ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 3: The Weak Axiom and the Law of Demand (MWG, Ch. 2, pp.28-36)

I. Introduction

Lecture 2 introduced demand functions, and examined what properties they have if they are homogenous of degree zero and satisfy Walras' Law (the assumption that the consumer spends all of his or her wealth). This was done without any reference to utility functions or even preferences.

In this lecture (and subsequent lectures) we bring back assumptions about choice and preferences and see what their implications are for demand functions. This lecture focuses on the weakest assumption about consistent choice, the weak axiom of revealed preference.

In this lecture we assume that the demand function x(p, w):

1. Is single-valued (a function, not a correspondence)

2. Is homogenous of degree zero (no "money illusion")

3. Satisfies Walras' Law (all wealth is spent)

II. Applying the Weak Axiom to Demand Functions

To get started, here is the weak axiom of revealed preference, as given in Lecture 1:

Definition: The choice structure ($\mathscr{B}$, C( )) satisfies the weak axiom of revealed preference if the following property holds:

> If for some $B \in \mathscr{B}$, with x, y $\in B$, we have x $\in$ C(B), then for any $B' \in \mathscr{B}$, with x, y $\in B'$, and y $\in$ C(B'), we must also have x $\in$ C(B').

In words, if it ever happens that x is chosen when y is available, then there can be no budget set containing both x and y for which y is chosen but x is not chosen.

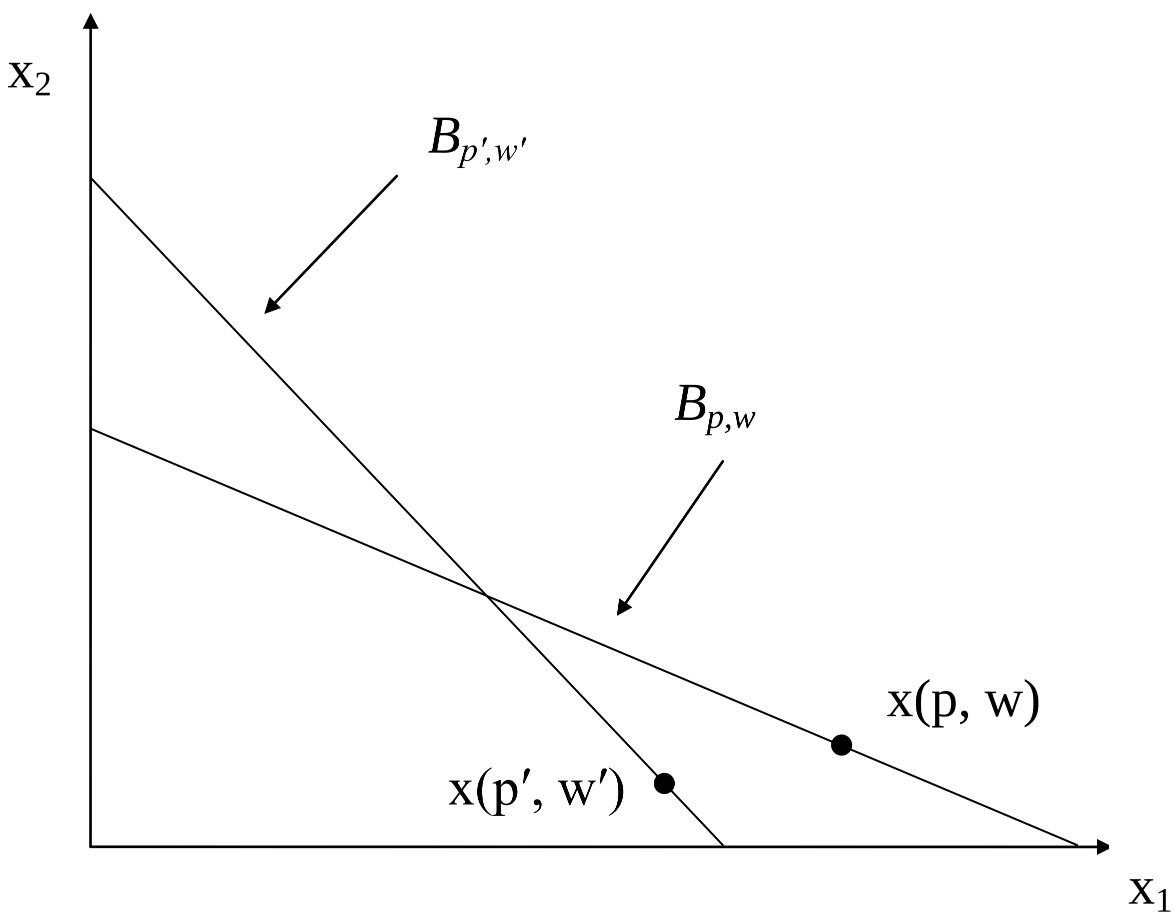
Now let's see what form this definition takes for Walrasian demand functions (which have unique values):

