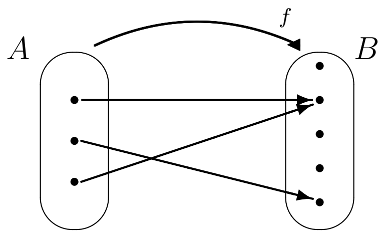
# Functions

Given sets $A$ and $B$, a **<u>function from $A$ to $B$</u>** is a <u>rule</u> $f$ that associates with <u>*each*</u> element of $A$ <u>*exactly one*</u> element of $B$.




If $f$ associates $x \in A$ with $y \in B$, then we write $\boxed{f(x) = y}$
and say <u>"$f$ of $x$ is $y$"</u>, or <u>"$f$ maps $x$ to $y$"</u>, or the <u>"value of $f$ at $x$ is $y$"</u>.

If $f$ is function from $A$ to $B$, then we write: $\boxed{f : A \to B}$.

We call $A$ the **<u>domain of $f$</u>**, and $B$ **<u>the codomain of $f$</u>**.

- **every** element of the domain has to be mapped somewhere in the codomain
- but **not everything** in the codomain has to be a value of a domain element
- one element **cannot** be mapped to 2 different places
- but it **can** happen that 2 different elements are mapped to the same place

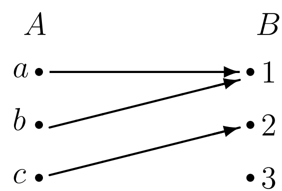
# Different ways of describing functions

FOR EXAMPLE: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.

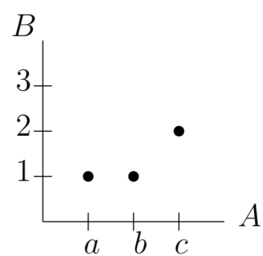
- We can describe a function $f : A \to B$ by listing all its associations:

$$f(a) = 1, \qquad f(b) = 1, \qquad f(c) = 2\,.$$

- We can describe the same $f$ by drawing points and arrows:




- We can describe the same $f$ by drawing its '<u>**graph**</u>':



# Functions and not functions

Let $H$ be the set of all humans, alive or dead. Let's make some $H \to H$ associations and discuss whether each is a $H \to H$ function or not.

- $\boxed{f(x) \text{ is a parent of } x}$

  This $f$ is NOT a function, because people have two parents.

- $\boxed{f(x) \text{ is the mother of } x}$

  This $f$ is a $H \to H$ function, because each person has exactly one mother.

- $\boxed{f(x) \text{ is the oldest child of } x}$

  This $f$ is NOT a $H \to H$ function, because some person has no children.

- $\boxed{f(x) \text{ is the set of all children of } x}$

  Though this $f$ is a function, it is NOT a $H \to H$ function, because each person is associated with a **set** of people rather than one person.
  (This $f$ is a $H \to P(H)$ function.)

## More examples of $f : A \to B$ functions

- A function is a rule. Sometimes we can describe this rule by a single formula:

  Let $A = B = \mathbf{Z}$. For every $x \in \mathbf{Z}$, let

  $$\boxed{f(x) \;=\; 6x + 28}$$

  Then, for example, $f(2) = 6 \cdot 2 + 28 = 40$, $f(0) = 28$, $f(-113) = -650$, ...

- Sometimes the rule can only be described by case distinction:

  Let $A = B = \mathbf{N}$. For every $n \in \mathbf{N}$, let

  $$\boxed{g(n) \;=\; \begin{cases} 2^n, & \text{if } n \text{ is odd,} \\ 3n^2 + n + 1, & \text{if } n \text{ is even} \end{cases}}$$

  Then, for example, $g(4) = 3 \cdot 4^2 + 4 + 1 = 53$, $g(3) = 2^3 = 8$, ...

- And the rule does not have to be described by a formula at all:

  Let $A = \{E \mid E \text{ is a healthy African elephant}\}$ and

  $B = \{e \mid e \text{ is an elephant ear}\}$.

