Solution to Algebra : Chapter 0 by Paolo Aluffi

macyayaya[1]

Last updated at March 1, 2020
v0.4.2a

[1] https://github.com/macyayaya/

Prologue

Over a few months I want to improve my skills in solving algebra problems. I tried to find a textbook that can serves me good and is good enough to use in self-study.

Eventually, this is what I felt the most "comfortable" book in my opinion. It doesn't contain that much unlike Dummit & Foote, but the writing style, the explanation, and the exercises really served me well.

So here is the solution to Algebra : Chapter 0. There are a few important points to note here:

- The solution is *only* hosted on my GitHub page https://github.com/macyayaya/algebra-chapter-0-solutions. If you find this document outside this page, you might have an outdated version of the solution which might have errors, so please be aware.
- I will update the solution irregularly.
- I'll try to write this beginner-friendly (as I am also a beginner), so the answer might be way too detailed/verbose. Sorry if you find this annoying.
- If you found an error in the solutions, typos, bad grammar or want to give an advise on LaTeX formatting, etc., don't hesitate to open an issue or a pull request on my repo.
- The questions I picked is completely random, so if you want to see some solution of a certain problem (but please not all of them), you can also open an issue to notify me.
- However, I currently do *not* accept any PRs to new solutions; this is more than my note on self-study rather than a complete solution set.

Thanks.

macyayaya @ https://github.com/macyayaya/
Department of Mathematics, National Taiwan University
February 16, 2020

Contents

Chapter I

Preliminaries: Set theory and categories

Throughout this solution manual, we will use the same notation (and convention) as in the book, with probably a little to none changes.

For your convenience, it is recommended to search your question via whatever your browser provides (e.g. F3). The format of questions are *Chapter*(in roman).*Section.Question*.

In the following, categories are denoted using the Sans-serif font, e.g. Set.

I.1

Problem I.1.1. Locate a discussion of Russel's paradox, and understand it.

Problem I.1.2. Prove that if $\sim$ is an equivalence relation on a set S, then the corresponding family $\mathscr{P}_\sim$ defined in §1.5 is indeed a partition of S.

Proof. The union of such class must contain S by definition, as at worse the elements can be in the equivalence class formed by themselves. It suffices to check disjointness: If $a \in [x], a \in [y]$ but $x \nsim y$, then transitivity implies $x \sim a, a \sim y \Rightarrow x \sim y$, a contradiction. ∎

I.2

Problem I.2.1. How many different bijection are there between a set S with n elements and itself?

Solution. The first number has n choices; to make the map a bijection, the next number has only $(n-1)$ choices remaining. By continuing choosing, we have $n!$ different bijections. ∎

Problem I.2.5. Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism*, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections.

Solution. Epimorphism are *right-cancelable*; that is,

A function $f : A \to B$ is a epimorphism if for all sets Z and all functions $\beta, \beta' : Z \to A$,

$$\beta \circ f = \beta' \circ f \Longrightarrow \beta = \beta'.$$

We shall prove the following:
Proposition. *A function is surjective if and only if it is an epimorphism.*
Proof.
($\Rightarrow$) Let f be surjective. By Proposition I.2.1, a surjective function has a right-inverse, which we

call it g. Then if $\beta, \beta' : B \to Z$ are arbitrary function such that $\beta \circ f = \beta' \circ f$, then by composion with g we obtain

$$(\beta \circ f) \circ g = (\beta' \circ f) \circ g \Rightarrow \beta \circ (f \circ g) = \beta' \circ (f \circ g) \Rightarrow \beta \circ id_A = \beta' \circ id_A \Rightarrow \beta = \beta'$$

as desired.
($\Leftarrow$) Let f be an epimorphism. We need to consider some special $\beta : B \to Z$ so we can prove the assertion. We done this by "labeling": define

$$\beta(b) = \begin{cases} 1, & b \in \operatorname{im} f \\ 0, & b \notin \operatorname{im} f \end{cases}, \quad \beta'(b) = 1$$

Then since

$$\beta \circ f = \beta' \circ f \Rightarrow \beta = \beta'$$

this implies that beta receives *only* values in $\operatorname{im} f$, so $\operatorname{im} f \supseteq B$. Since we have $\operatorname{im} f \subseteq B$ clearly for any function f, we conclude that $\operatorname{im} f = B$, which is the definition of surjectivity. ■

I.3

Problem I.3.1. Let C be a category. Consider a structure C^{op} with

- $\mathrm{Obj}(\mathsf{C}^{op}) = \mathrm{Obj}(\mathsf{C})$;
- for A, B objects of C^{op}, $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A)$.

Show how to make this into a category.

Solution. For $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B), g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, define the composite of morphisms by

$$g \circ f := fg$$

where fg is defined in the sense of the category C. Now we check the definition of category:

- 1_A exists as $\mathrm{Hom}_{\mathsf{C}^{op}}(A, A) := \mathrm{Hom}_{\mathsf{C}}(A, A) \ni 1_A$;
- The composition works as intended: the map on the right is a morphism from C to A;
- The composite law is checked as

$$(h \circ g) \circ f = gh \circ f = f(gh) = (fg)h = h \circ fg = h \circ (g \circ f);$$

- Idenitity morphism work as intended:

$$1_A \circ f = f1_A = f, \quad f \circ 1_A = 1_A f = f.$$

■

Problem I.3.11. Draw the relevant diagrams and define composition and identities for the category $\mathsf{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10.

Solution. By reversing the arrow of $\mathsf{C}_{A,B}$, we obtain:

- Objects of this category are diagrams

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
& Z \\
B \arrow[ur, "g"'] &
\end{tikzcd}$$

- morphisms are

$$\begin{tikzcd}
A \arrow[dr, "f_1"] & & & A \arrow[dr, "f_2"] & \\
& Z_1 & \longrightarrow & & Z_2 \\
B \arrow[ur, "g_1"'] & & & B \arrow[ur, "g_2"'] &
\end{tikzcd}$$

  which are commutative diagrams

$$\begin{tikzcd}
A \arrow[dr, "f_2"] \arrow[drr, bend left, "f_1"] & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[ur, "g_2"'] \arrow[urr, bend right, "g_1"'] & &
\end{tikzcd} .$$

For the case $\mathsf{C}^{\alpha,\beta}$:

- Objects are diagrams

$$\begin{tikzcd}
& A \arrow[dr, "f"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \\
& B \arrow[ur, "g"'] &
\end{tikzcd}$$

- morphisms are

$$\begin{tikzcd}
& A \arrow[dr, "f_1"] & & & & A \arrow[dr, "f_2"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 & \longrightarrow & C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_2 \\
& B \arrow[ur, "g_1"'] & & & & B \arrow[ur, "g_2"'] &
\end{tikzcd}$$

  which are commutative diagrams

$$\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \\
& B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd} .$$

composition and identity are defined analogously as in Example 3.5. ■

I.4

Problem I.4.3. Let A, B be objects of a category C, and let $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

Proof. Let g be the right inverse of f, i.e. $fg = 1$. Then for any morphism $h, h' \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$,

$$h \circ f = h' \circ f \Rightarrow h \circ f \circ g = h' \circ f \circ g \Rightarrow h \circ 1 = h' \circ 1 \Rightarrow h = h'$$

showing that f is an epimorphism. For a counterexample in which the converse does not hold, consider $\mathsf{C} = \mathbb{Z}$, objects are integers, and morphisms are the relation $\leq$ (c.f. p.p.27). Then

$$f : 1 \to 2$$

is an epimorphism, but there are no right inverse for f, since there are no morphisms in $\mathrm{Hom}_{\mathsf{C}}(2, 1)$. ■

I.5

Problem I.5.1. Prove that a final object in a category C is initial in the opposite category C^{op} (I.3.1).

Proof. Let F be a final object in C, which means that the set $\mathrm{Hom}_{\mathsf{C}}(A, F)$ is a singleton for all $A \in \mathrm{Obj}(\mathsf{C})$. Since

$$\mathrm{Hom}_{\mathsf{C}}(A, F) = \mathrm{Hom}_{\mathsf{C}^{\mathsf{op}}}(F, A)$$

we have that F is initial in C^{op}. ■

Problem I.5.12. Define the notions of *fibered products* and *fibered coproducts*, as terminal objects of the categories $\mathsf{C}_{\alpha,\beta}, \mathsf{C}^{\alpha,\beta}$ considered in Example 3.10 (cf. also I.3.11), by stating carefully the corresponding universal properties.

As it happens, Set has both fibered products and fibered coproducts. Define these objects 'concretely', in terms of naive set theory.

Solution. Fibered product is *final* in $\mathsf{C}_{\alpha,\beta}$; that is, there are only one morphism in

$$\mathrm{Hom}\left(\begin{tikzcd}[ampersand replacement=\&] \& A \arrow[dr, "\alpha"] \& \\ Z \arrow[ur, "f_a"] \arrow[dr, "f_b"'] \& \& C \\ \& B \arrow[ur, "\beta"'] \& \end{tikzcd}, \begin{tikzcd}[ampersand replacement=\&] \& A \arrow[dr, "\alpha"] \& \\ F \arrow[ur, "i_a"] \arrow[dr, "i_b"'] \& \& C \\ \& B \arrow[ur, "\beta"'] \& \end{tikzcd}\right)$$

for any choice of the triple (Z, f_a, f_b). Expand this to a diagram leads to the following universal property:

The triple $(F, i_a : F \to A, i_b : F \to B)$ is universal in the sense that for every triple $(Z, f_a : Z \to A, f_b : Z \to B)$, there exists a unique morphism $\varphi : Z \to F$ such that the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "f_a"] \arrow[ddr, bend right, "f_b"'] & & \\ & F \arrow[r, "i_a"'] \arrow[d, "i_b"] & A \arrow[d, "\alpha"] \\ & B \arrow[r, "\beta"'] & C \end{tikzcd}$$

commutes. Fibered product are also called *pullback.*

Fibered coproduct is *initial* in $\mathsf{C}^{\alpha,\beta}$. Following the same argument as above, we have the following universal property:

The triple $(I, i_A : A \to I, i_B : B \to I)$ is universal in the sense that for every triple $(Z, f_A : A \to Z, f_B : B \to Z)$, there exists a unique morphism $\varphi : I \to Z$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "i_A"'] \arrow[ddr, bend left, "f_A"] & \\
B \arrow[r, "i_B"] \arrow[drr, bend right, "f_B"'] & I \arrow[dr, dashed, "\exists!\varphi"] & \\
& & Z
\end{tikzcd}$$

commutes. Fibered coproduct are also called *pushout.*

Set has fibered products: Let us define

$$A \times_C B := I = \{(a, b) : a \in A, b \in B, \alpha(a) = \beta(b)\}$$

with projections i_a, i_b. We check that this satisfy the universal property: define

$$\varphi(z) := (f_a(z), f_b(z))$$

we check:

- $i_b\varphi = f_b$ (resp. $i_a\varphi = f_a$):

$$i_b\varphi(z) = i_b(f_a(z), f_b(z)) = f_b(z)$$

- $\alpha i_a = \beta i_b$:

$$\alpha i_a(a, b) = \alpha(a) \overset{!}{=} \beta(b) = \beta i_b(a, b).$$

  note that ! is true since I gurantees the existence of b.

Set also has fibered coproducts, but it's more complicated. We first define an equivalence relation: define

$$R = \{(\alpha(x), 0) \sim (\beta(x), 1) : x \in C\}$$

This gives an equivalence relation on $A \amalg B$, which gives a new structure $I = (A \amalg B)/\sim$. Let $i_A(a) = (a, 0), i_B(b) = (b, 1)$, then it is direct that $i_B\beta = i_A\alpha$. Now we define

$$\varphi[i = (x, c)] = \begin{cases} f_A(x) & \text{if } c = 0 \\ f_B(x) & \text{if } c = 1 \end{cases}$$

We need to check that it is well-defined, then it is direct that $\varphi\beta = f_B$ (resp. $\varphi\alpha = f_A$), proving the universal property. There are two cases to consider:

- Case $[(a, 0)] = [(a', 0)]$ (resp. $[(b, 1)] = [(b', 1)]$): If there are relations

$$a = \alpha(x) \sim \beta(x) = \beta(x') \sim \alpha(x') = a'$$

  then they evaluated to the same value since

$$\varphi[(a, 0)] = \varphi i_A(a) = \varphi i_A(\alpha(x)) = \varphi i_B(\beta(x)) = \varphi i_B(\beta(x')) = \varphi i_A(\alpha(x')) = \varphi i_A(a') = \varphi[(a', 0)]$$

- Case $[(a,0)] = [(b,1)]$: If there are relations

  $$a = \alpha(x) \sim \beta(x) = b$$

  then
  $$\varphi[(a,0)] = \varphi i_A(a) = \varphi i_A(\alpha(x)) = \varphi i_B(\beta(x)) = \varphi i_B(b) = \varphi[(b,1)]$$
  as desired.

By the above analysis, as all elements in the same equivalence class connects to the other by some chain
$$a = \alpha(x_1) \sim \beta(x_1) = \beta(x_2) \sim \alpha(x_2) = \alpha(x_3) \cdots = b,$$
and since every $\sim$ preserves the result, φ is well-defined. ■

Chapter II

Groups, first encounter

Unless otherwise specified, in the following G denotes a group, e denotes the identity of G. Some description and hints are omitted for simplicity.

II.1

Problem II.1.8. Let G be a finite abelian group with exactly one element f of order 2. Prove that $\prod_{g \in G} g = f$.

Proof. For all elements that is not of order 2, they have an inverse that is not itself, so they canceled out in the product $\prod_{g \in G} g$, leaving only elements that is of order 2, i.e. f. ■

Problem II.1.10. If the order of g is odd, what can you say about the order of g^2 ?

Solution. The order of g^2 is $|g|$ since the only number that divides $|g|$ and in $\{2, 4, ..., 2|g|\}$ is $2|g|$ if $|g|$ is odd. ■

Problem II.1.11. Prove that for all g, h in a group G, $|gh| = |hg|$.

Proof. Simply observe that $e = (gh)^{|gh|} = g(hg)^{(|gh|-1)}h$, therefore

$$g^{-1}h^{-1} = (hg)^{-1} = (hg)^{|gh|-1}$$

hence $(hg)^{|gh|} = e$. The other case $((gh)^{|hg|} = e)$is the same. ■

Problem II.1.13. Give an example showing that $|gh| \neq \operatorname{lcm}(|g|, |h|)$ even if g and h commute.

