# Constant Elasticity of Substituion Preferences: Utility, Demand, Indirect Utility and Expenditure Functions

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This worksheet illustrates the relationship between primal and dual representations of consumer preferences. We illustrate this with a constant elasticity of substitution (CES) utility function, of the form:

$$U(x,y) = (\alpha x^{\rho} + (1-\alpha)y^{\rho})^{1/\rho}$$

in which $x$ and $y$ represent consumption levels.

## Derivation of Demand Functions

Denote income as $M$ and the prices of the goods as $p_x$ and $p_y$. The consumer's budget constrained choice problem has the Lagrangian:

$$\mathcal{L} = U(x,y) - \lambda(M - p_x x - p_y y)$$

Setting the derivatives of $\mathcal{L}$ with respect to $x$ and $y$ to zero, we find that demand functions then satisfy the familiar tangency condition:

$$\frac{\mathrm{MRS_x}}{\mathrm{MRS_y}} = \frac{\alpha x^{\rho-1}}{(1-\alpha)y^{\rho-1}} = \frac{p_x}{p_y}$$

Solving this equation for $y$ in terms of $x$, we have:

$$y = x\left(\frac{(1-\alpha)p_x}{\alpha p_y}\right)^{\sigma}$$

where $\sigma = 1/(1-\rho)$. We then can substitute into the budget constraint to obtain:

$$p_x x + p_y y = p_x x + p_y x\left(\frac{(1-\alpha)p_x}{\alpha p_y}\right)^{\sigma} = M$$

or

$$x = \frac{M}{p_x + p_y\left(\frac{(1-\alpha)p_x}{\alpha p_y}\right)^{\sigma}} = \frac{(p_x/\alpha)^{-\sigma}}{\alpha^{\sigma}p_x^{1-\sigma} + (1-\alpha)^{\sigma}p_y^{1-\sigma}} \quad M$$

By symmetry, we then have the demand function for $y$ as:

$$y = \frac{(p_y/(1-\alpha))^{-\sigma}}{\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma}} \quad M$$

## Calibration

In many applications of consumer choice theory, we are given a reference equilibrium choice, and we must infer properties of of the consumer utility function given these values. In the present model, the benchmark would consist of reference prices ($\bar{p}_x$ and $\bar{p}_y$) and reference quantities ($\bar{x}$ and $\bar{y}$). The elasticity parameter is typically regarded as a *free parameter*, the value of which is exogenously specified. We then determine the implicit value of $\alpha$ which is consistent with these observations. From the demand functions, we can conclude that:

$$\frac{\bar{x}}{\bar{y}} = \left(\frac{\bar{p}_x/\alpha}{\bar{p}_y/(1-\alpha)}\right)^{-\sigma}$$

A bit of algebra solves this expression for $\alpha$ in terms of the given values:

$$\frac{\bar{p}_x/\alpha}{\bar{p}_y/(1-\alpha)} = \left(\frac{\bar{y}}{\bar{x}}\right)^{1/\sigma}$$

$$(1-\alpha)\bar{p}_x\bar{x}^{1/\sigma} = \alpha\bar{p}_y\bar{y}^{1/\sigma}$$

and

$$\alpha = \frac{\bar{p}_x\bar{x}^{1/\sigma}}{\bar{p}_x\bar{x}^{1/\sigma} + \bar{p}_y\bar{y}^{1/\sigma}}$$

## A Convenient Normalization

The reference utility level can then be evaluated as:

$$\bar{u} = (\alpha\bar{x}^\rho + (1-\alpha)\bar{y}^\rho)^{1/\rho}$$

Microeconomic theory is based on the concept of *ordinal* rather than *cardinal* preferences. Hence, the preferences described by utility function $u(x,y)$ is equivalent to that defined by utility function $\tilde{u}(x,y)$, provided that $\tilde{u}$ is a monotonic (order preserving) transformation of $u$. For example, one might define:

$$\tilde{u}(x,y) = \phi u(x,y)$$

where $\phi$ is a positive constant. A convenient choice of $\phi$ might be $1/\bar{u}$, so that $\tilde{u}(\bar{x},\bar{y}) = 1$.

