Counting finite sets

We denote the size of a finite set S by

$$\boxed{|\,S\,|}$$

FOR EXAMPLE:

- If $S = \{a, b, c\}$ then $|S| = |\{a, b, c\}| = 3$.

  We say that "the cardinality of S is 3", or simply "S has 3 elements".

- If $A = \{n \in \mathbf{N} \mid n$ is an odd number between 4 and $24\}$ then $|A| = 10$.

- $|\emptyset| = 0$.

The sum rule

If A and B are *disjoint* sets then $|A \cup B| = |A| + |B|$.

If $A_1, A_2, \ldots, A_n$ are n *parwise disjoint* sets then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| .$$

FOR EXAMPLE:

A student can choose a project from one of the project-lists of three lecturers, John, Bill and Eve. John's list has 11 projects, Bill's 10, and Eve's 8. No project occurs in two lists. How many possible projects are there to choose from?

SOLUTION: $A = \{p \mid p$ is a project on John's list$\}$

$B = \{p \mid p$ is a project on Bill's list$\}$

$C = \{p \mid p$ is a project on Eve's list$\}$

Then the number of possible projects is

$$|A \cup B \cup C| = |A| + |B| + |C| = 11 + 10 + 8 = 29 .$$

The inclusion-exclusion principle

$$\boxed{|A \cup B| \;=\; |A| + |B| - |A \cap B|}$$

FOR EXAMPLE: A company receives 350 applications from fresh graduates for a certain job. 220 of these people have studied computer science, 147 management, and 51 both computer science and management. How many of these applicants studied neither computer science nor management?

SOLUTION: Let $C \;=\; \{p \mid p$ is an applicant who studied CS$\}$,

$M \;=\; \{p \mid p$ is an applicant who studied management$\}$.

Then the number of those who studied at least one of the two is

$$|C \cup M| \;=\; |C| + |M| - |C \cap M| \;=\; 220 + 147 - 51 \;=\; 316\,.$$

So the number of those who studied neither computer science nor management is this number subtracted from the total number of applicants:

$$350 - 316 \;=\; 34\,.$$

Inclusion-exclusion principle for 3 sets

$$
\begin{aligned}
\boxed{|A \cup B \cup C|} &= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - \\
&\qquad - \big(|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|\big)
\end{aligned}
$$

$$
= \boxed{|A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|}
$$

The product rule

If there is a sequence of k tasks such that

there are n_1 ways to do the first task,

n_2 ways to do the second task, $\ldots$,

n_k ways to do the kth task,

then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways to do the whole sequence of k tasks.

$$\boxed{|A_1 \times A_2 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|}$$

FOR EXAMPLE:

If each number plate contains a sequence of three letters followed by three digits (and no such sequence is prohibited), then the number of available different number plates is:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17\,576\,000$$

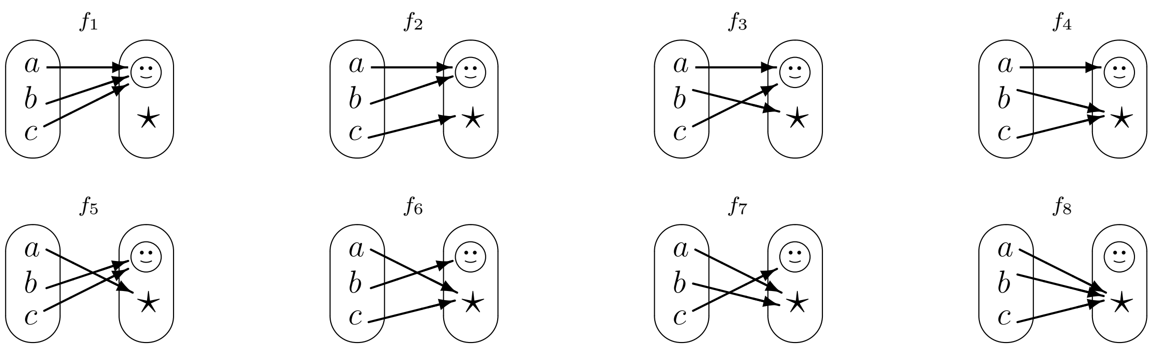
The product rule: another example

If $|A| = n$ and $|B| = m$ then the number of different $f : A \to B$ functions is:

$$\overbrace{m \cdot m \cdot \ldots \cdot m}^{n} = m^n$$

FOR EXAMPLE: $A = \{a, b, c\}$, $B = \{\text{☺}, \star\}$ $\rightsquigarrow$ $n = 3$, $m = 2$

- there are 2 ways to choose a value for a
- then there are 2 ways to choose a value for b
- then there are 2 ways to choose a value for c



Counting the subsets of a finite set

Let S be a finite set having n elements. How many subsets does S have?
In other words, what is the cardinality of $P(S)$?

SOLUTION: List the elements of S: $s_1, s_2, \ldots, s_n$.

We can choose a subset A of S by going through this list and decide, for each element in the list, whether we choose it to be in A or not.

