4CCS1ELA – Elementary Logic with Applications

Small Group Tutorial 1 (week 2) – Solutions

Propositional Logic

1. A police officer has collected witness statements that enables the following assertions, A_1, A_2, A_3 and A_4, to be made from the evidence gathered about four crime suspects Ahmed, Bob, Carol, and Dot:

a_1: *If Ahmed is telling the truth then so is Bob.*

a_2: *Bob and Carol cannot both be telling the truth.*

a_3: *Carol and Dot are not both lying.*

a_4: *If Dot is telling the truth then Bob is lying.*

Represent the information above in propositional logic.

Heuristic for formalising English. Pick the smallest statements without *and, or, if . . . then . . .* etc. about which you could answer the question 'Is it true or false?'. Using propositional variables to stand for these statements connect them with the relevant logical connectives $\neg$, $\wedge, \vee, \rightarrow$ etc.

SOLUTION

Let A denote that "Ahmed is telling the truth,
let B denote that "Bob is telling the truth,
let C denote that "Carol is telling the truth, and
let D denote that 'Dot is telling the truth.
Then, the information may be represented by the following propositions:
a_1: $A \rightarrow B$.
a_2: $\neg(B \wedge C)$. Equivalently, $\neg B \vee \neg C$.
a_3: $\neg(\neg C \wedge \neg D)$. Equivalently, $C \vee D$.
a_4: $D \rightarrow \neg B$.

2. Which of the following formulas are tautologies? Check using truth tables.

(i) $P \vee P$.

(ii) $P \vee (Q \wedge P)$.

(iii) $\neg\neg P \leftrightarrow P$.

(iv) $\neg P \rightarrow \neg P$.

<u>SOLUTION</u>

(i) $P \vee P$: no tautology. Take I with $I(P) = 0$.

(ii) $P \vee (Q \wedge P)$: no tautology. Take I with $I(P) = 0$.

(iii) $\neg\neg P \leftrightarrow P$ is a tautology.

(iv) $\neg P \rightarrow \neg P$ is a tautology.

3.

(i) If $\neg(P \leftrightarrow Q)$ is true then what can be said about the truth values of $P \wedge Q$ and $P \vee Q$?

(ii) If $P \rightarrow Q$ is false then what can be said about the truth value of $P \wedge \neg Q$?

(iii) If $P \rightarrow Q$ is true then what can be said about the truth value of $P \vee R \rightarrow Q \vee R$?

<u>SOLUTION</u>

(i) $P \wedge Q$ is false. $P \vee Q$ is true.

(ii) $P \wedge \neg Q$ is true.

(iii) $P \vee R \rightarrow Q \vee R$ is true whatever R is.

4. Determine whether the following proposition is a tautology, a contradiction, or neither:

$$(((P \to Q) \wedge (R \to S) \wedge (\neg Q \vee \neg S)) \to (\neg P \vee \neg R)).$$

SOLUTION

The sentence is a tautology. To demonstrate this, a truth table may be used or Quine's method or a higher-level argument. For the third one, suppose that the sentence may be false. Then, we require a truth value assignment I such that $I(P) = 1$ and $I(R) = 1$. Then, assuming the truth of the premises, for $I(P \to Q) = 1$ we must have $I(Q) = 1$ and so from $I(R \to S)$ we have $I(S) = 1$. However, then $I(\neg Q_{\vee} \neg S) = 0$. Hence, it is impossible for $(((P \to Q) \wedge (R \to S) \wedge (\neg Q \vee \neg S)) \to (\neg P \vee \neg R))$ to be false and thus the sentence is a tautology.

5. Consider the following formula

$$(P \wedge \neg Q) \to \neg(Q \vee \neg P)$$

(i) Draw up a truth table for this formula and determine whether this formula is a tautology, a contradiction or neither.

(ii) Read off from the truth table a disjunctive normal form (DNF) of this formula.

SOLUTION

(i)

P	Q	$P \wedge \neg Q$	$\to$	$\neg(Q \vee \neg P)$
1	1	0	1	0
1	0	1	1	1
0	1	0	1	0
0	0	0	1	0

This is a tautology (because principal column contains only 1s).

(ii) The DNF constructed from the truth table is

$(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

This DNF is equivalent to the truth constant **1**.