Lecture 5: Probabilistic reasoning over time

Frederik Mallmann-Trenn

Department of Informatics
King's College London

(Version 1.3)

Today

- Introduction
- Probabilistic Reasoning I
- Probabilistic Reasoning II
- Sequential Decision Making
- Probabilistic Reasoning over Time
- Game Theory
- Argumentation I
- Argumentation II
- (A peek at) Machine Learning
- AI & Ethics

Introduction

- So far we have looked at a number of ways of using probability to handle uncertainty
  - Bayesian networks
  - Decision theory
  - Markov Decision Processes
- The formalisms are static, and so have limited ability to handle changing information.
- This week we'll look at models that can handle such dynamic situations.
  - Based on Bayesian networks

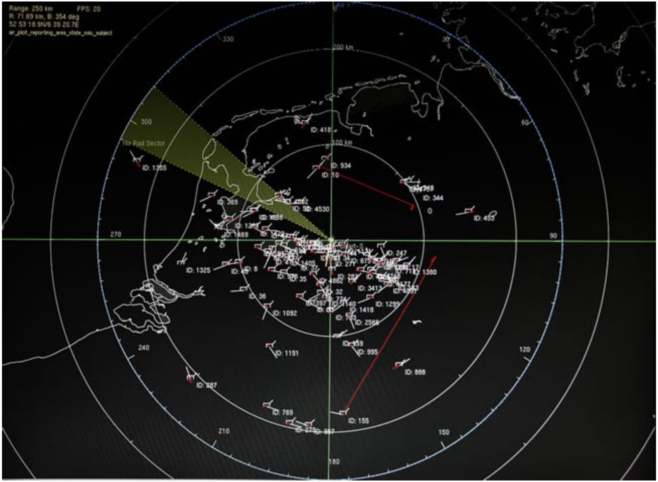
What?

(Pendleton Ward/Cartoon Network)

Robot localization

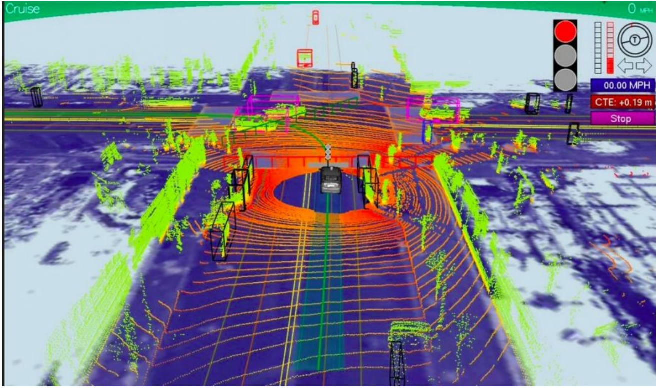
(Eric Schneider)

Tracking



(Thales)

Tracking



(Sebastian Thrun & Chris Urmson/Google)

States and observations

- The approach we'll look at considers the world to be a series of time slices.
- Each slice contains some variables:
  - The set $\mathbf{X}_t$ which we can't observe; and
  - The set $\mathbf{E}_t$ which we can observe.
- Note that $\mathbf{X}_t$ and $\mathbf{E}_t$ are **bold** to indicate that they may be **sets** of variables.
- At a given point in time we have an observation $\mathbf{E}_t = \mathbf{e}_t$.
- What would be an example?

States and observations

- Consider you live and work in some location without a window
  - Not so hard to imagine when you know KCL.

(indianapublicmedia.org/jgraham)

- You want to know whether it is raining.
- Your only information is looking at whether somebody who comes into your location each morning is carrying an umbrella.

States and observations

- Each day is one value of t.
- $\mathbf{E}_t$ contains the single variable U_t (or $Umbrella_t$).
  - Is the person carrying an umbrella?
- $\mathbf{X}_t$ contains the single variable R_t (or $Rain_t$)
  - Is it raining?

States and observations

- State sequence starts at $t = 0$, and the interval between slices in general depends on the problem.
  - Here it is one day
  - In robot localization it is pretty arbitrary
- First piece of evidence arrives at $t = 1$
- So, the umbrella world is:

$$R_0, R_1, R_2, \ldots$$

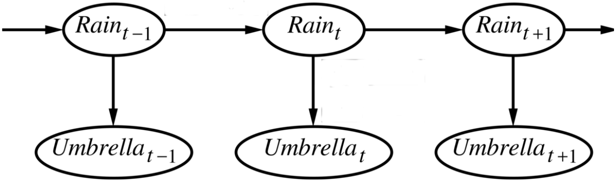
$$U_1, U_2, U_3, \ldots$$

- $a : b$ means the sequence of integers from a to b, so that $U_{2:4}$ is the sequence:

$$U_2, U_3, U_4$$

States and observations

- So the model is something like this:



Transition and sensor models

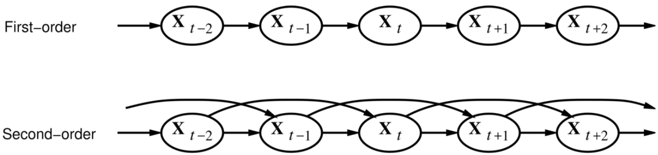
- We need to add two components to this backbone:
  - How the world evolves
    Transition model
  - What the evidence tells us
    Sensor model
- The transition model tells us:

$$\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{0:t-1})$$

what the probability is that it is raining today given the weather every previous day for as long as records have existed.

Transition and sensor models

- Luckily Professor Markov helps us out again.
- Make a Markov assumption that the value of the current state depends only on a finite fixed number of previous states.



- We commonly assume a first order Markov process, where the current state depends only on the previous state:

$$\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$$

Transition and sensor models

- What would the model look like for a second order Markov process?

$$\mathbf{P}(\mathbf{X}_t | \mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t | \mathbf{X}_{t-2}, \mathbf{X}_{t-1})$$

Transition and sensor models

- Even with the Markov assumption we have a potentially infinite set of conditional probabilities.

$$\mathbf{P}(\mathbf{X}_1|\mathbf{X}_0), \mathbf{P}(\mathbf{X}_2|\mathbf{X}_1), \mathbf{P}(\mathbf{X}_3|\mathbf{X}_2) \ldots$$

- Usually circumvent this by assuming a stationary process
  - The model doesn't change
  - But the state itself can
- Thus we only have one, general $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$

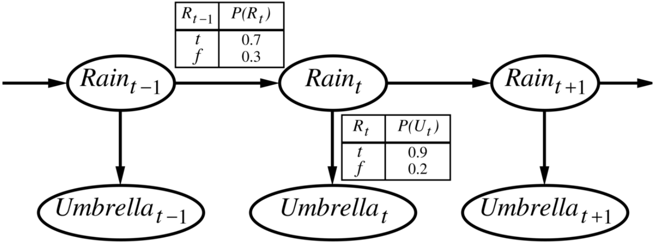
Sensor model

- The evidence variables $\mathbf{E}_t$ could depend on lots of previous variables.
- But we will assume the state is constructed in such a way that evidence only depends on the current state.
  - Again we can fix this by adding variables to the state.
- A Markov assumption for the sensor model:

$$\mathbf{P}(\mathbf{E}_t | \mathbf{X}_{0:t-1}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t | \mathbf{X}_t)$$

Transition and sensor models

- Here are the transition and sensor models for the umbrella world:



- As for previous Bayesian networks, arrows run from causes to effects.

