Compilers and Formal Languages (2)

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Slides & Progs:	KEATS (also homework is there)

Lets Implement an Efficient Regular Expression Matcher

Graphs: $a^{?\{n\}} \cdot a^{\{n\}}$ and strings $\underbrace{a \ldots a}_{n}$
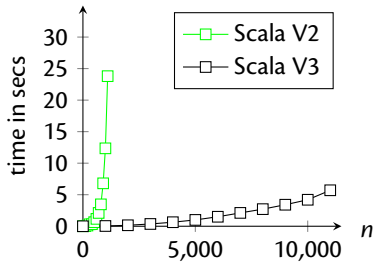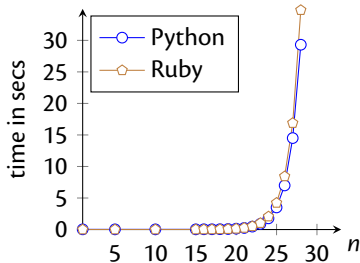
In the handouts is a similar graph for $(a^*)^* \cdot b$ and Java 8, JavaScript and Python.

(Basic) Regular Expressions

Their inductive definition:

$$r ::= \mathbf{0} \qquad \text{nothing}$$
$$\mid \mathbf{1} \qquad \text{empty string / " " / }[]$$
$$\mid c \qquad \text{character}$$
$$\mid r_1 + r_2 \qquad \text{alternative / choice}$$
$$\mid r_1 \cdot r_2 \qquad \text{sequence}$$
$$\mid r^* \qquad \text{star (zero or more)}$$

Q: What about $r \cdot \mathbf{0}$?

Languages (Sets of Strings)

- A **Language** is a set of strings, for example

$$\{[], hello, foobar, a, abc\}$$

- **Concatenation** for strings and languages

$$foo @ bar = foobar$$

$$A @ B \stackrel{\text{def}}{=} \{s_1 @ s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

For example $A = \{foo, bar\}$, $B = \{a, b\}$

$$A @ B = \{fooa, foob, bara, barb\}$$

Two Corner Cases

$$A @ \{[]\} = ?$$

Two Corner Cases

$$A @ \{[]\} = ?$$

$$A @ \{\} = ?$$

The Meaning of a Regular Expression

...all the strings a regular expression can match.

$$L(\mathbf{0}) \stackrel{\text{def}}{=} \{\}$$
$$L(\mathbf{1}) \stackrel{\text{def}}{=} \{[]\}$$
$$L(c) \stackrel{\text{def}}{=} \{[c]\}$$
$$L(r_1 + r_2) \stackrel{\text{def}}{=} L(r_1) \cup L(r_2)$$
$$L(r_1 \cdot r_2) \stackrel{\text{def}}{=} L(r_1) @ L(r_2)$$
$$L(r^*) \stackrel{\text{def}}{=}$$

L is a function from regular expressions to sets of strings (languages):

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

The Power Operation

- The **nth Power** of a language:

$$A^0 \overset{\text{def}}{=} \{[]\}$$
$$A^{n+1} \overset{\text{def}}{=} A @ A^n$$

For example

$$A^4 = A @ A @ A @ A \qquad (@\{[]\})$$
$$A^1 = A \qquad\qquad\qquad (@\{[]\})$$
$$A^0 = \{[]\}$$

Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

  How many strings are in A^4 ?

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- Say $A = \{[a], [b], [c], [d]\}$.

  How many strings are in A^4?

  What if $A = \{[a], [b], [c], []\}$;
  how many strings are then in A^4?

The Star Operation

- The **Kleene Star** of a language:

$$A\star \stackrel{\text{def}}{=} \bigcup_{0 \leq n} A^n$$

This expands to

$$A^0 \cup A^1 \cup A^2 \cup A^3 \cup A^4 \cup \dots$$

or

$$\{[]\} \cup A \cup A@A \cup A@A@A \cup A@A@A@A \cup \dots$$

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$$L(r_1 \cdot r_2) \overset{\text{def}}{=} L(r_1) @ L(r_2)$$
$$L(r^*) \overset{\text{def}}{=} (L(r))\star \overset{\text{def}}{=} \bigcup_{0 \leq n} L(r)^n$$

When Are Two Regular Expressions Equivalent?