Solution. In C_4, $|1 + 3| = |0| = 1$ but $\operatorname{lcm}(|1|, |3|) = 4$. Clearly C_4 is abelian. ■

Problem II.1.14. As a counterpoint of II.1.13, prove that if g and h commute and $\gcd(|g|, |h|) = 1$, then $|gh| = |g||h|$.

Proof. One has $|gh|$ divides $\operatorname{lcm}(|g|, |h|) = |g||h|$ by Proposition II.1.14, so it suffices to prove that $|g||h|$ divides $|gh|$. Let $N = |gh|$. By noting that $(gh)^N = g^N h^N$ since g and h commutes, we have

$$(gh)^{N|h|} = e^{|h|} = g^{N|h|}h^{N|h|} = g^{N|h|}$$

so $|g|$ divides $N|h|$, which implies $|g|$ divides N since $\gcd(|g|, |h|) = 1$. Similarly $|h|$ divides N, therefore $|g||h|$ divides $N = |gh|$, as desired. ■

Problem II.1.15. Let G be a commutative group, and let $g \in G$ be an element of maximal *finite* order. Prove that if h has finite order in G, then $|h|$ *divides* $|g|$.

Proof. Suppose that $|h|$ does not divide $|g|$, then we can assume that $|g| = p^m r$, $|h| = p^n s$, where p is a prime, r, s relatively prime to p and $m < n$. Since $|h|$ does not divide $|g|$, $\gcd(h, g) = 1$. Then by II.1.14 we can calculate the order of $g^{p^m}h^s$, which is $p^n r$. But this element has order bigger than g, which contradicts to the maximality of g. Hence $|h|$ must divide $|g|$. ■

II.2

Problem II.2.10. Prove that $\mathbb{Z}/n\mathbb{Z}$ consists of precisely n elements.

Proof. Trivial. ■

Problem II.2.14. Show that the multiplication in $\mathbb{Z}/n\mathbb{Z}$ is a well-defined action.

Proof. If $a \equiv a' \mod n$ and $b \equiv b' \mod n$, then $a = a' + kn$, $b = b' + ln$ for $k, l \in \mathbb{Z}$, therefore

$$(ab) - (a'b') = (a' + kn)(b' + ln) - a'b' = a'ln + b'kn + kln^2 \equiv 0 \mod n$$

as desired. ■

Problem II.2.16. Find the last digit of $1238237^{18238456}$.

Solution. $1238237^{18238456} \equiv 7^{18238456} = 49^{9119228} = 2401^{4559614} \equiv 1^{4559614} = 1 \mod 10.$ ■

Problem II.2.17. Show that if $m \equiv m' \mod n$, then $\gcd(m, n) = 1$ if and only if $\gcd(m', n) = 1$.

Proof. We can write $m = nk + m'$ for $n \in \mathbb{Z}$ and use Euclidean Algorithm to conclude. ■

II.3

Problem II.3.1. Let $\varphi : G \to H$ be a morphism in a category C with products. Explain why there is a unique morphism $(\varphi \times \varphi) : G \times G \to H \times H$ compatible in the evident way with the natural projections.

Solution. The compatibility of $(\varphi \times \varphi)$ comes from the commutative diagram

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \\
G \times G \arrow[u, "\pi_2"] \arrow[d, "\pi_1"] \arrow[r, "\exists!(\varphi\times\varphi)"] & H \times H \arrow[u, "\rho_2"] \arrow[d, "\rho_1"] \\
G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

which is easy to check. The uniqueness follows from the universal property of products that there is a unique homomorphism such that the diagram

$$\begin{tikzcd}
 & & H \\
G \times G \arrow[r, "\exists!(\varphi\times\varphi)"] \arrow[urr, bend left] \arrow[drr, bend right] & H \times H \arrow[ur] \arrow[dr] & \\
 & & H
\end{tikzcd}$$

commutes. ■

Problem II.3.3. Show that if G, H are abelian groups, then $G \times H$ satisfies the universal property for coproducts in Ab.

Proof. Let A be an arbitrary abelian group, f_G, f_H be homomorphisms, i_G, i_H be inclusions. We are required to prove the commutativity of the diagram

$$\begin{tikzcd}
G \arrow[dr, "i_G"'] \arrow[drr, bend left, "f_G"] & & \\
& G \times H \arrow[r, "\exists! \varphi"] & A \\
H \arrow[ur, "i_H"] \arrow[urr, bend right, "f_H"'] & &
\end{tikzcd}$$

To check the universal property, define $\varphi(g,h) := f_G(g)f_H(h)$. It is direct that the diagram commutes. Finally, φ is a homomorphism since for $g_1, g_2 \in G, h_1, h_2 \in H$,

$$\begin{aligned}
\varphi((g_1,h_1)(g_2,h_2)) = \varphi(g_1g_2, h_1h_2) = f_G(g_1g_2)f_H(h_1h_2) &= f_G(g_1)f_G(g_2)f_H(h_1)f_H(h_2) \\
&\overset{abelian}{=\!=\!=} f_G(g_1)f_H(h_1)f_G(g_2)f_H(h_2) = \varphi(g_1,h_1)\varphi(g_2,h_2)
\end{aligned}$$

as desired. ■

Problem II.3.6. Consider the product $C_2 \times C_3$, which is a coproduct in Ab. Show that it is *not* a coproduct of C_2 and C_3 in Grp.

Proof. If $C_2 \times C_3$ is a coproduct, then take $A = S_3$. Although there are injective homomorphisms

$$\begin{aligned}
\varphi_1 &: C_2 \to S_3 \text{ by } \varphi_1(1) = (12) \text{ or other two cycle} \\
\varphi_2 &: C_3 \to S_3 \text{ by } \varphi_2(1) = (123) \text{ or other three cycle}
\end{aligned}$$

but there are no homomorphisms $\varphi : C_2 \times C_3 \to S_3$ that satisfies the universal property of coproducts: Observe that any choice of cycles in φ_1 and φ_2 will exhaust all possible element of S_3, hence forces φ to be an isomorphism. But the element $\varphi(1,1)$ must be either a 2(or 3)-cycle (i.e. $\varphi^2(1,1)$ (or $\varphi^3(1,1)$) is zero), and neither $(1,1)^2$ nor $(1,1)^3$ are $(0,0)$, and φ will map a non-identity element to the identity, a contradiction (since φ is an isomorphism and must map $(0,0)$ to the trivial cycle). ■

II.4

Problem II.4.3. Prove that a group of order n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ if and only if it contains an element of order n.

Proof. Let G be such group.
($\Rightarrow$) Trivial.
($\Leftarrow$) Let g be an element of order n. Then consider a homomorphism $\varphi : G \to \mathbb{Z}/n\mathbb{Z}$ with $\varphi(g) = \bar{1}$. It is a direct check that this is an isomorphism. ■

Problem II.4.8. Let $g \in G$. Prove that the function $\gamma_g : G \to G$ defined by $\gamma_g(a) = gag^{-1}$ is an automorphism of G. Prove that the function $G \to \mathrm{Aut}(G)$ defined by $g \to \gamma_g$ is a homomorphism, and show that this homomorphism is trivial if and only if G is abelian.

Proof. γ_g is injective since if $gag^{-1} = gbg^{-1}$ then $a = b$; it is surjective since for $k \in G$ we can find $g^{-1}kg$ so that $\gamma_g(g^{-1}kg) = k$; it is a homomorphism since

$$\gamma_g(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \gamma_g(a)\gamma_g(b).$$

If G is abelian then the automorphism is simply $\gamma_g(a) = a$; conversely if $gag^{-1} = a$ then $ga = ag$ for all $a, g \in G$, hence abelian. ■

Problem II.4.9. Prove that if m, n are positive integers such that $\gcd(m, n) = 1$, then $C_{mn} \cong C_m \times C_n$.

Proof.
$$\varphi : C_{mn} \to C_m \times C_n,\ \varphi(a) = (a \bmod m, a \bmod n)$$
is a homomorphism and a bijection. ■

Problem II.4.11. Assuming the fact that the equation $x^d = 1$ can have at most d solutions in $\mathbb{Z}/p\mathbb{Z}$ for a prime p, prove that $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic.

Proof. Let g be an element of maximal order, and by II.1.15, all elements have degree that divides $|g|$, i.e. $|h|^{|g|} = 1$ for all $h \in G$. Using the fact, we have $|G| \leq |d|$, since only at most $|g|$ elements can be the solution to $h^{|g|} = 1$. Clearly we also have $|G| \geq |d|$, so $|G| = |d|$. Thus the proof is complete by II.4.3. ■

Problem II.4.13. Prove that $\mathrm{Aut}_{\mathsf{Grp}}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$.

Proof. To make an automorphism φ, φ must fix $(0, 0)$, leaving 6 possible permutations for elements $(0, 1), (1, 0), (1, 1)$. It suffices to check that all permutations of these elements are homomorphisms(hence isomorphisms). ■

Problem II.4.14. Prove that the order of the group of automorphisms of a cyclic group C_n is the number of positive integers $r \leq n$ that are *relatively prime* to n (cf. II.6.14).

Proof. We shall first show that every endomomorphism of cyclic group C is of form $\varphi_n(x) = x^n$ for some n. Indeed, if σ is a endomomorphism that $\sigma(x) = x^a = \varphi_a(x)$, then for every $x^b \in C$ we have
$$\sigma(x^b) = \sigma(x)^b = (x^a)^b = (x^b)^a = \varphi(x^b)$$
so every endomomorphism is of form $\varphi_n : x \mapsto x^n$ for some n. Now to make this into an automorphism, if k is not relatively prime to n, say $\gcd(n, k) = r > 1$, then for a generator $x \in C_n$, we have
$$\varphi_k(x^{n/r}) = x^{n/r \cdot k} = x^{n \cdot k/r} = (x^n)^{k/r} = e^{k/r} = e$$
and since n/r is not n, φ_k maps a non-identity element to e, in which it is already mapped by $e \in C_n$, so φ_k fails to be a bijection. Therefore the order of $\mathrm{Aut}(C_n)$ is the number of positive integers that is relatively prime to n. ■

Problem II.4.16. Prove the *Wilson's theorem*: for $p \in \mathbb{N}_{>1}$, p is a prime if and only if
$$(p-1)! \equiv -1 \mod p$$

Proof. ($\Rightarrow$) Assuming that the result of II.1.8 and II.4.11 is true, consider $G = (\mathbb{Z}/n\mathbb{Z})^*$. It is cyclic, and has exactly one element of order 2 since for $0 \leq k \leq p-2$,
$$(p-1-k)^2 \equiv 1 + 2k + k^2 \equiv 1 \mod p \iff k(k+2) \equiv 0 \mod p$$
and such solution can only be $k = 0$ or $p-2$ since p is a prime, which correspond to $p-1$ and 1 (identity). Therefore by II.1.8
$$\prod_{g \in G} g = (p-1)! \equiv (p-1) \equiv -1 \mod p$$
as desired.
($\Leftarrow$) If p is not a prime, then there exists $1 < k < p$ such that $k|p$. Since $k < p$ we have $k|(p-1)!$, i.e.
$$(p-1)! \equiv rk \mod p \text{ for some } r \in \mathbb{Z}$$
and clearly no choice of r will make $rk \equiv -1 \mod p$ by the fact that $k|p$. Therefore p must be a prime. ■

II.5

Problem II.5.3. Use the universal property of free groups to prove that the map $j : A \to F(A)$ is injective.

Proof. If there is $a, b \in A$ such that $j(a) = j(b)$ but $a \neq b$, then let f be a set function such that $f(a) \neq f(b)$; in particular, let $G = \mathbb{Z}$ and let $f(a) = 1, f(b) = 2$. Then there are no homomorphisms that will make the diagram commute, therefore j must be injective. ∎

Problem II.5.6. Prove that the group $F(\{x, y\})$ is a coproduct $\mathbb{Z} * \mathbb{Z}$ of $\mathbb{Z}$ by itself in the category Grp.

Proof. We are given the universal property of free group: for $j : \{x, y\} \to F(\{x, y\})$, $\exists G, f$ such that the diagram

$$\begin{tikzcd} F(\{x,y\}) \arrow[r, "\exists!\varphi"] & G \\ \{x,y\} \arrow[u, "j"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. To check that it is a coproduct, consider the coproduct diagram composed with above. Let $i(0) = x$, j be the inclusion, then we have the following diagram:

$$\begin{tikzcd} & \mathbb{Z} \arrow[dr, "i"] \arrow[drr, bend left, "f"] & & \\ \{x,y\} \arrow[ur, "\gamma"] \arrow[rr, "j"] \arrow[dr, "\gamma"'] \arrow[rrr, bend right=70, "h"'] & & F(\{x,y\}) \arrow[r, "\exists!\varphi"] & G \\ & \mathbb{Z} \arrow[ur, "i"'] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

Note that the arrows j, h, φ comes from the free group diagram. From this, we have $f \circ \gamma = \varphi \circ j$. To check the coproduct diagram commutes, it suffices to check $f = \varphi \circ i$ (the case $g = \varphi \circ i$ is identical). To do this, define $\gamma(x) = 0, \gamma(y) = 1$. Then

$$f \circ \gamma(x) = f(0) = \varphi(x) = \varphi \circ j(x), \quad f \circ \gamma(y) = f(1) = \varphi(y) = \varphi \circ j(y)$$

Since $f(1) = \varphi \circ i(1) = \varphi(y)$, the homomorphisms agree on the generator, hence are the same. ∎

II.6

Problem II.6.5. Let G be a *commutative* group, and let $n > 0$ be an integer. Prove that $\{g^n : g \in G\}$ is a subgroup of G. Prove that this is not necessarily the case if G is not commutative.

Proof. For any two elements a, b in the set, they can be represented as g^n and h^n respectively. Now

$$ab^{-1} = g^n h^{-n} = (gh^{-1})^n$$

which shows that ab^{-1} is also in the set, proving the set is a subgroup. A counterexample would be D_6, the dihedral group with 6 elements, with the choice $n = 3$. Let s denote the reflection, r denotes the rotation, we then have

$$\{g^3 : g \in D_3\} = \{1, r^3, r^{2\cdot 3}, s^3, (sr)^3, (sr^2)^3\} = \{1, 1, 1, s, sr, sr^2\}$$

this set is not a subgroup, as $s^{-1}sr = r$ is not an element of this set. ∎

Problem II.6.7. Show that inner automorphisms (the collection of γ_g in II.4.8) form a subgroup $\operatorname{Inn}(G)$ of $\operatorname{Aut}(G)$, and show that $\operatorname{Inn}(G)$ is cyclic if and only if $\operatorname{Inn}(G)$ is trivial if and only if G is abelian. Deduce that if $\operatorname{Aut}(G)$ is cyclic, then G is abelian.

