Solutions to Problem Set #1: Preferences and Utility

A.Please solve the following end-of-chapter questions:

1. (20 points) 3.12 CES utility

   a. $MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = \frac{\alpha x^{\delta-1}}{\beta y^{\delta-1}} = \frac{\alpha}{\beta}\left(\frac{y}{x}\right)^{1-\delta}$ so this function is homothetic.

   b. If $\delta = 1$, $MRS = \frac{\alpha}{\beta}$ and so is a constant. If $\delta = 0$, $MRS = \frac{\alpha}{\beta}\frac{y}{x}$. These MRS values match those for the perfect substitutes and Cobb-Douglas cases respectively.

   c. $\frac{\partial MRS}{\partial x} = (\delta - 1)\frac{\alpha}{\beta}y^{1-\delta}x^{\delta-2}$. This is negative iff $\delta < 1$.

   d. This follows from part a: if $x = y$, $MRS = \alpha/\beta$

   e. If $\delta = 0.5$, $MRS(0.9) = .949\alpha/\beta$ and $MRS(1.1) = 1.05\alpha/\beta$. With $\delta = -1$, $MRS(0.9) = .81\alpha/\beta$ and $MRS(1.1) = 1.21\alpha/\beta$. Hence, the MRS responds more strongly to changes inn y/x when $\delta = -1$ than when $\delta = 0.5$. The indifference curves are more sharply curved when δ is lower. When $\delta = -\infty$ the indifference curves are L-shaped, implying perfect complement (Leontief) preferences.

2. (20 points) 3.13 The quasi-linear function

   a. $MRS = y$

   b. For quasiconcavity, we need to have $f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 < 0$. We know that $f_x = f_1 = 1$, $f_y = f_2 = 1/y$, $f_{11} = 0$, $f_{22} = -1/y^2$, and $f_{12} = 0$.

      So, $f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 = 0 + 0 - 1/y^2 = -1/y^2$. For $y > 0$, the value is negative.

   c. $y = e^{C-x}$.

   d. Since the marginal utility of x is a constant at 1, while that of y is decreasing as y increases (as it is equal to $1/y$), we would expect consumers to shift more towards x and away from y when their income rises and allows them to buy more goods. This is because consumers will always try to maximize utility. They are better off buying more of the good with a higher marginal utility.

   e. Refer to Example 3.4 in NS. This function is usually used to describe the consumption of one commodity with respect to all other commodities. So, lny could represent the singular commodity while x could represent all the other goods consumed (which usually is to say that x just stands in for money).

B. Consider the following Cobb-Douglas utility function of a representative consumer:

$$u(x, y) = Ax^{0.35}y^{0.65}$$

where x_1, x_2 are goods, M denote consumer's expenditure and p_i denote good i's price.

1. (10 points) Show that $u(x)$ is homogeneous of degree 1.

- $u(tx, ty) = A(tx)^{0.35}(ty)^{0.65}$
- $u(tx, ty) = At^{0.35+0.65}x^{0.35}y^{0.65}$
- $u(tx, ty) = tu(x, y)$

Hence, by definition $u(x, y)$ is homogeneous of degree zero.

2. (10 points) Set up the consumer's utility maximization problem.

- $\underset{x_1,x_2}{\text{Max}} \ Ax^{0.35}y^{0.65}$ s.t. $p_x x + p_y y = M$
- $\underset{x,y,\lambda}{\text{Max}} \ L = Ax^{0.35}y^{0.65} + \lambda(M - p_x x + p_y y)$

3. (10 points) Derive and interpret the first order conditions.

