Applied Microeconomics: Firm and Household

Lecture 4: Utility Maximization

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Outline

Utility Maximization

- Deriving Marshallian Demand Curves (uncompensated demand)
- Properties of Marshallian Demand Curves
- An example with Cobb-Douglas utility
- The dual approach to utility maximization
    - Duality
    - The indirect utility function
    - Roy’s identity

Utility maximization

Consider a consumer's utility maximization problem subject to a budget constraint. Denote the utility function for n goods and the budget constraint as:

- $u = u(x_1, x_2, ..., x_n)$
- $\sum_i^n p_i x_i = M$

where p_i is the price of good i and M is money income. For the case of two goods we can set up the problem as

- $\underset{x_1, x_2}{\text{Max}} \quad u(x_1, x_2) \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = M$

To solve this constrained optimization problem we first set up the Lagrangian function

- $\underset{x_1, x_2}{\text{Max}} \quad L = u(x_1, x_2) + \lambda(M - p_1 x_1 - p_2 x_2)$

Utility maximization: FOCs

- $\underset{x_1,x_2}{\text{Max}} \quad L = u(x_1, x_2) + \lambda(M - p_1 x_1 - p_2 x_2)$

The necessary condition for an interior optimum is that the first partials of the Lagrangian equal zero.

1. $\frac{\partial L}{\partial x_1} = L_1 = u_1 - \lambda p_1 = 0$
2. $\frac{\partial L}{\partial x_2} = L_2 = u_2 - \lambda p_2 = 0$
3. $\frac{\partial L}{\partial \lambda} = L_\lambda = M - p_1 x_1 - p_2 x_2 = 0$

From FOC 1 and 2 we can derive the following equilibrium condition

- $\frac{u_1}{u_2} = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2}$
- $MRS_{12} = \frac{p_1}{p_2}$

The first order conditions imply that at the optimum the MRS of the two goods equals their price ratio.

Utility maximization: Graphical interpretation

- $MRS_{12} = \frac{u_1}{u_2} = \frac{p_1}{p_2}$

The condition for an optimum states that a utility-maximizing individual consumes goods x_1 and x_2 at levels where the ratio of prices of the goods equals to the ratio of marginal utilities of the goods (MRS).

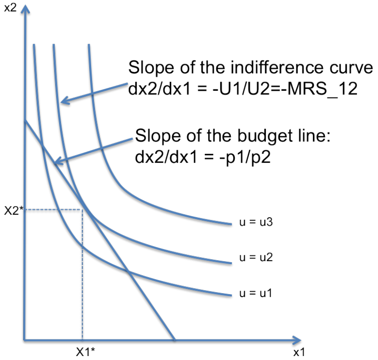

Figure: Graph of utility maximization

Utility maximization: Second order sufficient condition

The matrix of second partials of the utility maximization problem is a **bordered Hessian.** To form the bordered Hessian we differentiate the FOCs.

1. $L_1 = u_1 - \lambda p_1 = 0$
2. $L_2 = u_2 - \lambda p_2 = 0$
3. $L_\lambda = M - p_1 x_1 - p_2 x_2 = 0$

The second partials are:

- $L_{11} = u_{11}, \quad L_{12} = u_{12}, \quad L_{1\lambda} = -p_1$
- $L_{21} = u_{21}, \quad L_{22} = u_{22}, \quad L_{2\lambda} = -p_2$
- $L_{\lambda 1} = -p_1, L_{\lambda 2} = -p_2, L_{\lambda\lambda} = 0$

Utility maximization: Second order sufficient condition

Using the second partials we can form the bordered Hessian:

- $H_b = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{vmatrix}$

Check if $H_{b2} < 0$ and $H_{b3} > 0$ (SOC for a maximum). So long as the SOC holds and $H_b \neq 0$, we can solve for the system of FOC equations to obtain

- $x_1 = x_1^*(p_1, p_2, M)$
- $x_2 = x_2^*(p_1, p_2, M)$
- $\lambda = \lambda^*(p_1, p_2, M)$

x_1^* and x_2^* are the uncompensated demand functions. Also commonly referred as *Marshallian demand functions* or *Money-income-held-constant demand functions*.

Non-uniqueness of utility functions

Proposition: The demand functions derived from the utility maximization problem are identical when $u(x_1, x_2)$ is replaced by $v(u(x_1, x_2))$, where $V'(u) > 0$ (i.e., v is monotonic transformation of u).

Proof: When $u(.)$ is replaced by $v(.)$, the optimality condition obtained from the FOCs is

- $\frac{v_1}{v_2} = \frac{p_1}{p_2}$

However, using chain rule we note that

- $\frac{v_1}{v_2} = \frac{\frac{\partial v}{\partial u} u_1}{\frac{\partial v}{\partial u} u_2} = \frac{u_1}{u_2}$

because $\frac{v_1}{v_2}$ is identical to $\frac{u_1}{u_2}$ everywhere, the FOCs used to derive the demand functions are unchanged. Hence, the demand functions are identical.

