ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 12: Comparing Distributions, State-Dependent Utility and Subjective Probability Theory
(MWG, Ch. 6, pp.194-207)

I. Introduction

This lecture covers **three topics** on choice under uncertainty:

1. How to **compare distributions in terms of their returns and risk** without reference to a particular utility function.

2. **State-dependent utility**, which allows a person's utility function to vary according to the "state of nature" that the person is in.

3. **Subjective probability theory**, which examines what happens when individuals have their own ideas about what the probabilities are that different outcomes could occur, which may not be the same as the "real" probabilities that those outcomes occur.

II. Comparing Distributions' Returns and Risks

In **Lecture 11** we **compared different utility functions to see** which displayed risk aversion, and **whether one was "more risk averse" than another**.

In **this section** we focus on the distributions of the variable x (amount of money in a gamble/lottery), and **ask whether one distribution** of x "has a higher (or lower) return" than another and whether one **is "riskier" than another**. This is done **without reference to any specific utility function**, although these concepts will be related to very general classes of utility functions.

Comparing distributions to see whether one has a higher or lower return than another leads to the concept of **first-order stochastic dominance**, while comparisons of the riskiness of distributions leads to the concept of **second-order stochastic dominance**.

First-Order Stochastic Dominance

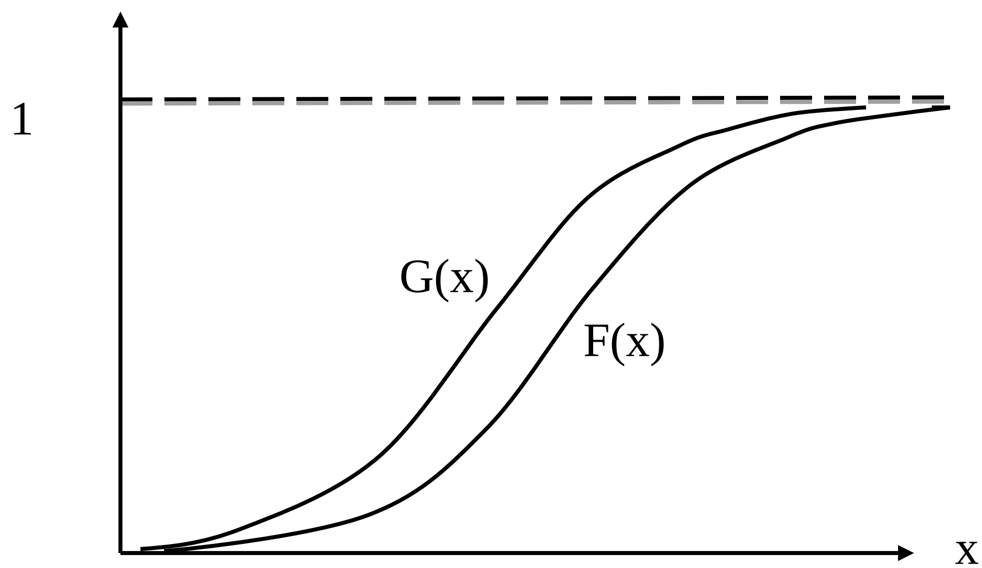
The goal is to attach a precise meaning to claims such as "distribution F( ) has an unambiguously higher return than distribution G( )".

Thinking about how to make this concept precise leads to (at least) **two possibilities**:

1. The distribution F( ) has a higher return than the distribution G( ) if, for all u(x) with $u'(x) \geq 0$ for all x, F( ) yields higher expected utility than G( ).

2. For any possible value of x, the probability that x is equal or greater than that value of x is higher for F( ) than it is for G( ). That is, $1 - F(x) \geq 1 - G(x)$ for all x, which implies that $F(x) \leq G(x)$ for all x.

The following diagram gives a visual depiction of the second possible definition of one distribution having a higher return than another. In this diagram, F( ) has a higher expected return than G( ).



This diagram shows that, for any value x, the probability that income (x) is less than that value is higher for G(x) than for F(x). So most people prefer F(x)!

