Econ 8010 HW6
Solutions

Nathan Yoder

University of Georgia

Fall 2017

1. Consider the following extensive form game of imperfect information played by players $i \in \{1, 2, 3\}$. The game takes place in five periods $t \in \{0, 1, 2, 3, 4\}$ and proceeds as follows.

   - At $t = 0$, Nature chooses a **state of the world** $\theta \in \{H, L\}$ according to the probability distribution ρ_0, where

     $$\rho_0(H) = \frac{1}{2}, \; \rho_0(L) = \frac{1}{2}.$$

     This state of the world corresponds to whether an investment will yield a **high** payoff (state H) or a **low** payoff (state L). It is not observed by any of the players.

   - At $t = 1$, Nature chooses a **signal** $r_1 \in \{h, \ell\}$ according to the probability distribution $\rho(\cdot|\theta)$, where

     $$\begin{aligned} \rho(h|H) &= \frac{3}{4}, & \rho(\ell|H) &= \frac{1}{4}, \\ \rho(h|L) &= \frac{1}{4}, & \rho(\ell|L) &= \frac{3}{4}. \end{aligned}$$

     This signal is observed by player 1 (only). Then, player 1 chooses an action $a_1 \in \{$*Invest, Don't Invest*$\}$. This action is observed by players 2 and 3.

- At $t = 2$, Nature chooses another signal $r_2 \in \{h, \ell\}$ according to $\rho(\cdot|\theta)$. This signal is observed by player 2 (only). Then, player 2 chooses an action $a_2 \in \{\textit{Invest, Don't Invest}\}$. This action is observed by player 3.
- At $t = 3$, Nature chooses yet another signal $r_3 \in \{h, \ell\}$ according to $\rho(\cdot|\theta)$. This signal is observed by player 3 (only). Then, player 3 chooses an action $a_3 \in \{\textit{Invest, Don't Invest}\}$.
- At $t = 4$, the game ends. Each player who chose *Invest* receives a payoff of 1 if $\theta = H$ and $-\frac{9}{10}$ if $\theta = L$. Each player who chose *Don't Invest* receives a payoff of zero, regardless of the state of the world.

Note that players' beliefs are only relevant insofar as they describe the probability placed on each state of the world. Thus, instead of writing beliefs as the probabilities of each decision node (e.g., $\mu_3(H, \ell, \textit{Don't Invest}, h, \textit{Invest}, \ell)$), we can simply write them as the probability of θ given the current information set (e.g., $\mu_3(H|\textit{Don't Invest}, \textit{Invest}, \ell)$).

(a) What are the unique Bayesian beliefs $\mu_1(H|h), \mu_1(H|\ell)$ for player 1 after $r_1 \in \{h, \ell\}$?

- $\mu_1(H|h) = \frac{3}{4}, \mu_1(H|\ell) = \frac{1}{4}$.

(b) In any weak sequential equilibrium, what action will player 1 take after signal h? After ℓ?

- $s_1(h) = \textit{Invest}, s_1(\ell) = \textit{Don't Invest}$.

(c) What are the unique Bayesian beliefs

$$\mu_2(H|\textit{Invest}, h), \mu_2(H|\textit{Invest}, \ell), \mu_2(H|\textit{Don't Invest}, h), \mu_2(H|\textit{Don't Invest}, \ell)?$$

- Player 2 can tell exactly what signal player 1 received by observing his action. So:

