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The Complexity of Hierarchical Systems

H. A. Ceccatto

Applied Physics Department, Stanford University, Stanford, CA. 94305, USA

and

B. A. Huberman

Xerox Palo Alto Research Center, Palo Alto, CA. 94304, USA

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Abstract

We introduce a procedure for coarse graining a given hierarchical structure and show how it leads to an effective saturation of the complexity value with increasing number of lower levels. Secondly, we verify that this coarse grained measure has the property of isolating the most diverse trees as the ones with maximal complexity. As a corollary, we cast the dynamical measure of complexity of Bachas and Huberman in terms of purely static properties of trees representing ultradiffusion. We also discuss the differences between the coarse-grained measure of complexity and that provided by relaxation processes.

1. Introduction

The description and understanding of the complexity of natural and artificial systems remains an open problem in science. In contrast with the relative simplicity implied by few degrees of freedom, on one hand, and the laws of large numbers of statistical mechanics on the other, systems whose behaviour cannot be simply understood from knowledge about the behaviour of their parts pose a number of questions about their dynamical and organizational principles.

Years of research on this topic however, have already provided a number of powerful insights on the subject. A very appealing one, that complex systems are often nearly decomposable, was articulated some time ago by Simon [1] in his survey of hierarchically organized structures. These hierarchies, which range from the structural layout of matter (the clustering of pieces according to the strengths of their interactions) to the organization diagrams represented by phylogenies and social organizations, allow for an effective isolation of a given level from both the rapid fluctuations of the lower echelons and the quasistatic constraints of the higher ones. This leads in turn to dynamical processes which bear the imprint of the underlying tree structure [2–4] and which can also be used for deciding on the complexity of the hierarchical system described by such trees [5].

Recently, and in the spirit that allows the macroscopic states of matter to be specified by a few parameters, Huberman and Hogg [6] introduced a complexity measure which, besides being quantifiable, encodes the relevant properties of hierarchical systems. This measure, related to the diversity or lack of self-similarity in trees, stands in sharp contrast to existing information theoretic ones [7, 8], which in their present form are more suited to the analysis of message and algorithms than to physical structures. Moreover, it has the property of being maximal for systems that are intermediate between ordered ones and random ones, in agreement with the intuitions one has about such systems. We should also

mention that there exist attempts at constructing entropy based complexity measures, with the above properties, both for ensembles of chaotic patterns [9], and for the information content of grammars describing physical systems [10].

More recently, a study of relaxation processes on structures with a hierarchical distribution of energy barriers [5] uncovered an important relationship between the exponent characterizing the decay of the autocorrelation function and the topology of the underlying tree. Specifically, it was shown that ultradiffusion is slowest for trees which lack self-similarity while fastest for both uniform and randomly multifurcating trees. Thus, in physical systems with an ultrametric topology, one can in principle determine the complexity of a given structure from the temperature dependence of the dynamical exponent.

One should point out however, that the results obtained for ultradiffusion pertain to structures that are effectively infinite, whereas the original definition of complexity applies to large, but finite trees. Thus, before having a quantitative connection between these two measures, one needs a procedure whereby the static measure of complexity can be generalized to infinite trees, while preserving its physically intuitive properties. Moreover, this procedure should also incorporate an effective coarse graining in the measure in order to remedy a problem found in its original formulation; i.e., the detailed organization of the lower levels heavily determines the overall complexity of the system. While perhaps relevant for small organizations, this effect should become negligible in large and strongly hierarchical systems.

The purpose of this paper is two-fold. We first introduce a procedure for coarse graining a given hierarchical structure and show how it leads to an effective saturation of the complexity value with increasing number of lower levels. Secondly, we verify that this coarse grained measure has the property of isolating the most diverse trees as the ones with maximal complexity. As a corollary, we cast the dynamical results of Bachas and Huberman in terms of purely static properties of trees. This in turn allows to establish the differences between the dynamical measure of complexity and the coarse-grained one we introduce in this paper.

