4CCS1ELA – ELEMENTARY LOGIC WITH APPLICATIONS

2 – SYNTACTICAL TRANSFORMATIONS

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Outline

1. Fundamental Logical Equivalences
2. Normal Forms
3. Complete Sets of Connectives
4. Substitution Instances
5. Quine's Method

Fundamental Logical Equivalences

Fundamental Logical Equivalences

$P \vee P$	$\equiv$	P	idempotency
$P \wedge P$	$\equiv$	P	idempotency
$P \vee Q$	$\equiv$	$Q \vee P$	commutativity
$P \wedge Q$	$\equiv$	$Q \wedge P$	commutativity
$P \vee (Q \vee R)$	$\equiv$	$(P \vee Q) \vee R$	associativity
$P \wedge (Q \wedge R)$	$\equiv$	$(P \wedge Q) \wedge R$	associativity
$P \wedge (Q \vee R)$	$\equiv$	$(P \wedge Q) \vee (P \wedge R)$	distributivity
$P \vee (Q \wedge R)$	$\equiv$	$(P \vee Q) \wedge (P \vee R)$	distributivity
$\neg(P \wedge Q)$	$\equiv$	$(\neg P \vee \neg Q)$	De Morgan's law
$\neg(P \vee Q)$	$\equiv$	$(\neg P \wedge \neg Q)$	De Morgan's law

Fundamental Logical Equivalences (cont)

$$
\begin{aligned}
P \vee \neg P &\equiv \mathbf{1} && \text{excluded middle}\\
P \wedge \neg P &\equiv \mathbf{0}\\
\neg\neg P &\equiv P\\
P \vee \mathbf{1} &\equiv \mathbf{1}\\
P \wedge \mathbf{1} &\equiv P\\
P \vee \mathbf{0} &\equiv P\\
P \wedge \mathbf{0} &\equiv \mathbf{0}\\
P \to Q &\equiv \neg P \vee Q\\
P \to Q &\equiv \neg Q \to \neg P && \text{contraposition}\\
\neg(P \to Q) &\equiv P \wedge \neg Q
\end{aligned}
$$

A Few More Useful Equivalences

$$
\begin{aligned}
\mathbf{1} \leftrightarrow P &\equiv P\\
\mathbf{0} \leftrightarrow P &\equiv \neg P\\
\mathbf{1} \to P &\equiv P\\
P \to \mathbf{1} &\equiv \mathbf{1}\\
\mathbf{0} \to P &\equiv \mathbf{1}\\
P \to \mathbf{0} &\equiv \neg P\\
P \leftrightarrow P &\equiv \mathbf{1}\\
P \leftrightarrow \neg Q &\equiv \neg(P \leftrightarrow Q)\\
P \vee (P \wedge Q) &\equiv P && \text{absorption}\\
P \wedge (P \vee Q) &\equiv P && \text{absorption}
\end{aligned}
$$

The Equivalence Replacement Rule

Any subformula can be replaced by an equivalent formula without changing the truth-value of its containing formula.

In other words, from

$$G \equiv H$$

we can conclude

$$A(\cdots G \cdots) \equiv A(\cdots H \cdots)$$

Example. From $\neg(Q \vee R) \equiv \neg Q \wedge \neg R$ we obtain, by replacement,

$$(P \to \underbrace{\neg(Q \vee R)}_{G}) \wedge \neg Q \equiv (P \to \underbrace{(\neg Q \wedge \neg R)}_{H}) \wedge \neg Q$$

Example

If we replace the second occurrence of the (sub)formula $P \vee Q$ in the formula A

$$(P \vee Q) \to (R \leftrightarrow \underline{(P \vee Q)})$$

by the equivalent formula $Q \vee P$, then we obtain the equivalent formula A'

$$(P \vee Q) \to (R \leftrightarrow \underline{(Q \vee P)})$$

Replacement Can Be Used to Simplify Formulae

The formula $(P \wedge Q) \vee \neg(\neg P \vee Q)$ can be simplified as follows.

$$\begin{aligned}
(P \wedge Q) \vee \neg(\neg P \vee Q) &\equiv (P \wedge Q) \vee (\neg\neg P \wedge \neg Q) \\
&\equiv (P \wedge Q) \vee (P \wedge \neg Q) \\
&\equiv P \wedge (Q \vee \neg Q) \\
&\equiv P \wedge \mathbf{1} \\
&\equiv P
\end{aligned}$$

Thus, the initial formula $(P \wedge Q) \vee \neg(\neg P \vee Q)$ is equivalent to the formula P.

