5CCS2FC2: Foundations of Computing II

Recursive Algorithms & Solving Recursion Relations

Week 6

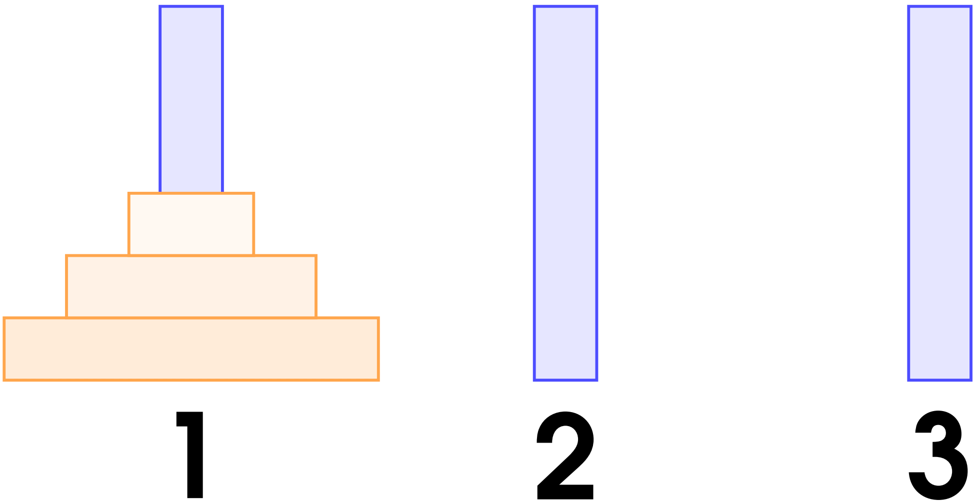
Dr Christopher Hampson

Department of Informatics

King's College London

Warm-up: The Towers of Hanoi

- Move the Tower of Blocks from **Post 1** to **Post 3**
- Can only move one block at a time,
- Cannot place any block on top of a smaller block.



- **The Towers of Hanoi**
  - A legend tells of **64 golden disks** stacked on **three posts** and sealed away in a monestry in Hanoi,
  - Acting out the command of an **ancient prophecy**, monks dilligently move the tower (one disk at a time) from one post to another, following the aforementioned prescribed rules,
  - Upon completion of their task...

...The world will end!

(the legend is likely fabricated by Édouard Lucas who devised the puzzle)

When will the World End?

Objectives for Today

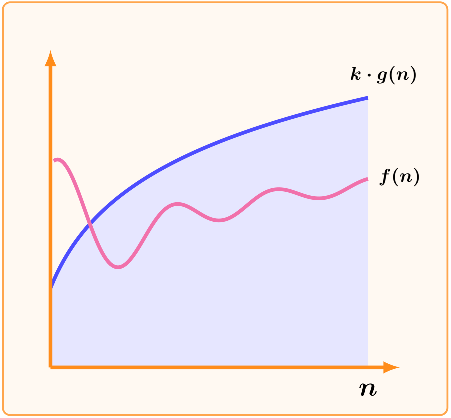
- To solve the **Towers of Hanoi** problem for $n = 64$,
- Be able to solve problems recursively using the **Divide-and-Conquer** technique,
- Be able to solve recurrence relations **inductively**
- Be able to solve common recurrence relations using **The Master Theorem**,

Asymptotic Notation

- **Big Oh Notation (upper bounds)**
  - Let $f(n)$ and $g(n)$ be any real-valued function. We say that g **eventually dominates** f if there is some constant $k > 0$ such that

$$f(n) \leq k \cdot g(n) \qquad \text{for all `large' } n$$

$$O(g(n)) \;=\; \left\{ \begin{array}{c} \text{All functions } f(n) \\ \text{that are \textit{eventually}} \\ \text{\textit{dominated} by } g(n) \end{array} \right\}$$

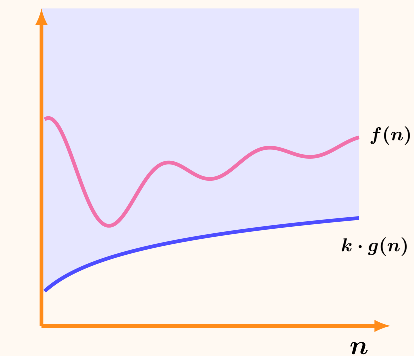


Asymptotic Notation – Big Omega

- **Big Omega Notation (lower bounds)**
  - Let $f(n)$ and $g(n)$ be any real-valued function. We say that g **eventually dominates** f if there is some constant $k > 0$ such that

$$f(n) \leq k \cdot g(n) \qquad \text{for all `large' } n$$

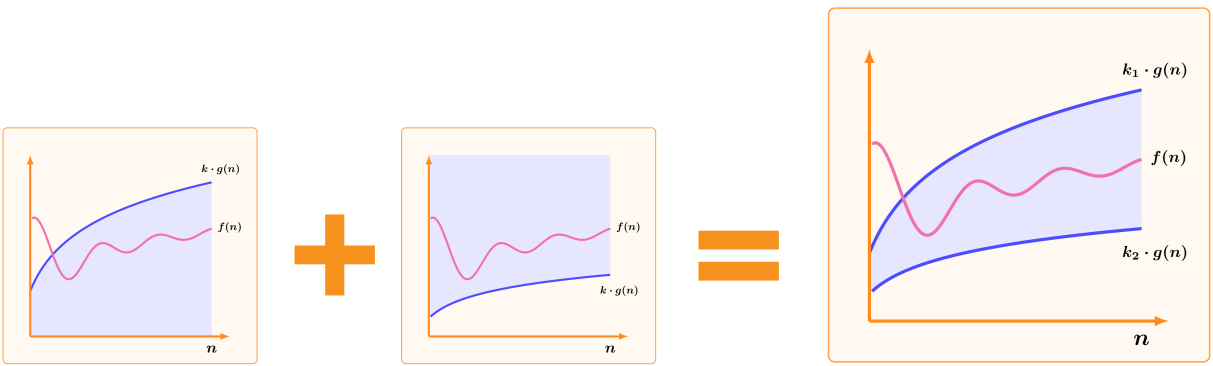
$$\Omega(g(n)) = \left\{ \begin{array}{c} \text{All functions } f(n) \\ \text{that } \textit{eventually} \\ \textit{dominate } g(n) \end{array} \right\}$$



