Dynamic Games of Complete Information

I. Introduction

A. Timing of moves is a crucial element in many models of interest ("Timing is everything").
B. Introduce notion of dynamic games where there is an order to the moves of players.
C. Strategic incentive: an action at one stage of the game can change the conditions in future stages. With dynamic games, there is an incentive for a player to consider how action will affect future conditions and how this will affect future play of rival players. This is an additional element of strategy not present in static games
D. Use extensive form representation for dynamic game – represents the order of moves.
E. Example of dynamic game and the importance of timing: Cournot vs. Stackelberg game
   1. Basic duopoly setup
      a. Inverse market demand function: $P(Q) = 12 - Q$, $Q = q_1 + q_2$
      b. Assume that costs are zero: $C_1(q_1) = C_2(q_2) = 0$
   2. Simultaneous move game - Cournot equilibrium: Profit maximization for firm i:

      $$Max\ (12 - q_1 - q_2)q_i$$

      Take the first order condition and solve to find the reaction function:

      $$q_i(q_j) = \frac{12 - q_i}{2}$$

      Solve for Nash equilibrium in the Cournot game (where the two reaction functions cross):

      $$q_1 = q_2 = 4$$

      Profit for each firm:

      $$\Pi_i = (12 - q_1 - q_2)q_i = (12 - 4 - 4)4 = 16$$

   3. Stackelberg game: suppose that firm 1 moves first (Stackelberg leader) followed by firm 2 (Stackelberg follower). The leader knows how firm 2 will react to a given level of output by the leader:

      $$q_2(q_1) = \frac{12 - q_1}{2}$$

      The Stackelberg leader's problem is then:

      $$Max\ (12 - q_1 - q_2(q_1))q_1$$

      $$= Max\ (12 - q_1 - \frac{12 - q_1}{2})q_1$$

      $$Max\ (\frac{12 - q_1}{2})q_1$$

      The first order condition yields:

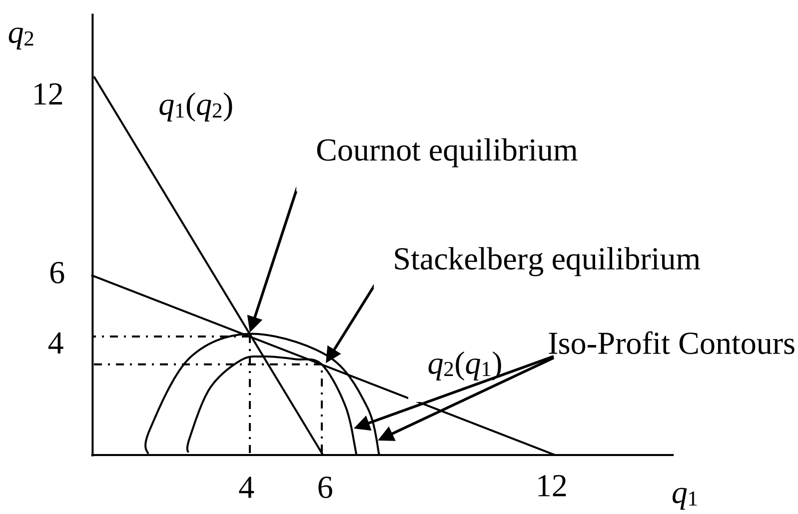
      $$\frac{12 - 2q_1}{2} = 0$$

      $$q_1 = 6$$

      Using firm 2's reaction function we can find the follower's output:

$$q_2(q_1) = \frac{12 - q_1}{2} = \frac{12 - 6}{2} = 3$$

4. Comparing Cournot and Stackelberg equilibrium
   a. Iso-profit contours: combinations of output of firm 1 and 2 that yield equal profit for firm *i*. For firm i: $\Pi_1(q_1, q_2) = \overline{\Pi}_1$.
   b. The slope of the iso-profit contour along the reaction function must be zero – by definition the reaction function is a best response. The iso-profit contour for firm 1 will just be tangent to the firm 2's reaction function at the Stackelberg equilibrium.
   c. Firm 1 gets to a higher value iso-profit contour at the Stackelberg equilibrium compared to the Cournot equilibrium.
   d. Firm 1 produces more output in the Stackelberg equilibrium and firm 2 produces less. Overall, production increases in Stackelberg equilibrium.
   e. Profit for firm 1 increases in Stackelberg equilibrium and profit for firm 2 decreases: $\Pi_1^S = (12 - 6 - 3)6 = 18$
      $\Pi_2^S = (12 - 6 - 3)3 = 9$



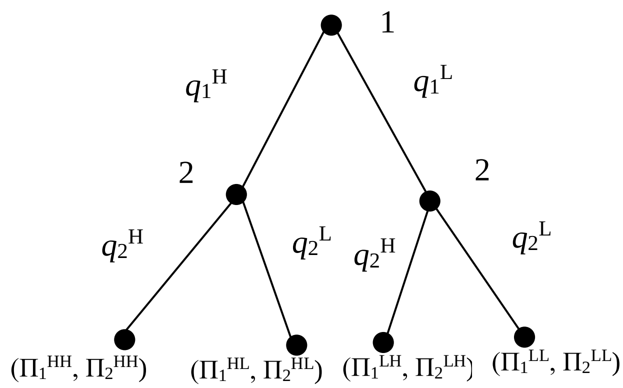
   f. Results:
      i. Timing matters for equilibrium outcomes
      ii. First mover advantage in this game. Commitment – stake out a position prior to other player. By producing more, the leader gets the follower to produce less.
      iii. Note: some games have a last mover advantage, e.g., matching pennies. It is not always advantageous to go first.

II. Extensive Form Representation of a Game: Game Tree

A. Elements of Game
   1. Set of players

2. Action/Strategies
3. Order of moves
4. Information structure – what do players know when they choose an action? (With games of complete information – this part is not important – won't discuss this until we get to games of incomplete information)
5. Payoffs

B. Simple example of a game tree
   1. Stackelberg game – 2 actions – high quantity, low quantity



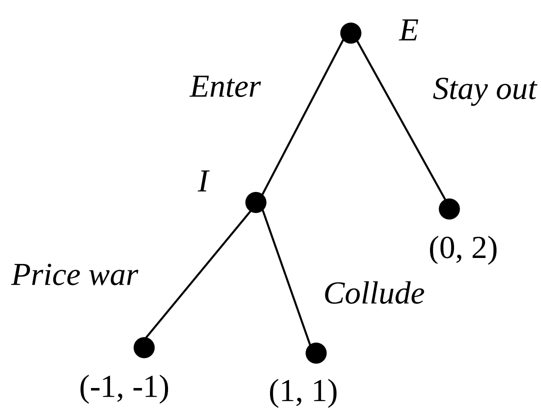
   2. Nodes
      a. Initial node – first decision in game
      b. Successor nodes – nodes that follow in the game tree
      c. Terminal nodes: end of game tree – associated with payoffs
      d. All nodes except terminal nodes are associated with a player
      e. Probabilities: moves of chance belong to a player called "nature"
   3. Branches: represent possible actions at a decision node. Trace the history of the game to the outcome
   4. Game tree is common knowledge.

