5CCS2FC2: Foundations of Computing II

Tutorial Sheet 7

Solutions

7.1 Use the greedy SAT algorithm to find a variable assignment that maximises the number of clauses satisfied from the following set:

$$(P \vee Q), \quad (P \vee \neg Q), \quad (\neg P \vee Q \vee R),$$
$$(\neg P \vee R), \quad (P \vee \neg Q \vee R), \quad (P \vee Q \vee R).$$

<u>SOLUTION:</u> Depending on our initial assignment, we may arrive at different paths. However, starting at the assignment

$$\boxed{P = 0, \qquad Q = 0, \qquad R = 0}$$

leads to the following path:

- **First Assignment** $(0,0,0)$**:** **Clauses satisfied** $= 4$

  The neighbouring assignments are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, which satisfy the following number of clauses

  - *Assignment* $(1,0,0)$*:* Clauses satisfied $= 4$
    (fails to satisfy $(\neg P \vee Q \vee R)$ and $(\neg P \vee R)$)
  - *Assignment* $(0,1,0)$*:* Clauses satisfied $= 4$
    (fails to satisfy $(P \vee \neg Q \vee R)$ and $(P \vee \neg Q)$)
  - *Assignment* $(0,0,1)$*:* Clauses satisfied $= 5$
    (fails to satisfy $(P \vee Q)$)

  Seeking to maximise the total number of satisfied clauses, we move to the assignment $(0,0,1)$.

- **Second Assignment** $(0,0,1)$**:** **Clauses satisfied** $= 5$

  The neighbouring assignments yet to have been visted are $(0,1,1)$ and $(1,0,1)$, which satisfy the following number of clauses

  - *Assignment* $(0,1,1)$*:* Clauses satisfied $= 5$
    (fails to satisfy $(P \vee Q)$)
  - *Assignment* $(1,0,1)$*:* Clauses satisfied $= 6$
    (satisfies all clauses)

  We have then found a satisfying assignment for P, Q and R

- **Final Assignment** $(1,0,1)$**:** **Clauses satisfied** $= 6$

$$\boxed{P = 1, \qquad Q = 0, \qquad R = 1}$$

Hence in this instance, the greedy SAT algorithm was able to quickly identify a solution.

7.2 Use the DPLL Algorithm to decide whether the following set of clauses is satisfiable? If so, what is a satisfying assignment?

$$
\begin{aligned}
&(P \vee Q), && (1) \qquad && (\neg P \vee R \vee \neg S), && (5)\\
&(P \vee R \vee S), && (2) && (\neg P \vee \neg R), && (6)\\
&(\neg Q \vee \neg R \vee S), && (3) && (\neg P \vee \neg Q \vee \neg S), && (7)\\
&(\neg Q \vee \neg R \vee \neg S), && (4) && (P \vee \neg Q \vee R \vee \neg S). && (8)
\end{aligned}
$$

<u>SOLUTION:</u> We will get different solutions depending on the ordering we choose for selecting the unassigned variables for branching. Assuming that we choose the lexicographic ordering $P < Q < R < S$, so that P is the first choice for branching and S is the last choice, we have the following computation:

Rule	Partial Assignment	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Initial	$\emptyset$								
Branch $\neg P$	$\neg P$					✓	✓	✓	
Unit clause (1)	$\neg P, Q$	✓				✓	✓	✓	
Branch $\neg R$	$\neg P, Q, \neg R$	✓		✓	✓	✓	✓	✓	
Conflict S	$\neg P, Q, \neg R$	✓	✗	✓	✓	✓	✓	✓	✗
Backtrack R	$\neg P, Q, R$	✓	✓			✓	✓	✓	✓
Conflict S	$\neg P, Q, R$	✓	✓	✗	✗	✓	✓	✓	✓
Backtrack P	P	✓	✓						✓
Unit clause (6)	$P, \neg R$	✓	✓	✓	✓		✓		✓
Unit clause (5)	$P, \neg R, \neg S$	✓	✓	✓	✓	✓	✓	✓	✓

Having eliminated all clauses from the set, our final satisfying assignment is given by

$$\boxed{P := 1, \qquad Q := 0 \text{ or } 1, \qquad R := 0, \qquad S := 0}$$

7.3 Consider the following instance of 2SAT:

$$F = (P \vee Q) \wedge (\neg R \vee S) \wedge (\neg Q \vee S) \wedge (\neg P \vee \neg S) \wedge (\neg S \vee \neg Q).$$

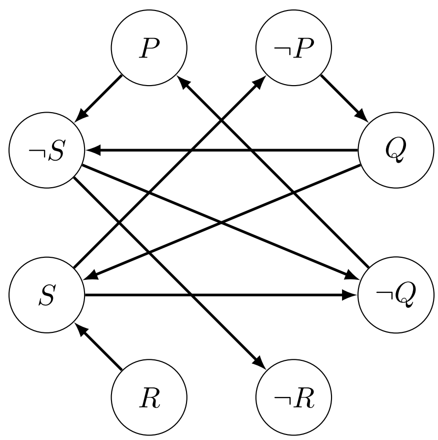
(i) Construct the *implication graph* for F,

(ii) Decide whether the F is satisfiable or not?

