Sequences

A **sequence** is a list of things, taken in a certain order.

To distinguish sequences from sets, we use brackets (instead of braces) when we list the elements of a sequence:

$(1, 5, 8, -2, 3)$ (*Cinderella*, *Tasmania*, *Tuesday*)

Important features of sequences:

- **The order of listing DOES matter:**
  $(1, 2, 3)$, $(1, 3, 2)$ and $(3, 1, 2)$ are different sequences.
- **Repeated occurrences DO matter:**
  (H, E, L, L, O), (H, H, H, E, L, L, O) and (H, E, L, O) are different sequences.

Tuples

Finite sequences are called **tuples**. A sequence with k elements is a **k-tuple**. The number k is called the **length** of the tuple.

Two tuples are **equal** if they have the same length and their corresponding elements are the same:

$\boxed{(x_1, x_2, \ldots, x_k) = (y_1, y_2, \ldots, y_n)}$ means that $k = n$ and

$x_1 = y_1$ and $x_2 = y_2$ $\ldots$ and $x_k = y_k$.

A 2-tuple is also called an **ordered pair**.

$\boxed{(a, b) = (c, d)}$ means that $a = c$ and $b = d$.

Cartesian product of sets

The <u>**Cartesian product of sets** A **and** B</u> is the set

$$\boxed{A \times B \;=\; \{(x,y) \mid x \in A \;\text{ and }\; y \in B\}}.$$

$A \times B$ consists of those <u>ordered</u> pairs (x, y) where $x \in A$ and $y \in B$.

FOR EXAMPLE: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$.
Then $A \times B \;=\; \{(1,a), (2,a), (1,b), (2,b), (1,c), (2,c)\}$.

The <u>**Cartesian product of sets** $A_1, A_2, \ldots, A_k$</u> is the set

$$\boxed{A_1 \times A_2 \times \cdots \times A_k \;=\; \{(x_1, x_2, \ldots, x_k) \mid x_i \in A_i \;\text{ for }\; i = 1, 2, \ldots, k\}}.$$

$A_1 \times A_2 \times \cdots \times A_k$ consists of those k-tuples $(x_1, x_2, \ldots, x_k)$
where $x_1 \in A_1$, $x_2 \in A_2$, $\ldots$, $x_k \in A_k$.

FOR EXAMPLE: Let $A = \{1, 2\}$, $B = \{a, b, c\}$ and $C = \{\Diamond, \Box\}$.

Then $A \times B \times C \;=\; \big\{(1,a,\Diamond), (1,a,\Box), (2,a,\Diamond), (2,a,\Box), (1,b,\Diamond), (1,b,\Box), (2,b,\Diamond),$
$(2,b,\Box), (1,c,\Diamond), (1,c,\Box), (2,c,\Diamond), (2,c,\Box)\big\}$.

Binary relations

For sets A and B, a **(binary) relation from A to B** is *any* subset R of the Cartesian product $A \times B$. (So a binary relation is a set consisting of some ordered pairs.)

We use notation $\boxed{aRb}$ to denote that $(a,b) \in R$, and say that **a is related to b**.
If $(a,b) \notin R$, then we write $\boxed{a \not\!R\, b}$.

FOR EXAMPLE:

Let P be the set of people, and C the set of football clubs in the English Premier League. Define a relation Fan_of by taking

$$\text{Fan\_of} \;=\; \{(x,y) \in P \times C \mid x \text{ is a fan of } y\}.$$

If I am a fan of MU, but can't stand Arsenal, then

$(Agi, MU) \in \text{Fan\_of}$ but $(Agi, Arsenal) \notin \text{Fan\_of}$

We can also write the same things as

Agi Fan_of *MU* and *Agi* $\not\text{Fan\_of}$ *Arsenal*

Relations on a set

A relation from a set A to A itself is called a **relation on A**.

In other words: a relation on a set A is a subset of $A \times A$

a relation on a set A is a set consisting *some* ordered pairs of elements from A

FOR EXAMPLE:

Let A be the set of all people in the lecture theatre.

$$R_1 = \{(u, v) \in A \times A \mid u \text{ likes } v\}$$

$$R_2 = \{(u, v) \in A \times A \mid u \text{ is taller than } v\}$$

Raj is 180 cm tall and likes Jill who is 165 cm. Unfortunately, Jill doesn't like Raj.