Asymptotic Notation – Big Theta

- **Big Theta Notation (exact bounds)**
  - A function $f(n)$ belongs to $\Theta(g(n))$ if it is eventually **bounded above** *and* **below** by contant multiples of $g(n)$.

$$\boxed{\Theta(g(n)) \;=\; O(g(n)) \;\cap\; \Omega(g(n))}$$

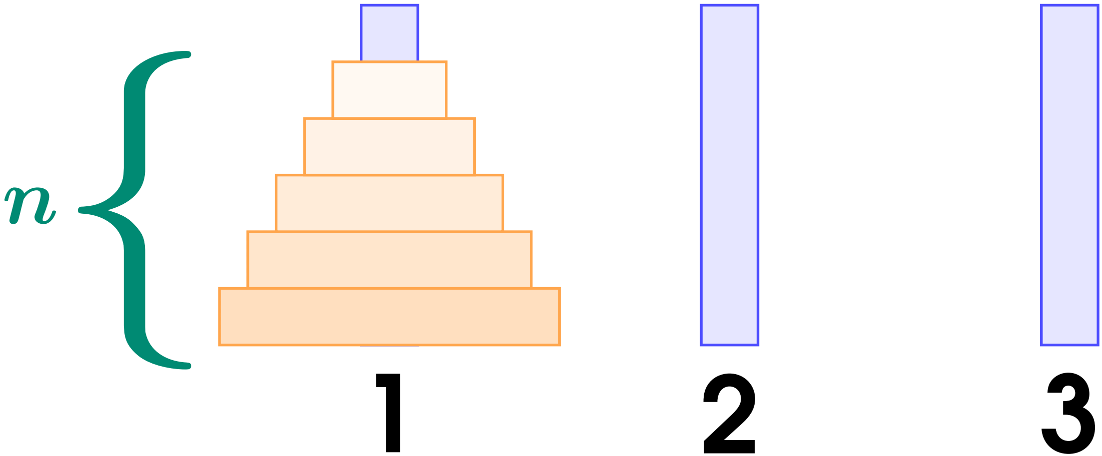


Solving Problems Recursively

The Towers of Hanoi

We would like an **general algorithm** that solves the Hanoi Tower problem for **any number of blocks**:

$$\boxed{\textsc{Move-Tower}\ (n,\ \text{post 1},\ \text{post 3})}$$



Divide-and-Conquer

- The Divide-and-Conquer technique is a useful technique for **designing** and **understanding** algorithms by diving them into easier sub-problems.

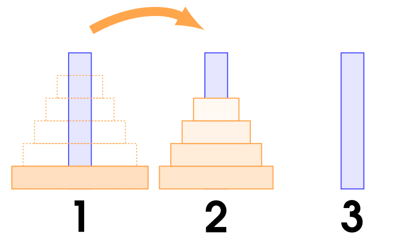
> **Divide-and-Conquer Technique**
>
> **Divide)** Divide the problem into several 'self-similar' but smaller sub-problems.
>
> **Conquer)** Solve these sub-problems recursively
>
> **Combine)** Recombine the sub-problems into a solution for the whole problem

Divide-and-Conquer

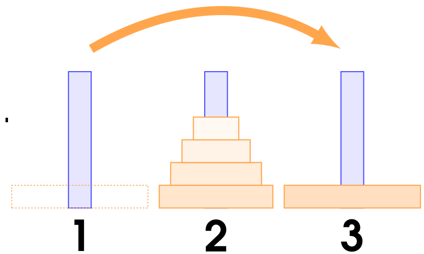
MOVE-TOWER $(n,\ i,\ j)$:

Step 1) MOVE-TOWER $(n-1,\ i,\ k)$

(move the top part of the tower out of the way)

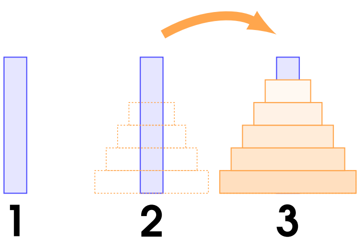


Step 2) Move the base of the tower from i to j.



Step 3) MOVE-TOWER $(n-1,\ k,\ j)$

(replace the top part of the tower)



Divide-and-Conquer

- How long does our MOVE-TOWER algorithm take?

$$\boxed{T(n) \;=\; \text{time to solve MOVE-TOWER } (n,\ i,\ j)}$$

(for any posts i and j)

- We can find a recurrence relation for $T(n)$ by examining the structure of the algorithm:

$$\begin{aligned} T(1) &= 1 \\ T(n) &= \underbrace{T(n-1)}_{\textbf{Step } 1} + \underbrace{1}_{\textbf{Step } 2} + \underbrace{T(n-1)}_{\textbf{Step } 3} \\ &= 2\,T(n-1) + 1 \end{aligned}$$

(to get the next value of $T(n)$ we multiply by 2 and add 1)

Divide-and-Conquer

- We can start to get an idea about the running time by **iterating** the first few values of $T(n)$

n	$T(n)$	
1	1	$\times 2 + 1$
2	3	$\times 2 + 1$
3	7	$\times 2 + 1$
4	15	$\times 2 + 1$
5	31	
$\vdots$	$\vdots$	
n	$2^n - 1$	

Divide-and-Conquer

- This gives us an **exact formula** for the running time...

$$\boxed{T(n) \;=\; 2^n - 1}$$

- ...but part that important for **scalability** is the 2^n,

$$\boxed{T(n) \;\in\; \Theta(2^n)}$$

(since $2^n - 1 < 2(2^n)$ for sufficiently large values of n)

- The running time for this algorithm quickly becomes **infeasible** to run!

