4CCS1ELA – Elementary Logic with Applications

3 – Model and proof theories

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Outline

Satisfiability

Satisfiability

A formula F is **satisfiable** if there is an interpretation v that makes the formula F true. In this case, we say that v **satisfies** F.

A set $\mathcal{S} = \{A_1, \ldots, A_n\}$ of propositional formulae is **satisfiable** (**consistent**) if there is an interpretation v satisfying *every* formula in $\mathcal{S}$.

	p_1	$\ldots$	p_m	A_1	$\ldots$	A_n
v	e_1	$\ldots$	e_m	1	$\ldots$	1

$p_1, \ldots, p_m$ are all the propositional symbols appearing in $\mathcal{S}$.

The set of formulae $\{A_1, \ldots, A_n\}$ is satisfiable if, and only if, the conjunction $A_1 \land \ldots \land A_n$ is satisfiable

Example

Let $A_1 = P{\to}Q$, $A_2 = Q{\to}R$ and $A_3 = R{\to}P$ and $\mathcal{S}$ be the set of formulae $\{A_1, A_2, A_3\}$. The combined truth-table for $\mathcal{S}$ is:

	p_1	p_2	p_3	A_1	A_2	A_3
	P	Q	R	$P \to Q$	$Q \to R$	$R \to P$
v_0	0	0	0	1	1	1
v_1	0	0	1	1	1	0
v_2	0	1	0	1	0	1
v_3	0	1	1	1	1	0
v_4	1	0	0	0	1	1
v_5	1	0	1	0	1	1
v_6	1	1	0	1	0	1
v_7	1	1	1	1	1	1

Thus, $\mathcal{S}$ is satisfiable, since v_0 and v_7 satisfy every formula in $\mathcal{S}$ (we can also say that $\mathcal{S}$ is *consistent*).

Models

A *model* is an interpretation that makes a formula (or set of formulae) true.

We denote the fact that v is a model of A by $v \Vdash A$.

The set of all models of a formula A is denoted by $\text{mod}(A)$. We use the same symbol for a set of formulae $\mathcal{S}$, as before.

Thus, in Example one we have that $\text{mod}(\mathcal{S}) = \{v_0, v_7\}$.

Example

Let $\mathcal{S}$ be the set of formulae $\{P \leftrightarrow \neg Q, Q \leftrightarrow R, R \leftrightarrow P\}$. The truth-table for its formulae is:

	P	Q	R	$P \leftrightarrow \neg Q$	$Q \leftrightarrow R$	$R \leftrightarrow P$
v_0	0	0	0	0	1	1
v_1	0	0	1	0	0	0
v_2	0	1	0	1	0	1
v_3	0	1	1	1	1	0
v_4	1	0	0	1	1	0
v_5	1	0	1	1	0	1
v_6	1	1	0	0	0	0
v_7	1	1	1	0	1	1

$\mathcal{S}$ is **not** satisfiable (i.e., $\mathcal{S}$ is inconsistent) and thus, $\text{mod}(\mathcal{S}) = \varnothing$.

Logical consequence

Logical (semantic) consequence

In a valid **argument**, we say informally that a conclusion B "follows" from a set of premises $A_1, \ldots, A_n$.

More formally, we say that the formula B is a *logical consequence* of the set of formulae $A_1, \ldots, A_n$, if the following implication holds for *every* interpretation v:

$$\text{If } v(A_i) = 1, \text{ for all } 1 \leq i \leq n, \text{ then } v(B) = 1.$$

Logical consequence

The following statements are equivalent.

- B is a logical consequence of $A_1, \ldots, A_n$.
- $A_1, \ldots, A_n \models B$.
- The argument $A_1, \ldots, A_n \models B$ is *valid*.
- B is *semantically entailed (or implied)* by $A_1, \ldots, A_n$
- B is a *valid consequence* of $A_1, \ldots, A_n$.