C. Strategies in Extensive Form Games
   1. Pure strategies: contingent plan of how to play at **each** node of the player. Specify action at **all** nodes (not just nodes on equilibrium path). OFF EQUILIBRIUM PLAY MATTERS!
   2. Behavior (mixed strategies): specifies probabilities of each action at all nodes of the player.
   3. Example: Stackelberg game – each player has two actions $\{H, L\}$. Strategies:
      a. Leader $\{H, L\}$
      b. Follower $\{HH, HL, LH, LL\}$ where HH represents "If firm 1 plays H, firm 2 will play H, If firm 1 plays L, firm 2 will play H", and similarly for other strategies…
   4. Strategies are complete contingent mapping saying what the player will do at all decision nodes of the player. Strategies can get complicated quickly!

III. Nash Equilibrium and Subgame Perfect Nash Equilibrium

A. Nash Equilibrium:

1. Definition: a strategy s^* is a Nash equilibrium if for all $i \in N$
   $U_i(s_i^*, s_{\sim i}^*) \geq U_i(s_i, s_{\sim i}^*)$ for all $s_i \in S_i$.
2. Problem – too many Nash Equilibria – and not all Nash equilibria make sense in a dynamic game context
3. Example: duopoly game
   a. Cournot equilibrium is a Nash equilibrium – pair of best responses
   b. Stackelberg equilibrium is a Nash equilibrium – follower plays a best response to the leaders strategy and leader plays its best strategy given that they know how the follower will react to the leader's strategy.
   c. If the game is really one with a leader and a follower then Cournot equilibrium shouldn't be an equilibrium – but it is a Nash equilibrium. In the numerical example given above, if follower plays $q_2 = 4$, then leader should also play $q_1 = 4$.
4. Game of Entry
   a. Two players: potential entrant (*E*) and incumbent (*I*)
   b. Potential entrant can decide to either enter or stay out
   c. Incumbent: if entrant enters, the incumbent chooses either a price war or collusion



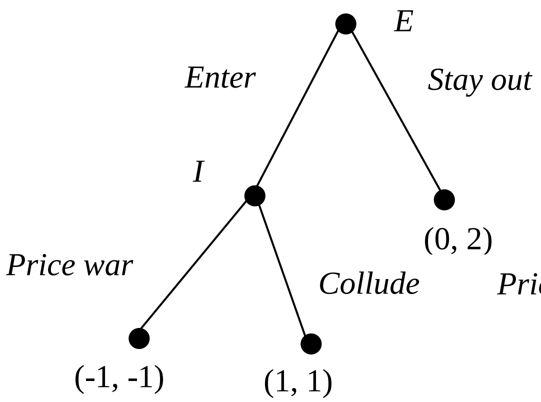
   4. Nash Equilibrium: (*enter*, *collude*) ; (*stay out*, *price war*)

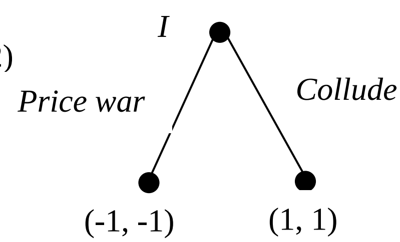
	Price war	*Collude*
Enter	-1, -1	1, 1
Stay out	0, 2	0, 2

   5. Are both Nash equilibria sensible? (*stay out*, *price war*) involves a threat that is not credible in the sense that it is not in the best interest of the incumbent to really have a price war if entry has occurred.

B. Subgame Perfect Nash Equilibrium
   1. Refinement of Nash Equilibrium: wish to rule out \ Nash equilibria that rely on incredible threats.
   2. Definition: A subgame of an extensive form game consists of a single node which is a singleton in every players' information set, that node's successor and the payoffs at the associated terminal nodes.
   3. Example: Game of entry Proper subgames:

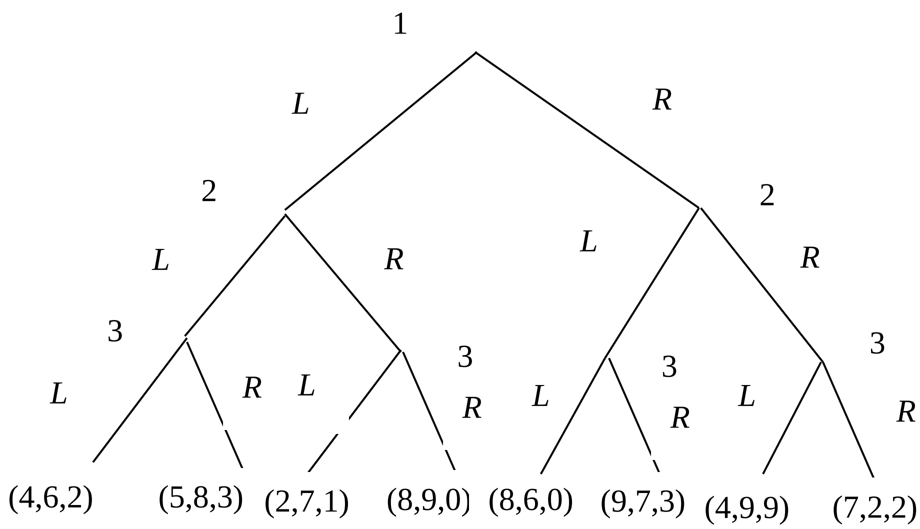
4. Definition: A set of strategies is a ***subgame perfect Nash Equilibrium*** if it is a Nash Equilibrium in all subgames.
5. Example: Game of entry.
   a. Two subgames: the game starting with the incumbent's node and the entire game
   b. In the subgame starting with the incumbent's decision, it is pretty easy to see that *collude* is the only Nash equilibrium.
   c. Therefore, (*enter*, *collude*) is the only subgame perfect Nash equilibrium for this game.