Two regular expressions r_1 and r_2 are equivalent provided:

$$r_1 \equiv r_2 \quad \overset{\text{def}}{=} \quad L(r_1) = L(r_2)$$

Some Concrete Equivalences

$$(a + b) + c \;\equiv\; a + (b + c)$$
$$a + a \;\equiv\; a$$
$$a + b \;\equiv\; b + a$$
$$(a \cdot b) \cdot c \;\equiv\; a \cdot (b \cdot c)$$
$$c \cdot (a + b) \;\equiv\; (c \cdot a) + (c \cdot b)$$

Some Concrete Equivalences

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$$a + a \equiv a$$
$$a + b \equiv b + a$$
$$(a \cdot b) \cdot c \equiv a \cdot (b \cdot c)$$
$$c \cdot (a + b) \equiv (c \cdot a) + (c \cdot b)$$

$$a \cdot a \not\equiv a$$
$$a + (b \cdot c) \not\equiv (a + b) \cdot (a + c)$$

Some Corner Cases

$$a \cdot \mathbf{0} \quad \not\equiv \quad a$$
$$a + \mathbf{1} \quad \not\equiv \quad a$$
$$\mathbf{1} \quad \equiv \quad \mathbf{0}^*$$
$$\mathbf{1}^* \quad \equiv \quad \mathbf{1}$$
$$\mathbf{0}^* \quad \not\equiv \quad \mathbf{0}$$

Some Simplification Rules

$$r + \mathbf{0} \equiv r$$
$$\mathbf{0} + r \equiv r$$
$$r \cdot \mathbf{1} \equiv r$$
$$\mathbf{1} \cdot r \equiv r$$
$$r \cdot \mathbf{0} \equiv \mathbf{0}$$
$$\mathbf{0} \cdot r \equiv \mathbf{0}$$
$$r + r \equiv r$$

The Specification for Matching

A regular expression *r* matches a string *s* provided:

$$s \in L(r)$$

…and the point of the this lecture is to decide this problem as fast as possible (unlike Python, Ruby, Java etc)

Questions?

homework (written exam 80%)
coursework (20%; first one today)
submission Fridays @ 18:00 – accepted until Mondays

Semantic Derivative

- The **Semantic Derivative** of a language w.r.t. to a character c:

$$Der \; c \; A \; \stackrel{def}{=} \; \{s \mid c::s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$Der \; f \; A \; = \; \{oo, rak\}$$
$$Der \; b \; A \; = \; \{ar\}$$
$$Der \; a \; A \; = \; \{\}$$

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$$Der\, a\, A = \{\}$$

We can extend this definition to strings

$$Ders\, s\, A = \{s' \mid s\,@\,s' \in A\}$$

Brzozowski's Algorithm (1)

…whether a regular expression can match the empty string:

$$nullable(\mathbf{0}) \stackrel{\text{def}}{=} \textit{false}$$

$$nullable(\mathbf{1}) \stackrel{\text{def}}{=} \textit{true}$$

$$nullable(c) \stackrel{\text{def}}{=} \textit{false}$$

$$nullable(r_1 + r_2) \stackrel{\text{def}}{=} nullable(r_1) \vee nullable(r_2)$$

$$nullable(r_1 \cdot r_2) \stackrel{\text{def}}{=} nullable(r_1) \wedge nullable(r_2)$$

$$nullable(r^*) \stackrel{\text{def}}{=} \textit{true}$$

The Derivative of a Rexp

If *r* matches the string *c* :: *s*, what is a regular expression that matches just *s*?

der c r gives the answer, Brzozowski 1964

The Derivative of a Rexp

$$der\, c\, (\mathbf{0}) \overset{\text{def}}{=} \mathbf{0}$$

$$der\, c\, (\mathbf{1}) \overset{\text{def}}{=} \mathbf{0}$$

$$der\, c\, (d) \overset{\text{def}}{=} \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}$$

$$der\, c\, (r_1 + r_2) \overset{\text{def}}{=} der\, c\, r_1 + der\, c\, r_2$$

$$der\, c\, (r_1 \cdot r_2) \overset{\text{def}}{=} \text{if } nullable(r_1)$$
$$\text{then } (der\, c\, r_1) \cdot r_2 + der\, c\, r_2$$
$$\text{else } (der\, c\, r_1) \cdot r_2$$