Definition/Proposition: The Walrasian demand function satisfies the weak axiom of revealed preference if the following holds for any two price-wealth situations (p, w) and (p′, w′): **[write on board]**

$$\text{If } p\cdot x(p', w') \le w \text{ and } x(p', w') \ne x(p, w),$$
$$\text{then } p'\cdot x(p, w) > w'$$

The logic of this definition is as follows. If $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$, then the consumer when faced with prices p and wealth w chose $x(p, w)$ even though he or she could have chosen $x(p' w')$. This "reveals" that the consumer will choose $x(p, w)$, not $x(p', w')$, when both are affordable. Thus if we see a consumer choosing $x(p', w')$ when faced with prices p' and wealth w' it must be the case that $x(p, w)$ was not affordable under p' and w'. [If $x(p, w)$ had been affordable, he or she would have chosen it instead of $x(p', w')$.]

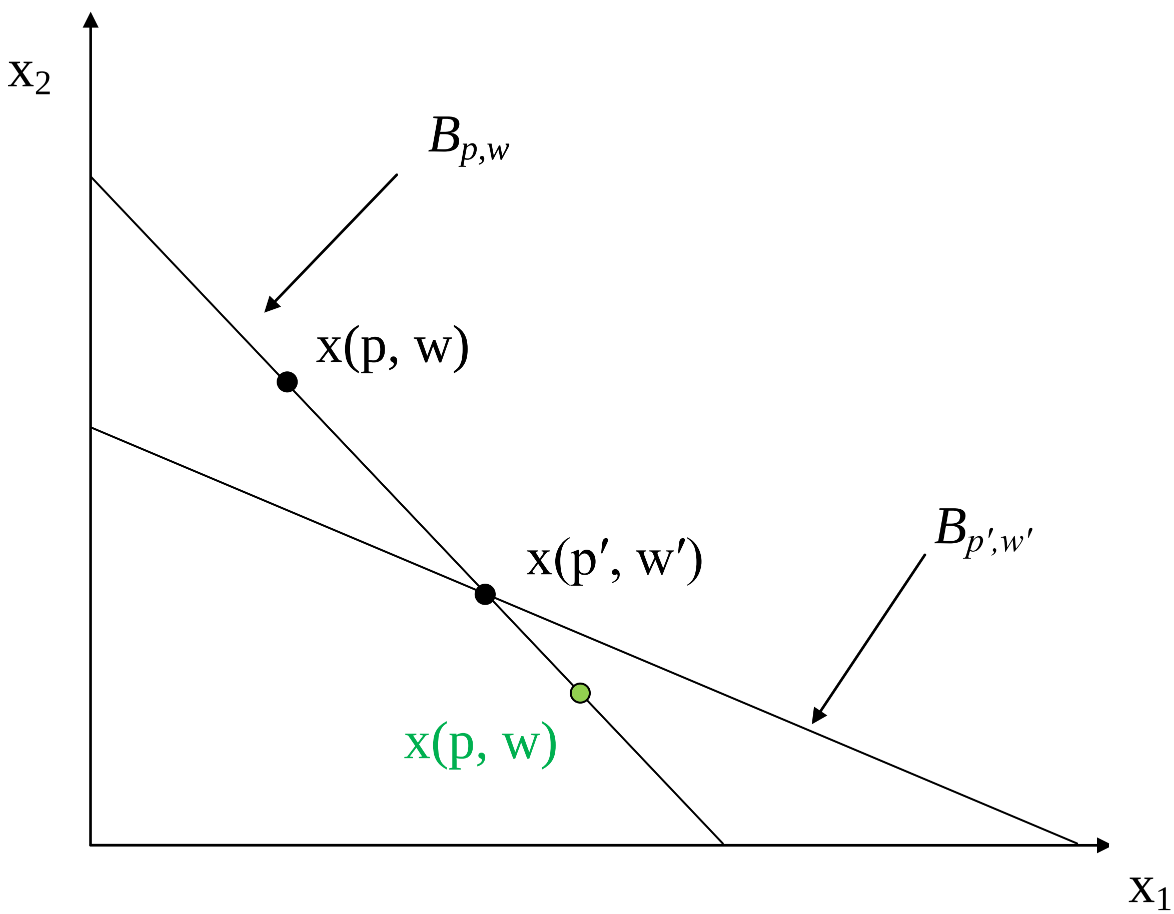
This diagram shows this for the case of 2 goods (L = 2):