Initial state

- Also need to say how things get started:

$$\mathbf{P}(\mathbf{X}_0)$$

The prior probability over the state

And so. . .

- With this, we can then compute the complete joint probability over all the time slices:

$$\mathbf{P}(\mathbf{X}_{0:t}, \mathbf{E}_{1:t}) = \mathbf{P}(\mathbf{X}_0) \prod_{i=1}^{t} \mathbf{P}(\mathbf{X}_i | \mathbf{X}_{i-1}) \mathbf{P}(\mathbf{E}_i | \mathbf{X}_i)$$

- This is just the joint probability over a Bayesian network:

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

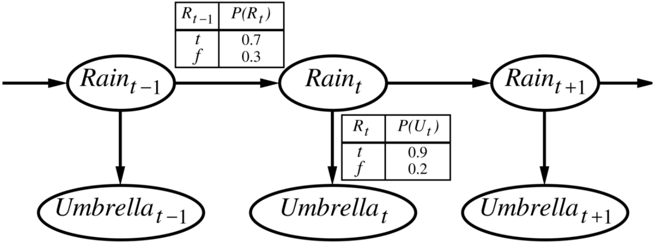
  from Lecture 3

- As we know from before, this is sufficient to compute anything we want.

Awesome!

(Pendleton Ward/Cartoon Network)

Dynamic Bayesian Network

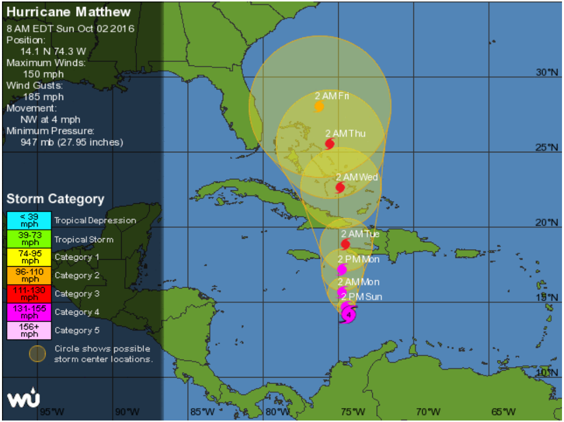


- Each slice contains some variables:
  - The set $\mathbf{X}_t$ which we can't observe; and
  - The set $\mathbf{E}_t$ which we can observe.
- Each slice is a Bayesian network.
- Whole thing is a Dynamic Bayesian Network (DBN).

Inference tasks

- What kinds of thing can we do with the model?
- Filtering: $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$
  - determine belief state—input to the decision process of a rational agent.
- Prediction: $\mathbf{P}(\mathbf{X}_{t'}|\mathbf{e}_{1:t})$ for $t' > t$
  - Evaluation of possible action sequences, like filtering without the evidence.
- Smoothing: $\mathbf{P}(\mathbf{X}_{t'}|\mathbf{e}_{1:t})$ for $0 \leqslant t' < t$
  - Better estimate of past states, essential for learning.
- We'll look at how to do all these.

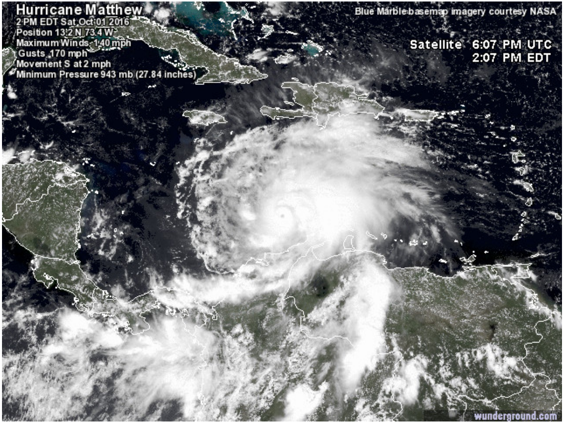
Inference tasks



(wunderground.com)

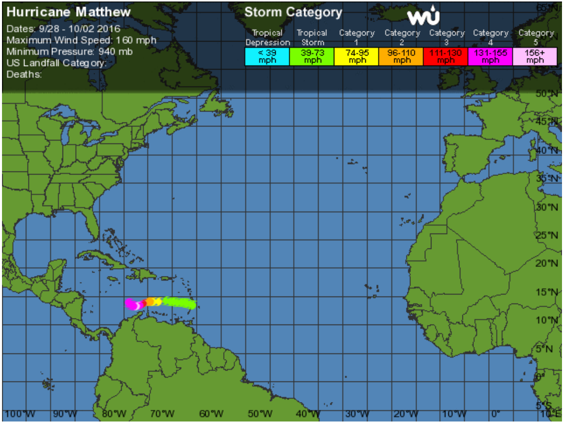
- Filtering
- Prediction

Inference tasks



(wunderground.com/NASA)

Inference tasks



(wunderground.com)

- Smoothing

Inference tasks

- Filtering
- Prediction
- Smoothing

Filtering

- A good algorithm for filtering will maintain a current state estimate and update it at each point.

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = f(\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t}), \mathbf{e}_{t+1}) \tag{1}$$

- So we are looking for a simple function $f(\cdot)$ that we can just use to get the next estimate based on the last estimate
- Saves recomputation.
- It turns out that this is easy enough to come up with.

Filtering

- Our goal is to rewrite the l.h.s. of Equation (1)

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1})$$

- First, we divide up the evidence:

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}, \mathbf{e}_{t+1})$$

- Then we apply Bayes rule (see next slide), remembering the use of the normalization factor α.

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}, \mathbf{e}_{1:t})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

Bayes rule?

