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Assignment 1
DS203: Programming for Data Sciences

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1

Let us define the following events,

$$\begin{aligned} A &\equiv \text{Product is manufactured in company A} \\ B &\equiv \text{Product is manufactured in company B} \\ E &\equiv \text{Product is defective} \end{aligned}$$

Given: $P(A) = 0.8, P(B) = 0.2, P(E|A) = 0.3, P(E|B) = 0.1$

1.1

From Law of Total Probability,

$$\begin{aligned} & P(E) = P(A)P(E|A) + P(B)P(E|B) \\ \implies & P(E) = 0.8 \times 0.3 + 0.2 \times 0.1 \\ & \quad\quad = 0.26 \end{aligned}$$

The probability that the sample is defective is **0.26**.

1.2

Required probability is $P(A|E)$
From Bayes' Formula,

$$P(A|E) = \frac{P(E|A)P(A)}{P(A)} = \frac{0.3 \times 0.8}{0.26} = 0.923$$

The probability that a defective sample in the market is manufactured at company A is **0.923**.

2

Given probabilities are *P(server is working)* = 0.8, *P(access attempt succeeds | server is working)* = 0.9
Thus,

$$\begin{aligned} \textit{P(server is not working)} &= 1 - \textit{P(server is working)} = 1 - 0.8 = 0.2 \\ \textit{P(access attempt fails | server is working)} &= 1 - \textit{P(access attempt succeeds | server is working)} = 1 - 0.9 = 0.1 \end{aligned}$$

2.1

$$\begin{aligned} P(\textit{first access attempt fails}) &= P(\textit{access attempt fails} \mid \textit{server is working})\ P(\textit{server is working}) + P(\textit{server is not working}) \\ &= 0.1 \times 0.8 + 0.2 = 0.28 \end{aligned}$$

2.2

$$\begin{aligned} P(\textit{server is working} \mid \textit{first attempt fails }) &= \frac{P(\textit{first attempt fails} \mid \textit{server working})\ P(\textit{server working})}{P(\textit{first attempt fails})} \\ &= \frac{0.1 \times 0.8}{0.28} = 0.286 \end{aligned}$$

2.3

$$\begin{aligned} P(\textit{second attempt fails} \mid \textit{first attempt fails }) &= \frac{P(\textit{first two attempts fail})}{P(\textit{first attempt fails})} \\ &= \frac{0.2 + 0.8 \times 0.1 \times 0.1}{0.28} = 0.783 \end{aligned}$$

(Each attempt is independent, therefore, we can multiply 0.1 twice for both failures)

2.4

$$\begin{aligned} P(\textit{server is working} \mid \textit{first two attempts fail }) &= \frac{P(\textit{server is working and first two attempts fail})}{P(\textit{first two attempts fail})} \\ &= \frac{0.8 \times 0.1 \times 0.1}{0.783} = 0.039 \end{aligned}$$

3

Let A be the event that at least one die shows six. Let B be the event where the dice show different faces.

$$P(B) = 1 - \frac{6}{36} = \frac{5}{6}$$

(Complement of same faces which happens 6 times out of 36 possibilities).

$$P(A \cap B) = \frac{10}{36}$$

(5 possibilities of only one six for each dice).

Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \mathbf{1/3}$$

4

$$
\begin{aligned}
\textit{P(man} \mid \textit{colorblind)} &= \frac{\textit{P(colorblind} \mid \textit{man)}}{\textit{P(colorblind} \mid \textit{man) P(colorblind} \mid \textit{woman)}} \\
&= \frac{0.05}{0.05 + 0.01} = \mathbf{0.83}
\end{aligned}
$$

5

5 (a)

$P(E)$ is independent of itself, i.e., $P(E \cap E) = P(E) \times P(E) \implies P(E) = P(E)^2 \implies P(E) = 0$ or 1.

5 (b)

$P(A) = 0.3$ and $P(B) = 0.4$
If A and B are independent, $P(A \cap B) = P(A)P(B) = 0.12$
Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.12 = \mathbf{0.58}$

If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B) = 0.3 + 0.4 = \mathbf{0.7}$

5 (c)

$P(A) = 0.6$ and $P(B) = 0.8$
If A and B are independent, $P(A \cap B) = P(A)P(B) = 0.48$
Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.8 - 0.48 = 0.92$, so they can be independent.

If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B) = 0.6 + 0.8 = 1.4$ which is greater than one and contradictory. So, they cannot be mutually exclusive.

6

6.1

It is a valid CDF.

$$
\begin{aligned}
P(X^2 > 5) &= P(X > \sqrt{5} \text{ and } X < \sqrt{5}) = 1 - P(-\sqrt{5} \leq X \leq \sqrt{5}) \\
&= 1 - (F(\sqrt{5}) - F(-\sqrt{5})) \\
&= 1 - ((1 - e^{-5)/4)-e^{-(-5)}}/4) \\
&= e^{-5}/2
\end{aligned}
$$

6.2

Not a valid CDF. **Reason:** for $0 \leq x < 3$, it is a decreasing function

6.3

It is a valid CDF.

$$
\begin{aligned}
P(X^2 > 5) &= P(X > \sqrt{5} \text{ and } X < \sqrt{5}) = 1 - P(-\sqrt{5} \leq X \leq \sqrt{5}) \\
&= 1 - (F(\sqrt{5}) - F(-\sqrt{5})) \\
&= 1 - (0.5 + \sqrt{5}/20) - 0) \\
&= 0.5 - \sqrt{5}/20
\end{aligned}
$$