Example

Show that $P, P \to Q \models P \wedge Q$.

Solution:

	P	Q	$P \to Q$	$P \wedge Q$
	0	0	1	0
	0	1	1	0
	1	0	0	0
$*$	1	1	1	1

The statement follows because in every row in which the columns for P and $P \to Q$ contain 1, so does the column for $P \wedge Q$.

In this case the only relevant row is the one with $*$.

Equivalent definitions of $\models$

$A_1, \ldots, A_n \models B$ if and only if

- $A_1 \wedge \ldots \wedge A_n \to B$ is a tautology (i.e., logically valid).
- $A_1 \wedge \ldots \wedge A_n \wedge \neg B$ is a contradiction (i.e., unsatisfiable)
- The set $\{A_1, \ldots, A_n, \neg B\}$ is inconsistent.

An argument that is not valid is said to be **invalid**. $A_1, \ldots, A_n \not\models B$ if there exists an interpretation v such that

$$v(A_i) = 1 \text{ for all } 1 \leq i \leq n, \text{ but } v(B) = 0.$$

Examples

1. Show that $P \not\models Q$, where P, Q are atoms.

   Solution: Take the interpretation v with

   $$v(P) = 1 \text{ and } v(Q) = 0.$$

2. Show that $P \to Q \not\models Q$, where P, Q are atoms.

   Solution: Take the interpretation v with

   $$v(P) = 0 \text{ and } v(Q) = 0.$$

   These are not solutions:

   - The interpretation v_1 with $v_1(P) = 1$, $v_1(Q) = 0$, because v_1 does not satisfy $P \to Q$.
   - The interpretation v_2 with $v_1(P) = 1$, $v_1(Q) = 1$, because v_2 does satisfy Q.

Validity

Checking the validity of arguments

If Jack takes a holiday, then Jill will be happy and she will not cry. Jack will take a holiday and if Jill is happy she will cry. **Therefore** *Jack will take a holiday.*

Let J stand for 'Jack will take a holiday'; H stand for 'Jill will be happy'; and C stand for 'Jill will cry'.

In semantical terms, the argument is $J \to (H \land \neg C), J \land (H \to C) \models J$:

$$\begin{array}{l} J \to (H \land \neg C) \\ J \land (H \to C) \\ \hline J \end{array}$$

The argument is valid because the set of formulae associated with the premises is unsatisfiable.

Alternative definition of $\models$

Let the symbol $\mathcal{I}$ denote the set of all interpretations.

Let $\mathcal{S} = \{A_1, \ldots, A_n\}$. We have that

$$\mathcal{S} \models B \text{ iff } \text{mod}\,(\mathcal{S}) \subseteq \text{mod}(B)$$

Notice that $\text{mod}(\neg B) = \mathcal{I} - \text{mod}(B)$ and hence $\text{mod}(B) \cap \text{mod}(\neg B) = \varnothing$.

Therefore, if $\mathcal{S} \models B$, then $\text{mod}(\mathcal{S}) \cap \text{mod}(\neg B) = \emptyset$ and hence $\mathcal{S} \cup \{\neg B\}$ is unsatisfiable.

Logical consequence and arguments

The Internet Encyclopedia of Philosophy defines an *argument* as a sequence of statements (*the premises, or the hypotheses*) which are intended to provide support, justification or evidence for the truth of another statement (the conclusion).

In the argument

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \\ \hline B \end{array}$$

$A_1, \ldots, A_n$ are the premises and B is the conclusion (you can read '$A_1, \ldots, A_n$, *therefore* B').

Exercise – Valid and invalid arguments

We have seen that we can prove the validity of an argument in several ways.

1. Using truth-tables demonstrate that $P \vee Q, \neg P \models Q$:
   - By showing that whenever $(P \vee Q) \wedge \neg P$ is true then Q is true.
   - By showing that $((P \vee Q) \wedge \neg P) \rightarrow Q)$ is a tautology.
   - By using the definition of $\models$ in terms of unsatisfiability, that is by showing that $(P \vee Q) \wedge \neg P \wedge \neg Q$ is unsatisfiable.