- We are (hopefully) used to Bayes rule in its standard format:

$$\mathbf{P}(A|C) = \frac{\mathbf{P}(C|A)\mathbf{P}(A)}{\mathbf{P}(C)}$$

- The same thing holds when everything is conditioned on B:

$$\begin{aligned}\mathbf{P}(A|B,C) &= \frac{\mathbf{P}(C|A,B)\mathbf{P}(A|B)}{\mathbf{P}(C|B)} \\ &= \alpha\mathbf{P}(C|A,B)\mathbf{P}(A|B)\end{aligned}$$

- $(\alpha = 1\mathbf{P}(C|B))$
- This gives (for $A = \mathbf{X}_{t+1}$, $B = \mathbf{e}_{1:t}$, and $C = \mathbf{e}_{t+1}$,):

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}, \mathbf{e}_{t+1}) = \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}, \mathbf{e}_{1:t})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

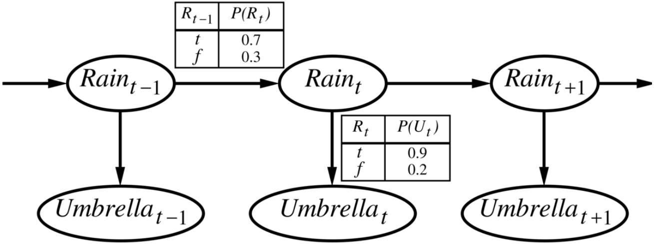
Filtering

- And after that we use the Markov assumption on the sensor model:

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}, \mathbf{e}_{1:t})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})\end{aligned}$$

- The result of this assumption is to make that first term on the right hand side ignore all the evidence before $t+1$ — the probability of the observation at $t+1$ only depends on the value of $\mathbf{X}_{t+1}$.

Filtering


R_{t-1} | $P(R_t)$
t 0.7
f 0.3
R_t | $P(U_t)$
t 0.9
f 0.2
$Rain_{t-1}$
$Rain_t$
$Rain_{t+1}$
$Umbrella_{t-1}$
$Umbrella_t$
$Umbrella_{t+1}$

Filtering

- Let's look at that expression some more:

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

- The first term on the right updates with the new evidence and the second term on the right is a one step prediction from the evidence up to t to the state at $t + 1$.

Filtering

- Next we condition (see next slide) on the current state $\mathbf{P}(\mathbf{X}_t)$:

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\sum_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t,\mathbf{e}_{1:t})P(\mathbf{x}_t|\mathbf{e}_{1:t})\end{aligned}$$

- Finally, we apply the Markov assumption again:

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\sum_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t}) \quad (2)$$

and this is how we obtain the filtered probability of $\mathbf{X}_{t+1}$ given evidence $\mathbf{e}_{1:t+1}$

Conditioning?

- By law of total probability:

$$\begin{aligned}\mathbf{P}(A) &= \sum_B \mathbf{P}(A, B)\\ &= \sum_B \mathbf{P}(A|B)\mathbf{P}(B)\\ \mathbf{P}(A|C) &= \sum_B \mathbf{P}(A|B, C)\mathbf{P}(B|C)\end{aligned}$$

- So

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t, \mathbf{e}_{1:t})P(\mathbf{x}_t|\mathbf{e}_{1:t})$$

where:

$$\begin{aligned}A &= \mathbf{X}_{t+1}\\ B &= \mathbf{x}_t\\ C &= \mathbf{e}_{1:t}\end{aligned}$$

Filtering

- If we define $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1})$ to be $\mathbf{f}_{1:t+1}$, then we have a recursive update.
- $\mathbf{f}_{1:t+1}$ is a combination of:
  - $\mathbf{f}_{1:t}$, the value for the previous step.
  - $\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})$, the sensor model.
  - $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_{t})$, the transition model.
- In other words, the probability distribution over the state variables at $t+1$ is: a function of:
  - the probability distribution over the state variables at t;
  - the sensor model;
  - and the transition model.
- Space and time constant, independent of t.
- This allows a limited agent to compute the current distribution for any length of sequence.

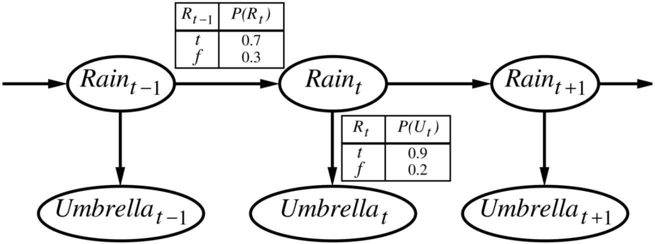
Filtering

- Since we will need it later, we'll write out $\mathbf{f}_{1:t}$:

$$\begin{aligned}\mathbf{f}_{1:t} &= \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t}) \\ &= \alpha \mathbf{P}(\mathbf{e}_t|\mathbf{X}_t) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_t|\mathbf{x}_{t-1}) P(\mathbf{x}_{t-1}|\mathbf{e}_{1:t-1})\end{aligned} \tag{3}$$

- Note that this is different to what we had on slide 34. That was for days from 1 to $t+1$. This is for days 1 to t.

Filtering the umbrella example



Filtering the umbrella example

- The prior for day 0, $P(R_0)$, is $\langle 0.5, 0.5 \rangle$.
- We can first *predict* whether it will rain on day 1 given what we already know:

$$
\begin{aligned}
\mathbf{P}(R_1) &= \sum_{r_0} \mathbf{P}(R_1 | r_0) P(r_0) \\
&= \langle 0.7, 0.3 \rangle \times 0.5 + \langle 0.3, 0.7 \rangle \times 0.5 \\
&= \langle 0.5, 0.5 \rangle
\end{aligned}
$$

- As we should expect, this just gives us the prior — that is the probability of rain when we don't have any evidence.

Filtering the umbrella example

- However, we have observed the umbrella, so that $U_1 = true$, and we can update using the sensor model:

$$\begin{aligned}\mathbf{P}(R_1|u_1) &= \alpha\mathbf{P}(u_1|R_1)\mathbf{P}(R_1)\\ &= \alpha\langle 0.9, 0.2\rangle\langle 0.5, 0.5\rangle\\ &= \alpha\langle 0.45, 0.1\rangle\\ &\approx \langle 0.818, 0.182\rangle\end{aligned}$$

- So, since umbrella is strong evidence for rain, the probability of rain is much higher once we take the observation into account.

Filtering the umbrella example

- We can then carry out the same computation for day 2, first *predict*ing whether it will rain given what we already know:

$$\begin{aligned}
\mathbf{P}(R_2|u_1) &= \sum_{r_1} \mathbf{P}(R_2|r_1)P(r_1|u_1) \\
&= \langle 0.7, 0.3\rangle \times 0.818 + \langle 0.3, 0.7\rangle \times 0.182 \\
&\approx \langle 0.627, 0.373\rangle
\end{aligned}$$

- So even without evidence of rain on the second day there is a higher probability of rain than the prior because rain tends to follow rain.
- In this model rain tends to persist.

