# Predicates

**Connectives** ⇒ prop. logic $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow)$.

Quantifiers ⇒ $\exists$ "exists" ⇒ Existential quantifier.

$\forall$ "for all" ⇒ Universal quantifier.

**Variables**

$S(x)$ ⇒ "x" is a student.

$\exists x\, S(x)$ ⇒ there is a student "x"

$\forall x\, S(x)$ ⇒ every student ...

$\forall x\, S(x)$ ⇒ everyone is a student.

$\exists x\, S(x)$ ⇒ there is someone who is a student.

$A(x, y)$ ⇒ "x" attends "y"

$L(x)$ ⇒ "x" is a lecture.

Every student who attends "ela" ....

$\forall x\, [S(x) \wedge A(x, ela) \rightarrow \_\_\_\_ ]$.

... attends "fc1"

$A(x, fc1)\,]$

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# Tutorial 4 Q2

DICTIONARY

$B(x)$ x is a bird

$W(x)$ x is a worm

$E(x,y)$ x eats y

Every bird eats every worm.

$$\forall x \forall y [B(x) \wedge W(y) \to E(x,y)]$$

Some birds do not eat some worms.

$$\exists x [B(x) \wedge \exists y [W(y) \wedge \neg E(x,y)]]$$

$$\forall x \forall y (B(x) \leftrightarrow \neg W(y)) \times \quad \forall x \neg [B(x) \wedge W(x)]$$

"Nothing is both a bird and a worm"

No bird is eaten by a worm.

$$\forall x \forall y [B(x) \wedge W(y) \to \neg E(\cancel{y}, \cancel{x})]$$

$$\neg \exists x \exists y (B(x) \wedge W(y) \wedge E(y,x))$$

Some worms do not get eaten by birds

$$\exists x ( W(x) \wedge \forall y ( B(y) \to \neg E(y,x)))$$

$$\forall x \exists y (B(x) \wedge W(y) \wedge \neg E(x,y))$$

Only birds eat worms.

$$\forall x \forall y (W(y) \wedge E(x,y) \to B(x)).$$

There exist at least two birds:

$$\exists x \exists y [B(x) \wedge B(y) \wedge \neg(x = y)$$

$$\exists x \exists y (B(x) \wedge B(y) \wedge \forall z (B(z) \to [(z=x) \vee (z=y)]))$$

Domain $D = \mathbb{N}$

$Odd(x)$ $\Rightarrow$ "$x$ is odd" $Odd^{I} \subset \mathbb{N}$

$Odd^{I} \subsetneq \mathbb{N}$ $\leftarrow$ $Odd^{I}$ is a proper subset of $\mathbb{N}$

$<(x,y)$ $\Rightarrow$ "$x<y$" "$x$ is smaller than $y$"

$\mathbb{N} \times \mathbb{N} = \{(0,0), (0,1), (0,2), (0,3) \ldots$
$(1,0), (1,1), (1,2), \ldots$
$(2,0 \ldots$
$\vdots$

$<(x,y)$ $x<y$.

$\subseteq \mathbb{N} \times \mathbb{N} = \{(0,1),(0,2),(0,3) \ldots$
(in fact $<^{I} \subset \mathbb{N}$) $(1,2),(1,3),(1,4) \ldots$
$\vdots$
$\}$.

$\uparrow$

pairs of natural numbers $(x,y)$ such that $x<y$

EXERCISE IN SLIDE (4) 44/51 

$$\underline{\underline{\exists x}} \underbrace{\forall y \; (P(y) \to x=y)}_{} \equiv F .$$

1. $D = \{a\}$ $\quad P(a) = \text{true}$ .

$x=a$ $\quad \forall y\,(P(y) \to a=y)$ F is true, if is true. .. if.

$y=a$ $\quad P(a) \to a=a$ .

$$\underbrace{\underset{T}{P(a)} \to \underset{T}{a=a}}_{T}$$

Formula is true under interpretation 1.

2. $D = \{a\}$ $\quad P(a) = \text{false}$ .

$P(a) \to a=a$ .

$$\underset{F}{P(a)}$$

$$\overline{\quad T \quad} .$$

Formula is true under interpretation 2.

$\exists x \forall y\,(P(y) \to x=y)$

3. $D = \{a, b\}$ $\quad P(a) = T$ $\quad P(b) = T$ .

for $\underline{\underline{x=a}}$ $\quad \forall y\,(P(y) \to a=y)$ ?