(it takes 'exponential time' to solve)

How long until the end of the world?

- If we have **64 golden disks** to move,

$$\boxed{n = 64}$$

- The number of moves required to complete the puzzle is, therefore:

$$\begin{aligned} T(64) &= 2^{64} - 1 \\ &= 18,446,744,073,709,551,615 \end{aligned}$$

(or 1000 moves every second for 5 billion years...)

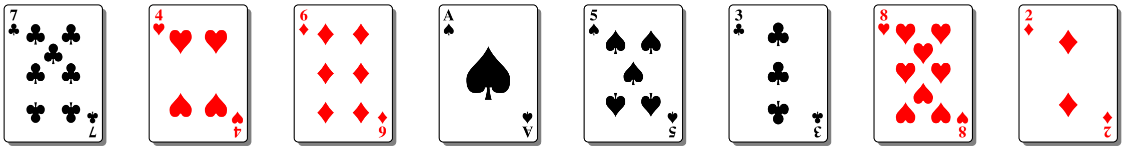
The world is safe for now!

Sorting Arrays with Divide-and-Conquer

Sorting Algorithms

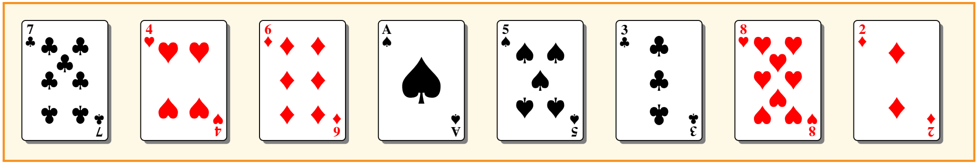
- Suppose we have an **array of integers (or cards)** of length n that we want to sort **ascending order**

(we are ignoring the suit)



Naïve Sorting

- A **naïve sorting algorithm** would be to locate each card **in order** placing then into a new pile,



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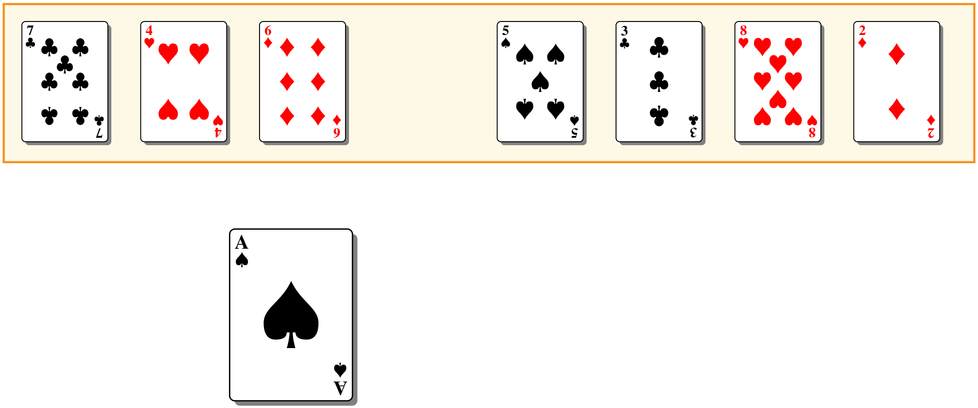

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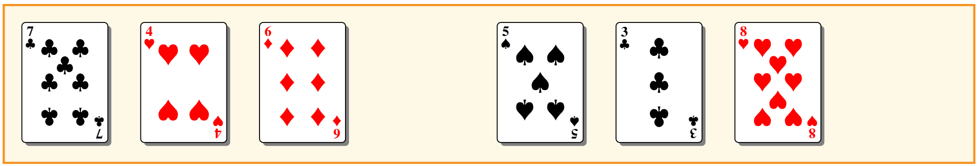

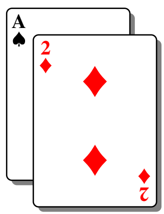

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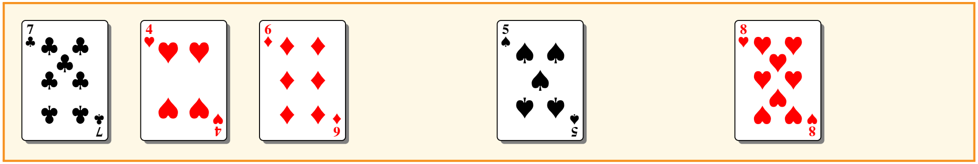

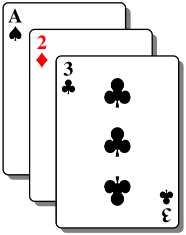

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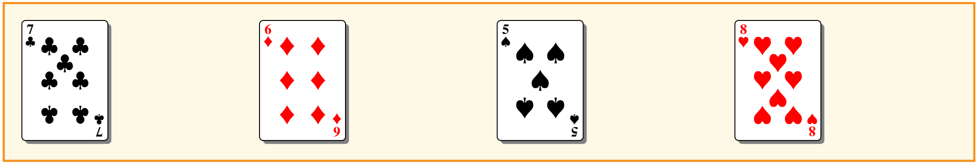