Filtering the umbrella example

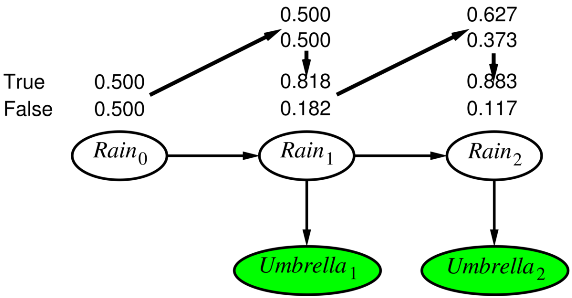
- Then we can repeat the evidence update, u_2 ($U_2 = \mathit{true}$), so:

$$\begin{aligned}\mathbf{P}(R_2|u_1, u_2) &= \alpha \mathbf{P}(u_2|R_2)\mathbf{P}(R_2|u_1) \\ &= \alpha\langle 0.9, 0.2\rangle\langle 0.627, 0.373\rangle \\ &= \alpha\langle 0.565, 0.075\rangle \\ &\approx \langle 0.883, 0.117\rangle\end{aligned}$$

- So, the probability of rain increases again, and is higher than on day 1.

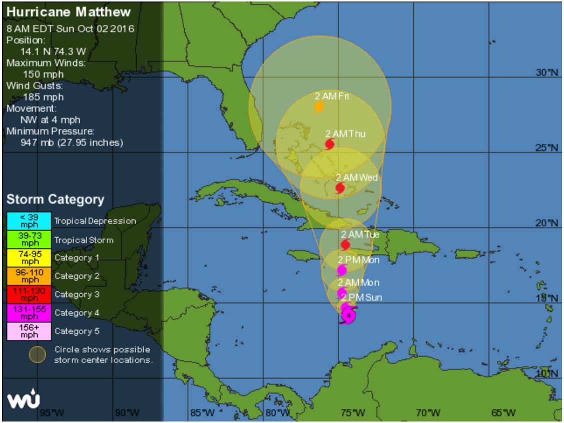
Filtering the umbrella example

- Put more succinctly:



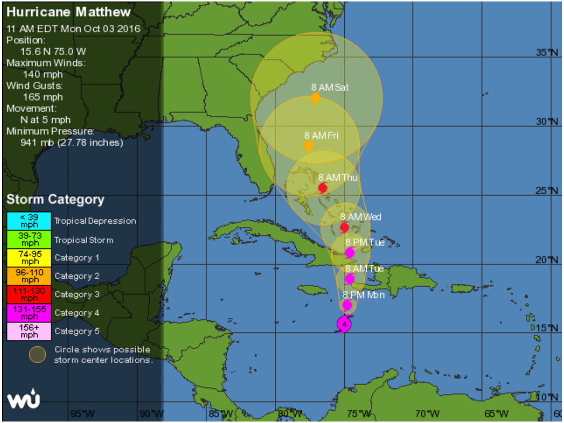
- We can think of the calculation as messages passed along the chain.

Inference tasks



(wunderground.com)

Inference tasks



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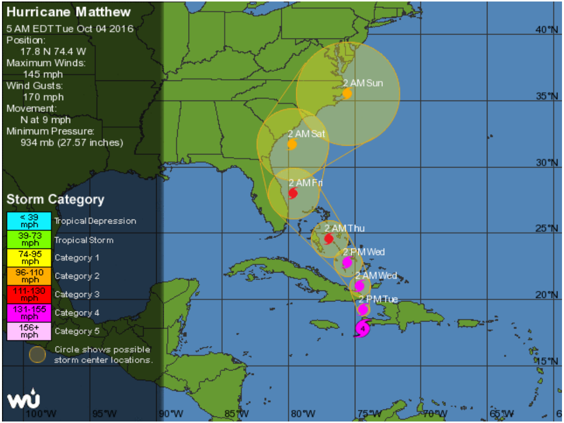
Inference tasks



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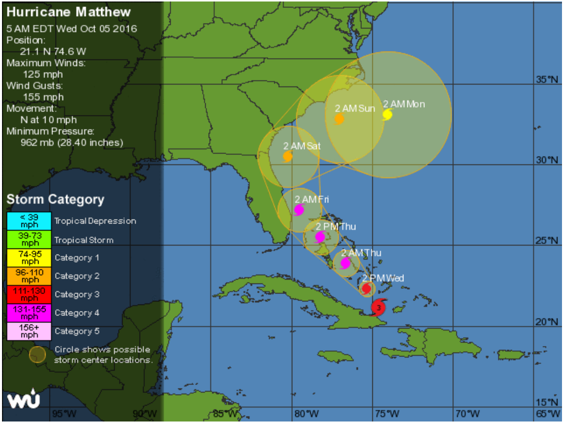
Inference tasks



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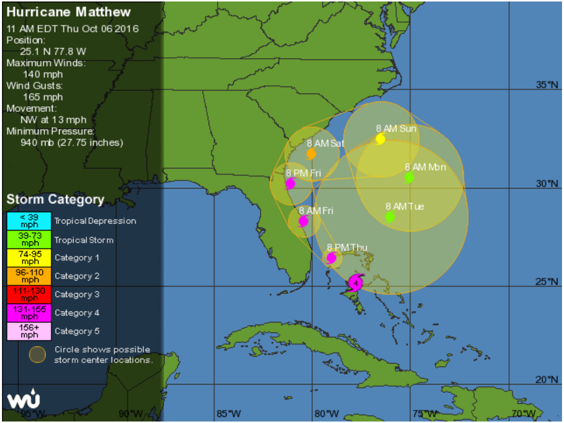
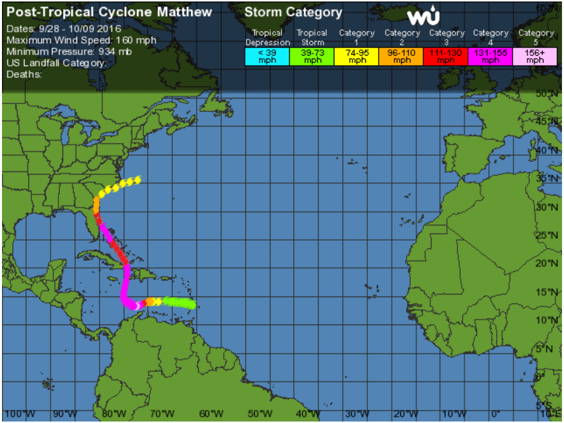
Inference tasks



(wunderground.com)

Inference tasks



(wunderground.com)

Inference tasks

- Filtering
- Prediction
- Smoothing

Prediction

- Prediction is filtering without new evidence.

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t|\mathbf{e}_{1:t})$$

- Given the current state, what does the future bring?

