Applied Microeconomics: Firm and Household

Lecture 5: Expenditure Minimization

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Outline

Expenditure Minimization

- Deriving Hicksian Demand Functions (compensated demand)
- Another example with Cobb-Douglas utility
- Duality: The expenditure function
- The relationship between the Indirect Utility Function and the Expenditure Function

Expenditure minimization

Consider a problem in which consumer is assumed to minimize the expenditure needed to achieve a given utility level, u^0.

- $\underset{x_1, x_2}{Min}\ E = p_1 x_1 + p_2 x_2$

subject to

- $u^0 = u(x_1, x_2)$

The Lagrangian is formed as

- $\underset{x_1, x_2, \lambda}{Min}\ L = p_1 x_1 + p_2 x_2 + \lambda(u^0 - u(x_1, x_2))$

the FOCs are

1. $L_1 = p_1 - \lambda u_1 = 0$
2. $L_2 = p_2 - \lambda u_2 = 0$
3. $L_\lambda = u^0 - u(x_1, x_2) = 0$

Expenditure minimization

From FOCs 1 and 2 we can derive the following equilibrium condition

- $\frac{\lambda u_1}{\lambda u_2} = \frac{p_1}{p_2}$
- $\frac{u_1}{u_2} = \frac{p_1}{p_2}$
- $MRS_{12} = \frac{p_1}{p_2}$

This is the same tangency condition as we derived in the utility maximization problem. In both cases, at the optimum the budget line must be tangent to the indifference curve.

Expenditure minimization: Graphical interpretation

- $MRS_{12} = \frac{u_1}{u_2} = \frac{p_1}{p_2}$

The condition for an optimum states that an expenditure-minimizing individual consumes goods x_1 and x_2 at levels where the ratio of the prices of the goods equals to the ratio of their marginal utilities (MRS).

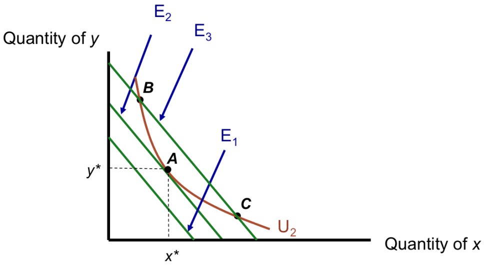


Figure: Graph of expenditure minimization

Expenditure minimization

The second-order sufficient condition for a **constrained minimum** is

- $H = \begin{vmatrix} -\lambda u_{11} & -\lambda u_{12} & -u_1 \\ -\lambda u_{21} & -\lambda u_{22} & -u_2 \\ -u_1 & -u_2 & 0 \end{vmatrix} < 0$

Assuming SOSC holds and that $H \neq 0$, the **Hicksian demand** functions are

- $x_1 = x_1^h(p_1, p_2, u^0)$
- $x_2 = x_2^h(p_1, p_2, u^0)$
- $\lambda = \lambda^h(p_1, p_2, u^0)$

x_1^h and x_2^h are the compensated demand functions. Also, commonly referred as Hicksian demand functions or real-income-held-constant demand functions.

Marshallian vs. Hicksian demand functions

- Marshallian demands are money-income-constant demand functions. Whereas, Hicksian demands are real-income-constant (i.e. utility constant) demand functions.

- Suppose that $u^*(x_1^*, x_2^*)$ is the maximum utility achieved from a utility maximization problem. Since the tangency condition for expenditure minimization is the same as the tangency condition for utility maximization, an expenditure minimizing consumer achieves the same outcome (x_1^*, x_2^*) by setting $u^0 = u^*$ with the same prices.

- Even though the same outcome is achieved in both utility maximization and expenditure minimization problems, the comparative statics are *not* the same! (We will show this in our next lecture)

- While the partial derivatives of Marshallian demands w.r.t. prices represent combined substitution and income effects, the partial derivatives of Hicksian demands w.r.t. prices represent the *pure substitution effect*.