In fact, **these two concepts are equivalent**. To see why, let's start with a definition of stochastic dominance:

Definition 6.D.1: The distribution F( ) **first-order stochastically dominates** G( ) if, for every non-decreasing utility function u( ), that is u′( ) ≥ 0, we have:

$$\int_0^\infty u(x)dF(x) \geq \int_0^\infty u(x)dG(x)$$

That is, expected utility of F(x) ≥ expected utility of G(x).

This definition is based on the first possibility. The following proposition shows that it is equivalent to the second possibility:

Proposition 6.D.1: The distribution of monetary payoffs F( ) first-order stochastically dominates the distribution G( ) if and only if F(x) ≤ G(x) for all x.

Proof: To show the "only if" part it suffices to show that if F(x) > G(x) for some x then there exists a utility function for which stochastic dominance does not hold. Let H(x) ≡ F(x) – G(x). For some $\bar{x}$, suppose H($\bar{x}$) > 0. Note that this contradicts the requirement that F(x) ≤ G(x) for all x. Define a non-decreasing utility function u( ) as u(x) = 1 for x > $\bar{x}$ and u(x) = 0 for x ≤ $\bar{x}$. For this utility function, you can show that $\int_0^\infty u(x)dH(x) = -H(\bar{x}) < 0$.

[$\int_0^\infty u(x)dH(x) = \int_0^{\bar{x}} u(x)dH(x) + \int_{\bar{x}}^\infty u(x)dH(x) = 0 + \int_{\bar{x}}^\infty dH(x)$ $= H(\infty) - H(\bar{x}) = (F(\infty) - G(\infty)) - H(\bar{x}) = -H(\bar{x})$.] Thus $\int_0^\infty u(x)dF(x) - \int_0^\infty u(x)dG(x) < 0$, so stochastic dominance, $\int_0^\infty u(x)dF(x) \geq \int_0^\infty u(x)dG(x)$, does not hold for this u(x).

[Draw picture of F(x) and G(x) at $\bar{x}$ to show intuition.]

To show the "if" part, we show this only for utility functions u( ) that are differentiable. Again, define H(x) ≡ F(x) – G(x). Integrating by parts gives:

$$\int_0^\infty u(x)dH(x) = [u(x)H(x)]_0^\infty - \int_0^\infty u'(x)H(x)dx$$

Since H(0) = 0 and H(∞) = 0 (since F(∞) = G(∞) = 1), the term in brackets is zero. Thus $\int_0^\infty u(x)dH(x)$, which equals $\int_0^\infty u(x)dF(x) - \int_0^\infty u(x)dG(x)$, is ≥ 0 if and only if the last term is ≤ 0. Since H(x) = F(x) – G(x) is ≤ 0, and u′( ) ≥ 0, for all x then $\int_0^\infty u'(x)H(x)dx \leq 0$. **Q.E.D.**

Note: If F(x) stochastically dominates G(x) then it must be that $E[x_F] > E[x_G]$. **However**, $E[x_F] > E[x_G]$ does **not** necessarily imply that F(x) stochastically dominates G(x).

See pp.196-197 of Mas Colell et al. for another way to describe stochastic dominance. If you have a variable x with distribution G( ), and define a new variable x + z, where z never takes negative values, then the distribution of x + z stochastically dominates the distribution of x.

Second-Order Stochastic Dominance

First-order stochastic dominance indicates that one distribution is clearly better than another no matter what a decision maker's risk aversion is. Thus it really **does not tell us anything about the risk** (or dispersion) **of one distribution relative to another** distribution.

This subsection will focus on risk. In order to focus on risk alone, the **discussion will be limited to distributions that have the same mean**.