In Section 2 we introduce the general coarse graining procedure, define the appropriate measure of complexity, and analytically compute it for trees with similar (but not identical) diversification at each level. We also introduce a scaling conjecture based on the fact that large trees have a complexity which increases at most linearly with the number of levels. Section 3 contains the detailed analytical evaluation of the

complexity for random trees and numerical results which confirm the theoretical predictions. Furthermore an evaluation of the distribution of trees in given classes as a function of complexity value yields a normal distribution with the maximum in between that of ordered trees and random ones. Section 4 compares this coarse grained measure of complexity with the dynamical one studied by Bachas and Huberman, and an Appendix outlines the calculation of higher order corrections to the complexity measure.

2. Hierarchies and coarse graining

We will consider hierarchical structures composed of N elementary units at the lowest level, described by trees of constant depth, M . For indistinguishable elementary units, the only distinction between nodes in the tree is in the structure of the subtrees, which defines the diversity of the global structure [6]. Thus we write the total diversity $D(T)$ as

$$D(T) = F(T) \prod_{j=1}^k D(T_j) \quad (2.1)$$

where $D(T_j)$ denotes the diversity of the j th subtree in the forest, k is the number of nonisomorphic subtrees, and $F(T)$ is the form factor of the root T of the forest. The recursive nature of this definition is well suited to a hierarchical tree structure, in which each subtree represents a particular stage in the global organization of the system.

The contribution of the T root, $F(T)$, is a function of the number k_T of nonisomorphic pieces, T_1, T_2, \dots, T_k in which T diversifies. This can be seen from the fact that the total number, I_T , of interactions among the distinct parts of the tree that take place through the node T is simply related to k_T by [6]

$$I_T = (2^{k_T} - 1) \quad (2.2)$$

where in general $k_T(m)$ depends on the number of lower levels that one considers in order to decide on whether or not two subtrees are isomorphic. In what follows we will consider the case $m = 1$ for simplicity.

Besides this dependence on k_T , the form factor should also contain some information on the relative importance of the T clustering level contribution to the global diversity. In other words, a node which subtends a fat tree should have a larger contribution to the total tree complexity than one at the same height but with only infertile branches. To weight this relative importance we have at our disposal only two global properties of the subtree i.e. its number of levels, M_T , and the number of final leaves, N_T , as shown in Fig. 1(a). Note that the recursive nature of the complexity measure requires that only M_T , N_T and the branching ratio, b_T of the T root, enter its definition; for they are the only properties of T which do not require a detailed inspection of its subtrees, as shown in Fig. 1(b). One can therefore write in general $F_T = F(k_T, M_T, N_T, b_T)$.

We can therefore define a complexity measure of a tree as

$$C(T) = \log_2 D(T) \quad (2.3)$$

so that the complexity of a forest composed of n nonisomorphic trees is given by

$$C\left[\bigcup_{i=1}^n T_i\right] = \sum_{i=1}^n C(T_i) + \log_2 F\left[\bigcup_{i=1}^n T_i\right] \quad (2.4)$$

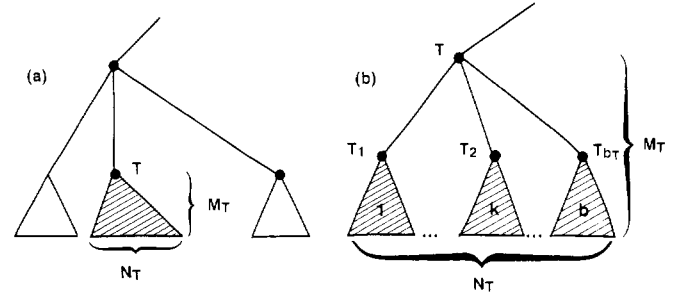


Fig. 1. (a) Generic forest with subtree generated by node T . (b) A blow-up section of the subtree T showing its own subtrees

