{a \in A} r'_a \chi(a)\right) &= \sum_{a \in A} r_a f(a) + \sum_{a \in A} r'_a f(a) \overset{\checkmark}{=} \sum_{a \in A} (r_a + r'_a) f(a) \\ &= \varphi\left(\sum_{a \in A} (r_a + r'_a)\chi(a)\right) = \varphi\left(\sum_{a \in A} r_a \chi(a) + \sum_{a \in A} r'_a \chi(a)\right) \end{aligned}$$

Note that R-module's definition gurantees the commutative of $\checkmark$. ■

Problem III.6.3. Let R be a ring, M an R-module, and $p : M \to M$ an R-module homomorphism such that $p^2 = p$. Prove that $M \cong \ker p \oplus \operatorname{im} p$.

Proof. We are required to prove that the diagram

$$\begin{tikzcd} \ker p \arrow[dr, "i_k"] \arrow[drr, bend left, "f_k"] & & \\ & M \arrow[r, "\exists! \varphi"] & N \\ \operatorname{im} p \arrow[ur, "i_m"'] \arrow[urr, bend right, "f_m"'] & & \end{tikzcd}$$

commutes. Notice that for $x \in \ker p, p(x) = 0$, and

$$\text{for } x \in \operatorname{im} p, x - p(x) = p(y) - p(p(y)) = p(y) - p(y) = 0$$

where $p(y) = x$. This suggest that we define φ as

$$\varphi(x) = f_k(x - p(x)) + f_m(p(x))$$

Indeed, if $x \in \ker p$, then $\varphi(x) = f_k(x)$; if $x \in \operatorname{im} p$, then $\varphi(x) = f_m(p(x)) = f_m(x)$ since for $x \in \operatorname{im} p$,

$$p(y) = x, p(p(y)) = p(y) \Rightarrow p(x) = x.$$

But what about $x \in \ker p \cap \operatorname{im} p$? In fact, the only element in the intersection is 0, as such x must have
$$x = p(y) = p(p(y)) = p(x) = 0$$
so φ is well-defined. Now it suffices to check that φ is a homomorphism, which is direct since p, f_k and f_m are both R-homomorphisms, so it preserves the action on M (check yourself if you're not convinced). Therefore by the universal property of coproduct, $\ker p \oplus \operatorname{im} p \cong M$. ■

Problem III.6.4. Let R be a ring, and let $n > 1$. View $R^{\oplus(n-1)}$ as a submodule of $R^{\oplus n}$, via the injective homomorphism $R^{\oplus(n-1)} \hookrightarrow R^{\oplus n}$ defined by

$$(r_1, \ldots, r_{n-1}) \hookrightarrow (r_1, \ldots, r_{n-1}, 0).$$

Give a one-line proof that
$$\frac{R^{\oplus n}}{R^{\oplus(n-1)}} \cong R.$$

Proof. The surjective map
$$(r_1, \ldots, r_{n-1}, r_n) \twoheadrightarrow r_n.$$
has kernel precisely to $R^{\oplus(n-1)}$, therefore by first isomorphism theorem

$$\frac{R^{\oplus n}}{R^{\oplus(n-1)}} \cong R.$$

■

This is the end of the solution manual as of February 19, 2020.
Please revisit
https://github.com/macyayaya/algebra-chapter-0-solutions/releases
for possible new releases.
Thanks for your reading.