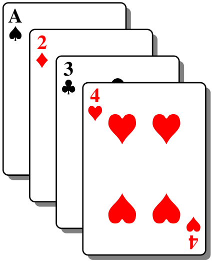

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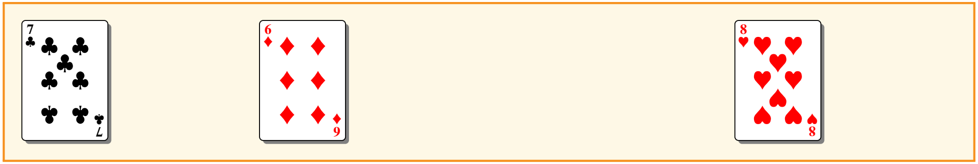

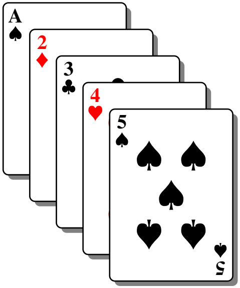

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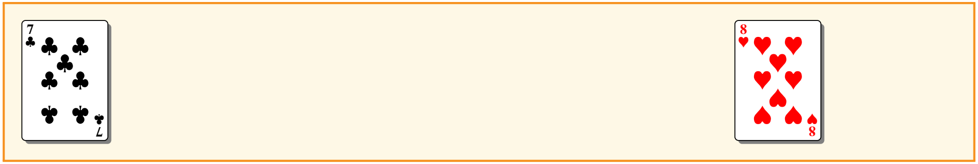

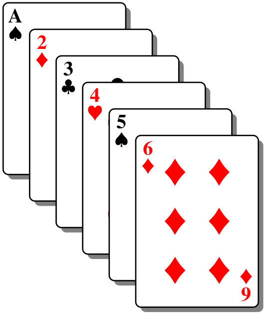

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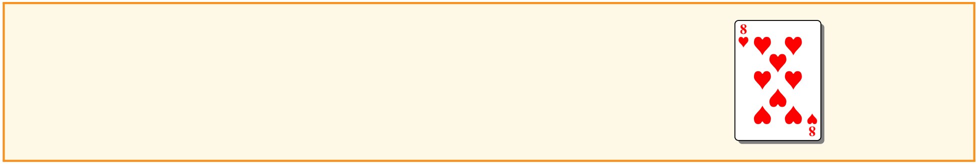

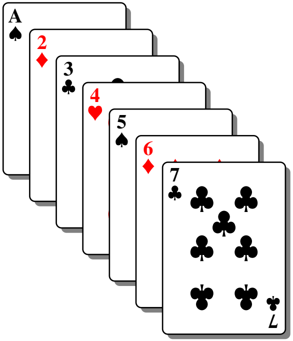

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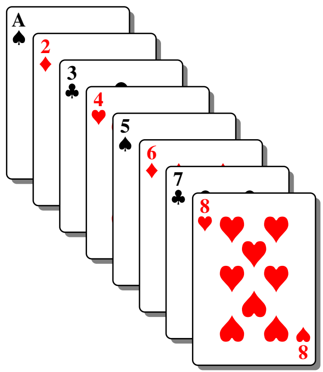


Naïve Sorting

- What is the **(worst case) running time** for the naïve sorting algorithm?

Step 1) Locating the first card may take at most n steps,

Step 2) Adding to the new pile takes 1 step,

(depending on the data structure)

Step 3) Locating the next card takes at most $(n - 1)$ steps plus 1 to add to the new pile,

(there is one fewer card to search through)

Step 4) *etc.*

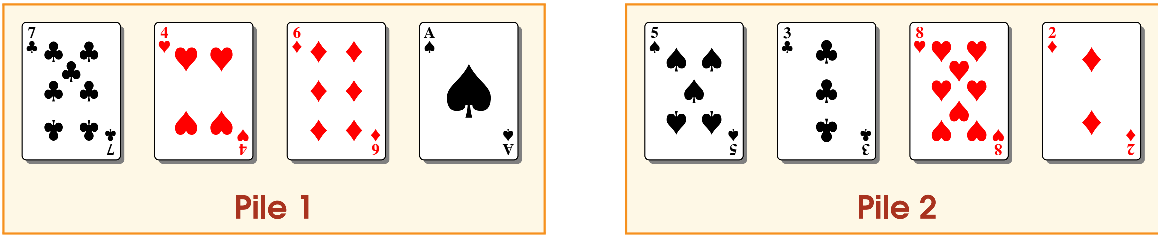
$$\begin{aligned} T(n) &\approx n + (n-1) + \cdots + 3 + 2 + 1 \\ &= \tfrac{1}{2}(n^2 + n) = \Theta(n^2) \end{aligned}$$

The Merge Sort Algorithm

MERGE-SORT($X[1:n]$):

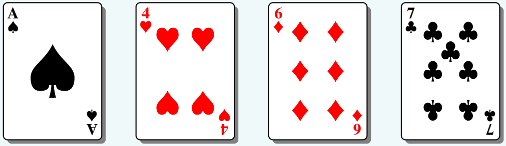
Step 1) Divide into two (roughly) even piles

(cannot divide perfectly if n is odd!)