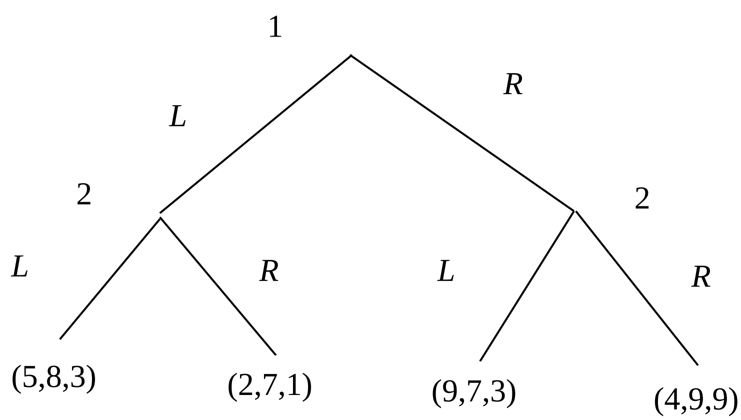


6. Example: Stackelberg game
   a. Subgames (lots of subgames):
      i. Entire game
      ii. Firm 2's choice given q_1.
      iii. Note: there is a different subgame for every choice of q_1.
   b. In a subgame starting with firm 2's choice after the choice of q_1, firm 2's best response is to play $q_2(q_1)$.
   c. So if $q_1 = 6$ in the initial example, then $q_2 = 3$; if $q_1 = 4$, then $q_2 = 4$.
   d. Given all possible combinations of q_1 and $q_2(q_1)$, firm 1 would choose $q_1 = 6$ in the first stage. The Stackelberg Equilibrium ($q_1 = 6$, $q_2(q_1)$) is the unique subgame perfect Nash Equilibrium.

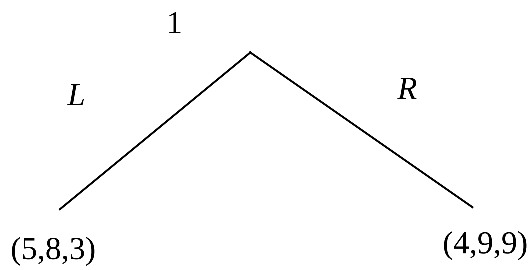
C. Method of Solution: Backward Induction
1. Replace any subgame with Nash Equilibrium payoffs for subgame. Start at the subgames closest to the end of the game and work backwards up the game tree to the initial node. Makes finding a solution (relatively) easy.
2. Example:



a. What is player 3's best choice at each node? Use the answer to this to replace the bottom four subgames with the equilibrium payoffs



b. Now repeat this for player 2's best responses.

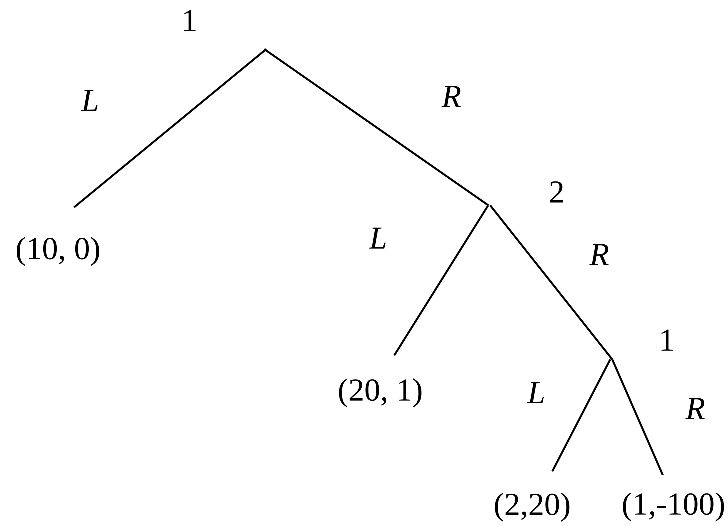


   c. Now repeat for player 1's best response. The best choice is *L*.
   d. We find that the subgame perfect Nash equilibrium is: (*L*; (*L*,*R*); (*R*,*L*,*R*,*L*))
   e. The subgame perfect Nash equilibrium payoffs are (5,8,3).
3. Theorem (Zermelo 1913; Kuhn, 1953) In a finite game of perfect information these exists a pure-strategy Nash Equilibrium. (Further it is a subgame perfect Nash equilibrium) Proof: Backward induction.

D. Comments:
1. Subgame perfection is used extensively. It captures the essence of timing of moves that simple Nash does not. It rules out incredible threats. Each action must be in a player's best interest to play at the time they are called upon to choose.
2. Off equilibrium play matters. A different play off the equilibrium path by some player could change the best response of another player somewhere else in the game tree. Need to specify actions at all nodes in the game tree.
3. The fact that off equilibrium path play matters can be somewhat problematic. Should never observe off-equilibrium play. But suppose you do observe it. What should you conclude?
4. The value of commitment
   a. Schelling: sometimes it does pay to "burn your bridges." Commit to an action that is dominated.
   b. Consider the following game in normal form. *U* is dominated by *D* for player 1.

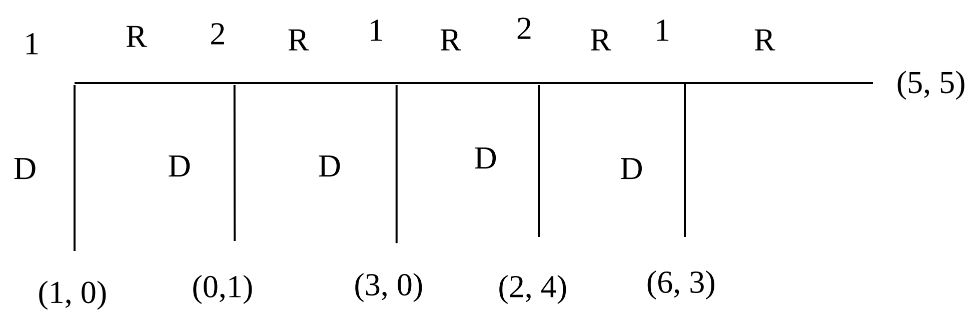
	L	*R*
U	4, 5	0, 4
D	5, 2	1, 3

   c. But suppose that player 1 gets to move first. Player 1 should in fact play *U*. The subgame perfect Nash equilibrium is (*U; L*, *R*).
5. Criticisms of Backward Induction and Subgame Perfection
   a. Crazy like a Fox Example: Subgame perfection places great emphasis on rational play. Schelling – perhaps it pays to make others question your rationality. (Give the crazy person what he/she wants.)



i. Subgame perfect Nash equilibrium: (*L*, *L; R*). The equilibrium payoffs are (10,0)
ii. What if player 2 actually gets a chance to play? Then player 1 must have "made a mistake." But if they made a mistake before, might they do so again?
iii. Perhaps it is better to avoid the possibility of making another mistake and ending up with a payoff of -100. Player 2 might instead play L, which yields payoffs of (20,1). (see Reny, *Journal of Economic Perspectives*, 1992)
iv. In a larger sense – was it irrational to play *R* at the first stage?

b. Centipede Game



i. Subgame perfect Nash equilibrium: choose D at every stage.

ii. But this ends the game right away – with low payoffs. Why not let things ride for a while to increase payoffs?

iii. Suppose I let the game go on 1 million rounds. Would you still want to predict that players would end immediately?

c. There are some weaknesses with subgame perfection – especially true if you require more than about two chains in the logic of backward induction.

d. Still – subgame perfect Nash equilibrium is a very useful concept. It is widely applied. It is the standard equilibrium notion for dynamic games of complete information.

