Econ 8010 Midterm
Solutions

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1. (30 points) A hospital is looking to fill **up to** three open positions in its residency program. There are three doctors who might apply: Alice (a), Bob (b), and Claire (c). Thus, its set of alternatives is the set of possible hiring decisions:

$$X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}.$$

For any $Y \subseteq \{a,b,c\}$, define the **power set of** Y as

$$2^Y \equiv \{Z \mid Z \subseteq Y\}.$$

2^Y is the set of hiring decisions that the hospital can make when it receives applications from the doctors in Y.

The hospital's budget sets $B \in \mathcal{B}$ are the sets of hiring decisions it can make after receiving applications from some combination of Alice, Bob, and Claire:

$$\mathcal{B} = \{2^Y \mid Y \subseteq \{a,b,c\}\}$$

- When it receives applications from Alice and Bob, it will choose to hire Bob (and not Alice):

$$C(2^{\{a,b\}}) = \{b\}$$

- When it receives applications from Bob and Claire, it will choose to hire Claire (and not Bob):

$$C(2^{\{b,c\}}) = \{c\}$$

(a) (15 points) What restrictions does the weak axiom place on the hospital's hiring decision $C(2^{\{a,b,c\}})$ when it receives applications from Alice, Bob, and Claire?

- **Answer:** $C(2^{\{a,b,c\}}) \in \{\{c\}, \{a,c\}, \{a,b,c\}\}$.
- When Alice and Bob applied, the hospital rejected a, $\{a,b\}$, and $\emptyset$ in favor of b, which is available. When Bob and Claire applied, the hospital rejected b, $\{b,c\}$, and $\emptyset$ in favor of c, which is available. When the hospital gets applications from all three, it can't choose anything we already rejected, since the options it chose are available. So the weak axiom restricts it to choose either c, $\{a,c\}$, or $\{a,b,c\}$.

(b) (15 points) What restrictions does the weak axiom place on the hospital's hiring decision $C(2^{\{a,c\}})$ when it receives applications from Alice and Claire?

- **Answer:** $C(2^{\{a,c\}}) \in \{\{c\}, \{a,c\}, \{a\}\}$.
- There are no restrictions, other than the hospital has to hire at least one doctor (since it chose c over $\emptyset$). The only budget set we've seen where it could have chosen something available here and rejected something available here other than the empty set was $2^{\{a,b,c\}}$. But if it chose $\{a,b,c\}$, which it could have (see solution to part a), that choice won't constrain $C(2^{\{a,c\}})$.

2. (35 points) A widget manufacturing company produces a single output, widgets q. In doing so, it uses two inputs: machines m and sprockets s. These inputs must be consumed in whole (integer) quantities: $m, s \in \mathbb{Z}_+$.

   When two sprockets are fed into a machine, it will produce one widget. More formally, the firm's production function $f(m,s)$ is given by

$$f(m,s) = \min\{m, 2s\}$$

(a) (15 points) Show that f is supermodular.

- **Answer:** Let $m' > m$. Then

$$\begin{aligned} f(m',s) - f(m,s) &= \min\{m', 2s\} - \min\{m, 2s\} \\ &= \begin{cases} m' - m, & 2s > m' > m \\ 2s - m, & m' > 2s > m \\ 0, & m > 2s \end{cases} \\ &= \min\{m' - m, \max\{2s - m, 0\}\} \end{aligned}$$

  which is increasing in s, so f has increasing differences in m, s.

- (As s increases, you either stay in the second case and the difference goes up, or you jump to a different case which has a larger difference than the one you were in.)

(b) (10 points) If the price of sprockets increases, will the firm use more or fewer machines? Why?

- **Answer: (weakly) fewer.** Since the (s, w_s) and (m, w_s) cross partial derivatives of the firm's objective function

$$g(m, s, p, w_s, w_m) = pf(m,s) - w_s s - w_m m$$

  are nonpositive, g has increasing differences in $((m,s), -w_s)$; since it is supermodular in m, s by part 1, this is a straightforward application of Topkis' theorem.

(c) (10 points) If the price of widgets increases, will the firm use more or fewer machines? Sprockets? Why?

- **Answer: (weakly) more machines and more sprockets — and therefore (weakly) more widgets.** g has increasing differences in $(m,s), p$: Suppose $m' > m$ and $s' > s$. Since f is weakly increasing in both inputs,

$$\begin{aligned} g(m', s, p, w_s, w_m) - g(m, s, p, w_s, w_m) &= p(f(m',s) - f(m,s)) \\ g(m, s', p, w_s, w_m) - g(m, s, p, w_s, w_m) &= p(f(m,s') - f(m,s)) \end{aligned}$$

  are both weakly increasing in p. Applying Topkis' theorem gets us our answer.

