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Rényi entropy and complexity measure for skew-gaussian distributions and related families

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HIGHLIGHTS

- I present the Rényi entropy and complexity measure for skew-gaussian distributions.
- I also derived closed expressions for extended and truncated skew-gaussian distributions.
- Additional inequalities for skew-gaussian and extended skew-gaussian entropies were reported.

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ABSTRACT

In this paper, we provide the Rényi entropy and complexity measure for a novel, flexible class of skew-gaussian distributions and their related families, as a characteristic form of the skew-gaussian Shannon entropy. We give closed expressions considering a more general class of closed skew-gaussian distributions and the weighted moments estimation method. In addition, closed expressions of Rényi entropy are presented for extended skew-gaussian and truncated skew-gaussian distributions. Finally, additional inequalities for skew-gaussian and extended skew-gaussian Rényi and Shannon entropies are reported.

1. Introduction

The family of skew-gaussian distributions has been popularized by Azzalini [1] and ever since it has been discussed extensively in the literature. Such discussions include a wide variety of skewed models in addition to having Gaussian distribution as a special case and flexibility in capturing skewness in the data [2–4]. In this sense, González-Faráas et al. [5] present the closed skew-gaussian distribution as an extension of the skew-gaussian case, but closed under operations such as sums, marginalization, and linear conditioning [6]. Another generalization of the skew-gaussian distribution is the extended skew-gaussian distribution [7] that adds a fourth real parameter to accommodate both skewness and heavy tails. In some cases where observed variables can be simultaneously skewed and restricted to a fixed interval, the truncated skew-gaussian distribution is a good choice for those applications, especially for environmental and biological variables in which the observations are positives [8].

In many applications, the empirical distribution of some observed variables was modelled by a skew-gaussian distribution. For example, the closed skew-gaussian distribution is used by Rezaie et al. [6] to simulate seismic amplitude variations. Contreras-Reyes and Arellano-Valle [9] consider the skew-gaussian distribution for seismic magnitudes of aftershocks catalogue of the 2010 Maule earthquake in Chile; Arellano-Valle et al. [10] for the optimization of ozone's monitoring network;

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and Figiel [11] for a digital reconstruction of nanocomposite morphologies from TEM (Transmission Electron Microscopy) images. An implementation of the extended skew-gaussian (in logarithmic form) can be found in Ref. [12] for pricing of both Asian and basket options. As mentioned above, Flecher et al. [8] consider the truncated skew-gaussian distribution to fit the daily relative humidity measurements. See more applications in Ref. [13].

More recently, Contreras-Reyes and Arellano-Valle [9] and Arellano-Valle et al. [10] compute the Kullback–Leibler divergence measure for skew-gaussian distribution and Shannon entropy for the full class of skew-elliptical distributions, respectively. They highlight that the Kullback–Leibler information measure should be represented in quadratic form, including a non-analytical expected value. In addition, they gave the Kullback–Leibler divergence of a multivariate skew-gaussian distribution with respect to multivariate Gaussian distribution. Information measure applications dealing with skewed data have been performed by Contreras-Reyes and Arellano-Valle [9], Arellano-Valle et al. [10], Contreras-Reyes [14] and references therein.

In this work, we focus on the Rényi entropy [15] as a characteristic form of the Shannon entropy to give a closed expression of skew-gaussian densities. Additionally, the LMC complexity measure [16] is derived by the difference between the extensive Rényi entropy and Shannon entropy [17]. To do this, we briefly describe the main properties of closed skew-gaussian distributions. Finally, we compute the Rényi entropy and complexity measure for the extended skew-gaussian and univariate truncated skew-gaussian densities.

2. Rényi entropy and complexity measure

Consider the αth-order Rényi entropy [15] of probability density $f(\mathbf{x})$ on a variable $\mathbf{x} \in \Delta \subset \mathbb{R}^d$:

$$R_\alpha[f] = \frac{1}{1-\alpha} \ln \int [f(\mathbf{x})]^\alpha \mathrm{d}\mathbf{x}, \tag{1}$$

where normalization to unity as given by $\int f(\mathbf{x})\mathrm{d}\mathbf{x} = 1$ [18]. Golshani and Pasha [19] provide some important properties of the Rényi entropy: 1. $R_\alpha[f]$ can be negative, 2. $R_\alpha[f]$ is invariant under a location transformation, 3. $R_\alpha[f]$ is not invariant under a scale transformation, and 4. for any $\alpha_1 < \alpha_2$, $\mathbf{x} \in \Delta$, we have $R_{\alpha_1}[f] \geq R_{\alpha_2}[f]$, which are equal if and only if $\mathbf{x}$ is uniformly distributed.

From (1), the Shannon entropy is obtained by the limit

$$S[f] = \lim_{\alpha \to 1} R_\alpha[f] = -\int f(\mathbf{x}) \ln f(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{2}$$

by applying l'Hôpital's rule to $R_\alpha[f]$ with respect to α [15]. This measure is the expected value of $g(\mathbf{x}) = -\ln f(\mathbf{x})$ with respect to $f(\mathbf{x})$, i.e., $S[f] = \langle g(\mathbf{x}) \rangle$ [20]. Hereafter, we will refer to this as the expected information of $g(\mathbf{x})$ in $\mathbf{x}$. See Ref. [21] for additional properties of the Shannon entropy.

