The Envelope Theorem: Shephard's Lemma, Hotelling's Lemma, etc.

Suppose $g\,(x,\,a)$ where a is a parameter.

Choose x to max the function.

In general $x^{*} = x\,(a)$.

Therefore

$g\,((x\,(a),\,a)$ is the maximum value of $g\,(\cdot)$ given a.

Call this a value function

$M(a) \equiv g\,(x\,(a),\,a)$.

The profit function $\pi\,(p,\,w) \equiv pf\,(x\,(p,\,w) - w'x\,(p,\,w))$ is an example.

Returning to the general form

① $M(a) \equiv g\,(x\,(a),\,a)$

Differentiate both sides of this identity with reference to a

② $$\frac{dM(a)}{da} = \frac{\partial g\,(x(a),a)}{\partial x}\,\frac{\partial x(a)}{\partial a} + \frac{\partial g\,(x(a),a)}{\partial a}$$

but,

③ $\dfrac{\partial g\,(x(a),a)}{\partial x} = 0$ because $g\,(x,a)$ is maximized when $x = x(a)$.

④ Substitute ③ into ②

$$\frac{dM(a)}{da} = \frac{\partial g(x(a), a)}{\partial a}$$ (This is the Envelope Theorem)

which we often write

$$\equiv \left. \frac{\partial g(x, a)}{\partial a} \right|_{x = x(a)}.$$

Return to the example value function $\pi(p, w)$. By the envelope theorem

$$\frac{\partial \pi(p, w)}{\partial p} = \left. \frac{\partial \lfloor pf(x) - w'x \rfloor}{\partial p} \right|_{x = x(p, w)}$$

$$= \left. f(x) \right|_{x = x(p, w)}$$

$$= f(x(p, w))$$

which tells us that $y^s = f(x(p, w)) = \dfrac{\partial \pi(p, w)}{\partial p}$. (this is Hotelling's Lemma)

Now consider the cost function

$$c = c(y, w)$$

which identifies the minimum cost of producing *y* given *w*.

Note that

$$c(y, w) = w' \underbrace{x(y, w)}_{\substack{\text{conditional input} \\ \text{demand function}}}$$

So, $c(y, w)$ is a value function (minimum not maximum).

For simplicity assume *x* is a scalar (only 1 input).

If the envelope theorem applies

$$\frac{\partial c(y, w)}{\partial w_i} = \frac{\partial \left[w' x(y, w) \right]}{\partial w_i} = \frac{\partial (w' x)}{\partial w_i} \Bigg|_{x = x(y, w)} = x_i((y, w)$$

which is the conditional input demand function for input *i*. (This is Shephard's Lemma)

We have just used the Envelope theorem to prove Shephard's Lemma and Hotelling's Lemma. Both are special cases of the Envelope theorem.