This equation, together with a prescription on how to compute the form factor $F(T)$, provides a way of calculating the complexity of a hierarchical system. In order to make it reflect the relative importance of each node in the structure, as measured by the number of final leaves it subtends, one may choose the last term of eq. (2.4) to be given by

$$\log_2 F(T) = N_T \log_2 (2^{k_T} - 1) \quad (2.5)$$

so that fat trees and their clustering will be the major contributors to the overall complexity of the system. Note that any other factorized form of the type $F(T) = f(T)(2^{k_T} - 1)$ would lead to an irrelevant additive contribution of the form $C(T) = \log_2 f(T) + \log_2 (2^{k_T} - 1)$. A further insight into this definition is provided by estimating the complexity per leaf of *homogeneous trees*, i.e., trees that have a similar diversification at each level. For these trees it is easy to show that

$$\begin{aligned} c_h &= \frac{C_h}{N} = \log_2 (2^{k_T} - 1) \left(1 + \frac{k}{b} + \left(\frac{k}{b} \right)^2 + \dots \right) \\ &= \frac{\log_2 (2^k - 1)}{1 - k/b} \end{aligned} \quad (2.6)$$

where k is the average number of nonisomorphic pieces generated at each branching point, and b is the average branching ratio (i.e., $N \simeq b^M$). This result suggests an alternative definition of $F(T)$ as

$$\log_2 F(T) = N_T \frac{\log_2 (2^{k_T} - 1)}{b_T} \quad (2.7)$$

so that, for $k \gg 1$, the analogue of eq. (2.6) becomes

$$c_h = \frac{k}{b} + \left(\frac{k}{b} \right)^2 + \dots = \frac{q}{1 - q} \quad (2.8)$$

where $q = k/b$ is the average fraction of branches that generate nonisomorphic subtrees at each node. This result allows one to write the complexity per leaf as

$$\frac{C_h}{N} = \sum_{\text{levels}} \frac{NI_{\text{level}}}{NB_{\text{level}}} \quad (2.9)$$

where $NI_m \simeq k^m$ is the number of non-isomorphic trees at level $m > 1$ ($m = 0$ is the root) and $NB_m \simeq b^m$ is the number of branches at this level. In other words, on average each level contributes to the complexity the fraction of its branches that generate non-isomorphic trees. Thus, for large M , very complex trees will have a complexity which at most increases linearly with the number of levels, so that c will exhibit saturation. Similarly, since for almost ordered trees $k_T \simeq 1$, and $\log_2 (2^{k_T} - 1) \simeq 0$, the diversity measure stays constant,

and in the large N limit c will approach zero. Thus one can conjecture that the complexity of any hierarchical system will scale like

$$\frac{C}{N} \simeq M^\beta \quad (2.10)$$

with $M \gg 1$ and $0 \leq \beta \leq 1$. In particular, $\beta = 0$ for ordered trees and $\beta = 1$ for very complex ones. The existence of such scaling law implies that one can characterize the complexity of any given tree by the exponent β .

The above discussion provides a general and absolute measure of the complexity of a hierarchical system. There are situations however, where rather than computing the most general complexity measure, one is interested in that of a well defined class of trees. Such is the case when the level hierarchy reflects some intrinsic property of the class, such as the ratio R between energy barriers in an ultradiffusion problem [2], to provide a physical example. In that case, one can define the form factor as

$$\log_2 F(T) = (R)^{-M_T} \log_2(2^{k_T} - 1) \quad (2.11)$$

which ensures that the relative importance of a given subtree T is measured by the height, $(1/R)^{M_T}$, of its root. As we did earlier, one can also choose to weight the complexity of every subtree by its diversification efficiency, $[\log_2(2^{k_T} - 1)]/b_T$, so that in analogy with eq. (2.7) the form factor is given by

$$\log_2 F(T) = \left(\frac{1}{R}\right)^{M_T} \frac{\log_2(2^{k_T} - 1)}{b_T} \quad (2.12)$$