Joe is 190 cm tall, and they don't like each other with Jill.

Then: (Raj , Jill) $\in R_1$ (Raj , Jill) $\in R_2$ (Jill , Raj) $\notin R_1$ (Jill , Raj) $\notin R_2$

(Joe , Jill) $\notin R_1$ (Joe , Jill) $\in R_2$ (Jill , Joe) $\notin R_1$ (Jill , Joe) $\notin R_2$

Relations on a set: some more examples

Here are some relations on the set **Z** of integers:

- **'smaller than':** $\boxed{< \;=\; \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is smaller than } y\}}$

  -1 is smaller than 0, so $(-1,0) \in <$ used more often: $-1 < 0$

  5 is smaller than 25, so $(5,25) \in <$ used more often: $5 < 25$

  -3 is smaller than -2, so $(-3,-2) \in <$ used more often: $-3 < -2$

  0 is not smaller than -1, so $(0,-1) \notin <$ used more often: $0 \not< -1$

  10 is not smaller than 2, so $(10,2) \notin <$ used more often: $10 \not< 2$

  -4 is not smaller than -4, so $(-4,-4) \notin <$ used more often: $-4 \not< -4$

- **'smaller than or equal to':** $\boxed{\leq \;=\; \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ is smaller than or equal to } y\}}$

  $-1 \leq -1, \quad 5 \leq 6,$ but $1 \not\leq 0, \quad -2 \not\leq -10$

  $\boxed{< \subset \leq}$

- **'divisibility':** $\boxed{\text{div} \;=\; \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x \text{ divides } y\}}$

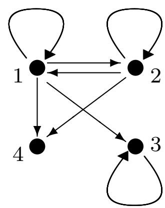
  $1 \text{ div } 25, \quad -3 \text{ div } 12, \quad 10 \text{ div } 1000,$ but $0 \not\text{div}\, 3, \quad 2 \not\text{div} -21, \quad 5 \not\text{div}\, 6$

Representing relations

Let $A = \{1, 2, 3, 4\}$ and take the following relation R on A:

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,4), (3,3)\}.$$

Then R can be represented by 'points and arrows':



directed graph

Or, by a table:

R	1	2	3	4
1	x	x	x	x
2	x	x		x
3			x	
4				

Or, by its **matrix:**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Properties of relations

- A relation R on a set A is called **reflexive**,

  if $(a, a) \in R$ for every element $a \in A$.

  FOR EXAMPLE:

  - reflexive: $\leq$, $\geq$, $=$, 'divisibility' on $\mathbf{N}^+$

    $R_1 = \{(1,1), (1,2), (3,1), (2,2), (3,3)\}$ on $\{1,2,3\}$
  - not reflexive: $R_2 = \{(1,1), (1,2), (3,1), (3,3)\}$ on $\{1,2,3\}$

- A relation R on a set A is called **irreflexive**,

  if $(a, a) \notin R$ for every element $a \in A$.

  FOR EXAMPLE: $<$, $>$, $\neq$ on $\mathbf{N}$ or on $\mathbf{Z}$

$\boxed{\text{Irreflexive is more than just `not reflexive' !}}$

Properties of relations (cont.)

- A relation R on a set A is called **symmetric**, if for all elements $a, b \in A$,

  $(b, a) \in R$ whenever $(a, b) \in R$.

  FOR EXAMPLE:

  - symmetric: $R_1 = \{(1,1),(1,2),(2,1)\}$ on $\{1,2,3\}$
    $R_2 = \{(1,1),(2,2),(3,3)\}$ on $\{1,2,3\}$
    $\equiv_3 = \{(x,y) \in \mathbf{N} \times \mathbf{N} \mid x = y \text{ (mod } 3)\}$ on $\mathbf{N}$
  - not symmetric: $R_3 = \{(1,1),(1,2),(3,1),(3,3)\}$ on $\{1,2,3\}$

- A relation R on a set A is called **antisymmetric**, if

  (a, b) and (b, a) cannot be both in R unless $a = b$.

  (It does NOT mean that (a, a) should be in R !)