Associativity and the Use of Parentheses

Since the associative law holds for $\vee$ and $\wedge$, it is common practice to drop parentheses in situations such as

$$P \wedge ((Q \wedge R) \wedge S),$$

yielding

$$P \wedge Q \wedge R \wedge S.$$

Likewise, we may write

$$P \vee Q \vee R \vee S,$$

instead of

$$(P \vee Q) \vee (R \vee S), \text{ or } P \vee (Q \vee (R \vee S)), \text{ or } ((P \vee Q) \vee R) \vee S, \text{ etc .}$$

Normal Forms

Literals

A **literal** is a propositional symbol or the negation of a propositional symbol.

Examples.

P, $\neg P$, Q, $\neg Q$, etc

Some formulae that are **not** literals.

$\neg\neg P$, $P \wedge Q$, $Q \vee Q$, etc

Literals can have at most one connective and it must be the negation

Disjunctive Normal Form

Disjunctive normal form (DNF)

A formula is in DNF if it is a disjunction of one or more formulae, each of which is a conjunction of one or more literals.

Examples.

$(P \wedge \neg Q) \vee (\neg P \wedge Q)$

P (special case, think of it as $P \vee P$)

$P \wedge Q$

$P \vee Q$

Every propositional formula is equivalent to some formula in DNF

Conjunctive Normal Form

Conjunctive normal form (CNF)

A formula is in CNF if it is a conjunction of one or more formulae, each of which is a disjunction of one or more literals.

Examples.

$(P \vee \neg Q) \wedge (\neg P \vee Q)$

P (special case, think of it as $P \wedge P$)

$P \vee \neg Q$

$P \wedge \neg Q$

Every propositional formula is equivalent to some formula in CNF

Rewrite Rules to Obtain a Normal Form – 1

To put a formula into **disjunctive** normal form (DNF), apply the following transformations:

$$
\begin{array}{rcl}
F \to G & \Longrightarrow & \neg F \vee G \\
F \leftrightarrow G & \Longrightarrow & (F \to G) \wedge (G \to F) \\
\neg (F \vee G) & \Longrightarrow & \neg F \wedge \neg G \\
\neg (F \wedge G) & \Longrightarrow & \neg F \vee \neg G \\
\neg \neg F & \Longrightarrow & F \\
F \wedge (G \vee H) & \Longrightarrow & (F \wedge G) \vee (F \wedge H) \\
(F \vee G) \wedge H & \Longrightarrow & (F \wedge H) \vee (G \wedge H)
\end{array}
$$

These rules are applied until no further applications are possible.

Rewrite Rules to Obtain a Normal Form – 2

To put a formula into **conjunctive** normal form (CNF), apply the following transformations:

$$
\begin{array}{rcl}
F \to G & \Longrightarrow & \neg F \vee G \\
F \leftrightarrow G & \Longrightarrow & (F \to G) \wedge (G \to F) \\
\neg (F \vee G) & \Longrightarrow & \neg F \wedge \neg G \\
\neg (F \wedge G) & \Longrightarrow & \neg F \vee \neg G \\
\neg \neg F & \Longrightarrow & F \\
F \vee (G \wedge H) & \Longrightarrow & (F \vee G) \wedge (F \vee H) \\
(F \wedge G) \vee H & \Longrightarrow & (F \vee H) \wedge (G \vee H)
\end{array}
$$

These rules are applied until no further applications are possible.