$$der\, c\, (r^*) \overset{\text{def}}{=} (der\, c\, r) \cdot (r^*)$$

The Derivative of a Rexp

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$$der\ c\ (r_1 \cdot r_2) \overset{\text{def}}{=} \text{if } nullable(r_1)$$
$$\text{then } (der\ c\ r_1) \cdot r_2 + der\ c\ r_2$$
$$\text{else } (der\ c\ r_1) \cdot r_2$$

$$der\ c\ (r^*) \overset{\text{def}}{=} (der\ c\ r) \cdot (r^*)$$

$$ders\ [\,]\ r \overset{\text{def}}{=} r$$

$$ders\ (c :: s)\ r \overset{\text{def}}{=} ders\ s\ (der\ c\ r)$$

Examples

Given $r \overset{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$der\, a\, r = ?$$
$$der\, b\, r = ?$$
$$der\, c\, r = ?$$

The Brzozowski Algorithm

$$matches\ r\ s \stackrel{\text{def}}{=} nullable(ders\ s\ r)$$

Brzozowski: An Example

Does r_1 match *abc*?

Step 1: build derivative of *a* and r_1 $(r_2 = der\ a\ r_1)$

Step 2: build derivative of *b* and r_2 $(r_3 = der\ b\ r_2)$

Step 3: build derivative of *c* and r_3 $(r_4 = der\ c\ r_3)$

Step 4: the string is exhausted: $(nullable(r_4))$
test whether r_4 can recognise
the empty string

Output: result of the test
$\Rightarrow$ *true* or *false*

The Idea of the Algorithm

If we want to recognise the string *abc* with regular expression r_1 then

1. *Der a* $(L(r_1))$

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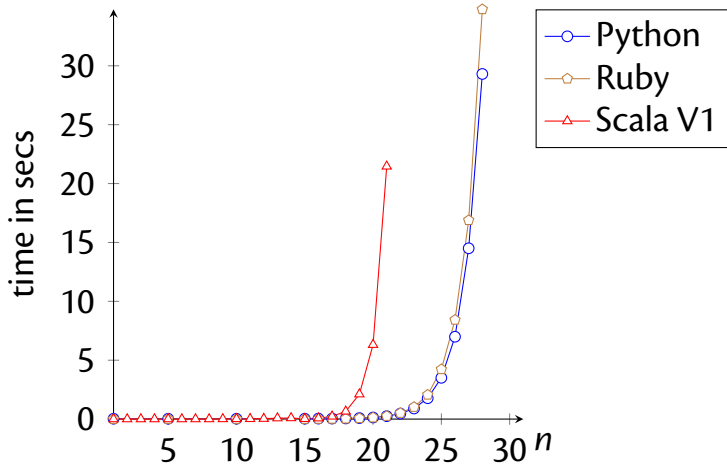
1. *Der a* $(L(r_1))$
2. *Der b* $(Der a (L(r_1)))$

The Idea of the Algorithm

If we want to recognise the string *abc* with regular expression r_1 then

1. *Der a* $(L(r_1))$
2. *Der b* $(Der\ a\ (L(r_1)))$
3. *Der c* $(Der\ b\ (Der\ a\ (L(r_1))))$

4. finally we test whether the empty string is in this set; same for *Ders abc* $(L(r_1))$.

The matching algorithm works similarly, just over regular expressions instead of sets.

Oops... $a^{?\{n\}} \cdot a^{\{n\}}$

A Problem

We represented the "n-times" $a^{\{n\}}$ as a sequence regular expression:

1: a

2: $a \cdot a$

3: $a \cdot a \cdot a$

...

13: $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

...