Suppose we have two distributions of x, F( ) and G( ), that have the same mean. One way to define the concept that, say, G( ) is clearly riskier than F( ) is to require that **any** decision maker who is risk averse prefers F( ) over G( ). Indeed, we will use this to define second-order stochastic dominance:

Definition 6.D.2: For any two distributions F( ) and G( ) with the same mean, F( ) **second-order stochastically dominates** (or **is less risky than**) G( ) if, for every non-decreasing ***concave*** function u( ), i.e. $u''(\ ) \leq 0$, we have:

$$\int_0^\infty u(x)dF(x) \geq \int_0^\infty u(x)dG(x)$$

Here is another, equally intuitive, way to define risk which is mathematically equivalent to second-order stochastic dominance:

Example 6.D.2: Mean-Preserving Spreads. Consider a compound lottery. In the first stage we have the "lottery" of x which has a distribution F( ). In the second stage a second lottery is done by adding a variable z to x, so that the final "payoff" is x + z, and z has the distribution function $H_x(\ )$ with E[z| x] = 0. Note that the distribution of z could be different for different "draws" of x in the first lottery, hence the x subscript for $H_x(\ )$. Denote the reduced (combined) lottery by G( ), and notice that $E[x_F] = E[x_G]$ since E[z| x] = 0 for all x. This G( ) is called a **mean preserving spread** of F( ).

An important property of mean preserving spreads is that, for any u( ) that is concave, we have:

$$\int_0^\infty u(x)dG(x) =$$

$$\int_0^\infty \left(\int_{-\infty}^\infty u(x+z)dH_x(z)\right) dF(x) \le \int_0^\infty u\left(\int_{-\infty}^\infty (x+z)dH_x(z)\right) dF(x)$$

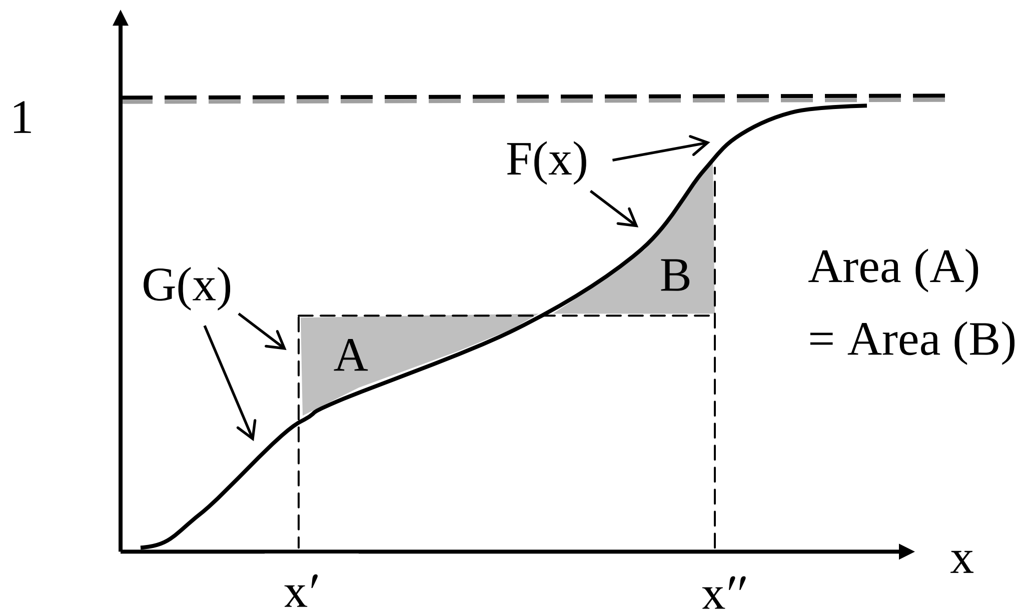
$$= \int_0^\infty u(x)dF(x)$$

The second line follows from Jensen's inequality.

Thus if G( ) is a mean-preserving spread of F( ), then F( ) second-order stochastically dominates G( ).

The converse is also true: if F( ) second-order stochastically dominates G( ), and $E[x_F] = E[x_G]$, then G( ) is a mean-preserving spread of F( ).

Here is an example, with an accompanying figure, of second-order stochastic dominance and a mean-preserving spread.



Example 6.D.3: An Elementary Increase in Risk. G( ) is an **elementary increase in risk** relative to F( ) if G( ) is generated from F( ) by taking all the mass that F( ) assigns to an interval [x′, x″] and transferring it to the endpoints x′ and x″ in a way that the mean is preserved.