Example 1 (*Dembo et al. [22] and Cover and Thomas [21]*)**.** Let $\mathbf{x}$ be a Gaussian with mean vector $\mu \in \mathbb{R}^d$ and $\mathbf{J}$ is a $d \times d$ variance matrix (with determinant $|\mathbf{J}| > 0$). Then, the Rényi and Shannon entropies of $\mathbf{x}$ are given by

$$R_\alpha[f] = \frac{1}{2} \ln[(2\pi)^d |\mathbf{J}|] + \frac{d \ln \alpha}{2(\alpha - 1)}, \quad 1 < \alpha < \infty, \tag{3}$$

$$S[f] = \frac{1}{2} \ln[(2\pi e)^d |\mathbf{J}|], \tag{4}$$

respectively.

Another important concept is the statistical complexity that measures the randomness and structural correlations of a known system [23]. López-Ruiz et al. [16] proposed a measure of statistical complexity (LMC) in order to determine the *disequilibrium* of the system attributed to entropy measure [24,18]. LMC measure is defined as the product

$$C_{LMC}[f] = e^{S[f] - R_2[f]}, \tag{5}$$

where $R_2[f]$ is the quadratic Rényi entropy of $\mathbf{x}$ ($\alpha = 2$). Yamano [17] provides an extensive entropy instead of an additive Shannon entropy in (5), characterized as a difference between the αth-order Rényi entropy and quadratic Rényi entropy as

$$C_\alpha[f] = e^{R_\alpha[f] - R_2[f]}. \tag{6}$$

Note that $C_\alpha[f]$ reflects the shape of the distribution of $\mathbf{x}$ and takes unity for all distributions when $\alpha = 2$. In addition, $\mathcal{C}_\alpha$ satisfies a great variety of interesting mathematical and physical properties. Let us just recall here the following properties: 1. $C_\alpha[f] > 1, \forall \alpha \leq 2$, and, $0 < C_\alpha[f] \leq 1, \ \forall \alpha > 2$; 2. $C_\alpha[f]$ is invariant under a location and scale transformation in the distribution of $\mathbf{x}$; and 3. is invariant under replications of the original distribution of $\mathbf{x}$.

3. Skew-gaussian distribution and related families

The closed skew-gaussian distribution has interesting properties inherited from the Gaussian distribution and corresponds to a generalization of the skew-gaussian distribution. We briefly describe some of its inferential properties and present the weighted moments method in Proposition 2 [25], necessary to calculate Rényi entropy of skew-gaussian random vectors.

3.1. Closed skew-gaussian distributions

Concerning the definition of Flecher et al. [25] and González-Farías et al. [5], let $\mathbf{y} \in \Delta \subset \mathbb{R}^d$ be a random vector with closed skew-gaussian distribution denoted as $CSN_{d,s}(\mu, \mathbf{J}, \mathbf{D}, \nu, \mathbf{A})$ and with density function

$$f_{d,s}(\mathbf{y}) = \phi_d(\mathbf{y}; \mu, \mathbf{J}) \frac{\Phi_s(\mathbf{D}^\top(\mathbf{y} - \mu); \nu, \mathbf{A})}{\Phi_s(\mathbf{0}; \nu, \mathbf{A} + \mathbf{D}^\top \mathbf{J}\mathbf{D})}, \tag{7}$$

where $\mu \in \mathbb{R}^d$, $\nu \in \mathbb{R}^s$, $\mathbf{J} \in \mathbb{R}^{d\times d}$ and $\mathbf{A} \in \mathbb{R}^{s\times s}$ are both covariance matrices, $\mathbf{D} \in \mathbb{R}^{d\times s}$, $\mathbf{D}^\top$ denotes the transposed $\mathbf{D}$ matrix,

$$\phi_d(\mathbf{y}; \mu, \mathbf{J}) = \frac{1}{(2\pi)^{d/2}|\mathbf{J}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mu)^\top \mathbf{J}^{-1}(\mathbf{y} - \mu)\right)$$

and $\Phi_d(\mathbf{y}; \mu, \mathbf{J})$ are the probability function (pdf) and cumulative distribution function, respectively, of the d-dimensional Gaussian distribution with mean vector μ and variance matrix $\mathbf{J}$. The closed skew-gaussian distribution is closed under translations, scalar multiplications, and full, row rank linear transformations [5,13]. Let $\mathbf{T} \in \mathbb{R}^{n\times d}$ be a matrix with rank n such that $d \le n$, then

$$\mathbf{T}\mathbf{y} = CSN_{n,s}(\mathbf{T}\mu, \tilde{\mathbf{J}}, \tilde{\mathbf{D}}, \nu, \tilde{\mathbf{T}}) \tag{8}$$

where $\tilde{\mathbf{J}} = \mathbf{T}^\top \mathbf{J}\mathbf{T}$, $\tilde{\mathbf{D}} = \mathbf{D}^\top \mathbf{J}\mathbf{T}\tilde{\mathbf{J}}^{-1}$, and $\tilde{\mathbf{T}} = \mathbf{A} + \mathbf{D}^\top \mathbf{J}\mathbf{D} - \tilde{\mathbf{D}}^\top \tilde{\mathbf{J}}\tilde{\mathbf{D}}$ [13, see Proposition 2.3.1].

A particular case of (8), is the standardized random vector $\mathbf{z}_0 = \mathbf{J}^{-1}(\mathbf{y} - \mu)$. In this case, Eq. (7) is rewritten as

$$f_{d,s}(\mathbf{z}_0) = \phi_d(\mathbf{z}_0) \frac{\Phi_s(\mathbf{D}^\top \mathbf{J}^{1/2}\mathbf{z}_0; \nu, \mathbf{A})}{\Phi_s(\mathbf{0}; \nu, \mathbf{A} + \mathbf{D}^\top \mathbf{J}\mathbf{D})}. \tag{9}$$

Given that the closed skew-gaussian distribution is closed under translations and by property (8), the standardized random vector $\mathbf{Z}_0$ follows $CSN_{d,s}(\mathbf{0}, \mathbf{I}_d, \mathbf{D}^\top \mathbf{J}^{1/2}, \nu, \mathbf{A})$, where $\mathbf{I}_d$ denotes the d-dimensional identity matrix. For more details, see Refs. [25,13].