II. Two-Stage Games

A. Introduction

1. Many economic situations have an "investment" stage followed by a strategic competition stage.
2. Large class of economic situations can be analyzed with two-stage games, or dynamic games with multiple periods Examples:
   a. Capital investment followed by competition among rival firms
   b. Location decision followed by competition
   c. Business contract or strategy followed by competition (e.g., design of software compatibility)
   d. Principal-agent relationships where principal sets contract in first stage and agents compete in second stage
   e. Government policy (taxes, tariffs, etc.) followed by firms decisions.
3. Often two stages are enough to show the economic phenomena of interest and are simpler to solve than multi-period games.
4. Incentives to invest:
   a. Direct incentive to invest: consider the effect of investment on payoffs
   b. Strategic incentive to invest: by investing a player changes the set of conditions in stage 2, which may change the play of rival players. Investing player has an incentive to invest to alter the strategy of the rival player in the second stage in ways that increase the payoff for the investing player

B. "Generic" Two Stage Model (Source: Shapiro, *Handbook of Industrial Organization*)

1. Model setup:
   a. Two players: {1, 2}
   b. Strategies: Player 1 can make an investment $\{K\}$ in stage 1. In stage 2, players 1 and 2 simultaneously choose a strategy $\{x_1, x_2\}$.
   c. Payoffs: $\pi_i(x_1, x_2, K)$ for $i = 1, 2$.
2. Solution method: subgame perfect equilibrium using backward induction.
   a. Solve for optimal choices in stage 2: $x_1^*(K)$ and $x_2^*(K)$.
   b. Note: Nash equilibrium solution in second stage will satisfy:

$$\frac{\partial \pi_i(x_1^*(K), x_2^*(K))}{\partial x_i} = 0$$

c. Now consider the first stage choice of investment in K. Firm 1 will solve:

$Max\ \pi_1(x_1{}^*(K), x_2{}^*(K), K)$
FOC:
$$\frac{\partial \pi_1}{\partial x_1}\frac{dx_1{}^*(K)}{dK} + \frac{\partial \pi_1}{\partial x_2}\frac{dx_2{}^*(K)}{dK} + \frac{\partial \pi_1}{\partial K} = 0$$
The first term equals zero $\left(\frac{\partial \pi_1}{\partial x_1} = 0\right)$, so we have
$$\frac{\partial \pi_1}{\partial x_2}\frac{dx_2{}^*(K)}{dK} + \frac{\partial \pi_1}{\partial K} = 0$$

3. Interpretation:
   a. $\frac{\partial \pi_1}{\partial K}$: direct effect of investment on profit
   b. $\frac{\partial \pi_1}{\partial x_2}\frac{dx_2{}^*(K)}{dK}$: strategic effect of investment. How does investment change other players' strategy and how does a change in other players' strategy affect profit.
4. Strategic effect can work in a variety of ways – causing either an increase or decrease in investment
   a. Examples: Cournot vs. Bertrand competition with investment (K) that lowers cost of production but does so at a decreasing rate. Assume that $\frac{\partial^2 \pi_1}{\partial K^2} < 0$
   b. *Cournot*: Strategic effect: $\frac{\partial \pi_1}{\partial x_2} < 0, \frac{dx_2{}^*(K)}{dK} < 0$; $\frac{\partial \pi_1}{\partial x_2}\frac{dx_2{}^*(K)}{dK}$ >0. Strategic effect is to expand investment, push investment to point where the direct effect is negative $\frac{\partial \pi_1}{\partial K}$ < 0. By expanding investment, a firm lowers its cost and becomes a better competitor. Firm 1 will expand output and in response Firm 2 will contract output. Lower output by Firm 2 will increase Firm 1's profit.
   c. *Bertrand*: Strategic effect: $\frac{\partial \pi_1}{\partial p_2} > 0\ \frac{dp_2{}^*(K)}{dK} < 0$; $\frac{\partial \pi_1}{\partial x_2}\frac{dx_2{}^*(K)}{dK}$ <0. Strategic effect is to contract investment, i.e., direct effect will be positive at optimal investment level: $\frac{\partial \pi_1}{\partial K}$ > 0. In this case, more investment will cause Firm 1 to wish to produce more and lower its price. But a lower price by Firm 1 will cause Firm 2 to lower its price, which will result in lower profits to Firm 1.

5. Further analysis of the strategic effect: $\frac{\partial \pi_1}{\partial x_2}\frac{dx_2{}^*(K)}{dK}$.

At second stage:
$$\frac{\partial \pi_1(x_1*(K), x_2*(K), K)}{\partial x_1} = 0$$
$$\frac{\partial \pi_2(x_1*(K), x_2*(K))}{\partial x_1} = 0$$

Differentiate each expression with respect to *K*:

$$\frac{\partial^2 \pi_1}{\partial x_1{}^2}\frac{dx_1*(K)}{dK} + \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2}\frac{dx_2*(K)}{dK} + \frac{\partial^2 \pi_1}{\partial x_1 \partial K} = 0$$

$$\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2}\frac{dx_1*(K)}{dK} + \frac{\partial^2 \pi_2}{\partial x_2{}^2}\frac{dx_2*(K)}{dK} = 0$$

Solve for $\frac{dx_2*(K)}{dK}$ using Cramer's Rule:

$$\frac{dx_2*(K)}{dK} = \frac{\begin{vmatrix} \frac{\partial^2 \pi_1}{\partial x_1{}^2} & -\frac{\partial^2 \pi_1}{\partial x_1 \partial K} \\ \frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{\partial^2 \pi_1}{\partial x_1{}^2} & -\frac{\partial^2 \pi_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi_2}{\partial x_1 \partial x_2} & \frac{\partial^2 \pi_2}{\partial x_2{}^2} \end{vmatrix}}$$

Note: the denominator must be positive to satisfy stability condition (SOC)

So, the sign of $\frac{dx_2*(K)}{dK}$ = sign of $\left(\frac{\partial^2 \pi_1}{\partial x_1 \partial K}\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2}\right)$

The sign of the strategic effect $\left(\frac{\partial \pi_1}{\partial x_2}\frac{dx_2*(K)}{dK}\right)$ depends upon the sign of $\left(\frac{\partial \pi_1}{\partial x_2}\frac{\partial^2 \pi_1}{\partial x_1 \partial K}\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2}\right)$

6. Three effects:

a. The effect of a change in Firm 2's output on Firm 1's profit: $\left(\frac{\partial \pi_1}{\partial x_2}\right)$

b. The effect of investment on the best response function of Firm 1: $\left(\frac{\partial^2 \pi_1}{\partial x_1 \partial K}\right) = \frac{\partial}{\partial K}\left(\frac{\partial \pi_1}{\partial x_1}\right)$

c. The effect of a change in Firm 1's output on Firm 2's output: $\left(\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial \pi_2}{\partial x_2}\right)$

