PROFIT MAXMIZATION

APEC 8002, Fall 2016 **Instructor: Terry Hurley**

INTRODUCTION

The cost minimization problem addresses the question of how to produce assuming we know how much we want to produce. The revenue maximization problem addresses how to produce assuming we know what inputs we want to use. Both use a competitive market assumption either on the input or output side of the market. We now turn to the joint question of how much to produce and with what. To address this question, we need two more assumptions in addition to A1, A2, A3.OW, and A3.IW. First, we will assume that a producer's objective is to maximize profit — an assumption that often comes under fire, but still provides a useful benchmark. Second, we will assume that both output and input markets are perfectly competitive, so a producer takes output and input prices as exogenous. Of course, this assumption is also not very appropriate in many circumstances, which provides fodder for the field of Industrial Organization. We will start with the case where we know what our outputs and inputs are ahead of time. We then turn to the case where what is an output and what is an input may change based on market prices. With this, we will develop the notion of the input demand, supply, and profit function, and discuss the properties of these objects. Note that I will often refer to the input demand and supply in the profit maximization problem as the unconditional input demand and unconditional supply to more clearly distinguish them from the conditional input demand and conditional supply. Again, throughout our discussion, we will assume existence.

PREDETERMINED OUTPUTS & INPUTS

When we know what our outputs and inputs are the profit maximization problem can be generally framed as

PM1 $\mathbf{Y}(\mathbf{p},\mathbf{r}) = \{(\mathbf{q},-\mathbf{z}) \in \mathbf{PPS}: \mathbf{p}\cdot\mathbf{q} - \mathbf{r}\cdot\mathbf{z} \geq \mathbf{p}\cdot\mathbf{q}' - \mathbf{r}\cdot\mathbf{z}' \text{ for all } (\mathbf{q}',-\mathbf{z}') \in \mathbf{PPS}\}$

where $\mathbf{p}\cdot\mathbf{q}$ reflects the revenue to the producer from the sale of output and $\mathbf{r}\cdot\mathbf{z}$ represents the producer's cost. $\mathbf{Y}(\mathbf{p},\mathbf{r})$ may be a set of vectors in general, so when we want to refer to any particular vector in this set, we will write $\mathbf{y}(\mathbf{p},\mathbf{r}) = \big(\mathbf{q}(\mathbf{p},\mathbf{r}), -\mathbf{z}(\mathbf{p},\mathbf{r})\big)$ where $\mathbf{q}(\mathbf{p},\mathbf{r})$ is the profit maximizing vector of outputs and $\mathbf{z}(\mathbf{p}, \mathbf{r})$ is the profit maximizing vector of inputs. Again with just this simple definition, we can make some important points about what we should observe from profit maximizing behavior in a competitive market. First, $\alpha\mathbf{p} \cdot \mathbf{q} - \alpha\mathbf{r} \cdot \mathbf{z} \geq \alpha\mathbf{p} \cdot \mathbf{q}' - \alpha\mathbf{r} \cdot \mathbf{z}'$ is identical to$\mathbf{p} \cdot \mathbf{q} - \mathbf{r} \cdot \mathbf{z} \geq \mathbf{p} \cdot \mathbf{q}' - \mathbf{r} \cdot \mathbf{z}'$ for all $\alpha > 0$, so $\mathbf{Y}(\mathbf{p}, \mathbf{r})$ is homogenous of degree zero in $\mathbf{p}$ and $\mathbf{r}$ — relative, not absolute, prices matter. Second,

$$\mathbf{p}^0 \cdot \mathbf{q}^0 - \mathbf{r}^0 \cdot \mathbf{z}^0 \geq \mathbf{p}^0 \cdot \mathbf{q}^1 - \mathbf{r}^0 \cdot \mathbf{z}^1 \text{ and}$$
$$\mathbf{p}^1 \cdot \mathbf{q}^1 - \mathbf{r}^1 \cdot \mathbf{z}^1 \geq \mathbf{p}^1 \cdot \mathbf{q}^0 - \mathbf{r}^1 \cdot \mathbf{z}^0$$