Prediction

- As for filtering, we can use the Markov assumption to give us:

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) &= \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t|\mathbf{e}_{1:t}) \\ &= \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t) P(\mathbf{x}_t|\mathbf{e}_{1:t})\end{aligned}$$

  which allows us to predict one step forward in time.

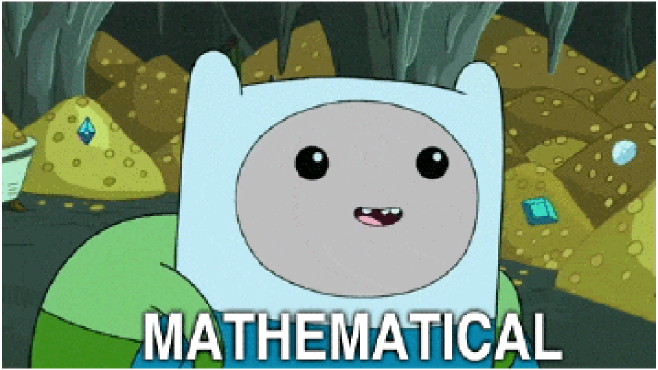
Predicting the Umbrella example

- In fact, we already did this.
- The first step in filtering was prediction.
- Given we saw an umbrella on day 1, what is the probability of rain on day 2?

$$
\begin{aligned}
\mathbf{P}(R_2|u_1) &= \sum_{r_1} \mathbf{P}(R_2|r_1)P(r_1|u_1) \\
&= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \\
&\approx \langle 0.627, 0.373 \rangle
\end{aligned}
$$

just as on slide 40.

Sorted



(Pendleton Ward/Cartoon Network)

Predicting and Filtering

- (2) gave us a one step method for computing the filtered probability, but the worked example first computed the predicted probability and then the filtered probability.
- What's going on?

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t}) \qquad \text{(Predict)}$$

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t}) \qquad \text{(Filter)}$$

- So if we write the predicted probability of $\mathbf{X}$ at $t+1$ given evidence $\mathbf{e}_{1:t}$ as $Pred(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$, filtering is:

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})Pred(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

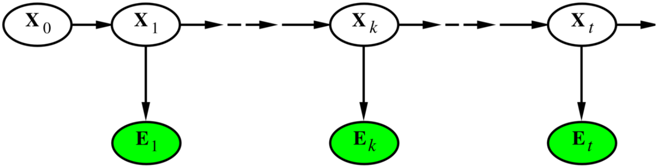
- Filtering is predicting and then updating with evidence.

Inference tasks

- Filtering
- Prediction
- Smoothing

Smoothing

- Smoothing is computing the distribution over past states given evidence up to the present.



- Want the probability over all states k, $0 \leqslant k < t$.

Smoothing

- Break the computation into two pieces, evidence from 0 to k and evidence from $k+1$ to t.
- Proceeding just as before:

$$
\begin{aligned}
\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{e}_{1:k}) && \text{Bayes' Rule (see over)} \\
&= \alpha \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) && \text{Cond. Independence} \\
&= \alpha \mathbf{f}_{1:k}\mathbf{b}_{k+1:t} && (4)
\end{aligned}
$$

- $\mathbf{f}$ is a “forward” message, computed just as we did for the filtering/prediction case.
- $\mathbf{b}$ is a backward message.

Bayes rule?

- Again we make use of:

$$\begin{aligned}\mathbf{P}(A|B,C) &= \frac{\mathbf{P}(A|B)\mathbf{P}(C|A,B)}{\mathbf{P}(C|B)} \\ &= \alpha\mathbf{P}(A|B)\mathbf{P}(C|A,B)\end{aligned}$$

- This gives:

$$\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k},\mathbf{e}_{k+1:t}) = \alpha\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k,\mathbf{e}_{1:k})$$

where:

$$\begin{aligned}A &= \mathbf{X}_k \\ B &= \mathbf{e}_{1:k} \\ C &= \mathbf{e}_{k+1:t}\end{aligned}$$

Smoothing

- To compute the backwards message we condition on $\mathbf{X}_{k+1}$:

$$\begin{aligned}\mathbf{b}_{k+1:t} &= \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k)\\ &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)\end{aligned}$$

- Then apply the Markov assumption:

$$\mathbf{b}_{k+1:t} = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)$$

- Rewriting $\mathbf{e}_{k+1:t}$:

$$\mathbf{b}_{k+1:t} = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)$$

Smoothing

- We had:

$$\mathbf{b}_{k+1:t} = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)$$

- Applying conditional independence again:

$$\mathbf{b}_{k+1:t} = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)$$

Smoothing

- So, we have:

$$\begin{aligned}\mathbf{b}_{k+1:t} &= \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) \\ &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1})P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)\end{aligned} \tag{5}$$

- Rewriting $\mathbf{P}(\mathbf{e}_{k+2:t}|\mathbf{X}_k)$ as $\mathbf{b}_{k+2:t}$:

$$\mathbf{b}_{k+1:t} = \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1})\mathbf{b}_{k+2:t}\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \tag{6}$$

- The first term on the right hand side is the sensor model.
  The third term is the transition model.
  The second term is the message from $k + 2$

Smoothing

- So in:

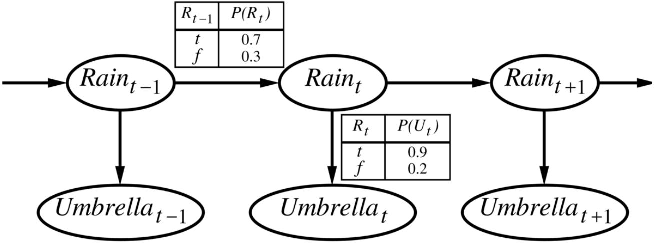
$$\mathbf{f}_{1:k}\mathbf{b}_{k+1:t}$$

  there are two recursive components.
- We have a forward component from 1 to k, and a backward component from t to k.
- The backward component is initialized with:

$$\mathbf{P}(\mathbf{e}_{t+1:t}|\mathbf{X}_t) = \mathbf{P}(\ |\mathbf{X}_t) = 1 \tag{7}$$

  since the probability of observing the set of no observations is always 1.

Smoothing the umbrella example



- Consider the umbrella world on day 1.
- Compute the smoothed probability of rain on day 1.
- Have seen umbrellas on day 1 and day 2.