Exercise – Valid and invalid arguments

2. Demonstrate that $P \to Q \not\models P$.

Inference systems

Inference systems

At the syntactical level, reasoning can be achieved by an **inference system (a deduction system)**, a *computational device* deriving formulae that follow from available premises.

$$A_1, \ldots, A_n \vdash B \text{ denotes } B \text{ is derived from } A_1, \ldots, A_n.$$

To be useful at all, an inference system must be **sound** (correct):

$$\text{if } A_1, \ldots, A_n \vdash B, \text{ then } A_1, \ldots, A_n \models B.$$

Ideally, an inference system should also be **complete:**

$$\text{if } A_1, \ldots, A_n \models B, \text{ then } A_1, \ldots, A_n \vdash B.$$

Note: $\boxed{\models\equiv\vdash}$ is the logo of the Association for Logic Programming.

Inference (or deduction) systems

An inference system is typically described by a set of **inference (or derivation) rules.** Each inference rule has the form:

$$\frac{A_1, \ldots, A_n}{B} \quad \begin{array}{ll} \Rightarrow & \textit{Premises} \\ \Rightarrow & \textit{Conclusion} \end{array}$$

This indicates that the conclusion follows from the premises above it.

In other words, the formula B can be generated as long as all of the formulae $A_1, \ldots, A_n$ appear before it (in any order) in the proof.

Soundness and completeness

In general one wants to relate the inference system to the semantics of the logic it is used for. There are two important properties describing this relationship.

Soundness: An inference system is sound if all of its rules are sound, i.e., whenever $A_1, \ldots, A_n \vdash B$, then $A_1, \ldots, A_n \models B$.

Completeness: An inference system is complete if whenever $A_1, \ldots, A_n \models B$, B can be derived from $A_1, \ldots, A_n$ using its inference rules.

Natural deduction

Natural deduction

Natural deduction is an inference system consisting of a number of rules that manipulate assumptions towards conclusions.

There are two ways of manipulating formulae:

- We either obtain new formulae by breaking down existing ones (via an **elimination rule**);
- or we combine some formulae with connectives to generate new ones (via an **introduction rule**);.

Natural deduction is a form of *forward reasoning* system

.

Introduction and elimination rules

Since our rules will either eliminate or introduce a connective, our *proof system* will have a pair rules for each connective of the language.

Θ-I will be used to designate the introduction rule of the connective Θ, and **Θ**-E will be used for its elimination rule.

For instance, $\wedge$-E will indicate the rule that eliminates the $\wedge$ connective and $\wedge$-I, the rule that introduces it, and so forth. If there is more than one rule for introduction or elimination, they will be numbered.

Proofs

Format of a natural deduction proof

Let $\mathcal{S}$ be a set of formulae. A natural deduction proof of $\mathcal{S} \vdash B$ is a sequential numbered list such that

- each item contains a main formula and its *justification*
- the main formula is obtained
  - either directly from $\mathcal{S}$ and justified as 'data';
  - or by manipulating previous formulae of the sequence according to one single rule – the rule and the formulae must be given as justifications;
- the last item in the sequence contains B as the main formula

Applying a rule

Before a rule can be applied its premises (requirements) must already appear in the proof.

The result of the application of the rule is a new item in the proof whose main formula is the conclusion of the rule.

The addition of the new item to the proof is justified by describing the rule used and the item numbers where the formulae in the rule's premises appear in the current proof.

Applying a rule

For example, consider the ($\vee$I) rule:

$$\frac{A}{A \vee B} \quad (\vee\mathrm{I})$$

This rule has A as a premise. So before it can be applied, you need to have a line in the proof with A as its main formula.