With these definitions, it is possible to provide a connection between hierarchical structures and the metric trees introduced by Bachas and Huberman in their study of complexity and ultradiffusion [5]. For instance if eqs. (2.5) and (2.11) are written as $\log_2 F(T) = \alpha_T \log_2(2^{k_T} - 1)$, one can give a geometric meaning to α_T by writing it as

$$\alpha_T = \exp\left(\frac{h_T}{L}\right) \quad (2.13)$$

with h_T the height of the root of T from its leaves, and L a scale factor. For the case $\alpha_T = N_T$, when drawing trees according to the definition $h_T/L = \ln(N_T)$, the branching levels are no longer horizontal, as shown in Fig. 2(a). Notice however, that the value of the complexity for the tree shown in this figure is the same as that of Fig. 2(b).

3. Random trees

In order to make further progress, we will restrict ourselves to the study of trees whose multifurcation factor, b_T , at any node T , is always less or equal than a maximum one, B . This restriction allows for quantitative statements to be made about the structure of the measure introduced in the previous section.

Consider a tree grown from its root by choosing at random the branching ratio of each node. One can ask about the probability $P_B(b, k)$ of having a b -branching node with k anisotropic subtrees. This problem is equivalent to computing the probability of obtaining k different balls from an urn with a total of B balls by sequentially picking b ones with replacement. The answer is given by

$$P_B(b, k) = \sum_{\{\lambda\}} \frac{N_B(\lambda)}{B N_B(b)} \quad (3.1)$$

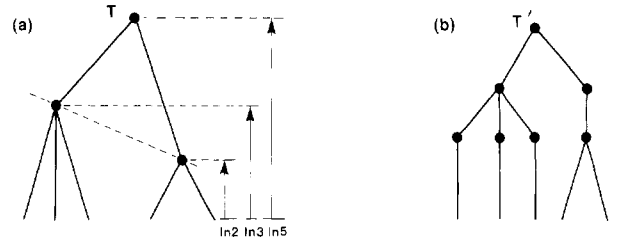


Fig. 2. (a) A metric tree illustrating the relative importance (as given by the number of leaves subtended) of given nodes. (b) Equivalent representation.

where

$$\begin{aligned} N_B(b) &= \sum_{k=1}^b \sum_{\{\lambda\}} N_B(\lambda) \\ &= \sum_{k=1}^b \sum_{\{\lambda\}} \frac{B!b!}{(b-k)! \left(\prod_{r=1}^b \lambda_r! \right) \prod_{r=1}^b (r!)^{\lambda_r}} \end{aligned} \quad (3.2)$$

and the sum $\sum_{\{\lambda\}}$ is taken over all partitions $\lambda(b, k)$ that satisfy the equalities

$$\sum_{r=1}^b \lambda_r = b; \quad \sum_{r=1}^b r \lambda_r = k \quad (3.3)$$

In terms of the ball example, λ_r is the number of different balls picked r -times each, in an experiment involving the extraction of b balls. For example, for $B = 3$ we get $P_3(1, 1) = 1/3$, $P_3(2, 1) = 1/9$, $P_3(2, 2) = 2/9$, $P_3(3, 1) = 1/27$, $P_3(3, 2) = 2/9$ and $P_3(3, 3) = 2/27$, so that

$$\langle b \rangle = \sum_{b,k} b P_3(b, k) = 2 \quad (3.4)$$

$$\langle k \rangle = \sum_{b,k} k P_3(b, k) = \frac{4}{3} = 1.593 \quad (3.5)$$

For this simple example, the complexity per leaf of the average homogeneous tree, as given by eq. (2.6) becomes

$$c_h = \frac{\log_2(2^{\langle k \rangle} - 1)}{(1 - \langle k \rangle / \langle b \rangle)} = 4.97 \quad (3.6)$$

