4CCS1ELA – Elementary Logic with Applications

Small Group Tutorial 2 (week 4) – Solutions

Propositional Logic II

1. A professor of logic meets 10 of his former students. Albert, Alice, Bob, Bertha, Clifford, Connie, David, Dora, Edgar, Edith who have become five married couples. When asked about their husbands, the ladies gave the following answers:

Alice: My husband is Clifford, and Bob has married Dora.

Bertha: My husband is Albert, and Bob has married Connie.

Connie: Clifford is my husband, Bertha's husband is Edgar.

Dora: My husband is Bob, and David has married Edith.

Edith: Yes, David is my husband. And Albert's wife is Alice.

Additional true information coming from the men was that every lady gave one correct and one wrong answer. This was sufficient to find out the truth. Reproduce the professor's argument.

(from the "Logic for Artificial Intelligence and Information Technology book)

<u>SOLUTION</u>

There are many ways of using Logic to solve this question. Here is a possibility:

There are 25 atomic propositions of the form 'a man whose name begins with X and a woman whose name begins with Y are married', which we denote by $P_{AA}, P_{AB}, \ldots, P_{EE}$. Thus, for instance, P_{AA} means 'Albert is married to Alice'; P_{CB} that 'Clifford is married to Bertha'; and so forth. The information at the professor's disposal is expressed by a rather long conjunction including clauses $(P_{XY} \to \neg P_{XZ})$ and $(P_{YX} \to \neg P_{ZX})$ for any values of X, Y, Z taken from $\{A, B, C\}$ where $Y \neq Z$, and also the following ones, since every lady gave one correct answer and one wrong answer. The requirement that $Y \neq Z$ is because a man in only allowed to marry one woman, so P_{AA} implies $\neg P_{AB}$ and $\neg P_{AC}$ and so forth.

(F1) $P_{CA} \leftrightarrow \neg P_{BD}$

(F2) $P_{AB} \leftrightarrow \neg P_{BC}$

(F3) $P_{CC} \leftrightarrow \neg P_{EB}$

(F4) $P_{BD} \leftrightarrow \neg P_{DE}$

(F5) $P_{DE} \leftrightarrow \neg P_{AA}$

For (F1) above, if the first answer given by Alice is the correct one, then 'Clifford is married to Alice' and furthermore 'Bob is not married to Dora' (and vice-versa).

It will be convenient to show the truth-values of the atoms in a 5×5 table. The unique wife requirement means that 1 occurs exactly once at each row and each column of the table.

Now either the value of P_{CA} is 1 or it is 0, in which case we have $\neg P_{CA}$. Let us assume the value of P_{CA} is 1 first. This will make all other columns X, $X \neq A$, take value 0 in the row C and and all rows Y, $Y \neq C$, take value 0 in the column A.

	A	B	C	D	E
A	0				
B	0				
C	1	0	0	0	0
D	0				
E	0				

From P_{CA} and (F1) we get $\neg P_{BD}$ is 1 and hence we get P_{BD} is 0 and then P_{DE} must be 1 from (F4). Therefore, P_{XE} must be 0 for all $X \neq D$ and P_{DY} must also be 0 for all $Y \neq E$:

	A	B	C	D	E
A	0				0
B	0			0	0
C	1	0	0	0	0
D	0	0	0	0	1
E	0				0

Using the fact that P_{CC} is 0 and (F3), we get that P_{EB} must be 1.

	A	B	C	D	E
A	0	0			0
B	0	0		0	0
C	1	0	0	0	0
D	0	0	0	0	1
E	0	1	0	0	0

If we proceed in this way, we eventually get

	A	B	C	D	E
A	0	0	0	1	0
B	0	0	1	0	0
C	1	0	0	0	0
D	0	0	0	0	1
E	0	1	0	0	0

We now need to check the other possibility, i.e., that P_{CA} is 0. Then if we fill in the table according to (F1), (F4) and (F5), we have

	A	B	C	D	E
A	1	0	0	0	0
B	0	0	0	1	0
C	0			0	
D	0			0	0
E	0			0	

and we observe that P_{BD} must be 1 (F1); hence P_{DE} must be 0 (F4); hence P_{AA} must be 1 (F5); hence P_{AB} and all other columns on that row must be 0 (wife uniqueness). But by (F2) P_{BC} must be 1 (F2), which is not possible, since in that row we already have that P_{BD} is 1 and it must be the only '1' value in that row.

Therefore we cannot have $P_{CA} = 0$ and the previous solution is unique.

2. Rewrite the formulas below into equivalent formulas in CNF and DNF.

1. $P \rightarrow (Q \wedge R)$
2. $S \vee (P \rightarrow \neg Q)$
3. $P \leftrightarrow Q$
4. $P \vee Q$

SOLUTION

	CNF	DNF
1.	$(\neg P \vee Q) \wedge (\neg P \vee R)$	$\neg P \vee (Q \wedge R)$
2.	$S \vee \neg P \vee \neg Q$	$S \vee \neg P \vee \neg Q$
3.	$(P \vee \neg Q) \wedge (\neg P \vee Q)$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$
4.	$P \vee Q$	$P \vee Q$

3. Use the equivalence rules to push all occurrences of the negation symbol $\neg$ next to the atoms in the formulas below:

- $\neg((P \to Q) \vee ((P \to R) \wedge \neg P))$
- $(\neg(P \wedge \neg Q)) \to P$

SOLUTION

- $\neg((P \to Q) \vee ((P \to R) \wedge \neg P))$
  $\neg(P \to Q) \wedge \neg((P \to R) \wedge \neg P)$
  $\neg(\neg P \vee Q) \wedge [\neg(P \to R) \vee \neg\neg P)]$
  $(\neg\neg P \wedge \neg Q) \wedge [\neg(\neg P \vee R) \vee P]$
  $(P \wedge \neg Q) \wedge [(\neg\neg P \wedge \neg R) \vee P]$
  $(P \wedge \neg Q) \wedge ((P \wedge \neg R) \vee P)$

- $(\neg(P \wedge \neg Q)) \to P$
  $(\neg P \vee \neg\neg Q) \to P$
  $(\neg P \vee Q) \to P$

4. Check the validity of the following arguments by using truth-tables.

- $(P \vee Q) \to R$, Q. Therefore R.
- $P \to (Q \vee R)$, P. Therefore R.

SOLUTION

The rows in which all premises are true are marked with ►. The conclusion is marked in bold font. The conclusion must be true (i.e., have value 1) whenever all premises are true for the argument to be valid.

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	P	Q	$\boldsymbol{R}$	$P \vee Q$	$(P \vee Q) \to R$	$((P \vee Q) \to R) \wedge Q$	Argument
	0	0	0	0	1	0	✓
	0	0	1	0	1	0	✓
	0	1	0	1	0	0	✓
▶	0	1	1	1	1	1	✓
	1	0	0	1	0	0	✓
	1	0	1	1	1	0	✓
	1	1	0	1	0	0	✓
▶	1	1	1	1	1	1	✓

  This argument is valid.

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	P	Q	$\boldsymbol{R}$	$Q \vee R$	$P \to (Q \vee R)$	$P \wedge (P \to (Q \vee R))$	Argument
	0	0	0	0	1	0	✓
	0	0	1	1	1	0	✓
	0	1	0	1	1	0	✓
	0	1	1	1	1	0	✓
	1	0	0	0	0	0	✓
▶	1	0	1	1	1	1	✓
▶	1	1	0	1	1	1	$\times$
▶	1	1	1	1	1	1	✓

  This argument is not valid. The row with the symbol $\times$ is a *counter-model* for the argument.