A side note on Lagrange multipliers

Recall the FOCs from the utility maximization problem:

1. $L_1 = u_1 - \lambda^m p_1 = 0$
2. $L_2 = u_2 - \lambda^m p_2 = 0$
3. $L_\lambda = M - p_1 x_1 - p_2 x_2 = 0$

We just derived the FOCs of the expenditure minimization as

1. $L_1 = p_1 - \lambda^h u_1 = 0$
2. $L_2 = p_2 - \lambda^h u_2 = 0$
3. $L_\lambda = u^0 - u(x_1, x_2) = 0$

These FOCs reveal that $\lambda^m = \frac{u_i}{p_i}$ and $\lambda^h = \frac{p_i}{u_i}$. That is,

- $\lambda^h = \frac{1}{\lambda^m}$

Therefore, the Lagrange multiplier in the expenditure minimization problem can be interpreted as the inverse of the marginal utility of income, λ^m.

Expenditure minimization: C-D example

Consider the same utility function $u = x_1^{0.5}x_2^{0.5}$ that we used in the utility maximization problem. This time we want to derive the Hicksian demand curves.

Given the utility function the expenditure minimization problem is

- $\underset{x_1,x_2}{\text{Min}} \quad E = p_1x_1 + p_2x_2 \quad \text{s.t.} \quad u^0 = x_1^{0.5}x_2^{0.5}$

and the Lagrangian function can be formed as

- $\underset{x_1,x_2,\lambda}{\text{Min}} \quad L = p_1x_1 + p_2x_2 + \lambda(u^0 - x_1^{0.5}x_2^{0.5})$

The FOCs are,

1. $\frac{\partial L}{\partial x_1} = L_1 = p_1 - \lambda 0.5(\frac{x_2}{x_1})^{0.5} = 0$
2. $\frac{\partial L}{\partial x_2} = L_2 = p_2 - \lambda 0.5(\frac{x_1}{x_2})^{0.5} = 0$
3. $\frac{\partial L}{\partial \lambda} = L_\lambda = u^0 - x_1^{0.5}x_2^{0.5} = 0$

Expenditure minimization: C-D example

To check the SOSC we derive the second partials as

- $L_{11} = \lambda 0.25 x_2^{0.5} x_1^{-1.5} > 0$
- $L_{22} = \lambda 0.25 x_1^{0.5} x_2^{-1.5} > 0$
- $L_{21} = L_{12} = -\lambda 0.25 x_2^{0.5} x_1^{0.5} < 0$
- $L_{\lambda 1} = L_{1\lambda} = -u_1 < 0, \qquad L_{\lambda 2} = L_{2\lambda} = -u_2 < 0, \qquad L_{\lambda\lambda} = 0$

Using the derived second partials we can form the bordered Hessian

- $H = \begin{vmatrix} L_{11} & L_{12} & -u_1 \\ L_{21} & L_{22} & -u_2 \\ -u_1 & -u_2 & 0 \end{vmatrix}$
- $|H|_3 = \underbrace{-u_1(-u_2 \underbrace{L_{21}}_{-} - (-u_1 \underbrace{L_{22}}_{+}))}_{-} + \underbrace{u_2(-u_2 \underbrace{L_{11}}_{+} - (-u_1 \underbrace{L_{12}}_{-}))}_{-} < 0$

Expenditure minimization: C-D example

1. $\frac{\partial L}{\partial x_1} = L_1 = p_1 - \lambda 0.5(\frac{x_2}{x_1})^{0.5} = 0$
2. $\frac{\partial L}{\partial x_2} = L_2 = p_2 - 0.5(\frac{x_1}{x_2})^{0.5} = 0$
3. $\frac{\partial L}{\partial \lambda} = L_\lambda = u^0 - x_1^{0.5} x_2^{0.5} = 0$

From FOCs 1 and 2 we can derive the following equilibrium condition:

- $\frac{L_1}{L_2} = \frac{p_1}{p_2} = \frac{x_2}{x_1} \rightarrow p_1 x_1 = p_2 x_2$

Note that this is the same tangency condition that we derived from the utility maximization problem. Once again, the equation $p_1 x_1 = p_2 x_2$ says that at the optimum the total amount spent on x_1 always equals the total amount spent on x_2, at any set of prices and fixed utility.