7. The sign of the third term is particularly important:

a. Strategic substitutes: $\left(\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2}\right) < 0$; Best response functions are downward sloping (e.g., Cournot competition)

b. Strategic complements: $\left(\frac{\partial^2 \pi_2}{\partial x_1 \partial x_2}\right) > 0$; Best response functions are upward sloping (e.g., Bertrand competition)

C. Application: Tariffs and imperfect international competition

1. Rules of the game
   a. Two countries: {1, 2}
   b. Each country has one domestic firm that produces output for home production (h_i) and export (e_i), $i = 1, 2$.
   c. Timing of moves:
      i. First stage: each government simultaneously sets a tariff on imports: t_i
      ii. Second stage: each firm simultaneously chooses domestic production and exports
   d. Inverse demand in each country net of production cost ($P - c$) is: $P_i = a - Q_i$, where $Q_i = h_i + e_j$ and $a = a^* - c$, a^* is the demand curve intercept and where c is constant marginal cost of production.
2. Analysis: Second stage
   a. In country i:
      Home producer: $Max_{h_i}\,(a - h_i - e_j)h_i$
      Importer: $Max_{e_j}\,(a - h_i - e_j - t_i)e_j$
   b. Cournot equilibrium in second stage:
      $h_i{*} = \frac{a + t_i}{3},\; e_j{*} = \frac{a - 2t_i}{3}$
3. First stage
   a. Government objective: maximize domestic social welfare, which is sum of domestic firm profit (π_i), consumer surplus (CS_i) and tariff revenue (T_i).
   b. Domestic firm profit: domestic profits plus export profits
      $$\pi_i = \frac{(a + t_i)^2}{9} + \frac{(a - 2t_j)^2}{9}$$
   c. Consumer surplus:
      $$CS_i = \int_0^Q (a - q - P)dq = (a - P)Q - Q^2/2 = Q^2/2 = \frac{(h_i + e_j)^2}{2} = \frac{(2a - t_i)^2}{18}$$
   d. Tariff revenue:
      $$T_i = t_i e_j = \frac{t_i(a - 2t_i)}{3}$$
   e. Government I's objective function is then:
      $$Max_{t_i}\; \frac{(a + t_i)^2}{9} + \frac{(a - 2t_j)^2}{9} + \frac{(2a - t_i)^2}{18} + \frac{t_i(a - 2t_i)}{3}$$

f. Note: in the first stage there is no strategic interaction among tariff terms (t_i and t_j are separable) so we can solve for t_i* by solving this maximization problem.

FOC: $\frac{2(a+t_i*)}{9}+\frac{-2(2a-t_i*)}{18}+\frac{a-4t_i*}{3}=0$

g. Subgame perfect equilibrium tariff: $t_i*=\frac{a}{3}$

h. Using the equilibrium tariff, we can solve for equilibrium home production and imports: $h_i*=\frac{4a}{9}$, $e_j*=\frac{a}{9}$

i. Total quantity and price: $Q=\frac{5a}{9}$; $P=\frac{4a}{9}$

j. Comparison with Cournot competition: $Q=\frac{6a}{9}=\frac{2a}{3}$; $P=\frac{3a}{9}=\frac{a}{3}$

k. Tariffs cause a reduction in quantity and an increase in price compared to Cournot competition. Welfare is reduced with tariffs compared to the case where tariffs are zero ("free trade is good"). This result occurs because tariffs cause a negative externality onto the welfare of the other country. By ignoring this effect, each country chooses to impose a higher than optimal tariff and we end up with a Prisoner's Dilemma.

D. Application: The Bank Run Game

1. Rules of the game
   a. Two investors who each deposit *D* in a bank
   b. These deposits must be left in the bank for two periods before they mature resulting in a return of 2*R* where *R* > *D*.
   c. If the bank is forced to liquidate the assets before maturity, the bank recovers 2*r* to return to its investors where *D* > *r* > *D*/2.
   d. The investors have two periods in which they can withdraw their deposits. If both investors withdraw their deposits in period 1, both receive *r*/2. If only 1 withdraws its investment in period 1, that investor receives *D*, while the other receives 2*r* – *D*. If both investors let their deposits ride, the game moves into period 2. In period 2, if both withdraw their investment they receive *R*. If only one investor withdraws its investment, that investor receives 2*R* – *D* while the other receives *D*. If neither investor withdraws their investment, the game ends with both receiving *R*.
2. Analysis
   a. This dynamic game has a single subgame (besides the game as a whole). The second period subgame is reached only if both investors choose not to withdraw their deposits in the first period.
   b. If the second period is reached, the players play the following game shown below in normal form where *W* denotes a decision to withdraw and *N* denotes a decision not to withdraw.

	W	N
W	R, R	$2R – D, D$
N	$D, 2R - D$	R, R

c. Since $R > D$, $2R - D > R$, there is a strictly dominant strategy to choose W and the unique pure strategy Nash equilibrium is (W, W).

d. Going back to the first period, if both players choose not to withdraw in the first period we know they will both withdraw in the second period and get a payoff of R. Therefore, in the first period, the players play the following game, which can be shown in normal form:

	W	N
W	r, r	$D, 2r - D$
N	$2r – D, D$	R, R

e. Since $r < D$, we then also know that $2r – D < r$. This game has two pure strategy Nash equilibria (there is also a mixed strategy equilibrium): (W, W) and (N, N).

f. In sum: there are two pure strategy Nash equilibria.
- Both players immediately withdraw in the first period (and for completeness note that they would both withdraw in period 2).
- Both players don't withdraw in the first period and they both withdraw in the second period.

E. Application: Tournaments (Lazear & Rosen, 1981)

1. Rules of the game
   a. Players: a boss and two workers
   b. At the first stage, the boss decides to motivate workers by offering to pay the most productive worker a wage of w_H. The least productive worker gets a wage of w_L, $w_H > w_L$.
   c. At the second stage, workers choose effort level simultaneously
   d. The output of worker i is $y_i = e_i + \varepsilon_i$ where e_i is effort and ε_i is a mean zero random noise
   e. ε_1 and ε_2 are independent draws from the distribution $f(\varepsilon)$
   f. The boss does not observe an individual's effort but does observe the individual's output.
   g. The boss's payoff is output minus the wages paid to the worker, $y_1 + y_2 – (w_H + w_L)$.
   h. A worker's payoff is equal to his wage minus the cost of his effort, $g(e_i)$ where $g'(e_i) > 0$ and $g''(e_i) > 0$.
2. Analysis
   a. Worker i's expected payoff as a function of the wage schedule and choice of effort is: $w_H\Pr(y_i(e_i) > y_j(e_j)) + w_L (1 - \Pr(y_i(e_i) > y_j(e_j))) - g(e_i) = (w_H - w_L)\Pr(y_i(e_i) > y_j(e_j)) + w_L - g(e_i)$.
   b. The resulting FOC is: $(w_H - w_L)\dfrac{\partial \Pr\left(y_i\left(e_i^*\right) > y_j\left(e_j\right)\right)}{\partial e_i} = g'\left(e_i^*\right)$.