Proof. $\operatorname{Inn}(G)$ is a subgroup since

$$\gamma_g \circ \gamma_{h^{-1}} = gh^{-1}ahg^{-1} = (gh^{-1})a(gh^{-1})^{-1} \in \operatorname{Inn}(G).$$

If $\operatorname{Inn}(G)$ is cyclic, then let $\gamma_g(a) = gag^{-1}$ be a generator of order n. Then for any $b \in G$, we have $\gamma_b(x) = \gamma_g^n(x)$, for some integer n. Then by plug in b into the homomorphism, we have $gbg^{-1} = b^n b b^{-n}$. This gives $gb = bg\ \forall b \in G$, so γ_g is in fact trivial. Since the generator is trivial, we conclude that $\operatorname{Inn}(G)$ is trivial. If $\operatorname{Inn}(G)$ is trivial, then the function given in II.4.8 can only be the trivial map, so G is abelian by II.4.8. Finally, if G is abelian, then all inner automorphisms are trivial, and clearly trivial group is cyclic.

The last statement follows from Proposition II.6.11 that every subgroup of cyclic group is cyclic. ∎

Problem II.6.9. Prove that an *abelian* group G is finitely generated if and only if there is a surjective homomorphism

$$\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}} \twoheadrightarrow G$$

for some n.

Proof.
($\Rightarrow$) As the group is abelian, for $G = \langle a_1, \cdots a_n \rangle$, we can represen an element g uniquely as

$$g = a_1^{p_1} \cdots a_n^{p_n}$$

where $p_i \in \mathbb{Z},\ i = 1, \cdots n$. Therefore we can explictly write down the surjective homomorphism

$$\varphi : \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}} \twoheadrightarrow G \quad \text{by} \quad \varphi(p_1, \cdots, p_n) = a_1^{p_1} \cdots a_n^{p_n} = g$$

as desired.
($\Leftarrow$) By the universal property of $\mathbb{Z}^{\oplus n}$ we have the following diagram that commutes:

$$\begin{tikzcd} \mathbb{Z}^{\oplus n} \arrow[r, "\exists!\varphi"] & G \\ \{1, \cdots, n\} \arrow[u, "j"] \arrow[ur, "f"'] & \end{tikzcd} \tag{*}$$

To prove, it suffices to "replace" the set $\{1, \cdots, n\}$ by a subset of G.

$$\begin{tikzcd} & \mathbb{Z}^{\oplus n} \arrow[r, "\exists!\varphi"] & G \\ \{1, \cdots, n\} \arrow[ur, "j"] \arrow[r, "f"] & A \arrow[u, "\tilde{j}"] \arrow[ur, "i"'] & \end{tikzcd}$$

By the diagram (*), we have $i \circ f = \varphi \circ j$. It is a fast check that the diagram formed by $\tilde{j}, i$ and φ commutes. Finally since A is a finite set and $\operatorname{im} \varphi = G$, it follows by definition that G is finitely generated. ∎

Problem II.6.14. Let ϕ be the Euler's ϕ-function. Prove that for $n \in \mathbb{N}$,

$$\sum_{m>0, m|n} \phi(m) = n.$$

Proof. Let $\langle x \rangle = C_n$. We have the trivial equation

$$\sum_{g \in C_n} 1 = n$$

Now note that every element in C_n generates a cyclic subgroup. To establish the result, we show that for every $d > 0$ that is a divisor of n, the subgroup of order d is *unique*, i.e. the unique subgroup is given by

$$\langle x^{n/d} \rangle = \{g \in G : g^d = 1\}$$

Indeed, if $g = x^{kn/d}$ for some positive integer k, then $g^d = x^{kn} = 1$. Conversely, if $g^d = 1$, then we have $g = x^m$ for some m since x is a generator. But this means that $x^{md} = 1$, and this implies $n|md$. Hence we have

$$g = x^m = x^{n/d \cdot dm/n} = x^{n/d} \in \langle x^{n/d} \rangle$$

as desired.

Now we count the generators of each subgroup of C_n, which is $\phi(d)$ for every d that is a divisor of n. Since every element in C_n generates a cyclic subgroup C_d, the sum of generator along each subgroup is exactly n, namely

$$\sum_{g \in C_n} 1 = \sum_{m : m|n} \phi(m) = n$$

which proved the assertion. ■

Problem II.6.15. Prove that if $\varphi : G \to G'$ has a left inverse, then φ is a monomorphism.

Proof. If $a, b \in G$ are distinct elements that satisfies $\varphi(a) = \varphi(b)$, then having left inverse means there exists a homomorphism ψ such that $\psi \circ \varphi = id_G$. Then we would have $\psi \circ \varphi(a) = \psi \circ \varphi(b)$, which means $a = b$, a contradiction. ■

II.7

Problem II.7.3. Verify that the equivalent conditions for normality given in §7.1 are indeed equivalent.

Proof. Let $g \in G$ be fixed.

- ($gng^{-1} \in N \Rightarrow gNg^{-1} \subseteq N$) is clear.
- ($gNg^{-1} \subseteq N \Rightarrow gNg^{-1} = N$): For $n \in N$, there is an element $g^{-1}ng \in N$ by normality, so $g(g^{-1}ng)g^{-1} = n$, showing that $gNg^{-1} \supseteq N$.
- ($gNg^{-1} = N \Rightarrow gN \subseteq Ng$): For $h \in gN$, there is $h = gn$ for some $n \in N$. By normality of N, there is some $n' \in N$ such that $gng^{-1} = n'$, or $gn = n'g$. Hence $h = n'g$, therefore $h \in Ng$.
- ($gN \subseteq Ng \Rightarrow gN = Ng$): If $gN \subseteq Ng$, then we also have $g^{-1}N \subseteq Ng^{-1}$, which is $Ng \subseteq gN$.
- ($gN = Ng \Rightarrow gng^{-1} \in N$): If $gn = n'g$, then $gng^{-1} = n'$. Since N is a subgroup, $gng^{-1} \in N$.

■

Problem II.7.7. Let n be a positive integer. Let $H \subset G$ be the subgroup generated by all elements of order n in G. Prove that H is normal.

Proof. For $a \in H, g \in G$, since $a^n = e$,

$$(gag^{-1})^n = ga^n g^{-1} = e$$

we have $gag^{-1} \in H$, hence normal. ∎

Problem II.7.11. Prove that the commutator subgroup $[G, G]$ is normal, and the quotient $G/[G, G]$ is commutative.

Proof. Observe

$$gaba^{-1}b^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) = xyx^{-1}y^{-1} \in [G, G]$$

for $x = gag^{-1}, y = gbg^{-1}$. The quotient is commutative since $aba^{-1}b^{-1}[G, G] = [G, G]$ implies $ab[G, G] = ba[G, G]$. ∎

Problem II.7.12. Let $F = F(A)$ be a free group, and let $f : A \to G$ be a set-function from the set A to a *commutative* group G. Prove that f induces a unique homomorphism $F/[F, F] \to G$, where $[F, F]$ is the commutator subgroup of F defined in Exercise 7.11. Conclude that $F/[F, F] \cong F^{ab}(A)$.

Proof. We need to define a proper homomorphism $\tilde{f} : F/[F, F] \to G$. By the universal property of free group, we have a unique homomorphism $\varphi : F \to G$ induced from f. Now observe that for $g, h \in A$,

$$\varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1} = \varphi(ghg^{-1}h^{-1}) = e$$

as G is commutative, we know that φ vanish on $[F, F]$. Now we just define

$$\tilde{f} : F/[F, F] \to G \quad \text{by} \quad \tilde{f}(x[F, F]) = \varphi(x).$$

It is a fast check that $\tilde{f}$ is the required homomorphism. This gives the following diagram.

$$\begin{tikzcd}
 & F/[F,F] \arrow[dr, "\exists! \tilde{f}"] & \\
F^{ab}(A) \arrow[ur, "\pi"] \arrow[rr, "\exists! \varphi"] & & G \\
A \arrow[u, "j"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

Since both triangles commutes, the "triangle" formed by the edges $\pi \circ j$, f and $\tilde{f}$ also commutes. By general nonsense (Proposition I.5.4), we conclude that $F/[F, F] \cong F^{ab}(A)$. ∎

II.8

Problem II.8.2. Extend Example 8.6 as follows. Suppose G is a group and $H \subseteq G$ is a subgroup of *index* 2, that is, such that there are precisely two (say, left-) cosets of H in G. Prove that H is normal in G.

Proof. Let $x \in H$, and we need to prove that $gxg^{-1} \in H$ for all $g \in G$. If $g \in H$ then there is nothing to prove, so assume that $g \in aH$, another coset of H in G. We can write $g = ah$ for some h, so it remains to study $ahxh^{-1}a^{-1}$. By noting that $ahxh^{-1} \in aH$, we know that $ahxh^{-1}$ does not belong to H, and in the sense of right cosets, $ahxh^{-1}$ must belong to Ha, so there exists $h' \in H$ such that $ahxh^{-1} = h'a$. Finally

$$gxg^{-1} = ahxh^{-1}a^{-1} = h'aa^{-1} = h' \in H$$

which shows that H is normal. ■

Problem II.8.7. Let $(A|\mathscr{R}), (A'|\mathscr{R}')$, be the presentation for groups G, G', respectively, and assume that A and A' are disjoint. Prove that

$$G * G' := (A \cup A' \mid \mathscr{R} \cup \mathscr{R}')$$

satisfies the universal property for the coproduct of G and G' in Grp.

Proof. Write $H = \mathscr{R} \cup \mathscr{R}'$. Let us construct a homomorphism from G to $G * G'$. As $G = F(A)/R$, by the universal property of quotient we have a commutative diagram

$$\begin{tikzcd} F(A) \arrow[rr, "f"] \arrow[dr, "\pi"'] & & G * G' \\ & F(A)/\mathscr{R} \arrow[ur, "\exists!\varphi_1"'] & \end{tikzcd}$$

In particular, we let f be an quotient map, i.e. $f(w) = wH$. Then naturally we have $\varphi_1(w\mathscr{R}) = wH$. Similarly, for G' we have another homomorphism $\varphi_2(v\mathscr{R}') = vH$.

Now it suffices to check the universal property. For every homomorphism that maps G and G' to a group K, which we call them f_1 and f_2, we can define $\phi : G * G' \to K$ by

$$\phi(wH) = \prod_{i=1}^{|w|} \left(f_1(w_i\mathscr{R})\chi_{F(A)}(w_i) + f_2(w_i\mathscr{R}')\chi_{F(A')}(w_i) \right)$$

where $w = w_1 \cdots w_n$, χ is the indicator function. The commutative of the coproduct diagram is clear, and ϕ is clearly a homomorphism since we can clearly combine two finite product to one. ■

Problem II.8.13. Let G be a finite group, and assume $|G|$ is odd. Prove that every element of G is a square.

Proof. Let $|G| = 2n - 1$, $n \in \mathbb{N}$. For every $g \in G$, we have

$$g = g \cdot g^{2n-1} = g^{2n} = (g^n)^2$$

which implies that every element in G is a square. ■

Problem II.8.14. Generalize the result of II.8.13: if G is a group of order n and k is an integer relatively prime to n, then the function $G \to G, g \to g^k$ is surjective.

Proof. By the prime condition, we can apply Bezout's identity, namely there exists integers a, b such that $an + bk = 1$. Then for every $g \in G$, we have

$$g = g \cdot g^{-an} = g^{1-an} = g^{bk} = (g^b)^k$$

which implies that every element in G is a k-power of some element in G. ■

Problem II.8.17. Assume that G is a finite abelian group, and let p be a prime divisor of $|G|$. Prove that there exists an element in G of order p.

Proof. We proceed by induction. Clearly if $|G| = 1$ then the statement is true. Now suppose for all abelian group with order less than n, we can find a element whose order is a prime and a divisor of G. Then for any group G that has order n, consider an element $g \in G$, and consider the subgroup generated by g, $H = \langle g \rangle$.

Clearly H is cyclic, so we can find a element $g^{|g|/q}$ of order q where q is a prime since

$$1 = g^{|g|} = (g^{|g|/q})^q$$

provided that $q \mid |g|$. Now if $q = p$, then we are done; otherwise, we replace G with $G/\langle h \rangle$, where $h = g^{|g|/q}$ (note that all subgroups are normal since G is abelian). Now this quotient has order less than n, and by induction, we can find an element of order p in it, which we call it $m\langle h \rangle$. Finally the element mh^q has order p, since

$$(mh^q)^p = m^p g^{p|g|} = 1$$

Note that the commutative is used here. ∎

Problem II.8.20. Assume that G is a finite abelian group, and let d be a divisor of $|G|$. Prove that there exists a *subgroup* $H \subseteq G$ of order d.

Proof. We proceed by induction. Clearly if $|G| = 1$ then the statement is true. Now suppose for all abelian group with order less than n, we can find a subgroup whose order is a divisor of $|G|$. Then if $|G| = n$, then by II.8.18, we have an element in G that is of order p, where p is a prime and a divisor of d. If $p = d$, then we are done. Otherwise, we consider the quotient $G/\langle p \rangle$. This group has order $|G|/p$, and by induction hypothesis, we can find a subgroup H in the quotient that is of order d/p. Now we claim that the set

$$H' = \{gp^n : n \in \{0, \cdots, p-1\}, g\langle p \rangle \in H\}$$

is a subgroup of order d. It is indeed a subgroup since for $g, h \in H'$,

$$gh^{-1} = ap^k b^{-1} p^{-l} = ab^{-1} p^{k-l} \in H'$$

for some a, b that is a coset representative ($ab^{-1}\langle p \rangle \in H$ since H is a subgroup). As the cosets are disjoint, there are precisely $p \cdot d/p = d$ elements in H', proving the assertion. ∎

Problem II.8.21. Let H, K be subgroups of a group G. Construct a bijection between the set of cosets hK with $h \in H$ and the set of left-cosets of $H \cap K$ in H. If H and K are finite, prove that

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

Proof. The map $hK \leftrightarrow h(K \cap H), h \in H$ is a bijection: it is well-defined since for $g, h \in H$, $gK = hK$ implies $gh^{-1} \in K$, and since $g, h \in H$, $gh^{-1} \in H \cap K$ and hence $g(H \cap K) = h(H \cap K)$. It is injective by reversing the above argument, and surjective by construction.

$$\{hK : h \in H\} \longleftrightarrow \{h(H \cap K) : h \in H\}$$

Now the set on the left has $|HK|/|H|$ elements in total, and the set on the right has $|H|/|H \cap K|$. A simple rearrangement gives the result. ∎

Problem II.8.22. Let $\varphi : G \to G'$ be a group homomorphism, and let N be the smallest normal subgroup containing im φ. Prove that G'/N satisfies the universal property of coker φ in Grp.