The result of the application of the rule is a new line with $A \vee B$ as its main formula and "from y. and ($\vee$I)" as its justification, where y is the number of the line in the proof where A appears.

y. A

$\vdots$

x. $A \vee B$ from y. and ($\vee$I)

Example

1. $A \wedge B$ data
2. $C \wedge D$ data
3. B from 1. and ($\wedge$E)
4. $\vdots$
5. C from 2. and ($\wedge$E)
6. $B \wedge C$ from 3., 5., and ($\wedge$I)
7. $\vdots$

Rules for conjunction ($\wedge$)

Look at the truth-table for $\wedge$:

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

$A \wedge B$ is only true when both A and B are true. Similarly, whenever we know that $A \wedge B$ is true, we also know that A and B are both true.

This gives us the ($\wedge$I) and ($\wedge$E) rules below.

$$\frac{A, B}{A \wedge B} \ (\wedge\mathrm{I}) \qquad \frac{A \wedge B}{A} \text{ and } \frac{A \wedge B}{B} \ (\wedge\mathrm{E})$$

Introduction of the disjunction ((∨I))

Look at the truth-table for ∨:

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

We can obtain $A \vee B$ as long as at least one of A and B is true. This gives us the introduction rules ((∨I)) below.

$$\frac{A}{A \vee B}\ (\vee I) \quad \text{and} \quad \frac{B}{A \vee B}\ (\vee I)$$

Example with the rules defined so far

Prove that $A \wedge B \vdash B \vee C$

1. $A \wedge B$ data
2. B from (1.) and (∧E)
3. $B \vee C$ from (2.) and (∨I)

Elimination of disjunction (∨E)

The first (∨E) we present is an indirect rule.

We can eliminate a disjunction by stating that some proposition that depends on each of its constituents is also true:

$$\frac{A \to C, B \to C, A \lor B}{C} \quad (\lor\text{E})$$

Elimination of the implication (→E)

The elimination of the implication rule can be easily understood from the connective's truth-table:

A	B	$A \to B$
0	0	1
0	1	1
1	0	0
1	1	1

If we know that an implication and its antecedent are both true, then we must also have that the consequent of the implication is also true.

$$\frac{A, A \to B}{B} \quad (\to\text{E})$$

Introduction of the implication ((→E))

The introduction of the implication is a little more complicated.

If we want to derive $A \rightarrow B$ from some premises, then we must show that these premises together with A imply B.

$$\frac{\text{premises}}{A \rightarrow B}\,, \qquad \text{if} \quad \frac{\text{premises}, A}{B} \qquad (\rightarrow \mathrm{I})$$

The proof that B follows from A (and premises) is done in a separate subcomputation box (explained next)

Subcomputations

Subcomputation boxes

A subcomputation box defines a sub-proof that is dependent on an extra assumption.

The box starts with the extra assumption that the antecedent of the implication is true and is used to define the scope in which this assumption can be used.

To introduce the implication, the box must constitute a valid sub-proof in its own right, using as many conclusions already obtained in the main proof (before the box's introduction) as necessary, and whose conclusion is the consequent of the implication.

Subcomputation boxes

The box demonstrates that the consequent is a valid conclusion of the current proof, provided that the antecedent is also true, and thus provides a justification for the derivation of the implication in the current proof. The reasoning is as follows.

Either the extra assumption used in the box is true or it is false.

If the assumption is true, the box provides a valid proof of the conclusion, so the implication holds. Otherwise, the implication also holds, since the implication is true when its antecedent is false (look at $\rightarrow$'s truth-table).

Therefore, if the box manages to show the conclusion, the implication will be true whether or not the assumption is true.

Example

Consider the following proof: $A \to B, B \to C \vdash A \to C$

1.	$A \to B$	data
2.	$B \to C$	data
3.	$A \to C$	($\to$I), from the subcomputation below

$\underline{C}$

3.1	A	assumption
3.2	B	from (3.1), (1.) and ($\to$E)
3.3	C	from (3.2), (2.) and ($\to$E)

You can think of the main box as a proof that $A \to B, B \to C \vdash A \to C$.

