4CCS1ELA: Tutorial list 2 – Sample Solutions

1. Consider the formula

$$(P \vee \neg R) \to \neg(\neg Q \vee R)$$

(i) 1. Write a conjunctive normal form (CNF) for this formula from its truth table.

2. Write a disjunctive normal form (DNF) for this formula from its truth table.

(ii) Transform this formula to a logically equivalent disjunctive normal form (DNF) using the rewrite rules.

(iii) Find a disjunctive normal form for this formula using Quine's tree.

SOLUTION

(i)

P	Q	R	$(P \vee \neg R)$	$\to$	$\neg(\neg Q \vee R)$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	0	0
0	1	1	0	1	0
0	1	0	1	1	1
0	0	1	0	1	0
0	0	0	1	0	0

1. A disjunction of three literals is created for each line (for each interpretation of propositional variables) with a **false** (i.e. 0) value of the formula. For each propositional variable P, the literal P is added to the disjunction if $I(P) = 0$, and $\neg P$ is added to the disjunction if $I(P) = 1$. Such disjunction is false for the interpretation given by this line.
The corresponding CNF is :

$$(\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee R)$$

Here each disjunction has exactly 3 literals, one for each variables appearing in the formula. Such CNFs are called **full** CNFs.

2. A conjunction of three literals is created for each line (for each interpretation of propositional variables) with a true (i.e., 1) value of the formula. For each propositional variable P, the literal $\neg P$ is added to the conjunction if $I(P) = 0$, and P is added to the conjunction is $I(P) = 1$. Such conjunction is true for the interpretation given by this line.
The corresponding DNF is :

$$(P \land Q \land \neg R) \lor (\neg P \land Q \land R) \lor (\neg P \land Q \land \neg R) \lor (\neg P \land \neg Q \land R)$$

Here each conjunction has exactly 3 literals, one for each variables appearing in the formula. Such DNFs are called **full** DNFs.

(ii)

$$\begin{array}{lcl}
(P \lor \neg R) \to \neg(\neg Q \lor R) & \Longrightarrow & (\text{ replacing } \to \text{ with } \lor) \\
\neg(P \lor \neg R) \lor \neg(\neg Q \lor R) & \Longrightarrow & (\text{De Morgan's laws}) \\
(\neg P \land \neg\neg R) \lor (\neg\neg Q \land \neg R) & \Longrightarrow & (\text{removing double negations}) \\
(\neg P \land R) \lor (Q \land \neg R) & (\text{DNF}) &
\end{array}$$

This is another DNF, which is logically equivalent to the previous one obtained in (i) 2:

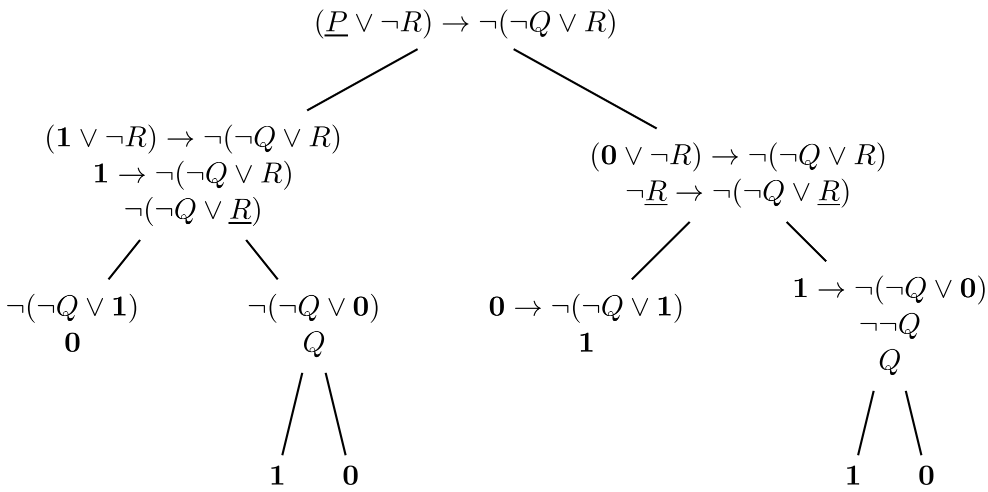
$$(P \land Q \land \neg R) \lor (\neg P \land Q \land R) \lor (\neg P \land Q \land \neg R) \lor (\neg P \land \neg Q \land R)$$

Indeed,

$$\begin{array}{l}
(P \land Q \land \neg R) \lor (\neg P \land Q \land \neg R) \equiv (P \lor \neg P) \land (Q \land \neg R) \equiv (Q \land \neg R) \\
(\neg P \land Q \land R) \lor (\neg P \land \neg Q \land R) \equiv (Q \lor \neg Q) \land (\neg P \land R) \equiv (\neg P \land R)
\end{array}$$

(The associativity, commutativity and distributivity laws for $\land$ and $\lor$ have been used without mentioning.)

(iii) Agreements below: the occurrences of the variable that is substituted with a truth-value constant are underlined; for each internal node, the child on the left corresponds to the substitution with **1**, the child on the right corresponds to the substitution with **0**.