## The Calibrated Share Form

It is a tedious algebraic exercise to show that when the constant elasticity utility function is scaled such that the utility level is equal to one at the reference point, it can be parameterized as:

$$\tilde{u}(x,y) = \left(\theta\left(\frac{x}{\bar{x}}\right)^{\rho} + (1-\theta)\left(\frac{y}{\bar{y}}\right)^{\rho}\right)^{1/\rho}$$

where

$$\theta = \frac{\bar{p}_x \bar{x}}{\bar{p}_x \bar{x} + \bar{p}_y \bar{y}}.$$

It also can be shown that with utility function $\tilde{u}$, demand functions can be written as:

$$x = \bar{x}\left(\frac{c}{p_x/\bar{p}_x}\right)^{\sigma} \frac{M}{c\bar{M}}$$

where $\bar{M}$ is benchmark expenditure, and $c$ is the "cost of living index":

$$c = \left(\theta\left(\frac{p_x}{\bar{p}_x}\right)^{1-\sigma} + (1-\theta)\left(\frac{p_y}{\bar{p}_y}\right)^{1-\sigma}\right)^{1/(1-\sigma)}$$

## Indirect Utility

Indirect utility takes goods prices and income as arguments, i.e. $v(p_x, p_y, M)$. The indirect utility function if formally defined as:

$$v(p_x, p_y, m) = \max u(x,y)$$

s.t.

$$p_x x + p_y y \leq m$$

Alternatively, the indirect utility function can be found by substituting the demand functions into the primal utility function:

$$v(p_y, p_y, M) = u(x(p_x, p_y, M), y(p_x, p_y, M))$$

It is then straight-forward but messy to show that given preference $u(x,y)$, we have:

$$v(p_x, p_y, M) = \left(\alpha^{1+\sigma} p_x^{-\rho\sigma} + (1-\alpha)^{1+\sigma} p_y^{-\rho\sigma}\right)^{1/\rho} \frac{M}{\alpha^{\sigma} p_x^{1-\sigma} + (1-\alpha)^{\sigma} p_y^{1-\sigma}}$$

When we work with the scaled utility function $\tilde{u}$, the indirect utility function is simply:

$$\tilde{v}(p_x, p_y, M) = \frac{M}{\bar{M} c(p_x, p_y)}$$

## The Expenditure Function

The expenditure function relates the minimum expenditure required to achieve a given utility level, here denoted $e(p_x, p_y, u)$. Formally, this is defined as:

$$e(p_x, p_y, u^*) = \min p_x x + p_y y$$

s.t.

$$u(x, y) \geq u^*$$

This function can be alternatively be derived by solving for $M$ in the expression:

$$v(p_x, p_y, M) = u^*$$

With CES prferences ($\tilde{u}$) we solve the equation:

$$\frac{M}{\bar{M} c(p_x, p_y)} = u$$

hence,

$$e(p_x, p_y, u) = \bar{M} u \left( \theta \left( \frac{p_x}{\bar{p}_x} \right)^{1-\sigma} + (1-\theta) \left( \frac{p_y}{\bar{p}_y} \right)^{1-\sigma} \right)^{1/(1-\sigma)}$$

At the benchmark point,

$$e(\bar{p}_x, \bar{p}_y, u = 1) = \bar{M} u \left( \theta \left( \frac{\bar{p}_x}{\bar{p}_x} \right)^{1-\sigma} + (1-\theta) \left( \frac{\bar{p}_y}{\bar{p}_y} \right)^{1-\sigma} \right)^{1/(1-\sigma)} = \bar{M}$$

## Duality

As noted in the previous lecture, there is a close connection between the primal and dual representation of the consumer demand model. In the primal model, we take the budget constraint as exogenous and choose the largest indifference curves. In the dual model, we fix the indifference curve and choose the smallest budget line. This defines the expenditure function:

$$e(p_x, p_y, u) = \min_{x,y} p_x x + p_y y$$

s.t.

$$u(x, y) = 1$$

Alternatively, we could define the cost minimization problem with prices rather than quantities as decision variables, given have a consumption bundle $(x, y)$ for which $\tilde{u}(x, y) = 1$ and solve:

$$\min_{p_x, p_y} p_x x + p_y y$$

s.t.

$$v(p_y, p_y, M) = 1$$

In the primal model, the first order conditions imply that the slope of the indifference curve at the optimal point, the *marginal rate of substitution*, is equal to the price ratio. In the dual model, we have a similar condition, namely:

$$\left.\frac{\partial p_y}{\partial p_x}\right|_{\tilde{u}=1} = \frac{\partial e(p_x, p_y, 1)/\partial p_x}{\partial e(p_x, p_y, 1)/\partial p_y} = \frac{\bar{x}}{\bar{y}}$$

Hence, the slope of the expenditure funciton level set is simply equal the ratio of the optimal choices at that price level.

What about the *curvature* of the indifference curve and the minimum expenditure curves? It turns out that the curvature of these two functions are inversely related: if the minimum expenditure curve is very curve, the indifference curve is rather flat and vice-versa. We can see this by considering a specific point $(p_x, p_y)$ on the expenditure curve and then moving this to some $(p'_x, p'_y)$ far away on the same expenditure curve. Suppose that we find the slope of the expenditure curve doesn't change very much, i.e. the minimum expenditure curve is fairly flat. The slope of the minimum expenditure curve is simply the ratio of the optimal demands, then if this slope remains constant, it implies that final demands do not change much. In the limit, where the demands are constant, the indifference curve must be L-shaped.