This result provides the lowest order approximation to the average complexity of the class. It can be improved by including corrections due to fluctuations around the values $\langle k \rangle$ and $\langle b \rangle$ at the root level. These fluctuations can be calculated by considering an infinite tree with a given branching b at the root and k anisomorphic subtrees, as shown in Fig. 1. With the definition for the form factor given by eq. (2.5), we have

$$c_T(b, k) = \log_2(2^k - 1) + \sum_{r=1}^k c_r \quad (3.7)$$

where c_r is the complexity of the r -th nonisomorphic subtree that grows from the first level under the root. Taking into account the fact that, on average, the mean number of final leaves belonging to each subtree is N/b , one can estimate $\bar{c}_r = c_T/b$, so that

$$\bar{c}(b, K) = \sum_{b,k} P_B(b, k) c_T(b, k) = \frac{\langle \log_2(2^k - 1) \rangle}{(1 - \langle k/b \rangle)} \quad (3.8)$$

From the values of P_3 obtained above, $\langle k/b \rangle = 137/162 = 0.846$ and $\langle \log_2(2^k - 1) \rangle = 0.912$, which in turn gives $\bar{c} = 5.92$.

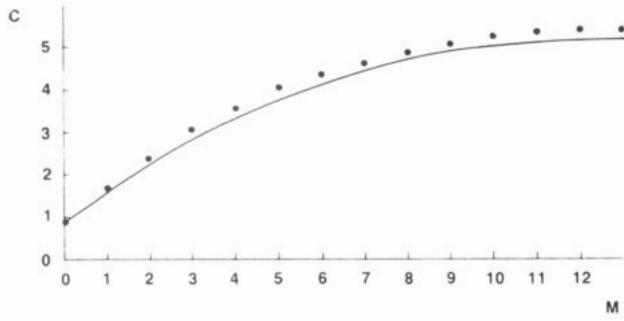


Fig. 3. Average complexity per leaf for 1000 trees of hierarchical levels. The full line denotes the theoretical prediction and the dots the measured data from a computer simulation.

Although further corrections to eq. (3.4) can be obtained in a similar way (see Appendix), we point out that this approximation gives results to within 10% even for finite trees. Fig. 3 shows the results of a numerical computation of the average complexity per leaf for 1000 trees of 12 hierarchical levels. The full line represents the predictions of eq. (3.8) expressed as

$$\bar{c}(b, k) = \langle \log_2(2^k - 1) \rangle (1 + \langle k/b \rangle + \dots \langle k/b \rangle^M) \quad (3.9)$$

which as expected, gives the exact contribution of the root ($M = 0$) to the complexity, while slightly underestimating the contributions of the next levels. Notice that the effect of coarse graining produced by the factor N_T , appears in the downward curvature and final saturation of $\bar{c}(M)$ in Fig. 3.

A more interesting quantity in the class is the distribution of trees as a function of complexity. Fig. 4 shows the histogram obtained by plotting the relative frequency of appearance of trees with complexity values between n and $n + 1$ ($n = 0, 1, \dots$) for a sample of 1000 trees. The normal-like distribution can be easily understood by noticing that the complexity is determined by a large sum over the contributions by the nonisomorphic nodes. Since the branching ratio of each node is a random variable, these contributions are almost independent of each other. Therefore, according to the central limit theorem, c must be normally distributed.

The width, σ_c , of the distribution can be estimated with the help of eq. (3.7). Squaring both sides of the equation, averaging over the whole class, and making the approxi-