$$\mu_2(H|Invest, h) = \frac{\frac{3}{4}\frac{3}{4}\frac{1}{2}}{\frac{3}{4}\frac{3}{4}\frac{1}{2} + \frac{1}{4}\frac{1}{4}\frac{1}{2}} = \frac{9}{10}$$
$$\mu_2(H|Invest, \ell) = \frac{\frac{3}{4}\frac{1}{4}\frac{1}{2}}{\frac{3}{4}\frac{1}{4}\frac{1}{2} + \frac{1}{4}\frac{3}{4}\frac{1}{2}} = \frac{1}{2}$$
$$\mu_2(H|Don't\ Invest, h) = \frac{\frac{1}{4}\frac{3}{4}\frac{1}{2}}{\frac{1}{4}\frac{3}{4}\frac{1}{2} + \frac{3}{4}\frac{1}{4}\frac{1}{2}} = \frac{1}{2}$$
$$\mu_2(H|Don't\ Invest, \ell) = \frac{\frac{1}{4}\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{4}\frac{1}{2} + \frac{3}{4}\frac{3}{4}\frac{1}{2}} = \frac{1}{10}$$

(d) In any weak sequential equilibrium, what action will player 2 take after $(Invest, h)$? After $(Invest, \ell)$? After $(Don't\ Invest, h)$? After $(Don't\ Invest, \ell)$?

- Solution:

$$s_2(Invest, h) = Invest$$
$$s_2(Invest, \ell) = Invest$$
$$s_2(Don't\ Invest, h) = Invest$$
$$s_2(Don't\ Invest, \ell) = Don't\ Invest$$

(e) Solve for all weak sequential equilibria. (You only need to solve for beliefs about the state, not about individual decision nodes that follow the same state.)

- We already solved for the equilibrium strategies and beliefs of players 1 and 2.
- The histories $Invest, Don't\ Invest, h$ and $Invest, Don't\ Invest, \ell$ are off the equilibrium path, so player 3's beliefs and strategies there can be anything.

- For the other histories:

$$\mu_3(H|Invest, Invest, h) = \frac{\frac{3}{4}\frac{3}{4}\frac{1}{2}}{\frac{3}{4}\frac{3}{4}\frac{1}{2} + \frac{1}{4}\frac{1}{4}\frac{1}{2}} = \frac{9}{10}$$

$$\mu_3(H|Invest, Invest, \ell) = \frac{\frac{3}{4}\frac{1}{4}\frac{1}{2}}{\frac{3}{4}\frac{1}{4}\frac{1}{2} + \frac{1}{4}\frac{3}{4}\frac{1}{2}} = \frac{1}{2}$$

$$\mu_3(H|Don't\ Invest, Invest, h) = \frac{\frac{1}{4}\frac{3}{4}\frac{3}{4}\frac{1}{2}}{\frac{1}{4}\frac{3}{4}\frac{3}{4}\frac{1}{2} + \frac{3}{4}\frac{1}{4}\frac{1}{4}\frac{1}{2}} = \frac{3}{4}$$

$$\mu_3(H|Don't\ Invest, Invest, \ell) = \frac{\frac{1}{4}\frac{3}{4}\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{3}{4}\frac{1}{4}\frac{1}{2} + \frac{3}{4}\frac{1}{4}\frac{3}{4}\frac{1}{2}} = \frac{1}{4}$$

$$\mu_3(H|Don't\ Invest, Don't\ Invest, \ell) = \frac{\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{2} + \frac{3}{4}\frac{3}{4}\frac{3}{4}\frac{1}{2}} = \frac{1}{28}$$

$$\mu_3(H|Don't\ Invest, Don't\ Invest, h) = \frac{\frac{1}{4}\frac{1}{4}\frac{3}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{4}\frac{3}{4}\frac{1}{2} + \frac{3}{4}\frac{3}{4}\frac{1}{4}\frac{1}{2}} = \frac{1}{4}$$

$$s_3(Invest, Invest, h) = Invest$$

$$s_3(Invest, Invest, \ell) = Invest$$

$$s_3(Don't\ Invest, Invest, h) = Invest$$

$$s_3(Don't\ Invest, Invest, \ell) = Don't\ Invest$$

$$s_3(Don't\ Invest, Don't\ Invest, h) = Don't\ Invest$$

$$s_3(Don't\ Invest, Don't\ Invest, \ell) = Don't\ Invest$$

(f) In any weak sequential equilibrium, what is the probability of a history (a_1, a_2) occurring after which

i. player 3 invests, no matter what her signal r_3 is?

