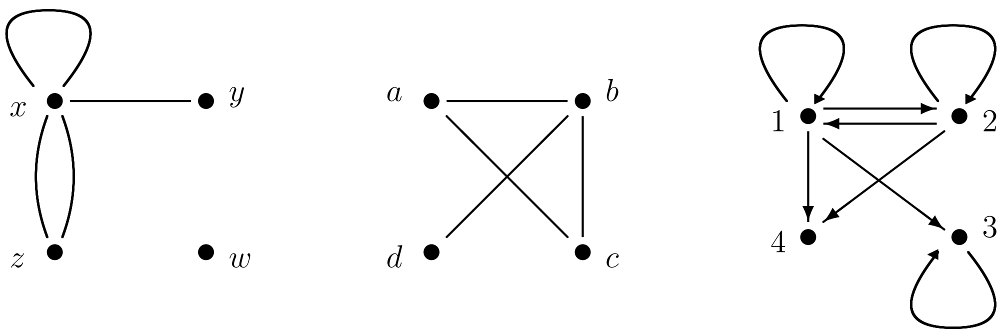
Graphs

Graphs are drawings with dots and (not necessarily straight) lines or arrows.



The dots are called **vertices** (or **nodes**).

The lines or arrows are called **edges**.

Different kinds of graphs

Type	Edges	Multiple edges	Loop edges
(simple) graph	undirected	no	no
multigraph	undirected	yes	yes
directed graph	directed	no	yes
...	...	...	...

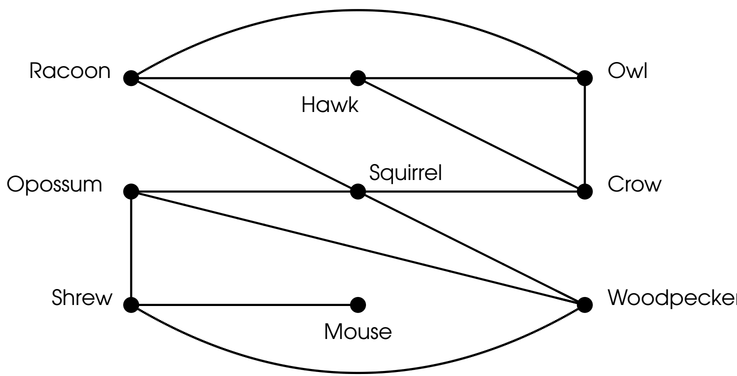
Because graphs have applications in a variety of disciplines, many different terminologies of graph theory have been introduced. You may find different ones in different books, areas, etc.

Example 1: Niche overlap graphs in ecology

Competitions between species in an ecosystem can be modelled using

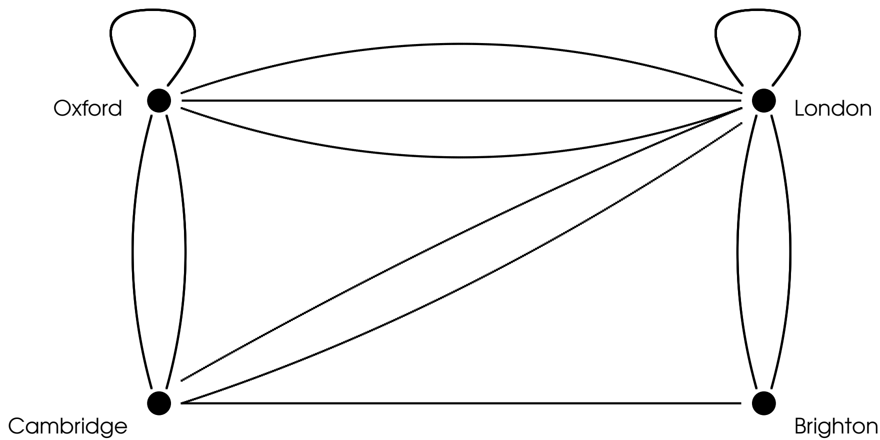
a **niche overlap graph**:

Each species is represented by a vertex. An edge connects two vertices if the two species represented by these vertices compete (that is, some of the food resources they use are the same).