20:

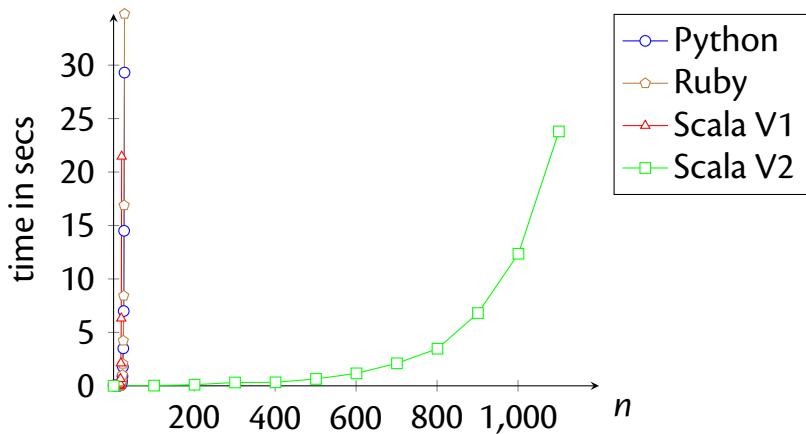This problem is aggravated with $a^{?}$ being represented as $a + \mathbf{1}$.

Solving the Problem

What happens if we extend our regular expressions with explicit constructors

$$
\begin{aligned}
r \quad ::= \quad & ... \\
| \quad & r^{\{n\}} \\
| \quad & r^?
\end{aligned}
$$

What is their meaning?

What are the cases for *nullable* and *der*?

Brzozowski: $a^{?\{n\}} \cdot a^{\{n\}}$

Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$der\, a\, r = ((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r$$
$$der\, b\, r = ((\mathbf{0} \cdot b) + \mathbf{1}) \cdot r$$
$$der\, c\, r = ((\mathbf{0} \cdot b) + \mathbf{0}) \cdot r$$

What are these regular expressions equivalent to?

Simplification Rules

$$r + \mathbf{0} \;\Rightarrow\; r$$
$$\mathbf{0} + r \;\Rightarrow\; r$$
$$r \cdot \mathbf{1} \;\Rightarrow\; r$$
$$\mathbf{1} \cdot r \;\Rightarrow\; r$$
$$r \cdot \mathbf{0} \;\Rightarrow\; \mathbf{0}$$
$$\mathbf{0} \cdot r \;\Rightarrow\; \mathbf{0}$$
$$r + r \;\Rightarrow\; r$$

```scala
def ders(s: List[Char], r: Rexp) : Rexp = s match {
  case Nil => r
  case c::s => ders(s, simp(der(c, r)))
}
```
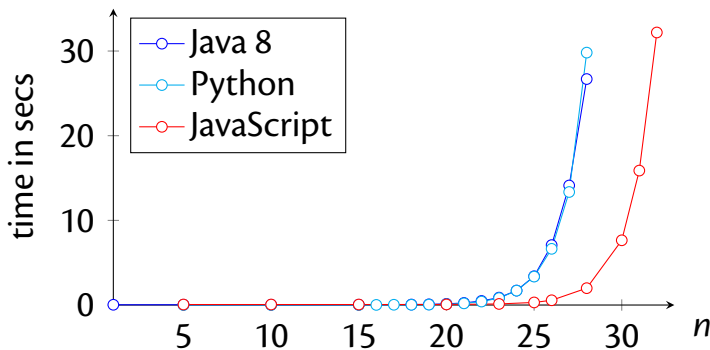
```scala
def simp(r: Rexp) : Rexp = r match {
  case ALT(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, r2s) => r2s
      case (r1s, ZERO) => r1s
      case (r1s, r2s) =>
        if (r1s == r2s) r1s else ALT(r1s, r2s)
    }
  }
  case SEQ(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, _) => ZERO
      case (_, ZERO) => ZERO
      case (ONE, r2s) => r2s
      case (r1s, ONE) => r1s
      case (r1s, r2s) => SEQ(r1s, r2s)
    }
  }
  case r => r
}
```
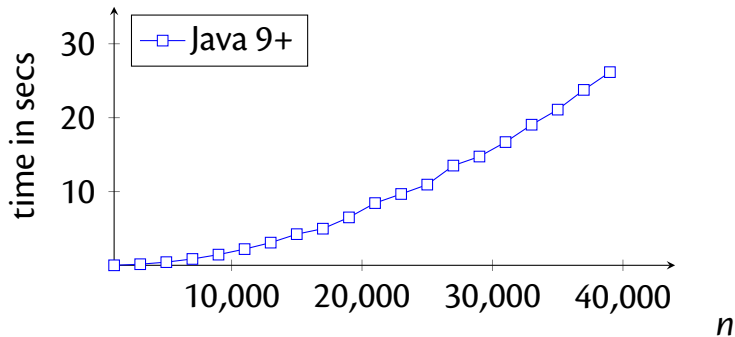
Brzozowski: $a^{?\{n\}} \cdot a^{\{n\}}$
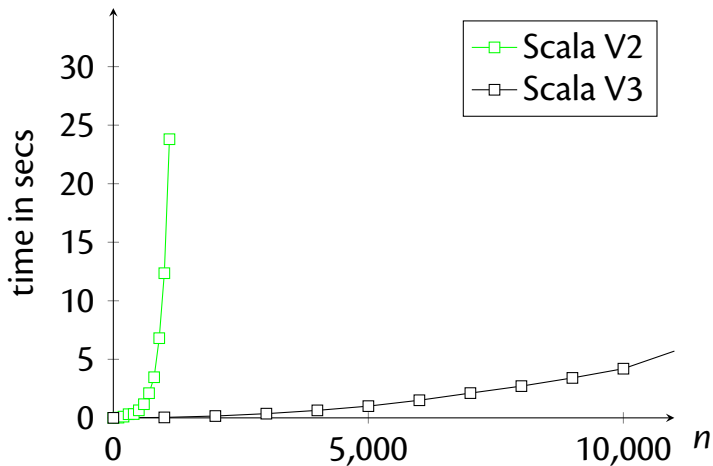
Another Example
in Java 8, Python and JavaScript

