5CCS2FC2: Foundations of Computing II

Tutorial Sheet 6

Solutions

6.1 Use the Master Theorem to identify the growth-rates of the following recurrence relations:

(i) $T(n) = 4\ T(n/2) + n^2$

(ii) $T(n) = 16\ T(n/4) + n!$

(iii) $T(n) = 3\ T(n/3) + \sqrt{n}$

(iv) $T(n) = 7\ T(n/3) + n^2$

(v) $T(n) = 4\ T(n/2) + n/\log_2 n$

SOLUTION:

6.1 We have that $a = 4$, $b = 2$, so that $k = \log_2 4 = 2$. Furthermore, $f(n) = n^2 \in \Theta(n^2)$, so we have that

$$T(n) = \Theta(n^2 \log_2 n) \qquad \text{(Case 2)}$$

6.2 We have that $a = 16$, $b = 4$, so that $k = \log_4 16 = 2$. Furthermore, $f(n) = n! \in \Omega(n^3)$, so we have that

$$T(n) = \Theta(n!) \qquad \text{(Case 3)}$$

(Indeed $n!$ eventually overtakes *every* polynomial function; *i.e.*, $n! \in \Omega(n^\ell)$ for *every* $\ell \geq 0$.)

6.3 We have that $a = 3$, $b = 3$, so that $k = \log_3 3 = 1$. Furthermore, $f(n) = \sqrt{n} \in O(n^{0.5})$, so we have that

$$T(n) = \Theta(n) \qquad \text{(Case 1)}$$

6.4 We have that $a = 7$, $b = 3$, so that $k = \log_3 7 \approx 1.77$. Furthermore, $f(n) = n^2 \in \Omega(n^2)$, so we have that

$$T(n) = \Theta(n^2) \qquad \text{(Case 3)}$$

6.5 We have that $a = 4$, $b = 2$, so that $k = \log_2 4 = 2$. Furthermore, $f(n) = n/\log_2 n \in O(n)$, so we have that

$$T(n) = \Theta(n^2) \qquad \text{(Case 1)}$$

6.2 Consider the following recurrence relation

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(n-1) + n \end{aligned}$$

Prove, by induction on n, that $T(n) = n(n+1)/2$, for all $n \geq 1$. What is the growth-rate for $T(n)$?

SOLUTION:

Base Case) For the base case, we have that

$$T(1) \ = \ 1 \quad = \quad \frac{1 \cdot (1+1)}{2}$$

Inductive Case) Suppose, for induction, that

$$T(k) \ = \ \frac{k(k+1)}{2} \qquad \text{(I.H.)}$$

for some $k \geq 1$. It then follows that

$$\begin{aligned} T(k+1) &= T(k) + (k+1) \\ &= \frac{k(k+1)}{2} \ + \ (k+1) \qquad \text{by (I.H.)} \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= (k+1)\left(\frac{k}{2} + \frac{2}{2}\right) \\ &= (k+1)\frac{(k+2)}{2} \ = \ \frac{(k+1)\big((k+1)+1\big)}{2} \end{aligned}$$

Conclusion) Therefore, if the formula forms for $n = k$ it also holds for $n = (k+1)$. Since the base case shows that the formula holds for $n = 1$, it must hold for *all* values of $n \geq 1$.

The growth-rate for $T(n)$ is $\Theta(n^2 + n) = \Theta(n^2)$, *i.e.* quadratic time.

6.3 Consider the following recurrence relation

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 8\,T(\lceil n/2 \rceil) + n^3 \end{aligned}$$

Prove, by induction on n, that $T(n) \geq 2n^3$, for all $n \geq 2$, thereby proving that $T(n) = \Omega(n^3)$.