- $\frac{\partial L}{\partial x_1} = L_1 = A0.35(y/x)^{0.65} - \lambda p_x = 0$
- $\frac{\partial L}{\partial y} = L_1 = A0.65(x/y)^{0.35} - \lambda p_y = 0$
- $\frac{\partial L}{\partial \lambda} = L_\lambda = M - p_x x - p_y y = 0$

From FOCs 1 and 2 we can find that $MRS = \frac{7y}{13x} = \frac{p_x}{p_y}$. The general interpretation holds. That is, consumer determines the optimal x and y where their marginal rate substitution equal to their price ratio. Furthermore, in this case we can write the optimum condition as $\frac{13}{7}p_x x = p_y y$, implying that at the optimum, the total expenditure on y is two times the total expenditure on x.

4. (10 points) Derive the demand curves and price elasticities of demand. Please interpret the elasticities.

Using FOC 3 and $x = \frac{7p_y y}{13p_x}$:

- $M = p_x \frac{7p_y y}{13p_x} + p_y y = \frac{20p_y}{13} y$
- $y = 13M/20p_y$. Plugging back in to the formula for x, $x = 7M/20p_x$

Elasticities for good x. $\epsilon_{x,p_x} = (\partial x/\partial p_x)(p_x/x) = -(7M/20p_x^2)(p_x/(7M/20p_x)) = -1$ and $\epsilon_{x,p_y} = 0$. Similarly, $\epsilon_{y,p_y} = -1$ and $\epsilon_{y,p_x} = 0$.

5. (10 points) Derive the indirect utility function. Show that this is a legitimate indirect utility function for a consumer with strictly positive consumption of x and y).

- $V(p_x, p_y, M) = A(\frac{7M}{20p_x})^{0.35}(\frac{13M}{20p_y})^{0.65}$.
- $V(p_x, p_y, M) = A(\frac{7M}{20p_x})^{0.35}(\frac{13M}{20p_y})^{0.65} = AM(\frac{7}{20p_x})^{0.35}(\frac{13}{20p_y})^{0.65} = AM(0.35)^{0.35}(0.65)^{0.65}p_x^{-0.35}p_y^{-0.65}$

The properties of an indirect utility function relate only to homogeneity and the direction of change in its arguments.

- $V(tp_x, tp_y, tM) = A(\frac{0.35tM}{tp_x})^{0.35}(\frac{0.65tM}{tp_y})^{0.65} = V(p_x, p_y, M)$. So, V is HD0.
- $\frac{\partial V}{\partial M} = A(\frac{0.35}{p_x})^{0.35}(\frac{0.65}{p_y})^{0.65} > 0, \forall p, M$. So, V is increasing (or non-decreasing) in M.
- $\frac{\partial V}{\partial p_x} = -0.35AM0.35^{0.35}0.65^{0.65}p_x^{-1.35}p_y^{-0.65} < 0, \forall p, M$. So, V is decreasing (or non-increasing) in p_x.
- $\frac{\partial V}{\partial p_y} = -0.65AM0.35^{0.35}0.65^{0.65}p_x^{-0.35}p_y^{-1.65} < 0, \forall p, M$. So, V is decreasing (or non-increasing) in p_y.

6. (10 points) Verify Roy's identity and comment on the homogeneity of the demand functions.
   Roy's Identity:
   - $x = -\frac{\partial V}{\partial p_x} / \frac{\partial V}{\partial M}$
   - $x = -\frac{-0.35AM0.35^{0.35}0.65^{0.65}p_x^{-1.35}p_y^{-0.65}}{A0.35^{0.35}0.65^{0.65}p_x^{-0.35}p_y^{-0.65}} = 0.35M/p_x$
   - $y = -\frac{-0.65AM0.35^{0.35}0.65^{0.65}p_x^{-0.35}p_y^{-1.65}}{A0.35^{0.35}0.65^{0.65}p_x^{-0.35}p_y^{-0.65}} = 0.65M/p_y$

   This verifies Roy's identity.

   If we double all the parameters of the demand functions (p_x,p_y, and M) then we can easily verify that the quantities demanded are unchanged, since own price enters linearly in the denominator, income enters linearly in the numerator, and cross price is not in the demand function. Hence the demand functions are homogeneous of degree zero in the parameters.