Proof. By universal property of quotient, for every homomorphism $\alpha : G' \to L$, the homomorphism $\bar{\alpha} : G'/N \to L$ exists and is unique. Now it suffices to check the universal property of cokernel. For any $\alpha : G' \to L$ such that $\alpha \circ \varphi = 0$, define $\bar{\alpha}(gN) = \alpha(g)$. We need to check that this is well defined. If $\bar{\alpha}(gN) = \bar{\alpha}(hN)$ but $\alpha(g) \neq \alpha(h)$, then $gh^{-1} \notin \ker \alpha$. However since $\alpha \circ \varphi = 0$, $\operatorname{im} \varphi \subseteq \ker \alpha$. By noting that N is normal and minimal, we have

$$\ker \alpha \supseteq N \ni gh^{-1}$$

since $gN = hN$. This is a contradiction, therefore $\alpha(g) = \alpha(h)$, showing the well-definedness of $\bar{\alpha}$. Then

$$\bar{\alpha}(\pi(\varphi(g)) = \bar{\alpha}(N) = \alpha(e) = e_L$$

for all $g \in G$. This shows $\bar{\alpha} \circ \pi \circ \varphi = 0$, and the assertion is proved. ■

Problem II.8.24. Show that epimorphisms in Grp do not necessarily have right-inverses.

Proof. Let

$$\varphi : \mathbb{Z} \to \mathbb{Z}_2, \quad \varphi(x) = x \mod 2$$

this map has no right inverses as any homomorphism from $\mathbb{Z}_2$ to $\mathbb{Z}$ can only be the identity map. ■

II.9

Problem II.9.7. Prove that stabilizers are indeed subgroups.

Proof. Assume G acts on A, and pick $a \in A$. For $g, h \in \operatorname{Stab}_G(a)$, we have

$$gh^{-1}a = g(h(h^{-1}a)) = ga = a$$

as required. ■

Problem II.9.11. Let G be a finite group, and let H be a subgroup of index p, where p is the *smallest prime dividing* $|G|$. Prove that H is normal in G.

Proof. We consider the left-multiplication action of G on the left cosets of H, which is $g \cdot hH = ghH$. This induces a homomorphism $\varphi : G \to S_p$, whose kernel includes H since

$$\text{if } g \in \ker \varphi, \text{ then } aH = gaH \; \forall a \in G \Rightarrow g = gH \Rightarrow g \in H.$$

Then $G/\ker \varphi \cong \operatorname{im} \varphi$, so $G/\ker \varphi$ is a subgroup of S_p, therefore it has order dividing $p!$. However by Lagrange, such order also divides $|G|$, and hence must be divisible by p, so $|G/\ker \varphi| = p$. Finally

$$p = [G : H] = [G : \ker \varphi][\ker \varphi : H] = p[\ker \varphi : H]$$

which leads to $[\ker \varphi : H] = 1$. Since $\ker \varphi \subseteq H$, $\ker \varphi = H$ by index consideration, proving the assertion. ■

Problem II.9.12. Let G be a group, and let $H \subseteq G$ be a subgroup of index n. Prove that H contains a subgroup K that is normal in G and such that $[G : K]$ divides the gcd of $|G|$ and $n!$. (In particular, $[G : K] \leq n!$.)

Proof. Following the same pattern from II.9.11, consider the left-multiplication action of G on the left cosets of H, which is $g \cdot hH = ghH$. This induces a homomorphism $\varphi : G \to S_n$ (as there are n left cosets), whose kernel includes H since

$$\text{if } g \in \ker\varphi, \text{ then } aH = gaH \ \forall a \in G \Rightarrow g = gH \Rightarrow g \in H.$$

Define $K = \ker\varphi$. Then $G/K \cong \operatorname{im}\varphi$, so G/K is a subgroup of S_n, therefore it has order dividing $n!$. By Lagrange, such order also divides $|G|$, so we've found the required K. ■

Problem II.9.13. Prove 'by hand' that that for all subgroups H of a group G and $\forall g \in G$, G/H and $G/(gHg^{-1})$ (endowed with the action of G by left-multiplication) are isomorphic in G-Set.

Proof. We want to find a *bijection* function $\varphi : G/H \to G/gHg^{-1}$ such that the diagram

$$\begin{tikzcd} G \times G/H \arrow[r, "id_G \times \varphi"] \arrow[d, "\rho"] & G \times G/gHg^{-1} \arrow[d, "\rho'"] \\ G/H \arrow[r, "\varphi"] & G/gHg^{-1} \end{tikzcd}$$

commutes. Indeed the most natural map would be $\varphi(xH) = (gxg^{-1})gHg^{-1}$. We check that this is well-defined; if $aH = bH$, then $gaHg^{-1} = gbHg^{-1}$ clearly. We now check that this is a bijection, by explicitly give the inverse

$$\phi : G/gHg^{-1} \to G/H, \quad \phi(xgHg^{-1}) = (g^{-1}xg)H$$

so $\varphi \circ \phi = id$. Therefore G/H and $G/(gHg^{-1})$ are isomorphic in G-Set. Note that if we assume $\varphi(xH) = xgHg^{-1}$, then H would need to be normal in order to be well-defined. ■

Problem II.9.17. Consider G as a G-set, by acting with left-multiplication. Prove that $\mathrm{Aut}_{G-\mathsf{Set}(G)} \cong G$.

Proof. The set of automorphisms on $G - \mathsf{Set}(G)$ are bijections that satisfies $g\varphi(h) = \varphi(gh)$. In particular we can define

$$\varphi_g(h) = g^{-1}h$$

this is clearly a bijection and forms a group structure by $\varphi_g\varphi_h = \varphi_{gh}$. We now consider the map $\psi : \mathrm{Aut}_{G-\mathsf{Set}(G)} \to G$ by $\psi(\varphi_g) = g$. We claim that this is an isomorphism. Indeed, its kernel is precisely φ_e, which is the identity of $\mathrm{Aut}_{G-\mathsf{Set}(G)}$. The map is clearly surjective, and it is an homomorphism by construction. Therefore $\mathrm{Aut}_{G-\mathsf{Set}(G)} \cong G$. ■

Chapter III

Rings and modules

Unless otherwise specified, in the following $R = (R, +, \cdot)$ denotes an arbitrary ring *with identity* (the book assumes this throughout this book), $0, 1$ denotes the additive and multiplicative identity of R, respectively. In the case of possible confusion, I will use $0_R, 1_R$ instead.

Some description and hints are omitted for simplicity.

III.1

Problem III.1.1. Prove that if $0 = 1$ in a ring R, then R is a zero ring.

Proof. If r is any element in R, then

$$r = r \cdot 1 = r \cdot 0 = 0$$

showing that $R = 0$. ∎

Problem III.1.6. Prove that if a and b are nilpotent in R and $ab = ba$, then so is $a + b$.

Proof. If $a^n = 0, b^m = 0$, then

$$(a+b)^{n+m} = a^{n+m} + \binom{n+m-1}{1} a^{n+m-1}b + ... + b^{n+m}$$

and all terms are zeros since every term either have a^n or b^m. If we do not assume that $ab = ba$, then the statement would be false, for example, in $M_n(\mathbb{Z})$,

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

are nilpotent of degree 3, but $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not nilpotent. ∎

Problem III.1.7. Prove that $[m]$ is nilpotent in $\mathbb{Z}/n\mathbb{Z}$ if and only if m is divisible by all prime factors of n.

Proof.
($\Rightarrow$) If $[m]^k = [0]$ for some integer k, then this implies $m^k = dn$ for some integer d. Now we write $n = p_1^{a_1} \cdots p_n^{a_n}$, where p_i are primes, and a_i are positive integers. Then

$$m^k = dp_1^{a_1} \cdots p_n^{a_n}$$

and it is clear to see that m must contain each p_i at least once.
($\Leftarrow$) If $n = p_1^{a_1} \cdots p_n^{a_n}$ where p_i are primes, and a_i are positive integers, then we can write

$$m = p_1^{b_1} \cdots p_n^{b_n} d$$

where b_i, d are positive integers, and $p_i \nmid d$ for all i. Define

$$f = \text{floor}\left(\max\left\{\frac{a_1}{b_1}, \cdots \frac{a_n}{b_n}\right\}\right)$$

then let $r = m^f/n$, which is an integer larger than 0 by the choice of f. Finally

$$m^f = nr = 0 \mod n$$

showing that m is nilpotent in $\mathbb{Z}/n\mathbb{Z}$. ■

Problem III.1.9. Prove Proposition 1.12, that is:

- *The inverse of a two-sided unit is unique;*
- *two-sided units form a group under multiplication.*

Proof. For a two-sided unit v, we have $uv = 1$ and $vw = 1$ for some $u, w \in R$. Then

$$w = 1 \cdot w = uvw = u \cdot 1 = u$$

showing that $w = u$, so the inverse can be uniquely defined as $v^{-1} = u$. Now as the inverse is unique, we can properly define a group structure, using the multiplication from the ring R. ■

Problem III.1.15. Prove that $R[x]$ is a domain if and only if R is a domain.

Proof.
($\Rightarrow$) Trivial since $R \subset R[x]$.
($\Leftarrow$) Assume the contrary that $R[x]$ is not a domain. Then we can find $f = \sum_{i=0}^n a_i x^i$, $g = \sum_{j=0}^m b_j x^j$, $f \neq 0, g \neq 0$ such that $fg = 0$. Then we would have $a_n b_m = 0$, and since R is a domain, either a_n or b_m is zero. Without loss of generality, we can reduce the case to $f = a_0 \neq 0$. Then by the same argument, we would arrive at $a_0 b_0 = 0$, since all higher terms must be zero. But this contradict to the assumption that R is a domain, since $f = a_0$ and $g = b_0$ are nonzero. Hence $R[x]$ must be a domain. ■

III.2

Problem III.2.1. Prove that if there is a homomorphism from a zero ring to a ring R, then R is a zero ring.

Proof. If 1_R is the multiplicative identity of R, then for any homomorphism $\varphi : 0 \to R$,

$$0_R = \varphi(0) = \varphi(1) = 1_R$$

and by III.1.1, R is a zero-ring. ■

Problem III.2.6. Verify the 'extension property' of polynomial ring:
Let $\alpha : R \to S$ be a fixed ring homomorphism, and let $s \in S$ be an element commuting with $\alpha(r)$ for all $r \in R$. Then there is a unique ring homomorphism $\bar{\alpha} : R[x] \to S$ extending α and sending x to s.

Proof. Indeed, for $\sum_{i\geq 0} a_i x^i \in R[x]$, we have no choice but to define

$$\bar{\alpha}\left(\sum_{i\geq 0} a_i x^i\right) = \sum_{i\geq 0} \alpha(a_i)s^i \tag{1}$$

so that $\bar{\alpha}(r) = \alpha(r)$ and x sends to s in this map. It is clearly a homomorphism (note that the commutativity of s is used in the proof of $\bar{\alpha}(fg) = \bar{\alpha}(f)\bar{\alpha}(g)$), so it suffices to check that $\bar{\alpha}$ is unique. But it is clear by the fact that any map that extends α and send x to s must have the same value evaluated as in (1). ■

Problem III.2.9. Prove that the center of R is a subring. Moreover, prove that the center of a division ring is a field.

Proof. A subset of a ring S is a subring if it is a subgroup of $(R, +)$, closed under multiplication, and 1 is in it. So we check that:

- it is a subgroup of $(R, +)$: for $a, b \in C$, for all $r \in R$,

$$(a - b)r = ar - br = ra - rb = r(a - b)$$

  showing that $a - b \in C$, hence a subgroup;
- closed under multiplication: for $a, b \in C$, for all $r \in R$,

$$abr = a(br) = a(rb) = (ar)b = (ra)b = rab$$

  showing that $ab \in C$;
- finally, 1 is in C since $1r = r1$ for all $r \in R$.

Clearly the center forms a commutative ring since for $a, b \in C$, $ab = ba$. Then it follows by definition that a commutative division ring is a field. ■

Problem III.2.10. Prove that the centralizer of a is a subring for every $a \in R$. Prove that the center is the intersection of all its centralizers, and prove that every centralizer of a division ring is a division ring.

Proof. We use the same test as above. Let C_x denotes the centralizer of x.

- It is a subgroup of $(R, +)$: for $a, b \in C_x$,

$$(a - b)x = ax - bx = xa - xb = x(a - b)$$

  showing that $a - b \in C_x$, hence a subgroup;
- closed under multiplication: for $a, b \in C_x$,

$$abx = a(bx) = a(xb) = (ax)b = (xa)b = xab$$

  showing that $ab \in C_x$;
- finally, 1 is in C_x since $1x = x1$.

It is easy that the center is the intersection of all its centralizers, since such elemet in the intersection must commute with the whole ring R. Finally, if R is a division ring, then for every element $a \in C_x$, we can show that $a^{-1} \in C_x$:

$$ax = xa \Rightarrow axa^{-1} = x \Rightarrow xa^{-1} = a^{-1}x$$

Therefore every element in C_x has a inverse, and by definition, C_x is a division ring. ■

Problem III.2.11. Prove that a division ring R which consists of p^2 elements where p is a prime, is commutative.

Proof. Suppose the contrary that R is not commutative. Then the center C must be a proper subring, which can only consist of p elements by Lagrange. Now let $r \in R\backslash C$. Then the centralizer of r will contain at least r and C by III.2.10, therefore the centralizer of r must be R itself (again by Lagrange), for every $r \in R\backslash C$. But then the intersection of all centralizer are now R (element of center has centralizer R clearly), which is a contradiction to that C is proper. Therefore R must be commutative, i.e. a field. ■

Problem III.2.12. Consider the inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$. Describe the cokernel of ι in Ab and its cokernel in Ring.