$\rightsquigarrow$ simple graph

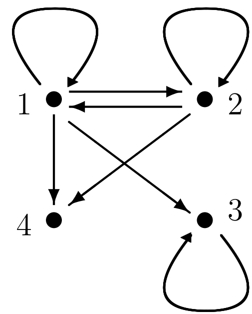
Example 2: Road networks



$\rightsquigarrow$ multigraph

Example 3: Representing binary relations

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,4),(3,3)\}.$$



$\rightsquigarrow$ directed graph

Undirected graphs: basic terminology

If there is an edge e between vertices u and v, we say that

- u and v are **adjacent**, and
- e is **incident with** u and v.

The **degree** of a vertex is the number of edges incident with it.

- A vertex of degree zero is called **isolated**.

  So an isolated vertex is not adjacent to any vertex.

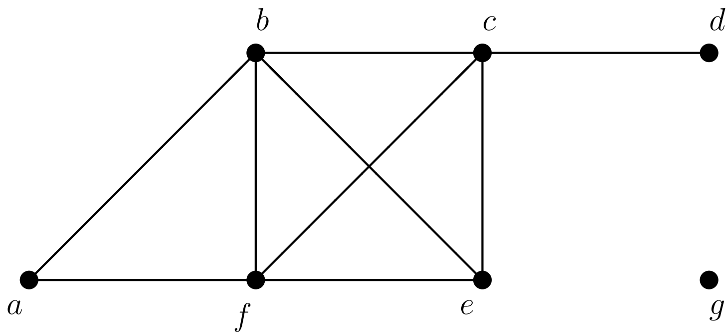
- A vertex of degree one is called **pendant**.

  So a pendant vertex is adjacent to exactly one other vertex.

Handshaking theorem:

$$\text{number of edges} = \frac{\text{sum of the degrees of vertices}}{2}$$

Degrees of vertices: example 1



- degree$(a) = 2$,
- degree$(b) =$ degree$(c) =$ degree$(f) = 4$,
- degree$(e) = 3$,
- degree$(d) = 1$, so d is pendant,
- degree$(g) = 0$, so g is isolated.

Directed graphs: basic terminology

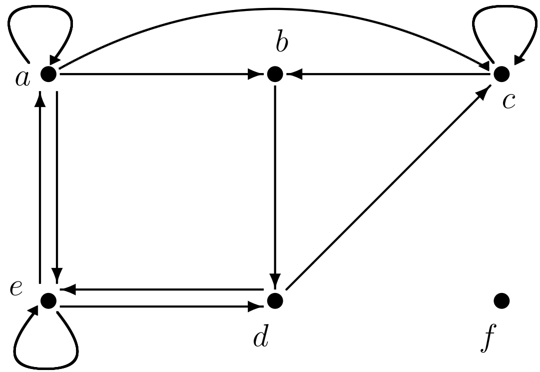
If there is an edge e going from vertex u to v, we say that

- u is **adjacent to** v,
- u is the **initial** or **start vertex** of e, and
- v is the **terminal** or **end vertex** of e.

The **in-degree** of a vertex v is the number of edges with v as their terminal vertex. The **out-degree** of a vertex v is the number of edges with v as their initial vertex. (A loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

$$\begin{aligned} \text{number of edges} &= \text{sum of the in-degrees of vertices} \\ &= \text{sum of the out-degrees of vertices.} \end{aligned}$$

Degrees of vertices: example 2



- in-degree(a) = in-degree(b) = in-degree$(d) = 2$,

  in-degree(c) = in-degree$(e) = 3$,

  in-degree$(f) = 0$,

- out-degree$(a) = 4$, out-degree$(b) = 1$,

  out-degree(c) = out-degree$(d) = 2$, out-degree$(e) = 3$,

  out-degree$(f) = 0$.