There is yet another way to characterize second-order stochastic dominance. Consider two distributions, F( ) and G( ), with the same mean. Assume that the maximum value of x in both distributions is $\bar{x}$. This implies that $F(\bar{x}) = G(\bar{x}) = 1$. Integrating by parts implies the following:

$$\int_0^{\bar{x}}(F(x) - G(x))dx = \int_0^{\bar{x}} x\, d(F(x) - G(x)) + (F(\bar{x}) - G(\bar{x}))\bar{x} = 0$$

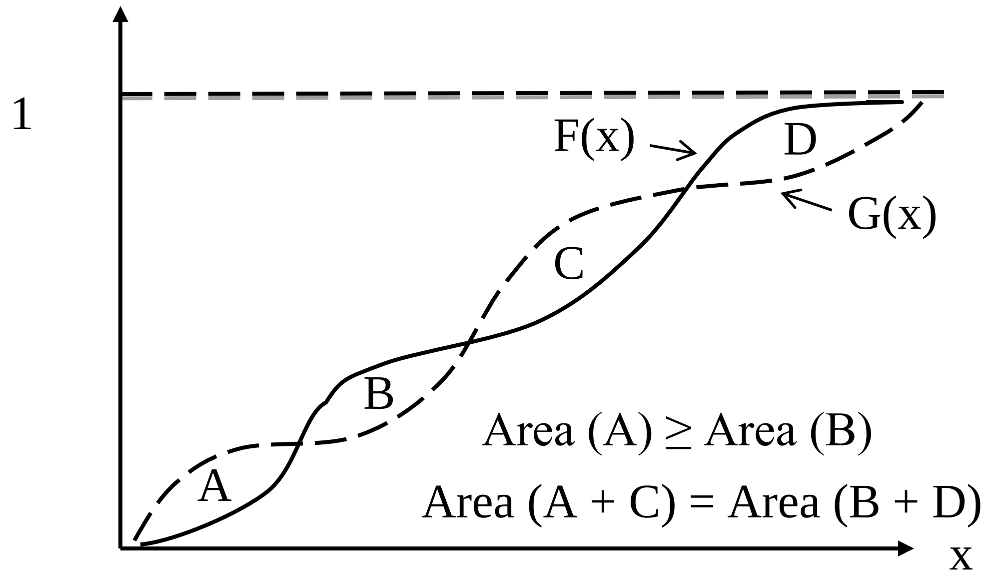
Question: What is $\int_0^{\bar{x}} x\, d(F(x) - G(x))$ equal to?

This means that the areas from 0 to $\bar{x}$ under the two distribution functions F( ) and G( ) are equal as long as F( ) and G( ) have the same mean. Thus the areas A and B in the figure on the previous page must have the same area. It is also the case, for that figure, that:

$$\int_0^x G(t)dt \geq \int_0^x F(t)dt, \text{ for all } x$$

More generally, **for any two distributions F( ) and G( ) with the same means, the above expression is equivalent to** the property that **F( ) second-order stochastically dominates G( )**.

A more elaborate example of this can be seen in the following figure:



Question: Why is it that Area (A + C) = Area (B + D)?

As long as the area A ≥ the area B then the condition near the bottom of p.9 will hold, and thus F(x) second-order stochastically dominates G( ).

The following proposition sums up what we have seen regarding second-order stochastic dominance:

Proposition 6.D.2: Consider 2 distributions, F( ) and G( ), with the same mean. The following statements are equivalent:

1. F( ) second-order stochastically dominates G( ).
2. G( ) is a mean preserving spread of F( )
3. $\int_0^x G(t)dt \geq \int_0^x F(t)dt$, for all x

III. State-Dependent Utility

Thus far, we have assumed that all **the decision maker** cares about is the money payoffs, and he or she **does not care about any underlying "state of nature"** that led to the payoff received. However, there could be situations where a person cares. For example, suppose that your income is low because you are ill. In this case your utility depends not just on the money you have but also on the state that "caused" your income to be low.

