4CCS1ELA – Elementary Logic with Applications

5 – Predicate Logic 2: Reasoning with Quantifiers

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Outline

1. Quantifier order
2. Relationship with $\wedge$ and $\vee$
3. Transformations and simplifications
4. Quantifier Rules
5. Proofs
6. Exercise

Quantifier order

Order of appearance of quantifiers

In a continuous sequence of quantifiers, the order of appearance of the same type of quantifiers is irrelevant:

$\forall x \forall y \mathcal{F} \equiv \forall y \forall x \mathcal{F}$

Example

$\forall x \forall y (x < y \to \exists z (x < z \land z < y)) \equiv$

$\forall y \forall x (x < y \to \exists z (x < z \land z < y))$

$\exists x \exists y \mathcal{F} \equiv \exists y \exists x \mathcal{F}$

Example

Order of appearance of quantifiers

When different types of quantifiers are involved, you need to be careful:

$\forall x \exists y \mathcal{F} \not\equiv \exists y \forall x \mathcal{F}$: $\exists y \forall x \mathcal{F} \models \forall x \exists y \mathcal{F}$, but $\forall x \exists y \mathcal{F} \not\models \exists y \forall x \mathcal{F}$.

To see that $\forall x \exists y \mathcal{F} \not\models \exists y \forall x \mathcal{F}$ interpret the predicate $loves(x, y)$ as "x loves y".

$\forall x \exists y\ loves(x, y)$ means "Everyone loves someone". If everyone loves someone, this does not mean that everyone will love the same person as in $\exists y \forall x\ loves(x, y)$.

Quantifier order

Order of appearance of quantifiers

Note that $\exists y \forall x \mathcal{F} \models \forall x \exists y \mathcal{F}$ **does** hold.

If there exists someone whom everyone loves (i.e., $\exists y \forall x\ loves(x, y)$), then everyone will love someone, i.e., that person: $\forall x \exists y\ loves(x, y)$.

RELATIONSHIP WITH $\wedge$ AND $\vee$

Distribution of quantifiers over $\wedge$ and $\vee$

Universal quantification distributes over conjunction and existential quantification distributes over disjunction:

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x(P(x)) \wedge \forall x(Q(x))$$

$$\exists x(P(x) \vee Q(x)) \equiv \exists x(P(x)) \vee \exists x(Q(x))$$

However, note the following:

$\exists x(P(x) \wedge Q(x)) \models \exists x\, P(x) \wedge \exists x\, Q(x)$, but not conversely!

$\forall x\, P(x) \vee \forall x\, Q(x) \models \forall x(P(x) \vee Q(x))$, but not conversely!

Home exercise: Show that the converse does not hold. Think of

Class Exercise

Show that $\exists x(P(x) \to Q(x)) \equiv \forall xP(x) \to \exists xQ(x)$

Hint: Turn the implication into disjunction and re-write the quantifers using the equivalences.

TRANSFORMATIONS AND SIMPLIFICATIONS

Vacuous quantification

- When the variable x has no occurrence in a formula $\mathcal{F}$, then binding x with a quantifier in that formula has no effect.

  Example:

  $$\forall x \forall y(P(y) \to Q(y)) \equiv \forall y(P(y) \to Q(y)) \equiv \exists x \forall y(P(y) \to Q(y))$$

- When the variable x is already bound by another quantifier in the formula $\mathcal{F}$, then binding x again has no effect.

  Examples:

  $$\forall x(\forall x(P(x) \to Q(x)) \equiv \forall x(P(x) \to Q(x))$$
  $$\exists x(\forall x(P(x) \to Q(x)) \equiv \forall x(P(x) \to Q(x))$$

Renaming Quantified Variables

If y is a new variable that does not occur in $\mathcal{F}$, then the following equivalences hold:

$$\exists x\mathcal{F} \equiv \exists y\mathcal{F}(x/y).$$
$$\forall x\mathcal{F} \equiv \forall y\mathcal{F}(x/y).$$

Recall that $\mathcal{F}(x/y)$ is obtained from $\mathcal{F}$ by replacing all **free** occurrences of x in it by y.

Example

$$\forall xP(x) \vee \forall xQ(x) \to \forall x(P(x) \vee Q(x)) \equiv \forall yP(y) \vee \forall zQ(z) \to \forall x(P(x) \vee Q(x))$$

On the second half of the equivalence above, all quantifiers use a different variable.