Representing graphs: adjacency matrix

- List the vertices in some order horizontally from left to right.
- Then, using the same order, list them vertically from top to bottom.
- The entry in the i^{th} row and the j^{th} column is the

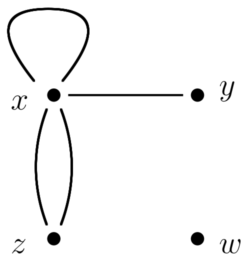
  <u>number of edges going from vertex i to vertex j.</u>

If the graph is undirected, then

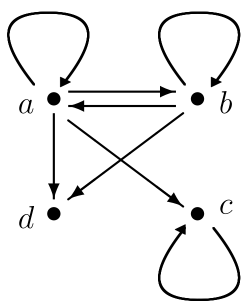
the number in the i^{th} row and j^{th} column

$=$ the number in the j^{th} row and i^{th} column.

Adjacency matrices: examples



	x	y	w	z
x	1	1	0	2
y	1	0	0	0
w	0	0	0	0
z	2	0	0	0

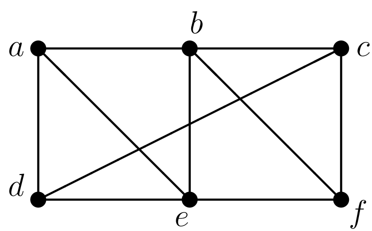


	a	b	c	d
a	1	1	1	1
b	1	1	0	1
c	0	0	1	0
d	0	0	0	0

Paths in simple graphs

- A **path** is a sequence of vertices travelling along edges.
- The **length** of a path is the number of edges in it.
- A path is called **simple** if it does not contain the same *edge* twice.
- A **Hamiltonian path** is a *simple* path passing through *every vertex exactly once*.

FOR EXAMPLE:



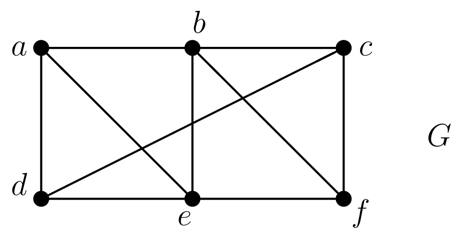
G

- (a, b, c, f, b, e) is a simple path in G of length 5.
- (d, c, a, e) is not a path in G.
- (a, b, e, d, a, b) is a path in G of length 5, but it is not simple.
- (d, a, e, b, f, c) is a Hamiltonian path in G.
- (a, b, c, f, b, e, d) is a simple path, but not a Hamiltonian path in G.

Cycles in simple graphs

- A **cycle** is a path beginning and ending with the same vertex.
- The **length** of a cycle is the number of edges in it.
- A cycle is called **simple** if it does not contain the same *edge* twice.
- A **Hamiltonian cycle** is a *simple* cycle passing through *every vertex exactly once.*

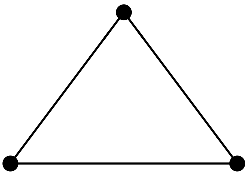
FOR EXAMPLE:



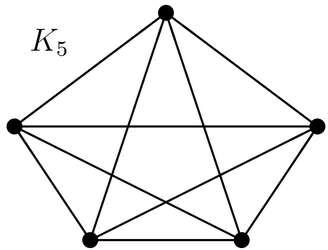
- (a, d, c, b, a) is a simple cycle in G of length 4.
- (b, e, d, a, e, b) is a cycle in G of length 5, but it is not simple.
- (c, f, e, d, a, b, c) is a Hamiltonian cycle in G.
- (e, d, c, f, e, b, a, e) is a simple cycle in G, but not Hamiltonian.

Special simple graphs

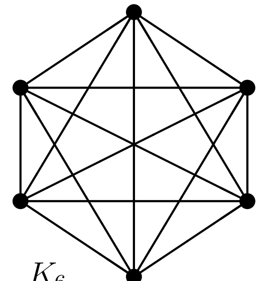
- The **complete graph on n vertices** (or n-**clique**), denoted by $\boxed{K_n}$, is the simple graph that contains an edge between each pair of distinct vertices.