- So we have one 'task' for each element: n tasks altogether.
- Each task can be done in two ways: IN or OUT.

Therefore, by the product rule:

$$\boxed{|P(S)| = \overbrace{2 \cdot 2 \cdot \ \ldots \ \cdot 2}^{n} = 2^n}$$

Mixing the principles

How many four-digit numbers begin either with 7 or with 42?

SOLUTION:

- Our set can be broken into two disjoint subsets:

  (1) four-digit numbers beginning with 7, and

  (2) four-digit numbers beginning with 42.

  We deal with these separately, then apply the sum rule.

- In case (1), if we begin with 7, then there are 3 digits still to be filled in, with 10 choices each. So, by the product rule, we obtain: $10 \cdot 10 \cdot 10 = 1000$ possibilities.

- Similarly, in case (2), there are 2 digits still to be filled in, with 10 choices each. So we have $10 \cdot 10 = 100$ possibilities.

- By the sum rule, altogether we have: $1000 + 100 = \boxed{1100}$

The pigeonhole principle

$\boxed{\text{If } k \in \mathbf{N}^+ \text{ and } k+1 \text{ or more objects are placed into } k \text{ boxes, then there is at least one box containing two or more objects.}}$

FOR EXAMPLE:

- Among any group of 367 people, there must be at least two with the same birthdays.
- If an exam is marked on a scale from 0 to 100, then among 102 students there must be at least two with the same mark.
- If A and B are finite sets with $|A| > |B|$ and $f : A \to B$ is a function, then f is <u>not one-to-one.</u>

  (*Hint:* Suppose that for each $b \in B$ we have a box that contains those $a \in A$ for which $f(a) = b$.)

The pigeonhole principle: another example

For every $n \in \mathbf{N}^+$ there is a multiple of n

that is not 0, and has only 0s and 1s among its digits.

SOLUTION: Take any $n \in \mathbf{N}^+$. Consider the $n+1$ many numbers

$$1, \quad 11, \quad 111, \quad \ldots, \quad \overbrace{111\ldots1}^{n+1}$$

- Observe that there are n possible different remainders when a natural number is divided by n (these are: $0, 1, \ldots,$ or $n-1$).
- Because there are $n+1$ numbers on the above list, by the pigeonhole principle there must be two with the same remainder when divided by n.
- Then "the larger of these numbers minus the smaller one"
  - is divisible by n, and so is a multiple of n,
  - it has only 0s and 1s among its digits.

The generalised pigeonhole principle

> If n objects are placed into k boxes, then
> there is at least one box containing at least $\lceil n/k \rceil$ objects.

FOR EXAMPLE:

- Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who where born in the same month.
- If the possible grades for a module are A, B, C, D, and F,

  then among 26 students there are always at least 6 with the same module grade (because $\lceil 26/5 \rceil = 6$).

Four ways of selecting items

A girl is given a bag containing three types of sweets:

aniseed drops (A), butter mints (B), and cherry drops (C).

In how many ways can she select two sweets?

It depends.

- If **order matters, repetition not allowed:**

  AB, AC, BA, BC, CA, CB

- If **order matters, repetition allowed:**

  AA, AB, AC, BA, BB, BC, CA, CB, CC

- If **order does not matter, repetition not allowed:**

  AB, AC, BC

- If **order does not matter, repetition allowed:**

  AA, AB, AC, BB, BC, CC

Order matters, repetition not allowed

How many ways can we select k persons from a group of n people to stand in line for a photo shoot?

There are:

- n ways to select the first person,
- $n-1$ ways to select the second person,
- $\ldots$ and so on, up to $n-(k-1)$ ways to select the kth person.

So, by the product rule, the overall number is:

$$\boxed{n\cdot(n-1)\cdot\ \ldots\ \cdot(n-k+1) = \frac{n\cdot(n-1)\cdot\ \ldots\ \cdot(n-k+1)\cdot(n-k)\cdot\ \ldots\ \cdot 2\cdot 1}{1\cdot 2\cdot\ \ldots\ \cdot(n-k)} = \frac{n\,!}{(n-k)\,!}}$$

SPECIAL CASE: $k=n$, the number of ways k objects can be ordered:

$$\frac{k\,!}{(k-k)\,!} = \frac{k\,!}{0\,!} = \frac{k\,!}{1} = \boxed{k\,!} = 1\cdot 2\cdot\ \ldots\ \cdot k$$

Order matters, repetition allowed

How many words of length k can be formed from the letters of an n letter alphabet?

There are:

- n ways to select the first letter,
- n ways to select the second letter,
- $\dots$ and so on, n ways to select the kth letter.

So, by the product rule, the overall number is:

$$\boxed{\overbrace{n \cdot n \cdot \ldots \cdot n}^{k} = n^k}$$

Order does not matter, repetition not allowed

How many ways can we select k persons from a group of n people if the order of selection does not matter?