FOR EXAMPLE: $<$, $>$, $\leq$, $\geq$ on $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, or $\mathbf{R}$

$R_2 = \{(1,1),(2,2),(3,3)\}$ on $\{1,2,3\}$

$\boxed{\text{Symmetric and antisymmetric are not opposites !}}$

Properties of relations (cont.)

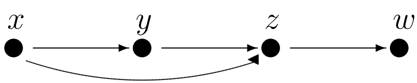
A relation R on a set A is called **transitive**, if the following hold, **for all elements** $a, b, c \in A$:

if **both** $(a,b) \in R$ **and** $(b,c) \in R$, **then** $(a,c) \in R$.

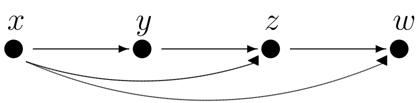
In the directed graph representing R :

R is transitive, if **every** two-step journey along arrows can be done in one step.

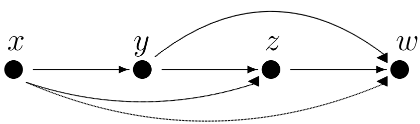
FOR EXAMPLE:



not transitive



still not transitive



transitive

Transitive relations: more examples and non-examples

- Transitive:

  $<$, $>$, $\leq$, $\geq$ on **N**, **Z**, **Q**, or **R**

  $R_1 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

  $R_2 = \{(3,4)\}$ (both R_1 and R_2 on $\{1,2,3,4\}$)

- Not transitive:

  $R_3 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

  $R_4 = \{(1,1),(1,2),(2,1)\}$

  $R_5 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

  (all R_3, R_4, R_5 on $\{1,2,3,4\}$)

Transitive closure

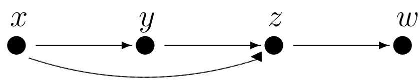
- Let R be a relation on a set A.

  The <u>**transitive closure of R**</u> is the smallest transitive relation on A containing R.

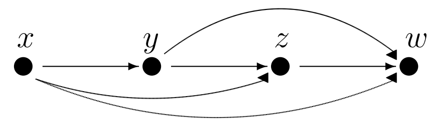
- We denote the transitive closure of R by $\boxed{R^*}$.

  FOR EXAMPLE:

R :



R^* :



- Given R, we can define R^* inductively:

  Basis: $R \subseteq R^*$.

  (we begin with pairs in R)

  Inductive step: If $(a,b) \in R^*$ and $(b,c) \in R^*$ then $(a,c) \in R^*$.

  (we add the missing pairs step-by-step)

- (If R is already transitive, then $R^* = R$.)

Computing the transitive closure: Warshall's algorithm

An **algorithm** is a finite sequence of precise step-by-step instructions.

Warshall's algorithm computes the matrix of the transitive closure R^* of R.

- Given a relation R on a set with n elements, we begin with its $n \times n$ matrix $\boxed{M_0}$.
- There are n rounds.

  In each round, we turn the previous matrix to a new matrix:

  $$M_0 \overset{\text{round 1}}{\rightsquigarrow} M_1 \overset{\text{round 2}}{\rightsquigarrow} M_2 \overset{\text{round 3}}{\rightsquigarrow} \quad \dots \quad \overset{\text{round n}}{\rightsquigarrow} M_n$$

  $\boxed{M_n}$ is the matrix of the transitive closure R^* of R.

  - 1st rule. we never change a 1 to 0
  - 2nd rule. rule for changing **some** 0s to 1s:

$$\underbrace{\begin{array}{cc} & \begin{array}{cccc} j & & k\text{th column} & \end{array} \\ \begin{array}{c} i \\ \\ k\text{th row} \\ \\ \end{array} & \begin{bmatrix} 0 & \rightarrow & 1 & \quad \\ \downarrow & & & \\ 1 & & & \\ & & & \end{bmatrix} \end{array}}_{M_{k-1}} \quad \overset{\text{round k}}{\rightsquigarrow} \quad \underbrace{\begin{array}{cc} & \begin{array}{ccc} j & & k \end{array} \\ \begin{array}{c} i \\ \\ k \\ \\ \end{array} & \begin{bmatrix} \underline{1} & & 1 \\ & & \\ 1 & & \\ & & \end{bmatrix} \end{array}}_{M_k}$$

Warshall's algorithm: an example

Let R be a relation on $\{a, b, c, d\}$: $\boxed{R \ = \ \{(a, d), (b, a), (b, c), (c, a), (c, d), (d, c)\}}$

As $\{a, b, c, d\}$ has 4 elements, $n = 4$. The matrix of R is the 4×4 matrix

$$M_0 \ = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

There will be 4 rounds.