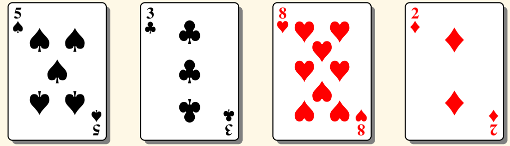


The Merge Sort Algorithm

Step 2) $\textsc{Merge-Sort}(\boldsymbol{X}[1 : \frac{1}{2}\boldsymbol{n}])$ (sort the first pile recursively)

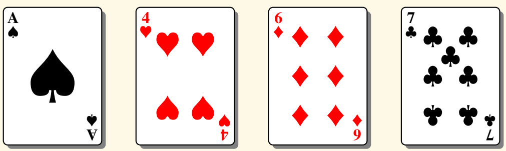


Sorted ✓



Pile 2

Step 3) $\textsc{Merge-Sort}(\boldsymbol{X}[\frac{1}{2}\boldsymbol{n} : \boldsymbol{n}])$ (sort the second pile recursively)



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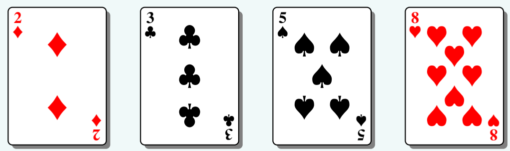
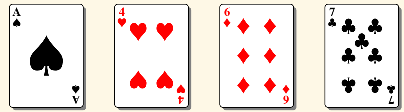


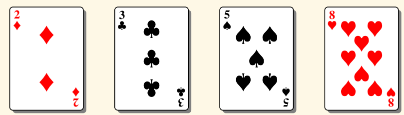
Sorted ✓

The Merge Sort Algorithm

Step 4) Merge the two sorted piles,



Pile 1

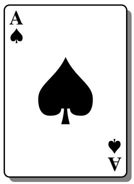


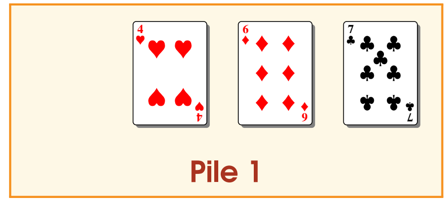
Pile 2

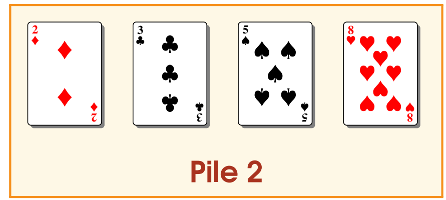
(we need only compare the first cards in each pile)

The Merge Sort Algorithm

Step 4) Merge the two sorted piles,







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The Merge Sort Algorithm

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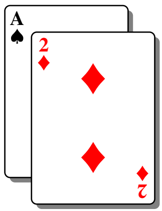

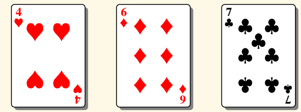

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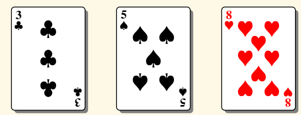

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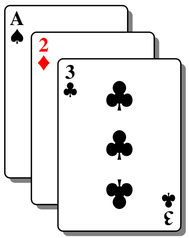

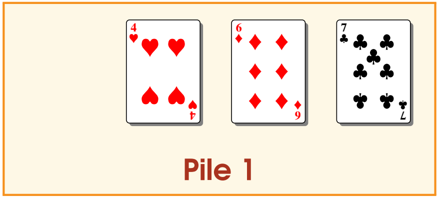

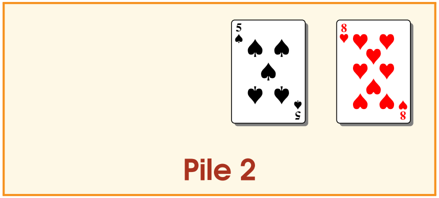

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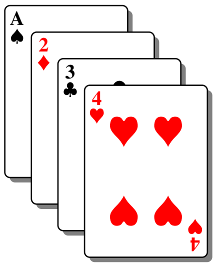

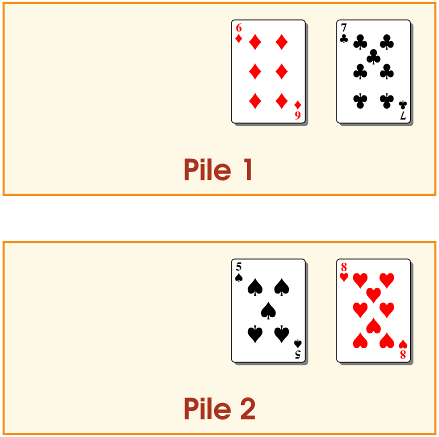

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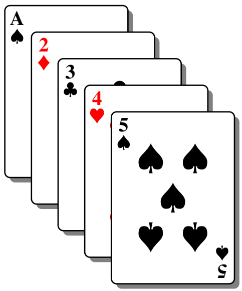

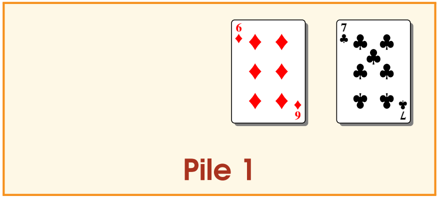

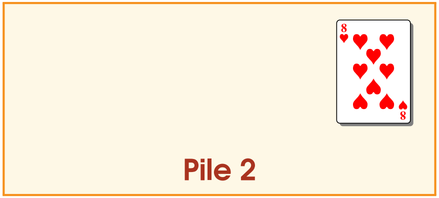

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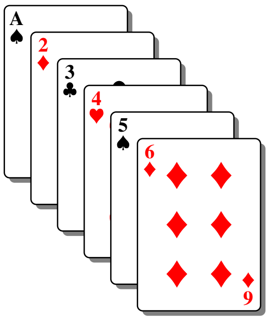
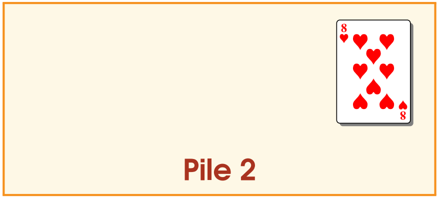
The Merge Sort Algorithm

