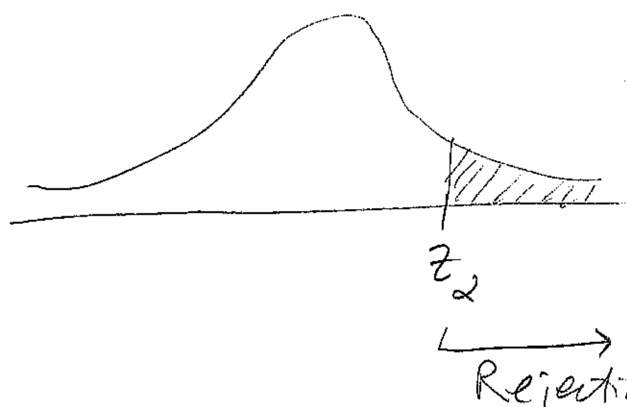
Relationship between hypothesis testing procedures and confidence intervals

<u>Upper one-sided alternative hypothesis</u>

$H_0: \theta = \theta_0, \quad H_1: \theta > \theta_0$



<u>Decision rule</u>: reject H_0 in favor of H_1 @ the α significance level if $Z > z_\alpha$

Under the null, $Z = \dfrac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$,

So reject if $\dfrac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} > z_\alpha$

$\Rightarrow \hat{\theta} - \theta_0 > z_\alpha \sigma_{\hat{\theta}}$

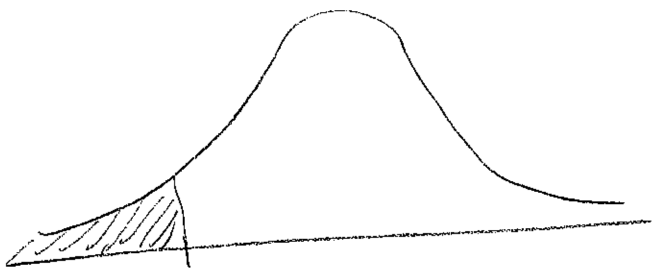
$\Rightarrow \theta_0 < \underbrace{\hat{\theta} - z_\alpha \sigma_{\hat{\theta}}}_{\text{This is } \hat{\theta}_L}$ $\Rightarrow$ reject H_0 in favor of H_1 at the α sig. level if $\theta_0 < \hat{\theta}_L$ where

$P(\theta \geq \hat{\theta}_L) = 1 - \alpha$ and

$[\hat{\theta}_L, \infty)$ is the $(1-\alpha) \times 100\%$ lower one-sided CI for θ

Lower one-sided alternative hypothesis

$H_0: \theta = \theta_0$, $H_1: \theta < \theta_0$



$-z_\alpha$

Rejection region

Decision rule: reject H_0 in favor of H_1 at the α significance level if $z < -z_\alpha$

Under the null, $z = \dfrac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$

So reject if $\dfrac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} < -z_\alpha$

$\Rightarrow \hat{\theta} - \theta_0 < -z_\alpha \sigma_{\hat{\theta}}$

$\Rightarrow \theta_0 > \underbrace{\hat{\theta} + z_\alpha \sigma_{\hat{\theta}}}_{\text{This is } \hat{\theta}_u}$ $\Rightarrow$ So reject H_0 in favor of H_1 at the α sig. level if $\theta_0 > \hat{\theta}_u$

where $P(\theta \leq \hat{\theta}_U) = 1 - \alpha$

and $(-\infty, \hat{\theta}_u]$ is the $(1-\alpha) \times 100\%$ upper one-sided CI for θ

Example – CIs and hypothesis testing

Suppose $H_0: \mu = 3{,}000$, $H_1: \mu < 3{,}000$, $\bar{Y} = 2959$, $S = 39.1$, $N = 8$.
Data are from an approximately normal distribution. $N < 30 \Rightarrow$ Use t-stat.
Want to test H_0 vs. H_1 at the $\alpha = 0.025$ significance level $\Rightarrow 1-\alpha = 0.975$ confidence level.

CI approach: H_1 is $\mu < \mu_0$, so use upper one-sided CI for μ.

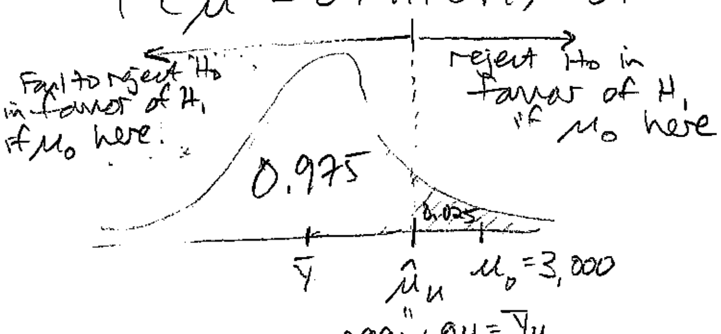
That is, find the $\hat{\mu}_U$ (or $\bar{Y}_U$) such that $P(\mu \le \hat{\mu}_U) = 1-\alpha = 0.975$

$$\hat{\mu}_u = \bar{Y}_u = \bar{Y} + t_\alpha \frac{S}{\sqrt{N}} = 2959 + \underbrace{2.365}_{t_{0.025} \text{ for } 7 \text{ d.f}} \left(\frac{39.1}{\sqrt{8}}\right)$$

$\Rightarrow \hat{\mu}_u = 2991.694$

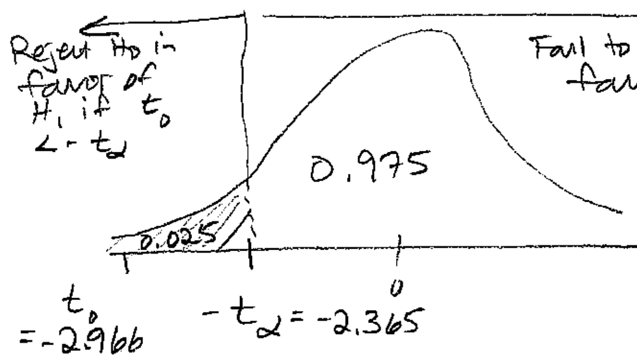
So $(-\infty, 2991.694]$ is the 97.5% upper one-sided CI for μ. That is,

$$P(\mu \le 2991.694) = 0.975$$



$\Rightarrow \mu_0 > \hat{\mu}_u$ so we reject $H_0: \mu = 3000$ (implicitly $H_0: \mu \geq 3000$) in favor of $H_1: \mu < 3000$ at $\alpha = 0.025$.
Given $P(\mu \le 2991.694) = 0.975$, it's very unlikely that $\mu \geq 3{,}000$.

Regular (without using CI) approach to hypothesis testing



where t_0 is the t-stat under the null

Construct the t-stat under the null:

$$t_0 = \frac{\bar{Y} - \mu_0}{S/\sqrt{N}} = \frac{2959 - 3000}{39.1/\sqrt{8}} = -2.966$$

$t_0 < t_\alpha \Rightarrow$ Reject H_0 in favor of H_1 at $\alpha = 0.025$.

Note that $-t_\alpha$ defines the rejection region here, and $\hat{\mu}_u = \bar{Y}_u$ defines it in CI approach.

But from, $\bar{Y}_u = \bar{Y} + t_\alpha \frac{S}{\sqrt{N}} \Rightarrow -t_\alpha = \frac{\bar{Y} - \bar{Y}_u}{S/\sqrt{N}}$

☆ The tail where the rejection region is flips for CI vs. "regular" hypothesis test