Lemma 1 (*Flecher et al. [25]*)**.** *Let $\mathbf{Y}$ be a $CSN_{d,s}(\mu, \mathbf{J}, \mathbf{D}, \mathbf{0}, \mathbf{A})$, r a positive integer and $h(\mathbf{y}) = h(y_1, \dots, y_d)$ be any real valued function such that $\langle h(\mathbf{Y})\rangle$ is finite, then*

$$\langle h(\mathbf{Y})[\Phi_d(\mathbf{Y}; \mathbf{0}, \mathbf{I}_d)]^r\rangle = \langle h(\tilde{\mathbf{Y}})\rangle \frac{\Phi_{rd+s}(\mathbf{0}; \tilde{\nu}, \tilde{\mathbf{A}} + \tilde{\mathbf{D}}^\top \mathbf{J}\tilde{\mathbf{D}})}{\Phi_s(\mathbf{0}; \mathbf{0}, \mathbf{A} + \mathbf{D}^\top \mathbf{J}\mathbf{D})}, \tag{10}$$

where $\tilde{\mathbf{Y}} \sim CSN_{d,rd+s}(\mu, \mathbf{J}, \tilde{\mathbf{D}}, \tilde{\nu}, \tilde{\mathbf{A}})$ with $\tilde{\mathbf{D}} = (\mathbf{E}^\top, \ \mathbf{D}^\top)$, $\mathbf{E}$ a $d \times rd$ matrix defined by $\mathbf{E} = (\mathbf{I}_d, \dots, \mathbf{I}_d)$, $\tilde{\nu} = (-\mu, \dots, -\mu, \mathbf{0}_s)$ a $(rd + s)$ vector and

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{I}_{rd} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}.$$

3.2. Skew-gaussian distribution

A special case of closed skew-gaussian is the Gaussian density when $\mathbf{D} = \mathbf{0}$. When $s = 1$, the skew-gaussian density function is obtained [2–4]. For simplicity, a slight variant of the original definition is considered here. In this work it is posited that a random vector $\mathbf{Z} \in \Delta \subset \mathbb{R}^d$ has a skew-gaussian distribution with mean vector $\mu \in \mathbb{R}^d$, variance matrix $\mathbf{J} \in \mathbb{R}^{d\times d}$ and shape/skewness parameter $\boldsymbol{\eta} \in \mathbb{R}^d$, denoted by $\mathbf{Z} \sim SN_d(\mu, \mathbf{J}, \boldsymbol{\eta})$, if its probability density function is

$$f(\mathbf{z}) = 2\phi_d(\mathbf{z}; \mu, \mathbf{J})\Phi_1[\boldsymbol{\eta}^\top(\mathbf{z} - \mu)]. \tag{11}$$

The mean vector and the variance matrix of $\mathbf{Z}$ are

$$\langle \mathbf{z}\rangle = \mu + \sqrt{\frac{2}{\pi}}\,\delta, \qquad \mathrm{Var}[z] = \mathbf{J} - \frac{2}{\pi}\delta\delta^\top,$$

respectively, where $\delta = \mathbf{J}\boldsymbol{\eta}/\sqrt{1 + \boldsymbol{\eta}^\top \mathbf{J}\boldsymbol{\eta}}$ [3,9].

Proposition 1. *Let* $\mathbf{Z}$ *be a* $SN_d(\mu, \mathbf{J}, \boldsymbol{\eta})$*. Then:*

$$\int [f(\mathbf{z})]^\alpha d\mathbf{z} = \psi_{\alpha,d}(\mathbf{J}) \frac{\Phi_{\alpha+1}(\mathbf{0}; \mathbf{0}, \tilde{\mathbf{J}})}{\Phi_1(0; 0, \sigma^2)}, \quad \alpha \in \mathbb{N}, \ \alpha > 1, \tag{12}$$

where

$$\psi_{\alpha,d}(\mathbf{J}) = \frac{2^\alpha}{\alpha^{d/2}} [(2\pi)^d |\mathbf{J}|]^{(1-\alpha)/2},$$

$\tilde{\mathbf{J}} = \mathbf{I}_{\alpha+1} + \|\tilde{\boldsymbol{\eta}}\|^2 \tilde{\mathbf{D}}^\top \tilde{\mathbf{D}}$, $\tilde{\mathbf{D}} = (\mathbf{1}_\alpha, \|\tilde{\boldsymbol{\eta}}\|)^\top$, $\mathbf{1}_\alpha$ *is the* α*-dimensional vector of ones,* $\sigma^2 = 1 + \|\tilde{\boldsymbol{\eta}}\|^4$, $\|\tilde{\boldsymbol{\eta}}\| = \tilde{\boldsymbol{\eta}}^\top \tilde{\boldsymbol{\eta}}$ *and* $\tilde{\boldsymbol{\eta}} = \alpha^{-1/2} \mathbf{J}^{1/2} \boldsymbol{\eta}$.