This section develops a framework that allows utility to depend not only on money but also on the "state of nature" that is associated with the random outcome that "causes" a specific amount of money to be received. For analytical convenience, we assume the possible number of money payoffs is finite.

To start, assume there are S states of nature, each of which is denoted by s = 1, 2, … S. Each state of nature has a well-defined probability, denoted by π_s, where $\sum_{s=1}^{S} \pi_s = 1$.

Formally, there is a **function g(s)** that **provides the monetary payoffs for each state s**, that is $\mathbf{x_s = g(s)}$. This leads to the following definition:

Definition 6.E.1: A **random variable** is a function g(s) that maps each state into a monetary outcome.

For each possible **random variable** g( ), that is each possible distribution of monetary outcomes over the possible states, **there is an associated monetary lottery** that has a distribution function F( ) that can be defined as:

$$F_{g(\,)}(x) = \sum_{s:g(s)\leq x} \pi_s, \text{ for all } x \quad \text{(I added "g( )" subscript)}$$

Note that **this lottery representation "loses information"** since it does not indicate which state was associated with the distribution of monetary outcomes. **Give a simple example.**

State-Dependent Preferences and the Extended Expected Utility Representation

The fundamental building block of choice theory is the rational preference relation $\succsim$ on the set $\mathbb{R}_+^S$ of non-negative random variables (x_1 to x_S), given probabilities $\pi_1, \pi_2 \ldots \pi_S$. However, it is very convenient to represent these preferences over money outcomes with a utility function that takes the **extended expected utility form**:

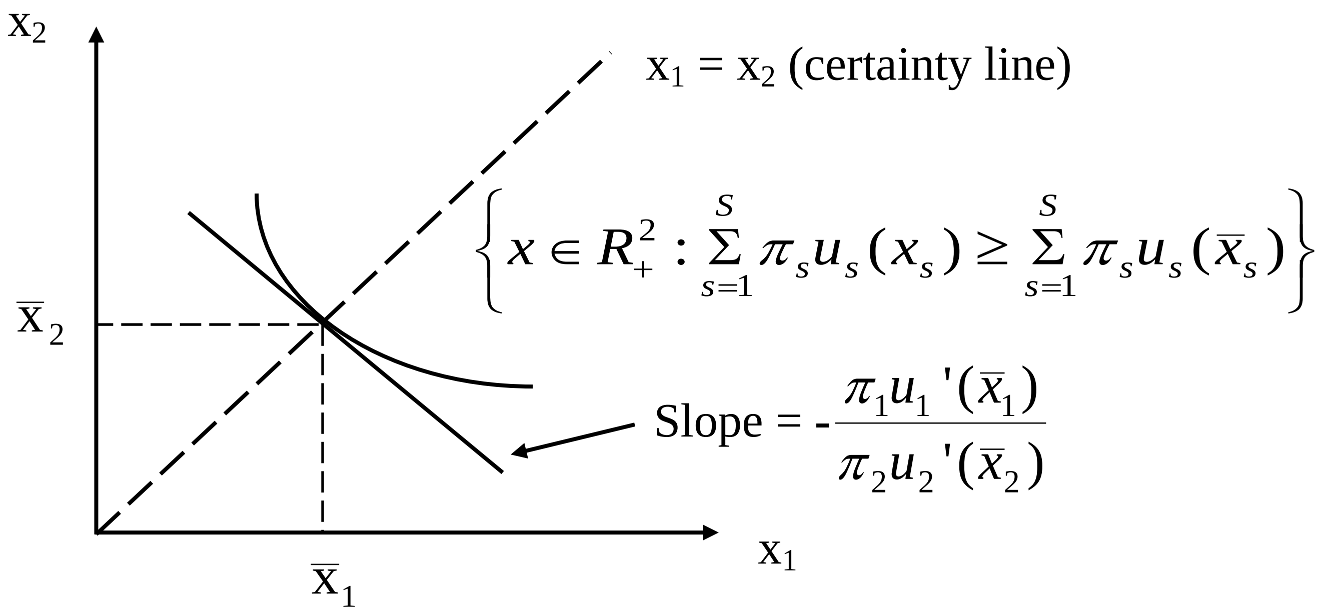
Definition 6.E.2: The preference relation $\succsim$ has an **extended expected utility representation** if, for every $s \in S$, there is a function $u_s(\;)$ such that for any $(x_1, x_2, \ldots x_S) \in \mathbb{R}_+^S$ and $(x_1', x_2', \ldots x_S') \in \mathbb{R}_+^S$, we have:

$$(x_1, x_2, \ldots x_S) \succsim (x_1', x_2', \ldots x_S') \text{ if and only if } \sum_{s=1}^{S} \pi_s u_s(x_s) \geq \sum_{s=1}^{S} \pi_s u_s(x_s')$$

At first glance this seems to be the same expected utility function that we have been using in the last two lectures. But there is an important difference, which is **the $u_s(\ \ )$ function has an s subscript**. That is, there could be a different utility function for each possible state s.

To see why the extended expected utility representation is a useful "tool", consider the **money certainty line**, which is simply the points in the $\mathbb{R}_+^S$ space that pay the same amount for all possible states. This is shown in the diagram below for the case when S = 2.

This graph does not show π_1 or π_2, but the indifference curve drawn reflects those values (the π's are constants).



The marginal rate of substitution (between the probabilistic payoffs x_1 and x_2) at the point where the indifference curve crosses the certainty line is:

$$\pi_1 u_1'(\bar{x})/\pi_2 u_2'(\bar{x})$$

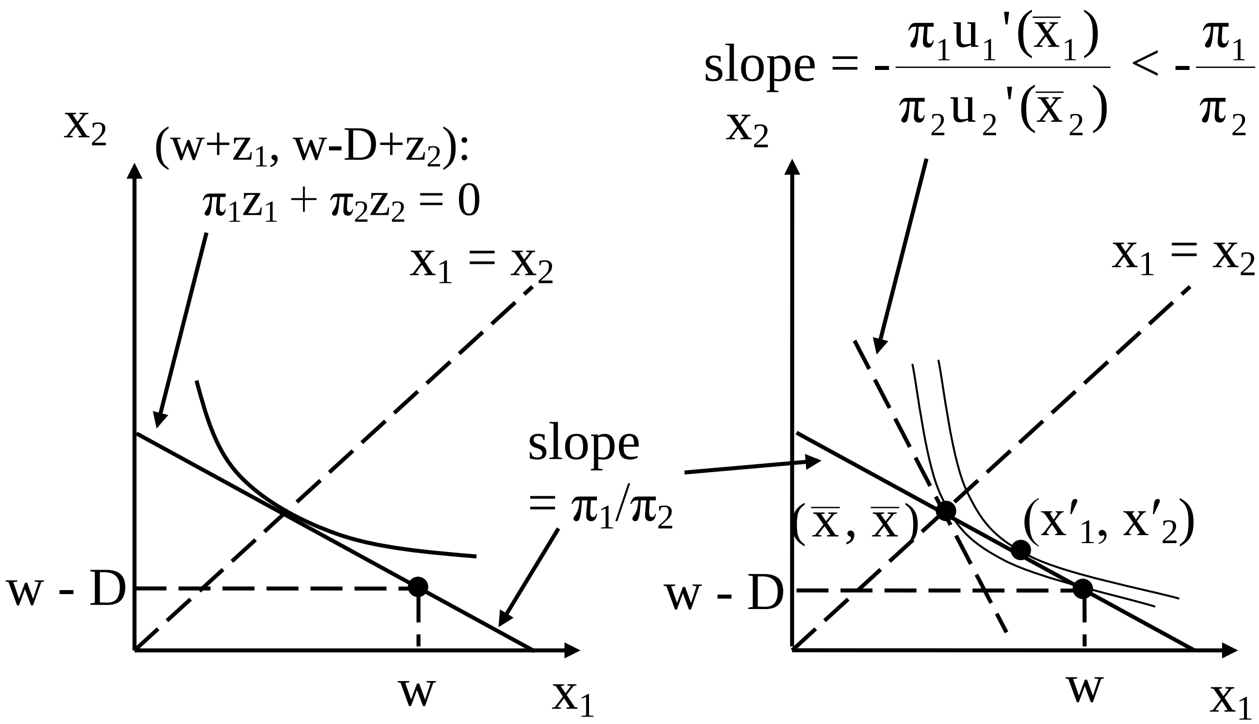
[Show that this is the slope of the indifference curve for any x_1 and x_2 by total differentiation.]

