4CCS1ELA – Elementary Logic with Applications

Small Group Tutorial 3 (week 6) – Solutions

1. Rewrite the following propositional formula in (i) a logically equivalent *conjunctive normal form*, and (ii) a logically equivalent *disjunctive normal form*:

$$\big(P \to (Q \wedge R)\big) \to S.$$

SOLUTION

$$\begin{array}{rl}
\big(P \to (Q \wedge R)\big) \to S & \Longrightarrow \\
\neg\big(P \to (Q \wedge R)\big) \vee S & \Longrightarrow \\
\big(P \wedge \neg(Q \wedge R)\big) \vee S & \Longrightarrow \\
\big(P \wedge (\neg Q \vee \neg R)\big) \vee S & \Longrightarrow \\
(P \vee S) \wedge (\neg Q \vee \neg R \vee S) & (CNF).
\end{array}$$

$$\begin{array}{rl}
\big(P \to (Q \wedge R)\big) \to S & \Longrightarrow \\
\neg\big(P \to (Q \wedge R)\big) \vee S & \Longrightarrow \\
\big(P \wedge \neg(Q \wedge R)\big) \vee S & \Longrightarrow \\
\big(P \wedge (\neg Q \vee \neg R)\big) \vee S & \Longrightarrow \\
(P \wedge \neg Q) \vee (P \wedge \neg R) \vee S & (DNF).
\end{array}$$

2. Formalise the following argument in propositional logic and demonstrate its validity using natural deduction.

> *"If I graduate this semester, then I will have passed physics. If I do not study physics for 10 hours a week, then I will not pass physics. If I study physics for 10 hours a week, then I cannot play volleyball. Therefore, I will not graduate this semester if I play volleyball.*

SOLUTION

Let A denote "*I will graduate this semester*, let B denote "*I will pass the physics course*", let C denote "I will study physics for 10 hours a week, and let D denote "*I will play volleyball.*

Then, we have:

$$A \to B,\ \neg C \to \neg B,\ C \to \neg D \models D \to \neg A.$$

Suppose that the argument is invalid. Then, there is a truth value assignment I under which $I(D \to \neg A) = 0$, i.e., $I(D) = I(A) = 1$. At the same time the premises are true under I. Then, from $A \to B = 1$, we must have $I(\neg B) = 0$, but then $I(C) = 1$ and for $I(C \to \neg D) = 1$ we must have $I(D) = 0$. Thus, the assumption of invalidity is false and the argument is valid.

We can prove the validity of the argument using natural deduction in the following way.

1. $A \to B$ data
2. $\neg C \to \neg B$ data
3. $C \to \neg D$ data
4. $B \to C$ subcomputation box below

		$\underline{C}$
4.1 B	assumption	
4.2 $\neg C \to B$	subcomputation box below	
4.2.1 $\neg C$	assumption	$\underline{B}$
4.2.2 B	4.1	
4.3 C	2, 4.1 and $\neg$E	

5. $A \to \neg D$ subcomputation box below

		$\underline{\neg D}$
5.1 A	assumption	
5.2 B	1., 5.1 and $\to$E	
5.3 C	4., 5.2 and $\to$E	
5.4 $\neg D$	3., 5.3 and $\to$E	

6. $D \to \neg A$ subcomputation box below

			$\underline{\neg A}$
6.1	D	assumption	
6.2	$A \to D$	subcomputation box below	
			$\underline{D}$
	6.2.1 A	assumption	
	6.2.2 D	6.1	
6.3	$\neg A$	5, 6.2 and $\neg$I	

Notice that a subcomputation box is the actual $\to$I rule. Two patterns emerge from the proof above:

1. that implication is *transitive*:

   From (1) $A \to B$, (4) $B \to C$ and (3) $C \to \neg D$, we get that (5) $A \to \neg D$ (the proof is in the subcomputation box for 5).

2. the *contrapositive* of the implication:

   $$P \to Q \vdash \neg Q \to \neg P \quad \text{and} \quad \neg P \to \neg Q \vdash Q \to P$$

   are *valid* inferences.

   In line 2. we had $\neg C \to \neg B$. We showed in the subcomputation box for line 4. that by only using this fact we can get that $B \to C$.

   The *modus tollens* rule is based on this fact:

   $$\frac{P \to Q, \neg Q}{\neg P} \ \text{(modus tollens)}$$

3. Consider the set of natural numbers $\mathbf{N} = \{0, 1, 2, \ldots\}$ with the predicate $<$ and the function $+$ with their usual interpretation in arithmetic.

Express the following first-order sentences in English and determine which of these sentences are true.

(a) $\forall x \exists y (x < y)$

(b) $\forall y \exists x (x < y)$

(c) $\exists x \forall y (x < y)$

(d) $\forall x \forall y (x < y)$

(e) $\exists x \exists y (x < y)$

(f) $\forall x \forall y((x < y) \rightarrow \exists z(x = y + z))$

(g) $\forall x \forall y \exists z(x = y + z)$

SOLUTION

(a) Every natural number is less than some natural number. (True)

(b) For every natural number there is a lesser one. (False: witness 0)

(c) There is a natural number that is less than less than every natural number. (False: although 0 is less than every other natural number, it is not less than itself)

(d) Every natural number is less than every natural number. (False: e.g. 3 is not less than 2)

(e) Some natural number is less than some natural number. (True: e.g. 2 is less than 3)

(f) False, witnesses $x = 1$ and $y = 2$. However, it becomes true if the atom $(x < y)$ in the premise to replace with the atom $(x > y)$.

(g) False. Witnesses $x = 1$ and $y = 2$: there is no natural number z such that $2 + z = 1$.

4. Let $Country(x)$ denote "x is a country; $Plane(x, y)$ denote the fact that one can travel from country x to country y by plane; $Train(x, y)$ denote the fact that one can travel from country x to country y by train; and $Boat(x, y\ )$ denote the fact that one can travel from country x to country y by boat.

Let $france$, uk, $germany$, $ireland$ and $switzerland$ be the constants interpreted as France, UK, Germany, Ireland and Switzerland, respectively.

(a) Using the dictionary defined above, represent the following in first-order logic.

1. One can travel from France to the United Kingdom by air, by train and by boat.
2. There is at least one country that can be reached by train from the United Kingdom.
3. Any country that can be reached by plane from France can also be reached by plane from the United Kingdom.

<u>SOLUTION</u>

We may have:

1. $Plane(france, uk) \wedge Train(france, uk) \wedge Boat(france, uk)$.
2. $\exists x(Country(x) \ \wedge \ Train(uk, x))$.
3. $\forall x(Country(x) \ \wedge \ Plane(france, x) \rightarrow Plane(uk, x))$.

(b) Translate the following sentences into equivalent English statements.

1. $\exists x(Country(x) \wedge Train(germany, x) \wedge \neg Train(ireland, x))$.
2. $\neg \exists x(country(x) \wedge Boat(switzerland, x))$.
3. $\forall x(Country(x) \rightarrow \forall y(Country(y) \wedge Plane(x, y) \rightarrow Boat(x, y)))$.

For part (b) we may have:

1. There is at least one country which can be reached by train from Germany but cannot reached by train from Ireland.
2. There is no country which can be reached by boat from Switzerland.
3. If one can travel from one country to another by plane, then one can make the same journey by boat.