Solution. In Ab, this is easy: it is just $\mathbb{Q}/\operatorname{im}\iota = \mathbb{Q}/\mathbb{Z}$. However in Ring, we notice that for any map $\alpha : \mathbb{Q} \to F$ that satisfy $\alpha \circ \iota = 0$, we have

$$0_F = \alpha(1) = \alpha \circ \iota(1) = \alpha(1) = 1_F$$

which shows that F must be the zero ring by III.1.1. Now the unique homomorphism $\bar{\alpha} : \operatorname{coker} \iota \to F$ must also be the zero map, and by the requirement $\bar{\alpha} \circ \pi \circ \iota = 0$, we finally have $\pi \circ \iota = 0$, and by the same argument as above, we have that the codomain of π is the zero ring, i.e. $\operatorname{coker} \iota = 0$. ■

III.3

Problem III.3.2. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of S. Prove that $\varphi^{-1}(J)$ is an ideal.

Proof. The ideal is clearly nonempty, so it suffices to check that $\varphi^{-1}(J)$ is a additive subgroup and satisfies the absorption property. For $x, y \in \varphi^{-1}(J)$, we have $\varphi(x), \varphi(y) \in J$, so $\varphi(x) - \varphi(y) = \varphi(x - y) \in J$, therefore $x - y \in \varphi^{-1}(J)$, showing that it is a subgroup of $(R, +)$.

Now for any $r \in R, a \in \varphi^{-1}(J)$, we have $\varphi(a) \in J$, so $\varphi(r)\varphi(a) = \varphi(ra) \in J$, and hence $ra \in \varphi^{-1}(J)$, showing the left-absorption property. The right case is the same. ■

Problem III.3.3. Let $\varphi : R \to S$ be a ring homomorphism, and let J be an ideal of R.

- Show that $\varphi(J)$ need not be an ideal of S.
- Assume that φ is surjective; then prove that $\varphi(J)$ *is* an ideal of S.
- Assume that φ is surjective, and let $I = \ker \varphi$. Let $\bar{J} = \varphi(J)$. Prove that

$$\frac{R/I}{\bar{J}} \cong \frac{R}{I + J}.$$

Proof. Let $\varphi : \mathbb{Z} \hookrightarrow \mathbb{R}$ be inclusion (and clearly a homomorphism). Then every ideal of $\mathbb{Z}$ will be directly transformed into $\mathbb{R}$. But since $\mathbb{R}$ is a field, by III.3.8 (which will be proved later) the possible ideal of $\mathbb{R}$ are only $\{0\}$ and $\mathbb{R}$ itself, so the image of a homomorphism need not to be an ideal.

However, If φ is surjective, Then $\varphi(J)$ is indeed an ideal: if $\varphi(x), \varphi(y) \in \varphi(J)$, then so is $\varphi(x) - \varphi(y) = \varphi(x - y) \in \varphi(J)$. The absorption property is also true since $\varphi(r)\varphi(x) = \varphi(rx) \in \varphi(J)$.

Finally, we consider the homomorphism

$$\phi : R/I \to R/(I + J), \quad \phi(a + I) = a + I + J$$

ϕ is clearly a surjective homomorphism, and by first isomorphism theorem

$$\frac{R/I}{\ker\phi} \cong \frac{R}{I+J}$$

so it remains to solve $\ker\phi$, which is

$$\begin{aligned}\ker\phi &= \{a+I : a+I+J = I+J\}\\ &= \{a+b+I : a \in I, b \in J\}\\ &= \{b+I : b \in J\}\\ &= \{\varphi(b) \in S : b \in J\} \quad (\text{regarding } R/I \text{ as } S)\\ &= \varphi(J) = \bar{J}\end{aligned}$$

therefore

$$\frac{R/I}{\bar{J}} \cong \frac{R}{I+J}$$

as required. ■

Problem III.3.7. Let R be a ring, and let $a \in R$. Prove that Ra is a left-ideal of R and aR is a right-ideal of R. Prove that a is a left-, resp. right-, unit if and only if $R = aR$, resp. $R = Ra$.

Proof. We prove only the left-ideal case since the same argument holds for right-ideal case. Ra is a subgroup of $(R, +)$ since for $ra, sa \in Ra$, $ra - sa = (r-s)a \in Ra$. The absorption property follows easily since $rsa = (rs)a \in Ra$.

If a is a right unit, then there exists u such that $ua = 1$. Then 1 is contained in Ra, and since for all $r \in R$, $r \cdot 1 \in Ra$, we conclude that $R = Ra$. ■

Problem III.3.8. Prove that R is a division ring if and only if its only left-ideals and right-ideals are $\{0\}$ and R.

In particular, a commutative ring R is a field if and only if the only ideals of R are $\{0\}$ and R.

Proof.
($\Rightarrow$) If a nonzero element a is in the left-ideal I, then so is 1 since

$$1 = a^{-1}a \in I \text{ by definition}$$

Therefore any nonzero left-ideals are automatically R itself. The right-ideal case is the same.
($\Leftarrow$) If a nonzero element a does not have a left inverse, then aR would be a proper right-ideal by III.3.7. Therefore all elements must have left(and hence right) inverse. ■

Problem III.3.10. Let $\varphi : k \to R$ be a ring homomorphism, where k is a field and R is a nonzero ring. Prove that φ is *injective*.

Proof. φ is injective if and only if $\ker\varphi = \{0\}$ by Proposition III.2.4. Also, the ideals of k are only $\{0\}$ and k by III.3.8. If $\ker\varphi = \{0\}$ then there is nothing to prove, so let $\ker\varphi = k$. But this means that $\varphi = 0$, so we have

$$1_R = \varphi(1) = 0 = \varphi(0) = 0_R$$

and by III.1.1, R is a zero ring, a contradiction to the hypothesis. Therefore $\ker\varphi = \{0\}$, showing that φ is injective. ■

Problem III.3.12. Let R be a *commutative* ring. Prove that the set of nilpotent elements forms an ideal of R. This ideal is called the *nilradical* of R.

Proof. From III.1.6 we already know that it forms a subgroup of $(R, +)$ by relpacing b with $-b$, so it remains to check that it is an ideal. Let I be such ideal. If $a \in R, r \in I$ and $r^n = 0$, then since

$$(ar)^n \overset{!}{=} a^n r^n = 0$$

in which ! is where commutative is used. Therefore $ar \in I$, proving the absorption property.

For an counter-example where R is not commutative, simply consider the example of III.1.6: it is not even a subgroup of $(R, +)$. ∎

Problem III.3.13. Let R be a commutative ring, and let N be its nilradical. Prove that R/N contains no nonzero nilpotent elements. Such a ring is said to be *reduced*.

Proof. Pick an element $a \in R \backslash N$. Then for every integer $n > 0$,

$$(a + N)^n = a^n + \binom{n}{1} a^{n-1} N + \cdots + N^n = a^n + N$$

Since a is not nilpotent, $a^n \neq 0$ for every n, showing that $a + N$ is not nilpotent for $a \in R \backslash N$. ∎

III.4

Problem III.4.1. Let R be a ring, and let $\{I_\alpha\}_{\alpha \in A}$ be a family of ideals of R. We let

$$\sum_{\alpha \in A} I_\alpha := \left\{ \sum_{\alpha \in A} r_\alpha \text{ such that } r_\alpha \in I_\alpha \text{ and } r_\alpha = 0 \text{ for all but finitely many } \alpha \right\}.$$

Prove that $\{I_\alpha\}_{\alpha \in A}$ is an ideal of R and that it is the smallest ideal containing all of the ideals I_α.

Proof. We only consider the case when $A = \{1, 2\}$: Any other A follows the same exact argument.

Let $I = I_1 + I_2$. I is a subgroup of $(R, +)$: the two elements in I can be represented as $r_1 + r_2$ and $r_1' + r_2'$, and clearly $(r_1 - r_1') + (r_2 - r_2')$ is in I. The absorption property is also clear, since $r(r_1 + r_2) = (rr_1 + rr_2) \in I$.

Now it suffice to show that I is minimal. For every ideal that contains I_1 and I_2, they must also contain $r_1 + r_2$ for $r_1 \in I_1$ and $r_2 \in I_2$, since ideal is a subgroup of $(R, +)$. Therefore every such ideal must also contain I, proving the minimality of I. ∎

Problem III.4.2. Prove that the homomorphic image of a Noetherian ring is Noetherian.

Proof. Let R be Noetherian, S be any ring, $\varphi : R \to S$ be a surjective ring homomorphism. Let J be an ideal of S. By III.3.2, the preimage is an ideal, which we call $I = \langle a_1, ...a_n \rangle$. We claim that $J = \langle \varphi(a_1), ...\varphi(a_n) \rangle$, so every finitely generated ideal will map to a finitely generated ideal, proving that S is Noetherian.

Indeed, since $a_i \in \varphi^{-1}(J)$, $\varphi(a_i) \in J$ for $i = 1, ..., n$, so $\langle \varphi(a_1), ...\varphi(a_n) \rangle \subseteq J$. On the other hand, for an element $j \in J$, there exists $i \in R$ such that $\varphi(i) = j$ by surjectivity, therefore $i \in I$, so i is generated by elements $a_1, ..., a_n$, i.e. $i = r_1 a_1 + ... + r_n a_n$. Then since φ is a homomorphism,

$$\varphi(i) = j = \varphi(r_1 a_1 + ... + r_n a_n) = s_1 \varphi(a_1) + ... + s_n \varphi(a_n)$$

so $J \subseteq \langle \varphi(a_1), ...\varphi(a_n) \rangle$, and the claim is proved. ∎

Problem III.4.3. Prove that the ideal $(2, x)$ of $\mathbb{Z}[x]$ is not principal.

Proof. Assume that $(f) = (2, x)$. Then there is some $q \in \mathbb{Z}[x]$ such that $fq = 2$. Then f, q are constant and f must be 2 since 1 is not in it. But we also have $fg = x$ for some $g \in Z[x]$, and there are no possible choice of g such that $2g = x$. Hence $(2, x)$ is not principal. ∎

Problem III.4.4. Prove that if k is a field, then $k[x]$ is a PID.

Proof. Let I be any ideal of $k[x]$. If $I = (0)$, then there is nothing to prove. Otherwise, there is some polynomial $f \in I$ that has minimal degree in I and is monic (since you can do scalar division). We claim that $I = (f)$. Indeed, for $g \in I$, we can use division algorithm to write

$$g(x) = f(x)q(x) + r(x)$$

where $\deg r(x) < \deg f(x)$. Since $k[x]$ is a subgroup, $r = g - fq \in I$, and by the minimality of f, $r(x) = 0$, so every element of I can be written as $g(x)f(x)$ for some $g \in k[x]$, showing that $k[x]$ is a PID. ∎

Problem III.4.5. Let I, J be ideals in a commutative ring R, such that $I + J = (1)$. Prove that $IJ = I \cap J$.

Proof. If $x \in IJ$, then it can be represented as ij for some $i \in I, j \in J$, and by the property of ideal, $ji \in I, ij \in J$, so $ij \in I \cap J$. Conversely, we have

$$I \cap J = (I \cap J)(1) = (I \cap J)(I + J) = (I \cap J)I + (I \cap J)J \subseteq IJ + IJ = IJ$$

showing the identity. ∎

Problem III.4.7. Let $R = k$ be a field. Prove that every nonzero (principle) ideal in $k[x]$ is generated by a unique *monic* polynomial.

Proof. From III.4.4 we already know that every ideal is generated by a single polynomial f. Since k is a field, we can do division, so there is a monic polynomial $f(x)/a$ where a is the coefficient of the largest degree in f. Then it's trivial that $(f) = (f/a)$. ∎

Problem III.4.11. Let R be a commutative ring, $a \in R$, and $f_1(x), \ldots, f_r(x) \in R[x]$.

- Prove the equality of ideals

$$(f_1(x), \ldots, f_r(x), x - a) = (f_1(a), \ldots, f_r(a), x - a).$$

- Prove the useful substitution trick

$$\frac{R[x]}{(f_1(x), \ldots, f_r(x), x - a)} \cong \frac{R}{(f_1(a), \ldots, f_r(a))}$$

Proof. We consider only the case $k = 1$; the other cases are just extending the same argument. We are required to prove that

$$(f(x), x - a) = (f(a), x - a)$$

For $f(x)$, we can apply division algorithm to get

$$f(x) = q(x)(x - a) + r$$

where $q(x) \in R[x], r \in R$. By plug in $x = a$, we obtain $r = f(a)$. Therefore $f(x)$ is generated by $f(a)$ and $(x - a)$, showing $f(x) \in (f(a), x - a)$. On the other hand, note the division algorithm also implies

$$f(a) = f(x) - q(x)(x - a) \in (f(x), x - a)$$

therefore $f(a) \in (f(x), x-a)$, so $(f(x), x-a) = (f(a), x-a)$. Now since $R[x]/(x-a) \cong R$, by III.3.3

$$\frac{R}{\varphi(J)} \cong \frac{R[x]}{\ker \varphi + J}$$

for an ideal $J \in R[x]$, $\varphi : R[x] \to R$ a surjective homomorphism. It is clear that how should we choose these: by taking

$$J = (f_1(x), \ldots, f_r(x)), \quad \varphi(f(x)) = f(a)$$

we have

$$\frac{R}{(f_1(a), \ldots, f_r(a))} \cong \frac{R[x]}{(f_1(x), \ldots, f_r(x), x-a)}$$

as desired (note that φ is surjective). ■

Problem III.4.13. Let R be an integral domain. For all $k = 1, \ldots, n$, prove that $(x_1, \ldots, x_k)$ is prime in $R[x_1, \ldots, x_n]$.

Proof. We proceed by induction. For the case $k = 1$, we have

$$\frac{R[x]}{(x)} \cong R \quad \text{(p.p.151)}$$

and since R is a domain, it follows by definition that (x) is a prime ideal. Suppose that for $k < n$, the argument holds. Then for $k = n$, choose

$$J = (x_1, \ldots, x_{n-1}), \quad \varphi : R[x_1, \ldots, x_n] \hookrightarrow R[x_1, \ldots, x_{n-1}]$$

where φ is the inclusion map and $\ker \varphi = (x_n)$. Then by III.3.3

$$\frac{R[x_1, \ldots, x_n]/(x_n)}{(x_1, \ldots, x_{n-1})} \cong \frac{R[x_1, \ldots, x_n]}{(x_1, \ldots, x_{n-1}) + (x_n)}$$

which simplifies to

$$\frac{R[x_1, \ldots, x_{n-1}]}{(x_1, \ldots, x_{n-1})} \cong \frac{R[x_1, \ldots, x_n]}{(x_1, \ldots, x_n)}$$

By induction hypothesis, the quotient on the left is a domain since $(x_1, \ldots, x_{n-1})$ is a prime ideal, therefore by definition, $(x_1, \ldots, x_n)$ is a prime ideal. ■

Problem III.4.16. Let R be a commutative ring, and let P be a prime ideal of R. Suppose 0 is the only zero-divisor of R contained in P. Prove that R is an integral domain.

