4CCS1ELA: Tutorial list 4 – Sample Solutions

1. Formalising scenarios. (The ambiguity of natural languages.)

Let	$S(x)$	represent	'x is a student'
	$L(x)$	represent	'x is a lecture'
	$A(x, y)$	represent	'x attended y'

Formalise the following sentence:
'At least one student attended every lecture.'

<u>*SOLUTION*</u>

This sentence can be understood in two different ways (because of the ambiguity of the *natural language*):

(i) Every lecture was attended (by at least one student).

(ii) There exists a student who attended every lecture (the same student).

Let us note, that the sentence **(i)** is a *logical consequence* of sentence **(ii)**, but **not** vice-versa. (In other words **(ii)** $\models$ **(i)** but **(i)** $\not\models$ **(ii)**).

<u>Consider the sentence **(i)**.</u> Its formal representation (in the given notations) is the following:

$$\forall x(L(x) \to \exists y(S(y) \land A(y, x)))$$

Given the dictionary above, this formal representation can be read as:

> 'For every x, if x is a lecture, then there exists y such that y is a student and y attended x.'

<u>Consider the sentence **(ii)**.</u> Its formal representation is:

$$\exists y(S(y) \land \forall x(L(x) \to A(y, x)))$$

Given the dictionary above, this formula can be read as:

> 'There exists y such that y is a student and, for every x, if x is a lecture then y attended x.'

2. Let $B(x)$ mean "x is a bird", let $W(x)$ mean "x is a worm", let $E(x, y)$ mean "x eats y". Using these predicates, represent in first-order logic each of the following statements:

(i) *Every bird eats every worm.*

(ii) *Some birds do not eat some worms.*

(iii) *No bird is eaten by a worm.*

(iv) *Some worms do not get eaten by birds.*

(v) *Only birds eat worms.*

<u>SOLUTION</u>

(i) *Every bird eats every worm:*
$\forall x(B(x) \to \forall y(W(y) \to E(x, y)))$.
Equivalently,
$\forall x \forall y(B(x) \land W(y) \to E(x, y))$.

(ii) *Some birds do not eat some worms:*
$\exists x(B(x) \land \exists y(W(y) \land \neg E(x, y)))$.
Equivalently,
$\exists x \exists y(B(x) \land W(y) \land \neg E(x, y))$.

(iii) *No bird is eaten by a worm:*
$\forall x(B(x) \to \forall y(W(y) \to \neg E(y, x)))$.
Equivalently,
$\neg \exists x \exists y(B(x) \land W(y) \land E(y, x))$.

(iv) *Some worms do not get eaten by birds:*
$\exists x(W(x) \land \forall y(B(y) \to \neg E(y, x)))$.
Equivalently,
$\exists x(W(x) \land \neg \exists y(B(y) \land E(y, x)))$.

(v) *Only birds eat worms:*
$\forall x(W(x) \to \forall y((E(y, x) \to B(y))))$.
Equivalently,
$\forall x \forall y(W(x) \land E(y, x) \to B(y))$.

3. Identify which occurrences of variables in the formulas below are free and which occurrences are bound. Justify you answers.

1. $y \geq 0 \; \wedge \forall x(N(x) \rightarrow x \geq y)$
2. $x \geq 0 \; \wedge \forall x(N(x) \rightarrow x \geq y)$
3. $\forall x(N(x) \rightarrow \exists y(N(y) \wedge x \geq y))$

Here N is a unary predicate symbol, $\geq$ is a binary predicate symbol in infix notation, and $x \geq y$ is an atom in infix notation.

SOLUTION

1. all occurrences of x are bound (therefore x is a bound variable in this formula). Both occurrences of y are free (therefore y is a free variable in this formula).

   The free occurrences are boxed:
   $\boxed{y} \geq 0 \; \wedge \forall x(N(x) \rightarrow x \geq \boxed{y}\,)$

2. the variable x is free and bound (i.e., there are free occurrences of x and there are bound occurrences of x. The variable y is free, i.e., all occurrences of y are free.

   The free occurrences are boxed:
   $\boxed{x} \geq 0 \; \wedge \forall x(N(x) \rightarrow x \geq \boxed{y})$

3. Both x and y are bound (i.e., all occurrences of x and y are bound).

4. Let $\mathcal{F}$ be a wff interpreted over D and $d \in D$. Then $\mathcal{F}(x/d)$ denotes the wff obtained from $\mathcal{F}$ by replacing all **free** occurrences of x by d.

Compute the following substitutions and determine the meaning (the truth-values) of the resulting sentences over natural numbers.

Here $N(x)$ denotes "x is a natural number", predicates $\geq$ and $>$ have their usual interpretation

1. $\big(y \geq 0 \ \wedge \forall x(N(x) \to x \geq y)\big)(y/3)$
2. $\big(x \geq 0 \ \wedge \exists y(N(y) \wedge x \geq y)\big)(x/3)$
3. $\big(\forall x(N(x) \to \exists y(N(y) \wedge x > y))\big)(x/3)$
4. $\big(\forall x(N(x) \to \exists y(N(y) \wedge y > x))\big)(y/3)$

<u>SOLUTION</u>

1. $(3 \geq 0 \ \wedge \forall x(N(x) \to x \geq 3)$ is false (witness $x = 1$).
2. $(3 \geq 0 \ \wedge \exists y(N(y) \wedge 3 \geq y)$ is true (witness $y = 3$).
3. $(\forall x(N(x) \to \exists y(N(y) \wedge x > y)))$ is false (witness $x = 0$).
4. $(\forall x(N(x) \to \exists y(N(y) \wedge y > x)))$ is true (witness $y = x + 1$).

Formula 3 does not contain free occurrences of x. Similarly, all occurrences of y in formula 4 are bound.