Non-uniqueness of utility functions

The implications of the previous result for demand functions are that

- The demand curves are independent of numbers assigned to utility.
  - i.e., the cardinal value of utility has nothing to do with the value or exchange of goods.
- Only the marginal evaluation of goods (that is, only the slope of the indifference curve) matters for the exchange of goods.

Homogeneity of demand functions

Proposition: The demand curves $x_i = x_i(p_1, p_2, M)$ are homogeneous of degree 0 in p_1, p_2, and M. That is, $x_i(tp_1, tp_2, tM) = x_i(p_1, p_2, M)$.

Proof: Suppose all prices and money income are multiplied by some constant, t. then the utility maximization problem becomes

- $\underset{x_1, x_2}{\text{Max}} \quad u(x_1, x_2) \quad \text{s.t.} \quad tp_1x_1 + tp_2x_2 = tM$

Note that the new budget constraint is equivalent to the old one (we can divide by t to recover the old BC). Since the maximization problem is unchanged, the FOCs and their solutions (i.e. the demand functions) are also unchanged. Therefore, demand functions are homogeneous of degree zero.

Homogeneity of demand functions

The zero-degree homogeneity of the demand functions imply that

- Only relative prices and real income matter for consumer decisions
  - i.e. not absolute prices or absolute income
- In other words, because $\frac{u_1}{u_2} = \frac{p_1}{p_2}$ it is the ratio of prices and the ratios of income to prices determine the marginal values of goods and therefore the predicted patterns of exchange.

Interpreting Lagrange multipliers

Question: How can we interpret the Lagrange multiplier?

The Lagrange multiplier is the marginal change in the objective function due to relaxing the constraint by one unit. Thus in this case, λ^* can be interpreted as the marginal utility of money income $\frac{\partial u^*(x_1^*, x_2^*)}{\partial M}$.

We can derive this result by differentiating the optimal utility, u^*, with respect to income, M.

- $u^* = u(x_1^*(p_1, p_2, M), x_2^*(p_1, p_2, M))$
- $\frac{\partial u^*}{\partial M} = \underbrace{u_1^*}_{\lambda^* p_1} \frac{\partial x_1^*}{\partial M} + \underbrace{u_2^*}_{\lambda^* p_2} \frac{\partial x_2^*}{\partial M}$

Interpretation of Lagrange multiplier

Substituting FOCs 1 and 2 into the above equation we obtain

- $\frac{\partial u^*}{\partial M} = \lambda^* p_1 \frac{\partial x_1^*}{\partial M} + \lambda^* p_2 \frac{\partial x_2^*}{\partial M}$
- $\frac{\partial u^*}{\partial M} = \lambda^* (p_1 \frac{\partial x_1^*}{\partial M} + p_2 \frac{\partial x_2^*}{\partial M})$

The term in parentheses is equal to 1 because of FOC 3

- $\frac{\partial u^*}{\partial M} = \lambda^* \underbrace{(p_1 \frac{\partial x_1^*}{\partial M} + p_2 \frac{\partial x_2^*}{\partial M})}_{=1}$
- $M = p_1 x_1^* - p_2 x_2^*$
- $\frac{\partial M}{\partial M} = p_1 \frac{\partial x_1^*}{\partial M} + p_2 \frac{\partial x_2^*}{\partial M}$
- $1 = p_1 \frac{\partial x_1^*}{\partial M} + p_2 \frac{\partial x_2^*}{\partial M}$

Thus $\lambda^* = \frac{\partial u^*}{\partial M}$

Example: Cobb-Douglas utility

Consider the Cobb-Douglas utility function $u = x_1^{0.5} x_2^{0.5}$. We seek to derive the corresponding Marshallian demand curves, and their price and income elasticities.

Given the utility function, the consumer's utility-maximization problem is

- $\underset{x_1, x_2}{\text{Max}} \quad x_1^{0.5} x_2^{0.5} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = M$

and the Lagrangian function can be formed as

- $\underset{x_1, x_2, \lambda}{\text{Max}} \quad L = x_1^{0.5} x_2^{0.5} + \lambda(M - p_1 x_1 - p_2 x_2)$

The FOCs are

1. $\frac{\partial L}{\partial x_1} = L_1 = 0.5(\frac{x_2}{x_1})^{0.5} - \lambda p_1 = 0$
2. $\frac{\partial L}{\partial x_2} = L_2 = 0.5(\frac{x_1}{x_2})^{0.5} - \lambda p_2 = 0$
3. $\frac{\partial L}{\partial \lambda} = L_\lambda = M - p_1 x_1 - p_2 x_2 = 0$