Regex: $\left(a^*\right)^* \cdot b$

Strings of the form $\underbrace{a \ldots a}_{n}$
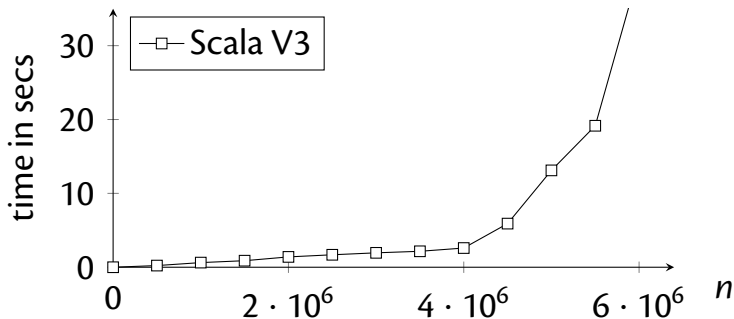
Same Example in Java 9+

Regex: $(a^*)^* \cdot b$

Strings of the form $\underbrace{a \ldots a}_{n}$

...and with Brzozowski

Regex: $(a^*)^* \cdot b$

Strings of the form $\underbrace{a \ldots a}_{n}$

What is good about this Alg.

- extends to most regular expressions, for example $\sim r$ (next slide)
- is easy to implement in a functional language (slide after)
- the algorithm is already quite old; there is still work to be done to use it as a tokenizer (that is relatively new work)
- we can prove its correctness...(several slides later)

Negation of Regular Expr's

- $\sim r$ (everything that r cannot recognise)

- $L(\sim r) \stackrel{\text{def}}{=} UNIV - L(r)$

- $nullable(\sim r) \stackrel{\text{def}}{=} not\,(nullable(r))$

- $der\,c\,(\sim r) \stackrel{\text{def}}{=} \sim (der\,c\,r)$

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- $der\ c\ (\sim r) \stackrel{\text{def}}{=} \sim (der\ c\ r)$

Used often for recognising comments:

$$/ \cdot * \cdot (\sim ([a\text{-}z]^* \cdot * \cdot / \cdot [a\text{-}z]^*)) \cdot * \cdot /$$

Coursework

Strand 1:

- Submission on Friday 11 October
  accepted until Monday 14 @ 18:00

- source code needs to be submitted as well

- you can re-use my Scala code from KEATS
  or use any programming language you like

- https://nms.kcl.ac.uk/christian.urban/ProgInScala2ed.pdf

Proofs about Rexps

Remember their inductive definition:

$$
\begin{aligned}
r \quad ::= \quad & \mathbf{0} \\
\mid \quad & \mathbf{1} \\
\mid \quad & c \\
\mid \quad & r_1 \cdot r_2 \\
\mid \quad & r_1 + r_2 \\
\mid \quad & r^*
\end{aligned}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for **0**, **1** and c

- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.

- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.

- P holds for r^* under the assumption that P already holds for r.

Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$nullable(r) \text{ if and only if } [] \in L(r)$$

Proofs about Rexp (4)

$$rev(\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0}$$
$$rev(\mathbf{1}) \stackrel{\text{def}}{=} \mathbf{1}$$
$$rev(c) \stackrel{\text{def}}{=} c$$
$$rev(r_1 + r_2) \stackrel{\text{def}}{=} rev(r_1) + rev(r_2)$$
$$rev(r_1 \cdot r_2) \stackrel{\text{def}}{=} rev(r_2) \cdot rev(r_1)$$
$$rev(r^*) \stackrel{\text{def}}{=} rev(r)^*$$

We can prove

$$L(rev(r)) = \{s^{-1} \mid s \in L(r)\}$$

by induction on r.

Correctness Proof for our Matcher

- We started from

$$s \in L(r)$$
$$\Leftrightarrow \quad [] \in \textit{Ders } s \left(L(r) \right)$$

Correctness Proof for our Matcher

- We started from

$$s \in L(r)$$
$$\Leftrightarrow \quad [] \in Ders\, s\, (L(r))$$

- if we can show $Ders\, s\, (L(r)) = L(ders\, s\, r)$ we have

$$\Leftrightarrow \quad [] \in L(ders\, s\, r)$$
$$\Leftrightarrow \quad nullable(ders\, s\, r)$$
$$\stackrel{\text{def}}{=} \quad matches\, s\, r$$

Proofs about Rexp (5)

Let $Der\ c\ A$ be the set defined as

$$Der\ c\ A \stackrel{\text{def}}{=} \{s \mid c :: s \in A\}$$

We can prove

$$L(der\ c\ r) = Der\ c\ (L(r))$$

by induction on r.

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and

- P holds for the string $c :: s$ under the assumption that P already holds for s
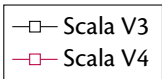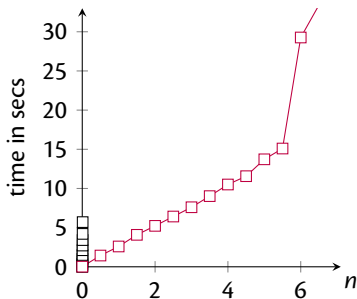
Proofs about Strings (2)

We can then prove

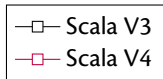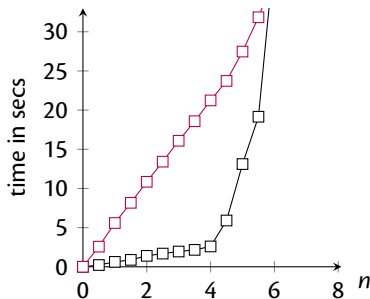$$Ders\ s\ (L(r)) = L(ders\ s\ r)$$

We can finally prove

$$matches\ s\ r \text{ if and only if } s \in L(r)$$

Epilogue
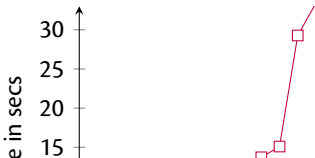
Graph: $a^{?\{n\}} \cdot a^{\{n\}}$
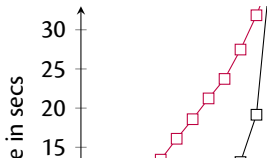

Graph: $(a^*)^* \cdot b$

Epilogue

Graph: $a^{?\{n\}} \cdot a^{\{n\}}$

Graph: $(a^*)^* \cdot b$


```
def ders2(s: List[Char], r: Rexp) : Rexp = (s, r) match {
  case (Nil, r) => r
  case (s, ZERO) => ZERO
  case (s, ONE) => if (s == Nil) ONE else ZERO
  case (s, CHAR(c)) => if (s == List(c)) ONE else
                       if (s == Nil) CHAR(c) else ZERO
  case (s, ALT(r1, r2)) => ALT(ders2(s, r2), ders2(s, r2))
  case (c::s, r) => ders2(s, simp(der(c, r)))
}
```

Another Homework Question

- How many basic regular expressions are there to match the string *abcd* ?

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- How many if they cannot include **1** and **0**?

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- How many if they cannot include **1** and **0**?
- How many if they are also not allowed to contain stars?
- How many if they are also not allowed to contain _ $+$ _?