K_3

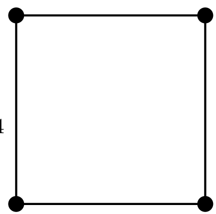


K_5

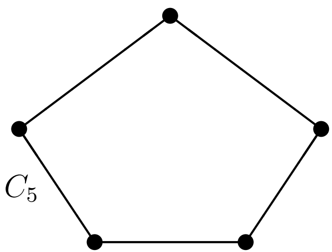


K_6

- The n-**cycle** is denoted by $\boxed{C_n}$, for $n \geq 4$.



C_4



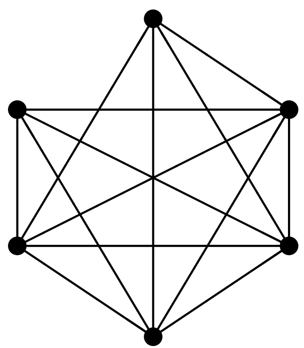
C_5

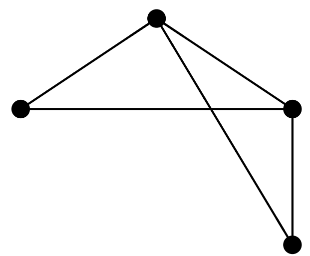
Subgraphs of graphs

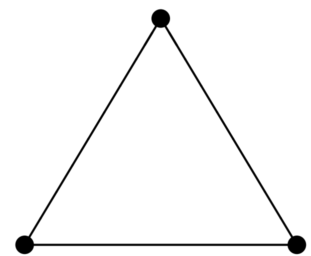
When edges and vertices are removed from a graph, without removing end-points of any remaining edges, a smaller graph is obtained.

Such a graph is called a **subgraph** of the original graph.

FOR EXAMPLE: Each of the following 3 graphs





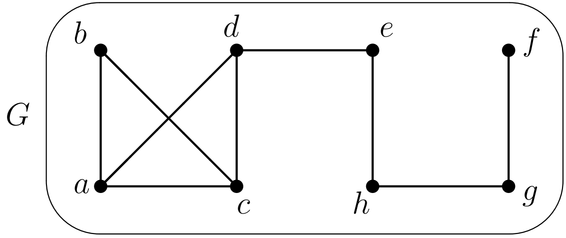


is a subgraph of K_6.

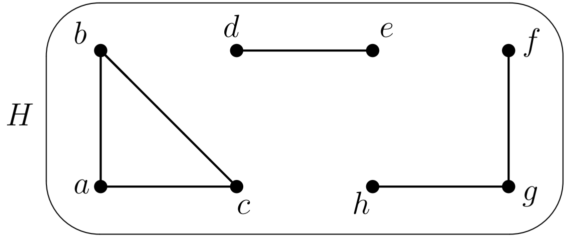
Connected graphs

A simple graph is called **connected** if there is a path between every pair of distinct vertices.

FOR EXAMPLE:



is connected

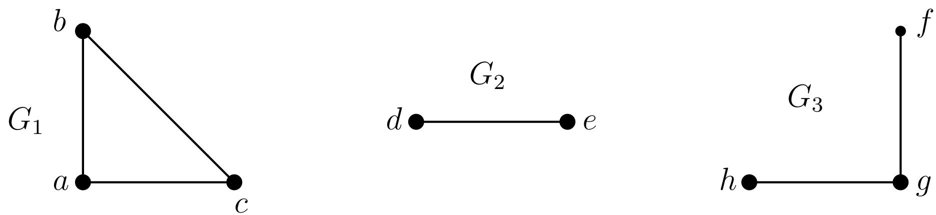


is not connected

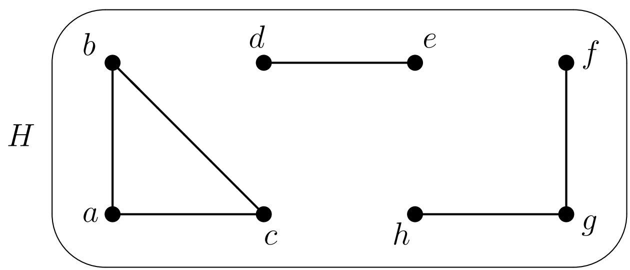
Connected components of graphs

A <u>**connected component**</u> of a graph is a maximal connected subgraph.