- As we have seen, if the order of selection does matter, then there are $n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \dfrac{n\,!}{(n-k)\,!}$ ways.
- But then for each set A of k people, we counted the people in A many times. How many? As many as the number of ways k persons can be ordered: $k\,! = 1 \cdot 2 \cdot \ldots \cdot k$
- So, if the selection order does not matter, then the number of ways is:

$$\frac{\frac{n\,!}{(n-k)\,!}}{k\,!} = \frac{n\,!}{(n-k)\,! \cdot k\,!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = \boxed{\binom{n}{k}}$$

n **choose** k

(Other notations in the literature: C^n_k, ${}_nC_k$, $C(n,k)$)

Properties of the 'choose numbers'

- $$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ \ldots \ \cdot (n-k+1)}{1 \cdot 2 \cdot \ \ldots \ \cdot k} \qquad \frac{\leftarrow\ k \text{ `backward steps'}}{\leftarrow\ k \text{ `forward steps'}}$$

  FOR EXAMPLE:

  $$\binom{128}{3} = \frac{128 \cdot 127 \cdot 126}{1 \cdot 2 \cdot 3} \qquad \frac{\leftarrow\ 3 \text{ `backward steps'}}{\leftarrow\ 3 \text{ `forward steps'}}$$

- 'choose numbers' are also called **binomial coefficients**

- $$\binom{n}{k} = \binom{n}{n-k}$$

  WHY? choosing k elements is the same as 'leaving' $n-k$

Pascal's identity

$$\boxed{\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}}$$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & \boxed{20} & & 15 & & 6 & & 1 & \ldots
\end{array}$$

row 6 $\rightarrow$ $1 \quad 6 \quad 15 \quad \boxed{20} \quad 15 \quad 6 \quad 1 \quad \ldots$

$\uparrow$

entry 3: $\binom{6}{3} = \binom{5}{2} + \binom{5}{3} = \frac{5 \cdot 4}{1 \cdot 2} + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 20$

(counting of rows and entries starts at 0)

Order does not matter, repetition allowed

Suppose we have an unlimited supply of 3 types of fruits:

apples (A), oranges (O), and peaches (P).

How many ways are there to select 4 pieces of fruit, if the order of selection doesn't matter, and only the type of fruit and not the individual piece matters?

Tricky. Let us try to reformulate. Suppose we have a rectangular box capable of storing 4 fruit pieces. The box has 3 compartments, for storing As, Os, and Ps. These compartments are divided by 2 movable dividers that can be shifted, depending on how many pieces of fruit of each type we want to store in the box. For example:

- 2 apples, 1 orange, 1 peach: $\boxed{A\,A \mid O \mid P}$
- 4 oranges: $\boxed{\mid O\,O\,O\,O \mid}$
- 1 apple, 3 peaches: $\boxed{A \mid\mid P\,P\,P}$
- 4 apples: $\boxed{A\,A\,A\,A \mid\mid}$

Order does not matter, repetition allowed (cont.)

So if we have to choose k objects from a set of n objects, with repetitions allowed, then we need a box with

- k places for the chosen objects, and
- $n-1$ places for the dividers dividing the box to n compartments.

$$\boxed{* * \mid\mid * * * \mid * \mid\mid} \qquad \text{altogether } k+n-1 \text{ places}$$

The number of ways we can choose our k objects is the number of ways we can distribute the dividers in the box: Out of the $k+n-1$ places we have to choose $n-1$ for the dividers. The number of ways doing this is:

$$\boxed{\binom{k+n-1}{n-1}} = \binom{k+n-1}{k}$$

IN THE 'FRUIT SELECTION' EXAMPLE: There are

$$\binom{4+3-1}{3-1} = \binom{6}{2} = \frac{6\,!}{4\,!\cdot 2\,!} = \frac{6\cdot 5}{2} = 15 \qquad \text{ways.}$$

Counting selections: a summary

The number of ways of selecting k items from a set of n items:

	Order matters: **permutations**	Order doesn't matter: **combinations**
repetitions not allowed	$n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$	$\binom{n}{k}$
repetitions allowed	n^k	$\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$

Exercise 4.1

How many different words can be made by reordering the letters of the word *SUCCESS* ?

SOLUTION: This word contains three *S*s, two *C*s, one *U*, and one *E*.
When we reorder these letters, the new word will also contain seven letters.

- The three *S*s can be placed among the seven positions $\binom{7}{3}$ different ways, leaving four positions free.
- Then the two *C*s can be placed in $\binom{4}{2}$ ways, leaving two free places.
- Then the *U* can be placed in $\binom{2}{1}$ ways, leaving just one position free.
- Hence *E* can be placed in $\binom{1}{1}$ way.

Therefore, by the product rule, the number of different words that can be made:

$$\boxed{\binom{7}{3} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = \frac{7!}{3! \cdot 4!} \cdot \frac{4!}{2! \cdot 2!} \cdot \frac{2!}{1! \cdot 1!} \cdot \frac{1!}{1! \cdot 0!} = \frac{7!}{3! \cdot 2! \cdot 1! \cdot 1!} = \frac{5040}{12} = 420}$$