4CCS1ELA: Tutorial list 3 – Sample Solutions

1. Represent the following system specification in propositional logic and determine whether this specification is consistent.

> *Whenever the system software is being upgraded, users cannot access the file system. If users can access the file system, then they can save new files. If users cannot save new files, then the system software is not being upgraded.*

SOLUTION

Let U denote "the system software is being upgraded"; let A denote "users can access the system"; and let S denote "users can save new files".

In propositional form, we have the following representation:

$$U \to \neg A,\ A \to S,\ \neg S \to \neg U.$$

This system specification is consistent because there is some assignment I of truth values to the propositional letters such that all three formulas become true. For instance, $I(U) = I(A) = I(S) = 0$.

Another possible interpretation making all three formulas true is
$I(U) = I(S) = 1,\ I(A) = 0$.

2. Which of the following are true? Explain your answers carefully.

(i) $P,\ P \to Q\ \models\ Q$;

(ii) $P \to Q\ \models\ Q \to P$;

(iii) $P \vee \neg Q\ \models\ P$;

(iv) $P \wedge \neg P\ \models\ Q$.

SOLUTION

(i) $P,\ P \to Q\ \models\ Q$ holds.
One of the reasons: the formula $\big(P \wedge (P \to Q)\big) \to Q$ is a tautology (show this).

(ii) $P \to Q\ \models\ Q \to P$ does not hold.
Take I such that $I(P) = 0$ and $I(Q) = 1$. Then $I(P \to Q) = 1$ but $I(Q \to P) = 0$.

(iii) $P \vee \neg Q \models P$ does not hold either.
Take I such that $I(P) = 0$ and $I(Q) = 0$. Then $I(P \vee \neg Q) = 1$ but $I(P) = 0$.

(iv) $P \wedge \neg P \models Q$ does hold.
The premise of this logical consequence is a contradiction. Therefore it is true (it does not matter in this case what the conclusion is). To see this note that the truth-table for $P \wedge \neg P$ has no rows with the value 1 for the formula, therefore "in every row in which the value of $P \wedge \neg P$ is 1, so is the value of any conclusion."

3. Determine which of these arguments are valid. If an argument is valid, indicate the rule of inference being used. If it is not, what logical error occurs?

(i) If n is a real number such that $n > 1$, then $n^2 > 1$.
Suppose, $n^2 > 1$. Then $n > 1$.

(ii) If n is a real number such that $n > 3$, then $n^2 > 9$.
Suppose, $n^2 \leq 9$. Then $n \leq 3$.

(iii) If n is a real number such that $n > 2$, then $n^2 > 4$.
Suppose, $n \leq 2$. Then $n^2 \leq 4$.

SOLUTION

(i) If n is a real number such that $n > 1$, then $n^2 > 1$.
Suppose, $n^2 > 1$. Then $n > 1$.

This argument is not valid, because the formula $\big((P \rightarrow Q) \wedge Q\big) \rightarrow P$ is not a tautology. In other words, $(P \rightarrow Q),\ Q \not\models P$.

Such type of incorrect reasoning is called the **fallacy of affirming the conclusion.**

(ii) If n is a real number such that $n > 3$, then $n^2 > 9$.
Suppose, $n^2 \leq 9$. Then $n \leq 3$.

This argument is valid. The Modus Tollens (MT) rule has been used.

(iii) If n is a real number such that $n > 2$, then $n^2 > 4$.
Suppose, $n \leq 2$. Then $n^2 \leq 4$.

This argument is not valid, because the formula
$\big((P \to Q) \wedge \neg P\big) \to \neg Q$ is not a tautology.
In other words, $(P \to Q),\ \neg P \not\models \neg Q$.

Such type of incorrect reasoning is called the **fallacy of denying the hypothesis.**

4. Let $\mathcal{A}$, $\mathcal{B}$ be propositional formulas. Demonstrate that if there exists a propositional formula $\mathcal{C}$ such that $\mathcal{A}$ is a logical consequence of $\mathcal{C}$ and $\mathcal{B}$ is a logical consequence of $\neg\,\mathcal{C}$, then formula $\mathcal{A} \vee \mathcal{B}$ is a tautology.

<u>SOLUTION</u>

By contradiction, suppose that $\mathcal{C} \models A$ and $\neg\mathcal{C} \models B$ but $\mathcal{A} \ \vee \ \mathcal{B}$ is not a tautology. Since $\mathcal{A} \ \vee \ \mathcal{B}$ is not a tautology, then there is an interpretation I such that $I(\mathcal{A}) = 0$ and $I(\mathcal{B}) = 0$. It follows that $I(\mathcal{C}) = 0$, because $\mathcal{C} \models A$. Hence, $I(\neg\mathcal{C}) = 1$ and $\neg\mathcal{C} \not\models B$. Therefore, we have a contradiction.

