

## PRODUCTION UNDER UNCERTAINTY EXAMPLES

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With all this new risk and uncertainty machinery described, we are finally ready to ask the question: When faced with uncertainty, how much should a producer produce and how?

We will start to think about this problem from the perspective of a *risk neutral* producer because it serves as a useful benchmark. Risk neutrality implies the producer's objective is to maximize expected profit:

$$\text{PU10} \quad \max_{\mathbf{R} \geq 0} \boldsymbol{\phi} \cdot (\mathbf{R} - C(\mathbf{p}, \mathbf{r}, \mathbf{R}) \mathbf{1}^S),$$

which has the first-order conditions

$$\text{PU11} \quad \phi_s - \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} \leq 0 \text{ and } \left( \phi_s - \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} \right) R_s^* = 0 \text{ for } s = 1, \dots, S.$$

For  $R_s^* > 0$  and  $R_t^* > 0$ , equation PU11 implies  $\frac{\phi_s}{\phi_t} = \frac{\frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s}}{\frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_t}}$ . The producer sets its marginal rate of substitution, which equals the ratio of probabilities, equal to the marginal rate of transformation. Equation PU11 also implies  $\sum_{s=1}^S \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} \geq 1$  (CQ refer to all  $\mathbf{R}$  that satisfy  $\sum_{s=1}^S \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} \geq 1$  as the *risk efficient* set), and  $\sum_{s=1}^S \phi_s R_s^* = \sum_{s=1}^S \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} R_s^*$ . The solution to this problem are revenue demands that depend on prices:  $\mathbf{R}(\mathbf{p}, \mathbf{r})$ . These revenue demands yield the cost function  $C(\mathbf{p}, \mathbf{r}) = C(\mathbf{p}, \mathbf{r}, \mathbf{R}(\mathbf{p}, \mathbf{r}))$  and the distribution of profits  $\pi(\mathbf{p}, \mathbf{r}) = \mathbf{R}(\mathbf{p}, \mathbf{r}) - C(\mathbf{p}, \mathbf{r}, \mathbf{R}(\mathbf{p}, \mathbf{r})) \mathbf{1}^S$ . Also note that once we have our revenue demands we can figure out how much output we should produce in each state and how much inputs we should use:

$$\text{PU12} \quad q_m^s(\mathbf{p}, \mathbf{r}) = q_m^s(\mathbf{p}, \mathbf{r}, \mathbf{R}(\mathbf{p}, \mathbf{r})) \text{ and } z_n(\mathbf{p}, \mathbf{r}) = z_n(\mathbf{p}, \mathbf{r}, \mathbf{R}(\mathbf{p}, \mathbf{r})).$$

It is important to realize that even for a nice interior solution

$$\text{PU13} \quad \frac{\partial C(\mathbf{p}, \mathbf{r})}{\partial r_n} = z_n(\mathbf{p}, \mathbf{r}) + \sum_{t=1}^S \phi_t \frac{\partial R_t(\mathbf{p}, \mathbf{r})}{\partial r_n} \text{ and}$$

$$\text{PU14} \quad -\frac{\frac{\partial C(\mathbf{p}, \mathbf{r})}{\partial p_m^s}}{\frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}(\mathbf{p}, \mathbf{r}))}{\partial R_s}} = q_m^s(\mathbf{p}, \mathbf{r}, \mathbf{R}(\mathbf{p}, \mathbf{r})) - \sum_{t=1}^S \frac{\phi_t}{\phi_s} \frac{\partial R_t(\mathbf{p}, \mathbf{r})}{\partial p_m^s},$$

which means we cannot easily derive  $q_m^s(\mathbf{p}, \mathbf{r})$  and  $z_n(\mathbf{p}, \mathbf{r})$  directly from  $C(\mathbf{p}, \mathbf{r})$  without knowing  $\mathbf{R}(\mathbf{p}, \mathbf{r})$ . This differs from what we were able to do without uncertainty. A result that does carry through from our earlier analysis of production without uncertainty, at least for the case of risk neutrality, is

$$\text{PU15} \quad \sum_{s=1}^S \phi_s (\mathbf{p}^{s0} - \mathbf{p}^{s1}) \cdot (\mathbf{q}^{s0} - \mathbf{q}^{s1}) + (\mathbf{r}^1 - \mathbf{r}^0) \cdot (\mathbf{z}^0 - \mathbf{z}^1) \geq 0$$

where  $\mathbf{q}^i = \mathbf{q}(\mathbf{p}^i, \mathbf{r}^i)$  and  $\mathbf{z}^i = \mathbf{z}(\mathbf{p}^i, \mathbf{r}^i)$  are the supplies and inputs demands given prices  $\mathbf{p}^i$  and  $\mathbf{r}^i$  for  $i = 0, 1$ . Equation PU15 follows from the definition of expected profit maximization and implies the supplies will be non-decreasing, while input demands will be non-increasing in their own price.