- In the observational learning literature, this is referred to as an **informational cascade on *Invest***. After an informational cascade occurs on an action, all subsequent players play that action, regardless of their private signal. This can lead to learning the correct state (in which case observational learning **succeeds**) or learning the incorrect state (in which case observational learning **fails**).

- This happens after *Invest*, *Invest*. The probability of this history occurring is $\frac{3}{4}$ when $\theta = H$ and $\frac{1}{4}$ when $\theta = L$. Each state is equally likely, so **the probability that an informational cascade on *Invest* occurs after two players is $\frac{1}{2}$.**

ii. player 3 does not invest, no matter what her signal r_3 is?

- This happens after *Don't Invest*, *Don't Invest*. The probability of this history occurring is $\frac{1}{16}$ when $\theta = H$ and $\frac{9}{16}$ when $\theta = L$. Each state is equally likely, so **the probability that an informational cascade on *Don't Invest* occurs after two players is $\frac{5}{16}$.**

iii. player 3 plays the "correct" action (i.e., *Invest* if the state is H or *Don't Invest* if the state is L), no matter what her signal r_3 is?

- This happens after H, *Invest*, *Invest*, which occurs with probability $\frac{3}{8}$, and L, *Don't Invest*, *Don't Invest*, which occurs with probability $\frac{9}{32}$. So **observational learning succeeds after two players with probability $\frac{21}{32}$.**

iv. player 3 plays the "incorrect" action (i.e., *Don't Invest* if the state is H or *Invest* if the state is L), no matter what her signal r_3 is?

- This happens after L, *Invest*, *Invest*, which occurs with probability $\frac{1}{8}$, and H, *Don't Invest*, *Don't Invest*, which occurs with probability $\frac{1}{32}$. So **observational learning fails after two players with probability $\frac{5}{32}$.**

2. Consider the following extensive form game of imperfect information played by two players: a pharmaceutical firm (player 1) and a regulator (player 2). Play proceeds as follows.

- **First,** nature chooses whether the firm's new drug has high effectiveness (θ_h) or low effectiveness (θ_ℓ) with equal probability. The firm observes the effectiveness of this choice, but the regulator does not.
- **Second,** the firm chooses whether to conduct a costly test of the drug's effectiveness (T) or not (N). If the firm declines to test, the game ends and both players get a payoff of zero. If the firm tests, it costs the firm \$1 million.

- If the firm chose to test the drug, Nature chooses whether the test succeeds (s) or fails (f) according to the probability distribution $\rho(\cdot|\theta)$, where

$$\rho(s|\theta_h) = \beta, \qquad \rho(f|\theta_h) = 1 - \beta,$$
$$\rho(s|\theta_\ell) = .05, \qquad \rho(f|\theta_\ell) = .95.$$

  for $\beta \in (.5, 1)$.

- After seeing whether the test is a success or failure, the regulator decides whether to approve the drug or deny approval. If the regulator approves the drug, the firm receives \$5 million in monopoly profits from selling it.

  In making its decision, the regulator works to maximize consumer welfare. If the regulator approves a drug which has high effectiveness, consumers are \$1 million better off. If the regulator approves a drug which has low effectiveness, consumers are \$1 million worse off.

This game can be thought of as a signaling game (with messages T and N) where the receiver receives additional information about the sender's type after T.

(a) Find all sequential equilibria of the game. (Hint: like standard signaling games, all weak sequential equilibria of this game are sequential equilibria.)