Step 4) Merge the two sorted piles,





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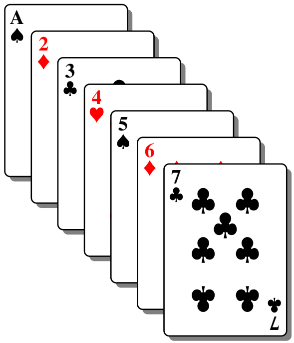
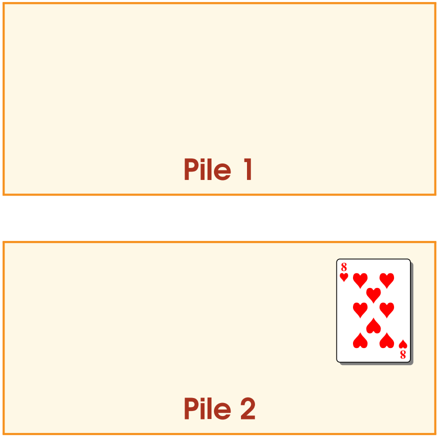
The Merge Sort Algorithm

Step 4) Merge the two sorted piles,





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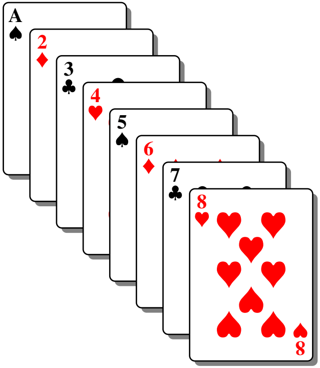

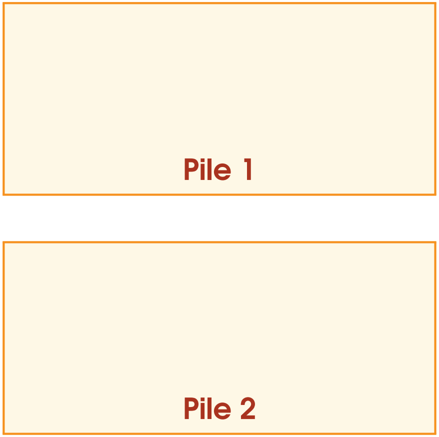

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The Merge Sort Algorithm

- Again, we can ask: how long does our algorithm take?

$$\boxed{\boldsymbol{T(n)} \;=\; \text{time to solve } \textsc{Merge-Sort}(\boldsymbol{X})}$$

(for any array X of length n)

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- Again, we can ask: how long does our algorithm take?

$$\boldsymbol{T}(\boldsymbol{n}) \;=\; \text{time to solve } \textsc{Merge-Sort}(\boldsymbol{X})$$

(for any array X of length n)

- We can find a **recurrence relation** for $\boldsymbol{T}(\boldsymbol{n})$ by examining the structure of the algorithm:

$$\boldsymbol{T}(1) \;=\; 1$$

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(for any array X of length n)

- We can find a **recurrence relation** for $T(n)$ by examining the structure of the algorithm:

$$\begin{aligned} T(1) &= 1 \\ T(n) &\approx \underbrace{n}_{\text{Step 1}} + \underbrace{T(\lceil n/2 \rceil)}_{\text{Step 2}} + \underbrace{T(\lceil n/2 \rceil)}_{\text{Step 3}} + \underbrace{n}_{\text{Step 4}} \end{aligned}$$

The Merge Sort Algorithm

- Again, we can ask: how long does our algorithm take?

$$\boxed{T(n) \;=\; \text{time to solve } \textsc{Merge-Sort}(X)}$$

(for any array X of length n)

- We can find a **recurrence relation** for $T(n)$ by examining the structure of the algorithm:

$$\begin{aligned} T(1) &= 1 \\ T(n) &\approx \underbrace{n}_{\textbf{Step } 1} + \underbrace{T(\lceil n/2 \rceil)}_{\textbf{Step } 2} + \underbrace{T(\lceil n/2 \rceil)}_{\textbf{Step } 3} + \underbrace{n}_{\textbf{Step } 4} \\ &= 2\,T(\lceil n/2 \rceil) + 2n \end{aligned}$$

The Merge Sort Algorithm

- Again, we can ask: how long does our algorithm take?

$$\boldsymbol{T}(\boldsymbol{n}) \;=\; \text{time to solve } \textsc{Merge-Sort}(\boldsymbol{X})$$

(for any array X of length n)

- We can find a **recurrence relation** for $\boldsymbol{T}(\boldsymbol{n})$ by examining the structure of the algorithm:

$$\begin{aligned}
\boldsymbol{T}(1) &= 1 \\
\boldsymbol{T}(\boldsymbol{n}) &\approx \underbrace{\boldsymbol{n}}_{\textbf{Step } 1} + \underbrace{\boldsymbol{T}(\lceil \boldsymbol{n}/2 \rceil)}_{\textbf{Step } 2} + \underbrace{\boldsymbol{T}(\lceil \boldsymbol{n}/2 \rceil)}_{\textbf{Step } 3} + \underbrace{\boldsymbol{n}}_{\textbf{Step } 4} \\
&= 2\,\boldsymbol{T}(\lceil \boldsymbol{n}/2 \rceil) + 2\boldsymbol{n}
\end{aligned}$$

(where $\lceil x \rceil$ is the *ceiling function* of x)

The Merge Sort Algorithm

- Again, we can **iterate** the first few values of $T(n)$

(easier if we look only at powers of two!)

n	$T(n)$	
1	1	
		$\times 2 + 4$
2	6	
		$\times 2 + 8$
4	20	
		$\times 2 + 16$
8	56	
		$\times 2 + 32$
16	144	
		$\times 2 + 64$
32	352	
$\vdots$	$\vdots$	

(it is not quite so easy to see what the growth-rate is...)