- If a graph is connected, then it has only 1 connected component, <u>*itself.*</u>
- But if it is not connected, it can have more:



are the 3 connected components of

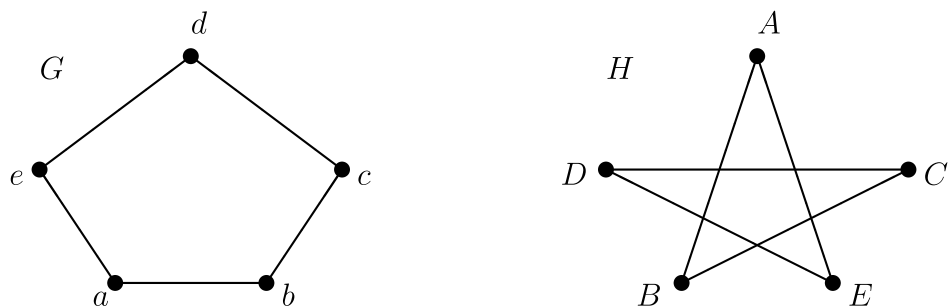


Isomorphism of graphs: an example

The following instructions were told to two persons:

> "Draw and label five vertices with a, b, c, d, and e.
> Connect a and b, b and c, c and d, d and e, and a and e."

They drew the graphs:

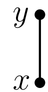


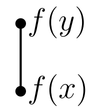
Surely, these drawings describe the same situation, though the graphs G and H appear dissimilar.

Isomorphism of graphs

Graphs G and H are **isomorphic** if there is an **isomorphism** between them:
A function f from the vertices of G to the vertices of H such that

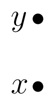
- f is a bijection (one-to-one and onto)
- f 'takes' edges to edges:

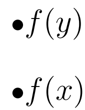
  for all vertices x, y in G, if 

then 

in H

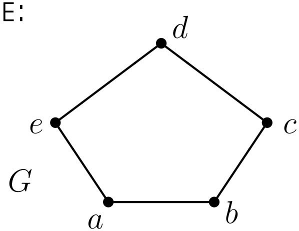
- f 'takes' non-edges to non-edges:

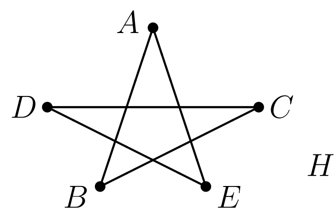
  for all vertices x, y in G, if 

then 

in H

FOR EXAMPLE:



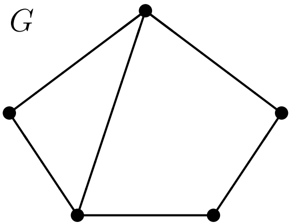


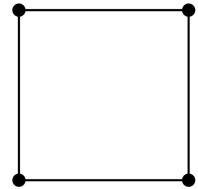
The function f defined by taking

$$f(a) = A, \qquad f(b) = B, \qquad f(c) = C, \qquad f(d) = D, \qquad f(e) = E$$

is an isomorphism, showing that graphs G and H are isomorphic.

Isomorphic or not — how can we decide?





TASK: Determine whether two graphs G and H are isomorphic or not.

isomorphic = there is an isomorphism
not isomorphic = there is no isomorphism

- We can try **all possible functions** from G to H, and check whether any of them is an isomorphism.
- But this might take a lot of time: there are $4^5 = 1024$ possible functions even for this simple example (see slide 80). So for larger graphs it is hopeless.

Is there some quicker way?

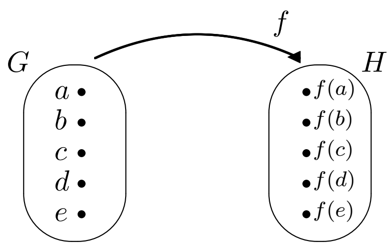
Isomorphism of graphs: invariants

A property $\mathcal{P}$ of graphs is called an **invariant** if it is 'preserved under isomorphisms': If G and H are isomorphic graphs, and G has property $\mathcal{P}$, then H has property $\mathcal{P}$ as well.