We can see from the tree that our formula is true under following interpretations:

$I(P) = 1,\ I(R) = 0,\ I(Q) = 1$
$I(P) = 0,\ I(R) = 1,$ any value of Q
$I(P) = 0,\ I(R) = 0,\ I(Q) = 1$

This conclusion leads to the following DNF

$$(P \wedge \neg R \wedge Q) \vee (\neg P \wedge R) \vee (\neg P \wedge \neg R \wedge Q),$$

which is equivalent to

$$(\neg R \wedge Q) \vee (\neg P \wedge R).$$

2. Rewrite the following propositional formula in (i) a logically equivalent *conjunctive normal form*, and (ii) a logically equivalent *disjunctive normal form*:

$$(P \to Q) \ \land \ \neg(S \ \to \ R).$$

<u>SOLUTION</u>

$$\begin{array}{ll} (P \to Q) \ \land \ \neg(S \ \to \ R) & \Longrightarrow \\ (\neg P \lor Q) \ \land \ \neg(S \ \to \ R) & \Longrightarrow \\ (\neg P \lor Q) \ \land \ (S \ \land \ \neg R) & \Longrightarrow \\ (\neg P \lor Q) \ \land \ S \ \land \ \neg R & \text{(CNF)} \end{array}$$

$$\begin{array}{ll} (P \to Q) \ \land \ \neg(S \ \to \ R) & \Longrightarrow \\ (\neg P \lor Q) \ \land \ \neg(\neg S \ \lor \ R) & \Longrightarrow \\ (\neg P \lor Q) \ \land \ (S \ \land \ \neg R) & \Longrightarrow \\ (\neg P \ \land \ S \ \land \ \neg R) \ \lor \ (Q \ \land \ S \ \land \ \neg R) & \text{(DNF)} \end{array}$$

3. Which of the following propositional formulas are substitution instances of the formula

$P \to (Q \to P)$?

If a formula is indeed a substitution instance, give the formulas substituted for P, Q.

(i) $\neg R \to (R \to \neg R)$

(ii) $\neg R \to (\neg R \to \neg R)$

(iii) $\neg R \to (\neg R \to R)$

(iv) $(P \land Q \to P) \to ((Q \to P) \to (P \land Q \to P))$

(v) $((P \to P) \to P) \to ((P \to (P \to (P \to P))))$?

<u>SOLUTION</u>

(i) **Yes**. $\neg R$ is substituted for P, R is substituted for Q, i.e., the substitution is $(P/\neg R,\ Q/R)$.

Let us note that since the formula $\neg R \to (R \to \neg R)$ is a tautology, every substitutional instance of this formula is also tautology, including $\neg R \to (R \to \neg R)$.

(ii) **Yes**. $\neg R$ is substituted for P, $\neg R$ is substituted for Q, i.e., the substitution is $(P/\neg R,\ Q/\neg R)$. The result is a tautology.

(iii) **No**. Two different occurrences of the same variable P are replaced with different formulas $\neg R$ and R.

(iv) **Yes**. $(P \wedge Q \to R)$ is substituted for P, $(Q \to R)$ is substituted for Q, i.e., the substitution is $(P/(P \wedge Q \to R),\ Q/(Q \to R))$. The result is a tautology.

(v) **No**. Two different occurrences of the same variable P are replaced with different formulas, $((P \to P) \to P)$ and $(P \to (P \to P))$. Let us note, that the second formula $(P \to (P \to P))$ is a tautology, but the first one is not.

4. Let $P|Q$ be defined as the wff with P and Q having the truth-table below:

P	Q	$P\|Q$
1	1	0
0	1	1
1	0	1
0	0	1

Define $\wedge$, $\vee$, $\neg$ and $\to$ using $|$.

Hint: Look at the truth-table for $P|P$ too!

SOLUTION

As we have seen, the symbol $|$ is known as the *Sheffer stroke.*

It is easy to see that that the truth-table for $|$ is the negated version of the truth-table for $\wedge$, i.e., $v(P|Q) = v(\neg(P \wedge Q))$. That is why in boolean algebra this operation is known as “NAND”.

This solves part of our problem, but we still need to eliminate the $\neg$ in $\neg(P \wedge Q)$ in order to re-write $P \wedge Q$ in terms of $|$ only. We can do this by negating it again, since $\neg\neg P \equiv P$.

The hint says that $P|P$ behaves like $\neg P$. Therefore, to negate $\neg(P \wedge Q)$ (i.e., $P|Q$), all we have to do is to apply $|$ again. This will give us $(P|Q)|(P|Q)$. In order to check that $P \wedge Q$ can indeed be written as

$(P|Q)|(P|Q)$, check that they produce exactly the same values using a truth-table.

Now the rest is easy. We know that $P \vee Q$ is equivalent to $\neg(\neg P \wedge \neg Q)$, which is then re-written as $(P|P)|(Q|Q)$. Similarly, we know that $P \to Q$ is equivalent to $\neg P \vee Q$, so we want $((P|P)|(P|P))|(Q|Q)$ for $P \to Q$. (There are other (and shorter) ways of re-writing $\to$ using the Sheffer stroke, for instance, notice that $P \to Q \equiv \neg(P \wedge \neg Q)$).