Proof. Let $a, b \in R$ such that $ab = 0$. Then since $0 \in P$, $ab \in P$, so either $a \in P$ or $b \in P$. Without loss of generality, let $a \in P$. If $a = 0$, then we are done; otherwise, $a \neq 0$, and since $ab = 0$, we must have $b = 0$ as a is not a zero divisor (0 is the only zero-divisor in P). In both cases, we show that $ab = 0$ implies $a = 0$ or $b = 0$, showing that R is a domain. ■

Problem III.4.18. Let R be a commutative ring, and let N be its nilradical (III.3.12). Prove that N is contained in every prime ideal of R.

Proof. Let $x^n = 0$ for some positive integer n, and P a prime ideal. Then since $0 \in P$, we have

$$P \ni 0 = x^n = x \cdot x^{n-1}$$

By the property of prime ideal, either $x \in P$ or x^{n-1} in P. If the former case is true, then we are done; else, we can reduce to the case where either $x \in P$ or $x^{n-2} \in P$. By continuing this process, we will arrive at either $x \in P$ or $x \in P$, showing that in any cases, $x \in P$. Therefore all nilpotent elements are in P, proving the statement. ■

Problem III.4.21. Let k be an algebraic closed field, and let $I \subseteq k[x]$ be an ideal. Prove that I is maximal if and only if $I = (x - c)$ for some $c \in k$.

Proof.
($\Leftarrow$) We have

$$\frac{k[x]}{(x-c)} \cong k \quad \text{(p.p.151)}$$

and since k is a field, it follows by definition that $(x - c)$ is maximal.
($\Rightarrow$) Let J be a maximal ideal. By III.4.4, $k[x]$ is a PID, hence every ideal is being generated by a single *monic* polynomial $f(x) \in k[x]$ (III.4.7). Since k is algebraic closed, we can write $f(x) = q(x)(x - c)$ for some $q(x) \in k[x]$, $c \in k$. Then

$$J = (f(x)) = (q(x)(x-c)) \subseteq (x-c)$$

and by Proposition III.4.11, either $J = (x - c)$ or $J = k[x]$. The latter case could not happen since the maximal can not be $k[x]$ itself, therefore $J = (x - c)$, as desired. ∎

Unless otherwise specified, in the following M denotes a (left-)module over R.

III.5

Problem III.5.2. Prove claim 5.1.

Proof. Let $\sigma : R \to \mathrm{End}_{\mathsf{Ab}}(M)$ be a ring homomorphism and $\rho : R \times M \to M$ a function. We verify the following properties:

- $\rho(r, m + n) = \rho(r, m) + \rho(r, n)$.
  Note that $\sigma(r)$ is a endomorphism on M. Then

$$\rho(r, m+n) = \sigma(r)(m+n) = \sigma(r)(m) + \sigma(r)(n) = \rho(r, m) + \rho(r, n)$$

- $\rho(r + s, m) = \rho(r, m) + \rho(s, m)$.

$$\rho(r+s, m) = \sigma(r+s)(m) = \sigma(r)(m) + \sigma(s)(m) = \rho(r, m) + \rho(s, m)$$

- $\rho(rs, m) = \rho(r, \rho(s, m))$.

$$\rho(rs, m) = \sigma(rs)(m) = \sigma(r)\sigma(s)(m) = \sigma(r)\rho(s, m) = \rho(r, \rho(s, m))$$

- $\rho(1, m) = m$.

$$\rho(1, m) = \sigma(1)(m) = 1(m) = m$$

∎

Problem III.5.3. Prove that $0 \cdot m = 0$ and that $(-1) \cdot m = -m$ for all $m \in M$.

Proof. Since $0m = (0 + 0)m = 0m + 0m, 0m = 0$. Since $0 = 0m = (-1 + 1)m = (-1)m + m, (-1)m = -m$. ∎

Problem III.5.11. Let R be commutative, and let M be an R-module. Prove that there is a natural bijection between the set of $R[x]$-module structures on M (extending the given R-module structure) and $\mathrm{End}_{R-\mathsf{Mod}}(M)$.

Proof. If $f \in \mathrm{End}_{R-\mathsf{Mod}}(M)$, then we have to show that there are some suitable maps

$$\begin{aligned} R[x] \times M &\to M \\ (f(x),\, m) &\to ? \end{aligned}$$

that makes M into a $R[x]$-module. We consider $(g(x), m) \to g(f)(m)$, where if $g(x) = \sum_i a_i x^i$, then

$$\sigma(f, m) = \sum_i a_i f^i(m) \text{ where } f^i = \underbrace{f \circ \cdots \circ f}_{i \text{ times}}$$

We can easily check by definition that M is a $R[x]$-module. Conversely, if M is a $R[x]$-module, then define $f(m) = xm$. Then f is indeed an endomorphism (note that the commutativity of R ensures that $rxm = xrm$ for $r \in R$, so f is an endomorphism), proving the statement. ■

Problem III.5.12. Let M, N be R-modules, and let $\varphi : M \to N$ be a homomorphism of R-modules which has a inverse (therefore a bijection). Prove that φ^{-1} is also a homomorphism of R-modules. Conclude that a bijective R-module homomorphism is a R-module isomorphism.

Proof. Since

$$\varphi(\varphi^{-1}(m) + \varphi^{-1}(n)) = m + n = \varphi(\varphi^{-1}(m + n))$$

we have $\varphi^{-1}(m) + \varphi^{-1}(n) = \varphi^{-1}(m + n)$. And

$$\varphi(r\varphi^{-1}(m)) = r\varphi(\varphi^{-1}(m)) = rm = \varphi(\varphi^{-1}(rm))$$

so $r\varphi^{-1}(m) = \varphi^{-1}(rm)$ indeed. ■

Problem III.5.14. Prove Proposition 5.18, that is:

Let N, P be submodules of an R-module M. Then

- *$N + P$ is a submodule of M;*
- *$N \cap P$ is a submodule of P, and*

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}.$$

Proof. Every element of $N + P$ can be written as $n + p$ where $n \in N, p \in P$. Then it is clear that $r(n + p) = rn + rp \in N + P$ for $r \in M$. For the intersection $N \cap P$, it is also clear that for $p \in P, n \in N \cap P$, $pr \in N$ since $r \in N$, and $pr \in P$ since $p \in P$.

The proof for the second isomorphism theorem follows exactly the same as in groups (Proposition II.8.11). Consider the homomorphism

$$\varphi : P \to \frac{N + P}{N}, \quad \varphi(p) = pN$$

it is surjective since for every $(n + p)N$, there is a corresponding p. Then

$$\ker \varphi = \{p \in P : p \in N\} = P \cap N$$

finally it follows by first isomorphism theorem that

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}.$$

■

III.6

Problem III.6.1. Prove Claim 6.3, that is, $F^R(A) \cong R^{\oplus A}$.

Proof. Observe that every element in $R^{\oplus A}$ can be uniquely written as

$$\sum_{a \in A} r_a \chi(a)$$

where $\chi(a) = \chi_a(x)$, the indicator function of a, and $r_a \in R$ for $a \in A$. Then it suffices to check the universal property of free modules: given a function $f : A \to M$ where M is a module, we show that the following diagram

$$\begin{tikzcd} R^{\oplus A} \arrow[r, "\exists!\varphi"] & M \\ A \arrow[u, "\chi"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. Indeed, we define

$$\varphi\left(\sum_{a\in A} r_a\chi(a)\right) = \sum_{a\in A} r_a f(a)$$

then the diagram clearly commutes (and is unique). Finally, φ is a $R-\mathsf{Mod}$ homomorphism since

$$\varphi\left(\sum_{a\in A} r_a\chi(a)\right) + \varphi\left(\sum_{a\in A} r'_a\chi(a)\right) = \sum_{a\in A} r_a f(a) + \sum_{a\in A} r'_a f(a) \overset{\checkmark}{=} \sum_{a\in A}(r_a + r'_a) f(a)$$
$$= \varphi\left(\sum_{a\in A}(r_a + r'_a)\chi(a)\right) = \varphi\left(\sum_{a\in A} r_a\chi(a) + \sum_{a\in A} r'_a\chi(a)\right)$$

Note that R-module's definition gurantees the commutative of ✓ (scalar multiplication is direct). ■

Problem III.6.3. Let R be a ring, M an R-module, and $p : M \to M$ an R-module homomorphism such that $p^2 = p$. Prove that $M \cong \ker p \oplus \operatorname{im} p$.

Proof. We are required to prove that the diagram

$$\begin{tikzcd} \ker p \arrow[dr, "i_k"] \arrow[drr, bend left, "f_k"] & & \\ & M \arrow[r, "\exists!\varphi"] & N \\ \operatorname{im} p \arrow[ur, "i_m"'] \arrow[urr, bend right, "f_m"'] & & \end{tikzcd}$$

commutes. Notice that for $x \in \ker p, p(x) = 0$, and

$$\text{for } x \in \operatorname{im} p, x - p(x) = p(y) - p(p(y)) = p(y) - p(y) = 0$$

where $p(y) = x$. This suggest that we define φ as

$$\varphi(x) = f_k(x - p(x)) + f_m(p(x))$$

Indeed, if $x \in \ker p$, then $\varphi(x) = f_k(x)$; if $x \in \operatorname{im} p$, then $\varphi(x) = f_m(p(x)) = f_m(x)$ since for $x \in \operatorname{im} p$,

$$p(y) = x, p(p(y)) = p(y) \Rightarrow p(x) = x.$$

But what about $x \in \ker p \cap \operatorname{im} p$? In fact, the only element in the intersection is 0, as such x must have

$$x = p(y) = p(p(y)) = p(x) = 0$$

so φ is well-defined. Now it suffices to check that φ is a homomorphism, which is direct since p, f_k and f_m are both R-homomorphisms, so it preserves the action on M (check yourself if you're not convinced). Therefore by the universal property of coproduct, $\ker p \oplus \operatorname{im} p \cong M$. ∎

Problem III.6.4. Let R be a ring, and let $n > 1$. View $R^{\oplus(n-1)}$ as a submodule of $R^{\oplus n}$, via the injective homomorphism $R^{\oplus(n-1)} \hookrightarrow R^{\oplus n}$ defined by

$$(r_1, \ldots, r_{n-1}) \hookrightarrow (r_1, \ldots, r_{n-1}, 0).$$

Give a one-line proof that

$$\frac{R^{\oplus n}}{R^{\oplus(n-1)}} \cong R.$$

Proof. The surjective map

$$(r_1, \ldots, r_{n-1}, r_n) \twoheadrightarrow r_n.$$

has kernel precisely $R^{\oplus(n-1)}$, therefore by first isomorphism theorem

$$\frac{R^{\oplus n}}{R^{\oplus(n-1)}} \cong R.$$

∎

Problem III.6.5. For any ring R and any two sets A_1, A_2, prove that $(R^{\oplus A_1})^{\oplus A_2} \cong R^{\oplus(A_1 \times A_2)}$.

Proof. By III.6.1, it is equivalent to prove the following diagram commutes:

$$\begin{tikzcd}
(R^{\oplus A_1})^{\oplus A_2} \arrow[r, "\exists!\varphi"] & M \\
A_1 \times A_2 \arrow[u, "j"] \arrow[ur, "f"'] &
\end{tikzcd}$$

To do this, note that an element in $(R^{\oplus A_1})^{\oplus A_2}$ is a function $g : A_2 \to R^{\oplus A_1}$, in which we send an element $a_2 \in A_2$ to

$$j_{a_1,a_2}(x) := \begin{cases} 1 & \text{if } x = a_1 \\ 0 & \text{if } x \neq a_1 \end{cases} \quad \text{(p.p.168)}$$

this suggests us to define

$$j(a_1, a_2) \mapsto (j_{a_1,a_2}(b_2))(b_1) = \chi_{a_1}(b_1)\chi_{a_2}(b_2)$$

where χ is the indicator function. Then it follows the same pattern as in III.6.1: for $f : A_1 \times A_2 \to M$ given and any element $\sum_{a_1 \in A_1, a_2 \in A_2} r_{a_1,a_2}(j_{a_1,a_2}(b_2))(b_1) \in (R^{\oplus A_1})^{\oplus A_2}$, define

$$\varphi\left(\sum_{a_1 \in A_1, a_2 \in A_2} r_{a_1,a_2}(j_{a_1,a_2}(b_2))(b_1)\right) = \sum_{a_1 \in A_1, a_2 \in A_2} r_{a_1,a_2} f(a_1, a_2)$$

The commutative of diagram is direct. Finally, the check for φ is a $R - \mathsf{Mod}$ homomorphism is the same as in III.6.1. ∎

Problem III.6.7. Let A be any set, and for any module M over a ring R, define

$$M^A := \prod_{a \in A} M, \quad M^{\oplus A} := \bigoplus_{a \in A} M.$$

Prove that $\mathbb{Z}^{\mathbb{N}} \not\cong \mathbb{Z}^{\oplus \mathbb{N}}$.

Proof. Note that $\mathbb{Z}^{\mathbb{N}}$ can be regarded as the collection of functions

$$f : \mathbb{Z} \to \mathbb{N}$$

which is the collection of all infinite sequences in $\mathbb{Z}$. This set has uncountably many elements (as one can argue using Cantor's diagonal argument). On the other hand, $\mathbb{Z}^{\oplus \mathbb{N}}$ is also the collection of these function, but with the additional criterion that

$$f(n) = 0 \text{ for all but finitely many } n \in Z$$

which says that this set collects all finite sequence in $\mathbb{Z}$, and as we know (i.e. can construct a bijection to $\mathbb{Z}$), this set is countable. As the cardinality does not match, $\mathbb{Z}^{\mathbb{N}} \not\cong \mathbb{Z}^{\oplus \mathbb{N}}$, as required. ■

Problem III.6.9. Let R be a ring, F a nonzero free R-module, and let $\varphi : M \to N$ be a homomorphism of R-modules. Prove that φ is onto if and only if for all R-module homomorphisms $\alpha : F \to N$ there exists an R-module homomorphism $\beta : F \to M$ such that $\alpha = \varphi \circ \beta$.