By (1) and (12), the Rényi entropy of a random variable $\mathbf{Z} \sim SN_d(\mu, \mathbf{J}, \boldsymbol{\eta})$ is retrieved. Taking $\boldsymbol{\eta} = \mathbf{0}$ in (12), the Rényi entropy of the Gaussian distribution given by (3) is obtained. Lemma 1 allows the computing of the expected value of the cumulative density function of a gaussian density. Considering the standardized closed skew-gaussian variable in (9), the Proposition 1 is solved by (10), by setting $\nu = \mathbf{0}$ and $\mathbf{A} = \mathbf{I}_d$, with $d = s = 1$. However, the case $\nu \neq \mathbf{0}$ and $\mathbf{A} \neq \mathbf{I}_d$, $d > 1$, is still an open problem and, it is useful to find the Rényi entropy for closed skew-gaussian distributions. By (1) and (9), the Shannon entropy for closed skew-gaussian distributions is rewritten as

$$\begin{aligned} S[f] &= -\langle \ln[f_{d,s}(\mathbf{Y})] \rangle \\ &= \frac{1}{2} \ln |\mathbf{J}| - \ln[\Phi_s(\mathbf{0}; \nu, \mathbf{A} + \mathbf{D}^\top \mathbf{J} \mathbf{D})] - \langle \ln[\phi_d(\mathbf{Z}_0) \Phi_s(\tilde{\mathbf{D}}^\top \mathbf{Z}_0; \nu, \mathbf{A})] \rangle \\ &= S[f_0] - \ln[\Phi_s(\mathbf{0}; \nu, \mathbf{A} + \mathbf{D}^\top \mathbf{J} \mathbf{D})] - \langle \ln[\Phi_s(\tilde{\mathbf{D}}^\top \mathbf{Z}_0; \nu, \mathbf{A})] \rangle, \end{aligned} \tag{13}$$

where f_0 is the standardized gaussian distribution and $S[f_0] = (1/2) \ln(2\pi e)$.

Corollary 1. *Let* $\mathbf{Z} \sim SN_d(\mu, \mathbf{J}, \boldsymbol{\eta})$, $\mathbf{Z}_N \sim N_d(\mu, \mathbf{J})$, $\|\tilde{\boldsymbol{\eta}}\| = \tilde{\boldsymbol{\eta}}^\top \tilde{\boldsymbol{\eta}}$ *and* $\tilde{\boldsymbol{\eta}} = \mathbf{J}^{1/2} \boldsymbol{\eta}$*. Then,*

(i) $R_\alpha[f] = R_\alpha[f_0] - N_\alpha[f]$, $\alpha \in \mathbb{N}$, $\alpha > 1$, *where*

$$N_\alpha[f] = \frac{1}{\alpha - 1} \ln \left[2^\alpha \frac{\Phi_{\alpha+1}(\mathbf{0}; \mathbf{0}, \tilde{\mathbf{J}})}{\Phi_1(0; 0, \sigma^2)} \right]$$

is the so-called Negentropy, $R_\alpha[f_0]$ *is given by* (3)*, and* $\tilde{\mathbf{J}}$ *and* σ^2 *are defined as in* Proposition 1.

(ii) $\lim_{\alpha \to 1} N_\alpha[f] = \langle \ln[2\Phi_1(\|\tilde{\boldsymbol{\eta}}\| W)] \rangle$.

(iii) $S[f] = S[f_0] - \langle \ln[2\Phi_1(\|\tilde{\boldsymbol{\eta}}\| W)] \rangle$, *where* $S[f_0]$ *is given by* (4) *and* $W \sim SN_1(0, 1, \|\tilde{\boldsymbol{\eta}}\|)$.

(iv) $S[f_0] - \ln(4e) \leq S[f] \leq S[f_0]$, $\forall \boldsymbol{\eta}$.

Contreras-Reyes and Arellano-Valle [9] define the negentropy as the departure from gaussianity of the distribution of $\mathbf{Z}$. Therefore, the skew-gaussian Rényi entropy corresponds to the difference between gaussian Rényi entropy and negentropy, that depends on the skewness parameter $\boldsymbol{\eta}$. On the another hand, by setting $\nu = \mathbf{0}$ and $\mathbf{A} = \mathbf{I}_d$ in (13) with $d = s = 1$, we obtain the property (ii) of Corollary 1.

By properties (iii) and (iv), $-0.967 \leq S[f_0] - \log(4e) \leq S[f]$ because, the minimum value of normal Shannon entropy is obtained for $d = 1$ and, $0 \leq \langle \ln[2\Phi_1(\|\tilde{\boldsymbol{\eta}}\| W)] \rangle \leq 2.386$, for all $\boldsymbol{\eta}$. In addition, Contreras-Reyes and Arellano-Valle [9] reported a maximum value of this expected value equal to 2.339, using numerical approximations. Considering (1), (6) and (12); the complexity measure for skew-gaussian distribution is obtained.

3.3. Extended skew-gaussian distributions

Consider a slight variant of the extended skew-gaussian distribution proposed by Capitanio et al. [7]. Let $\mathbf{Z} \sim ESN_d(\mu, \mathbf{J}, \boldsymbol{\eta}, \tau)$, $\mathbf{Z} \in \Delta \subset \mathbb{R}^d$, with mean vector $\mu \in \mathbb{R}^d$, variance matrix $\mathbf{J} \in \mathbb{R}^{d \times d}$, shape/skewness parameter $\boldsymbol{\eta} \in \mathbb{R}^d$, extended parameter $\tau \in \mathbb{R}$, and with pdf given by:

$$p(\mathbf{z}) = \frac{1}{\Phi_1(\tau)} \phi_d(\mathbf{z}; \mu, \mathbf{J}) \Phi_1[\boldsymbol{\eta}^\top (\mathbf{z} - \mu) + \tilde{\tau}], \tag{14}$$

where $\mathbf{z} \in \mathbb{R}^d$ and $\tilde{\tau} = \tau \sqrt{1 + \boldsymbol{\eta}^\top \mathbf{J} \boldsymbol{\eta}}$. The mean vector and the variance matrix of $\mathbf{Z}$ are

$$\langle \mathbf{z} \rangle = \mu + \delta \zeta_1(\tau), \tag{15}$$

$$\mathrm{Var}[z] = \mathbf{J} - \zeta_1(\tau)[\tau + \zeta_1(\tau)] \delta \delta^\top, \tag{16}$$

respectively; where $\zeta_1(\mathbf{z}) = \phi(\mathbf{z}) / \Phi_1(\mathbf{z})$ is the *zeta* function [3,7].