<u>SOLUTION:</u>

Base Case) For the base case, we have that must verify that the formula holds for the minimum value $n = 2$,

$$T(2) \;=\; 8(1) + 2^3 \;=\; 16 \;\geq\; 2(2^3)$$

Inductive Case) Suppose, for induction, that

$$T(m) \;\geq\; 2m^3 \tag{I.H.}$$

for all $2 \leq m \leq k$, for some $k \geq 2$. It then follows that

$$\begin{aligned} T(k+1) &= 8\,T\left(\left\lceil \frac{k+1}{2} \right\rceil\right) + (k+1)^3 \\ &\geq 8 \cdot 2\left(\left\lceil \frac{k+1}{2} \right\rceil\right)^3 + (k+1)^3 \qquad \text{by (I.H.)} \\ &\geq 16\left(\frac{k+1}{2}\right)^3 + (k+1)^3 \\ &\geq 16\frac{(k+1)^3}{8} + (k+1)^3 \\ &= 3(k+1)^3 \\ &\geq 2(k+1)^3 \end{aligned}$$

Conclusion) Therefore, $T(n) \geq 2n^3$ for all $n \geq 2$, which is to say that $T(n) = \Omega(n^3)$.

6.4 *(Tricky!)* Let $F(n)$ denote the nth Fibonacci number, given by the recurrence relation

$$\begin{aligned} F(0) &= 0, \qquad F(1) = 1 \\ F(n) &= F(n-1) + F(n-2) \end{aligned}$$

for all $n \geq 2$.

(i) Calculuate the first 10 Fibonacci numbers.

(ii) Prove, by induction on n, that the nth Fibonacci number can be calculated directly with the formula

$$F(n) = \frac{1}{\sqrt{5}}(A^n - B^n)$$

for all $n \geq 0$, where A and B are two solutions to the quadratic equation $x^2 = x + 1$. You should:

- Show that the above formula is correct for $n = 0$ and $n = 1$.
- Assume the formulas holds for all $m \leq k$ for some $k \geq 1$, and substitute your induction hypothesis to find an expression for $F(k+1)$,
- Simplify your expression to show $F(k+1) = \frac{1}{\sqrt{5}}(A^{k+1} - B^{k+1})$.

(iii) What is the growth-rate of $F(n)$?

SOLUTION:

(i) The first ten Fibonacci numbers are given in the following table

n	0	1	2	3	4	5	6	7	8	9	10
$F(n)$	0	1	1	2	3	5	8	13	21	34	55

(ii) Base Case) Since the recurrence relation involve *two* recursive calls to both $F(n-1)$ and $F(n-2)$, we require two base cases! We must show that our formula give the correct value for $F(0)$ and $F(1)$. The first is straightforward since

$$\frac{1}{\sqrt{5}}(A^0 - B^0) = \frac{1}{\sqrt{5}}(1-1) = 0 \quad = F(0)$$

since $A^0 = B^0 = 1$. For the second we have that

$$\begin{aligned}
\frac{1}{\sqrt{5}}(A^1 - B^1) &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right) = 1 \quad = F(1)
\end{aligned}$$

Inductive Case) Suppose, for induction, that

$$F(m) = \frac{1}{\sqrt{5}}(A^m - B^m) \qquad \text{(I.H.)}$$

for all $m \leq k$, for some $k \geq 2$. It then follows that

$$\begin{aligned}
F(k+1) &= F(k) + F(k-1) \\
&= \frac{1}{\sqrt{5}}(A^k - B^k) + \frac{1}{\sqrt{5}}(A^{k-1} - B^{k-1}) && \text{by (I.H.)} \\
&= \frac{1}{\sqrt{5}}\Big((A^k + A^{k-1}) - (B^k + B^{k-1})\Big) \\
&= \frac{1}{\sqrt{5}}\Big(A^{k-1}(A+1) - B^{k-1}(B+1)\Big) \\
&= \frac{1}{\sqrt{5}}\big(A^{k-1}A^2 - B^{k-1}B^2\big) && \text{since } x^2 - x - 1 = 0 \text{ for } x = A, B \\
&= \frac{1}{\sqrt{5}}\big(A^{k+1} - B^{k+1}\big)
\end{aligned}$$

Conclusion) Therefore, it follows by induction on n that our formula holds for all $n \geq 0$, as required.

We can verify that our formula holds by comparing the output values with the values will calculated for part (i).

(iii) We can verify $B < 1$, so for large values of n, B^n will be negligible (it is never greater than 1).
Hence we have that

$$F(n) \approx \frac{1}{\sqrt{5}}A^n$$

Since, we do not care about the constant factor of $1/\sqrt{5}$, we have that $F(n) = \Theta(A^n)$.