Relaxing the assumption of risk neutrality, the general problem can be written as

$$\text{PU10}' \quad \max_{\mathbf{R} \geq 0} W(\mathbf{R} - C(\mathbf{p}, \mathbf{r}, \mathbf{R}) \mathbf{1}^S),$$

which has the first-order conditions

$$\text{PU11}' \quad \frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_s} - \sum_{t=1}^S \frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_t} \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} \leq 0 \text{ and } \left( \frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_s} - \sum_{t=1}^S \frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_t} \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} \right) R_s^* = 0$$

for  $s = 1, \dots, S$ . For  $R_s^* > 0$  and  $R_t^* > 0$ , equation PU11' implies  $\frac{\frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_s}}{\frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_t}} = \frac{\frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s}}{\frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_t}}$ . The

marginal rate of substitution will equal the marginal rate of transformation, which may or may not equal the ratio of probabilities (assuming probabilities exist). It also implies  $\sum_{s=1}^S \frac{\partial C(\mathbf{p}, \mathbf{r}, \mathbf{R}^*)}{\partial R_s} \geq$

1, which means the solution is risk efficient as with risk neutrality, and  $\sum_{s=1}^S \frac{\frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_s}}{\sum_{t=1}^S \frac{\partial W(\boldsymbol{\pi}^*)}{\partial \pi_t}} R_s^* =$

$\sum_{s=1}^S \frac{\partial C(p, r, R^*)}{\partial R_s} R_s^*$ , which is a little different than with risk neutrality. Solving equation PU11', again yields optimal revenue demands, which can be used to get a cost function and profit distribution. Our state-contingent supplies and input demands can be found as in equation PU12. Modifying equations PU13 and PU14 for our more general preferences does not change the fundamental result that we cannot use  $C(\mathbf{p}, \mathbf{r})$  directly to recover our supplies and input demands like we did without uncertainty because we also need to know the optimal revenues. Finally, it is important to note that PU15 cannot be shown in general nor can we show even more specifically that supplies must be non-decreasing and input demands non-increasing in their own price. That is, with our more general preferences, the law of supply and demand can break down under uncertainty. Can we get these laws back if we add more structure to preferences? Generalized Schur concavity and EUT are not enough to get back these laws.

The implications of equations PU13 and PU14 present quite a challenge for empirical work. Recall that with choice under certainty if we observed price data and cost or profit data, which are often readily available, we can estimate a cost or profit function and then derive input demands or input demands and supplies. We could further improve our estimation if we also observed input demands and supplies, which are also often readily available. Furthermore, we had lots of ways to test whether our assumptions were consistent with the observed data. Equations PU13 and PU14 imply that if we have data on state-contingent output prices (which are tougher to get) and input prices, we could estimate a cost function, but we can't necessarily use it to derive the input demands. To be able to use cost and price data like we did with production under uncertainty, we also need to be able to observe state-contingent revenues or profits, which can be quite problematic with the exception of the observed state.

Another empirical challenge that has emerged with the state-contingent approach is that states are treated as discrete, when in actuality many sources of uncertainty are more continuous in nature (e.g., rainfall and temperature). What attempts have been made to do empirical work have typically used artificial partitions of continuous states and a limited number at that.

### *Some Applications*

For the remainder of our discussion of production under uncertainty, it is worth taking a look at some applications, to see how results established under EUT can still hold under the state-contingent approach. The first question we will ask is how do fixed costs affect production under uncertainty? EUT's answer is that with CARA there is no effect, with DARA the optimal output is lower, and with IARA the optimal output is higher.