Proposition 2. *Let* $\mathbf{Z}$ *be a* $ESN_d(\mu, \mathbf{J}, \boldsymbol{\eta}, \tau)$, $\mathbf{z} \in \mathbb{R}^d$. *Then:*

$$\int [f(\mathbf{z})]^{\alpha} \mathrm{d}\mathbf{z} = \psi_{\alpha,d}(\mathbf{J}) \left\langle \left[\frac{\Phi_1(W)}{2\Phi_1(\tau)} \right]^{\alpha} \right\rangle, \quad \alpha \in \mathbb{N}, \ \alpha > 1, \tag{17}$$

where $\psi_{\alpha,d}(\mathbf{J})$ *is defined as in Proposition 1 and* $W = \tilde{\boldsymbol{\eta}}^{\top}\mathbf{Z}_0 + \tilde{\tau} \sim ESN_1(\tilde{\tau}, \|\tilde{\boldsymbol{\eta}}\|^2, \|\tilde{\boldsymbol{\eta}}\|, \tau)$, $\|\tilde{\boldsymbol{\eta}}\| = \tilde{\boldsymbol{\eta}}^{\top}\tilde{\boldsymbol{\eta}}$, *and* $\tilde{\boldsymbol{\eta}} = \mathbf{J}^{1/2}\boldsymbol{\eta}$.

Corollary 2. *Let* $\mathbf{Z} \sim ESN_d(\mu, \mathbf{J}, \boldsymbol{\eta}, \tau)$, $\mathbf{Z}_N \sim N_d(\mu, \mathbf{J})$ *and* W *are defined as in Proposition 2. Then,*

(i) $R_\alpha[f] = R_\alpha[f_0] - N_\alpha[f]$, $\alpha \in \mathbb{N}$, $\alpha > 1$, *where*

$$N_\alpha[f] = \frac{1}{\alpha - 1} \ln \left\langle \left[\frac{\Phi_1(W)}{\Phi_1(\tau)} \right]^{\alpha} \right\rangle,$$

and $R_\alpha[f_0]$ *is given by (3).*

(ii) $R_\alpha[f] \le R_\alpha[f_0] + \frac{\alpha}{1-\alpha} \ln \left[\frac{\Phi_1(\tilde{\tau} + \tilde{\delta}\zeta_1(\tau))}{\Phi_1(\tau)} \right]$,

where $\tilde{\delta} = \|\tilde{\boldsymbol{\eta}}\|^3 / \sqrt{1 + \|\tilde{\boldsymbol{\eta}}\|^4}$.

(iii) $S[f] = S[f_0] - \left\langle \ln \left[\frac{\Phi_1(W)}{\Phi_1(\tau)} \right] \right\rangle$.

(iv) $S[f_0] + \ln[\Phi_1(\tau)] - \Phi_1 \left(\frac{\tilde{\tau}}{\sqrt{1 + \boldsymbol{\eta}\boldsymbol{\eta}^{\top}}} \right) \le S[f] \le \frac{1}{2} \ln \left[(2\pi e)^d \left| \mathbf{J} - \zeta_1(\tau)[\tau + \zeta_1(\tau)]\delta\delta^{\top} \right| \right]$, $\forall \boldsymbol{\eta}$.

(v) $\lim_{\alpha \to 1} N_\alpha[f] = \left\langle \ln \left[\frac{\Phi_1(W)}{\Phi_1(\tau)} \right] \right\rangle$.

Pourahmadi [26] illustrated the behaviour of $\zeta_1(\tau)$, $\tau \in \mathbb{R}$. This function is strictly decreasing for any $\tau \in \mathbb{R}$, tends to 0 when $\tau \to +\infty$, and diverge when $\tau \to -\infty$. For $\tau = 0$, the property (iv) of Corollary 2 becomes property (iii) of Corollary 1. By properties (iii) of Corollary 2 and (ii) of Corollary 1, the negentropy of an extended skew-gaussian random vector is always larger than the negentropy of a skew-gaussian random vector. Therefore, we obtain the following relationship among the Shannon entropies of gaussian ($f_0(\mathbf{z})$), skew-gaussian ($g(\mathbf{y})$), and extended skew-gaussian ($f(\mathbf{x})$) distributions: $S[f_0] \ge S[g] \ge S[f]$. Considering (1), (6) and (17); the complexity measure for extended skew-gaussian distribution is obtained.

3.4. Truncated skew-gaussian distributions

The truncated skew-gaussian pdf given by Flecher et al. [8], consider the random variable $Z \sim SN_1(\mu, \omega, \lambda)$, $\mathbf{Z} \in \Delta \subset \mathbb{R}$, and the definition given in (11) for the case $d = 1$. Flecher et al. [8] give the expressions of the higher order and weighted moments of truncated skew-gaussian distributions. We also consider the following definition based on (11) for a truncated skew-gaussian random variable $W \in [a, b] \subset \mathbb{R}$, denoted by $W \sim TSN(\mu, J, \lambda)$, and with density

$$g(w) = \frac{f(w)}{[F(w)]_a^b}, \quad a < w \le b, \tag{18}$$

where $f(z)$ is defined in (11) for $d = 1$ with $\mathbf{J} = J$, $\boldsymbol{\eta} = \lambda$; and $F(z)$ is the cumulative density function of Z with

$$[F(w)]_a^b = F(b) - F(a) = \int_a^b f(u)\mathrm{d}u.$$

The following remark allows the computation of $[F(w)]_a^b$ in terms of the gaussian cumulative density function and a bivariate integral term.