  - I realized after I wrote the assignment that this isn't true — in the equilibria where neither type tests, sequential equilibrium places some restrictions on the beliefs and actions of the regulator that weak sequential equilibrium does not. But restricting attention to sequential equilibria does not change the substantive predictions of the model. **Correctly giving either the set of sequential equilibria or the set of weak sequential equilibria will give you full credit on this problem.**
  - The first component of the set of weak sequential equilibria is where nei-

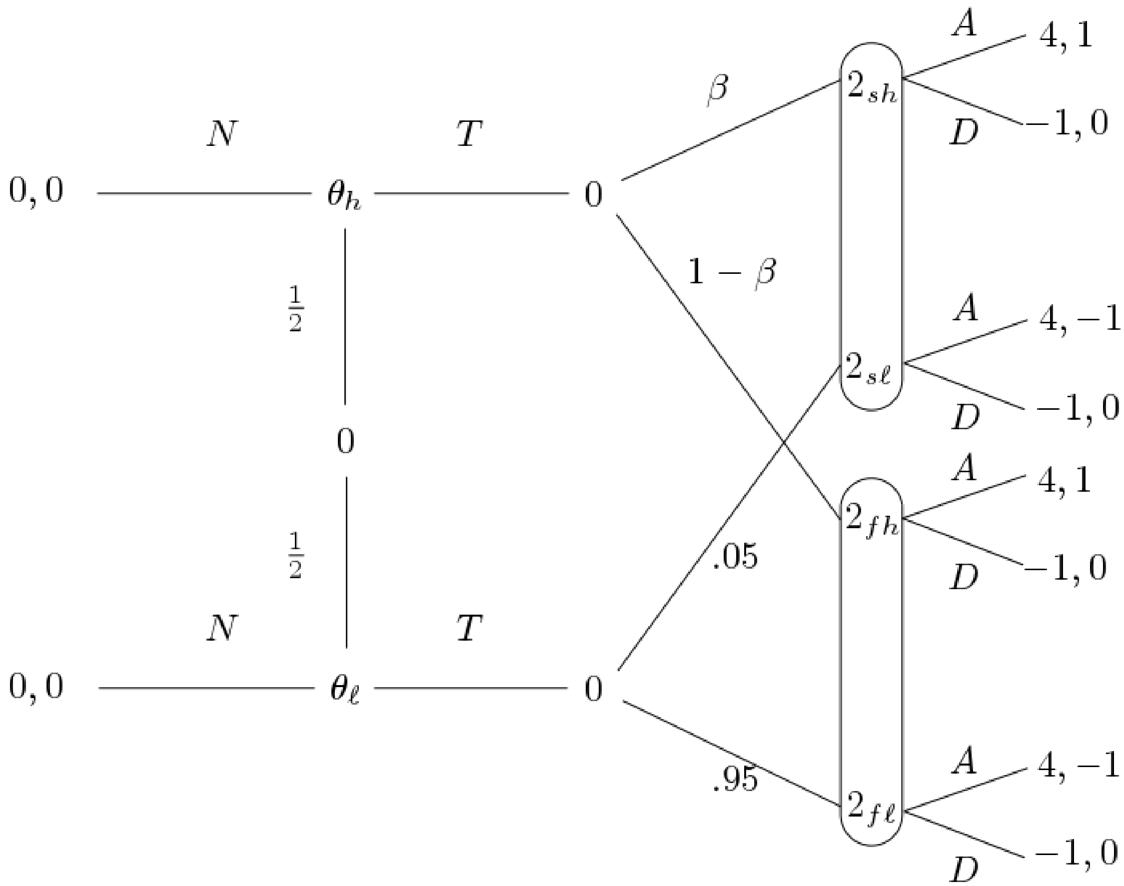

Figure 1: The game tree for the game described in Exercise 2.