Assume we have a producer with the cost function  $C(q) = C_v(q) + C_f$  where  $q \in \mathbb{R}_+$  is an output,  $C_v(q)$  is the variable cost such that  $C_v'(q) \geq 0$  and  $C_v''(q) \geq 0$ , and  $C_f$  is the fixed cost. Therefore, we are assuming that output is certain. Where uncertainty enters the model is the price:  $p_s > 0$  is the price of output in state  $s$  and  $\mathbf{p}$  is a vector of state-contingent prices. Note that this specification is of the output cubicle type. The producer's optimization problem can be written as

$$\mathbf{PU16} \quad \max_{q \geq 0} W(\boldsymbol{\pi}(q)) = W(\mathbf{p}q - (C_v(q) + C_f)\mathbf{1}^S),$$

which yields the first-order condition for an interior solution of

$$\mathbf{PU17} \quad \sum_{s=1}^S \frac{\partial W(\boldsymbol{\pi}(q^*))}{\partial \pi_s} (p_s - C_v'(q^*)) = 0.$$

This first-order condition depends on  $C^f$  through  $\boldsymbol{\pi}(q^*)$ , so in general fixed cost can affect production under uncertainty, unlike production without uncertainty, but we will come back to this shortly.

It is also worth noting that equation PU17 implies

$$\mathbf{PU18} \quad \frac{\sum_{s=1}^S \frac{\partial W(\boldsymbol{\pi}(q^*))}{\partial \pi_s} p_s}{\sum_{s=1}^S \frac{\partial W(\boldsymbol{\pi}(q^*))}{\partial \pi_s}} = C_v'(q^*)$$

in general and

**PU18'**  $\sum_{k=1}^S \phi_k p_k = C_v'(q^{rn})$

for a risk neutral producer. Since  $C_v''(q^*) \geq 0$ , this means that  $q^{rn} \geq (\leq) q^*$  if  $\sum_{k=1}^S \phi_s p_s \geq$

$(\leq) \frac{\sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s} p_s}{\sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s}}$ . In general, we cannot say for certain whether a risk neutral producer with the

same subjective probabilities produces more or less. However, if we have a risk averse producer with generalized Schur-concave preferences, then property (ii) from above implies

$\sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s} (\pi_s(q^*) - \sum_{t=1}^S \phi_t \pi_t(q^*)) \leq 0$ . Substituting in profit then yields

$$\sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s} (p_s q^* - C_v(q^*) - C^f - \sum_{t=1}^S \phi_t (p_t q^* - C_v(q^*) - C^f)) \leq 0,$$

which for positive  $q^*$  implies  $\sum_{k=1}^S \phi_k p_k \geq \frac{\sum_{s=1}^S \frac{\partial W(\pi)}{\partial \pi_s} p_s}{\sum_{s=1}^S \frac{\partial W(\pi)}{\partial \pi_s}}$  such that  $q^{rn} \geq q^*$  — a risk neutral

producer produces more than a risk averse one when output price is risky. This is the same result established with expected utility theory by Sandmo (1971).

It is useful to take a look at how we might establish this result with EUT preferences and continuous prices. Let  $p \in [p_L, p_U]$  with the cumulative distribution  $F(p)$  where  $p_L > 0$ . We will also assume the producer's objective is to maximize the expected utility of profit where its preferences are characterized by the strictly increasing, strictly concave, and twice differentiable function  $u(\cdot)$ .

The firm's objective can be written as

$$\max_{q \geq 0} U(q) = \int_{p_L}^{p_U} u(\pi(q)) dF(p) = E(\pi(q))$$

where  $\pi(q) = pq - C_v(q) - C_f$ . The first-order condition for an optimum is

**PU19** 
$$E\left((p - C_v'(q^*))u'(\pi(q^*))\right) = 0,$$

which we can rewrite as

**PU20** 
$$E(p) + \frac{Cov(p, u'(\pi(q^*)))}{E(u'(\pi(q^*)))} = C_v'(q^*).$$

We know  $E(u'(\pi(q^*))) > 0$  and  $Cov(p, u'(\pi(q^*))) < 0$  because  $\frac{\partial p}{\partial p} = 1 > 0$  and  $\frac{\partial u'(\pi(q^*))}{\partial p} = q^* u''(\pi(q^*)) < 0$ . Therefore, at an optimum  $E(p) > C_v'(q^*)$ . The uncertain firm will produce where the expected price exceeds the marginal cost of production. For a producer that maximizes expected profit, we know from above that  $E(p) = C_v'(q^{rn})$ . Therefore,  $q^{rn} \geq q^*$  because  $C_v''(q^*) \geq 0$ .