$\to$ $\underline{\underline{\text{FALSE}}}$

$P(a) \to a=a \overset{?}{=} T$ ✓ and.

$P(b) \to a=b \overset{?}{=} T$ ✗

$\Rightarrow F$

for $\underline{\underline{x=b}}$ $\quad \forall y\,(P(y) \to b=y)$ ?

$\underline{\text{FALSE}}$

$P(b) \to b=b \overset{?}{=} T$ ✓ .

$\underbrace{P(a) \to b=a}_{F} \overset{?}{\Rightarrow} T$ ✗

Formula is $\underline{\text{false}}$ under interpretation 3.

(4)

4. $D = \{a, b\}$. $P(a) = F$ $P(b) = F$.

$$\exists x \forall y [P(y) \to x = y].$$

$\underline{\underline{x = a}}$ $\forall y [P(y) \to a = y]$ (Implication is true when antecedent is false)

$y = a$ $P(a) \xrightarrow{F} a = a \Rightarrow T$

$y = b$ $P(b) \xrightarrow{F} a = b \Rightarrow T$

$\underline{\underline{x = a}}$ $\underline{\underline{\text{witness}}}$ (shows that the formula is true under the interpretation)

5. $D = \{a, b\}$ $P(a) = T$ $P(b) = F$.

$\underline{\underline{\exists x}}$

$\underline{\underline{x = a}}$. $\forall y [P(y) \to a = y]$.

$y = a$ $P(a) \to a = a$ $T$.

$y = b$ $P(b) \to a = b$ $T$

Sufficient to show that there exists $x$ such that $\forall y [P(y) \to x = y]$

$x = b$

$\forall y (P(y) \to b = y]$.

$P(a) \to b = a$. $F$.

$F$ would have failed for $x = b$ but we only need one to be true ($x = a$ in this case)

$\exists x \; \underline{F}$ $\Rightarrow$ $x = a$ $\vee$

Behaves "like" a big $\underline{\underline{\vee}}$ $x = b$ $\vee$

$x = c$ $\vee$

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$\forall x \; F$ $\Rightarrow$ $x = a$ $\wedge$

Behaves "like" a big $\underline{\underline{\wedge}}$ $x = b$ $\wedge$

$\wedge$

# TUTORIAL LIST 5

5

$\exists x \forall y \; P(x,y)$ true.

(c) $\forall x \forall y \; P(x,y)$ ?

$I$

$D = \{a, b\}$.

$P(a,a) = T$

$P(a,b) = T$

$x = a$

$I(\exists x \forall y \; P(x,y)) =$ $\forall y P(a,y) \leftarrow$ $P(a,a) = T$, $P(a,b) = T$

$\Downarrow$

$\exists x \forall y \; P(x,y) = T$

(c) CAN BE FALSE!

$I(\forall x \forall y \; P(x,y)) = F$

$x = a \Rightarrow \forall y \; P(a,y) = T$

$x = b \Rightarrow \forall y \; (b,y) = F$

$\exists x \Rightarrow$ works "as" a possibly infinite disjunction.

$\forall x \Rightarrow$ works "as" a possibly infinite conjunction.

If

$\forall x F \Rightarrow$ is false.

then $\neg \forall x F \Rightarrow$ is true.

then $\exists x \neg F \Rightarrow$ is true.

$$\forall x F \equiv \neg \exists x \neg F$$

$$\exists x F \equiv \neg \forall x \neg F$$

$$\neg \forall x F \equiv \neg\neg \exists x \neg F \equiv \exists x \neg F .$$

$\exists x \forall y \, P(x,y)$

(iii) $\exists x \exists y \, P(x,y)$.

If $\exists x \forall y \, P(x,y)$ is true:

all possible values for y are true. Any of these can be used to show $\exists x \exists y \, P(x,y)$

$x \to y_1$ $\quad P(x, y_1) = T$

$x \to y_2$ $\quad P(x, y_2) = T$

$x \to y_3$ $\quad P(x, y_3) = T$.

$\vdots$

$\exists x \forall y \, P(x,y) \models \exists x \exists y \, P(x,y)$.

$\exists x \exists y \, P(x,y) \overset{?}{\not\models} \exists x \forall y \, P(x,y)$

(iv) can be false. Using the same interpretation given for (i)

for $x=b$ there is no $y$ such that $P(b,y)$ is true.