ther type tests: $\sigma_h(T) = \sigma_\ell(T) = 0$. This requires

$$\beta(4\sigma(A|s) - (1 - \sigma(A|s))) + (1-\beta)(4\sigma(A|f) - (1 - \sigma(A|f))) \leq 0$$

$$(\beta\sigma(A|s) + (1-\beta)\sigma(A|f)) \leq \frac{1}{5} \quad (1)$$

$$.05(4\sigma(A|s) - (1 - \sigma(A|s))) + .95(4\sigma(A|f) - (1 - \sigma(A|f))) \leq 0$$

$$(.05\sigma(A|s) + .95\sigma(A|f)) \leq \frac{1}{5} \quad (2)$$

This can only be true if both $\sigma(A|s)$ and $\sigma(A|f)$ are less than one, so we must have $\mu_2(\theta_h|s) \leq \frac{1}{2}$ **and** $\mu_2(\theta_h|f) \leq \frac{1}{2}$**, and further** $\mu_2(\theta_h|s) = \frac{1}{2}$ **if** $\sigma(A|s) > 0$ **and** $\mu_2(\theta_h|f) = \frac{1}{2}$ **if** $\sigma(A|f) > 0$**. Together with (1), (2), and** $\sigma_h(T) = \sigma_\ell(T) = 0$**, this describes the first component of the set of weak sequential equilibria.**

- In addition, sequential equilibrium requires $\mu_2(\theta_h|s) < \mu_2(\theta_h|f)$ unless

$\mu_2(\theta_h|s) = \mu_2(\theta_h|f) = 0$. It follows that $\sigma(A|f) = 0$ and so (1) and (2) reduce to $\sigma(A|s) \leq \frac{1}{5\beta}$.

- Now we look for equilibria in which one or both types test.
- If the high type does not test, the regulator must believe in equilibrium that the drug is low effectiveness regardless of the outcome of the test, and so never approves the drug. Thus testing is suboptimal for the low type as well, a contradiction.
- If the low type does not test but the high type does, the regulator must believe in equilibrium that the drug is high effectiveness regardless of the outcome of the test, and so always approves the drug. Thus not testing is suboptimal for the low type, a contradiction.
- If both types test, the regulator must accept with positive probability after both success and failure, and so we must have $\mu_2(\theta_h|s) \geq \frac{1}{2}$ and $\mu_2(\theta_h|f) \geq \frac{1}{2}$. The unique Bayesian beliefs in this case are given by

$$\mu_2(\theta_h|s) = \frac{\beta\sigma_h(T)}{\beta\sigma_h(T) + .05\sigma_\ell(T)} = \frac{1}{1 + \frac{.05\sigma_\ell(T)}{\beta\sigma_h(T)}}$$
$$\mu_2(\theta_h|f) = \frac{(1-\beta)\sigma_\ell(T)}{(1-\beta)\sigma_\ell(T) + .95\sigma_\ell(T)} = \frac{1}{1 + \frac{.95\sigma_\ell(T)}{(1-\beta)\sigma_h(T)}}$$

  We have $\beta\sigma_h(T) > (1-\beta)\sigma_h(T)$ and $.05\sigma_\ell(T) < .95\sigma_\ell(T)$, so

$$\frac{.95\sigma_\ell(T)}{(1-\beta)\sigma_h(T)} > \frac{.05\sigma_\ell(T)}{\beta\sigma_h(T)}$$

  and thus we have $\mu_2(\theta_h|s) > \mu_2(\theta_h|f) \geq \frac{1}{2}$ in any equilibrium where both types test with positive probability. This means that:

  - $\sigma_2(A|s) = 1$, so $\sigma_h(T) = 1$
  - if $\sigma_\ell(T) = 1$, then we must have

$$\frac{1}{1 + \frac{.95}{(1-\beta)}} \geq \frac{1}{2}$$
$$1 \geq \frac{.95}{(1-\beta)}$$
$$1 - \beta \geq .95$$

a contradiction.