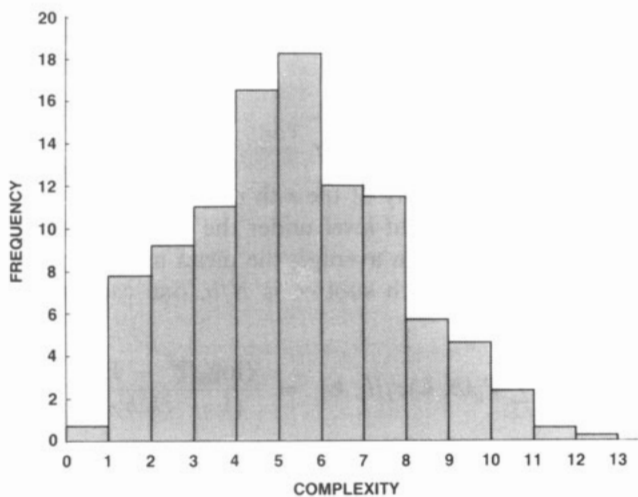


Fig. 4. Relative frequencies of appearance of trees with complexity values between n and $n + 1$ for a sample of 1000 trees.

mations that $\bar{c}_r = \bar{c}_1/b$, and $(\sigma_c)^2 = (\sigma_c)^2/b^2$ we obtain

$$c^2 = \langle [\log_2(2^k - 1) + (k/b)c]^2 \rangle + \langle k/b^2 \rangle (\sigma_c)^2 \quad (3.10)$$

which in turn yields

$$\sigma_c^2 = \langle c^2 \rangle - \langle c \rangle^2 = \frac{\langle [\log_2(2^k - 1) + (k/b)C_T]^2 \rangle - C_T^2}{(1 - \langle k/b^2 \rangle)} \quad (3.11)$$

For $B = 3$, the computed values of P_3 give $\langle k/b^2 \rangle = 0.53$, so that $\sigma_c^2 = 5.82$, or equivalently, $\sigma = 2.41$, which is in good agreement with the actual dispersion observed in Fig. 4.

4. Static complexity vs. dynamic complexity

In the previous section, we calculated the mean complexity, and its dispersion, for a given class of trees, using the form factor of eq. (2.5). According to the definition given by eq. (2.5), the contribution of each subtree to the overall complexity of a given forest is weighted by the number of its final leaves. As discussed in Section 2 there are cases however, where the contribution of each level of clustering is determined by some intrinsic parameter, $\alpha = 1/R$, in which case eq. (2.5) is replaced by eq. (2.11). The result equivalent to eq. (3.8) then reads:

$$\bar{c} = \frac{\langle \log_2(2^k - 1) \rangle}{(1 - k/\alpha)} \quad (4.1)$$

where \bar{c} is now the complexity of a tree with the weight of its root normalized to unity, i.e., $C = C/(\alpha^M)$. In similar fashion, the dispersion, σ_c is given by eq. (3.11), with b replaced by α .

One can now relate the above equation to the ultradiffusive process studied by Bachas and Huberman. This can be simply done by casting their dynamical results in terms of static topological properties of trees.

Consider the case where the complexity of a given tree is determined by every node, irrespective of their relative isomorphism. One would thus have

$$C = \sum_{\text{nodes}} W_{\text{node}} \quad (4.2)$$

where the contribution of the node B in Fig. 4 can be obtained recursively from that of node B_1 in the following way

$$W(B) = F(B)W(B_1) \quad (4.3)$$

and where coarse graining becomes equivalent to requiring that $F(B) < 1$. Taking logarithms on both sides of this equation and choosing $\ln F(B)$ to be given by

$$\ln F(B) = -(1 - R) \left(\frac{N_B}{N} \right) e^{(h_{\text{root}} - h_B)} \quad (4.4)$$

with the normalization condition $\ln F(\text{root}) = -1$, it is easy to see that this definition leads to a complexity per leaf equal to $c = (Ne^{-h_{\text{root}}})^v$, where v is the dynamical relaxation exponent of Bachas and Huberman.

