4CCS1ELA: Tutorial list 5 – Sample Solutions

1. Assume that $\exists x \forall y P(x, y)$ is true. Which of the following formulas must also to be true? If the formula is true, explain. Otherwise, give a counterexample.

(i) $\forall x \forall y P(x, y)$.

(ii) $\forall x \exists y P(x, y)$.

(iii) $\exists x \exists y P(x, y)$.

SOLUTION

(i) $\forall x \forall y P(x, y)$. This formula can be false, i.e. $\exists x \forall y P(x, y) \not\models \forall x \forall y P(x, y)$.
Let $D = \{a, b\}$ and $\mathcal{I}(P(a, a)) = \mathcal{I}(P(a, b)) = 1$ but $\mathcal{I}(P(b, a)) = \mathcal{I}(P(b, b)) = 0$.
Then $\exists x \forall y P(x, y)$ is true under the interpretations $\mathcal{I}$, but $\forall x \forall y P(x, y)$ is false under $\mathcal{I}$.

Another counterexample. Let the domain be the set of positive integers. let $P(x, y)$ means $x \leq y$. Then $\exists x \forall y P(x, y)$ is true (witness $x = 1$) but $\forall x \forall y P(x, y)$ is false (witness $x = 2, y = 1$).

(ii) $\forall x \exists y P(x, y)$. This formula can be false, i.e. $\exists x \forall y P(x, y) \not\models \forall x \exists y P(x, y)$. The interpretation given in the answer (i) is a counterexample for this case too.

(iii) $\exists x \exists y P(x, y)$. This formula must be true, i.e. $\exists x \forall y P(x, y) \models \exists x \exists y P(x, y)$.
If $\exists x \forall y P(x, y)$ is true under an interpretation $\mathcal{I}$, then there is some value for x, say d, such that $\forall y P(d, y)$ is true under the interpretation $\mathcal{I}$. Choosing any value for y whatsoever, including the same d, makes $P(d, y)$ true. Therefore, $\exists x \exists y P(x, y)$ is true under $\mathcal{I}$.

2. Consider the formula $\mathcal{F} = \neg \forall x \exists y P(x, y)$. Determine which of the following formulas is logically equivalent to $\mathcal{F}$ and which is not. If the formula is equivalent to $\mathcal{F}$, then show it using successive equivalences.

(i) $\exists x \neg \forall y P(x, y)$.

(ii) $\forall x \neg \exists y P(x, y)$.

(iii) $\exists x \forall y \neg P(x, y)$.

(iv) $\exists x \exists y \neg P(x, y)$.

SOLUTION

$\mathcal{F} = \neg \forall x \exists y P(x, y) \equiv \exists x \neg \exists y P(x, y) \equiv \underline{\exists x \forall y \neg P(x, y)}$ (quantifier interchange twice).

(i) No. $\exists x \neg \forall y P(x, y) \equiv \exists x \exists y \neg P(x, y)$.

(ii) No. $\forall x \neg \exists y P(x, y) \equiv \forall x \forall y \neg P(x, y)$.

(iii) Yes.

(iv) No.

3. Give a reason based on interpretations and the meaning of quantifiers why

(i) the following first-order sentence is valid:
$\exists x P(x) \wedge \forall y (P(y) \rightarrow Q(y)) \rightarrow \exists z Q(z)$

(ii) the following first-order sentence is not valid:
$\exists x P(x) \wedge \forall y (P(y) \rightarrow Q(y)) \rightarrow \forall z \neg Q(z)$

<u>SOLUTION</u>

(i) Let $\mathcal{I}$ be an interpretation for this formula, with domain D. Assume the antecedent of this formula $\exists x P(x) \wedge \forall y (P(y) \rightarrow Q(y))$ is true under $\mathcal{I}$. Then $P(d)$ is true for some $d \in D$ and the formula $P(d) \rightarrow Q(d)$ is true, so $Q(d)$ is true. Therefore, the consequent $\exists z Q(z)$ is true under $\mathcal{I}$, so the implication is true. Since $\mathcal{I}$ is an arbitrary interpretation, the sentence is valid.

(ii) Following the arguments given in the solution to (i), we conclude that if the premise of the main implication is true then there exists $d \in D$ such that $Q(d)$ is true. It follows that the consequent $\forall z \neg Q(z)$ is false.

4. Use successive equivalences, showing your work, to show that the formula

$\neg \exists x \forall y (\neg P(x) \wedge (Q(y) \rightarrow R(x, y)))$ is logically equivalent to $\forall x (P(x) \vee \exists y (Q(y) \wedge \neg R(x, y)))$.

<u>SOLUTION</u>

$$
\begin{array}{rcll}
\neg \exists x \forall y (\neg P(x) \wedge (Q(y) \rightarrow R(x, y))) & \equiv & \forall x \exists y \neg (\neg P(x) \wedge (Q(y) \rightarrow R(x, y))) & \text{quantifier interchange twice} \\
& \equiv & \forall x \exists y (\neg\neg P(x) \vee \neg (Q(y) \rightarrow R(x, y))) & \text{de Morgan} \\
& \equiv & \forall x \exists y (P(x) \vee \neg (Q(y) \rightarrow R(x, y))) & \text{double negation} \\
& \equiv & \forall x \exists y (P(x) \vee (Q(y) \wedge \neg R(x, y))) & \text{tautological equivalence} \\
& \equiv & \forall x (\exists y (P(x)) \vee \exists y (Q(y) \wedge \neg R(x, y))) & \text{distribution} \\
& \equiv & \forall x (P(x) \vee \exists y (Q(y) \wedge \neg R(x, y))) & \text{vacuous quantification}
\end{array}
$$

5. Determine whether the formula $\mathcal{F}$

$$\exists x \forall y (P(x) \rightarrow x = y)$$

is true or false under each of the following interpretations over the domain $D = \{a, b\}$.

(i) both $P(a)$ and $P(b)$ are true;

(ii) both $P(a)$ and $P(b)$ are false;

(iii) $P(a)$ is true and $P(b)$ is false.

<u>SOLUTION</u>

(i) both $P(a)$ and $P(b)$ are true.
Then the formula is false.
Indeed, $P(a) \rightarrow a = b$ is false, and $P(b) \rightarrow b = a$ is false.

(ii) both $P(a)$ and $P(b)$ are false. Then the formula is true.

(iii) $P(a)$ is true and $P(b)$ is false. Then the formula is true.
Indeed, both $P(b) \to b = a$ and $P(b) \to b = b$ are true.

In fact, this formula is true for any domain D for which $P(d)$ is false for at least one element $d \in D$.