– Thus $\sigma_\ell(T) < 1$, so

$$\begin{aligned} 4(.05 + .95\sigma_2(A|f)) - .95(1 - \sigma_2(A|f)) &= 0 \\ 5(\frac{19}{20}\sigma_2(A|f)) &= \frac{15}{20} \\ \sigma_2(A|f) &= \frac{3}{19} \end{aligned}$$

which requires

$$\begin{aligned} \frac{19}{20}\sigma_\ell(T) &= 1 - \beta \\ \sigma_\ell(T) &= \frac{1-\beta}{19/20} \end{aligned}$$

- **Thus the second component of the set of weak sequential equilibria is given by**

$$\begin{aligned} \sigma_h(T) &= 1 \\ \sigma_\ell(T) &= \frac{1-\beta}{19/20} \\ \sigma_2(A|s) &= 1 \\ \sigma_2(A|f) &= \frac{3}{19} \\ \mu_2(\theta_h|s) &= \frac{\beta}{\beta + \frac{1-\beta}{19}} \\ \mu_2(\theta_h|f) &= \frac{1}{2} \end{aligned}$$

Clearly, this is also a sequential equilibrium.

(b) Explain how a change in β affects

i. the equilibrium probability that a high-effectiveness drug is accepted;

- I wasn't as clear as I could have been about what I was asking for here. I was asking for $\frac{d}{d\beta}P(A|\theta_h)$, but if you put $\frac{d}{d\beta}P(A, \theta_H) = \frac{d}{d\beta}P(A|\theta_h)/2$ that's fine.
- Clearly, β does not affect the probability of a high-effectiveness drug being accepted in the no-test equilibria.

- In the other equilibrium, $P(A|\theta_h) = \beta + (1-\beta)\frac{3}{19}$. So $\frac{d}{d\beta}P(A|\theta_h) = \frac{16}{19}$. That is, an increase in the probability that a high-effectiveness drug tests successfully increases the equilibrium probability that it is accepted by less than one.

ii. the equilibrium probability that a low-effectiveness drug is accepted;

- Again, β does not affect the probability of a low-effectiveness drug being accepted in the no-test equilibria.
- In the other equilibrium, $P(A|\theta_\ell) = \frac{1-\beta}{19/20}(\frac{1}{20} + \frac{19}{20}\frac{3}{19}) = (1-\beta)\frac{4}{19}$. So $\frac{d}{d\beta}P(A|\theta_\ell) = -\frac{4}{19}$. That is, an increase in the probability that a high-effectiveness drug tests successfully decreases the equilibrium probability that it a low-effectiveness drug is accepted, despite having no impact on whether a test of a low-effectiveness drug will be successful.

iii. the regulator's expected payoff in equilibrium.

- Again, β does not affect the regulator's expected payoff in the no-test equilibria.
- The regulator's expected payoff is just $\frac{1}{2}P(A|\theta_h) - \frac{1}{2}P(A|\theta_\ell)$. So based on (i) and (ii), $\frac{d}{d\beta}\frac{1}{2}P(A|\theta_h) - \frac{1}{2}P(A|\theta_\ell) = \frac{10}{19}$.

(c) Explain how the "intuitive criterion" of Cho and Kreps should be applied to this game. Which of the sequential equilibria satisfy it?

- There are a couple ways we could apply the intuitive criterion here, so I won't have this part of the question graded. Below is one:
- For $I \subseteq \{\theta_\ell, \theta_h\}$, let $BR^T(I)$ be the set of mixed strategies $(\sigma_2(A|s), \sigma_2(A|f))$ for the regulator after T that are a best response to some set of beliefs $(\mu_2(\theta_h|s), \mu_2(\theta_h|f))$ which **are Bayesian after some strategy profile** $(\sigma_h(T), \sigma_\ell(T))$ **such that** $\sigma_i(T) > 0$ **for all** $\theta_i \in I$ **and** $\sigma_i(T) = 0$ **for all** $\theta_i \notin I$**.** The rest of the definition is as usual, but with this definition of BR^T and with actions replaced by mixed strategies.
- In the no-test equilibrium, since $(0,0) \in BR^T(\{\theta_\ell, \theta_h\})$, D^T is empty, so the equilibrium satisfies the modified intuitive criterion.

- T is used in the other equilibrium, so it also satisfies the modified intuitive criterion.