Econ 8010 HW4
Solutions

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1. Consider the following normal form game.

			2	
		a	b	c
	A	$4,5$	$0,2$	$0,0$
1	B	$3,2$	$3,0$	$3,2$
	C	$0,0$	$0,2$	$4,4$

   (a) Find all rationalizable strategies.

   - We proceed by iterated removal of never-best-responses. First note that b is strictly dominated by $\frac{3}{7}a + \frac{4}{7}c$, since

$$\begin{aligned} u_2(\frac{3}{7}a + \frac{4}{7}c, \mu_2) &= \frac{15}{7}\mu_2(A) + 2\mu_2(B) + \frac{16}{7}\mu_2(C) \\ &> u_2(b, \mu_2) = 2\mu_2(A) + 2\mu_2(C) \end{aligned}$$

   for all beliefs μ_2 of player 2. So we can remove b. Consider the incentives of player 1 in the game which remains after removal of b.

$$\begin{aligned}
u_1(B,\mu_1) \geq u_1(A,\mu_1) &\Leftrightarrow 3(\mu_1(a)+\mu_1(c)) \geq 4\mu_1(a) \\
&\Leftrightarrow 3 \geq 4\mu_1(a) \\
&\Leftrightarrow \frac{3}{4} \geq \mu_1(a) \\
u_1(B,\mu_1) \geq u_1(C,\mu_1) &\Leftrightarrow 3(\mu_1(a)+\mu_1(c)) \geq 4\mu_1(c) \\
&\Leftrightarrow 3 \geq 4-4\mu_1(a) \\
&\Leftrightarrow \mu_1(a) \geq \frac{1}{4} \\
u_1(A,\mu_1) \geq u_1(C,\mu_1) &\Leftrightarrow \mu_1(a) \geq \mu_1(c) \\
&\Leftrightarrow \mu_1(a) \geq \frac{1}{2}
\end{aligned}$$

We see that

- Each of player 1's pure strategies is a best response to some beliefs. So each survive this round of removal.
- There are no beliefs at which player 1 is indifferent between A and C. Thus, any mixed strategy which places positive probability on both A and C is never a best response, and we can remove it.
- Mixed strategies which place positive probability on A and B and those which place positive probability on B and C are best responses to $\mu_1(a) = \frac{3}{4}$ and $\mu_1(a) = \frac{1}{4}$, respectively.

Note that any beliefs are still reasonable for player 2, since all three of player 1's pure strategies remain. Consider player 2's incentives. We have

$$\begin{aligned}
u_2(a,\mu_2) \geq u_2(c,\mu_2) &\Leftrightarrow 5\mu_2(A)+2\mu_2(B) \geq 2\mu_2(B)+4\mu_2(C) \\
&\Leftrightarrow \mu_2(A) \geq \frac{4}{5}\mu_2(C)
\end{aligned}$$

So each of player 2's remaining strategies (pure and mixed) is a best response to some beliefs which place positive probability only on player 1's remaining strategies.

- Thus, all remaining strategies are rationalizable:

$$R_1 = \{\sigma_1 \in \Delta S_1 \mid \sigma_1(A) = 0 \text{ or } \sigma_1(C) = 0\}$$
$$R_2 = \{\sigma_2 \in \Delta S_2 \mid \sigma_2(b) = 0\}$$

(b) Find all Nash equilibria.

- Pure strategy equilibria are (A, a) and (C, c).
- If player 2 plays a mixed strategy in equilibrium, we must have

$$\sigma_1(A) = \frac{4}{5}\sigma_1(C)$$

  Since strategies for player 1 which place positive probability on both A and C are not rationalizable, this requires $\sigma_1 = B$. This is a best response for player 1 to σ_2 with $\frac{1}{4} \leq \sigma_2(a) \leq \frac{3}{4}$.
- If player 2 plays a pure strategy in equilibrium, player 1's best responses are unique. So no other mixed strategy equilibria exist.
- Thus the set of Nash equilibria is given by

$$\left\{(A, a), (C, c), (B, \alpha a + (1-\alpha)c) \mid \alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$$

2. Consider the following normal form game.

		2		
		a	b	c
	A	4, 4	0, 2	0, 0
1	B	3, 2	3, 0	3, 2
	C	0, 0	0, 2	4, 4

(a) Find all rationalizable strategies.

- Consider once again elimination of never-best-responses.
- b is no longer strictly dominated — only weakly dominated. Thus, we can no longer eliminate it in the first round.