Equivalences with Restrictions

Provided that x **does not occur in** $\mathcal{F}$ (or is bound by another quantifier), then the following equivalences hold:

Simplification

$$\forall x \mathcal{F} \equiv \exists x \mathcal{F} \equiv \mathcal{F}.$$

Disjunction

$$\forall x\big(\mathcal{F} \vee \mathcal{G}(x)\big) \equiv \mathcal{F} \vee \forall x \mathcal{G}(x).$$

$$\exists x\big(\mathcal{F} \vee \mathcal{G}(x)\big) \equiv \mathcal{F} \vee \exists x \mathcal{G}(x).$$

Conjunction

$$\forall x\big(\mathcal{F} \wedge \mathcal{G}(x)\big) \equiv \mathcal{F} \wedge \forall x \mathcal{G}(x).$$

Equivalences with Restrictions Involving Implication

Provided x **does not occur in** $\mathcal{F}$ (or is bound by another quantifier):

$$\forall x\big(\mathcal{F} \to \mathcal{G}(x)\big) \equiv \forall x\big(\neg\mathcal{F} \vee \mathcal{G}(x)\big) \equiv \neg\mathcal{F} \vee \forall x\, \mathcal{G}(x) \equiv \mathcal{F} \to \forall x\, \mathcal{G}(x)$$

$$\exists x\big(\mathcal{F} \to \mathcal{G}(x)\big) \equiv \exists x\big(\neg\mathcal{F} \vee \mathcal{G}(x)\big) \equiv \neg\mathcal{F} \vee \exists x\, \mathcal{G}(x) \equiv \mathcal{F} \to \exists x\, \mathcal{G}(x)$$

$$\forall x\big(\mathcal{G}(x) \to \mathcal{F}\big) \equiv \forall x\big(\neg\mathcal{G}(x) \vee \mathcal{F}\big) \equiv \forall x \neg\mathcal{G}(x) \vee \mathcal{F} \equiv \neg\exists x\, \mathcal{G}(x) \vee \mathcal{F} \equiv \exists x \mathcal{G}(x) \to \mathcal{F}$$

$$\exists x\big(\mathcal{G}(x) \to \mathcal{F}\big) \equiv \exists x\big(\neg\mathcal{G}(x) \vee \mathcal{F}\big) \equiv \exists x \neg\mathcal{G}(x) \vee \mathcal{F} \equiv \neg\forall x\, \mathcal{G}(x) \vee \mathcal{F} \equiv \forall x \mathcal{G}(x) \to \mathcal{F}$$

QUANTIFIER RULES

Rule of Inference for Quantified Statements ($\forall$)

All natural deduction rules involving the propositional connectives still hold as before. In addition, the new rules for the quantifiers are given below.

Universal Instantiation: given the premise $\forall x\mathcal{F}$, we may derive the conclusion $\mathcal{F}(x/d)$, where d is any element of the domain:

$$\frac{\forall x\mathcal{F}}{\mathcal{F}(x/d)}\ \text{(UI)}$$

"If $\mathcal{F}$ holds for all elements of the domain, it holds in particular for the element d as well."

You can think of this as $\forall$-elimination

Rules of Inferences for Quantified Statements ($\forall$)

Universal Generalisation: given the premise $\mathcal{F}(x/d)$ takes place for any element d in the domain, we may derive $\forall x\mathcal{F}$.

$$\frac{\mathcal{F}(x/d)}{\forall x\mathcal{F}}\ \text{(UG)}$$

Note. The element *d* in the premise of (UG) must not be specific but **arbitrary**, i.e., we cannot make any assumptions about *d* other that it comes from the domain.

You can think of this rule as $\forall$-introduction

Rule of Inference for Quantified Statements ($\exists$)

Existential Instantiation : given the premise $\exists x\mathcal{F}$, we may derive $\mathcal{F}(x/e)$ where *e* is a special element of the domain:

$$\frac{\exists x\mathcal{F}}{\mathcal{F}(x/e)}\ \text{(EI)}$$

Note. We cannot select an arbitrary element of the domain in (EI), but rather *it must be an element e for which $\mathcal{F}(x/e)$ is true*.

Usually, we have no knowledge of what the actual element is, only that it does exist. Because it exists, we can give it a *new* name, not used anywhere before, say, *e* and continue our derivation.

You can think of this rule as $\exists$-elimination

Rule of Inferences for Quantified Statements ($\exists$)

Existential Generalisation: given that $\mathcal{F}(x/d)$ is known to hold for a particular element d of the domain, we may derive $\exists x\mathcal{F}$.

$$\frac{\mathcal{F}(x/d)}{\exists x\mathcal{F}}\ \text{(EG)}$$

You can think of this rule as $\exists$-introduction

PROOFS

Example of Reasoning Using These Rules

Consider the following argument:

All barbers in the village of Podunk are men.

All barbers in the village of Podunk shave all the men who do not shave themselves, and only those men.

Therefore. There is no barber in the village of Podunk.

How can we check whether the argument is valid?

Example (contd.)

Let	$B(x)$	represent	"x is a barber"
	$M(x)$	represent	"x is man"
	$S(x,y)$	represent	"x shaves y"

Now the argument can be represented as follows:

$$\begin{array}{ll} \mathcal{F}_1 : & \forall x(B(x) \to M(x)) \\ \mathcal{F}_2 : & \forall x \forall y\big((B(x) \wedge M(y)) \to (\neg S(y,y) \to S(x,y))\big) \\ \mathcal{F}_3 : & \forall x \forall y\big((B(x) \wedge M(y)) \to (S(x,y) \to \neg S(y,y))\big) \\ \hline \mathcal{G} : & \neg \exists x B(x) \end{array}$$

<u>Note</u> We could introduce one more predicate $P(x)$ for "x is an inhabitant of Podunk". However this is not necessary if we assume that the domain is all inhabitants of Podunk).