5. Suppose we have the two propositions (with symbols to represent them):

It is raining (R) or I work in the yard (W).
It is not raining ($\neg R$) or I go to the library (L).

What conclusion can we draw from them?

<u>SOLUTION</u>

The statements can be translated as: $R \vee W$ and $\neg R \vee L$. We cannot have them both being true without either W or L being true. For if they both were false, then we would have to have that both R and $\neg R$ are both true, which is impossible. Therefore, we must have that either W or L is true, i.e., $W \vee L$.

Let us check that this is indeed the case, using natural deduction.

1. $R \vee W$ data
2. $\neg R \vee L$ data

3. $\neg R \to (W \vee L)$ subcomputation box below

		$\underline{W \vee L}$
5.1 $\neg R$	assumption	
5.2 W	5.1 and $\vee$E1	
5.3 $W \vee L$	5.2 and $\vee$I	

4. $L \to (W \vee L)$ subcomputation box below

		$\underline{W \vee L}$
4.1 L	assumption	
4.2 $W \vee L$	4.1 and $\vee$I	

5. $W \vee L$ 2., 3., 4., and $\vee$E

In fact, this pattern of inference is so useful that it has a special deduction rule called "Resolution" is given for it.

$$\frac{\neg P \vee Q,\ \ P \vee R}{Q \vee R}\ (Res)$$

6. Determine whether this argument is valid, using natural deduction:

Lynn works part time or full time.
If Lynn does not play on the team, then she does not work part time.
If Lynn plays on the team, she is busy.
Lynn does not work full time.
Therefore, Lynn is busy.

<u>SOLUTION</u>

Using the variables:
P: Lynn works part time
F: Lynn works full time
A: Lynn plays on the team
B: Lynn is busy
the argument can be written in symbols:

$$P \vee F, \neg A \to \neg P, A \to B, \neg F \vdash B$$

And we want to find out if it is valid. In semantical terms we would be asking whether

$$P \vee F, \neg A \to \neg P, A \to B, \neg F \models B$$

and this could be worked out via the truth-tables (this is done at the end for illustration purposes).

So our problem is to construct a natural deduction proof of B from $\{P \vee F, \neg A \to \neg P, A \to B, \neg F\}$.

1. $P \vee F$ data
2. $\neg A \to \neg P$ data
3. $A \to B$ data
4. $\neg F$ data
5. $F \to P$ subcomputation box below

> $\underline{P}$
>
> 5.1 F assumption
>
> 5.2 $\neg P \to F$ subcomputation box below
>
> > $\underline{F}$
> >
> > 5.2.1 $\neg P$ assumption
> >
> > 5.2.2 F 5.1
>
> 5.3 $\neg P \to \neg F$ subcomputation box below
>
> > $\underline{\neg F}$
> >
> > 5.3.1 $\neg P$ assumption
> >
> > 5.3.2 $\neg F$ 4
>
> 8.4 P 5.2., 5.3., and $\neg$E

6. $P \to P$ subcomputation box below

> $\underline{P}$
>
> 6.1 P assumption
>
> 6.2 P 6.2

7. P 1., 5., 6., and $\vee$E
8. $P \to A$ subcomputation box below

		$\underline{A}$
8.1	P	assumption
8.2	$\neg A \to P$	subcomputation box below

		$\underline{P}$
8.2.1	$\neg A$	assumption
8.2.2	P	8.1

8.3	A	2., 8.2., and $\neg$E

9.	A	7., 8., and $\to$E
10.	B	9., 3., and $\to$E

Note: Embedded in the proof above is the proof of the variant rule:

$$\frac{G \vee H, \neg H}{G} \qquad (\vee\mathsf{E2})$$

The proof uses lines 1. and 4. as premises (as in the rule above) and the conclusion is given in line 7. The actual proof steps are given in 5., 6., and 7. This shows that the variant rule ($\vee\mathsf{E2}$) can be obtained from the basic set of rules.

As we mentioned before, it is also possible to check the validity of the argument *semantically*, i.e., via the truth-tables. The two results must coincide!

We need to examine all cases where all hypotheses are true. There is only one such case (line 5), and in that case the conclusion is also true. Therefore, the argument is valid, as expected.