7

From the graph, we can write the CDF as,

$$
F(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \leq x \leq 1 \\ x/2 & 1 \leq x \leq 2 \\ 1 & x \geq 2 \end{cases}
$$

7.1

$$
\begin{aligned}
P(X \leq 0.8) &= F(0.8) \\
&= \mathbf{0.5}
\end{aligned}
$$

7.2

Let's use the following property (See this.):

$$\frac{1}{k}E[X^k] = -\int_{-\infty}^{0} x^{k-1}F(x)dx + \int_{0}^{\infty} x^{k-1}(1-F(x))dx$$

for k=1, we have,

$$\begin{aligned}
E[X] &= -\int_{-\infty}^{0} F(x)dx + \int_{0}^{\infty}(1-F(x))dx \\
&= 0 + \int_{0}^{1} 0.5dx + \int_{1}^{2}(1-x/2)dx + 0 \\
&= 0.5 + [x - x^2/4]_1^2 \\
&= 0.5 + 1 - 0.75 \\
&= \mathbf{0.75}
\end{aligned}$$

7.3

for k=2, we have,

$$\begin{aligned}
\frac{1}{2}E[X^2] &= -\int_{-\infty}^{0} xF(x)dx + \int_{0}^{\infty} x(1-F(x))dx \\
&= 0 + \int_{0}^{1}(x/2)dx + \int_{1}^{2}(x - x^2/2)dx + 0 \\
&= 0.25 + [x^2/2 - x^3/6]_1^2 \\
&= 0.25 + 0.667 - 0.333 \\
&= 0.583 \\
\implies E[X^2] &= 1.167
\end{aligned}$$

Finally,

$$Var[X] = E[X^2] - E[X]^2 = 1.167 - 0.5625 = \mathbf{0.605}$$

8

Since, $f(x)$ is a p.d.f, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\begin{aligned}
\implies \int_{0}^{\infty} ce^{-2x}dx &= 1 \\
\implies c[e^{-2x}/(-2)]_0^{\infty} &= 1 \\
\implies c[0 - (-1/2)] &= 1 \\
\implies c &= \mathbf{2}
\end{aligned}$$

Now,

$$\begin{aligned}
P(X > 2) &= \int_2^{\infty} f(x)dx \\
&= \int_2^{\infty} 2e^{-2x}dx \\
&= [-e^{-2x}]_2^{\infty} \\
&= 0 + e^{-4} \\
&= \mathbf{0.018}
\end{aligned}$$

9

Event	X
TTT	0
TTH	1
THT	1
HTT	1
THH	2
HTH	2
HHT	2
HHH	3

X	pmf		
0	0.3^3	$=$	0.027
1	$(3)(0.7)(0.3^2)$	$=$	0.189
2	$(3)(0.7^2)(0.3)$	$=$	0.441
3	0.7^3	$=$	0.343

10

$$\begin{aligned}
P(X \geq 0.4 | X \leq 0.8) &= \frac{P(0.4 \leq X \leq 0.8)}{P(X \leq 0.8)} \\
&= \frac{\int_{0.4}^{0.8} f(x)dx}{\int_{-\infty}^{0.8} f(x)dx} \\
&= \frac{\int_{0.4}^{0.8} 2xdx}{\int_{0}^{0.8} 2xdx} \\
&= \frac{[x^2]_{0.4}^{0.8}}{[x^2]_{0}^{0.8}} \\
&= \frac{0.64 - 0.16}{0.64} \\
&= \mathbf{\frac{3}{4}}
\end{aligned}$$

11

$F(x) = 1 - e^{-\lambda x}$ when $X \sim exp(\lambda)$

$$\begin{aligned} P(X \geq a+b | X \geq a) &= \frac{1 - P(X \leq a+b)}{1 - P(X \leq a)} \\ &= \frac{1 - F(a+b)}{1 - F(a)} \\ &= \frac{1 - (1 - e^{-\lambda(a+b)})}{1 - (1 - e^{-\lambda a})} \\ &= e^{\lambda a - \lambda(a+b)} \\ &= \mathbf{e^{-\lambda(b)}} \end{aligned}$$

12

The only outcome in the original sample space for which event E occurs is **HHHHH**

$$P(I_E = 1) = P(E) = \mathbf{1/2^5}$$

13

$$P(X = x_i) = F(x_i) - lim_{x \to x_i^-} F(x_i)$$

X	PMF
0	$\frac{1}{2} - 0 = 0.5$
1	$1 - \frac{1}{2} = 0.5$

14

Since, there is replacement of balls, every pick is independent of each other.

$$P(white) = 3/6 = 1/2, P(black) = 3/6 = 1/2$$

So, probability of drawing exactly two white balls out of fours =

$$\binom{4}{2}(1/2)^2(1/2)^2 = \mathbf{3/8}$$

15

Every flip is independent of the other. Thus, getting x heads out of n flips $\sim b(n, p)$.
$P(X = n)$ can also be interpreted as getting $r - 1$ heads in the first $n - 1$ flips and getting the rth head in the the nth flip, i.e.,

$$\begin{aligned} P(X = n) &= P(\textit{r-1 heads out of n-1 flips})P(head) \\ &= \binom{n-1}{r-1} p^{r-1}(1-p)^{[(n-1)-(r-1)]}(p) \\ &= \binom{n-1}{r-1} p^{r}(1-p)^{n-r} \end{aligned}$$

16

Let X be a random variable (the number of errors on the page). $X \sim exp(1)$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1} = 0.63$$

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