c. The worker sets the expected increase in its wage equal to the marginal disutility of working harder. Note that $\Pr(y_i(e_i) > y_j(e_j)) = \Pr(e_i + \varepsilon_i > e_j + \varepsilon_j) = \Pr(\varepsilon_i > e_j - e_i + \varepsilon_j)$ or $\Pr\left(\varepsilon_i > e_j - e_i + \varepsilon_j\right) = \int\left(1 - F\left(e_j - e_i + \varepsilon_j\right)\right) f\left(\varepsilon_j\right) d\varepsilon_j$ where $F(\cdot)$ is the CDF of $f(\varepsilon)$.

d. The FOC can now be written as $(w_H - w_L)\int f(e_j - e_i + \varepsilon_j) f(\varepsilon_j) d\varepsilon_j = g'(e_i)$.

e. Solving the FOC for both players is a difficult task given the ugliness of these conditions. It is hard to say whether or not a unique solution will exist given the potential nonlinearity of the FOCs in e_i and e_j. So what is an applied economist to do? Since both players are identical, why not assume they will behave identically, $e_i = e_j = e^*$? If this is the case, the first order condition becomes: $(w_H - w_L)\int f(\varepsilon_j)^2 d\varepsilon_j = g'(e^*)$.

f. With this expression one can show that effort is increasing in the difference in wages $\left(\dfrac{de^*}{d(w_H - w_L)} = \dfrac{\int f(\varepsilon_j)^2 d\varepsilon_j}{g''(e^*)} > 0\right)$ and decreasing in the variance of the productivity shock when that shock is normally distributed (note that $\int f(\varepsilon_j)^2 d\varepsilon_j$

$$= \int\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\varepsilon_j^2}{2\sigma^2}}\right)^2 d\varepsilon_j = \int \frac{1}{2\pi\sigma^2} e^{-\frac{\varepsilon_j^2}{\sigma^2}} d\varepsilon_j = \frac{\sqrt{2\pi}\,\frac{\sigma}{\sqrt{2}}}{2\pi\sigma^2}\int \frac{1}{\sqrt{2\pi}\,\frac{\sigma}{\sqrt{2}}} e^{-\frac{\varepsilon_j^2}{2\left(\frac{\sigma}{\sqrt{2}}\right)^2}} d\varepsilon_j =$$

$\dfrac{1}{2\sqrt{\pi}\sigma}$).

g. It is also worth noting that the solution will only depend on the difference in the wage rates and not the individual wages rates such that $e^* = e(\Delta w)$ where $\Delta w = w_H - w_L$.

h. In the symmetric equilibrium each individual has the same chance of winning the tournament. If U_a is the value to each worker of alternative employment, this result implies the boss must choose wages such that $(w_H + w_L)/2 - g(e^*) \geq U_a$ or else its workers will go elsewhere.

i. The boss's problem is then to maximize $2e(\Delta w) - (w_H + w_L)$ subject to $(w_H + w_L)/2 - g(e(\Delta w)) \geq U_a$. The Lagrangian for this problem can be written as

$$\begin{aligned} L &= 2e(\Delta w) - (w_H + w_L) + \lambda\left(\frac{w_H + w_L}{2} - g(e(\Delta w)) - U_a\right) \\ &= 2e(\Delta w) - (\Delta w + 2w_L) + \lambda\left(\frac{\Delta w + 2w_L}{2} - g(e(\Delta w)) - U_a\right)' \end{aligned}$$

which yields the first order conditions with respect to Δw and w_L:

$$\frac{\partial L}{\partial \Delta w} = 2e'\left(\Delta w^*\right) - 1 + \frac{\lambda^*}{2} - \lambda^* g'\left(e\left(\Delta w^*\right)\right)e'\left(\Delta w^*\right) \le 0\,, \quad \Delta w^* \frac{\partial L}{\partial \Delta w} = 0\,,\ \Delta w^* \ge 0\,,$$

$$\frac{\partial L}{\partial w_L} = -2 + \lambda^* \le 0\,,\ w_L{}^* \frac{\partial L}{\partial w_L} = 0\,,\ w_L{}^* \ge 0\,,$$

$$\frac{\partial L}{\partial \lambda} = \frac{\Delta w^* + 2w_L{}^*}{2} - g\left(e\left(\Delta w^*\right)\right) - U_a \ge 0\,,\ \lambda^* \frac{\partial L}{\partial \lambda} = 0\,, \text{ and } \lambda^* \ge 0\,.$$

j. Assuming we have an interior solution (i.e. $\Delta w^* > 0$ and $w_L{}^* > 0$), these equation imply $\lambda^* = 2$, $g'(e(\Delta w)) = 1$, and $(w_H{}^* + w_L{}^*)/2 - g(e(\Delta w)) = U_a$. Since $g'(e(\Delta w)) = 1$ and $\left(w_H{}^* - w_L{}^*\right)\int f\left(\varepsilon_j\right)^2 d\varepsilon_j = g'\left(e\left(\Delta w^*\right)\right)$, $\Delta w^* = w_H{}^* - w_L{}^* = \dfrac{1}{\int f\left(\varepsilon_j\right)^2 d\varepsilon_j}$ or

$$w_H{}^* = w_L{}^* + \frac{1}{\int f\left(\varepsilon_j\right)^2 d\varepsilon_j}\,.$$

k. Substituting into $\dfrac{\Delta w^* + 2w_L{}^*}{2} - g\left(e\left(\Delta w^*\right)\right) = U_a$, then yields

$$w_L{}^* = U_a + g\left(e\left(\Delta w^*\right)\right) - \frac{1}{2\int f\left(\varepsilon_j\right)^2 d\varepsilon_j} \text{ and } w_H{}^* = U_a + g\left(e\left(\Delta w^*\right)\right) + \frac{1}{2\int f\left(\varepsilon_j\right)^2 d\varepsilon_j}\,.$$

l. For example, suppose $g(e) = \mathrm{e}^{\alpha e}$ for $1 \ge \alpha > 0$, such that $g'(e) = \alpha \mathrm{e}^{\alpha e} = 1$ or $e = -\ln(\alpha)/\alpha$. The resulting equilibrium wage scale is then $w_L{}^* = U_a + \dfrac{1}{\alpha} - \dfrac{1}{2\int f(\varepsilon)^2 d\varepsilon}$ and $w_H{}^* = U_a + \dfrac{1}{\alpha} + \dfrac{1}{2\int f(\varepsilon)^2 d\varepsilon}$. Or, if the shock is normally distributed with mean 0 and variance σ^2, $w_L{}^* = U_a + \dfrac{1}{\alpha} - \sqrt{\pi}\sigma$ and $w_H{}^* = U_a + \dfrac{1}{\alpha} + \sqrt{\pi}\sigma$.

m. This solution implies that increasing the cost of effort by increasing α decrease the equilibrium wage for high and low productivity workers, but does not affect the equilibrium difference in the wage. Increasing the amount a worker can earn elsewhere, U_a, increases equilibrium wages, but not the difference in equilibrium wages. Finally, increasing the variability of productivity shocks, σ, increase the high equilibrium wage, decreases the low equilibrium wage, and increases the equilibrium wages difference.