I	P	F	A	B	$P \vee F$	$\neg A \to \neg P$	$A \to B$	$\neg F$	B
1.	1	1	1	1	1	1	1	0	1
2.	1	1	1	0	1	1	0	0	0
3.	1	1	0	1	1	0	1	0	1
4.	1	1	0	0	1	0	1	0	0
5.	1	0	1	1	1	1	1	1	1
6.	1	0	1	0	1	1	0	1	0
7.	1	0	0	1	1	0	1	1	1
8.	1	0	0	0	1	0	1	1	0
9.	0	1	1	1	1	1	1	0	1
10.	0	1	1	0	1	1	0	0	0
11.	0	1	0	1	1	1	1	0	1
12.	0	1	0	0	1	1	1	0	0
13.	0	0	1	1	0	1	1	1	1
14.	0	0	1	0	0	1	0	1	0
15.	0	0	0	1	0	1	1	1	1
16.	0	0	0	0	0	1	1	1	0

Note: The natural deduction method can provide a relatively quick way to verify that an argument is valid. However, we need to be ingenious enough to construct a derivation (i.e., *proof*) that leads from the hypotheses (premises) to the conclusion. However, the natural deduction technique cannot directly tell us that an argument is *not* valid. If we use the derivation rules on an argument and are unable to reach the conclusion, it may be simply because we could not construct the proof ourselves (maybe someone else could).

The truth-table technique provides a fool-proof way to determine whether or not the argument is valid – either the conclusion is true whenever the hypotheses are all true (argument is valid), or else there is a case where the hypotheses are true but the conclusion is false (and then the argument is not valid). This particular case, if it exists, constitutes a *counter-model* of the argument. If the argument *is* valid, then the completeness of the natural deduction rules tells us that *there will be* a proof of the conclusion from the hypothesis.

7. Show that the following hold using natural deduction:

1. $P \to Q, \neg Q \vdash \neg P$
2. $(P \to Q) \to Q, Q \to P \vdash P$
3. $\neg(P \land \neg Q) \vdash P \to Q$

<u>SOLUTION</u>

1. $P \to Q, \neg Q \vdash \neg P$

1.	$P{\to}Q$	data
2.	$\neg Q$	data
3.	$P{\to}\neg Q$	subcomputation box below

		$\underline{\neg Q}$
3.1	P	assumption
3.2	$\neg Q$	2.

4.	$\neg P$	1. and 3. and $\neg$I

2. $(P \to Q) \to Q, Q \to P \vdash P$

1. $(P{\to}Q){\to}Q$ data
2. $Q{\to}P$ data
3. $\neg P{\to}P$ subcomputation box below

> $\underline{P}$
>
> 3.1 $\neg P$ assumption
>
> 3.2 $P{\to}Q$ subcomputation box below
>
> > $\underline{Q}$
> >
> > 3.2.1 P assumption
> >
> > 3.2.2 $\neg Q{\to}P$ subcomputation box below
> >
> > > $\underline{P}$
> > >
> > > 3.2.2.1 $\neg Q$ assumption
> > >
> > > 3.2.2.2 P 3.2.1
> >
> > 3.2.3 $\neg Q{\to}\neg P$ subcomputation box below
> >
> > > $\underline{\neg P}$
> > >
> > > 3.2.3.1 $\neg Q$ assumption
> > >
> > > 3.2.3.2 $\neg P$ 3.1
> >
> > 3.2.4 Q 3.2.2, 3.2.3 and $\neg$E
>
> 3.3 Q 1, 3.2, and $\to$E
>
> 3.4 P 3.3, 2 and $\to$E

4. $\neg P{\to}\neg P$ subcomputation box below

> $\underline{\neg P}$
>
> 4.1 $\neg P$ assumption
>
> 4.2 $\neg P$ 4.1

5. P 3. and 4. and $\neg$E

3. $\neg(P \wedge \neg Q) \vdash P \to Q$

1.	$\neg(P \wedge \neg Q)$	data
2.	$P \to Q$	subcomputation box below

> $\underline{Q}$
>
> 2.1 P assumption
>
> 2.2 $\neg Q \to (P \wedge \neg Q)$ subcomputation box below
>
> > $\underline{P \wedge \neg Q}$
> >
> > 2.2.1 $\neg Q$ assumption
> >
> > 2.2.2 $P \wedge \neg Q$ 2.1, 2.2.1 and $\wedge$I
>
> 2.3 $\neg Q \to \neg(P \wedge \neg Q)$ subcomputation box below
>
> > $\underline{\neg(P \wedge \neg Q)}$
> >
> > 2.3.1 $\neg Q$ assumption
> >
> > 2.3.2 $\neg(P \wedge \neg Q)$ 1
>
> 2.4 Q 2.2, 2.3 and $\neg$E