Hicksian demand curves

To derive Hicksian demand curves we can substitute $x_2 = \frac{p_1 x_1}{p_2}$ into FOC 3,

- $x_1^{0.5}(\frac{p_1 x_1}{p_2})^{0.5} = u^0$
- $x_1(\frac{p_1}{p_2})^{0.5} = u^0$
- $x_1^h = (\frac{p_2}{p_1})^{0.5} u^0$, similarly, $x_2^h = (\frac{p_1}{p_2})^{0.5} u^0$

Exercise: Confirm that

- $\frac{\partial x_i^h}{\partial p_i} < 0$ for $i = 1, 2$
- $\frac{\partial x_i^h}{\partial p_j} > 0$ for $i, j \in \{1, 2\}$ and $i \neq j$

That is, Hicksian demand curves are downward sloping and in the case of two goods the goods are *net substitutes*.

The expenditure function

The expenditure function is obtained by substituting the Hicksian demand functions into the total expenditure equation:

- $E^*(p_1, p_2, u^0) = p_1 x_1^h(p_1, p_2, u^0) + p_2 x_2^h(p_1, p_2, u^0))$

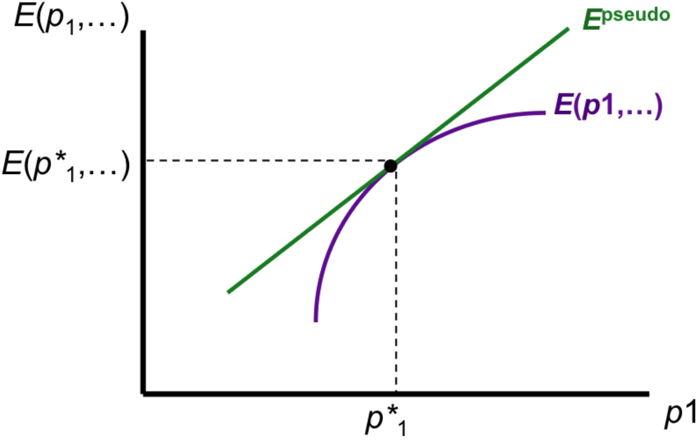
The expenditure function $E^*(p_1, p_2, u^0)$ gives the minimum expenditure to achieve an arbitrarily given utility level u^0 for any set of given prices.

Note that, just like the Hicksian demand functions, the expenditure function is a function of prices and the constant utility (model parameters.)

Properties of expenditure functions

1. Nondecreasing in prices, p.
   - if $p_1 \geqslant p_2$ then $E(p_1, u) \geqslant E(p_2, u)$
2. $E(p, u)$ is concave in prices, p.
3. $E(p, u)$ is linearly homogeneous (homogeneous of degree 1) in prices:
   - $E(tp, u) = tE(p, u)$

Concavity of the expenditure function



Concavity of expenditure function

Consider the figure in the previous slide, where the expenditure function is plotted against the price of good 1.

- Suppose at p_1^* the individual's optimal expenditure is $E(p_1^*,)$
- As p_1 changes without changing the bundle x, the expenditure will change linearly (the E^{pseudo} curve)
- Thus, the linear curve shows the *absolute worst* that an individual can do in response to a price change
- However, an expenditure-minimizing individual would adjust the consumption bundle x optimally in response to a change in p_1. Thus the individual would achieve lower expenditure than E^{pseudo}. That is, the expenditure function will lie everywhere below the linear curve.

Shephard's lemma (the envelope theorem)

If an expenditure function is differentiable in p, then the Hicksian demand functions can be derived by:

- $x_i^h(p, u) = \frac{\partial E^*(p,u)}{\partial p_i}$

Shephard's Lemma is very useful in applied work.

Note that because expenditure function is linearly homogeneous in prices, its first derivative with respect to prices is homogeneous of degree zero. This is just as we would expect for the Hicksian demand function.