III. Repeated Games

A. Introduction
1. Many situations in life are repeated – multiple interactions not just one time events.
2. With repeated interactions:
   a. Players can observe what other players did in prior rounds/stages: observed actions
   b. Make current action conditional on prior play: history dependent strategies
3. History dependent strategies allows for a rich set of interactions among players: retaliation, reciprocity…
4. History dependent strategies offer rich array of possible equilibrium outcomes.

B. Terminology
1. The stage game, G, is repeated T times (T can be finite or infinite). $G(T)$ is the repeated game (sometimes called the supergame).
2. Players $i = 1, 2, \ldots, N$. Let a_{it} be action of player i in time t. Let $a_t = (a_{1t}, a_{2t}, \ldots, a_{Nt})$.
3. History at stage t: $h_t = (a_1, a_2, \ldots, a_{t-1})$
4. History dependent strategies: $s_{it}(h_t)$, determines choice of action in stage t as a function of history (choice of actions of players in stages 0, 1, 2, …, $t-1$).
5. Example: consider two stage game where each player has a choice of two actions $\{x, y\}$ in each stage. Complete strategy profile for player i must specify: choice of action in stage 1, choice of action in stage 2 conditional on choices in stage 1: $\{a_{i1}; a_{i2}(x,x), a_{i2}(x,y), a_{i2}(y,x), a_{i2}(y,y)\}$
6. Note: set of potential strategies expands rapidly

C. Finitely repeated games
1. Games with a unique equilibrium to the stage game: the unique subgame perfect equilibrium to the repeated game $G(T)$ is to play the Nash equilibrium in every stage.
2. Example: Repeated Prisoner's Dilemma

	C	**D**
C	2, 2	0, 3
D	3, 0	1, 1

   a. Analyze from stage T: Nash equilibrium is (D,D).
   b. Next to last stage: we know what will happen in last round, so can't influence the future by action. So play (D,D) in this round, and so on…
3. "Chain Store Paradox": example by Selten of repeated game of entry with a single incumbent firm facing a series of rival potential entrants in different markets. Might expect the incumbent to "act tough" to develop a reputation to deter entry. But subgame perfect equilibrium is (entry, collude) in every round.
4. Games with multiple stage game Nash equilibria (Benoit and Krishna, *Econometrica* 1985)
   a. Set of subgame perfect equilibria expands beyond playing stage game Nash equilibria
   b. Example: expanded prisoner's dilemma

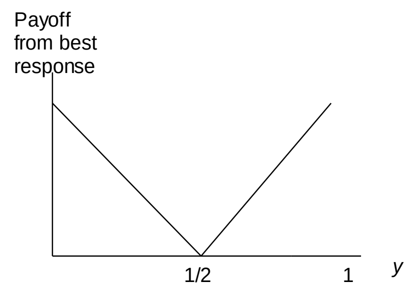
	A_2	B_2	C_2
A_1	3,3	1,4	0.0
B_1	4,1	2,2	0,0
C_1	0,0	0,0	1,1

c. Can (A_1, A_2) be supported as part of subgame perfect equilibrium strategy?
d. Suppose $G(T)$ payoffs equal the simple sum of payoffs from the two stages, and let $T = 2$.
e. History dependent strategy: play A_i in stage 1; play B_i in stage 2 if (A_1, A_2) played in stage 1, play C_i otherwise.
f. Check for profitable deviations: none in the final stage as each history dependent strategy is a Nash equilibrium in the final stage. In the initial stage: could deviate to either B_i or C_i. Clearly C_i is a bad choice. If choose B_i, the payoffs are 4 + 1 = 5. But equilibrium strategy gives payoffs of 3+2 = 5. So initial strategy is a best response. This history dependent strategy constitutes a subgame perfect equilibrium.

D. Infinitely repeated games
1. With infinite repetition, many outcomes can emerge as subgame perfect equilibria.
2. Interest in this question sparked in the 1980's – can players cooperate in repeated prisoner's dilemma game?
3. Axelrod (1984): *Evolution of cooperation*. Computer tournament, asked scholars to submit strategies. Tit-for-tat strategy emerged as best strategy (very simple history dependent strategy – that exhibits reciprocity).
4. Folk Theorem
   a. Intuitive definition: any individually rational payoffs can be supported as a subgame perfect equilibrium in an infinitely repeated game
   b. Definition of individually rational
      i. Min-max value for player i: $\underline{v}_i = \min_{\sigma_{\sim i}} \{\max_{\sigma_i} [U_i(\sigma)]\}$
      ii. Example: matching pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

      iii. Best response: $x = \begin{cases} 0 \; for \; y \le 1/2 \\ [0,1] \; for \; y = 1/2 \\ 1 \; for \; y \ge 1/2 \end{cases}$

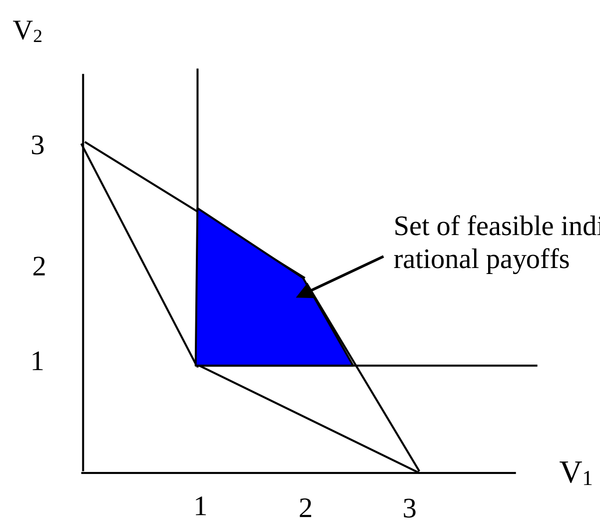


   iv. Min-max value is 0
   v. Individually rational payoffs for player i are any payoffs v_i such that $v_i \geq \underline{v}_i$

c. Nash –Threat version of the Folk Theorem: (Friedman 1971): Let α^* be a static equilibrium (a Nash equilibrium of the stage game) with payoff vector e. Then for any feasible payoff vector v with $v_i > e_i$ for all players i, there is a $\underline{\delta}$ such that for all $\delta \in (\underline{\delta},1)$ there is a subgame perfect equilibrium with payoffs v for the infinite horizon game with discount factor δ.

d. Example: Prisoner's Dilemma: can you support payoffs of (2,2)?