Example: Cobb-Douglas utility

To check the second-order sufficient condition (SOSC) we derive the second partials as

- $L_{11} = -0.25x_2^{0.5}x_1^{-1.5}$
- $L_{22} = -0.25x_1^{0.5}x_2^{-1.5}$
- $L_{21} = L_{12} = 0.25x_2^{-0.5}x_1^{-0.5}$
- $L_{\lambda 1} = L_{1\lambda} = -p_1, \quad L_{\lambda 2} = L_{2\lambda} = -p_2, \quad L_{\lambda\lambda} = 0$

Example: Cobb-Douglas utility

Using the derived second partials we can form the bordered Hessian

- $H = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{12} & u_{22} \end{vmatrix}$
- $|H|_2 = u_{11} * 0 - (-p_1) * (-p_1) = -p_1 p_1 < 0$
- $|H|_3 = \underbrace{0(u_{11}u_{22} - u_{21}^2)}_{0} \underbrace{-(-p_1)((\underbrace{-p_1 u_{22}}_{+}) - ((\underbrace{-p_2)u_{21}}_{-})) +}_{+}$

$\underbrace{-p_2((\underbrace{-p_1 u_{12}}_{-}) - (\underbrace{-p_2 u_{11}}_{+}))}_{+} > 0$

Income-Consumption path, and the demand curves

The SOSC holds if we assume strictly positive prices and quantities. From FOCs 1 and 2 we can derive the following equilibrium condition:

- $\frac{L_1}{L_2} = \frac{x_2}{x_1} = \frac{p_1}{p_2} \rightarrow p_1 x_1 = p_2 x_2$

The equation $p_1 x_1 = p_2 x_2$ says that at the optimum the total amount spent on x_1 always equals to the total amount spent on x_2, at any set of prices and income. (Note that we did not use FOC 3 yet, hence this equation holds for all possible income levels.)

Question: What can we say about price and income elasticities of demand?

The Income-Consumption path

The income-consumption path is the locus of all the tangency points of the indifference curves to various budget constraints.

As income M is increased the consumption bundle moves away from the origin along the ray line.

In this example, since the I-C path is a straight line, a given percent increase in M leads same percent increase in consumption of both goods.

Therefore, we expect demand curves derived from $u = x_1^{0.5} x_2^{0.5}$ to possess unitary income elasticities.

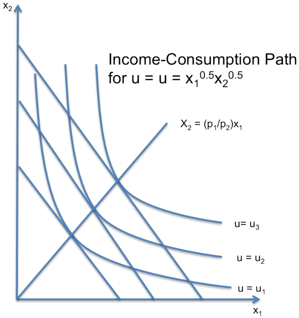

Figure: Graph of income-consumption path

C-D example continued: The demand curves

To derive the demand curves we substitute $p_1 x_1 = p_2 x_2$ into the budget constraint (FOC 3):

- $p_1 x_1 + p_1 x_1 = M$
- $x_1^* = \frac{M}{2p_1}$, Similarly, $x_2^* = \frac{M}{2p_2}$

The own- and cross-price elasticities of demand are

- $\epsilon_{ij} = \frac{\partial x_i}{\partial p_j} \frac{p_j}{x_i}$ for $i, j = 1, 2, ...N$

The income elasticities of demand are

- $\epsilon_{iM} = \frac{\partial x_i}{\partial M} \frac{M}{x_i}$ for $i = 1, 2, ...N$

C-D example continued: The elasticities

The own-price elasticity of each demand curve is given by

- $\epsilon_{ii} = -\frac{M}{2p_i^2}\frac{p_i}{x_i}$, substitute $x_i^* = \frac{M}{2p_i}$
- $\epsilon_{ii} = -\frac{M}{2p_i^2}\frac{2p_i^2}{M} = -1$

Both demand curves are unitary elastic w.r.t own-price and downward sloping.

The cross-price elasticites are both zero, since x_i, is not a function of p_j (if $i \neq j$).

The income elasticities are:

- $\epsilon_{iM} = \frac{1}{2p_i}\frac{M}{x_i}$, substitute $x_i^* = \frac{M}{2p_i}$
- $\epsilon_{iM} = \frac{1}{2p_i}\frac{M2p_i}{M} = 1$

Both demand curves have unit income elasticity.

The dual approach - motivation

- Economic optimization problems generally rely on the underlying preferences (e.g. for consumption decisions) or the udnerlying technology of the process being considered (e.g. for production decisions).
- Duality theory refers to the notion that the characterization of the preferences or technology can be recovered by estimating optimal functions (indirect functions) that are generated from solving the primal problem.
- Duality is a very powerful approach to economic analysis and is applied extensively in empirical literature.