FOR EXAMPLE: "*Having* 5 *vertices*" is an invariant:
If G and H are isomorphic graphs, and G has 5 vertices,
then H has 5 vertices as well.

WHY? If G and H are isomorphic, then there is an isomorphism f between them.

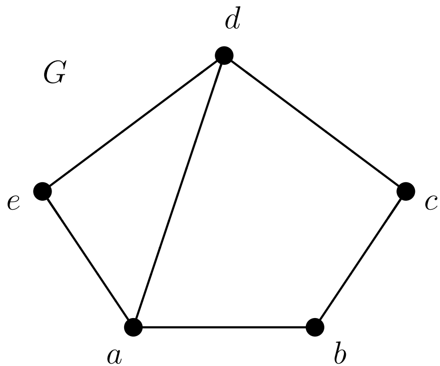
- f is a *bijection* from the vertices of G (domain) to the vertices of H (codomain).

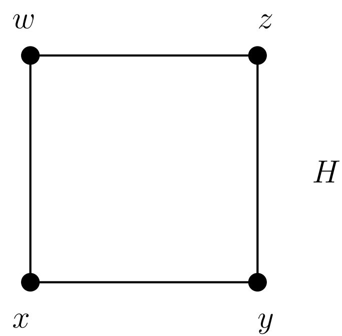


- As f is *one-to-one,* H has at least 5 vertices.
- And as f is *onto,* H has at most 5 vertices.

Exercise 6.1

Determine whether G and H are isomorphic or not:





SOLUTION: We've just seen that the property "having 5 vertices" is an invariant. This property holds for G, but not for H. Therefore, G and H are not isomorphic.

Some more examples of invariants

- the number of edges
- the number of vertices of each degree
- containing a triangle (K_3) as a subgraph
- containing two K_4s as disjoint subgraphs
- containing a simple cycle of length 4
- having a path of length 2 between two vertices of degree 2
- $\ldots$

Exercise 6.2

Explain why the property of "*having* 25 *edges*" is an invariant.

SOLUTION: Let G be a graph having 25 edges, and suppose that f is an isomorphism between G and another graph H.

$$G \xrightarrow{\quad f \quad} H$$

- As f takes edges to edges, every edge in G is "f-mapped" to an edge in H:

$$\begin{tikzcd} & y \arrow[d, dash] & & f(y) \arrow[d, dash] & \\ G & x \arrow[r, phantom, ""] & \arrow[r, bend left, "f"] & f(x) & H \end{tikzcd}$$

  So the number of edges in H is not less than 25.

- As f is onto and takes non-edges to non-edges, every edge in H is the "f-value" of some edge in G:

$$\begin{tikzcd} & y & & f(y) & \\ G & x & \arrow[r, bend left, "f"] & f(x) & H \end{tikzcd}$$

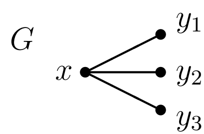
  So the number of edges in H is not more than 25.

Exercise 6.3

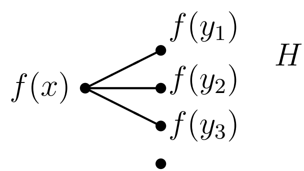
Show that if f is an isomorphism between graphs G and H, and x is a vertex in G of degree 3, then $f(x)$ in H is also of degree 3.

SOLUTION:

- If x is of degree 3, then x is connected in G to 3 distinct vertices, say, y_1, y_2 and y_3.