Simplifying Rewrite rules

$$
\begin{aligned}
\mathbf{0} \vee G &\Longrightarrow G \\
\mathbf{0} \wedge G &\Longrightarrow \mathbf{0} \\
\neg \mathbf{1} &\Longrightarrow \mathbf{0} \\
\cdots \wedge G \wedge \cdots \wedge \neg G \wedge \cdots &\Longrightarrow \mathbf{0} \\
\cdots \wedge G \wedge \cdots \wedge G \wedge \cdots &\Longrightarrow \cdots \wedge G \wedge \cdots \wedge \ldots
\end{aligned}
$$

$\mathbf{0}$ (false) can be eliminated in DNF.

Example:
$(P \wedge Q) \vee (R \wedge S \wedge \neg R) \equiv (P \wedge Q) \vee (\mathbf{0} \wedge S) \equiv (P \wedge Q) \vee \mathbf{0} \equiv P \wedge Q$.

Rewriting to DNF – Example

$$
\begin{aligned}
P \wedge (P \to Q) &\Longrightarrow P \wedge (\neg P \vee Q) \\
&\Longrightarrow (P \wedge \neg P) \vee (P \wedge Q)
\end{aligned}
$$

Now this formula clearly is in DNF. However, we can simplify it considerably.

Note that $P \wedge \neg P \equiv 0$ and $0 \vee (P \wedge Q) \equiv (P \wedge Q)$.

Therefore, $P \wedge (P \to Q)$ is equivalent to the formula $P \wedge Q$ (which is in DNF).

Special DNF and CNF Cases

Any contradictory formula F is equivalent to the single conjunction $P \wedge \neg P$ (in DNF and CNF), which we abbreviated to **0** .

Any tautology F is equivalent to the single disjunction $P \vee \neg P$ (in DNF and CNF), which we abbreviated to **1**.

COMPLETE SETS OF CONNECTIVES

Complete Sets of Connectives

A set of connectives is called *complete* (or *adequate*) if every formula of propositional logic is equivalent to a formula using only connectives from this set.

Since every formula has a disjunctive normal form, the set $\{\neg, \wedge, \vee\}$ **is complete.**

From the De Morgan's laws we have

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$
$$P \wedge Q \equiv \neg(\neg P \vee \neg Q).$$

Therefore **both** sets of connectives $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are **complete**.

Complete Sets of Connectives (cont)

To show that a given set of connective is complete all we need to do is to **express it in terms of a known complete set of connectives**.

Example. The set $\{\neg, \rightarrow\}$ is complete because $P \rightarrow Q \equiv \neg P \vee Q$. Thus, the set $\{\neg, \rightarrow\}$ is expressed in terms of the complete set $\{\neg, \vee\}$.

No singleton set from the *standard set of connectives* $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is complete.

However, the (non-standard) set $\{\mathbf{0}, \rightarrow\}$ is complete, because $\neg P \equiv P \rightarrow \mathbf{0}$, and hence it can be expressed in terms of the known complete set $\{\neg, \rightarrow\}$.

Truth-functions and Normal Forms

A **truth-function** is a function whose arguments can take only the values *true* (or 1) and *false* (or 0).

Any wff defines a truth-function, and vice-versa.

Example. Let f be the truth-function defined as follows:

$$f(P, Q, R) = 1 \text{ iff either } P = Q = 0 \text{ or } Q = R = 1$$

Then f is equal to 1 in exactly the following four cases:

$$f(0, 0, 1), f(0, 0, 0), f(1, 1, 1), f(0, 1, 1)$$

f can be represented by the formula

$$(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$$

Substitution Instances

Substitution

Uniform substitution of formulae for propositional variables

Let $W, H_1, \ldots, H_n$ be formulae and $P_1, \ldots, P_n$ be propositional variables.

Then the expression $W(P_1/H_1, \ldots, P_n/H_n)$ denotes the formula obtained by replacing *simultaneously* all occurrences of P_1 in W by the formula H_1, ..., and all occurrences of P_n by the formula H_n.