Proof. As M is free, it is generated by a set $X = \{x_i\}$ (not necessarily finite).
($\Rightarrow$) Let $\{n_i\} \in N$ be such that $\varphi(x_i) = n_i$. If φ is onto, then each n_i corresponds to a $m_i \in M$ such that $\varphi(m_i) = n_i$. We then just define $\beta(x_i) = m_i$, and the commutativity is clear (note that β might not be unique, but that's fine).
($\Leftarrow$) If φ is not onto, i.e. there exists $n \in N$ such that $n \notin \operatorname{im} \varphi$, then this also means that $n \notin \operatorname{im}(\varphi \circ \beta)$ for any β. Now we choose a suitable α so $\alpha = \varphi \circ \beta$ does not hold. Indeed, we can define

$$\alpha(x_i) = n$$

for all i. Then the commutativity does not hold for any choice of β, a contradiction. Therefore φ must be surjective. ■

Problem III.6.10. Let M, N, and Z be R-modules, and let $\mu : M \to Z, \nu : N \to Z$ be homomorphism of R-modules. Prove that $R-\mathsf{Mod}$ has 'fibered products'(I.5.12).

Proof. As in the case Set(I.5.12), we define fibered coproduct by the set of elements that agrees on Z after being pushed by μ and ν:

$$M \times_Z N := \{(m, n) \in M \oplus N : m \in M, n \in N, \mu(m) = \nu(n)\}$$

By the universal property of fibered product on Set, the diagram with the choice $\varphi(z) := (f_M(z), f_N(z))$ makes the following diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "f_N"] \arrow[dr, dashed, "\exists!\varphi"] \arrow[ddr, bend right, "f_M"'] & & \\
& M \times_Z N \arrow[r, "\pi_N"'] \arrow[d, "\pi_M"] & N \arrow[d, "\nu"] \\
& M \arrow[r, "\mu"'] & Z
\end{tikzcd}$$

commutes, regarding in Set. Now we check that $M \times_Z N$ indeed is a submodule of $M \oplus N$: for $(m, n) \in M \times_Z N$, $r(m, n) = (rm, rn)$, and since $\mu(m) = \nu(n)$, $r\mu(m) = \mu(rm) = \nu(rn) = r\nu(n)$, so $(rm, rn) \in M \times_Z N$ as required.

Now it remains to check φ is a R-module homomorphism, which is direct. ■

Problem III.6.11. Define a notion of *fibered coproduct* of two R-modules M, N, along an R-module A, in the style of III.6.10 (and cf. I.5.12).

Prove that fibered coproducts exist in R-Mod. The fibered coproduct $M \oplus_A N$ is called the *push-out* of M along ν (or of N along μ).

Proof. The universal property is as the same stated in I.5.12, but by replacing every set with R-modules and every morphism with R-Mod homomorphisms. We now show that the fibered coproduct is almost the same in Set: define an equivalence relation

$$S = \{(\mu(x), \nu(x)) \in M \oplus N : x \in A\}$$

on $M \oplus N$, and let $M \oplus_A N := (M \oplus N)/S$. We show that R is a submodule, so the quotient make sense. For $(m, n) \in S$,

$$r(m, n) = r(\mu(x), \nu(x)) = (r\mu(x), r\nu(x)) = (\mu(rx), \nu(rx)) \in S$$

which shows that S is indeed an R-module. Now define

$$\varphi((m, n) + R) = f_M(m) + f_N(n)$$

It is a simple check that φ is a R-module homomorphism, and φ is well-defined, using the same argument as in Set(I.5.12). This makes the following diagram

$$\begin{tikzcd}
A \arrow[r, "\nu"] \arrow[d, "\mu"'] & N \arrow[d, "i_N"] \arrow[ddr, bend left, "f_N"] & \\
M \arrow[r, "i_M"'] \arrow[drr, bend right, "f_M"] & M \oplus_A N \arrow[dr, dashed, "\exists!\varphi"] & \\
& & Z
\end{tikzcd}$$

commutes, as we check:

- $i_N\nu = i_M\mu$:
$$i_N\nu(x) = (0, \nu(x)) + S = (\mu(x), 0) + S = i_M\mu(x)$$
- $f_M = \varphi i_M$ (resp. $f_N = \varphi i_N$):
$$\varphi i_M(m) = \varphi((m, 0) + S) = f_M(m).$$

■

Problem III.6.14. Prove that the ideal $(x_1, x_2, \dots)$ of the ring $R = \mathbb{Z}[x_1, x_2, \dots]$ is not finitely generated (as an ideal, i.e. as an R-module).

Proof. If it were, then there exists a surjective R-Mod homomorphism

$$\varphi : R^{\oplus n} \twoheadrightarrow (x_1, x_2, \dots).$$

Then we collect the polynomials

$$\{\varphi(0, \dots, \underset{i\text{-th place}}{1}, \dots, 0)\}_{i=1}^{n}$$

Since each polynomials can only contain finitely many indeterminates, and there are only finite polynomials, there must be some indeterminates x_j that is not in the domain of φ (as there are countably many indeterminates in the ideal), contradicting to the surjectivity of φ. Therefore $(x_1, x_2, \dots)$ is not finitely generated. ■

Problem III.6.18. Let M be an R-module, and let N be a submodule of M. Prove that if N and M/N are both finitely generated, then M is finitely generated.

Proof. Let $\{a_i + N\}_{i=1}^m$ be generators of M/N, and $\{b_i\}_{i=1}^n$ be generators of N. Then for every $m \in M$, we consider

$$m + N = \sum_{i=1}^m r_i(a_i + N) = \sum_{i=1}^m r_i a_i + N$$

this says that $m - \sum_{i=1}^m r_i a_i \in N$, and therefore we can again write $m - \sum_{i=1}^m r_i a_i = \sum_{j=1}^n s_i b_i$. To this point we showed that every element in M can be generated by $\{a_i, b_j\}_{1\leq i\leq m, 1\leq j\leq n}$, showing that M is finitely generated. ■

III.7

Problem III.7.1. Assume that the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

is exact. Prove that $M \cong 0$.

Proof.

$$0 = \operatorname{im}(0 \longrightarrow M) = \ker(M \longrightarrow 0) = M.$$

■

Problem III.7.2. Assume that the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow M' \longrightarrow 0 \longrightarrow \cdots$$

is exact. Prove that $M \cong M'$.

Proof. The map $(M \longrightarrow M')$ is both a monomorphism and an epimorphism by Example III.7.1 and Example III.7.2. By definition, the map is an isomorphism. ■

Problem III.7.3. Assume that the complex

$$\cdots \longrightarrow 0 \longrightarrow L \longrightarrow M \xrightarrow{\varphi} M' \longrightarrow N \longrightarrow 0 \longrightarrow \cdots$$

is exact. Show that, up to natural identifications, $L = \ker\varphi$ and $N = \operatorname{coker}\varphi$.

Proof. The map $(L \longrightarrow M)$ is a monomorphism, so by canonical decomposition

$$L = \frac{L}{\ker(L \longrightarrow M)} \cong \operatorname{im}(L \longrightarrow M) = \ker(M \longrightarrow M') = \ker\varphi.$$

The map $(M' \longrightarrow N)$ is an epimorphism, so it follows by first isomorphism theorem that

$$\operatorname{coker}\varphi = \frac{M'}{\operatorname{im}\varphi} = \frac{M'}{\operatorname{im}(M \longrightarrow M')} = \frac{M'}{\ker(M' \longrightarrow N)} \cong N.$$

■

Problem III.7.6. Prove the 'split epimorphism part pf Proposition 7.5, that is,

φ has a right-inverse if and only if the sequence

$$0 \longrightarrow \ker\varphi \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0 \quad \textit{splits.}$$

Proof.
($\Leftarrow$) If the sequence splits, then by identifying φ with the projection map from $\ker\varphi \oplus N$ to N, we can let $\psi : N \to \ker\varphi \oplus N$ to be the inclusion, and it gives a right-inverse.
($\Rightarrow$) Assume that φ has a right inverse, which says that

$$\begin{tikzcd} N \arrow[r, "\psi"] \arrow[dr, "id"'] & M \arrow[d, "\varphi"] \\ & N \end{tikzcd}$$

To prove the statement, we claim that $M \cong \ker\varphi \oplus N$. This isomorphism is given by

$$(k, n) \mapsto k + \psi(n)$$

it has inverse

$$m \mapsto (m - \psi\varphi(m), \varphi(m))$$

Indeed, we check

$$m \mapsto (m - \psi\varphi(m), \varphi(m)) \mapsto m - \psi\varphi(m) + \psi\varphi(m) = m$$

and $m - \psi\varphi(m)$ is in $\ker\varphi$ since

$$\varphi(m - \psi\varphi(m)) = \varphi(m) - \varphi\psi\varphi(m) = 0$$

and the claim is proved. ■

Problem III.7.8. Prove that every exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow 0$$

of R-modules, with F *free*, splits.

Proof. By exactness, $\varphi : N \longrightarrow F$ is surjective. Therefore by III.6.9, for every $\alpha : F \to F$, there is $\beta : F \to N$ such that $\alpha = \varphi \circ \beta$. In particular, let $\alpha = id_F$, then $\varphi \circ \beta = id_F$.

$$\begin{tikzcd} & & & F \arrow[dl, "\beta"'] \arrow[d, "1"] & \\ 0 \arrow[r] & M \arrow[r, "i"] & N \arrow[r, "\varphi"] & F \arrow[r] & 0 \end{tikzcd}$$

With this, we now show that $M \oplus F \cong N$. Define

$$h : M \oplus F \to N, \quad h(m, f) = i(m) + \beta(f)$$

h is clearly an R-module homomorphism, so it remains to show that it is an isomorphism. h is injective: if $h(m, f) = 0$, then

$$i(m) + \beta(f) = 0 \ \Rightarrow \varphi i(m) + \varphi\beta(f) = 0 \ \Rightarrow 0 \text{ (definition of chain complex)} + f = 0$$

showing that $f = 0$. Then $i(m) = 0$, so we must have $m = 0$. h is surjective: we want to find m, f such that $i(m) + \beta(f) = n$ for $n \in N$. By applying φ we have

$$\varphi i(m) + \varphi\beta(f) = 0 + f = \varphi(n)$$

so we have the candidate of f. Now it remains to decide m in which $i(m) = n - \beta(\varphi(n))$: notice that by exactness, $\operatorname{im} i = \ker\varphi$, so we check that $\varphi(n - \beta(\varphi(n))) = 0$ to gurantee the existence of m:

$$\varphi(n - \beta(\varphi(n)) = \varphi(n) - \varphi \circ \beta \circ \varphi(n) = \varphi(n) - \varphi(n) = 0$$

Hence h is an isomorphism, and by definition, the sequence splits. ■

Chapter IV

Groups, second encounter

Unless otherwise specified, in the following G denotes a group, e denotes the identity of G. The conjugacy class of an element g is denoted by $[g]$. Some description and hints are omitted for simplicity.

Unless otherwise specified, all groups in this chapter are *finite*.

IV.1

Problem IV.1.1. Let p be a prime integer, let G be a p-group, and let S be a set such that $|S| \neq 0 \bmod p$. If G acts on S, prove that the action must have fixed points.

Proof. This is direct by Corollary IV.1.3: since $|S| \neq 0 \bmod p$, the set of fixed points Z satisfies $|S| \equiv |Z| \neq 0$. ∎

Problem IV.1.4. Let G be a group, and let N be a subgroup of $Z(G)$. Prove that N is normal in G.

Proof. For $g \in G$, $n \in N$,

$$gng^{-1} = gg^{-1}n = n \in N.$$

One should note that *normal is not transitive*: if $G \trianglelefteq H$ and $H \trianglelefteq I$, it is in general not true that $G \trianglelefteq I$. ∎

Problem IV.1.5. Let G be a group. Prove that $G/Z(G)$ is isomorphic to the group $\mathrm{Inn}(G)$ (II.6.7). Then prove Lemma 1.5 again.

Proof. Let $\varphi : G \to \mathrm{Inn}(G), \varphi(g) = \gamma_g(a) := gag^{-1}$ be a homomorphism (II.4.8). By construction it is clearly surjective, and the kernel is

$$\ker \varphi = \{g : gag^{-1} = a\} \Rightarrow \{g : ga = ag\} = Z(G)$$

therefore by first isomorphism theorem, $G/Z(G) \cong \mathrm{Inn}(G)$. If $G/Z(G)$ is cyclic, then by II.6.7 G is commutative. ∎

Problem IV.1.6. Let p, q be prime integers, and let G be a group of order pq. Prove that either G is commutative or the center of G is trivial. Conclude that every group of order p^2, for a prime p, is commutative.

Proof. The subgroups can only be of order $1, p, q$ or pq by Lagrange, and $|Z(G)|$ can be only one of these four. If $|Z(G)| = 1$, then there is nothing to prove; if $|Z(G)| = p(\text{or } q)$, then the quotient is cyclic, so it follows by Lemma IV.1.5 that G is commutative; if $|Z(G)| = pq$, then G is clearly commutative.

By Corollary IV.1.9, the center of a nontrivial p-group is nontrivial, so the order of the center for $|G| = p^2$ can not be 1. Then by above, all the remaining cases will conclude that G is commutative. ∎

Problem IV.1.8. Let p be a prime number, and let G be a p-group: $|G| = p^r$. Prove that G contains a normal subgroup of order p^k for every nonnegative $k \leq r$.

Proof. We proceed by induction. If $r = 1$ then there is nothing to prove, so we assume that for $n < r$, the p-group with order p^n has a normal subgroup of order p^k for $k \leq n$.

Now consider the center of G: it is abelian and is a nontrivial p-group by Corollary IV.1.9, so by II.8.20, there exists a (normal) subgroup N that is of order p in $Z(G)$. By IV.1.4, N is normal in G, so we can consider the quotient G/N. The quotient is a p-group and has order p^{r-1}, so by induction hypothesis, G/N has normal subgroups of order p^k for $k \leq r-1$, which we name them H_k for each k. By noting that H_k contains N, we can identify each H_k by H_k/N via Proposition II.8.9. Finally, since $|H_k/N| = p^k$, $|H_k| = p^{k+1}$, so we've found normal subgroup of order p^k for $k \leq r$, proving the statement. ∎

Problem IV.1.9. Let p be a prime number, G a p-group, and H a nontrivial normal subgroup of G. Prove that $H \cap Z(G) \neq \{e\}$.