where $\mathbf{q}^0 = \mathbf{q}(\mathbf{p}^0, \mathbf{r}^0)$, $\mathbf{q}^1 = \mathbf{q}(\mathbf{p}^1, \mathbf{r}^1)$, $\mathbf{z}^0 = \mathbf{z}(\mathbf{p}^0, \mathbf{r}^0)$, and $\mathbf{z}^1 = \mathbf{z}(\mathbf{p}^1, \mathbf{r}^1)$ such that

$$(\mathbf{p}^1 - \mathbf{p}^0) \cdot (\mathbf{q}^1 - \mathbf{q}^0) + (\mathbf{r}^1 \cdot -\mathbf{r}^0) \cdot (\mathbf{z}^0 - \mathbf{z}^1) \geq 0,$$

so supply will be non-decreasing in its own price and input demand will be non-increasing in its own price.

Recall that with weak free disposal of inputs and outputs $D_I(\mathbf{q}, \mathbf{z}) \geq 1$ and $D_O(\mathbf{q}, \mathbf{z}) \leq 1$ when $(\mathbf{q}, -\mathbf{z}) \in \mathbf{PPS}$. Therefore, if $D_I(\mathbf{q}, \mathbf{z})$ or $D_O(\mathbf{q}, \mathbf{z})$ is nicely differentiable in $\mathbf{q}$ and $\mathbf{z}$, we can use Lagrange methods to solve this problem. For a nicely differentiable $D_I(\mathbf{q}, \mathbf{z})$, we can write the problem as

PM1' $\max_{\mathbf{q}\geq 0, \mathbf{z}\geq 0} \mathbf{p} \cdot \mathbf{q} - \mathbf{r} \cdot \mathbf{z}$ subject to $D_I(\mathbf{q}, \mathbf{z}) \geq 1$,

which has the Lagrangian

PM2' $L = \mathbf{p} \cdot \mathbf{q} - \mathbf{r} \cdot \mathbf{z} + \gamma_I(D_I(\mathbf{q}, \mathbf{z}) - 1)$,

and first-order conditions

PM3' $\frac{\partial L}{\partial q_m} = p_m + \gamma_I{}^* \frac{\partial D_I(\mathbf{q}^*, \mathbf{z}^*)}{\partial q_m} \leq 0, \frac{\partial L}{\partial q_m} q_m{}^* \equiv 0$, and $q_m{}^* \geq 0$ for $m = 1, \ldots, M$,

$\frac{\partial L}{\partial z_n} = -r_n + \gamma_I{}^* \frac{\partial D_I(\mathbf{q}^*,\mathbf{z}^*)}{\partial z_n} \leq 0, \frac{\partial L}{\partial z_n} z_n{}^* \equiv 0$, and $z_n{}^* \geq 0$ for $n = 1,\ldots, N$,

PM4’ $\frac{\partial L}{\partial \gamma_I} = D_I(\mathbf{q}^*, \mathbf{z}^*) - 1 \geq 0, \frac{\partial L}{\partial \gamma_I} \gamma_I{}^* \equiv 0$, and $\gamma_I{}^* \geq 0$.