Solving Recurrence Relations

The Substitution Method

Proof by Induction

Base Case) Show that your solution holds for $n = 1$,

Inductive Case) **(i)** Assume your result holds for $n = k$,

(ii) Substitute to confirm that it also holds for $n = (k + 1)$.

- Like knocking over an **infinite stack of dominoes**:
  - The *base case* knocks over the **first domino**,
  - The *inductive case* shows that the dominoes are **spaced close enough** that the kth domino always knocks down the $(k + 1)$st domino!

(therefore ALL dominoes will fall!)

The Substitution Method

- **Example:** Show that the solution to the recurrence relation

$$\boxed{\begin{aligned} T(1) &= 1 \\ T(n) &= 2\,T(n-1) + 1 \end{aligned}}$$

is given by $T(n) = 2^n - 1$.

Base Case) We just need to check that our formula gives the correct value for $n = 1$.

$$\boxed{T(1) = 1 = 2^1 - 1}$$

The Substitution Method

Induction Case) Assume that

$$\boxed{T(k) = 2^k - 1 \qquad \text{for some } k \geq 1}$$

(this is known as the *'Induction Hypothesis'*)

We can substitute into the recurrence relation to find $T(k+1)$

$$\begin{aligned} T(k+1) &= 2 \cdot T(k) + 1 \\ &= 2 \cdot \boxed{(2^k - 1)} + 1 \\ &= 2 \cdot 2^k - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

Conclusion) Since this has the same form as the Induction Hypothesis, the formula must hold for **ALL** values of n. **Q.E.D**

The Substitution Method

- How about a more complicated recurrence relation such as

$$\boxed{\begin{aligned} T(1) &= 1 \\ T(n) &= 2\,T(\lceil n/2 \rceil) + 2n \end{aligned}}$$

(this was the approximate running time for the merge-sort algorithm)

- Since $T(n)$ depends on $T(n/2)$ rather than the immediate predecessor $T(n-1)$, we need a **slightly stronger** version of induction!
  - It is not enough to consider the spacing between **neighbouring dominos**,

The Substitution Method

Proof by Induction (Strong)

Base Case) Show that your solution holds for $n = 1$,

Inductive Case) **(i)** Assume your result holds for *all* $m \leq k$ for some k,

(ii) Substitute to confirm that it also holds for $n = (k + 1)$.

- We may rely not just on the previous 'domino' but on all those that have fallen before!
- When is this useful?
  - If your recurrence relation does not depend on the **previous value**,
  - Or if your recurrence relation involves **multiple calls** to itself, *e.g.*

$$\boxed{F(n) \;=\; F(n-1) + F(n-2)}$$

(this recurrence relation generates the Fibonacci numbers)

The Substitution Method

- **Example:** Show that the solution to the recurrence relation

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 2\,T(\lceil n/2 \rceil) + 2n \end{aligned}$$

bounded above by $T(n) \geq n \log_2 n$, for $n \geq 1$

Base Case) Again, we just need to check that our formula gives the correct value for $n = 1$.

$$\boxed{T(1) \;=\; 1 \;\geq\; 0 \;=\; 1 \log_2 1}$$

The Substitution Method

Induction Case) Assume that

$$\boxed{T(m) \geq m \log_2 m \qquad \text{for all } m \leq k \text{ for some } k \geq 1}$$

We can substitute into the recurrence relation to find $T(k+1)$

$$\begin{aligned}
T(k+1) &= 2 \cdot T\left(\left\lceil \frac{k+1}{2} \right\rceil\right) + 2(k+1) \\
&\geq 2 \cdot T\left(\frac{k+1}{2}\right) + 2(k+1) \\
&\geq 2 \cdot \boxed{\left(\frac{k+1}{2}\right) \log_2 \frac{k+1}{2}} + 2(k+1) \\
&= (k+1)\Big[\log_2(k+1) - 1\Big] + 2(k+1) \\
&= (k+1)\log_2(k+1) + (k+1) \\
&\geq (k+1)\log_2(k+1)
\end{aligned}$$

The Substitution Method

Conclusion) Hence, it follows that

$$\boxed{T(n) \;\geq\; n \log_2 n}$$

for **ALL** values of $n \geq 1$.

Q.E.D

- Hence it follows that the Merge-sort algorithm runs belongs to the class

$$\boxed{T(n) \;=\; \Omega(n \log_2 n)}$$

(we can similarly, show that $T(n) = \Theta(n \log_2 n)$, as well)

The Substitution Method

- **Remarks on Proof by Induction:**
    - Often easier to establish **upper** and **lower** bounds than to prove an exact formula.
    - You need to **correctly 'guess'** the correct formula before you start!
        - If the algorithm is similar to one whose growth-rate is known, try that!
        - If your first guess does not work, adjust accordingly!

          (if you can't bound above by a quadratic, try a cubic, *etc*..)
        - If the first few values **'misbehave'**, use a bigger base case!