  For every $E \in A$, let $\boxed{\ell(E) \;=\; E\text{'s left ear}}$

# Some useful functions

- The **floor function** $\boxed{\lfloor\ \rfloor : \mathbf{R} \to \mathbf{Z}}$ assigns to any real number $x$ the largest integer that is less than or equal to $x$.

FOR EXAMPLE: $\lfloor \frac{1}{2} \rfloor = 0$, $\lfloor -\frac{3}{2} \rfloor = -2$, $\lfloor 3.2 \rfloor = 3$, $\lfloor 9 \rfloor = 9$.

- The **ceiling function** $\boxed{\lceil\ \rceil : \mathbf{R} \to \mathbf{Z}}$ assigns to any real number $x$ the smallest integer that is greater than or equal to $x$.

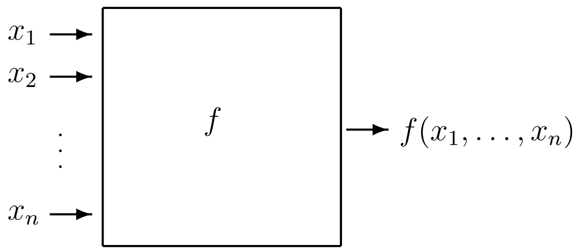
FOR EXAMPLE: $\lceil \frac{1}{3} \rceil = 1$, $\lceil -\frac{5}{4} \rceil = -1$, $\lceil 5.3 \rceil = 6$, $\lceil 7 \rceil = 7$.

$$\boxed{\begin{gathered} x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1 \\ \lfloor -x \rfloor = -\lceil x \rceil \\ \lceil -x \rceil = -\lfloor x \rfloor \end{gathered}}$$

# Functions with multiple arguments

If the domain of a function $f$ is a Cartesian product $A_1 \times \cdots \times A_n$,
we say that **$f$ has arity $n$**, or **$f$ is an $n$-ary function**, or **$f$ has $n$ arguments**.
In this case, for each $n$-tuple $(x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$,

$\boxed{f(x_1, \ldots, x_n)}$ denotes the value of $f$ at $(x_1, \ldots, x_n)$.




A function $f$ with two arguments is also called a **binary function.**

For binary functions, we have the option of writing $f(x, y) = z$
in the form $\boxed{x\ f\ y = z}$ (such as, $4 + 5 = 9$ instead of $+(4, 5) = 9$ ).

## Tuples and sequences are functions

- A tuple can be thought of as a function.

  FOR EXAMPLE: The $5$-tuple $(22, 14, 55, 1, 700)$ can be thought of as a listing of the values of the function $f : \{0, 1, 2, 3, 4\} \to \mathbf{N}$ defined by

$$f(0) = 22, \qquad f(1) = 14, \qquad f(2) = 55, \qquad f(3) = 1, \qquad f(4) = 700.$$

- Similarly, an infinite sequence of objects can also be thought of as a function.

  FOR EXAMPLE: Suppose that $(b_0, b_1, \ldots, b_n, \ldots)$ is an infinite sequence of objects from a set $S$. Then this sequence can be thought of as a listing of the values of the function $f : \mathbf{N} \to S$ defined by

$$f(n) = b_n.$$

# Functions are special binary relations

A function $f : A \to B$ can be considered as a relation from $A$ to $B$:

$$\boxed{\{(a,b) \in A \times B \mid f(a) = b\}}$$

Relations that are also functions have two special properties:

- for every $a \in A$ there is some $b \in B$ with $(a,b)$ being in the relation
  (**every** element of the domain has to be mapped somewhere in the codomain)
- and no two ordered pairs in the relation have the same first element
  (one element **cannot** be mapped to 2 different places)

FOR EXAMPLE:

$\{(x,y) \in \mathbf{N} \times \mathbf{N} \mid y = x - 1\}$ is NOT a function.

$\{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid y = x - 1\}$ is a function.

$\{(x,y) \in \mathbf{N} \times \mathbf{R} \mid x = y^2\}$ is NOT a function.

$\{(x,y) \in \mathbf{N} \times \mathbf{N} \mid x = y^2\}$ is NOT a function.

