17-10-16

P

1 If $\underline{n \text{ is a real number such that } n>1}$, then $\underline{n^2>1}$. Q

Suppose $\underline{n^2>1}$. Q Then $\underline{n>1}$. P

$$\Delta \models F$$

(1) (2) (3)

$$\underbrace{P \to Q,\ Q}\ \models\ \underbrace{P}$$

premises conclusion

↳ $\Delta \models F$ → a single formula.

Set of formulae

	P	Q	$P \to Q$	$(P \to Q) \wedge Q$
(1)	0	0	1	0
(2) (1)	0	1	1	1
	1	0	0	0
(1)	1	1	1.	1

Counter-model

(3)

(3)

$v(P)=0$

$v(P \to Q)=1.$

$v(Q)=1$

$\models$ behaves like $\to$

17-10-16.

(2) If n is a real number such that $n>3$, then $n^2>9$. $P \to Q$

Suppose $\underline{n^2 \leq 9}$. Then $n \leq 3$.

$n>3 \Rightarrow \underline{\underline{P}} \Rightarrow \neg P \Rightarrow \neg(n>3)$

$n^2>9 \Rightarrow \underline{\underline{Q}}$ $\qquad \underline{\underline{n \leq 3}} \Rightarrow \neg P$

$\Rightarrow \neg Q \Rightarrow \neg(n^2>9)$

$\underline{\underline{n^2 \leq 9}}$.

$P \to Q, \neg Q \models \neg P$ $\qquad \neg Q$

<u>Valid</u>

$P \to Q \equiv \neg Q \to \neg P$

$$\begin{array}{l} \neg Q \\ \hline \neg P \end{array}$$

(3) If n is a real number $n>2$, then $n^2>4$.

Suppose $n \leq 2$. Then $n^2 \leq 4$.

$n>2 \Rightarrow P \Rightarrow \neg P \Rightarrow n \leq 2$.

$n^2>4 \Rightarrow Q. \Rightarrow \neg Q \Rightarrow n^2 \leq 4$.

$P \to Q, \neg P \models \neg Q$? <u>NOT VALID</u>

$v(\neg Q) = 0$

$v(Q) = 1$.

$v(\neg P) = 1 \qquad v(P) = 0$.

$v(P \to Q) = 1$.

$v(\neg P) = 1 \qquad$ but $\qquad v(\neg Q) = 0$.

③

A, B, C propositional formulae.

If $\overbrace{C \models A}^{P}$ and $\underbrace{\neg C \models B}_{Q}$, then $\underbrace{A \vee B}_{R}$ is a tautology.

$P \wedge Q \to R$

$\neg R \to \neg(P \wedge Q)$

$\neg P \vee \neg Q$.

$A \vee B$ is not a tautology.

Suppose $v(A \vee B) = 0$

$v(A) = 0$

$v(B) = 0$.

If $C \models A$ then $v(C) = 1$ implies $v(A) = 1$

therefore $v(C) = 0$

$v(\neg C) = 1$

$\neg C \models B$ $\quad v(B) = 1$.

$v(A \vee B) = 1$ a contradiction.

So $C \models A$ and $\neg C \models B$ imply $A \vee B$ is indeed a tautology.

$D \models F$ $\quad F_1 \models F_2$.

$\emptyset \models F$ $\quad \models F$

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$R \vee W$ ①

$\neg R \vee L$ ② $\Rightarrow R \to L$

Conclusion? $W \vee L$

① Show that $R \vee W, \neg R \vee L \models W \vee L$

② Reasoning by cases

CASE 1

R

$+$ ②

L

CASE 2

$\neg R$

$+$ ①

W

either L is true

or W is true.

$W \vee L$

(5) data (premises) conclusion.

$P, Q, (P \wedge Q) \to R \models R$ **Valid**

Premises

1. P data
2. Q data.
3. $\underbrace{(P \wedge Q)}_{A} \to R$ data.
4. $P \wedge Q$ (A) (1), (2) $\wedge I$
5. R (3), (4) $\to E$

$$\frac{A, B}{A \wedge B} \wedge I.$$

$$\frac{A, A \to B}{B} \to E$$

$P, Q, (P \wedge Q) \to R \vdash R$

$\vdash$ iff $\models$

$\downarrow$ $\downarrow$

proof theory. (syntactical)

model theory (semantic (al))

Exercises

(1) $A \wedge B \vdash B \wedge A$. Data $\vdash$ conclusion

1. $A \wedge B$ data.
2. A 1. and $\wedge E$
3. B 1. and $\wedge E$.
4. $B \wedge A$ 3., 2., $\wedge I$.

