5CCS2FC2: Foundations of Computing II

Tutorial Sheet 8

Solutions

8.1 A carpenter produces tables and chairs.

- Each table takes 6 hours of labour, and takes up 2m^2 of storage space, and can be sold for a profit of £30,
- Each chair takes only 3 hours of labour, takes up 0.5m^2 of storage space, and can be sold for a profit of £10.

Customer demand requires that they produce at least three times as many chairs as tables, but there is a maximum storage space for at most four tables (8m^2). Given that the carpenter wishes not to spend more than 40 hours per week, how much of time should be taken up producing tables, and how much should be taken up producing chairs, in order to maximise profits.

Express this problem as a Linear Program, and find a solution using the online LP tool.

`http://www.zweigmedia.com/utilities/lpg/`

SOLUTION: Let x denote the number of tables produced, and let y denote the number of chairs produced. The objective function is the profit which we aim to maximise, which is given by

$$C \; = \; 30x + 10y$$

The customer demands require that the number of chairs is greater than three times the number of tables, *i.e.*, $y \geq 3x$, which we can express alternatively as

$$3x - y \; \leq \; 0$$

The requirement that we spend no longer than 40 hours per week means that

$$6x + 3y \; \leq \; 40$$

and the storage demands require than we use up no more than four tables worth of space, which is 8m^2. Each table takes up 2m^2 and each chair takes up 0.5m^2, so we have the additional constraint that

$$2x + 0.5y \ \leq \ 8$$

(If we wish to specify our linear program with integeral coefficients, we can just multiply both side by 2 to give the equivalent $4x + y \leq 16$).

Hence the full Linear Program can be specified as

$$\boxed{\begin{array}{ll} \text{Maximise:} & 30x + 10y \\ \text{Subject to:} & \begin{array}{rcl} 3x - y & \leq & 0 \\ 6x + 3y & \leq & 40 \\ 2x + 0.5y & \leq & 8 \\ x, y & \geq & 0 \end{array} \end{array}}$$

This have an optimal solution of

$$\boxed{x = 1.33 \qquad \text{and} \qquad y = 10.67}$$

for a optimal profit of $C =$ £146.67. Of course, in practise we cannot produce fractional quantities of tables and chairs, so this problem would be more accurately described as an Integer Program, the solution for which would be

$$\boxed{x = 2 \quad \text{and} \quad y = 8} \quad \text{or} \quad \boxed{x = 1 \quad \text{and} \quad y = 11}$$

both producing an optimal profit of $C =$ £140.

N.B. This example was taken from a problem set created by J.E.Beasley at Brunel University. The full list contains many great exams: `http://people.brunel.ac.uk/~mastjjb/jeb/or/morelp.html`

8.2 Use the Simplex Algorithm to find the optimal solutions to the following Linear Programs:

(i)

Maximise: $x + 3y$

Subject to:
$$\begin{aligned} 3y - x &\leq 9 \\ 4x + 6y &\leq 27 \\ 5x - 6y &\leq 1 \\ x, y &\geq 0 \end{aligned}$$

(ii)

Minimise: $y - x$

Subject to:
$$\begin{aligned} x + y &\geq 3 \\ x + 3y &\leq 6 \\ x - 3y &\leq 3 \\ x, y &\geq 0 \end{aligned}$$

<u>SOLUTION:</u>

(i) The initial tableau is given by

$$\left[\begin{array}{cccccc|c} -1 & 3 & 1 & 0 & 0 & 0 & 9 \\ 4 & 6 & 0 & 1 & 0 & 0 & 27 \\ 5 & -6 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

- In the first iteration, the pivot column is $\texttt{column}(2)$, the respective row quotients are therefore:

$$\boxed{\texttt{row}(1) = \frac{9}{3} = 3, \qquad \texttt{row}(2) = \frac{27}{6} = 4.5, \qquad \texttt{row}(3) = \frac{1}{-6} \approx -0.167}$$

  Therefore the pivot row is $\texttt{row}(1)$, since it has the smallest positive row quotient. The next tableau is therefore given by the following row transformation:

$$\left[\begin{array}{cccccc|c} -0.33 & 1 & 0.33 & 0 & 0 & 0 & 3 \\ 6 & 0 & -2 & 1 & 0 & 0 & 9 \\ 3 & 0 & 2 & 0 & 1 & 0 & 19 \\ \hline -2 & 0 & 1 & 0 & 0 & 1 & 9 \end{array}\right]$$

$$\begin{array}{lcl} \texttt{row}(1) & \leftarrow & \frac{1}{3}\texttt{row}(1) \\ \texttt{row}(2) & \leftarrow & \texttt{row}(2) - 2\texttt{row}(1) \\ \texttt{row}(3) & \leftarrow & \texttt{row}(3) + 2\texttt{row}(1) \\ \texttt{row}(4) & \leftarrow & \texttt{row}(4) + \texttt{row}(1) \end{array}$$

- In the second iteration, the pivot column is $\texttt{column}(1)$, which gives the row quotients:

$$\boxed{\texttt{row}(1) = \frac{3}{-0.33} = -1, \qquad \texttt{row}(2) = \frac{9}{6} = 1.5, \qquad \texttt{row}(3) = \frac{18}{3} \approx 6.33}$$