As in the cost minimization problem, weak free disposal of inputs implies production will be efficient at the optimum such that $D_I(\mathbf{q}^*, \mathbf{z}^*) = 1$. With weak free disposal of output, $D_I(\mathbf{q}^*, \mathbf{z}^*) = 1$ also implies $D_O(\mathbf{q}^*, \mathbf{z}^*) = 1$. While convexity was useful for ignoring the second order conditions in the cost minimization and revenue maximization problems, it is not as helpful here. For example, suppose we have non-decreasing returns to scale technology. Furthermore, assume that there is some $(\mathbf{q}^*, -\mathbf{z}^*) \in$ **PPS** such that $\mathbf{p}\cdot\mathbf{q}^* - \mathbf{r}\cdot\mathbf{z}^* > 0$. Now for non-decreasing returns to scale, $(\alpha\mathbf{q}^*, -\alpha\mathbf{z}^*) \in$ **PPS** for all $\alpha \geq 1$ such that $\mathbf{p}\cdot\alpha\mathbf{q}^* - \mathbf{r}\cdot\alpha\mathbf{z}^* > p\cdot\mathbf{q}^* - r\cdot\mathbf{z}^* > 0$ for any $\alpha > 1$. But this implies that if positive profits are possible, there will not be a profit maximum when we have non-decreasing returns to scale technology, regardless of convexity. Therefore, we will usually want to be more careful about checking second order conditions or adding extra assumptions when we are dealing with profit maximization. We know the input distance function is homogeneous of degree one in inputs, so we can again use $\frac{\partial L}{\partial z_n} z_n{}^* \equiv 0$ to show that $\gamma_I(\mathbf{p}, \mathbf{r}) = \mathbf{r} \cdot \mathbf{z}(\mathbf{p}, \mathbf{r})$. That is the Lagrange multiplier is equal to our cost at the maximum. There are other properties of $\mathbf{q}(\mathbf{p}, \mathbf{r})$ and $\mathbf{z}(\mathbf{p}, \mathbf{r})$ that we could talk about, but we will hold off on this until after talking about the more general case.

For any $z_n{}^* > 0$ and $z_{n\prime}{}^* \geq 0$, equation PM3’ implies $\frac{r_{n\prime}}{r_n} \geq \frac{\frac{\partial D_I(\mathbf{q}^*,\mathbf{z}^*)}{\partial z_{n\prime}}}{\frac{\partial D_I(\mathbf{q}^*,\mathbf{z}^*)}{\partial z_n}}$, which holds with equality when $z_{n\prime}{}^* > 0$, so the *MRTS* will never exceed the ratio of input prices and will equal it when two inputs are used in positive quantities. For any $q_m{}^* > 0$ and $q_{m\prime}{}^* \geq 0$, equation PM3’ implies $\frac{p_{m\prime}}{p_m} \leq \frac{\frac{\partial D_I(\mathbf{q}^*,\mathbf{z}^*)}{\partial q_{m\prime}}}{\frac{\partial D_I(\mathbf{q}^*,\mathbf{z}^*)}{\partial q_m}}$, which holds with equality for $q_{m\prime}{}^* > 0$, so the ratio of output prices will never exceed *MRT* and will equal it for any two outputs produced in positive quantities. For any

${z_n}^* > 0$ and ${q_m}^* > 0$, equation PM3' implies $-\frac{\frac{\partial D_I(\mathbf{q}^*,\mathbf{z}^*)}{\partial z_n}}{\frac{\partial D_I(\mathbf{q}^*,\mathbf{z}^*)}{\partial q_m}} = \frac{r_n}{p_m}$, which again says the marginal rate of transformation between an input and output also equals the ratio of prices when the input and output are positive.

The profit function is defined by

PM5 $\pi(\mathbf{p},\mathbf{r}) = \mathbf{p}\cdot\mathbf{q}(\mathbf{p},\mathbf{r}) - \mathbf{r}\cdot\mathbf{z}(\mathbf{p},\mathbf{r})$.

It should be clear that it is homogenous of degree one in **p** and **r** since supply and input demand are homogeneous of degree 0 in **p** and **r**. Again, there are lots of other properties to talk about that we will hold off on until the more general case.