(iii) *[Bonus]* If F is satisfiable, can you identify a satisfing assignment? Does the structure of the implication graph help you?

<u>SOLUTION:</u>

(i) The implication graph for F is as follows:



(ii) The implication graph has *four* strongly connected components, which are as follows

$$\{R\}, \qquad \{\neg P, Q, S\}, \qquad \{P, \neg Q, \neg S\}, \qquad \{\neg R\}$$

Since none of these components conatian both a literal and its negation, the formula must be satisfiable.

(iii) *[Bonus]* Note that there is a natural *ordering* to the strongly connected components,

$$\{R\} \longrightarrow \{\neg P, Q, S\} \longrightarrow \{P, \neg Q, \neg S\} \longrightarrow \{\neg R\}$$

Each component contains an edge pointing into the next component, but there are no edges pointing right-to-left.

If we assign something on the *left* to be true, then we are obliged to make everything 'downstream' also true (since otherwise we would have $1 \to 0 = 0$). However, assigning something on the *right* to be true does *not* require us to make everything 'upstream' true (since both $0 \to 1 = 1$ and $1 \to 1 = 1$).

Furthermore, since we always add edges to our implication graph in pairs (for example $R \to S$ and its contrapositive $\neg S \to \neg R$), the ordering of clusters has some symmetry: the cluster right-most cluster is $\{\neg R\}$, while the left-most cluster is $\{R\}$, the second right-most cluster is $\{P, \neg Q, \neg S\}$, while the second left-most cluster is $\{\neg P, Q, S\}$.

This means that if we assign a right-most cluster to be true, it forces the left-most cluster to be false. Proceeding from each end we can assign all the clusters on the right to be true and all the clusters on the left to be false, meeting in the middle.

Hence a satisfing valuation is given by

$$\boxed{P = 1, \quad Q = 0, \quad R = 0, \quad S = 0}$$

7.4 Using just the Unit Propogation and Pure Literal Elimination rules, decide whether the following set of Horn clauses is satisfiable

$$(P \vee \neg Q \vee \neg S), \quad (\neg P \vee \neg S \vee T), \quad (\neg Q \vee \neg R \vee \neg S \vee \neg W),$$
$$(P \vee \neg R), \quad (Q), \quad (T \vee \neg W), \quad (\neg Q \vee S), \quad (\neg P \vee \neg R \vee \neg T \vee \neg W).$$

<u>SOLUTION:</u> The literal Q is a unit clause, so we must assign $Q := 1$. We can then simply all clases containing the literal $\neg Q$.

$$(P \vee \neg S), \quad (\neg P \vee \neg S \vee T), \quad (\neg R \vee \neg S \vee \neg W),$$
$$(P \vee \neg R), \quad \cancel{(Q)}, \quad (T \vee \neg W), \quad (S), \quad (\neg P \vee \neg R \vee \neg T \vee \neg W)$$

The literal S is now a unit clause, which must also be assigned to true $S := 1$. Simplifying, we have

$$(P), \quad (\neg P \vee T), \quad (\neg R \vee \neg W),$$
$$(P \vee \neg R), \quad \cancel{(Q)}, \quad (T \vee \neg W), \quad \cancel{(S)}, \quad (\neg P \vee \neg R \vee \neg T \vee \neg W)$$

The literal P is now a unit clause. Letting $P := 1$ gives us

$$\cancel{(P)}, \quad (T), \quad (\neg R \vee \neg W),$$
$$\cancel{(P \vee \neg R)}, \quad \cancel{(Q)}, \quad (T \vee \neg W), \quad \cancel{(S)}, \quad (\neg R \vee \neg T \vee \neg W)$$

Finally, assigning the unit clause $T := 1$ gives us

$$\cancel{(P)}, \quad \cancel{(T)}, \quad (\neg R \vee \neg W),$$
$$\cancel{(P \vee \neg R)}, \quad \cancel{(Q)}, \quad \cancel{(T \vee \neg W)}, \quad \cancel{(S)}, \quad (\neg R \vee \neg W)$$

This leaves us with two pure literals $\neg R$ and $\neg W$ which can both be assigned false. The final assignment is therefore:

$$\boxed{P := 1, \quad Q := 1, \quad R := 0, \quad S := 1, \quad T := 1, \quad W := 0}$$