Remark 1. Let $Z \sim SN_1(\mu, J, \lambda)$, Owen [27] and Azzalini [1] give the expressions to compute $F(z)$ as follows

$$F(z) = 2\int_z^{-\infty} \int_{-\infty}^{\lambda s} \phi(s)\phi(t)\, \mathrm{d}t\, \mathrm{d}s = \Phi_1(z) - 2\int_z^{\infty} \int_0^{\lambda s} \phi(s)\phi(t)\, \mathrm{d}t\, \mathrm{d}s. \tag{19}$$

Then, by replacing (19) in $[F(w)]_a^b$ we obtain

$$[F(w)]_a^b = \Phi_1(b) - \Phi_1(a) - 2\int_a^b \int_0^{\lambda s} \phi(s)\phi(t)\mathrm{d}t\mathrm{d}s.$$

Proposition 3. *Let Z, W be a $SN_1(\mu, J, \lambda)$ and $TSN_1(\mu, J, \lambda)$, respectively, $\lambda \neq 0$. Then:*

$$\int_a^b [g(w)]^\alpha \mathrm{d}w = 2\psi_{\alpha,1}(J)\, \Phi_{\alpha+1}(\mathbf{0}; \mathbf{0}, \tilde{\mathbf{J}}) \frac{[H(v)]_{a_0}^{b_0}}{([F(z)]_a^b)^\alpha}, \tag{20}$$

where $\psi_{\alpha,1}(J)$ is defined as in Proposition 1 with $d = 1$ and $\mathbf{J} = J$; $\tilde{\mathbf{J}} = \mathbf{I}_{\alpha+1} + \tilde{\lambda}^2\tilde{\mathbf{D}}^\top\tilde{\mathbf{D}}$, $\tilde{\lambda}^2 = \omega\lambda^2/\alpha$, $\tilde{\mathbf{D}} = (\mathbf{1}_\alpha, \tilde{\lambda})^\top$ and $V \sim CSN_{1,2}(0, \tilde{\lambda}^2, \tilde{\mathbf{B}}, \mathbf{0}, \mathbf{I}_2)$ with cumulative density function $H(v)$, $\tilde{\mathbf{B}} = (1, \tilde{\lambda})^\top$, $a_0 = \lambda(a-\mu)/\omega$ and $b_0 = \lambda(b-\mu)/\omega$.

Remark 2. By Lemma 2.2.1 of Genton [13], $H(v)$ is easily computable by a tri-variate gaussian cumulative density function as

$$H(v) = \frac{\Phi_3\left[\begin{pmatrix} v \\ \mathbf{0}\end{pmatrix}; \begin{pmatrix} 0 \\ \mathbf{0}\end{pmatrix}, \begin{pmatrix} \tilde{\lambda}^2 & -\tilde{\lambda}^2\tilde{\mathbf{B}} \\ -\tilde{\lambda}^2\tilde{\mathbf{B}}^\top & \mathbf{I}_2 + \tilde{\lambda}^2\tilde{\mathbf{B}}^\top\tilde{\mathbf{B}}\end{pmatrix}\right]}{\Phi_2(\mathbf{0}; \mathbf{0}, \mathbf{I}_2 + \tilde{\lambda}^2\tilde{\mathbf{B}}^\top\tilde{\mathbf{B}})},$$

where $\tilde{\lambda}$ and $\tilde{\mathbf{B}}$ are defined as in Proposition 3.

Considering (1), (6) and (20); the complexity measure for extended skew-gaussian distribution is obtained.

4. Conclusions

In this paper, we have presented some solutions to compute the Rényi entropy with discrete α-order and for a wide range of asymmetric distributions. Specifically, we find a closed expression for skew-gaussian, extended skew-gaussian, and truncated skew-gaussian distributions. Finally, additional inequalities for skew-gaussian and extended skew-gaussian entropies were reported.

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Appendix

Proof of Proposition 1. To compute the integral $\int [f(\mathbf{z})]^\alpha \mathrm{d}\mathbf{z}$, we use the change of variables $\mathbf{J}_\alpha = \alpha^{-1}\mathbf{J}$ and $\mathbf{Z}_0 = \mathbf{J}_\alpha^{-1/2}(\mathbf{Z} - \mu)$, $\mathbf{Z}_0 \sim SN_d(\mathbf{0}, \mathbf{I}_d, \tilde{\eta})$, $\tilde{\eta} = \mathbf{J}_\alpha^{1/2}\eta$. We shall use the fact that $|\mathbf{J}_\alpha| = \alpha^{-d}|\mathbf{J}|$ for d-dimensional matrices [28]. Then, according to Lemma 2 of Arellano-Valle et al. [10], the integral $\int [f(\mathbf{z})]^\alpha \mathrm{d}\mathbf{z}$ should be rewritten in terms of an expected value with respect to a standardized gaussian density as

$$\begin{aligned}\int [f(\mathbf{z})]^\alpha \mathrm{d}\mathbf{z} &= \frac{2^\alpha}{|\mathbf{J}|^{\alpha/2}} |\mathbf{J}_\alpha|^{1/2} (2\pi)^{(1-\alpha)d/2} \langle [\Phi_1(\tilde{\eta}^\top \mathbf{Z}_0)]^\alpha \rangle \\ &= \frac{2^\alpha}{\alpha^{d/2}} (2\pi)^{(1-\alpha)d/2} |\mathbf{J}|^{(1-\alpha)/2} \langle [\Phi_1(W)]^\alpha \rangle\end{aligned}$$