Smoothing the umbrella example

- Now (4) tells us that:

$$\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) = \alpha \mathbf{f}_{1:k}\mathbf{b}_{k+1:t}$$

- We want probability on day 1 given umbrellas n day 1 and 2. So $k = 1$ and $t = 2$.
- We get:

$$\mathbf{P}(\mathbf{X}_1|\mathbf{e}_{1:2}) = \alpha \mathbf{f}_{1:1}\mathbf{b}_{2:2}$$

- Or in umbrella notation, with $\mathbf{X} = R$ and $\mathbf{e} = u$:

$$\mathbf{P}(R_1|u_{1:2}) = \alpha \mathbf{f}_{1:1}\mathbf{b}_{2:2}$$

Smoothing the umbrella example

Now

$$\mathbf{P}(R_1|u_{1:2}) = \alpha \mathbf{f}_{1:1}\mathbf{b}_{2:2}$$

and (3) tells us that:

$$\mathbf{f}_{1:1} = \mathbf{P}(\mathbf{X}_1|\mathbf{e}_{1:1})$$

which in umbrella notation is:

$$\mathbf{f}_{1:1} = \mathbf{P}(R_1|u_{1:1})$$

- Similarly, (5) gives us:

$$\begin{aligned}\mathbf{b}_{2:2} &= \mathbf{P}(e_{2:2}|X_1)\\ &= \mathbf{P}(u_{2:2}|R_1)\end{aligned}$$

- Overall:

$$\mathbf{P}(R_1|u_{1:2}) = \alpha \mathbf{P}(R_1|\mathbf{u}_{1:1})\mathbf{P}(u_{2:2}|R_1)$$

Smoothing the umbrella example

- This might be easier to read without the sequence notation.
- We have:

$$\begin{aligned}\mathbf{P}(R_1|u_{1:2}) &= \alpha \mathbf{f}_{1:1}\mathbf{b}_{2:2} \\ &= \alpha \mathbf{P}(R_1|u_{1:1})\mathbf{P}(u_{2:2}|R_1)\end{aligned}$$

which is just:

$$\begin{aligned}\mathbf{P}(R_1|u_1, u_2) &= \alpha \mathbf{f}_{1:1}\mathbf{b}_{2:2} \\ &= \alpha \mathbf{P}(R_1|u_1)\mathbf{P}(u_2|R_1)\end{aligned}$$

Smoothing the umbrella example

- For the forward message, we have:

$$\begin{aligned}\mathbf{f}_{1:1} &= \mathbf{P}(R_1|u_1)\\ &= \alpha\mathbf{P}(u_1|R_1)\sum_{R_1}\mathbf{P}(R_1|r_0)P(r_0|u_0)\end{aligned}$$

Smoothing the umbrella example

- We already computed the forward message:

$$\mathbf{f}_{1:1} = \mathbf{P}(R_1|u_1)$$

since this is the filtered probability of rain on day 1 when we observed an umbrella on day 1:

$$\begin{aligned}\mathbf{f}_{1:1} &= \mathbf{P}(R_1|u_1) \\ &= \langle 0.818, 0.182 \rangle\end{aligned}$$

which we calculated on slide 40.

Smoothing the umbrella example

- For the backward message we have:

$$
\begin{aligned}
\mathbf{b}_{2:2} &= \mathbf{P}(u_2|R_1) \\
&= \sum_{r_2} P(u_2|r_2)P(u_{3:2}|r_2)\mathbf{P}(r_2|R_1)
\end{aligned}
$$

- Now, from (7) we know that:

$$
P(u_{3:2}|r_2) = P(|r_2) = 1
$$

- And so:

$$
\begin{aligned}
\mathbf{b}_{2:2} &= \sum_{r_2} P(u_2|r_2)P(u_{3:2}|r_2)\mathbf{P}(r_2|R_1) \\
&= (0.9 \times 1 \times \langle 0.7, 0.3 \rangle) \\
&\quad + (0.2 \times 1 \times \langle 0.3, 0.7 \rangle) \\
&= \langle 0.69, 0.41 \rangle
\end{aligned}
$$

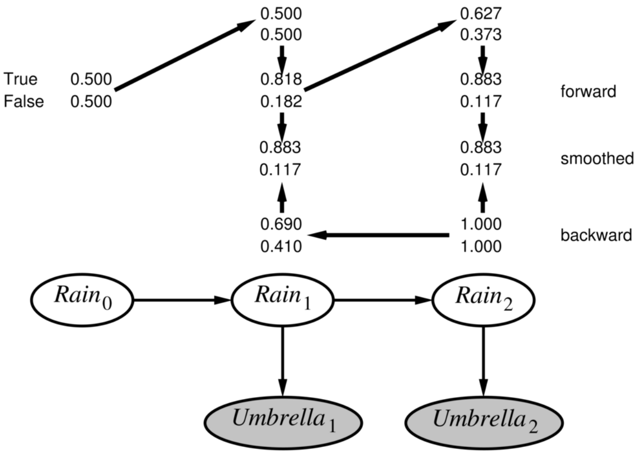
Smoothing the umbrella example

- Plugging the values for forward and backward messages back into the expression we started with gives:

$$
\begin{aligned}
\mathbf{P}(R_1 | u_1, u_2) &= \alpha \mathbf{f}_{1:1} \mathbf{b}_{2:2} \\
&= \alpha \langle 0.818, 0.182 \rangle \times \langle 0.69, 0.41 \rangle \\
&= \alpha \langle 0.56, 0.075 \rangle \\
&\approx \langle 0.882, 0.118 \rangle
\end{aligned}
$$

- Again we can think of this as message passing (next slide)

Smoothing the umbrella example



Smoothing the umbrella example

- The smoothed probability of rain on day 1 is higher than the filtered estimate because the umbrella on day 2 makes it more likely to have rained on day 1.

(indianapublicmedia.org/jgraham)

- Again these updates use constant time and space.

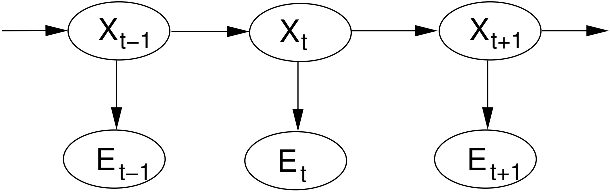
Where we are?

(Pendleton Ward/Cartoon Network)

Where we are?

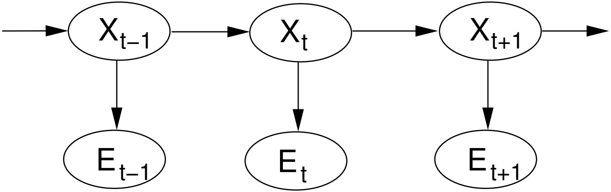
- We have a general approach to all the inference problems without thinking much about the specific details of the models.
- When we get specific, we find we can solve several common classes of problem.