$\{(x,y) \in \mathbf{N} \times \mathbf{N} \mid y = x^2\}$ is a function.

WHY?

- $0$ is not mapped
- $f(n) = n - 1$
- $(25,5)$ and $(25,-5)$ are both in
- $2$ is not mapped
- $f(n) = n^2$

# Properties of functions

- A function $f : A \to B$ is called **one-to-one** (or **injective**)

  if it maps distinct elements of $A$ to distinct elements of $B$.

  Another way to say this: $f$ is one-to-one if

  $x \neq y$ implies $f(x) \neq f(y)$.

- A function $f : A \to B$ is called **onto** (or **surjective**)

  if each element $b$ of $B$ can be obtained as $b = f(a)$ for some $a$ in $A$.

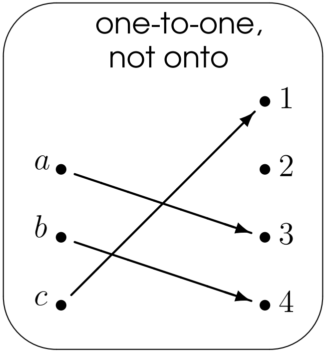
  In general, this might not be the case. If $f$ is an $A \to B$ function, this just means that for every $a \in A$, $f(a) \in B$. But the **range of $f$**

  $$\boxed{\text{range}(f) \;=\; \{f(a) \in B \mid a \in A\}}$$

  can be a **proper subset** of $B$.

- A function is called a **bijection** if it is both one-to-one and onto.

# Examples


one-to-one,
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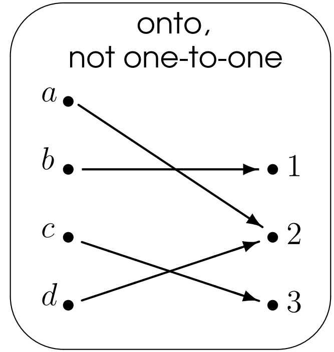
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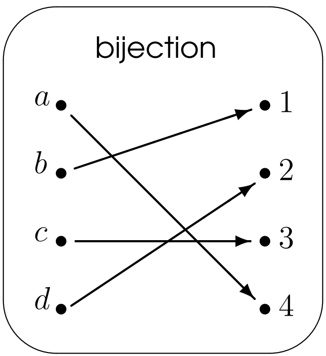
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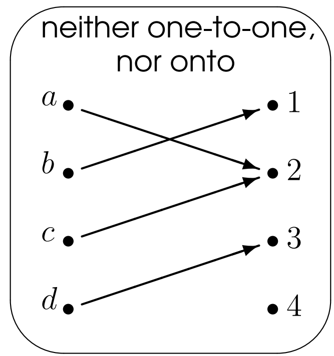
neither one-to-one,
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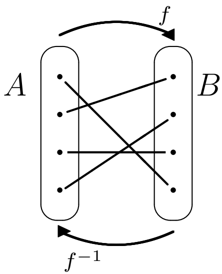
# Bijections and inverses

Bijections always come in pairs. If $f : A \to B$ is a bijection, then there is a function $\boxed{f^{-1} : B \to A}$, called the **inverse of $f$** defined by




$$\boxed{f^{-1}(b) = a \qquad \text{whenever} \qquad f(a) = b}$$

Then $f^{-1}$ is also a bijection, and we have

- $f^{-1}\big(f(a)\big) = a$, for every $a \in A$, $\qquad f\big(f^{-1}(b)\big) = b$, for every $b \in B$.

FOR EXAMPLE: Let *Odd* and *Even* be the sets of odd and even natural numbers, respectively. Define a function $f$ : *Odd* $\to$ *Even* by

$$f(n) = n - 1\,.$$

Then $f$ is a bijection and its inverse $f^{-1}$ : *Even* $\to$ *Odd* is defined by

$$f^{-1}(n) = n + 1\,.$$

## Some important functions

- For every set $A$, its **identity function** $\mathrm{id}_A : A \to A$

  is defined by, for all $a \in A$,

  $$\boxed{\mathrm{id}_A(a) = a}$$

  Then $\mathrm{id}_A$ is a bijection and its inverse is itself.