Example

Let W be $P \rightarrow (Q \rightarrow P)$, then $W(P/\neg P \vee R, Q/\neg P)$ is $\neg P \vee R \rightarrow (\neg P \rightarrow \neg P \vee R)$.

We say that the formula $\neg P \vee R \rightarrow (\neg P \rightarrow \neg P \vee R)$ is a *substitution instance* of $P \rightarrow (Q \rightarrow P)$.

Questions.

What type of formula is $P \rightarrow (Q \rightarrow P)$?
What does that make $\neg P \vee R \rightarrow (\neg P \rightarrow \neg P \vee R)$?

Exercises

Which of the following propositional formulae are substitution instances of the formula $P \to (Q \to P)$?

If a formula is indeed a substitution instance, give the formulae substituted for P and Q.

1. $\neg R \to (R \to \neg R)$
2. $\neg R \to (\neg R \to \neg R)$
3. $\neg R \to (\neg R \to R)$
4. $(P \wedge Q \to P) \to ((Q \to P) \to (P \wedge Q \to P))$
5. $((P \to P) \to P) \to ((P \to (P \to (P \to P))))$

Substitution Properties

From $F \equiv G$ we can conclude

$$F(P_1/H_1, \ldots, P_n/H_n) \equiv G(P_1/H_1, \ldots, P_n/H_n).$$

Example. We can obtain a new equivalence as a substitution instance of the De Morgan law:

From $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$, we have

$$\neg(\underline{(P \to R)} \vee \underline{(R \leftrightarrow Q)}) \equiv \neg\underline{(P \to R)} \wedge \neg\underline{(R \leftrightarrow Q)}.$$

Thus, a new tautology is generated:

$$\neg(\underline{(P \to R)} \vee \underline{(R \leftrightarrow Q)}) \leftrightarrow \neg\underline{(P \to R)} \wedge \neg\underline{(R \leftrightarrow Q)}.$$

QUINE'S METHOD

Quine's Method

For any formula W and propositional variable P:

- W is a tautology if and only if $W(P/\mathbf{0})$ and $W(P/\mathbf{1})$ are tautologies.
- W is a contradiction if and only if $W(P/\mathbf{0})$ and $W(P/\mathbf{1})$ are contradictions.

Quine's method can be described graphically with a binary tree (Hein, Section 6.2).

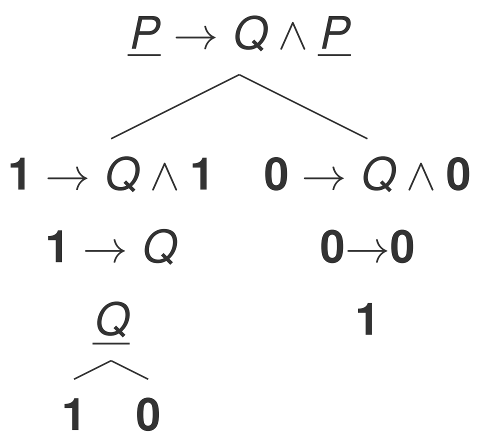
Constructing Quine's Tree for a Formula W

1. Start with W as the root of the tree.
2. Take the first level in the tree with a propositional symbol, say P, in any of the level's nodes n. If none are left, then finish.
3. Let the left child of n be $n(P/\mathbf{1})$ and let its right child be $n(P/\mathbf{0})$.
4. Repeat from 2.

When no propositional symbols remain:

- W is a tautology if all of the leaves in the tree are true (i.e., $\mathbf{1}$)
- W is a contradiction if all leaves in the tree are false (i.e., $\mathbf{0}$)
- Otherwise, W is a *contingency* (i.e., sometimes true, sometimes false)

Example



Conclusion. The formula $P \to Q \wedge P$ is a contingency, because neither all leaves in the tree are true, nor all of them are false.

To know more...

- Most of the material in this session can be found in detail in Chapter 1 of 'Elementary Logic with Applications' – the only exception is Quine's method.