Econ 8010 HW3

Solutions

Nathan Yoder

University of Georgia

Fall 2017

1. The Acme Widget Corporation produces a single output (widgets) $q \in \mathbb{R}_+$ using inputs $z \in \mathbb{R}^3_+$. Its production set is given by

$$\{(q,z) : 1 - q^{\frac{1}{\alpha}}(\beta z_1^\rho + \gamma z_2^\rho + \delta z_3^\rho)^{\frac{1}{\rho}} \leq 0\}$$

for $\alpha < 0$, $\rho < 1$, and $\beta, \gamma, \delta > 0$.

   (a) Find Acme Widget Corporation's production function.

   - $f(z) = (\beta z_1^\rho + \gamma z_2^\rho + \delta z_3^\rho)^{-\frac{\alpha}{\rho}}$.

   (b) Can we assume without loss of generality that $\alpha = -1$? That $\alpha = -\rho$? Why or why not?

   - No. Suppose we scale z by $a > 1$. Then we have

$$f(az) = (a^\rho)^{\frac{-\alpha}{\rho}} f(z) = a^{-\alpha} f(z)$$

     Different values for α necessarily mean different returns to scale (and thus a different production technology) — regardless of the other parameters.

   (c) Can we assume without loss of generality that $\beta + \gamma + \delta = 1$? Why or why not?

- No. Consider the response of production to an increase in z_1:

$$\frac{\partial f}{\partial z_1} = \beta\rho z_1^{\rho-1}\frac{-\alpha}{\rho}f(z)^{\frac{\alpha+\rho}{\alpha}}$$

  We already know that any change in α necessarily changes the production function, regardless of any changes to the other parameters. Suppose we change ρ to ρ' and β to β' so that $f(z)$ (and its derivatives) remains unchanged for some fixed z: Then

$$\begin{aligned}\alpha\beta z_1^{\rho-1}f(z)^{\frac{\alpha+\rho}{\alpha}} &= \alpha\beta' z_1^{\rho'-1}f(z)^{\frac{\alpha+\rho'}{\alpha}}\\ \frac{\beta}{\beta'} &= z_1^{\rho'-\rho}f(z)^{\frac{\rho'-\rho}{\alpha}}\end{aligned}$$

  But this has to hold for every z. Taking a derivative of both sides yields

$$\begin{aligned}0 &= (\rho'-\rho)z_1^{\rho'-\rho-1}f(z)^{\frac{\rho'-\rho}{\alpha}} - \frac{\rho'-\rho}{\alpha}z_1^{\rho'-\rho}f(z)^{\frac{\rho'-\rho}{\alpha}-1}\alpha\beta z_1^{\rho-1}f(z)^{\frac{\alpha+\rho}{\alpha}}\\ 0 &= (\rho'-\rho)z_1^{\rho'-\rho-1}f(z)^{\frac{\rho'-\rho}{\alpha}} - (\rho'-\rho)z_1^{\rho'-1}f(z)^{\frac{\rho'}{\alpha}}\beta\\ 0 &= z_1^{-\rho}f(z)^{-\rho/\alpha} - \beta\end{aligned}$$

  which need not hold for any z, and cannot hold for every z. So we cannot change any parameters and leave f unchanged.

(d) A production function f is said to have

- **increasing returns to scale** if $f(\alpha z) > \alpha f(z)$ for all $z \in \mathbb{R}^L$ and $\alpha > 1$
- **decreasing returns to scale** if $f(\alpha z) < \alpha f(z)$ for all $z \in \mathbb{R}^L$ and $\alpha > 1$
- **constant returns to scale** if it is homogeneous of degree one.

Under what circumstances does Acme Widget Corporation's production function have increasing returns to scale? Constant returns to scale?

- IRTS: $\alpha < -1$. CRS: $\alpha = 1$.

(e) Find Acme's cost function $c(w, q)$ and conditional factor demands $z(w, q)$. If its production function has increasing returns to scale, what does that tell you about $\frac{\partial^2}{\partial q^2}c(w, q)$?

- Let

$$g(w) \equiv \left(\left(\frac{w_1^\rho}{\beta} \right)^{\frac{1}{\rho-1}} + \left(\frac{w_2^\rho}{\gamma} \right)^{\frac{1}{\rho-1}} + \left(\frac{w_3^\rho}{\delta} \right)^{\frac{1}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}$$

- Then

$$\begin{aligned}
z_1(w,q) &= \left(\frac{w_1}{g(w)\beta} \right)^{\frac{1}{\rho-1}} q^{-\frac{1}{\alpha}} \\
z_2(w,q) &= \left(\frac{w_2}{g(w)\gamma} \right)^{\frac{1}{\rho-1}} q^{-\frac{1}{\alpha}} \\
z_3(w,q) &= \left(\frac{w_3}{g(w)\delta} \right)^{\frac{1}{\rho-1}} q^{-\frac{1}{\alpha}} \\
c(w,q) &= g(w) q^{-\frac{1}{\alpha}} \\
\frac{\partial^2}{\partial q^2} c(w,q) &= g(w) \left(\frac{1+\alpha}{\alpha^2} \right) q^{-\frac{1+2\alpha}{\alpha}}
\end{aligned}$$

When f has IRTS, $\alpha < -1$ and so $\frac{\partial^2}{\partial q^2} c(w,q) < 0$.

(f) Find Acme's output supply correspondence $q(w,p)$.

- When $\alpha > -1$:

$$q(w,p) = \left(\frac{g(w)}{-\alpha} \right)^{\frac{\alpha}{1+\alpha}}$$

- When $\alpha = -1$:

$$q(w,p) = \begin{cases} 0, & p < g(w) \\ \mathbb{R}_+, & p = g(w) \\ \emptyset, & p > g(w) \end{cases}$$

- When $\alpha < -1$: $q(w,p) = \emptyset$.

2. A consumer has Cobb-Douglas utility

$$u(x_1, x_2) = \frac{3}{4} \log x_1 + \frac{1}{4} \log x_2$$

Suppose that p_1 increases from 1 to 2.

(a) What other information do you need to calculate the EV and CV of this price change? Why?

- p_2, and initial wealth or initial utility level. Hicksian demand depends upon both p_1 and p_2.

(b) Suppose that $p_2 = 2$ and $w = 12$. Calculate the EV and CV of this price change.

- Using $h_1(p, \bar{u}) = e^{\bar{u}} \left(\frac{\alpha p_2}{(1-\alpha)p_1} \right)^{1-\alpha}$:

$$
\begin{aligned}
CV &= \exp(v((1,2),12)) \int_2^1 6p^{-1/4} dp \\
&= 3^{7/4} 2^{-1/4} * 8 * (1 - 2^{3/4}) = 2^{11/4} 3^{7/4} (1 - 2^{3/4}) \\
CV &= \exp(v((2,2),12)) \int_2^1 6p^{-1/4} dp \\
&= 2 * 3^{7/4} * 8 * (1 - 2^{3/4}) = 16 * 3^{7/4} (1 - 2^{3/4})
\end{aligned}
$$