The definition of the weak axiom (of revealed preference) implies that, for any two price-wealth pairs (p, w) and (p′, w′), such that $x(p, w) \neq x(p', w')$, it is not possible that both $p' \cdot x(p, w) \leq w'$ and $p \cdot x(p', w') \leq w$.

Question: Is it possible for these two inequalities to hold if $x(p, w) = x(p', w')$?

More questions: In the diagram below: 1. Do the two demands x(p, w) and x(p′, w′) satisfy the weak axiom? 2. What about x(p, w) and x(p′, w′)?



III. Implications of the Weak Axiom

The **weak axiom** of revealed preference, when combined with homogeneity of degree zero and Walras' law (and the assumption that $x(p, w)$ takes only one value), have **direct implications for the effects of prices on demand**. However, to see this we **need to define a price change that does not have any "wealth effect"**.

In general, price changes affect demand in two ways:

1. The relative costs of some (or all) goods are changed.

2. "Real" wealth changes (e.g., if the price of one good increases, the budget set shrinks, while if that price decreases, the budget set grows).

To focus on the first effect, we will define a price change in a way that "neutralizes" the "wealth effect" (sometimes called the "income effect").

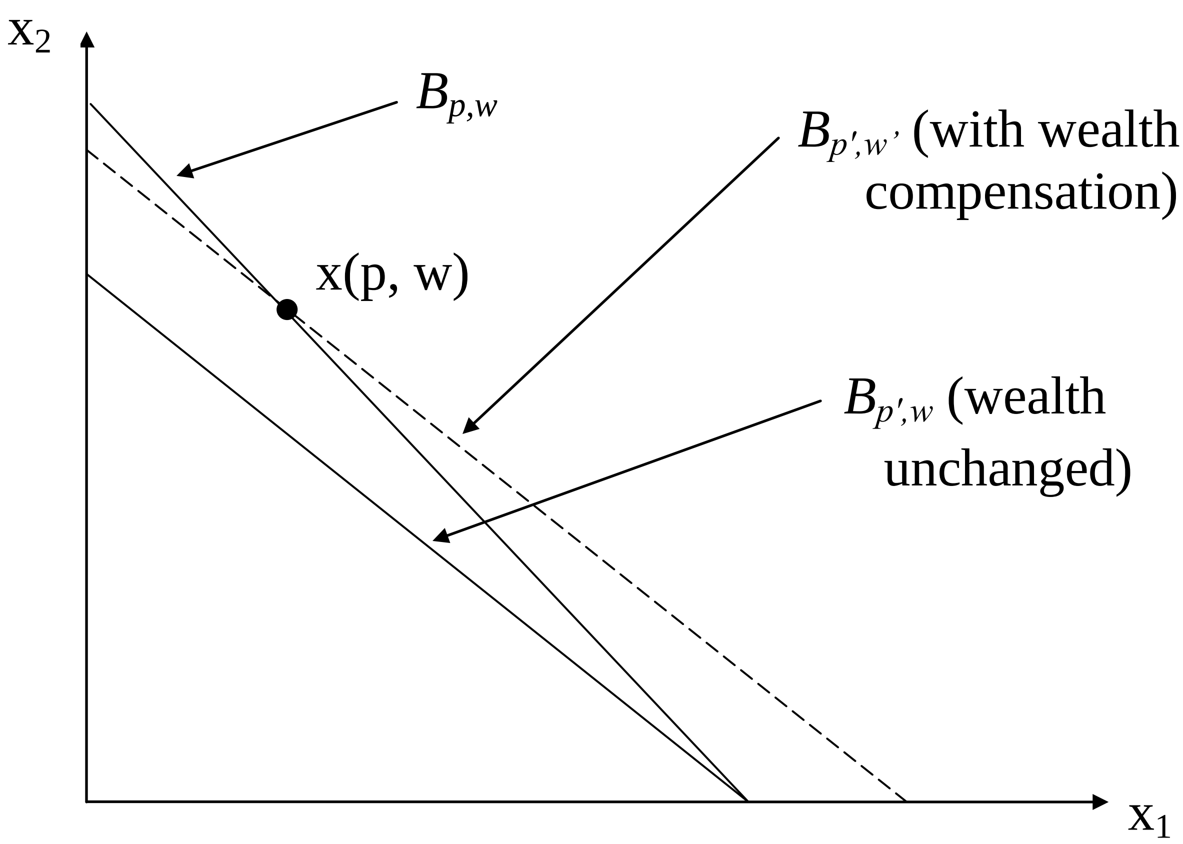
The **simplest way** to do this is to require that whenever prices change **wealth should also change so that the bundle** of goods **chosen before the price change is** (just) **affordable after the price change**. That is, if initial prices were p and initial wealth was w, so that initial demand was $x(p, w)$, then for a new set of prices, denoted by p', we require that the new wealth, w' satisfy:

$$w' = p' \cdot x(p, w)$$

This requirement **can also be expressed as *changes* in prices and wealth**. The initial situation was $w = p \cdot x(p, w)$. Combining this with the equation above gives:

$$\Delta w = \Delta p \cdot x(p, w)$$

where $\Delta w = w' - w$ and $\Delta p = p' - p$. This wealth adjustment is called the **Slutsky wealth compensation**. Price changes that are accompanied by the Slutsky wealth compensation are called **(Slutsky) compensated price changes**. This can be shown for 2 goods in a diagram:



It turns out that the weak axiom of revealed preference for demand functions can be stated in terms of the way demand responds to compensated price changes:

Proposition 2.F.1: Suppose that the Walrasian demand function $x(p, w)$ is homogenous of degree zero and satisfies Walras' law. Then $x(p, w)$ satisfies the weak axiom **if and only if** the following property holds:

For any *compensated* price change from an initial situation (p, w) to a new price-wealth pair (p', w') that satisfies $w' = p' \cdot x(p, w)$, we have:

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

with the strict inequality holding if $x(p', w') \neq x(p, w)$.

Proof: First, we show that the weak axiom implies the above inequality. Consider the case where $x(p', w') = x(p, w)$. Clearly, this implies $[x(p', w') - x(p, w)] = 0$ so the above holds as an equality. Next, consider the case where $x(p', w') \neq x(p, w)$. The above expression can be rewritten as:

$$(p' - p) \cdot [x(p', w') - x(p, w)] =$$

$$p' \cdot [x(p', w') - x(p, w)] - p \cdot [x(p', w') - x(p, w)]$$

Consider the first term in this expression. Since the move from p to p′ is a *compensated* price change, it must be that $p' \cdot x(p, w) = w'$ (the previous demand is "just" affordable given the new prices). Also, by Walras' law we have $p' \cdot x(p', w') = w'$. This implies that this first term equals 0:

$$p' \cdot [x(p', w') - x(p, w)] = 0$$

What about the second term: $p \cdot [x(p', w') - x(p, w)]$? Note that $p' \cdot x(p, w) = w'$ is affordable under the new set of prices and wealth (p′ and w′), **but** x(p, w) was not chosen under those prices. The weak axiom implies that the new bundle, x(p′, w′), was not affordable under the old prices. Thus $p \cdot x(p', w') > w$. Since Walras' law was also satisfied under the old prices we have $p \cdot x(p, w) = w$. Combining these two expressions implies that:

$$p \cdot [x(p', w') - x(p, w)] > 0$$

Putting all of this together gives the result we are looking for: $p' \cdot [x(p', w') - x(p, w)] - p \cdot [x(p', w') - x(p, w)] < 0$.

Second, we show that the weak axiom is implied by the inequality $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$. This proceeds in two steps. First, we show that **the weak axiom holds if and only if it holds for all COMPENSATED price changes**. That is, the weak axiom holds *in general* if, for *any* two price-wealth pairs, (p, w) and (p′, w′), whenever

$p \cdot x(p', w') = w$ and $x(p', w') \neq x(p, w)$, then $p' \cdot x(p, w) > w'$. [Here the price-wealth change is a move from p' and w' to p and w.] The proof for this is on pp.31-32 of Mas-Colell et al.

Thus to show that $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$ implies the weak axiom, we need only consider changes in price-wealth pairs that are compensated changes. We will use a proof by contradiction; i.e. of the weak axiom does not hold then the above inequality cannot hold.

Suppose the weak axiom does not hold. Then there exists a compensated price change from some (p', w') to some (p, w) such that $x(p', w') \neq x(p, w)$, $p \cdot x(p', w') = w$, and $p' \cdot x(p, w) \leq w'$. But since all values of x satisfy Walras' law, these two expressions imply:

$$p \cdot [x(p', w') - x(p, w)] = 0 \quad \textbf{and} \quad p' \cdot [x(p', w') - x(p, w)] \geq 0$$

These imply that:

$$(p' - p) \cdot [x(p', w') - x(p, w)] \geq 0 \quad \textbf{and} \quad x(p', w') \neq x(p, w)$$

But this contradicts the inequality being considered, that is $(p' - p) \cdot [x(p', w') - x(p, w)] < 0$ if $x(p', w') \neq x(p, w)$. **Q.E.D.**

IV. The Compensated Law of Demand

To see a key implication of Proposition 2.F.1, write it as:

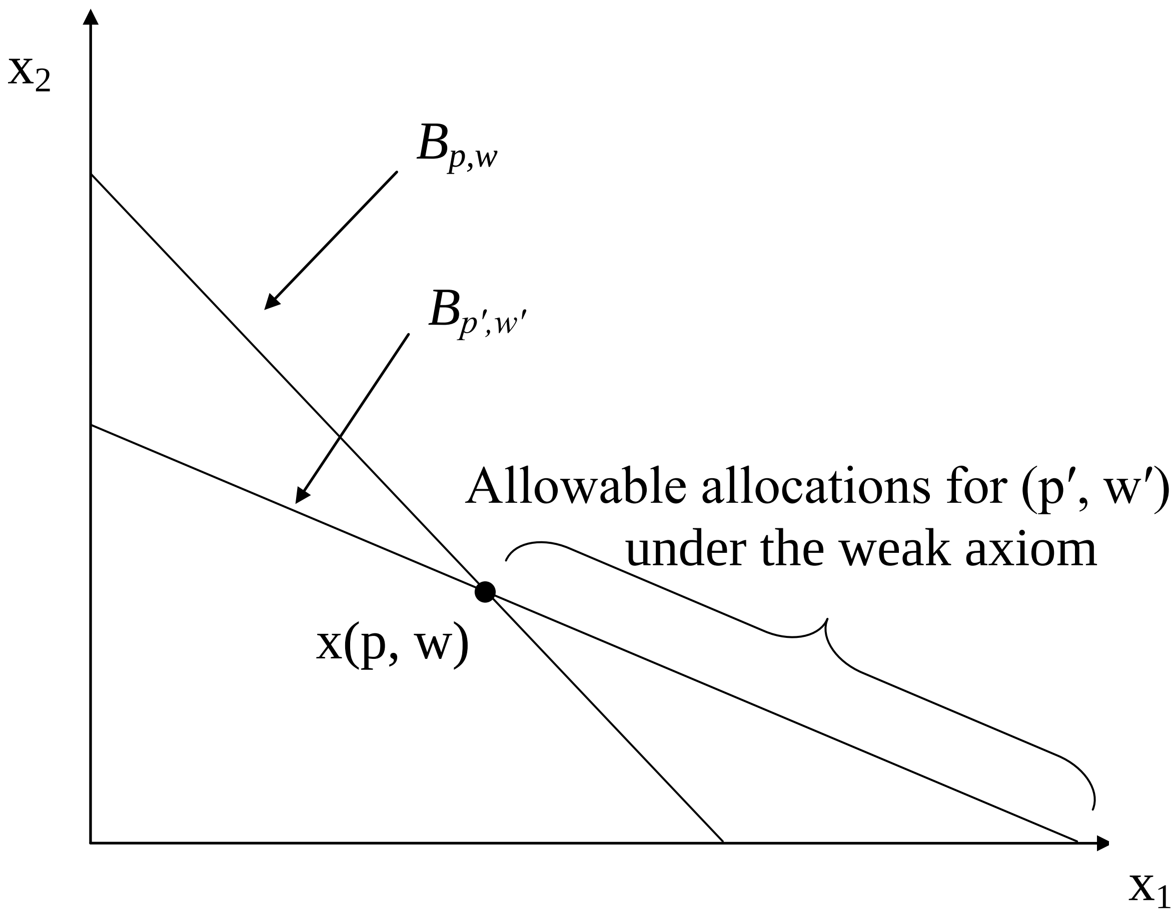
$$\Delta p \cdot \Delta x \leq 0$$

where $\Delta p = p' - p$ and $\Delta x = x(p', w') - x(p, w)$. This inequality demonstrates that **demand and price tend to move in opposite directions if the change in price is compensated** for by a change in wealth that allows the original bundle of goods to be purchased under the new prices. The intuition is: an increase in price reduces demand. Thus this inequality will be referred to as the **compensated law of demand**.

In the simplest case there is only one good that changes price, good ℓ. Then $\Delta p_i = 0$ for all $i \neq \ell$, and the compensated law of demand becomes $\Delta p_\ell \Delta x_\ell \leq 0$.

Question: For this simplest case, is it possible that $\Delta p_\ell \Delta x_\ell = 0$, or must it be that $\Delta p_\ell \Delta x_\ell < 0$?