To remember the objective of the box, it is useful to annotate its top right corner with the conclusion of the implication.

Using subcomputation boxes

In order to correctly perform a subcomputation, follow these rules

- the assumption added at the beginning of a subcomputation box may only be used inside that box
- we may use inside a box any data available previously including assumptions and formulae derived in any one of its enclosing boxes
- the conclusion obtained at the end of the box generally depends on the the box's initial assumption and cannot be used on its own.
  In the previous example, C cannot be used on its own, since its proof depended on A's assumption (line 3.1).

Rules for negation ($\neg$)

The rules for negation are done indirectly and are a bit trickier.

In order to conclude $\neg A$, we must show that if A were the case, then we would reach a contradiction (e.g., B and $\neg B$). Thus,

$$\frac{A \to B, A \to \neg B}{\neg A} \quad (\neg\mathrm{I})$$

Subcomputations

Rules for negation ($\neg$)

The rules for negation are done indirectly and are a bit trickier.

In order to conclude $\neg A$, we must show that if A were the case, then we would reach a contradiction (e.g., B and $\neg B$). Thus,

$$\frac{A \to B, A \to \neg B}{\neg A} \quad (\neg\mathrm{I})$$

$\neg$ is eliminated through the same reasoning, except we start with the negated formula ($\neg A$).

$$\frac{\neg A \to B, \neg A \to \neg B}{A} \quad (\neg\mathrm{E})$$

Exercises

Class exercise

Prove that $A \wedge B \vdash B \wedge A$

Class exercise

Prove that $(A \vee B) \rightarrow C \vdash (A \rightarrow C) \wedge (B \rightarrow C)$

Class exercise

Prove that $\vdash \neg(A \wedge \neg A)$

(This shows that $\neg(A \wedge \neg A)$ is a tautology. Why?)

Class exercise

Prove that $A \to B \vdash \neg(A \wedge \neg B)$ and $\neg(A \wedge \neg B) \vdash A \to B$

(This shows that $A \to B \equiv \neg(A \wedge \neg B)$. Why?)

Variant rules

Some of the rules presented have *variants*. Their use may simplify some proofs.

Disjunction $\dfrac{A \vee B, \neg A}{B}$ $(\vee E1)$ $\dfrac{A \vee B, \neg B}{A}$ $(\vee E2)$

Negation $\dfrac{\neg\neg A}{A}$ $(\neg E1)$ $\dfrac{\neg A \to B, A \to B}{B}$ $(\neg E2)$

Implication $\dfrac{\neg A}{A \to B}$ $(\to I1)$ $\dfrac{B}{A \to B}$ $(\to I2)$

$\dfrac{A \to B}{\neg A \vee B}$ $(\to E1)$

All variants rules can be obtained from the basic set of rules.

Home exercise

Show that each variant rule can be obtained from the basic set of rules.

Correctness and completeness

Correctness and completeness are two important properties of systems in general

In loose terms ...

- Correctness is related to “following the expected behaviour”
- Completeness is related to being able to do *all* that is expected

In logic, we will usually talk about these concepts with respect to the proof and model theories. A proof system is correct if it only produces conclusions that agree with the semantics and it is complete if it can produce all valid conclusions.

How to practice natural deduction proofs

Now you have everything you need to provide a natural deduction proof to every valid argument in propositional logic.

These are some of the things you can prove:

- All arguments shown to be valid using truth-tables.
- Both directions of every logical equivalence we have seen.
- That all tautologies are valid.

Practice makes perfect!

Exercises

To know more. . .

In the "Elementary Logic with Applications" book

- Satisfiability, logical consequence and validity can be found in Sections 1.2 and 1.3; and
- Natural deduction is explained in detail in Chapter 3.