We could have also framed the problem as

PM1'' $\max_{\mathbf{q}\geq 0,\mathbf{z}\geq 0} \mathbf{p}\cdot\mathbf{q} - \mathbf{r}\cdot\mathbf{z}$ subject to $D_O(\mathbf{q},\mathbf{z}) \leq 1$,

which has the Lagrangian

PM2'' $L = \mathbf{p}\cdot\mathbf{q} - \mathbf{r}\cdot\mathbf{z} + \gamma_O\big(1 - D_O(\mathbf{q},\mathbf{z})\big)$,

with the first-order conditions

PM3'' $\frac{\partial L}{\partial q_m} = p_m - {\gamma_O}^*\frac{\partial D_O(\mathbf{q}^*,\mathbf{z}^*)}{\partial q_m} \leq 0, \frac{\partial L}{\partial q_m}{q_m}^* = 0$, and ${q_m}^* \geq 0$ for $m = 1,\ldots, M$,

$\frac{\partial L}{\partial z_n} = -r_n - {\gamma_O}^*\frac{\partial D_O(\mathbf{q}^*,\mathbf{z}^*)}{\partial z_n} \leq 0, \frac{\partial L}{\partial z_n}{z_n}^* = 0$, and ${z_n}^* \geq 0$ for $n = 1,\ldots, N$,

PM4'' $\frac{\partial L}{\partial \gamma_O} = 1 - D_O(\mathbf{q}^*,\mathbf{z}^*) \geq 0, \frac{\partial L}{\partial \gamma_O}{\gamma_O}^* = 0$, and ${\gamma_O}^* \geq 0$.

This problem is identical to equations PM1' – PM4', with one important difference. We use the output distance function to describe production possibilities rather than the input distance function. The solution in terms of $\mathbf{q}(\mathbf{p}, \mathrm{r})$ and $\mathbf{z}(\mathbf{p}, \mathbf{r})$ will be identical, but $\gamma_O(\mathbf{p}, \mathbf{r})$ need not equal $\gamma_I(\mathbf{p}, \mathbf{r})$. Note that $D_O(\mathbf{q}, \mathbf{z})$ is homogeneous of degree one in $\mathbf{q}$ not $\mathbf{z}$, so $\frac{\partial L}{\partial q_m} q_m{}^* \equiv 0$ implies $\gamma_O(\mathbf{p}, \mathbf{r}) = \mathbf{p} \cdot \mathbf{q}(\mathbf{p}, \mathbf{r})$. That is, when we use the output distance function to describe our production possibilities, the Lagrange multiplier equals revenue at the maximum instead of costs at the maximum.

Returning again to our example production possibilities set, we have

PM1^exp^ $\max_{\mathbf{q} \geq 0, \mathbf{z} \geq 0} p_1 q_1 + p_2 q_2 - r_1 z_1 - r_2 z_2$ subject to $\frac{\sqrt{z_1 z_2}}{q_1{}^2 + q_2{}^2} \geq 1$,

which has the Lagrangian

PM2^exp^ $L = p_1 q_1 + p_2 q_2 - r_1 z_1 - r_2 z_2 + \gamma_I \left(\frac{\sqrt{z_1 z_2}}{q_1{}^2 + q_2{}^2} - 1 \right)$,

with the first-order conditions

PM3^exp^ $\frac{\partial L}{\partial q_1} = p_1 - 2\gamma_I{}^* q_1{}^* \frac{\sqrt{z_1{}^* z_2{}^*}}{\left(q_1{}^{*2} + q_2{}^{*2}\right)^2} \leq 0, \frac{\partial L}{\partial q_1} q_1{}^* \equiv 0$, and $q_1{}^* \geq 0$,

$\frac{\partial L}{\partial q_2} = p_2 - 2\gamma_I{}^* q_2{}^* \frac{\sqrt{z_1{}^* z_2{}^*}}{\left(q_1{}^{*2} + q_2{}^{*2}\right)^2} \leq 0, \frac{\partial L}{\partial q_2} q_2{}^* \equiv 0$, and $q_2{}^* \geq 0$,

$\frac{\partial L}{\partial z_1} = -r_1 + \gamma_I{}^* \frac{\sqrt{z_2{}^*}}{2\sqrt{z_1{}^*}\left(q_1{}^{*2} + q_2{}^{*2}\right)} \leq 0, \frac{\partial L}{\partial z_1} z_1{}^* \equiv 0$, and $z_1{}^* \geq 0$,