- Consider the incentives of player 1.

$$\begin{aligned}
u_1(B,\mu_1) \geq u_1(A,\mu_1) &\Leftrightarrow 3(\mu_1(a)+\mu_1(b)+\mu_1(c)) \geq 4\mu_1(a) \\
&\Leftrightarrow \frac{3}{4} \geq \mu_1(a) \\
u_1(B,\mu_1) \geq u_1(C,\mu_1) &\Leftrightarrow 3(\mu_1(a)+\mu_1(b)+\mu_1(c)) \geq 4\mu_1(c) \\
&\Leftrightarrow \frac{3}{4} \geq \mu_1(c) \\
u_1(A,\mu_1) \geq u_1(C,\mu_1) &\Leftrightarrow \mu_1(a) \geq \mu_1(c)
\end{aligned}$$

  Once again, A and C are never simultaneously best responses to any beliefs. So any mixed strategy which places positive probability on both is not rationalizable. But once again, any other mixed strategy is a best response to some beliefs.

- For player 2, observe that any mixed strategy is a best response to $\mu_2 = \frac{1}{2}A + \frac{1}{2}C$.
- So the set of rationalizable strategies is given by

$$\begin{aligned}
&R_1 = \{\sigma_1 \in \Delta S_1 \mid \sigma_1(A) = 0 \text{ or } \sigma_1(C) = 0\} \\
&R_2 = \Delta S_2
\end{aligned}$$

(b) Find all Nash equilibria.

- Again, pure strategy equilibria are (A,a) and (C,c).
- If player 2 mixes between a, b, and c, then

$$\begin{aligned}
u_2(a,\sigma_1) = u_2(b,\sigma_1) &\Leftrightarrow 4\sigma_1(A) + 2\sigma_1(B) = 2\sigma_1(A) + 2\sigma_1(C) \\
&\Leftrightarrow 2\sigma_1(A) + 2\sigma_1(B) = 2\sigma_1(C) \\
u_2(c,\sigma_1) = u_2(b,\sigma_1) &\Leftrightarrow 4\sigma_1(C) + 2\sigma_1(B) = 2\sigma_1(A) + 2\sigma_1(C) \\
&\Leftrightarrow 2\sigma_1(C) + 2\sigma_1(B) = 2\sigma_1(A) \\
\Rightarrow 2\sigma_1(C) + 4\sigma_1(B) &= 2\sigma_1(C) \\
\Rightarrow \sigma_1(B) &= 0 \\
\Rightarrow \sigma_1(A) &= \sigma_1(C) = \frac{1}{2}
\end{aligned}$$

which is not rationalizable, and so cannot be played in any Nash equilibrium. So there is no equilibrium where player 2 mixes between all three strategies.

- If player 2 mixes between a and b, then

$$\begin{aligned} u_2(a,\sigma_1) = u_2(b,\sigma_1) &\Leftrightarrow 4\sigma_1(A) + 2\sigma_1(B) = 2\sigma_1(A) + 2\sigma_1(C) \\ &\Leftrightarrow 2\sigma_1(A) + 2\sigma_1(B) = 2\sigma_1(C) \end{aligned}$$

  Since mixed strategies which place positive probability on both A and C are not rationalizable, this implies $\sigma_1(B) = \sigma_1(C) = \frac{1}{2}$. But mixing between B and C is a best response for player 1 only when $\sigma_2(c) = \frac{3}{4}$. So there is no equilibrium where player 2 mixes between a and b.

- By symmetry, there is no equilibrium where player 2 mixes between b and c.

- If player 2 mixes between between a and c, we must have

$$\sigma_1(A) = \sigma_1(C)$$

  Since strategies for player 1 which place positive probability on both A and C are not rationalizable, this requires $\sigma_1 = B$. This is a best response for player 1 to σ_2 with $\frac{1}{4} \leq \sigma_2(a) \leq \frac{3}{4}$.

- If player 2 plays a pure strategy in equilibrium, player 1's best responses are unique. So no other mixed strategy equilibria exist.

- Thus the set of Nash equilibria is once again given by

$$\left\{ (A,a), (C,c), (B, \alpha a + (1-\alpha)c) \mid \alpha \in \left[\frac{1}{4}, \frac{3}{4}\right] \right\}$$

3. Consider the following normal form game.

				2		
		a	b	c	d
	A	6,0	3,3	1,5	6,3
1	B	4,1	4,1	2,2	7,1
	C	2,4	2,3	5,2	2,0
	D	5,3	2,3	0,4	5,3

(a) Find all rationalizable strategies.