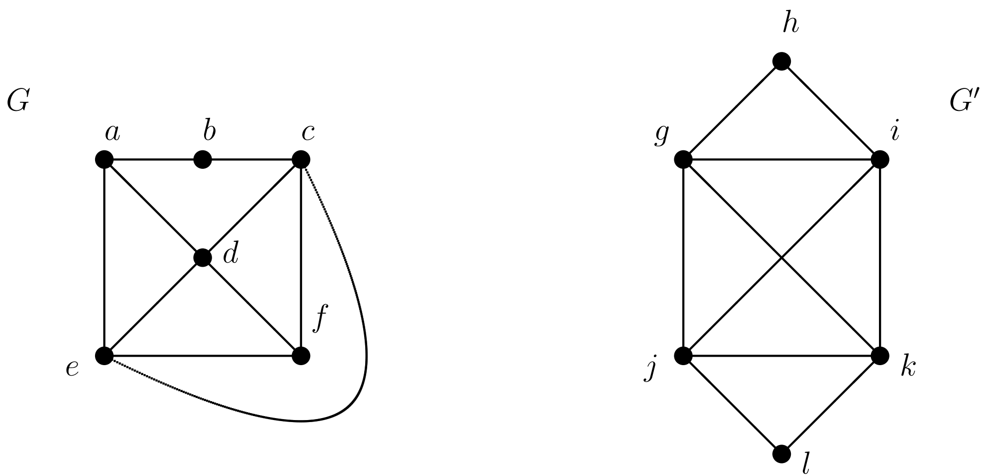
- As f is one-to-one, $f(y_1)$, $f(y_2)$ and $f(y_3)$ are 3 distinct vertices in H.
- As f takes edges to edges, there are edges connecting $f(x)$ to each of $f(y_1)$, $f(y_2)$ and $f(y_3)$.



- As f is onto and takes non-edges to non-edges, $f(x)$ cannot be connected to any other vertex in H.

Exercise 6.4

Determine whether G and G' are isomorphic or not:

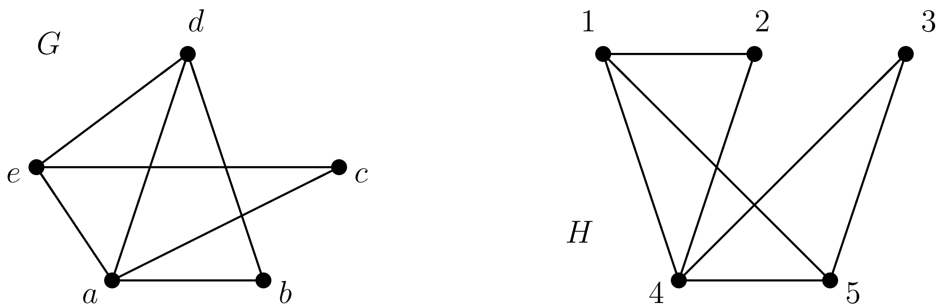


SOLUTION: Now it is a bit harder to find an invariant. Both graphs have 6 vertices and 10 edges. But we have just seen that "having a vertex of degree 3" is an invariant.

This property holds in G (say, degree$(a) = 3$), but does not hold in G', showing that G and G' are not isomorphic.

Exercise 6.5

Determine whether G and H are isomorphic or not:



SOLUTION: The function f defined by taking

$$f(a) = 4, \qquad f(b) = 2, \qquad f(c) = 3, \qquad f(d) = 1, \qquad f(e) = 5$$

is an isomorphism (because f is a bijection, takes edges to edges, and non-edges to non-edges). This shows that graphs G and H are isomorphic.

HOW TO FIND AN ISOMORPHISM? **Hint: always keep in mind Exercise 6.3 on slide 141**

Isomorphic or not — so how can we decide?

TASK: Determine whether two graphs G and H are isomorphic or not.

SOLUTION: There is no easy way. We have to try in parallel:

- To describe a bijection between the vertices of G and H that 'takes' edges to edges, non-edges to non-edges.
  $\leadsto$ If we succeed, the answer is YES.
- To find an invariant $\mathcal{P}$
  and show that G has $\mathcal{P}$ but H doesn't, or the other way round.
  $\leadsto$ If we succeed, the answer is NO.

It would be nice to have a list of easily checkable invariants that isomorphic graphs and *only* isomorphic graphs share. Then we would just have to check those.

Unfortunately, no one has yet succeeded in finding such a list of invariants, so determining whether two graphs are *not* isomorphic might require some creative thinking.