Hidden Markov models



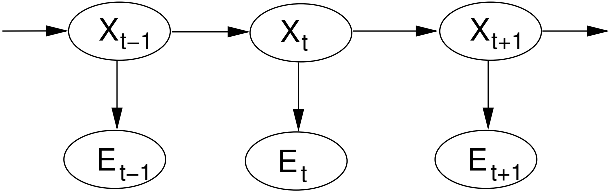
- HMMs have $\mathbf{X}_t$ as a single, discrete variable
- Usually $\mathbf{E}_t$ is also discrete
- Domain of $\mathbf{X}_t$ is $\{1, \ldots, S\}$

Hidden Markov models



- Transition matrix $\mathbf{T}_{ij} = P(X_t = j | X_{t-1} = i)$
  - e.g., $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$
- Sensor matrix $\mathbf{O}_t$ for each time step, diagonal elements $P(e_t | X_t = i)$
  - $\mathbf{O}_t = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$ for U = true.
  - $\mathbf{O}_t = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.8 \end{pmatrix}$ for U = false.

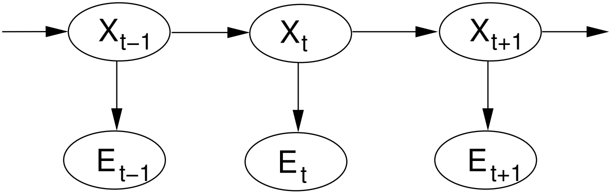
Hidden Markov models



- Forward and backward messages as column vectors:

$$
\begin{aligned}
\mathbf{f}_{1:t+1} &= \alpha \mathbf{O}_{t+1} \mathbf{T}^{\top} \mathbf{f}_{1:t} \\
\mathbf{b}_{k+1:t} &= \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t}
\end{aligned}
$$

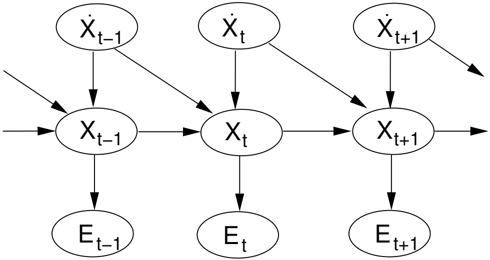
Hidden Markov models



- Forward-backward algorithm needs time $O(S^2 t)$ and space $O(St)$
- Matrix description points the way to easy implementation.

Kalman filters

- Modelling systems described by a set of continuous variables
- Gaussian prior, linear Gaussian transition model and sensor model

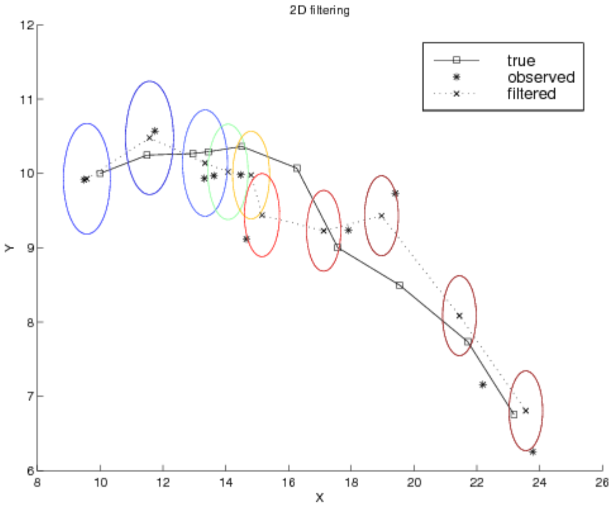


- Above is a Kalman filter for tracking in one dimension.

Kalman filters

- Robot tracking —$\mathbf{X}_t = X, Y, \dot{X}, \dot{Y}$
- Airplanes, ecosystems, economies, chemical plants, etc

Kalman filters

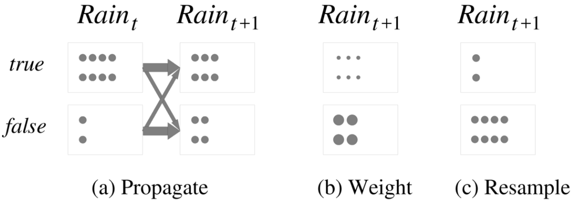


Back to DBNs

- The time slices we have looked at so far have been rather simple.
- Not a restriction on the method.
- Can have arbitrarily complex X and E.
- But our existing inference methods may have problems with computational complexity.

Particle filters

- Technique for approximate solution of a DBN.
- Basic idea: ensure that the population of samples ("particles") tracks the high-likelihood regions of the state-space
- Replicate particles proportional to likelihood for $\mathbf{e}_t$



- Time/space complexity linear in the number of particles.

Particle filters

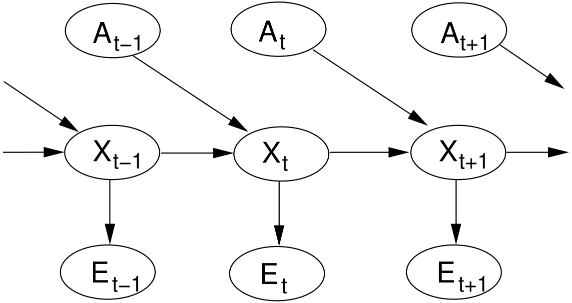
- Widely used for tracking nonlinear systems, esp. in vision

Chad Jenkins

- Also used for simultaneous localization and mapping in mobile robots
  - 10^5-dimensional state space
- Approximation error of particle filtering remains bounded over time.
- At least empirically—theoretical analysis is difficult.

Particle filters

- Slightly more complex DBN than an HMM.



- A is action.

Particle filter

- We approximate $P(x_t)$ by a set of samples:

$$P(x_t) \approx \{x_t^{(i)}, w_t^{(i)}\}_{i=1,\ldots,m}$$

- Each $x_t^{(i)}$ is a possible value of x, and each $w_t^{(i)}$ is the probability of that value (also called an importance factor).
- Initially we have a set of samples (typically uniform) that give us $P(x_0)$.
- Then we update with the following algorithm.

Particle filter

$x_{t+1} = \varnothing$

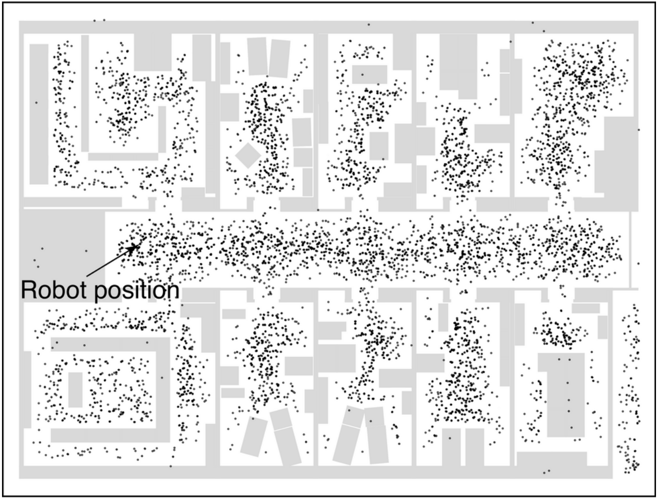
for $j = 1$ to m
// `apply the transition model`
generate a new sample $x_{t+1}^{(j)}$ from $x_t^{(j)}$, a_t and $\Pr(x_{t+1} \mid x_t, a_t)$
// `apply the sensor model`
compute the weight $w_{t+1}^{(j)} = \Pr(e_{t+1} \mid x_{t+1})$

// `pick points randomly but biased by their weight`
for $j = 1$ to m
pick a random $x_{t+1}^{(i)}$ from x_{t+1} according to $w_{t+1}^{(1)}, \ldots, w_{t+1}^{(m)}$

normalize w_{t+1} in x_{t+1}
return x_{t+1}

- And that is all it takes.