$\frac{\partial L}{\partial z_2} = -r_2 + \gamma_I{}^* \frac{\sqrt{z_1{}^*}}{2\sqrt{z_2{}^*}\left(q_1{}^{*2} + q_2{}^{*2}\right)} \leq 0, \frac{\partial L}{\partial z_2} z_2{}^* \equiv 0$, and $z_2{}^* \geq 0$,

PM4$^{\text{exp}}$ $\frac{\partial L}{\partial \gamma_I} = \frac{\sqrt{z_1{}^* z_2{}^*}}{q_1{}^{*2} + q_2{}^{*2}} - 1 \geq 0$, $\frac{\partial L}{\partial \gamma_I}\gamma_I{}^* \equiv 0$, and $\gamma_I{}^* \geq 0$.

Equation **PM3$^{\text{exp}}$** and a little algebra yields $z_2{}^* = \frac{r_1}{r_2} z_1{}^*$, $q_1{}^* = 4z_1{}^* \frac{p_1 r_1}{(p_1{}^2 + p_2{}^2)}$, and $q_2{}^* = 4z_1{}^* \frac{p_2 r_1}{(p_1{}^2 + p_2{}^2)}$ assuming the solution is interior. Substitution into equation **PM4$^{\text{exp}}$** then yields $z_1(p_1, p_2, r_1, r_2) = \frac{p_1{}^2 + p_2{}^2}{16 r_1^{\frac{3}{2}} r_2^{\frac{1}{2}}}$ such that $z_2(p_1, p_2, r_1, r_2) = \frac{p_1{}^2 + p_2{}^2}{16 r_1^{\frac{1}{2}} r_2^{\frac{3}{2}}}$, $q_1(p_1, p_2, r_1, r_2) = \frac{p_1}{4 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}}$, and $q_2(p_1, p_2, r_1, r_2) = \frac{p_2}{4 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}}$. The profit function is then

$$\pi(p_1, p_2, r_1, r_2) = p_1 \frac{p_1}{4 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}} + p_2 \frac{p_2}{4 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}} - r_1 \frac{p_1{}^2 + p_2{}^2}{16 r_1^{\frac{3}{2}} r_2^{\frac{1}{2}}} - r_2 \frac{p_1{}^2 + p_2{}^2}{16 r_1^{\frac{1}{2}} r_2^{\frac{3}{2}}} = \frac{p_1{}^2 + p_2{}^2}{8 r_1^{\frac{1}{2}} r_2^{\frac{1}{2}}}.$$

GENERAL PROFIT MAXIMIZATION

The analysis above assumes that we know what outputs we want to produce and what inputs we want to use to produce these outputs. However, these decisions could be influenced by prices. Fortunately, everything we have done so far can be generalized in a straightforward manner.

In more general terms, the profit maximization problem can be framed as

PM6 $\mathbf{Y}(\mathbf{p}) = \{\mathbf{y} \in \mathbf{PPS} : \mathbf{p} \cdot \mathbf{y} \geq \mathbf{p} \cdot \mathbf{y}' \text{ for all } \mathbf{y}' \in \mathbf{PPS}\}$

where $\mathbf{p}$ is now a $1 \times L$ vector of strictly positive prices and $\mathbf{Y}(\mathbf{p})$ is referred to as supply. Again, $\mathbf{Y}(\mathbf{p})$ in general can be a set of vectors, so if we want to refer to any particular vector in this set, we can write $\mathbf{y}(\mathbf{p})$. Homogeneity of degree zero in $\mathbf{p}$ follows from this definition just as before. So does the condition

$$(\mathbf{p}^1 - \mathbf{p}^0) \cdot (\mathbf{y}^1 - \mathbf{y}^0) \geq 0,$$

for $\mathbf{y}^0 = \mathbf{y}(\mathbf{p}^0)$ and $\mathbf{y}^1 = \mathbf{y}(\mathbf{p}^1)$, so supply will be non-decreasing in its own price.