Therefore the pivot row is $\texttt{row}(2)$, and the next tableau is given by

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0.22 & 0.056 & 0 & 0 & 3.5 \\ 1 & 0 & -0.33 & 0.17 & 0 & 0 & 1.5 \\ 0 & 0 & 3 & -0.5 & 1 & 0 & 14.5 \\ \hline 0 & 0 & 0.33 & 0.33 & 0 & 1 & 12 \end{array}\right]$$

$$\boxed{\begin{array}{lll} \texttt{row}(1) & \leftarrow & \texttt{row}(1) + \frac{1}{18}\texttt{row}(2) \\ \texttt{row}(2) & \leftarrow & \frac{1}{6}\texttt{row}(2) \\ \texttt{row}(3) & \leftarrow & \texttt{row}(3) - \frac{1}{2}\texttt{row}(2) \\ \texttt{row}(4) & \leftarrow & \texttt{row}(4) + \frac{1}{3}\texttt{row}(2) \end{array}}$$

- Since all the variable coefficients in the final row are positive, we cannot improve the value of the objective function any further, since

  $$C = 12 - 0.33s_1 - 0.33s_2$$

  Hence, the maximum value of C must be obtained when s_1 and s_2 are both zero, which is to say that $C = 12$. It then follows that

  $$\boxed{x = 1.5, \qquad y = 3.5, \qquad \text{and} \qquad s_3 = 14.5}$$

(ii) The initial tableau is given by

$$\left[\begin{array}{cccccc|c} -1 & -1 & 1 & 0 & 0 & 0 & -3 \\ 1 & 3 & 0 & 1 & 0 & 0 & 6 \\ 1 & -3 & 0 & 0 & 1 & 0 & 3 \\ \hline -1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

- In the first iteration, the pivot column is $\texttt{column}(1)$, the respective row quotients are therefore:

  $$\boxed{\texttt{row}(1) = \frac{-3}{-1} = 3, \qquad \texttt{row}(2) = \frac{6}{1} = 6, \qquad \texttt{row}(3) = \frac{-3}{-1} = 3}$$

  We have two possible options for the pivot column $\texttt{row}(1)$ or $\texttt{row}(3)$. If we take $\texttt{row}(3)$ to be the pivot row, the next tableau is therefore given by the following row transformation:

$$\left[\begin{array}{cccccc|c} 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 1 & -1 & 0 & 3 \\ 1 & -3 & 0 & 0 & 1 & 0 & 3 \\ \hline 0 & -2 & 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$$\boxed{\begin{array}{lcl} \texttt{row}(1) & \leftarrow & \texttt{row}(1) + \texttt{row}(3) \\ \texttt{row}(2) & \leftarrow & \texttt{row}(2) - \texttt{row}(3) \\ \texttt{row}(3) & \leftarrow & \texttt{row}(3) \\ \texttt{row}(4) & \leftarrow & \texttt{row}(4) + \texttt{row}(3) \end{array}}$$

- In the second iteration, the pivot column is $\texttt{column}(2)$, which gives the row quotients:

  $$\boxed{\texttt{row}(1) = \frac{0}{-4} = 0, \qquad \texttt{row}(2) = \frac{3}{6} = 0.5, \qquad \texttt{row}(3) = \frac{3}{-3} = -1}$$

  Therefore the pivot row is $\texttt{row}(2)$ (remember that we are only interested in *positive*, *i.e.*, non-zero row quotients). The next tableau is given by

$$\left[\begin{array}{cccccc|c} 0 & 0 & 1 & 0.66 & 0.33 & 0 & 2 \\ 0 & 1 & 0 & 0.17 & -0.17 & 0 & 0.5 \\ 1 & 0 & 0 & 0.5 & 0.5 & 0 & 4.5 \\ \hline 0 & 0 & 0 & 0.33 & 0.66 & 1 & 4 \end{array}\right]$$

$$\boxed{\begin{array}{lcl} \texttt{row}(1) & \leftarrow & \texttt{row}(1) + \frac{2}{3}\texttt{row}(2) \\ \texttt{row}(2) & \leftarrow & \frac{1}{6}\texttt{row}(2) \\ \texttt{row}(3) & \leftarrow & \texttt{row}(3) + \frac{1}{2}\texttt{row}(2) \\ \texttt{row}(4) & \leftarrow & \texttt{row}(4) + \frac{1}{3}\texttt{row}(2) \end{array}}$$

- Since all the variable coefficients in the final row are positive, we cannot improve the value of the objective function any further, since

  $$C = 4 - 0.33s_1 - 0.66s_2$$

  Hence, the maximum value of C must be obtained when s_1 and s_2 are both zero, which is to say that $C = 4$. It then follows that

  $$\boxed{x = 4.5, \qquad y = 0.5, \qquad \text{and} \qquad s_3 = 2}$$

8.3 Apply the Branch-and-Bound method to solve the following Integer Program:

$$
\boxed{
\begin{array}{lrcl}
\text{Maximise:} & x + 2y & & \\
\text{Subject to:} & 2y - x & \leq & 4 \\
 & 3x + 2y & \geq & 6 \\
 & 4x + 5y & \leq & 20 \\
 & x, y & \geq & 0 \\
 & x, y & \in & \mathbb{Z}
\end{array}
}
$$

You should use the online tool to provide the solutions to the linear relaxations given at each stage:

`http://www.zweigmedia.com/utilities/lpg/`

<u>SOLUTION:</u> The initial solution to the linear relaxation is given by

$$\boxed{x = 1.5385 \quad y = 2.7692 \quad C = 7.0769}$$

The full branch-and-bound tree is given below: