MICHIGAN STATE UNIVERSITY

The OLS estimators for β_0 and β_1

$$\boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{N}(x_i-\bar{x})^2}} \qquad \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}}$$

Work through proof. **This is a proof you should know.**

or

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N}\sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2}$$

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$$\min_{\hat{\beta}_0, \hat{\beta}_1} \sum_{i=1}^{N} \hat{u}_i^2 = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{N} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

F.O.C.

$$\frac{\partial(\cdot)}{\partial \hat{\beta}_0} = \sum_{i=1}^{N} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-1) = -2\sum_{i=1}^{N}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

Aside:

$$\frac{1}{N}\sum_{i=1}^{N} y_i = \bar{y} \Rightarrow \sum_{i=1}^{N} y_i = N\bar{y}$$

likewise, $\sum_{i=1}^{N} x_i = N\bar{x}$

$$\sum_{i=1}^{N}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\underbrace{\left(\sum_{i=1}^{N} y_i\right)}_{N\bar{y}} - N\hat{\beta}_0 - \hat{\beta}_1 \underbrace{\sum_{i=1}^{N} x_i}_{N\bar{x}} = 0$$

$$\bar{y} - \hat{\beta}_0 - \hat{\beta}_1\bar{x} = 0 \Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}}$$

See next page for $\hat{\beta}_1$

$$\frac{\partial(\cdot)}{\partial \hat{\beta}_1} = \sum_{i=1}^{N} 2(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)(-x_i) = 0$$

$$= -2\sum_{i=1}^{N} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\sum_{i=1}^{N} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$$

$$\sum_{i=1}^{N} x_i y_i - \hat{\beta}_0 \sum_{i=1}^{N} x_i - \hat{\beta}_1 \sum_{i=1}^{N} x_i^2 = 0$$

$$\sum_{i=1}^{N} x_i y_i = \underbrace{\hat{\beta}_0}_{=\bar{y}-\hat{\beta}_1\bar{x}} \sum_{i=1}^{N} x_i + \hat{\beta}_1 \sum_{i=1}^{N} x_i^2$$

$$\sum_{i=1}^{N} x_i y_i = (\bar{y} - \hat{\beta}_1 \bar{x}) \sum_{i=1}^{N} x_i + \hat{\beta}_1 \sum_{i=1}^{N} x_i^2$$

$$\sum_{i=1}^{N} x_i y_i = \bar{y} \sum_{i=1}^{N} x_i - \hat{\beta}_1 \bar{x} \sum_{i=1}^{N} x_i + \hat{\beta}_1 \sum_{i=1}^{N} x_i^2$$

Solve for $\hat{\beta}_1$:

$$\sum_{i=1}^{N} x_i y_i - \bar{y} \sum_{i=1}^{N} x_i = \hat{\beta}_1 \left[\sum_{i=1}^{N} x_i^2 - \bar{x} \sum_{i=1}^{N} x_i \right]$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N} x_i y_i - \bar{y} \sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i^2 - \bar{x} \sum_{i=1}^{N} x_i} = \frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N} \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N} \left(\sum_{i=1}^{N} x_i \right)^2}$$

b/c $\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i$

$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$

Can be simplified to:

$$\boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}}$$

See 1 on next page for $\sum x_i y_i - \bar{y} \sum x_i = \sum (x_i - \bar{x})(y_i - \bar{y})$

see 2 on next page for $\sum x_i^2 - \bar{x} \sum x_i = \sum (x_i - \bar{x})^2$

Intermediate steps for showing the equivalence of our two formulas for $\hat{\beta}_0$ and $\hat{\beta}_1$

1.

$$\sum_{i=1}^{n}(y_i-\overline{y})(x_i-\overline{x}) = \sum_{i=1}^{n}(y_ix_i-\overline{x}y_i-\overline{y}x_i+\overline{yx}) = \sum_{i=1}^{n}y_ix_i-\overline{x}\sum_{i=1}^{n}y_i-\overline{y}\sum_{i=1}^{n}x_i+n\overline{yx}$$

$$=\sum_{i=1}^{n}y_ix_i-n\overline{xy}-\overline{y}\sum_{i=1}^{n}x_i+n\overline{yx}=\sum_{i=1}^{n}y_ix_i-\overline{y}\sum_{i=1}^{n}x_i$$

2.

$$\sum_{i=1}^{n}(x_i-\overline{x})^2=\sum_{i=1}^{n}(x_i^2-2\overline{x}x_i+\overline{xx})^2=\sum_{i=1}^{n}x_i^2-2\overline{x}\sum_{i=1}^{n}x_i+n\overline{xx}$$

$$=\sum_{i=1}^{n}x_i^2-2n\overline{x}^2+n\overline{x}^2=\sum_{i=1}^{n}x_i^2-\overline{x}\sum_{i=1}^{n}x_i$$