4CCS1ELA – Elementary Logic with Applications

## Small Group Tutorial 4

**1.** [MEDIUM, 10 mins] Consider the following first-order sentence

$$\forall x \forall y (N(x) \wedge N(y) \rightarrow \exists z ((Q(z) \wedge ((x < z \wedge z < y) \vee (y < z \wedge z < x))))$$

and an interpretation over the set of real numbers such that

$N(x)$ means '$x$ is a natural number',
$Q(x)$ means '$x$ is a rational number',
$x < y$ means '$x$ is less than $y$'.

(i) Give an English translation of this sentence.

(ii) Determine if this sentence is true or false under the given interpretation.

**2.** [HARD, 15 mins] Assume that $\exists x \exists y P(x, y)$ is true. Which of the following formulae must also to be true? If the formula is true, explain why. Otherwise, give a counterexample.

(i) $\forall x \forall y P(x, y)$.

(ii) $\forall x \exists y P(x, y)$.

(iii) $\exists x \forall y P(x, y)$.

**3.** [HARD, 15 mins] Find a counterexample to show that the following argument is not valid:

$$\exists x P(x),\ \exists x (P(x) \rightarrow Q(x)) \ \models \exists x Q(x).$$

**4.** [EASY, 5 mins] Find an error in the following attempt to give a formal proof for the (invalid) argument in the previous exercise.

$$\exists x P(x),\ \exists x(P(x) \to Q(x)) \ \models\ \exists x Q(x).$$

| | | | |
|---|---|---|---|
| 1. | $\exists x P(x)$ | | $P$ |
| 2. | $\exists x(P(x) \to Q(x))$ | | $P$ |
| 3. | $P(a)$ | 1 | $EI$ Existential Instantiation |
| 4. | $P(a) \to Q(a)$ | 2 | $EI$ Existential Instantiation |
| 5. | $Q(a)$ | $3, 4$ | $MP$ Modus ponens |
| 6. | $\exists x Q(x)$ | 10 | $EG$ Existential Generalisation |

Here, $a$ is a new constant.

**5. (a)** [MEDIUM, 10 mins] Express the following specification in the language of predicate logic, using the dictionary below:

> *If every component works properly and all interfaces are functioning then every test-run will terminate.*
>
> Dictionary :
>
> $C(x)$: $x$ is a component
> $W(x)$: $x$ works properly
> $I(x)$ : $x$ is an interface
> $F(x)$: $x$ is functioning
> $R(x)$: $x$ is a test-run
> $T(x)$ : $x$ will terminate

**(b)** [MEDIUM, 5 mins] Negate the expression that you constructed, and then use the principles of quantifier interchange, together with equivalences of propositional logic, to move negation signs inwards as far as you can.

**(c)** [EASY, 3 mins] Translate the result of your work in (b) back into English.