Proof. Let G act on itself by conjugation. Since H is normal, it is the union of some conjugacy class and some element of $Z(G)$, with each conjugacy class of order p^n for some n by Corollary II.9.10. If $H \cap Z(G) = \{e\}$, then this means that H only take e from $Z(G)$, and since the order of all conjugacy classes in H are divisible by p, we would arrive at $|H| \equiv 1 \mod p$, a contradiction since $|H|$ must be a multiple of p. ∎

Problem IV.1.10. Prove that if G is a group of odd order and $g \in G$ is conjugate to g^{-1}, then $g = e$.

Proof. Suppose $g \neq e$. Since $[g]$ contains g^{-1}, there are two cases:

- If $g = g^{-1}$, then $g^2 = 1$, so $|g| = 2$. But this is impossible since $|g|$ does not divide $|G|$, a contradiction.
- If $g \neq g^{-1}$, then since $[g]$ must be odd order, there is some $y \in [g]$ such that $g = xyx^{-1}$. But this implies $g^{-1} = xy^{-1}x^{-1}$, so $y^{-1} \in [g]$, and $y \neq y^{-1}$ by above. So this says that $[g]$ must contain even number of elements(so must have even order), which again is impossible.

By above, we must have $g = e$, proving the assertion. ∎

Problem IV.1.14. Let G be a group, and assume $[G : Z(G)] = n$ is finite. Let $A \subseteq G$ be any subset. Prove that the number of conjugates of A is at most n.

Proof. We claim that there is a surjective set function from $G/Z(G)$ to $\{gAg^{-1}\}_{g\in G}$. Define

$$\varphi : G/Z(G) \to \{gAg^{-1}\}_{g\in G}, \quad \varphi(gZ) = gAg^{-1}$$

We check that it is well defined: If $gZ = hZ$, then $gh^{-1} \in Z$. Now for any element $\alpha = gAg^{-1}$ we have $\alpha = gag^{-1}$ for some $a \in A$, so we have $g^{-1}\alpha g = a$, and $hg^{-1}\alpha gh^{-1} = hah^{-1}$. Since $gh^{-1} \in Z$, $hg^{-1}\alpha gh^{-1} = hg^{-1}gh^{-1}\alpha = \alpha$, so $\alpha \in hAh^{-1}$, hence $gAg^{-1} = hAh^{-1}$, which showed the well-definedness. Clearly the map is surjective by construction, and by above, there can be only at most $[G : Z(G)] = n$ distinct conjugates of A, which proved the assertion. ∎

Problem IV.1.17. Let H be a proper subgroup of a finite group G. Prove that G is *not* the union of the conjugates of H.

Proof. By Lemma IV.1.13, the numbers of conjugates of H is $[G : N_G(H)]$. Since $H \subseteq N_G(H)$, $[G : N_G(H)]|H| \leq [G : H]|H| = |G|$. Even if the equality might hold, by noting that every conjugate is a subgroup and e is a common element for all subgroup, there are in fact at most $([G : N_G(H)]|H| - |H| + 1)$ distinct elements in the union of all conjugates of H. Since this number is strictly less than G, G will never be the union of conjugates of H. ■

Problem IV.1.18. Let S be a set endowed with a transitive action of finite group G, and assume $|S| \geq 2$. Prove that there exists a $g \in G$ without fixed points in S, that is, such that $gs \neq s$ for all $s \in S$.

Proof. In the sense of Proposition II.9.9, we can assume that $S = G/H$ (*left cosets, not quotient!*) where $H = \mathrm{Stab}_G(s)$ for some $s \in S$, with H proper in G (as $|S| \geq 2$). Suppose the contrary, i.e. every g satisfies $gkH = kH$ for some k. This means $k^{-1}gk \in H$, or equivalently, $g \in kHk^{-1}$. So every element in G is in some conjugacy class of H, which is a contradiction to IV.1.17 that G cannot be exhausted by conjugates of H. Hence G must have some elements that has no fixed points on S, as desired. ■

Problem IV.1.21. Let H, K be subgroups of a group G, with $H \subseteq N_G(K)$. Verify that the function $\gamma : H \to \mathrm{Aut}_{\mathsf{Grp}}(K)$ defined by conjugation is a homomorphism of group and that $\ker \gamma = H \cap Z_G(K)$, where $Z_G(K)$ is the centralizer of K.

Proof. Let γ maps h to a automorphism $\varphi_h(k) = hkh^{-1}$. It is a group homomorphism since

$$\gamma(g)\gamma(h) \mapsto \varphi_g\varphi_h(k) = ghkh^{-1}g^{-1} = \varphi(gh) \mapsto \gamma(gh).$$

The kernel of this map is

$$\ker \gamma = \{h \in H : hkh^{-1} = k \ \forall k \in K\} = \{h \in H : hk = kh \ \forall k \in K\} = H \cap Z_G(K).$$

■

Problem IV.1.22. Let G be a finite group, and let H be a cyclic subgroup og G of order p. Assume that p is the smallest prime dividing the order of G and that H is normal in G. Prove that H is contained in the center of G.

Proof. In the sense of IV.1.21, we have a homomorphism $\gamma : G \to \mathrm{Aut}_{\mathsf{Grp}}(H)$ since $H \subseteq N_G(G) = G$. By II.4.14, $\mathrm{Aut}_{\mathsf{Grp}}(H)$ has order $\phi(p) = p - 1$. But since G does *not* contain an element of order $p-1$ by the minimality of p, γ can only be the trivial homomorphism, so it has kernel equal to G. But by IV.1.21, $\ker \gamma = G \cap Z_G(H) = Z_G(H)$, so we must have $Z_G(H) = G$, which means that the element that commutes with h are the whole G, i.e. $H \subseteq Z(G)$, as desired. ■

IV.2

Problem IV.2.1. Prove Claim 2.2: *Let G be a finite group, let p be a prime divisor of $|G|$, and let N be the number of cyclic subgroups of G of order p. Then $N \equiv 1 \mod p$.*

Proof. We proceed with the same argument as in Theorem IV.2.1. Let S be a set that collects the p-tuple

$$(a_1, \dots, a_p)$$

such that $a_1 \cdots a_p = 1$. It is clear that $|S| = |G|^{p-1}$, and since $a_2 \cdots a_p a_1 = 1$, we can consider the action of $\mathbb{Z}/p\mathbb{Z}$ on S, by

$$\alpha_m : (a_1, \ldots, a_n) \mapsto (a_{m+1}, \ldots, a_p, a_1, \ldots, a_m)$$

By Corollary IV.1.3, $|Z| \equiv |S| \mod 0$, where Z is the fixed points under $\mathbb{Z}/p\mathbb{Z}$. The fixed points are of form $(a, \ldots, a)$ for $a \in G$, and since $(e, \ldots, e) \in Z$ and p divides $|Z|$, $|Z| > 1$. Now notice that for each $a \in G$ such that $(a, \ldots, a) \in Z$, a is a generator for some cyclic group of order p, so there are $N(p-1) + 1$(identity) elements in Z. But since $|Z| \equiv 0 \mod p$, we have

$$Np - N + 1 \equiv 0 \mod p \Longrightarrow N \equiv 1 \mod p$$

as desired. ■

Problem IV.2.2. Let G be a group. A subgroup H of G is *characteristic* if $\varphi(H) \subseteq H$ for every automorphism φ of G.

- Prove that every characteristic subgroups are normal.
- Let $H \subseteq K \subseteq G$, with H characteristic in K and K normal in G. Prove that H is normal in G.
- Let G, K be groups, and assume that G contains a single subgroup H isomorphic to K. Prove that H is normal in G.
- Let K be a normal subgroup of a finite group G, and assume that $|K|$ and $|G/K|$ are relatively prime. Prove that K is characteristic in G.

Proof.

- Consider $\gamma_g(h) := ghg^{-1}$ for all $g \in G$. Then $gHg^{-1} \subseteq H$ by characteristic property of H, so H is normal.
- By normalness of K, we have $gKg^{-1} = K$, so γ_g is an automorphism on K. Then since $\gamma_g(H) \subseteq H$, $gHg^{-1} \subseteq H$, so H is normal.
- Let φ be any automorphism of G. Then $\varphi(H) \cong H \cong K$ since φ is an isomorphism. But since H is the only subgroup that is isomorphic to K, $\varphi(H) = H$, so H is characteristic, hence normal.
- Let φ be any automorphism of G, and let $\pi : G \to G/K$ be the quotient homomorphism. Let $K' = \varphi(K)$. Then $\pi(K')$ is a subgroup of G/K, so $|\pi(K')|$ divides $|G/K|$. Also, by first isomorphism theorem, $K'/\ker \pi \cong \operatorname{im} \pi = \pi(K')$, so $|\pi(K')|$ divides $|K'| = |K|$. Since $|K|$ and $|G/K|$ are relatively prime, we can only have $|\pi(K')| = 1$, i.e. $\pi(K') = e_{G/H}$. Combining with $\ker \pi = K$, we have

  $$\varphi(K) = K' \subseteq \ker \pi = K$$

  as desired.

■

Problem IV.2.4. Prove that a nontrivial group G is simple if and only if its only homomorphic image are the trivial group and G itself (up to isomorphism).

Proof.
($\Rightarrow$) Let $\varphi : G \to G'$ be a surjective homomorphism. By first isomorphism theorem, $G/\ker \varphi \cong G'$. But since kernel is a normal subgroup, the only possibility of G' are $G/\{e\} = G$ or $G/G = \{e\}$.
($\Leftarrow$) If G is not simple, i.e. there are some nontrivial normal subgroup of G, which we call it H, then $\varphi : G \to G/H, g \mapsto gH$ is a surjective homomorphism, and G/H is neither $\{e\}$ nor G(up to isomorphism), a contradiction. ■

Problem IV.2.5. Let G be a *simple* group, and assume $\varphi : G \to G'$ is a nontrivial group homomorphism. Prove that φ is injective.

Proof. $\ker \varphi$ can only be $\{0\}$ or G by simpleness. If $\ker \varphi = \{0\}$ the we are done; if $\ker \varphi = G$ then $\varphi = 0$, which can't be by hypothesis. ■

Problem IV.2.6. Prove that there are no simple groups of order $4, 8, 9, 16, 25, 27, 32$ or 49. In fact, prove that no p-group of order $\geq p^2$ is simple.

Proof. The center of p-group, by Corollary IV.1.9, is nontrivial. Since center is a normal subgroup, no group of order p^n for $n \geq 2$ is simple. ■

Problem IV.2.8. Let G be a finite group, p a prime integer, and let N be the intersection of the p-Sylow subgroups of G. Prove that N is a *normal* p-subgroup of G and that every normal p-subgroup of G is contained in N.

Proof. Let P be a p-Sylow, then we can let $N = \bigcap_{g\in G} gPg^{-1}$. The conjugate of N is $pNp^{-1} = \bigcap_{g\in G} pgP(pg)^{-1}$, which is again N, so N is normal. Now if N' is a normal p-subgroup, then by Sylow II we can assume that $N \subseteq P$. Then for all $g \in G$, $N' = gN'g^{-1} \subseteq gPg^{-1}$, so $N' \subseteq \bigcap_{g\in G} gPg^{-1} = N$, and N' is in N, as required. ■

Problem IV.2.9. Let P be a p-Sylow subgroup of a finite group G, and let $H \subseteq G$ be a p-subgroup. Assume $H \subseteq N_G(P)$. Prove that $H \subseteq P$.

Proof. By noting that P is normal in $N_G(P)$ (Remark IV.1.12), we consider PH, which is a subgroup of $N_G(P)$ by Proposition II.8.11. Then by second isomorphism theorem

$$\frac{PH}{P} \cong \frac{H}{P \cap H}$$

Now $|PH| = \frac{|P||H|}{|P\cap H|}$ by II.8.21, and since either $|P \cap H| = 1$ or $|H|$ by Sylow II, PH is a p-group, and it must be P since P is the maximal p-subgroup of G. Then we have $H \subseteq P$ since $PH = P \Leftrightarrow H \subseteq P$. ■

Problem IV.2.10. Let P be a p-Sylow subgroup of a finite group G, and act with P by conjugation on the set of p-Sylow subgroups of G. Show that P is the unique fixed point of this action.

Proof. Let S be the collection of p-Sylow subgroups of G, and let P act on S by conjugation. If H is any p-Sylow that is fixed by P, then we have $H \subseteq N_G(P)$ ($PHP^{-1} = H \Rightarrow HPH^{-1} = P$), so we can apply IV.2.9 and obtain $H \subseteq P$. But by Sylow II, H must be P, proving the statement. ■

Problem IV.2.12. Let P be a p-Sylow subgroup of a finite group G, and let $H \subseteq G$ be a subgroup containing the normalizer $N_G(P)$. Prove that $[G : H] \equiv 1 \mod p$.

Proof. By Sylow III, $[G : N_G(P)] \equiv 1 \mod p$. Since H contains P, P is also a p-Sylow of H. Since $H \supseteq N_G(P)$, the normalizer of P in H is also $N_G(P)$, so $N_H(P) = N_G(P)$. Then clearly $[G : N_G(P)] = [G : N_H(P)] \equiv 1 \mod p$. Finally

$$[G : H] = \frac{[G : N_G(P)]}{[H : N_G(P)]} = \frac{[G : N_G(P)]}{[H : N_H(P)]}$$

and since both numerator and the denominator are both congruent to 1 mod p, $[G : H] \equiv 1 \mod p$. ■

Problem IV.2.13. Let P be a p-Sylow subgroup of a finite group G.

- Prove that if P is normal in G, then it is in fact characteristic in G.
- Let $H \subseteq G$ be a subgroup containing the Sylow subgroup P. Assume P is normal in H and H is normal in G. Prove that P is normal in G.
- Prove that $N_G(N_G(P)) = N_G(P)$.

Proof.

- Since $\gcd(|P|, |G/P|) = 1$ as P is Sylow, by the 4th point of IV.2.2, P is characteristic in G.
- By above, P is characteristic in H, so by 2nd point of IV.2.2, P is normal in G.
- We have the normal chain
$$P \trianglelefteq N_G(P) \trianglelefteq N_G(N_G(P))$$
and by above, P is normal in $N_G(N_G(P))$, so for any $g \in N_G(N_G(P)), gPg^{-1} = P$, i.e. $g \in N_G(P)$. Since the other inclusion is clear, we conclude that $N_G(N_G(P)) = N_G(P)$.

■

This is the end of the solution manual as of March 1, 2020.
Please revisit
https://github.com/macyayaya/algebra-chapter-0-solutions/releases
for possible new releases.
Thanks for your reading.