A closer look at Shephard's lemma

Rewrite the expenditure function as

- $E(p,u) = \sum_i^N p_i x_i^h(p,u) + \lambda\underbrace{(u^0 - u(x^h(p,u))}_{foc=0}$

Note that the second term is zero because the constraint **binds** (holds with equality) at the optimum. Taking the derivative with respect to the price, p_i,

- $\frac{\partial E(p,u)}{\partial p_i} = x_i^h(p,u) + p_i \frac{\partial x^h(p,u)}{\partial p_i} - \lambda u_x \frac{\partial x^h(p,u)}{\partial p_i}$
- $\frac{\partial E(p,u)}{\partial p_i} = x_i^h(p,u) + \underbrace{(p - \lambda u_x)}_{foc=0} \frac{\partial x^h(p,u)}{\partial p_i}$
- $\frac{\partial E(p,u)}{\partial p_i} = x_i^h(p,u)$

The result establishes Shephard's lemma: the first derivative of the expenditure function with respect to price yields the Hicksian demand.

C-D example continued

Continuing with the C-D example we can derive the expenditure function by substituting the optimal quantities into the objective function.

- $E^*(p_1, p_2, u^0) = p_1 x_1^h + p_2 x_2^h = p_1 (\frac{p_2}{p_1})^{0.5} u^0 + p_2 (\frac{p_1}{p_2})^{0.5} u^0$
- $E^*(p_1, p_2, u^0) = (p_2 p_1)^{0.5} u^0 + (p_1 p_2)^{0.5} u^0$
- $E^*(p_1, p_2, u^0) = 2 p_1^{0.5} p_2^{0.5} u^0$

We can verify Shephard's lemma,

- $\frac{\partial E^*}{\partial p_1} = x_1^h = (\frac{p_2}{p_1})^{0.5} u^0$
- $\frac{\partial E^*}{\partial p_2} = x_2^h = (\frac{p_1}{p_2})^{0.5} u^0$

Indirect utility and expenditure functions

Recall that because the tangency condition in both problems is identical, if $u^*(x_1^*, x_2^*)$ is the maximum utility achieved from a utility maximization problem, then an expenditure minimization problem subject to $u^0 = u^*$ leads to the same consumption bundle. That is, at the tangency of a budget line and an indifference curve the following relation holds:

- $x^*(p_1, p_2, M) = x^h(p_1, p_2, u^0)$

There is a direct relationship between the indirect utility and expenditure functions, since these functions are obtained by substituting the optimal consumption bundles into the respective objective function.This relationship is characterized as,

- $E^*(p_1, p_2, u^0) = M = u^{*-1}(p_1, p_2, M)$ (considering p_1 and p_2 as constants).

That is, given an indirect utility function we can obtain the expenditure function by solving for M. Similarly, given an expenditure function we can obtain the indirect utility function by solving for u.

Indirect utility and expenditure functions

Consider again our C-D example, $u = x_1^{0.5} x_2^{0.5}$. From the utility maximization and expenditure minimization problems, we obtained the following indirect utility and expenditure functions:

- $u^*(p_1, p_2, M) = \frac{M}{2p_1^{0.5}p_2^{0.5}}$
- $E^*(p_1, p_2, u^0) = M = 2p_1^{0.5}p_2^{0.5}u^0$

Hence, the relation $E^*(p_1, p_2, u^0) = u^{*-1}(p_1, p_2, M)$ is easily verified.

The primary implication of this result is that we can obtain both Marshallian and Hicksian demands given any of the indirect utility or expenditure functions. For example, if we know an expenditure function, $E(p_1, p_2, u^0)$, then:

- we can obtain the i^{th} Hicksian demand from Shephard's lemma, $x_i^h = \frac{\partial E}{\partial p_i}$
- we can obtain indirect utility by solving for u, $u = M^{-1}(p_1, p_2, u^0)$
- finally, we can obtain the Marshallian demands by applying Roy's identity, $x_i = \frac{\partial u / \partial p_i}{\partial u / \partial M}$.