The Master Theorem

The Master Theorem

Let $T(n)$ be a *monotonically increasing* recurrence relation such that

$$\boxed{T(n) \;=\; a\,T\left(\frac{n}{b}\right) + f(n)}$$

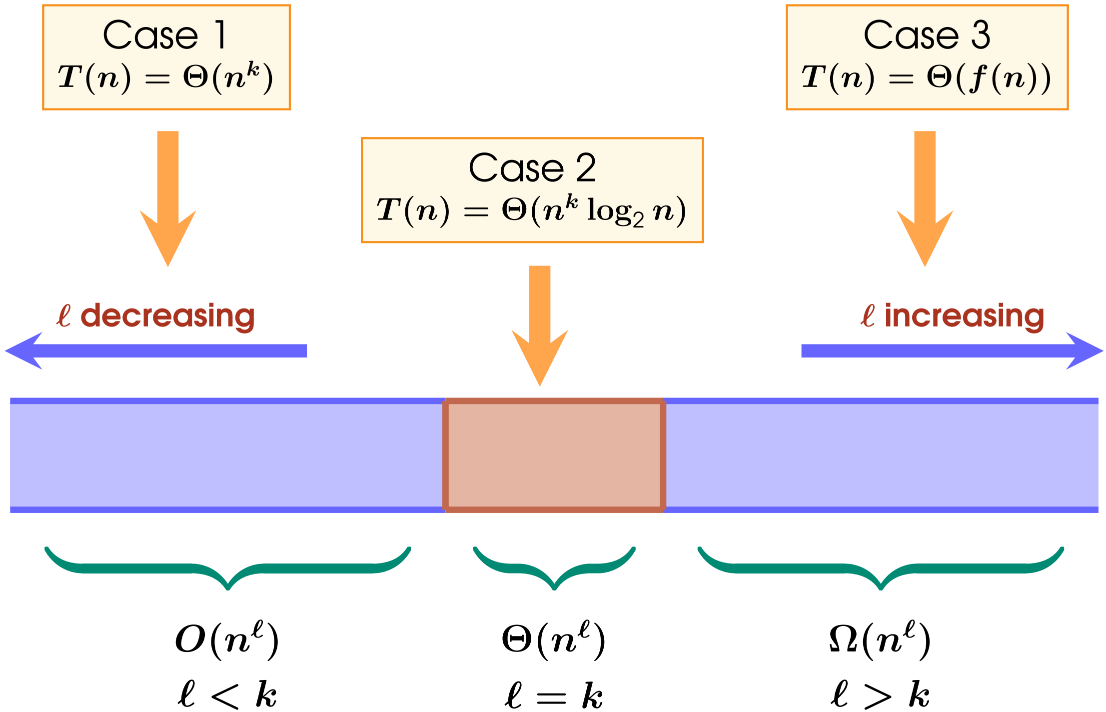
(for some constants $a \geq 1$, $b \geq 2$.)

Then

$$\boxed{T(n) \;\in\; \begin{cases} \Theta(n^k) & \text{if } f(n) \in O(n^\ell) \text{ for } \ell < k \\ \Theta(n^k \log_2 n) & \text{if } f(n) \in \Theta(n^\ell) \text{ for } \ell = k \\ \Theta(f(n)) & \text{if } f(n) \in \Omega(n^\ell) \text{ for } \ell > k \end{cases}}$$

where $\boxed{k = \log_b a \;=\; \log_{10} a / \log_{10} b}$

The Master Theorem



The Master Theorem : Some Examples

- **Example 1:** Let $T(n)$ be given by the following recurrence relation

$$\boxed{T(n) \;=\; 4\,T\left(\frac{n}{2}\right) + 2^n}$$

Step 1) Identify the parameters:

$$\boxed{a = 4, \quad b = 2, \quad \text{therefore} \quad k = \log_2 4 = 2}$$

Step 2) Identify the growth rate of $f(n)$

$$\boxed{f(n) \;=\; 2^n \;\in\; \Omega(n^3)}$$

(bounded below by a cubic, and $k < 3$)

Step 3) Therefore, Case 3 applies, and we have:

$$\boxed{T(n) \;\in\; \Theta(f(n)) \;=\; \Theta(2^n)}$$

The Master Theorem : Some Examples

- **Example 2:** Let $T(n)$ be given by the following recurrence relation

$$T(n) \;=\; 2\,T\left(\frac{n}{4}\right) + \sqrt{n+1}$$

Step 1) Identify the parameters:

$$a = 2, \quad b = 4, \quad \text{therefore} \quad k = \log_4 2 = 0.5$$

Step 2) Identify the growth rate of $f(n)$

$$f(n) \;=\; \sqrt{n+1} \;\in\; \Theta(\sqrt{n})$$

(bounded above and below by a square root, and $k = 0.5$)

Step 3) Therefore, Case 2 applies, and we have:

$$T(n) \;\in\; \Theta(n^k \log_2 n) \;=\; \Theta(\sqrt{n}\,\log_2 n)$$

The Master Theorem : Some Examples

- **Example 3:** Let $T(n)$ be given by the following recurrence relation

$$\boxed{T(n) \;=\; 3\,T\left(\frac{n}{2}\right) + \log_2 n}$$

Step 1) Identify the parameters:

$$\boxed{a = 3, \quad b = 2, \quad \text{therefore} \quad k = \log_2 3 \approx 1.5849}$$

Step 2) Identify the growth rate of $f(n)$

$$\boxed{f(n) \;=\; \log_2 n \;\in\; \Theta(n)}$$

(bounded above by a linear function, and $k > 1$)

Step 3) Therefore, Case 1 applies, and we have:

$$\boxed{T(n) \;\in\; \Theta(n^k) \;=\; \Theta(n^{1.5849})}$$

The Master Theorem : Some Examples

WARNING! WARNING!

- There is a **mistake** is the previous slide!
- Every approximation, no matter how accurate, will **eventually diverge!**

$$\boxed{\Theta(n^{1.58}) \neq \Theta(n^{1.584}) \neq \Theta(n^{1.5849}) \neq \cdots \neq \Theta(n^{\log_2 3})}$$

(we cannot use approximations when writing grown rates)

- The **correct** growth-rate for $T(n)$ should be

$$\boxed{T(n) \;\in\; \Theta(n^{\log_2 3})}$$

Next Time...

- SAT Solving
  - Greedy Algorithms,
  - The DPLL Algorithm
- Solving simple instances of SAT.

End of Slides!