This simple case can be shown in a diagram for the case of two goods (L = 2). Start from an initial demand of x(p, w). Suppose that the price of good 1 decreases, but that w also decreases so that the consumer is just able to purchase x(p, w) under the new price and wealth pair (p′, w′). This is shown in the following diagram:



The choices along the budget line $B_{p',w'}$ that are to the upper left of the point x(p, w) were not chosen under the budget line $B_{p,w}$, so there is no reason for the consumer to choose them now under the budget line $B_{p',w'}$. Thus the weak axiom of revealed preference (together with Walras' law) stipulates that the choice under the budget line $B_{p',w'}$ includes only the point x(p, w) and points along $B_{p',w'}$ to the lower right of x(p, w). **The weak axiom implies that the (compensated) demand for some good x cannot decrease if its price, p, decreases**; it must either stay the same or (more likely) increase.

Question: What if the reduction in the price for good 1 had **not** been compensated, so that the demand function had been $x(p', w)$? Is it possible for the demand for good 1 to decrease when its price decreases? Hint: Draw the new budget line and see what the weak axiom has to say about that situation.

V. Differentiable Demand Functions

We can derive a **very useful result if** we assume that **the demand function x(p, w) is differentiable in both prices and wealth**. Let's start with a price-wealth pair (p, w) and see what happens with "small" changes in each element of the vector p, which we can denote by dp.

The first thing to note is that to **make this price change a compensated price change** we need to require the following change in w: $\mathbf{dw = x(p, w) \cdot dp}$. From Proposition 2.F.1 we have:

$$dp \cdot dx \leq 0$$

We can use the chain rule (total differentiation) to determine what happens to x(p, w) when we have a compensated price change:

$$dx = D_p x(p, w)dp + D_w x(p, w)dw$$

$$= D_p x(p, w)dp + D_w x(p, w)[x(p, w)\cdot dp]$$

$$= [D_p x(p, w) + D_w x(p, w)x(p, w)^T]dp$$

Inserting this expression for dx into $dp\cdot dx \leq 0$ yields:

$$dp\cdot[D_p x(p, w) + D_w x(p, w)x(p, w)^T]dp \leq 0$$

The expression in square brackets is an L×L matrix known as the **Slutsky matrix**, denoted by S(p, w):

$$S(p, w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

where $s_{\ell k}(p, w) = \dfrac{\partial x_\ell(p,w)}{\partial p_k} + \dfrac{\partial x_\ell(p,w)}{\partial w} x_k(p, w)$.

The Slutsky matrix is also called the **substitution matrix**, and each element in it is called a **substitution effect**.

These effects are called substitution effects because they show how demand changes in response to a price change **after adjusting wealth** so that the consumer can still purchase the original bundle. Recall (p.5 above) that price changes have two effects, on relative prices and on "real" wealth. The substitution effect "controls for" the wealth effect and thus can be seen as a "pure" price effect.

More formally, the effect of a change in the price of good k on the demand for good ℓ that does not allow for changes in wealth is simply $(\partial x_\ell(p, w)/\partial p_k)dp_k$. This is the first part of the substitution effect $s_{\ell k}(p, w)$. **To change the consumer's wealth so that he or she can just afford the old bundle at the new prices requires a wealth change of the amount $x_k(p, w)dp_k$.** The effect of this wealth change on the demand for good ℓ is then $(\partial x_\ell(p, w)/\partial w)\times[x_k(p, w)dp_k]$. This is the second part of the substitution effect. In other words, combining these two effects yields $s_{\ell k}(p, w)dp_k$.

These results can be expressed as follows:

Proposition 2.F.2: If a differentiable Walrasian demand function x(p, w) satisfies Walras' law, homogeneity of degree zero, and the weak axiom of revealed preference, then for any price-wealth pair (p, w) the Slutsky matrix S(p, w) satisfies $v \cdot S(p, w)v \leq 0$ for any $v \in \mathbb{R}^L$.

Any matrix S(p, w) satisfying the property $v \cdot S(p, w)v \leq 0$ is **negative semidefinite** (it is **negative definite** if " $\leq$ " is replaced by " $<$ "). One implication of S(p, w) being negative semidefinite is that all the diagonal elements of that matrix, denoted by $s_{\ell\ell}(p, w)$, must be ≤ 0. In words,

the substitution effect of a good with respect to its own prices is always nonpositive. Another implication is that a Giffen good must always be inferior: $\partial x(p, w)/\partial w < 0$.