The key to this approach is the result  $Cov(p, u'(\pi(q^*))) < 0$  because  $\frac{\partial u'(\pi(q^*))}{\partial p} = q^* u''(\pi(q^*)) < 0$ . Under very general conditions, it can be shown that if  $x$  is a random variable,  $h(x)$  is an increasing function ( $h'(x) > 0$ ), and  $g(x)$  is a decreasing function ( $g'(x) < 0$ ), then  $Cov(h(x), g(x)) < 0$ . Alternatively, if  $h'(x) > 0$  and  $g'(x) > 0$  or  $h'(x) < 0$  and  $g'(x) < 0$ , then  $Cov(h(x), g(x)) > 0$ . This is very useful in analyzing EUT problems because you often get terms that represent the covariance of two functions of a random variable that can be signed one way or another.

We now return to the question of how fixed costs influence the optimal output with price uncertainty. Sandmo (1971) developed this result based on expected utility, but we will try to generalize his result. Totally differentiating equation PU17 with respect to  $q^*$  and  $C_f$  yields

**PU21** 
$$\frac{dq^*}{dC^f} = - \frac{-\sum_{s=1}^S (p_s - C_v'(q^*)) \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t}}{\sum_{s=1}^S \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} (p_t - C_v'(q^*)) (p_s - C_v'(q^*)) - \sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s} C_v''(q^*)}.$$

The denominator in equation PU21 is negative assuming the second order condition holds, so the sign of  $\frac{dq^*}{dcf}$  will depend on the sign of the numerator. To sign the numerator, we will define

$$\varphi_s(\boldsymbol{\pi}) = -\frac{\sum_{t=1}^S \frac{\partial^2 W(\boldsymbol{\pi})}{\partial \pi_s \partial \pi_t}}{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}} \text{ and consider three distinct cases:}$$

- (i)  $\varphi_s(\boldsymbol{\pi}) = \varphi(\boldsymbol{\pi})$  for all  $s = 1, \dots, S$  and all  $\boldsymbol{\pi}$ , which reduces to Arrow-Pratt CARA with expected utility preferences.
- (ii)  $\varphi_s(\boldsymbol{\pi}) \geq \varphi_t(\boldsymbol{\pi})$  if  $\pi_s \geq \pi_t$  for all  $s, t = 1, \dots, S$  and all  $\boldsymbol{\pi}$ , which essentially reduces to Arrow-Pratt IARA with expected utility preferences.
- (iii)  $\varphi_s(\boldsymbol{\pi}) \geq \varphi_t(\boldsymbol{\pi})$  if  $\pi_t \geq \pi_s$  for all  $s, t = 1, \dots, S$  and all  $\boldsymbol{\pi}$ , which essentially reduces to Arrow-Pratt DARA with expected utility preferences.

Let us start with case (i). Since  $\varphi(\boldsymbol{\pi})$  is independent of the state, the numerator of PU21 is

$$\varphi(\boldsymbol{\pi}(q^*)) \sum_{s=1}^S \frac{\partial W(\boldsymbol{\pi}(q^*))}{\partial \pi_s} (p_s - C_v'(q^*)) = 0$$

by the first order condition. So for this generalization of Arrow-Pratt CARA,  $\frac{dq^*}{dcf} = 0$  such that increasing fixed costs has no effect on output.

For case (ii), first note that  $\frac{\partial \pi_s(q^*)}{\partial p_s} = q^* > 0$  and define  $\pi^o = p^o q^* - c_v(q^*) - c_f$  where  $p^o = C_v'(q^*)$ . Also note that there exists a  $\varphi^o \in \mathbb{R}$  such that  $\varphi_s(\boldsymbol{\pi}(q^*)) \geq \varphi^o$  when  $\pi_s(q^*) \geq \pi^o$  and  $\varphi_s(\boldsymbol{\pi}(q^*)) \leq \varphi^o$  when  $\pi_s(q^*) \leq \pi^o$ . For  $p_s \geq C_v'(q^*)$ ,  $\pi_s(q^*) \geq \pi^o$  such that

$$\textbf{PU22} \quad \varphi_s(\boldsymbol{\pi}(q^*)) \geq \varphi^o.$$

Multiplying both sides of equation PU22 by  $-\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*)) \leq 0$  then yields

$$\textbf{PU23} \quad -\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*))\varphi_s(\pi(q^*)) \leq -\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*))\varphi^0 \text{ or}$$

$$\textbf{PU24} \quad (p_s - C_v'(q^*)) \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} \leq -\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*))\varphi^0.$$