Particle filter

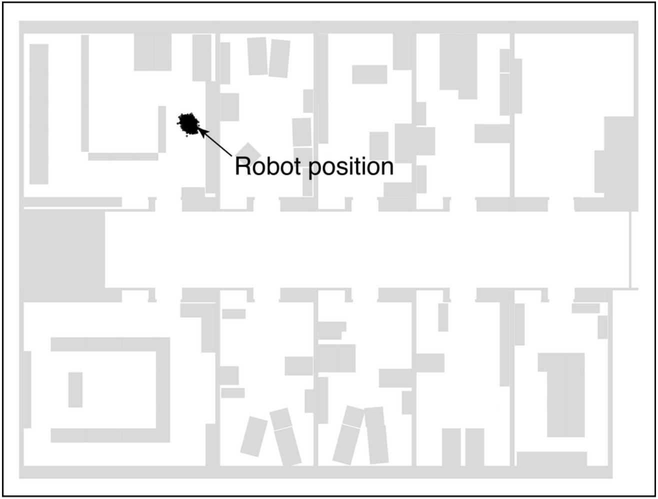


(a)

Particle filter

(b)

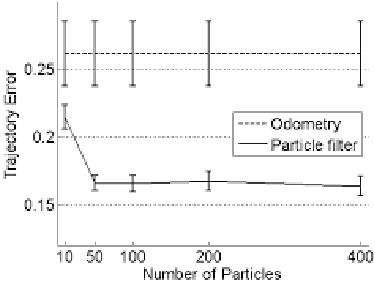
Particle filter



(c)

Particle filter

- Relation between error and number of particles:

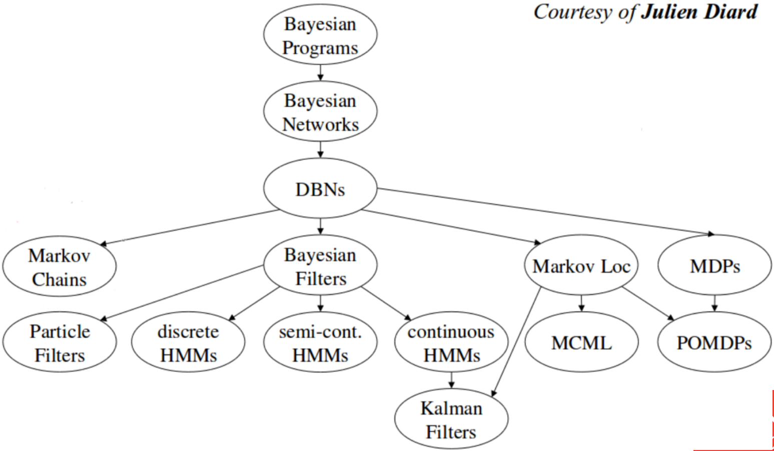


(Burguera, González and Oliver)

- Error would be reduced with more accurate sensing (laser not sonar).

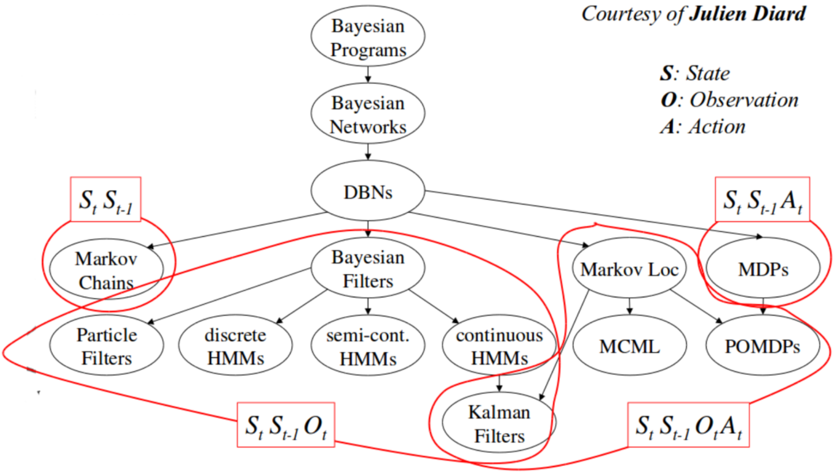
DBNs

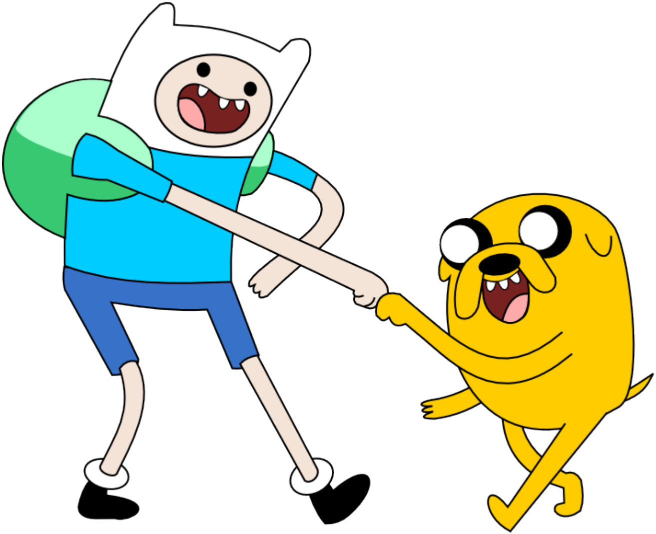
A general classification of dynamic probabilistic models



DBNs

A general classification of dynamic probabilistic models





(Pendleton Ward/Cartoon Network)

Summary

- This lecture moved from the static view of the world encorporated in Bayesian networks to something more dynamic.
- Dynamic Bayesian networks.
- We looked at the general types of inference possible in DBNs and showed how the necessary computations could be done.
- We also looked at some specific classes of problem that can be captured by DBNs.