These arguments are based on the choice of the form factor given by eq. (4.4). First, we point out that the minus sign guarantees that $F(B) < 1$. Furthermore, since $R = e^{-\Delta h}$, where Δh is the difference between the heights of successive levels in Fig. 5, we see that $(1 - R)$ in that equation gives $F(B)$ of order 1 for subtrees closer to the root, i.e. there are no important coarse graining effects in a shallow forest. Finally, the factor $(N_B/N)e^{(h_{\text{root}} - h_B)}$ indicates that the relevant

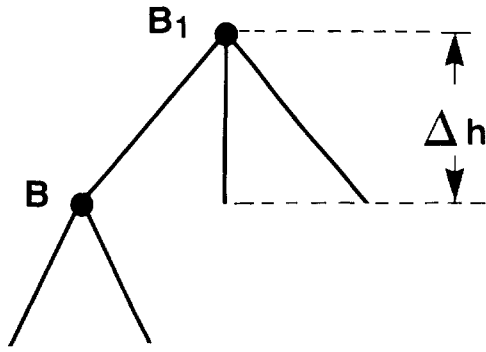


Fig. 5. Illustrating the difference in heights between successive levels.

contributions to eq. (4.2) come from trees with small silhouette ("brooms") which in our notation correspond to $N_B/N \ll 1$ and $h_B = h_{\text{root}}$.

In closing, we should point out that, in a sense, the static measure of complexity we introduced in the previous sections is different from that of Bachas and Huberman. Namely, whereas the former emphasizes the diversity of the upper nodes of the structure, along with their fertility, the other reflects the relative amount of infertile branches (or brooms). This is due to the fact that the highest barriers in ultradiffusion are responsible for the long time behaviour of relaxation in hierarchical systems.

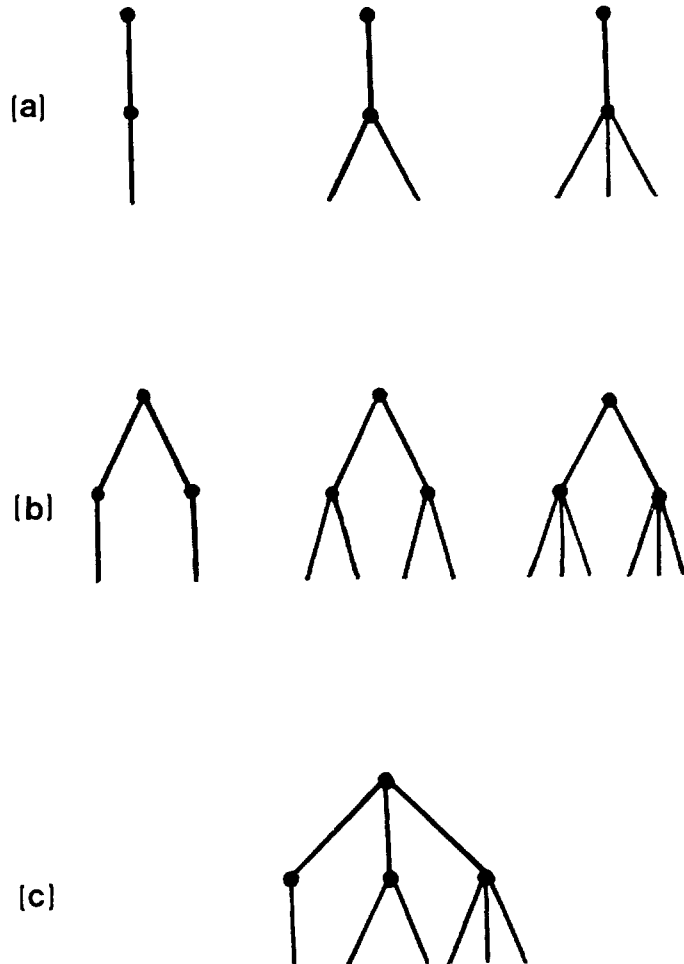


Fig. 6. Subtrees entering in the calculation of higher order corrections to eq. (3.6).