If the transformation function $T(\mathbf{y}) = 0$ is nicely differentiable, the problem can be written as

PM7 $\max_{\mathbf{y}} \mathbf{p} \cdot \mathbf{y}$ subject to $T(\mathbf{y}) = 0$

and solved using Lagrange methods:

PM8 $L = \mathbf{p}\cdot\mathbf{y} + \gamma T(\mathbf{y})$

with the first-order conditions

PM9 $\frac{\partial L}{\partial y_l} = p_l + \gamma^* \frac{\partial T(\mathbf{y}^*)}{\partial y_l} = 0$ for $l = 1,\ldots, L$

PM10 $\frac{\partial L}{\partial \gamma} = T(\mathbf{y}^*) = 0.$

Equation PM9 implies

PM11 $\frac{\frac{\partial T(\mathbf{y}^*)}{\partial y_l}}{\frac{\partial T(\mathbf{y}^*)}{\partial y_k}} = \frac{p_l}{p_k}$ for all l and k.

The left-hand side of equation PM11 is our *MRT* if $y_l^* > 0$ and $y_k^* > 0$. Alternatively, it is just the *MRTS* if $y_l^* < 0$ and $y_k^* < 0$. The profit function is defined as $\pi(\mathbf{p}) = \mathbf{p} \cdot \mathbf{y}(\mathbf{p})$, which will be homogeneous of degree one in $\mathbf{p}$. Indeed, if $\mathbf{Y}(\mathbf{p})$ is supply and $\pi(\mathbf{p})$ is the profit function derived for a production possibilities set that is nonempty, closed, and satisfies free disposal, then

(i) $\mathbf{Y}(\mathbf{p})$ is homogeneous of degree zero and $\pi(\mathbf{p})$ is homogeneous of degree 1 in $\mathbf{p}$;

(ii) $(\mathbf{p}^1 - \mathbf{p}^0) \cdot (\mathbf{y}(\mathbf{p}^1) - \mathbf{y}(\mathbf{p}^0)) \geq 0$ for all $\mathbf{p}^1$ and $\mathbf{p}^0$;

(iii) $\pi(\mathbf{p})$ is convex and continuous in $\mathbf{p}$;

(iv) $\mathbf{Y}(\mathbf{p})$ is a convex set for all $\mathbf{p}$ if **PPS** is convex;

(v) $\pi(\mathbf{p})$ is differentiable with respect to $\mathbf{p}$ at $\mathbf{p}^0$ and $\frac{\partial \pi(\mathbf{p}^0)}{\partial p_l} = y_l(\mathbf{p}^0)$ for $l = 1,\ldots, L$ if $Y(\mathbf{p}^0)$ is a singleton set (Hotelling's Lemma);

(vi) $\mathbf{D_p y}(\mathbf{p}^0) = \mathbf{D_p}^2\pi(\mathbf{p}^0)$ is a symmetric and positive semi-definite matrix with $\mathbf{D_p y}(\mathbf{p}^0)\, \mathbf{p}^0 = \mathbf{D_p}^2\pi(\mathbf{p}^0)\, \mathbf{p}^0 = \mathbf{0}_L$ if $\mathbf{y}(\mathbf{p}^0)$ is differentiable at $\mathbf{p}^0$.

None of these properties should be terribly surprising by now given what we have learned about cost minimization and revenue maximization.

LONG-RUN VERSUS SHORT-RUN

Previously, we explored differences in long-run and short-run costs by dividing our inputs $\mathbf{z}$ into variable and fixed inputs: $\mathbf{z}^v$ and $\mathbf{z}^f$. We also divided our price vector $\mathbf{r}$ into the prices paid for variable and fixed inputs: $\mathbf{r}^v$ and $\mathbf{r}^f$. We can do the same to compare long-run and short-run profit maximization:

PM12 $\max_{\mathbf{y}^v} \mathbf{p}^v \cdot \mathbf{y}^v + \mathbf{p}^f \cdot \mathbf{y}^f$ subject to $T(\mathbf{y}^v, \mathbf{y}^f) = 0,$

where $\mathbf{p}^v$ and $\mathbf{y}^v$ are $1\times L^v$ vectors of variable prices and supply, and $\mathbf{p}^f$ and $\mathbf{y}^f$ are $1\times L^f$ vectors of fixed prices and supply such that $L = L^v + L^f$. The Lagrangian and first order conditions are

PM13 $L = \mathbf{p}^v \cdot \mathbf{y}^v + \mathbf{p}^f \cdot \mathbf{y}^f + \gamma^{sr} T(\mathbf{y}^v, \mathbf{y}^f),$

PM14 $\frac{\partial L}{\partial y_l^v} = p_l^v + \gamma^{sr*} \frac{\partial T(\mathbf{y}^{v*}, \mathbf{y}^{f*})}{\partial y_l^v} = 0$ for $l = 1,\ldots, L^v$,

PM15 $\frac{\partial L}{\partial \gamma^{sr}} = T(\mathbf{y}^{v*}, \mathbf{y}^{f*}) = 0.$

The solution to this problem can be written as the set $\mathbf{Y}^v(\mathbf{p}^v, \mathbf{y}^f)$ with any particular solution in this set referred to as $\mathbf{y}^v(\mathbf{p}^v, \mathbf{y}^f)$. This solution doesn't depend on the prices of fixed supply. The short-run profit function is $\pi(\mathbf{p}, \mathbf{y}^f) = \mathbf{p}^v \cdot \mathbf{y}^v(\mathbf{p}^v, \mathbf{y}^f) + \mathbf{p}^f \cdot \mathbf{y}^f$. Recalling the solution to equation PM7, it is possible to show that $\mathbf{y}(\mathbf{p}) = \mathbf{y}^v(\mathbf{p}^v, \mathbf{y}^f(\mathbf{p}))$, which implies $\pi(\mathbf{p}) = \pi(\mathbf{p}, \mathbf{y}^f(\mathbf{p}))$. It also implies

PM16 $$g(\mathbf{p}, \mathbf{z}^f) = \pi(\mathbf{p}) - \pi(\mathbf{p}, \mathbf{y}^f) \geq 0$$

for all $\mathbf{z}^f$. Equation PM16 holds with equality for $\mathbf{y}^{f0} = \mathbf{y}^f(\mathbf{p})$, so $g(\mathbf{p}, \mathbf{z}^f)$ is at a minimum. Differentiating with respect to $\mathbf{p}_l^v$, then yields the first and second order conditions that must be satisfied:

PM17 $$\frac{\partial \pi(p)}{\partial p_l^v} - \frac{\partial \pi(p, y^{f0})}{\partial p_l^v} = y_l(\mathbf{p}) - y_l(\mathbf{p}^v, \mathbf{y}^{f0}) = 0$$

PM18 $$\frac{\partial^2 \pi(\mathbf{p})}{\partial p_l^{v2}} - \frac{\partial^2 \pi(\mathbf{p}, \mathbf{y}^{f0})}{\partial p_l^{v2}} = \frac{\partial y_l(\mathbf{p})}{\partial p_l^v} - \frac{\partial y_l^v(\mathbf{p}^v, \mathbf{y}^{f0})}{\partial p_l^v} \geq 0$$

Equation PM17 doesn't tell us anything we didn't already know. Equation PM18 tells us supply will be more elastic in the long-run: $\frac{\partial y_l(\mathbf{p})}{\partial p_l^v} \frac{p_l^v}{y_l(\mathbf{p})} \geq \frac{\partial y_l^v(\mathbf{p}^v, \mathbf{y}^{f0})}{\partial p_l^v} \frac{p_l^v}{y_l^v(\mathbf{p}^v, \mathbf{y}^{f0})}$ for $y_l^v(\mathbf{p}^v, \mathbf{y}^{f0}) = y_l^v(\mathbf{p}) > 0$. Intuitively, in the long-run, we have more options for responding to price changes, so our supplies will be more price responsive.