- Let $S$ be a set. For every subset $A \subseteq S$, its **characteristic function**

  $f_A : S \to \{0,1\}$ is defined by, for all $x \in S$,

  $$\boxed{f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A \end{cases}}$$

  Say, if $A \subseteq \mathbf{N}$ then $f_A : \mathbf{N} \to \{0,1\}$ can be represented by an infinite 0-1 sequence.

# Describing N $\to$ N functions by induction

- The **factorial function**:

  *Basis:* $f(0) = 1$ and $f(1) = 1$.

  *Inductive step:* If $n > 1$ then $f(n) = f(n-1) \cdot n$.

  We used to write $\boxed{n!}$ for $f(n)$.

- The **Fibonacci function**:

  *Leonardo Fibonacci* asked in 1202: Let's start with a pair of rabbits that needs one month to mature, and assume that every month each pair produces a new pair that becomes productive after one month. How many **new pairs** are produced each month?

  *Basis:* $f(0) = 0$ and $f(1) = 1$.

  *Inductive step:* If $n > 1$ then $f(n) = f(n-2) + f(n-1)$.

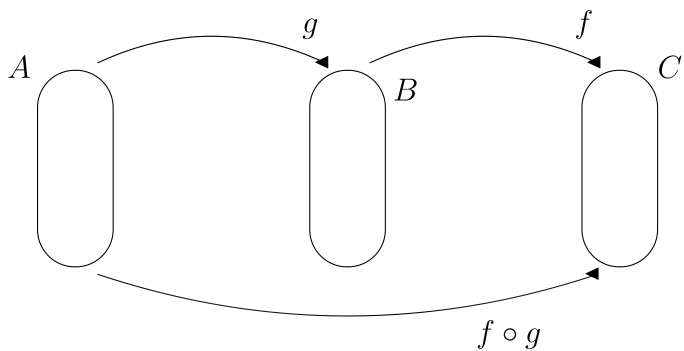
  $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$

# Combining functions: composition

Let $g : A \to B$ and $f : B \to C$ be functions.

The **composition of $f$ and $g$** is the function $\boxed{(f \circ g) : A \to C}$ defined by

$$\boxed{(f \circ g)(a) \;=\; f\big(g(a)\big)} \qquad \text{for each } a \in A.$$




The composition $f \circ g$ is **only** defined when

"domain of $f$" $=$ "codomain of $g$" !

## Composition of functions: examples

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Let function $g : X \to X$ be defined by

$$g(a) = b, \qquad g(b) = c, \qquad g(c) = a,$$

and function $f : X \to Y$ be defined by

$$f(a) = 3, \qquad f(b) = 2, \qquad f(c) = 1.$$

Then:

- $(f \circ g)(a) = f\big(g(a)\big) = f(b) = 2$,
  $(f \circ g)(b) = f\big(g(b)\big) = f(c) = 1$,
  $(f \circ g)(c) = f\big(g(c)\big) = f(a) = 3$.
- $(g \circ g)(a) = g\big(g(a)\big) = g(b) = c$,
  $(g \circ g)(b) = g\big(g(b)\big) = g(c) = a$,
  $(g \circ g)(c) = g\big(g(c)\big) = g(a) = b$.
- Watch out: $f \circ f$ and $g \circ f$ are not defined!

# Properties of composition

- Even if both are defined, $f \circ g$ and $g \circ f$ can be different:
  Let $f$ and $g$ be both $\mathbf{Z} \to \mathbf{Z}$ functions, defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. Then:

$$
\begin{aligned}
(f \circ g)(x) &= f\big(g(x)\big) = f(3x+2) = 2(3x+2)+3 = 6x+7, \\
(g \circ f)(x) &= g\big(f(x)\big) = g(2x+3) = 3(2x+3)+2 = 6x+11.
\end{aligned}
$$

- $\circ$ is associative: $\boxed{f \circ (g \circ h) = (f \circ g) \circ h}$

- If $f : A \to B$ is a bijection then

$$\boxed{f^{-1} \circ f = \mathrm{id}_A \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}_B}$$