<u>Round 1.</u>

$$\begin{array}{cc} & \text{1st column} \\ \text{1st row} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \underline{0} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ & M_0 \end{array} \quad \overset{\text{round 1}}{\leadsto} \quad \begin{array}{c} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \underline{\mathbf{1}} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ M_1 \end{array}$$

Warshall's algorithm: an example (cont.)

Round 2.

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \overset{\text{round 2}}{\rightsquigarrow} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = M_2 \quad \text{(no change)}$$

Round 3.

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \underline{0} & 0 & 1 & \underline{0} \end{bmatrix} \overset{\text{round 3}}{\rightsquigarrow} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ \underline{\mathbf{1}} & 0 & 1 & \underline{\mathbf{1}} \end{bmatrix} = M_3$$

Round 4.

$$M_3 = \begin{bmatrix} \underline{0} & 0 & \underline{0} & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & \underline{0} & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \overset{\text{round 4}}{\rightsquigarrow} \begin{bmatrix} \underline{\mathbf{1}} & 0 & \underline{\mathbf{1}} & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & \underline{\mathbf{1}} & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = M_4$$

$$\boxed{R^* = \{(a,a),(a,c),(a,d),(b,a),(b,c),(b,d),(c,a),(c,c),(c,d),(d,a),(d,c),(d,d)\}}$$

Equivalence relations

A relation R on a set A is called an **equivalence relation** if it is

- reflexive,
- symmetric, and
- transitive.

FOR EXAMPLE:

- $=$ on any set,
- $\equiv_4 = \{(x,y) \in \mathbf{Z} \times \mathbf{Z} \mid x = y \pmod 4\}$ on $\mathbf{Z}$

Partial orders

A relation R on a set A is called a **partial order** if it is

- reflexive,
- antisymmetric, and
- transitive.

FOR EXAMPLE: $\leq$, $\geq$, and 'divisibility' on $\mathbf{N}^+$

EXERCISE 2.1: Show that $\subseteq$ is a partial order on the power set $P(S)$ of a set S.

SOLUTION:

- $\subseteq$ is reflexive: It is because $A \subseteq A$ for every set A.
- $\subseteq$ is antisymmetric: If $A \subseteq B$ and $B \subseteq A$, then $A = B$ holds.
- $\subseteq$ is transitive: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ holds.

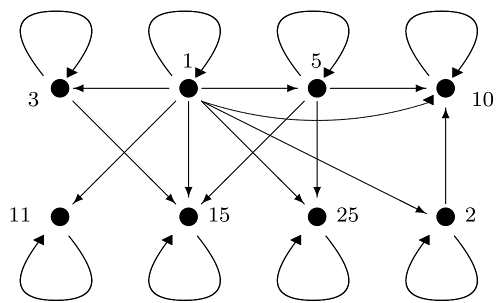
Representing partial orders: Hasse diagrams

If we know that a relation is a partial order, then there is a more 'economical' way of representing it than by a directed graph.

Say, take the 'divisibility' relation on the set $\{1, 2, 3, 5, 10, 11, 15, 25\}$:

$$\{(1,1),(1,2),(1,3),(1,5),(1,10),(1,11),(1,15),(1,25),(2,2),(2,10),(3,3),\\ (3,15),(5,5),(5,10),(5,15),(5,25),(10,10),(11,11),(15,15),(25,25)\}$$

As this is a relation, it can be represented by a directed graph:

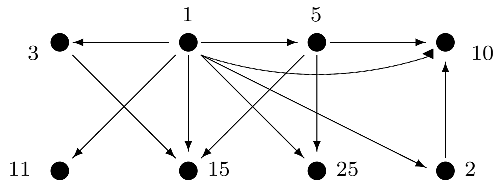


But we can do better.

Constructing Hasse diagrams (cont.)

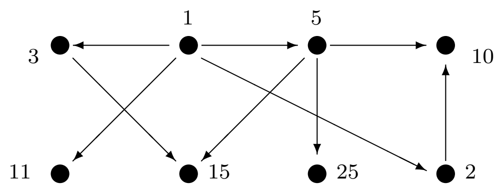
As partial orders are always **reflexive,** a loop is always present at every point.