	C	*D*
C	2, 2	0, 3
D	3, 0	1, 1



e. Consider a "trigger strategy" – play C in the first round. Continue to play *C* as long as the other player plays *C*. Play *D* forever after if the other player plays *D* in a prior round.

f. Proof of subgame perfection:
   i. Second stage (punishment phase): *(D, D)* is a Nash equilibrium so playing (*D, D*) in all stages is a subgame perfect equilibrium strategy.
   ii. Initial stage:
      - playing *C* gives a payoff of 2 each stage: $\frac{2}{1-\delta}$
      - playing D gives 3 in first stage followed by 1 thereafter: $3+\frac{\delta}{1-\delta}$
      - playing *C* is best response if and only if: $\frac{2}{1-\delta} \geq 3+\frac{\delta}{1-\delta}$
        $\delta \geq 1/2$
      - So for any $\delta \geq ½$, the payoff (2, 2) can be supported in a subgame perfect equilibrium.

g. General trigger strategy:
   i. Let π_i^* = equilibrium payoffs (per stage)
   ii. Let π_i^D = defection payoffs (given that other players play the equilibrium strategy)
   iii. Let π_i^P = punishment payoffs (Nash eq. payoff per stage)
   iv. $\pi_i^D > \pi_i^* > \pi_i^P$
   v. Cheating does not pay when:

$$\frac{\pi_i^*}{1-\delta} \geq \pi_i^D + \frac{\delta\pi_i^P}{1-\delta}$$

$$\pi_i^* \geq (1-\delta)\pi_i^D + \delta\pi_i^P$$

$$\delta \geq \frac{\pi_i^D - \pi_i^*}{\pi_i^D - \pi_i^P}$$

   vi. Note: since $\pi_i^* > \pi_i^P$ there exists $\underline{\delta} = \frac{\pi_i^D - \pi_i^*}{\pi_i^D - \pi_i^P} < 1$.

h. Subgame perfect Folk Theorem (Fudenberg and Maskin 1986): Assume that the dimension of the set *V* of feasible payoff vectors equals the number of players. Then for any feasible payoff vector $v \in V$ with $v_i > \underline{v}_i$ for all players *i*, there is a $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$ there is a subgame perfect equilibrium with payoffs *v* for the infinite horizon game with discount factor δ.

i. Two part punishment schemes and worst perfect equilibrium.
   i. In order to get strongest deterrence against cheating, want the harshest punishment possible.
   ii. But punishment still has to be subgame perfect (i.e., players still have to find it in their best interest to carry out the punishment).
   iii. Worst perfect equilibrium: gives a player the lowest possible payoff as part of a subgame perfect equilibrium.
   iv. Way to achieve the worst perfect equilibrium - Two part punishment schemes (optimal penal codes) – Abreu (*Econometrica*, 1988): front load punishment

in the first stage and then get higher payoffs in the second stage. In the two-part punishment (worst perfect equilibrium) make it so that present value of punishment phase is the min-max value for the player that cheated.

j. Example from Abreu

	L	*M*	*H*
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

i. Can (L, L) be supported as subgame perfect equilibrium payoffs? For what range of δ?

ii. Conventional trigger strategy with reversion to Nash equilibrium strategy (M, M): $\delta \geq \frac{\pi_i^D - \pi_i^*}{\pi_i^D - \pi_i^P} = \frac{15-10}{15-7} = \frac{5}{8}$.

iii. Two-part punishment and worst perfect equilibrium: in the worst perfect equilibrium the player that cheated should be driven to their min-max value. In this game, the min-max value is 0 (by playing L, can guarantee at least 0 as a payoff).

iv. Proposed strategy:
- Phase I: play L
- If player 1 deviates in round t then:
- Phase II_1: play (M, H) in round t+1
- Phase III_1: (L, M) for t+2, t+3, ….

v. Check whether penalty phase (starting from II_1) is not below the min-max value (otherwise player would just play L forever and not get punishment of playing M and getting -4):

$$-4 + \frac{3\delta}{(1-\delta)} \geq 0$$

$$\delta \geq \frac{4}{7}$$

vi. Check for possible deviations:
- Phase I:

$$\frac{10}{1-\delta} \geq 15 - 4\delta + \frac{3\delta^2}{1-\delta}$$

$$\delta \geq \frac{3}{10}$$

- Phase II_1: Player 1 – deviate by playing L instead of M, which then means punishment phase starts again in the following period:

$$-4 + \frac{3\delta}{1-\delta} \geq 0 - 4\delta + \frac{3\delta^2}{1-\delta}$$

$$\delta \leq 1$$

- Phase II_1: Player 2 – deviate by playing M not H:

$$5+\frac{15\delta}{1-\delta}\geq 7-4\delta+\frac{3\delta^{2}}{1-\delta}$$
$$\delta \geq 0.0984$$

- Phase III_1: Player 1 – deviate by playing M not L:

$$\frac{3}{1-\delta}\geq 7-4\delta+\frac{3\delta^{2}}{1-\delta}$$
$$\delta \geq \frac{4}{7}$$

- Phase III_1: Player 2 – there can't be a profitable deviation since player 2 is getting the best possible payoff (15)

vii. Summary: no profitable deviation so this strategy is a subgame perfect equilibrium strategy

k. Worst perfect equilibrium strategy uses highest possible punishment to sustain cooperation under widest set of circumstances.

E. Summary
1. Folk Theorem results are satisfactory is one sense: way to resolve paradox of supporting "sensible" outcomes in games, such as cooperation in prisoner's dilemma, in a subgame perfect equilibrium. Infinite horizon prevents unraveling.
2. Unsatisfactory results in another sense: *ANY* outcome that is individually rational can be supported. How should we predict which of the infinitely many possible outcomes will emerge? Game theory has no clear answer.
3. Application to anti-trust policy directed at preventing collusion among firms: strategies of firms that may look pro-competitive may in fact make collusion easier to support. To support collusion (cooperation) want harsh punishments, like promises of retaliation for price cutting. Meeting competition clauses – "we'll match your best offer" - may actually work to facilitate collusion among firms.
4. What if players can meet and rethink strategies? "Renegotiate"
   a. Would players then wish to carry out the punishment strategy?
   b. Example: simple trigger strategy in Prisoner's dilemma. Would you really sacrifice all of those future payments possible from cooperation simply because someone made a mistake in one stage? Perhaps not.
   c. But if not, then punishment is not really credible and must rethink whether deterrence is sufficient to stop people from cheating.
   d. "Renegotiation-proof" strategies – restrict punishment to strategy vector that yield payoffs on the Pareto frontier. In general, this will be a weaker set of deterrence than otherwise available. May not be able to support as many subgame perfect equilibria.
   e. Two-part punishment schemes where punishment is front-loaded may be reasonable. Punishment for short time (off Pareto frontier) but then go back to Pareto frontier after first phase of punishment is over.
5. Experimental evidence: people willingly punish those who defect, but are willing to forgive (eventually). See work by Ernst Fehr and colleagues, also by Elinor Ostrom and colleagues. This strengthens the case of thinking about strong temporary punishment.