## The Slutsky Equation

The Slutsky equation decomposes the impact of price changes on demand into substitution and income-related components. For our two-good model, the Slutsky equation describing the impact of the price of $y$ on the demand for $x$ can be written as:

$$\frac{\partial x}{\partial p_y} = \frac{\partial h_x(p_x, p_y, u^*)}{\partial p_y} - \frac{\partial x}{\partial m} y$$

where $h_x(p_x, p_y, u^*)$ is the *compensated demand* function.

Derivation of the Slutsky equation is a simple matter when we employ the expenditure function to define compensated demand. Let $(x^*, y^*)$ maximize utility at $(p^*_x, p^*_y, m^*)$, and let $u^* = u(x^*, y^*)$. Given our definition of the expenditure function the following equation is identically true:

$$h_x(p_x, p_y, u^*) = x(p_x, p_y,\ e(p_x, p_y, u^*))$$

Differentiate this function with respect to $p_y$ and evaluate this derivative at $(p^*_x, p^*_y)$ to get:

$$\frac{\partial h_x}{\partial p_y} = \frac{\partial x}{\partial p_y} + \frac{\partial x(p^*_x, p^*_y, m)}{\partial m} \frac{\partial e(p^*_x, p^*_y, u^*}{\partial p_y}$$

By the envelope theorem, we have

$$\frac{\partial e(p^*_x, p^*_y, u^*}{\partial p_y} = y^*$$

Rearranging, the Slutsky equation follows directly. This expression decomposes demand changes induced by a price change $\Delta p_y$ into two separate effects: the substitution effect and the income effect:

$$\Delta x \approx \underbrace{\frac{\partial x}{\partial p_y}\Delta p_y}_{\text{changeindemand}} = \underbrace{\frac{\partial h_x}{\partial p_y}\Delta p_y}_{\text{Substitutioneffect}} - \underbrace{\frac{\partial x}{\partial m} y \Delta p_y}_{\text{Incomeeffect}}$$

This is readily portrayed on an indifference curve diagram as the combination of a shift *along and indifference curve* (compensated demand) and a shift *outward* (income effect).

## Measuring Welfare Changes

When prices or income changes, a consumer is made better or worse off. Economists can monetize the welfare impact through introduction of the concept of *compensating* or *equivalent change in income.* Suppose the consumer faces an initial price vector $\bar{p} = (\bar{p}_x, \bar{p}_y)$, and a final vector $p' = (p'_x, p'_y)$. The consumer has the same monetary income $m$ in each situation. The consumer demand $(\bar{x}, \bar{y})$ in the first situation and $(x', y')$ in the second situation. How much would we have to pay the consumer at prices $p'$ to make her just as happy as she was at prices $\bar{p}$? This number, $C$, is called the *compensating variation* in income. It can be formally defined by use of the indrect utility and expenditure functions:

$$v(p'_x, p'_y, m + C) = v(\bar{p}_x, \bar{p}_y, m)$$

or

$$C = e(p'_x, p'_y, \bar{v}) - m$$

where

$$\bar{v} = v(\bar{p}_x, \bar{p}_y, m)$$

Working with the calibrated share form, we have $\bar{v} = 1$, and the compesating variation can be written as:

$$C = m\left[\left(\theta\left(\frac{p'_x}{\bar{p}_x}\right)^{1-\sigma} + (1-\theta)\left(\frac{p'_y}{\bar{p}_y}\right)^{1-\sigma}\right)^{1/(1-\sigma)} - 1\right]$$

An alternative (virtually equivalent) measure of the welfare impact of a price change is given by the *equivalent variation.* this number $E$ is defined to be the amount of income that one would have to take away from the consumer at prices $\bar{p}$ to make him as well off as he would be at prices $p'$. this is clearly given by:

$$v(\bar{p}, m - E) = v(p', m)$$

or

$$E = e(p', v') - e(\bar{p}, v')$$

where

$$v' = v(p', m).$$

Working with the calibrated share form the equivalent variation can be written as:

$$E = m \left[ 1 - \frac{1}{\left( \theta \left( \frac{p'_x}{\bar{p}_x} \right)^{1-\sigma} + (1-\theta) \left( \frac{p'_y}{\bar{p}_y} \right)^{1-\sigma} \right)^{1/(1-\sigma)}} \right]$$

Note that $C$ and $E$ can be positive or negative depending on whether prices rise or fall. Notice also that $C$ and $E$ are exact masures of welfare change due to a price change: they differ only in the choice of the comparison point.