So by removing these loops we don't lose info:



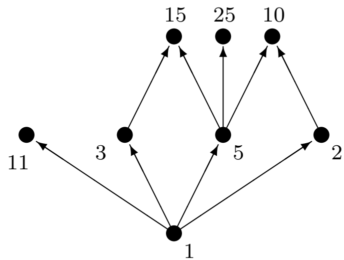
Partial orders are always **transitive.** Say, if $\begin{tikzcd} 1 \arrow[r] & 5 \arrow[r] & 10 \end{tikzcd}$ is part of our diagram, then we know that we must also have $\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend right] & 5 \arrow[r] & 10 \end{tikzcd}$

So we don't lose info by indicating only 'one-step' arrows, and removing the rest:

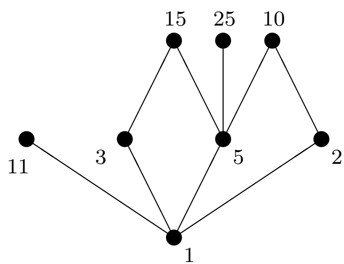


Constructing Hasse diagrams (cont.)

Partial orders are always **antisymmetric.** This means that between any two points there can be an arrow one way only, NOT both. So we can rearrange the points such that all the arrows 'point' from a lower position 'upwards':



So we don't lose info by removing the arrow-heads, and using lines instead:

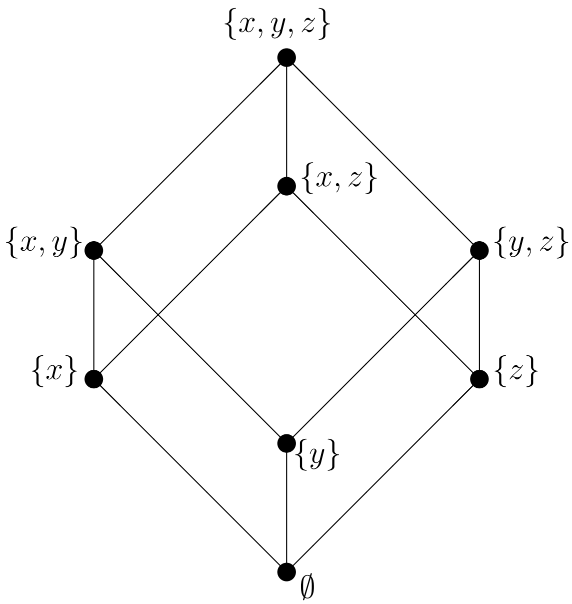


Hasse diagram
of the 'divisibility' relation
on $\{1, 2, 3, 5, 10, 11, 15, 25\}$

(Overall shape does not matter, but WATCH OUT: 'horizontal' lines are NO GOOD!)

Hasse diagrams: another example

The Hasse diagram of $\subseteq$ on the power set $P(\{x, y, z\})$ of $\{x, y, z\}$:



Linear orders

A relation R on a set A is called a **linear order** (or **total order**) if

- R is a partial order, and
- for all $a, b \in A$, either $(a, b) \in R$ or $(b, a) \in R$
  (that is, every pair of elements is 'comparable' this way or the other according to R).

FOR EXAMPLE:

- $\leq$ and $\geq$ on **N**, **Z**, **Q**, or **Z**
- BUT: 'divisibility' and $\subseteq$ are partial orders, but NOT linear orders
  (see previous two slides)

Exercise 2.2

Let S be a set with more than one element.

Show that $\subseteq$ is **not** a linear order on $P(S)$.

SOLUTION:

If S has more than one element, then there are at least two different elements in S, let's call them x and y. Then:

- $x \in S$, so $\{x\} \subseteq S$, and so $\{x\} \in P(S)$.
- $y \in S$, so $\{y\} \subseteq S$, and so $\{y\} \in P(S)$.
- Neither $\{x\} \subseteq \{y\}$, nor $\{y\} \subseteq \{x\}$ holds, as x and y are different.

So $\{x\}$ and $\{y\}$ are two elements in $P(S)$ that are incomparable according to $\subseteq$.