For  $p_s \leq C_v'(q^*)$ ,  $\pi_s(q^*) \leq \pi^0$  such that

$$\textbf{PU25} \quad \varphi_s(\pi(q^*)) \leq \varphi^0.$$

Multiplying both sides of equation PU25 by  $-\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*)) \geq 0$  then yields

$$\textbf{PU26} \quad -\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*))\varphi_s(\pi(q^*)) \leq -\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*))\varphi^0 \text{ or}$$

$$\textbf{PU27} \quad (p_s - C_v'(q^*)) \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} \leq -\frac{\partial W(\pi(q^*))}{\partial \pi_s}(p_s - C_v'(q^*))\varphi^0.$$

Equations PU24 and PU27 are identical for all  $s$ , so summing implies

$$\textbf{PU28} \quad \sum_{s=1}^S (p_s - C_v'(q^*)) \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} \leq -\varphi^0 \sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s} (p_s - C_v'(q^*)) = 0 \text{ or}$$

$$\sum_{s=1}^S (p_s - C_v'(q^*)) \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} \leq 0.$$

Equations PU21 and PU28 then imply  $\frac{dq^*}{dc^f} \geq 0$  such that for this generalization of Arrow-Pratt IARA fixed costs can increase output. Intuitively, increasing fixed costs lowers profits in every state equally, so that with IARA the producer's level of risk aversion decreases and it is willing to produce more. Analogous arguments can be used for case (iii) to show that with this



generalization of Arrow-Pratt DARA fixed costs can decrease output. Intuitively, increasing fixed costs lowers profits in every state equally, so that with DARA the producer's level of risk aversion increases and it is willing to produce less.

Before, we said it was not possible to show that the supply was non-decreasing in the output price. To get a better understanding why, let us consider what happens if we increase the price of output equally in all states (i.e., we can add  $\alpha$  to the price in each state, derive  $\frac{dq^*}{d\alpha}$ , and then set  $\alpha = 0$ ):

**PU29** 
$$\frac{dq^*}{d\alpha} = - \frac{q^* \sum_{s=1}^S (p_s - c_v'(q^*)) \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} + \sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s}}{\sum_{s=1}^S \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} (p_t - c_v'(q^*)) (p_s - c_v'(q^*)) - \sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s} c_{v''}(q^*)}.$$

As before, the sign of the denominator is negative, so the sign of  $\frac{dq^*}{d\alpha}$  will only depend on the sign of the numerator. Based on our analysis of the fixed costs, we now know that

$$\sum_{s=1}^S (p_s - c_v'(q^*)) \sum_{t=1}^S \frac{\partial^2 W(\pi(q^*))}{\partial \pi_s \partial \pi_t} \geq / = / \leq 0$$

for our Arrow-Pratt DARA/CARA/IARA. We also know by assumption that  $\sum_{s=1}^S \frac{\partial W(\pi(q^*))}{\partial \pi_s} \geq 0$ . Therefore, with DARA or CARA,  $\frac{dq^*}{d\alpha} > 0$  and the supply will be non-decreasing as the price increases the same amount in each state. With IARA, it is possible for the supply to be decreasing as price increases proportionally. Intuitively, equation PU29 is similar to the Slutsky equation in consumer theory where the first term in the numerator represents an income effect (e.g., increasing prices increases income which may make the producer more or less risk averse depending on its risk attitudes) and a substitution effect (e.g., increasing the price makes output more valuable so the producer demands more of it in exchange for higher production costs).

Many have advocated for EUT due to its tractability. CQ and others have now made great strides by showing that many of the results obtained based on EUT are really quite generalizable through the state contingent perspective. An interesting question that remains for

applied economists is how to use the implications of this theory more effectively in their empirical work.

## **References**

Sandmo, A. (1971). On the Theory of the Competitive Firm under Price Uncertainty. *American Economic Review* 61(1): 65-73.