- We proceed by iterated strict dominance.
- d is strictly dominated by c.
- In the game that remains, D is strictly dominated by A.
- No other pure strategies are strictly dominated.
- Consider now the incentives of player 1 in the game that remains.

$$\begin{aligned}
u_1(B,\mu_1) \geq u_1(A,\mu_1) &\Leftrightarrow 4\mu_1(a)+4\mu_1(b)+2\mu_1(c) \geq 6\mu_1(a)+3\mu_1(b)+\mu_1(c) \\
&\Leftrightarrow \mu_1(b)+\mu_1(c) \geq 2\mu_1(a) \\
&\Leftrightarrow 1-\mu_1(a) \geq 2\mu_1(a) \\
&\Leftrightarrow \frac{1}{3} \geq \mu_1(a) \\
u_1(B,\mu_1) \geq u_1(C,\mu_1) &\Leftrightarrow 4\mu_1(a)+4\mu_1(b)+2\mu_1(c) \geq 2\mu_1(a)+2\mu_1(b)+5\mu_1(c) \\
&\Leftrightarrow 2\mu_1(a)+2\mu_1(b) \geq 3\mu_1(c) \\
&\Leftrightarrow 2-2\mu_1(c) \geq 3\mu_1(c) \\
&\Leftrightarrow \frac{2}{5} \geq \mu_1(c) \\
u_1(A,\mu_1) \geq u_1(C,\mu_1) &\Leftrightarrow 6\mu_1(a)+3\mu_1(b)+\mu_1(c) \geq 2\mu_1(a)+2\mu_1(b)+5\mu_1(c) \\
&\Leftrightarrow 4\mu_1(a)+\mu_1(b) \geq 4\mu_1(c) \\
&\Leftrightarrow 4\mu_1(a)+1-\mu_1(a)-\mu_1(c) \geq 4\mu_1(c) \\
&\Leftrightarrow 3\mu_1(a)+1 \geq 5\mu_1(c)
\end{aligned}$$

So at $\mu_1 = (\frac{1}{3}, \frac{4}{15}, \frac{2}{5})$, player 1 is indifferent between all pure strategies. So all mixed strategies are best responses and survive this round of removal.

- Consider the incentives of player 2 in the game that remains.

$$\begin{aligned}
u_2(b, \mu_2) \geq u_2(a, \mu_2) &\Leftrightarrow 3\mu_2(A) + \mu_2(B) + 3\mu_2(C) \geq \mu_2(B) + 4\mu_2(C) \\
&\Leftrightarrow 3\mu_2(A) \geq \mu_2(C) \\
u_2(b, \mu_2) \geq u_2(c, \mu_2) &\Leftrightarrow 3\mu_2(A) + \mu_2(B) + 3\mu_2(C) \geq 5\mu_2(A) + 2\mu_2(B) + 2\mu_2(C) \\
&\Leftrightarrow \mu_2(C) \geq 2\mu_2(A) + \mu_2(B) \\
u_2(a, \mu_2) \geq u_2(c, \mu_2) &\Leftrightarrow \mu_2(B) + 4\mu_2(C) \geq 5\mu_2(A) + 2\mu_2(B) + 2\mu_2(C) \\
&\Leftrightarrow 2\mu_2(C) \geq 5\mu_2(A) + \mu_2(B)
\end{aligned}$$

So at $\mu_2 = (\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$, player 2 is indifferent between all pure strategies. So all mixed strategies are best responses and survive this round of removal.

- Thus, all remaining strategies are rationalizable:

$$\begin{aligned}
R_1 &= \{\sigma_1 \in \Delta S_1 \mid \sigma_1(D) = 0\} \\
R_2 &= \{\sigma_2 \in \Delta S_2 \mid \sigma_2(d) = 0\}
\end{aligned}$$

(b) Find all Nash equilibria.

- There are no pure strategy equilibria.
- If player 1 mixes between A, B, and C, we must have $\sigma_2 = \frac{1}{3}a + \frac{4}{15}b + \frac{2}{5}c$. This means we must have $\sigma_1 = \frac{1}{5}A + \frac{1}{5}B + \frac{3}{5}C$.
- If player 1 mixes between A and B, we must have $\sigma_2(a) = \frac{1}{3}$ and $\sigma_2(c) \leq \frac{2}{5}$. We know that player 2 will not mix between a, b, and c unless player 1 does as well, so we must have $\sigma_2(b) = \frac{2}{3}$. For this to be a best response for player 2 requires $3\sigma_1(A) = \sigma_2(C)$. So there is no Nash equilibrium where player 1 mixes between A and B (only).
- If player 1 mixes between B and C, we must have $\sigma_2(c) = \frac{2}{5}$ and $\sigma_2(a) \leq \frac{1}{3}$. We know that player 2 will not mix between a, b, and c unless player 1 does as well, so we must have $\sigma_2(b) = \frac{3}{5}$. For this to be a best response for

player 2 requires $3\sigma_2(A) \geq \sigma_2(C)$. So there is no Nash equilibrium where player 1 mixes between B and C (only).