where $W \sim SN_1(\mathbf{0}, \|\tilde{\eta}\|^2, \|\tilde{\eta}\|)$ with $\|\tilde{\eta}\| = \tilde{\eta}^\top\tilde{\eta}$ [9,10], i.e., the expected value $\langle [\Phi_1(\tilde{\eta}^\top \mathbf{Z}_0)]^\alpha \rangle$ is reduced from d dimensions to one dimension [10,14]. By Lemma 1 and setting $\mu = \mathbf{0}$, $\mathbf{J} = \|\tilde{\eta}\|^2$, $\mathbf{D} = \|\tilde{\eta}\|$, $r = \alpha$, $\mathbf{A} = s = h(w) = 1$; we obtain $\tilde{\mathbf{A}} = \mathbf{I}_{\alpha+1}$ and $\tilde{\mathbf{D}} = (\mathbf{1}_\alpha, \|\tilde{\eta}\|)^\top$. Therefore, the expected value of the integral is reduced to

$$\langle [\Phi_1(W)]^\alpha \rangle = \frac{\Phi_{\alpha+1}(\mathbf{0}; \mathbf{0}, \mathbf{I}_{\alpha+1} + \|\tilde{\eta}\|^2\tilde{\mathbf{D}}^\top\tilde{\mathbf{D}})}{\Phi_1(0; 0, 1 + \|\tilde{\eta}\|^4)}. \quad \square$$

Proof of Corollary 1. (i) Follows from (3) and Proposition 1.

(ii) See Proposition 2 of Arellano-Valle et al. [10].

(iii) Right side: see Ref. [9]. Left side: consider the nonsymmetrical entropy of Liu [29] given by

$$S(\mathbf{u}) = -\int f(\mathbf{u}) \ln[\beta(\mathbf{u}) f(\mathbf{u})]\, \mathrm{d}\mathbf{u},$$

where $f(\mathbf{u})$ is the probability density function of a gaussian variable $\mathbf{u}$. By choosing $\beta(\mathbf{u}) = 2\Phi_1(\eta^\top \mathbf{J}^{-1/2}(\mathbf{u} - \mu))$, $\mathbf{u} = \mathbf{Z}_N$, it follows that $\langle \ln \beta(\mathbf{Z}_N) \rangle = \ln(2) + \Phi_1(0) = (1/2)\ln(4e)$ (see Proposition 4 of Azzalini and Dalla-Valle [2]). Then, as $\langle \ln \beta(\mathbf{Z}) \rangle \leq 2\langle \ln \beta(\mathbf{Z}_N) \rangle$, the result is obtained.

(iv) Follows from properties (i), (ii) and (1). $\square$

Proof of Proposition 2. By (14), $\phi_d(\mathbf{y}; \mu, \mathbf{J}) = |\mathbf{J}|^{-1/2}\phi_d\left(\mathbf{J}^{-1/2}(\mathbf{y} - \mu)\right)$, where $\phi_d(\mathbf{z})$ is the probability density function of $N_d(\mathbf{0}, \mathbf{I}_d)$. Then, as in (1), to compute the integral $\int [f(\mathbf{z})]^\alpha \mathrm{d}\mathbf{z}$ we use the change of variables $\mathbf{J}_\alpha = \alpha^{-1}\mathbf{J}$ and $\mathbf{Z}_0 = \mathbf{J}_\alpha^{-1/2}(\mathbf{Z} - \mu)$.

In this case, $\mathbf{Z}_0 \sim ESN_d(\mathbf{0}, \mathbf{I}_d, \tilde{\boldsymbol{\eta}}, \tau)$ with $\tilde{\boldsymbol{\eta}} = \mathbf{J}_\alpha^{1/2}\boldsymbol{\eta}$. We shall use the fact that $|\mathbf{J}_\alpha| = \alpha^{-d}|\mathbf{J}|$ for d-dimensional matrices [28]. Then, according to Lemma 2 of Arellano-Valle et al. [10], the integral $\int [f(\mathbf{z})]^\alpha d\mathbf{z}$ should be rewritten in terms of an expected value with respect to a standardized gaussian density as

$$\int [f(\mathbf{z})]^\alpha d\mathbf{z} = \frac{1}{[\Phi_1(\tau)]^\alpha} |\mathbf{J}|^{-\frac{\alpha}{2}} |\mathbf{J}_\alpha|^{1/2} (2\pi)^{(1-\alpha)\frac{d}{2}} \langle [\Phi_1(\tilde{\boldsymbol{\eta}}^\top \mathbf{z}_0 + \tilde{\tau})]^\alpha \rangle$$
$$= \frac{1}{[\Phi_1(\tau)]^\alpha} \alpha^{-d} (2\pi)^{(1-\alpha)d/2} |\mathbf{J}|^{(1-\alpha)/2} \langle [\Phi_1(W)]^\alpha \rangle$$

where $W = \tilde{\boldsymbol{\eta}}^\top \mathbf{Z}_0 + \tilde{\tau} \sim ESN_1(\tilde{\tau}, \|\tilde{\boldsymbol{\eta}}\|^2, \|\tilde{\boldsymbol{\eta}}\|, \tau)$ with $\|\tilde{\boldsymbol{\eta}}\| = \tilde{\boldsymbol{\eta}}^\top \tilde{\boldsymbol{\eta}}$ [9,10], i.e., the expected value $\langle [\Phi_1(\tilde{\boldsymbol{\eta}}^\top \mathbf{z}_0 + \tilde{\tau})]^\alpha \rangle$ is reduced from d dimensions to one dimension [10,14]. □