5. Summary

In this paper we introduced a procedure for coarse graining a given hierarchical structure and showed how it leads to an effective saturation of the complexity value with increasing number of lower levels. This procedure remedied the problem found in its original formulation, i.e., that the detailed organization of the lower levels heavily determined the overall complexity of the system. While perhaps relevant for small organizations, this effect now becomes negligible in large and strongly hierarchical systems. We also verified that this coarse grained measure has the property of isolating the most diverse trees as the ones with maximal complexity. As a corollary, we were able to cast the dynamical measure of complexity of Bachas and Huberman in terms of purely static properties of trees representing ultradiffusion, and therefore compare it with the coarse grained measure introduced in this paper.

While making progress towards the description of complex hierarchical systems, the results of this paper leave unanswered a number of questions. Among them, the relationship between complexity and dynamics of the structure itself, and the procedures whereby systems become represented by given trees. Nevertheless, these results allow for treating very large, or effectively infinite, hierarchies while preserving the physical intuitions about order, complexity and disorder, and open the door for the quantitative analysis of adaptability in complex systems.

Appendix

In this section we outline the calculation of the first order correction to eq. (3.6). Higher order corrections can be obtained in a similar way.

We start by noticing that the complexity per leaf, c , is a function of the branching ratios, i.e., $c = c(b_0, b_1, \dots, b_M)$, where b_0 is the root branching, $b_1 = \{b_1^1, b_1^2, b_1^3\}$ is the set of branchings at the first level under the root, and so on.

Denoting by $c_1(b_0, b_1, \dots, b_M)$ the complexity of the nonisomorphic subtrees under the root, we have

$$c(b_0, b_1, \dots, b_M) = \log_2(2^{k(b_0, b_1)} - 1) + c_1(b_0, b_1, \dots, b_M) \quad (\text{A.1})$$

where the c_1 dependence on b_0 comes from the global factor $1/N$. For $b_0 = 1$ and $k(1, b_1) = 1$ (Fig. 1(a)), and by averaging eq. (A.1) over b_r ($r = 2, 3, \dots, M$) we get

$$c(1, b_1) = c_1(1, b_1) = c(b_1), \quad b_1 = 1, 2, 3 \quad (\text{A.1})$$

where the second equality comes from the fact that the subtree growing from the first level under the root looks like the whole tree, albeit with root branching b_1 .

In analogous fashion, for $b_0 = 2$ and $k[2, (b_1^1, b_1^2)] = 1$ (see Fig. 6) we have

$$c[2, (b_1^1, b_1^2)] = c_1[2, (b_1^1, b_1^2)] = 0.5c(b_1) \quad (\text{A.2})$$

with $b_1^1 = b_1^2 = b_1 = 1, 2, 3$, and the factor 0.5 taking into account the fact that each subtree under the root has, on average, $N/2$ final leaves.

A further example is provided by the case $b = 3, k = 3$,

as shown in Fig. 6(c). We then have

$$\begin{aligned} c[3, (1, 2, 3)] &= \log_2(7) + c_1[3, (1, 2, 3)] \\ &= \log_2(7) + \frac{c(1)}{6} + \frac{2}{6}c(2) + \frac{3}{6}c(3) \end{aligned} \quad (\text{A.3})$$

Considering the six possibilities $b_0 = 1, 2, 3$; $1 < k(b_0, b_1) < b_0$, and averaging the left hand side of the corresponding equation over b_1 , one obtains the system

$$\begin{aligned} 2c(1) - c(2) - c(3) &= 0 \\ -50c(1) + 191c(2) - 96c(3) &= 180 \log_2(3) \\ -3118c(1) - 5578c(2) + 15263c(3) \\ &= 15120 \log_2(3) + 5040 \log_2(7) \end{aligned} \quad (\text{A.4})$$

which yields the result

$$c = \frac{1}{3} \sum_{r=1}^3 c(r) = 5.99 \quad (\text{A.5})$$

to be compared with our previous result $c = 5.92$ obtained in Section 3. This improvement is due to the fact that this procedure incorporates additional information about the tree, i.e., the actual branching values of the k generic nonisomorphic nodes considered in Section III. A further refinement of this result can be obtained by explicitly taking into account the branching values b_2 at the second level under the root.

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