3. (35 points) A consumer's preferences are described by a utility function that is homogeneous of degree two: For all $\alpha > 0$ and $x \in \mathbb{R}^L_+$, $u(\alpha x) = \alpha^2 u(x)$.

   (a) (7 points) Are this consumer's preferences homothetic? Show that they are or give a counterexample.

   - **Answer: Yes.**

$$x \sim y \Leftrightarrow u(x) = u(y) \Leftrightarrow \alpha^2 u(x) = \alpha^2 u(y) \Leftrightarrow u(\alpha x) = u(\alpha y) \Leftrightarrow \alpha x \sim \alpha y.$$

   - Alternatively, you could have said that $\sqrt{u(x)}$ represents the same preferences and is HD1, therefore the underlying preferences are homothetic.

   (b) (7 points) Show that this consumer's Walrasian demand is multiplicatively separable in prices and wealth: $x(p,w) = y_0(w)y_1(p)$ for some $y_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and $y_1 : \mathbb{R}^L_+ \to \mathbb{R}^L_+$. What is $y_0(w)$?

   - We did this in class and in the lecture notes.
   - **Answer:** By definition,

$$x(p,w) = \arg\max_x \{u(x) \text{ s.t. } p \cdot x \leq w\}$$

   Let $y = \frac{1}{w}x$.

$$\begin{aligned} \frac{1}{w}x(p,w) &= \arg\max_y \{u(wy) \text{ s.t. } p \cdot yw \leq w\} \\ &= \arg\max_y \{w^2 u(y) \text{ s.t. } p \cdot y \leq 1\} \\ &= \arg\max_y \{u(y) \text{ s.t. } p \cdot y \leq 1\} \end{aligned}$$

   Thus $y_0(w) = w$ and $y_1(p) = \arg\max_y \{u(y) \text{ s.t. } p \cdot y \leq 1\}$.

   (c) (7 points) Show that this consumer's indirect utility is multiplicatively separable in prices and wealth: $v(p,w) = v_0(w)v_1(p)$ for some $v_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and $v_1 : \mathbb{R}^L_+ \to \mathbb{R}_+$. What is $v_0(w)$?

   - Even if you didn't get part 2, this one was still feasible, and you should get full credit if you got the answer in terms of y_0 and y_1.

- **Answer:**

$$
\begin{aligned}
v(p,w) &= u(x(p,w)) \\
&= u(y_0(w)y_1(p)) \\
&= (y_0(w))^2 u(y_1(p))
\end{aligned}
$$

Thus $v_0(w) = (y_0(w))^2 = w^2$ and $v_1(p) = u(y_1(p))$.

(d) (7 points) Show that this consumer's Hicksian demand is multiplicatively separable in prices and required utility level: $h(p,\bar{u}) = g_0(\bar{u})g_1(p)$ for some $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and $g_1 : \mathbb{R}^L_+ \to \mathbb{R}^L_+$. What is $g_0(\bar{u})$?

- This one is similar to part 2.
- **Answer:** By definition,

$$
h(p,u) = \arg\min_x \{p \cdot x \text{ s.t. } u(x) \geq \bar{u}\}
$$

Let $y = \frac{1}{\sqrt{\bar{u}}}x$.

$$
\begin{aligned}
\sqrt{u}h(p,u) &= \arg\min_y \{p \cdot y\sqrt{\bar{u}} \text{ s.t. } u(\sqrt{\bar{u}}y) \geq \bar{u}\} \\
&= \arg\min_y \{p \cdot y\sqrt{\bar{u}} \text{ s.t. } \bar{u}u(y) \geq \bar{u}\} \\
&= \arg\min_y \{p \cdot y \text{ s.t. } u(y) \geq 1\}
\end{aligned}
$$

Thus $g_0(\bar{u}) = \sqrt{\bar{u}}$ and $g_1(p) = \arg\min_y \{p \cdot y \text{ s.t. } u(y) \geq 1\}$

(e) (7 points) Show that for all p, $g_1(p)v_1(p) = y_1(p)$.

- There's a typo in this question: it should read $g_1(p)\sqrt{v_1(p)} = y_1(p)$.
- **Answer:** From duality,

$$
x(p,w) = h(p, v(p,w))
$$

Thus,

$$
\begin{aligned}
wy_1(p) &= g_0(v(p,w))g_1(p) \\
wy_1(p) &= \sqrt{v(p,w)}g_1(p) \\
wy_1(p) &= \sqrt{w^2 v_1(p)}g_1(p) \\
y_1(p) &= \sqrt{v_1(p)}g_1(p)
\end{aligned}
$$