5CCS2FC2: Foundations of Computing II

Tutorial Sheet 6

6.1 Use the Master Theorem to identify the growth-rates of the following recurrence relations:

(i) $T(n) = 4\ T(n/2) + n^2$

(ii) $T(n) = 16\ T(n/4) + n!$

(iii) $T(n) = 3\ T(n/3) + \sqrt{n}$

(iv) $T(n) = 7\ T(n/3) + n^2$

(v) $T(n) = 4\ T(n/2) + n/\log_2 n$

6.2 Consider the following recurrence relation

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + n \end{aligned}$$

Prove, by induction on n, that $T(n) = n(n+1)/2$, for all $n \geq 1$. What is the growth-rate for $T(n)$?

6.3 Consider the following recurrence relation

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 8\ T(\lceil n/2 \rceil) + n^3 \end{aligned}$$

Prove, by induction on n, that $T(n) \geq 2n^3$, for all $n \geq 2$, thereby proving that $T(n) = \Omega(n^3)$.

6.4 *(Tricky!)* Let $F(n)$ denote the nth Fibonacci number, given by the recurrence relation

$$\begin{aligned} F(0) &= 0, \qquad F(1) = 1 \\ F(n) &= F(n-1) + F(n-2) \end{aligned}$$

for all $n \geq 2$.

(i) Calculuate the first 10 Fibonacci numbers.

(ii) Prove, by induction on n, that the nth Fibonacci number can be calculated directly with the formula

$$F(n) = \frac{1}{\sqrt{5}}(A^n - B^n)$$

for all $n \geq 0$, where A and B are two solutions to the quadratic equation $x^2 = x + 1$. You should:

- Show that the above formula is correct for $n = 0$ and $n = 1$.
- Assume the formulas holds for all $m \leq k$ for some $k \geq 1$, and substitute your induction hypothesis to find an expression for $F(k+1)$,
- Simplify your expression to show $F(k+1) = \frac{1}{\sqrt{5}}(A^{k+1} - B^{k+1})$.

(iii) What is the growth-rate of $F(n)$?