- If player 1 mixes between A and C, we must have $3\sigma_2(a) + 1 = 5\sigma_2(c)$ and $\sigma_2(a) \geq \frac{1}{3}$. This means that player 2 must mix between a and c. This would require

$$\begin{aligned}
u_2(a, \sigma_1) \geq u_2(b, \sigma_1) &\Leftrightarrow \sigma_1(C) \geq 3\sigma_1(A) \\
&\Leftrightarrow 1 - \sigma_1(A) \geq 3\sigma_1(A) \\
&\Leftrightarrow \frac{1}{4} \geq \sigma_1(A) \\
u_2(a, \sigma_1) = u_2(c, \sigma_1) &\Leftrightarrow 2\sigma_1(C) = 5\sigma_1(A) \\
&\Leftrightarrow 2 - 2\sigma_1(A) = 5\sigma_1(A) \\
&\Leftrightarrow \frac{2}{7} = \sigma_1(A)
\end{aligned}$$

  which is impossible. So there is no Nash equilibrium where player 1 mixes between A and C (only).

- If player 1 plays a pure strategy as part of a Nash equilibrium, that Nash equilibrium must be in pure strategies since player 2's best responses to pure strategies are unique. But we already know no such pure strategy equilibrium exists.

- Thus, the unique Nash equilibrium is

$$\left(\frac{1}{5}A + \frac{1}{5}B + \frac{3}{5}C, \frac{1}{3}a + \frac{4}{15}b + \frac{2}{5}c\right)$$

4. Two firms $i \in \{1, 2\}$ engage in price competition in a differentiated product market. That is, their strategies are prices for their product $p_i \geq 0$. Consumers view the two firms' products as substitutes (but not perfect substitutes). The demand for firm 1's product is given by

$$Q_1(p_1, p_2) = \max\{12 - 2p_1 + p_2, 0\}$$

and the demand for firm 2's product is given by

$$Q_2(p_1, p_2) = \max\{12 - 2p_2 + p_1, 0\}$$

Firm 1 and firm 2 each produce at constant marginal cost of 4. Thus, their payoffs when they play (p_1, p_2) are

$$\pi_1(p_1, p_2) = (p_1 - 4) \max\{12 - 2p_1 + p_2, 0\}$$
$$\pi_2(p_1, p_2) = (p_2 - 4) \max\{12 - 2p_2 + p_1, 0\}$$

Solve for the pure strategy Nash equilibrium.

- First, find each player's best response. We can do so by
  - (a) taking a first-order condition;
  - (b) making sure that the resulting price is nonnegative, for any of the other firm's prices;
  - (c) making sure that the resulting profits are nonnegative, so that the best response is not a corner solution.
- Firm i's first-order condition is

$$\begin{aligned} 12 - 2p_i + p_j - 2(p_i - 4) = 0 \\ 20 + p_j = 4p_i \\ 5 + \frac{p_j}{4} = p_i \end{aligned}$$

  This is indeed nonnegative for any p_j. It also provides profits which are strictly positive:

$$\begin{aligned} \pi_i(p_i, B_i(p_j)) &= (1 + p_j/4) \max\{12 - 10 - p_j/2 + p_j, 0\} \\ &= (1 + p_j/4) \max\{2 + p_j/2, 0\} \\ &= (1 + p_j/4)(2 + p_j/2) > 0 \end{aligned}$$

  So the best response is an interior solution to the firm's problem, and so is given by the firm's FOC.

- Solving for the intersection of the best response functions yields

$$\begin{aligned} 5 + \frac{5 + \frac{p_i}{4}}{4} &= p_i \\ \frac{25}{4} &= \frac{15}{16} p_i \\ 5 &= \frac{3}{4} p_i \\ \frac{20}{3} &= p_i \end{aligned}$$

So the unique pure strategy Nash equilibrium is $\left(\frac{20}{3}, \frac{20}{3}\right)$.