Note that nothing assumed so far implies that the substitution matrix $S(p, w)$ is symmetric. (It must be symmetric if $L = 2$, but not if $L > 2$.) We will see in future lectures that if we assume rational preferences, which is slightly stronger than the weak axiom, then we do get symmetry of the substitution matrix.

There is one more property of the substitution matrix that is worth pointing out:

Proposition 2.F.3: Suppose that a Walrasian demand function $x(p, w)$ is differentiable, homogenous of degree zero, and satisfies Walras' law. Then $p \cdot S(p, w) = 0$ and $S(p, w)p = 0$ for any price-wealth pair (p, w).

The proof may be a homework exercise.

This implies that the substitution matrix $S(p, w)$ is **singular**, i.e. it has a rank less than L (which means that one or more if its columns is a linear combination of the others, and the same it true for the rows). This means that the **substitution matrix can never be negative definite**. (There exists a vector v, i.e. p, for which $v \cdot S(p, w)v = 0$.)

A final point is that although the weak axiom implies that the substitution matrix S(p, w) is negative semidefinite it is not necessarily the case that a system of demand functions for which the substitution matrix S(p, w) is negative semidefinite always satisfies the weak axiom of revealed preference. The sufficient condition for this is that $v \cdot S(p, w)v < 0$ for any $v \neq \alpha p$, where α is any real number. See p.35 of Mas-Colell et al. for further discussion.

V. The Weak Law and Maximization of Preferences

How does a theory of consumer demand that is based only on homogeneity of degree zero, Walras' law and the weak axiom of revealed preference compare to a demand theory based on maximizing rational preferences?

Recalling Proposition 1.D.2 from Lecture 1, you might think that the two are equivalent. But in fact that proposition does not apply because Walrasian budgets do not include all possible budgets, in particular they may not include all possible sets with three bundles of commodities. More generally, **Walrasian demand functions that** are homogenous of degree zero, satisfy Walras' law and **satisfy the weak axiom do not imply demand functions derived from maximizing rational preferences.** The former are based on weaker assumptions than the latter. The following example makes this point:

Example: Consider a world of three commodities, with three budget sets that correspond to three prices vectors:

$$p^1 = (2, 1, 2)$$
$$p^2 = (2, 2, 1)$$
$$p^3 = (1, 2, 2)$$

Let wealth (w) = 8. Suppose that the demands corresponding to these prices (and this wealth) are

$$x^1 = (1, 2, 2)$$
$$x^2 = (2, 1, 2)$$
$$x^3 = (2, 2, 1)$$

You can show that any two of these pairs of choices follow the weak axiom of revealed preference, but that these choices are **not rational** in that x^3 is “revealed preferred” to x^2, x^2 is “revealed preferred” to x^1, and x^1 is “revealed preferred” to x^3. (See the definition of “revealed preferred” on p.9 of Lecture 1.) This issue will be discussed again in a later lecture.

VI. The Main Conclusions from This Lecture

The 3 main conclusions to remember from this lecture are:

1. The weak axiom of revealed preference, when combined with a set of demand functions that are homogenous of degree zero and satisfy Walras' law, are equivalent to the **compensated law of demand**:

   For any compensated price change from an initial situation (p, w) to a new price-wealth pair (p', w') that satisfies $w' = p' \cdot x(p, w)$, we have:

   $$(p' - p) \cdot [x(p', w') - x(p, w)] = \Delta p \cdot \Delta x \leq 0$$

   with the strict inequality holding if $x(p', w') \neq x(p, w)$.

2. The compensated law of demand implies that the substitution matrix $S(p, w)$ is negative semidefinite.

3. However, these assumptions do **not** imply that the substitution matrix $S(p, w)$ is symmetric, although it is symmetric when there are only two goods ($L = 2$).