- For any function $f : A \to B$,

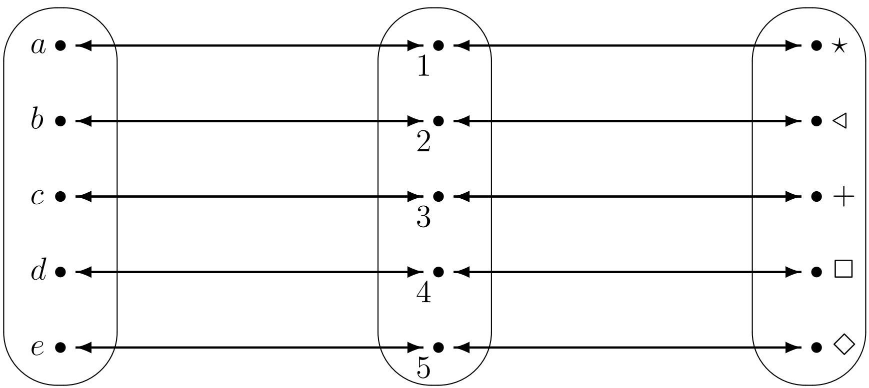
$$\boxed{f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f}$$

# Comparing finite sets

It is not hard to compare the sizes of finite sets:

we simply count the number of elements in each.

If finite sets have the same number of elements, then there is always a *bijection* between them.



## And infinite sets?

*Idea:* Two infinite sets have the same size if there is bijection between them.

We call a set $A$ **countable** if it is either finite
or there is a bijection between $A$ and **N**.

FOR EXAMPLE:

- As $\mathrm{id}_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ is a bijection, **N** is countable.
- Let *Odd* be the set of odd natural numbers.

  Strangely enough, even if $\mathit{Odd} \subset \mathbf{N}$, it has the same size as **N**:

  The function $f : \mathbf{N} \to \mathit{Odd}$ defined by

$$f(x) = 2x + 1$$

  is a bijection.

# $\mathbf{N} \times \mathbf{N}$ is countable

We need to describe a bijection between $\mathbf{N} \times \mathbf{N}$ and $\mathbf{N}$.

We arrange the ordered pairs in $\mathbf{N} \times \mathbf{N}$ in such a way that they can be easily counted:

$$
\begin{array}{lcl}
(0,0) & \longleftrightarrow & 0, \\
(0,1),\ (1,0), & \longleftrightarrow & 1,\ 2, \\
(0,2),\ (1,1),\ (2,0), & \longleftrightarrow & 3,\ 4,\ 5, \\
(0,3),\ (1,2),\ (2,1),\ (3,0), & \longleftrightarrow & 6,\ 7,\ 8,\ 9, \\
\quad\ldots & & \quad\ldots
\end{array}
$$

We can describe this $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$ bijection by

$$(m,n) \qquad \longleftrightarrow \qquad (1+2+\cdots+n)+m\,.$$

# But not all sets are countable

The power set $P(\mathbf{N})$ of $\mathbf{N}$ is NOT countable.

In other words, *there are more subsets of numbers than numbers.*

Why? Let $f : \mathbf{N} \to P(\mathbf{N})$ be any function.

Then, for every $n \in \mathbf{N}$, $f(n)$ is a subset of $\mathbf{N}$.

Now take the following subset $D$ of $\mathbf{N}$: $\boxed{D \;=\; \{n \in \mathbf{N} \mid n \notin f(n)\}}$

We show that $D$ **is not in the range of** $f$, that is, for every $n \in \mathbf{N}$, $D \neq f(n)$.

Indeed, the number $n$ ‘distinguishes’ the sets $D$ and $f(n)$ :

- If $n \in D$, then $n$ should have the property describing $D$, so $n \notin f(n)$.
- If $n \in f(n)$, then the property describing $D$ does not hold for $n$, so $n \notin D$.

As $D$ is not in the range of $f$, we obtain that $f$ is **not onto** and so **can’t be a bijection**.