Proof of Corollary 2. (i) From Proposition 2, we obtain directly

$$R_\alpha[f] = \frac{1}{1-\alpha} \left(\ln[\psi_{\alpha,d}(\mathbf{J})] - \alpha \ln[2\Phi_1(\tau)] + \ln[\langle [\Phi_1(W)]^\alpha \rangle] \right),$$
$$= R_\alpha[f_0] + \frac{\alpha}{1-\alpha} \ln\left[\frac{1}{\Phi_1(\tau)}\right] + \frac{1}{1-\alpha} \ln[\langle [\Phi_1(W)]^\alpha \rangle].$$

(ii) Considering Jensen's inequality, we obtain $\langle [\Phi_1(W)]^\alpha \rangle \geq [\Phi_1(\langle W \rangle)]^\alpha$. Then, (ii) is straightforward from (15).

(iii) By (1), it follows that

$$S[f] = -\left\langle \ln\left[\phi_d(\mathbf{Z}_0) \frac{\Phi_1(\tilde{\boldsymbol{\eta}}^\top \mathbf{Z}_0 + \tilde{\tau})}{\Phi_1(\tau)} \right] \right\rangle = S[f_0] - \left\langle \ln\left[\frac{\Phi_1(W)}{\Phi_1(\tau)} \right] \right\rangle,$$

where, as in Proposition 2, $\mathbf{Z}_0 = \mathbf{J}^{-1/2}(\mathbf{Z} - \mu) \sim ESN_d(\mathbf{0}, \mathbf{I}_d, \tilde{\boldsymbol{\eta}}, \tau)$ and $W = \tilde{\boldsymbol{\eta}}^\top \mathbf{Z}_0 + \tilde{\tau} \sim ESN_1(\tilde{\tau}, \|\tilde{\boldsymbol{\eta}}\|^2, \|\tilde{\boldsymbol{\eta}}\|, \tau)$.

(iv) Right side: by Cover and Thomas [21], for any density $g(\mathbf{x})$ of a random vector $\mathbf{x} \in \Delta \subset \mathbb{R}^d$ (not necessary gaussian) with zero mean and variance $\mathbf{J} = \langle \mathbf{X}\mathbf{X}^\top \rangle$, the Shannon entropy of $\mathbf{x}$ is maximized under gaussianity as $S[g] \leq (1/2) \ln[(2\pi e)^d |\mathbf{J}|]$. Then, the result is obtained from (16). Left side: as in Corollary 1(iii), by choosing $\beta(\mathbf{u}) = \Phi_1(\boldsymbol{\eta}^\top \mathbf{J}^{-1/2}(\mathbf{u} - \mu) + \tilde{\tau})/\Phi_1(\tau)$ in the nonsymmetrical entropy, it follows that

$$\langle \ln \beta(\mathbf{Z}_N) \rangle = \Phi_1\left(\frac{\tilde{\tau}}{\sqrt{1 + \|\boldsymbol{\eta}\|}} \right) - \ln[\Phi_1(\tau)]$$

(see Proposition 4 of Azzalini and Dalla-Valle [2]). Then, as $\langle \ln \beta(\mathbf{Z}) \rangle \leq \langle \ln \beta(\mathbf{Z}_N) \rangle / \Phi_1(\tau)$, the result is obtained.

(v) Follows from properties (i), (iii) and (1). □

Proof of Proposition 3. By (18), it follows that

$$\int_a^b [g(w)]^\alpha dw = \frac{1}{([F(z)]_a^b)^\alpha} \int_a^b [f(w)]^\alpha dw$$

and, by Proposition 1, the integral $\int_a^b [f(w)]^\alpha dw$ should be rewritten in terms of an expected value as

$$\int_a^b [f(w)]^\alpha dw = \psi_{\alpha,1}(J) \langle [\Phi_1(u)]^\alpha | a_0 < u \leq b_0 \rangle,$$

where $U \sim SN_1(0, \tilde{\lambda}^2, \tilde{\lambda})$, $\tilde{\lambda}^2 = \omega\lambda^2/\alpha$, $a_0 = \lambda(a - \mu)/\omega$ and $b_0 = \lambda(b - \mu)/\omega$. Again, by Lemma 1 and setting $\mu = 0$, $J = \tilde{\lambda}^2$, $r = \alpha$, $d = s = \mathbf{A} = h(u) = 1$; we obtain $\tilde{\mathbf{A}} = \mathbf{I}_{\alpha+1}$, $\tilde{\mathbf{D}} = (\mathbf{1}_\alpha, \tilde{\lambda})^\top$ and $\tilde{\mathbf{J}} = \mathbf{I}_{\alpha+1} + \tilde{\lambda}^2 \tilde{\mathbf{D}}^\top \tilde{\mathbf{D}}$. Then, the expected value is

$$\langle [\Phi_1(u)]^\alpha | a_0 < u \leq b_0 \rangle = 2\Phi_{\alpha+1}(\mathbf{0}; \mathbf{0}, \tilde{\mathbf{J}})[H(v)]_{a_0}^{b_0},$$

where $H(v)$ is the cumulative density function of a closed skew-gaussian variable $V \sim CSN_{1,2}(0, \tilde{\lambda}^2, \tilde{\mathbf{B}}, \mathbf{0}, \mathbf{I}_2)$ with $\tilde{\mathbf{B}} = (1, \tilde{\lambda})^\top$ (see Proposition 3 of Flecher et al. [8]). □

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