6CCS3AIN, 2019, Tutorial 02 Answers (Version 1.0)

1. (a)

$$
\begin{aligned}
P(cavity) &= \sum_{\omega \models cavity} P(\omega) \\
&= P(cavity \wedge catch \wedge toothache) + P(cavity \wedge \neg catch \wedge toothache) \\
&\quad + P(cavity \wedge catch \wedge \neg toothache) + P(cavity \wedge \neg catch \wedge \neg toothache) \\
&= 0.108 + 0.012 + 0.072 + 0.008 \\
&= 0.2
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{P}(Toothache) &= \langle P(toothache), P(\neg toothache) \rangle \\
P(toothache) &= P(cavity \wedge catch \wedge toothache) + P(cavity \wedge \neg catch \wedge toothache) \\
&\quad + P(\neg cavity \wedge catch \wedge toothache) + P(\neg cavity \wedge \neg catch \wedge toothache) \\
&= 0.108 + 0.012 + 0.016 + 0.064 \\
&= 0.2 \\
\mathbf{P}(Toothache) &= \langle 0.2, 0.8 \rangle
\end{aligned}
$$

The second probability is computed either in the same way as the first, or by knowing that the two sum to 1.

(c)

$$
\begin{aligned}
P(toothache|cavity) &= \frac{P(toothache \wedge cavity}{P(cavity)} \\
P(toothache \wedge cavity) &= P(cavity \wedge catch \wedge toothache) + P(cavity \wedge \neg catch \wedge toothache) \\
&= 0.108 + 0.012 \\
&= 0.12
\end{aligned}
$$

and so:

$$P(toothache|cavity) = 0.6$$

Similarly (or by subtraction):

$$P(\neg toothache|cavity) = 0.4$$

and:

$$\mathbf{P}(Toothache|cavity) = \langle 0.6, 0.4 \rangle$$

(d)

$$
\begin{aligned}
P(catch \vee cavity) &= P(cavity \wedge catch \wedge toothache) + P(cavity \wedge catch \wedge \neg toothache) \\
&\quad + P(\neg cavity \wedge catch \wedge toothache) + P(\neg cavity \wedge catch \wedge \neg toothache) \\
&\quad + P(cavity \wedge \neg catch \wedge toothache) + P(cavity \wedge \neg catch \wedge \neg toothache) \\
&= 0.108 + 0.072 + 0.016 + 0.144 + 0.012 + 0.008 \\
&= 0.36
\end{aligned}
$$

(e)

$$\mathbf{P}(Cavity|toothache \vee catch) = \frac{\mathbf{P}(Cavity \wedge (toothache \vee catch))}{P(toothache \vee catch)}$$

In a similar way to the previous question, we can compute:

$$P(toothache \vee catch) = 0.416$$

(in this case we sum up all the values on slide 32 except those in the last column where we have $\neg cavity$ and $\neg toothache$). Then:

$$\begin{aligned}
P(cavity \wedge (toothache \vee catch)) &= 0.108 + 0.012 + 0.072 \\
&= 0.192 \\
P(cavity|toothache \vee catch) &= 0.462 \\
\mathbf{P}(Cavity|toothache \vee catch) &= \langle 0.462, 0.538 \rangle
\end{aligned}$$

2. The problem tells us that:

$$\begin{aligned}
P(t|d) &= 0.95 \\
P(\neg t|\neg d) &= 0.95 \\
P(d) &= 0.0001
\end{aligned}$$

And:

$$\begin{aligned}
P(t|\neg d) &= 1 - P(\neg t|\neg d) \\
&= 0.05
\end{aligned}$$

Now,

$$\begin{aligned}
P(d|t) &= \frac{P(d \wedge t)}{P(t)} \\
&= \frac{P(t|d)P(d)}{P(t|d)P(d) + P(t|\neg d)P(\neg d)} \\
&= \frac{0.95 \times 0.0001}{0.95 \times 0.0001 + 0.05 \times 0.9999} \\
&= 0.00189
\end{aligned}$$

So the probability of having the disease once you have the positive test is 0.00189, which is not large, despite the accuracy of the test. The reason is that the disease is very unlikely, meaning that the prior probability is low.

Repeating the calculation with $P(d) = 0.01$ gives a probability of having the disease of 0.16, which is obviously much larger (though still much less than 0.95, which many people will say, without doing the calculation, is the chance of having the disease if the test comes back positive.)

3. We have:

$$\begin{aligned}
P(a|v) &= 0.95 & \qquad P(b|v) &= 0.9 \\
P(a|\neg v) &= 0.1 & \qquad P(b|\neg v) &= 0.05
\end{aligned}$$

Now, for Joe we want:

$$\begin{aligned}
P(v|a) &= \frac{P(a|v)P(v)}{P(a|v)P(v) + P(a|\neg v)P(\neg v)} \\
&= \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.99 \times 0.1} \\
&= 0.0876
\end{aligned}$$

and, as before, because the virus is unlikely even after an accurate test, there is not that great a chance of having the virus.

Carrying out a similar calculation for Bob, we find that:

$$P(v|b) = 0.154$$

so even though the second test is less accurate, the lower false positive rate means that Bob is about twice as likely to have the virus as Joe.

4. We have, from page 76 of the slides for lecture 2:

$$\begin{aligned} P(m) &= 0.0001 \\ P(s|m) &= 0.8 \end{aligned}$$

We also know that $P(s|\neg m) = 0.1$. Then:

$$\begin{aligned} \mathbf{P}(M|s) &= \alpha \mathbf{P}(s|M)\mathbf{P}(M) \\ \mathbf{P}(M|s) &= \alpha\langle P(s|m), P(s|\neg m)\rangle\langle P(m), P(\neg m)\rangle \\ &= \alpha\langle 0.8, 0.1\rangle\langle 0.0001, 0.9999\rangle \\ &= \alpha\langle 0.8 \times 0.0001, 0.1 \times 0.9999\rangle \\ &= \alpha\langle 0.00008, 0.09999\rangle \\ &= \langle 0.0008, 0.9992\rangle \end{aligned}$$

5. If you did this bit of the tutorial, you should be able to check that your code works by checking the answers against the calculations above.