We will search for a proof by contradiction

Semantical argument:

$$\begin{array}{l} \forall x(B(x) \to M(x)), \\ \forall x \forall y((B(x) \land M(y)) \to (\neg S(y,y) \to S(x,y))), \\ \forall x \forall y((B(x) \land M(y)) \to (S(x,y) \to \neg S(y,y))) \quad \models \quad \neg \exists x B(x) \end{array}$$

If the argument is valid, then the set

$$\left\{ \begin{array}{ll} (F1): & \forall x(B(x) \to M(x)), \\ (F2): & \forall x \forall y((B(x) \land M(y)) \to (\neg S(y,y) \to S(x,y))), \\ (F3): & \forall x \forall y((B(x) \land M(y)) \to (S(x,y) \to \neg S(y,y))), \\ (F4): & \exists x B(x) \end{array} \right\}$$

is inconsistent. We will therefore try to derive a contradiction from the set to help understand the full proof that will follow.

Reasoning steps (semantical)

1. Suppose there is a barber in the village, say Bob (F4)
2. It follows that Bob is a man (since all barbers are men) (F1)
3. Now Bob is both a barber and a man. (use in (F2) and (F3)), but
   - if Bob, the man, does not shave himself (F2), then Bob, the barber will shave him (and then Bob will shave himself, a contradiction)
   - otherwise Bob, the man, does shave himself (by modus tollens on (F3)), then Bob the barber will not shave him, but this is also a contradiction, because the barber and the man are the same
4. It follows that there cannot be any barber in the village.

This suggests a proof with introduction of $\neg$ as outlined above.

Full proof

Stage one

The proof is actually quite simple, but one may get lost in the details.

The strategy is as follows.

1.	$\forall x(B(x) \to M(x))$	data
2.	$\forall x \forall y\big((B(x) \wedge M(y)) \to (\neg S(y,y) \to S(x,y))\big)$	data
3.	$\forall x \forall y\big((B(x) \wedge M(y)) \to (S(x,y) \to \neg S(y,y))\big)$	data
4.	$\exists x B(x) \to (S(b,b) \wedge \neg S(b,b))$	To be completed
5.	$\exists x B(x) \to S(b,b)$	To be completed
6.	$\exists x B(x) \to \neg S(b,b)$	To be completed
7.	$\neg \exists x B(x)$	5. and 6. and $\neg$I

Stage two

4. $\exists x B(x) \to (S(b,b) \wedge \neg S(b,b))$ subcomputation box below

$\underline{S(b,b) \wedge \neg S(b,b)}$

4.1	$\exists x B(x)$	assumption
4.2	$B(b)$	4.1 and $\exists$-E
4.3	$B(b) \to M(b)$	1. and $\forall$-E (x/b)
4.4	$M(b)$	4.2, 4.3 and $\to$-E
4.5	$B(b) \wedge M(b)$	4.2, 4.4 and $\wedge$-I
4.6	$(B(b) \wedge M(b)) \to (\neg S(b,b) \to S(b,b))$	2. and 2 apps of $\forall$-E
4.7	$\neg S(b,b) \to S(b,b)$	4.5, 4.6 and $\to$-E
4.8	$\neg S(b,b) \to \neg S(b,b)$	easy!
4.9	$S(b,b)$	4.7, 4.8 and $\neg$-E

Stage two (cont.)

4.10	$(B(b) \wedge M(b)) \to (S(b,b) \to \neg S(b,b))$	3. and 2 apps of $\forall$-E
4.11	$S(b,b) \to \neg S(b,b)$	4.5, 4.10 and $\to$-E
4.12	$S(b,b) \to S(b,b)$	easy!
4.13	$\neg S(b,b)$	4.11 and 4.12 and $\neg$-I
4.14	$S(b,b) \wedge \neg S(b,b)$	4.9 and 4.13 and $\wedge$-I

Final Stage

5. $\exists x B(x) \to S(b,b)$ subcomputation box below

$\underline{S(b,b)}$

5.1	$\exists x B(x)$	assumption
5.2	$S(b,b) \wedge \neg S(b,b)$	5.1 and 4. and $\to$-E
5.3	$S(b,b)$	5.2 and $\wedge$-E

6. $\exists x B(x) \to \neg S(b,b)$ subcomputation box below

$\underline{\neg S(b,b)}$

6.1	$\exists x B(x)$	assumption
6.2	$S(b,b) \wedge \neg S(b,b)$	6.1 and 4. and $\to$-E
6.3	$\neg S(b,b)$	6.2 and $\wedge$-E

7. $\neg \exists x B(x)$ 5. and 6. and $\neg$-I

EXERCISE

Class exercise

Check whether the following argument is valid:

> *"All interrupt commands are undesirable. Some control commands are interrupt commands. Therefore some control commands are undesirable."*

Assume our language has the following predicate symbols:

- $\mathit{interrupt}(x)$: x is an interrupt command
- $\mathit{control}(x)$: x is a control command
- $\mathit{desirable}(x)$: x is desirable

To know more...

In the “Elementary Logic with Applications” book

- Chapter 6.

End of Part One ...