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$$(A \vee B) \to C \vdash (A \to C) \wedge (B \to C)$$

1. $(A \vee B) \to C$ given.
2. $A \to C$ from subcomputation box below.

2.1	A	assumption	$\underline{\underline{C}}$
2.2	$A \vee B$	2.1, $\vee I$.	
2.3	C	2.2, 1, $\to E$	

3. $B \to C$ from subcomputation box.

3.1.	B	assumption	$\underline{\underline{C}}$
3.2.	$A \vee B$	3.1, $\vee I$.	
3.3	C	3.2, 1, $\to E$	

4. $(A \to C) \wedge (B \to C)$ $\wedge I$, 2, 3. Proof complete!

If $\dfrac{\text{assumptions, } A}{C}$ (box) then.

$\dfrac{\text{assumptions}}{A \to C}$.

$$\vdash \neg(A \wedge \neg A) \qquad \underline{\underline{\neg I}}$$

$$\dfrac{\dfrac{\neg X}{(A \wedge \neg A) \to Y,\ (A \wedge \neg A) \to \neg Y}}{\neg(A \wedge \neg A)} \leftarrow \dfrac{X \to Y,\ X \to \neg Y}{\neg X}\ (\neg I)$$

(7)

Tutorial week 4 17-10-16.

Class Exercise 3

$\emptyset \vdash \neg(A \wedge \neg A)$. $\neg(A \wedge \neg A)$ always true.

$X \to Y$

1. $(A \wedge \neg A) \to A$ from subcomputation box.

1.1.	$A \wedge \neg A$	assumption.	$\overline{A}$
1.2	A	1.1, $\wedge E$	

$$\frac{X \to Y,\ X \to \neg Y}{\neg X}\ \neg I.$$

$X \to \neg Y$

2. $(A \wedge \neg A) \to \neg A$. from subcomp box.

2.1.	$A \wedge \neg A$	assumption.	$\overline{\neg A}$
2.2	$\neg A$	2.1., $\wedge E$	

3. $\neg(A \wedge \neg A)$ 1, 2, $\neg I$.

$\neg(A \wedge \neg A) \equiv \neg A \vee \neg\neg A \equiv \neg A \vee A$ which is always true.

⑧ $\models$

① $P \to Q,\ \neg Q \vdash \neg P$.

$$\frac{P \to Q \quad P \to \neg Q \quad X \to Y,\ X \to \neg Y}{\neg X} \neg I$$
$$\neg P$$

1. $P \to Q$ dztz.
2. $\neg Q$ dztz.
3. $P \to \neg Q$ subcomp.

3.1.	P	assume $\neg Q$
3.2	$\neg Q$	2.

4. $\neg P$ 1, 3, $\neg I$. <u>proof complete.</u>

③ $\neg(P \wedge \neg Q) \vdash P \to Q$.

(home exercise.
$P \to Q \vdash \neg(P \wedge \neg Q)$)

$\hookrightarrow \neg P \vee \neg\neg Q \equiv$

$\neg P \vee Q \equiv P \to Q$.

$\neg(P \wedge \neg Q) \models P \to Q$.

$P \to Q \models \neg(P \wedge \neg Q)$

$\neg(P \wedge \neg Q)$ $\quad$ $P \to Q$

P	$\neg Q \to (P \wedge \neg Q)$	
$\neg Q \to X$		$(P \wedge \neg Q)$
$\neg Q \to \neg X$.		$\neg(P \wedge \neg Q)$
	$\neg Q \to \neg(P \wedge \neg Q)$	
Q		

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$\neg(P \wedge \neg Q) \vdash P \to Q$.

1. $\neg(P \wedge \neg Q)$ data.

2. $P \to Q$ subcomp box.

2.1. P assumption Q

2.2 $\neg Q \to (P \wedge \neg Q)$ subcomp

$P \wedge \neg Q$

2.2.1. $\neg Q$ assumption

2.2.2 $P \wedge \neg Q$ 2.1, 2.2.1, $\wedge I$

2.3 $\neg Q \to \neg(P \wedge \neg Q)$ subcomp

2.3.1 $\neg Q$ assumption $\neg(P \wedge \neg Q)$

2.3.2 $\neg(P \wedge \neg Q)$ (1)

2.4 Q 2.2, 2.3, $\neg E$

$$\frac{\neg X \to Y, \quad \neg X \to \neg Y}{X}$$

Proof complete.

$P \to Q \vdash \neg(P \wedge \neg Q)$ $\quad X \equiv Y$

$X \vdash Y$

$Y \vdash X$