Direct and indirect functions

Consider a general maximization problem with two variables x_1, x_2 and a parameter α

- $\underset{x_1, x_2}{\text{Max}} \; f(x_1, x_2; \alpha)$

In this problem the objective function $f(.)$ is referred as the **direct function.** Assuming the sufficient second order conditions hold, we derive the solutions $x_1^*(\alpha)$ and $x_2^*(\alpha)$ by solving the first order necessary conditions, $f_1 = 0$ and $f_2 = 0$.

If we substitute the solutions back into the objective function we obtain the **indirect function.**

- $\phi(\alpha) = f(x_1^*(\alpha), x_2^*(\alpha); \alpha)$

$\phi(\alpha)$ represents the maximum (i.e. optimized) value of the objective function for any specified value of α.

Contrasting the primal and dual approaches

Primal Approach

Theoretical Aspects

- Optimize direct functions
- Decision makers (producers, consumers)
- Choice variables are usually quantities
- Comparative statics from primal problem FOCs

Dual Approach

Theoretical Aspects

- Work with indirect functions
- Shadow entities (central planner, invisible hand)
- Choice variables are model parameters (prices, income)
- Comparative statics from dual problem FOCs (usually easier)

A contrast of primal and dual approaches

Primal Approach

Applied Aspects

- Estimate derived demand functions from primal FOCs
- Primal problem doesn't always yield a closed-form solution. Therefore, analytic solutions for demand function sometimes don't exist.

Dual Approach

Applied Aspects

- Estimate derived demand functions from dual FOCs (envelope theorem)
- Almost always yields reduced form demand functions.
- Facilitates estimation of a system of equations (i.e., SUR)
- Allows us to introduce complexity in specifications of indirect functions.

Summary: How will duality assist us?

- The dual approach is a consistent paradigm (consistent with underlying economic theory) to derive testable hypotheses.
- It facilitates a system approach to modeling which allows us to impose restrictions from economic theory (i.e., homogeneity, symmetry).
  - This is not an absolute necessity. However, it makes it a lot easier to defend your results and convince the audience.
- The dual approach makes it easier to incorporate higher order approximations of the true underlying behavioral functions. It expands the set and complexity of technologies and preference representations we can explore (facilitates the use of flexible functional forms).

The indirect utility function

In general, the indirect utility function is obtained by substituting the demand functions into the utility function

- $u^* = u(x_1^*(p_1, p_2, M), x_2^*(p_1, p_2, M))$

Note that, just like the demand functions, the indirect utility function is a function of prices and income (model parameters.)

Because, it is the utility-maximizing quantities x_1^* and x_2^* that are substituted in the utility function, $u^*(p_1, p_2, M)$ gives the maximum value of the utility for any given prices and money income, p_1, p_2, M.

The indirect utility function

For each $(p, M) >> 0$ the indirect utility function is

- $u(p, M) \in R = u(x^*);\ \forall x^* \in x(p, M)$

where x^* is the optimal bundle derived from utility maximizing problem.
Properties of an indirect utility function:

- Homogeneous of degree 0 in prices and income
- Strictly increasing in M and non-increasing in p
- Quasiconvex in prices and income
- Continuous in p and M

We can take the dual approach to demand analysis by starting from an arbitrary indirect utility function (approximation via a functional form) and then deriving the system of demand equations via Roy's identity.

Roy's identity

Given an arbitrary indirect utility function we can derive the utility-maximizing Marshallian demand functions using **Roy's Identity**:

- $x_i^* = -\frac{\partial u^* / \partial p_i}{\partial u^* / \partial M}$

A very important result that has strong implications for applied work.

Comes from the **Implicit Function Theorem** – a powerful tool for microeconomic analysis in general.

C-D example continued: Roy's identity

Following on the C-D example we can derive the indirect utility function by substituting the optimal quantities

- $u^*(p_1, p_2, M) = (x_1^*)^{0.5}(x_2^*)^{0.5} = (\frac{M}{2p_1})^{0.5}(\frac{M}{2p_2})^{0.5}$
- $u^*(p_1, p_2, M) = \frac{M}{2p_1^{0.5}p_2^{0.5}}$

We can verify Roy's identity as follows:

- $\frac{\partial u^*}{\partial p_i} = \frac{-0.25M}{p_i^{1.5}p_j^{0.5}} = \frac{-M}{4p_i^{1.5}p_j^{0.5}}$
- $\frac{\partial u^*}{\partial M} = \frac{1}{2p_i^{0.5}p_j^{0.5}}$

Therefore,

- $x_i^* = -\frac{\partial u^*/\partial p_i}{\partial u^*/\partial M} = -\frac{-M}{4p_i^{1.5}p_j^{0.5}} * 2p_i^{0.5}p_j^{0.5}$
- $x_i^* = \frac{M}{2p_i}$