Question: Why does $\bar{x}$ not have a 1 or 2 subscript?

Note that the slope depends not only on the probabilities but also on the $u_1(\ )$ and $u_2(\ )$ functions. **If there were no state dependence of utility** (which is called "state-independent" or "state-uniform" utility) **then the slope of the indifference curve would simply be π_1/π_2.**

Example 6.E.1: Insurance with State-Dependent Utility
Recall the insurance example in Lecture 11, which showed that any risk averse person would "fully" insure risk when faced with "actuarially fair" insurance. Is this still the case with state-dependent utility? Without insurance, the situation faced is one with a random variable that shows the wealth in the 2 states: (w, w – D).

The **figure below on the left** shows this. In it, an insurance contract is a random variable (z_1, z_2) specifying the net payment by the insurance company in the two states. It is actuarially fair if $\pi_1 z_1 + \pi_2 z_2 = 0$. We saw in Lecture 11 that a person with ***state-independent*** utility would by insurance to the point where $x_1 = x_2$.



The right figure is the case of state-dependent preferences. The insurance contracts are the same, so the budget set is the same, but in this case it is possible that the slope of the indifference curve at point $(\bar{x}, \bar{x})$, i.e. $\pi_1 u_1'(\bar{x})/\pi_2 u_2'(\bar{x})$, is not π_1/π_2. As drawn, there is a higher payment if state 1 occurs, since $u_1'(\bar{x})$ is higher than $u_2'(\bar{x})$.

Question: Does this person overinsure or underinsure?

Existence of an Extended Expected Utility Representation

The following definition and theorem shows what **assumptions** are **needed** regarding preferences **to assure the existence of an extended expected utility function**. This is very similar to the assumptions required to obtain the "standard" expected utility function.

To start, note that extended expected utility implies that:

$$(x_1, x_2, \ldots x_S) \succsim (x'_1, x'_2, \ldots x'_S) \text{ if and only if } \sum_{s=1}^{S} u_s(x_s) \geq \sum_{s=1}^{S} u_s(x_s')$$

Why are there no π's in this? Because we can (re)define $u_s(\ )$ for each s to include π_s.

To extend expected utility theory to situations of state-dependent utility, let $\mathbf{x_s}$ for each s be **not a fixed amount of money** but a **random variable** with an associated **distribution $\mathbf{F_s(\ )}$**. All of these distribution functions together can be denoted by $L = (F_1(\ ), F_2(\ ), \ldots F_S(\ ))$. That is, L is like a compound lottery that assigns money amounts contingent on the realization of state s. **Denote by $\mathcal{L}$ all possible such lotteries**.

A corresponding reduced lottery between L and L′ can be written as:

$$\alpha L + (1\text{-}\alpha)L' = \alpha F_1(\ \ ) + \alpha F_2(\ \ ), + \ldots + \alpha F_S(\ \ )$$
$$+ (1\text{-}\alpha)F'_1(\ \ ) + (1\text{-}\alpha)F'_2(\ \ ), + \ldots + (1\text{-}\alpha)F'_S(\ \ )$$

Now we can define the extended independence axiom:
Definition 6.E.3: The preference relation $\succsim$ on $\mathscr{L}$ satisfies the **extended independence axiom** if, for all L, L′ and L″ $\in \mathscr{L}$, and $\alpha \in (0, 1)$, we have:

$$L \succsim L' \text{ if and only if } \alpha L + (1\text{-}\alpha)L'' \succsim \alpha L' + (1\text{-}\alpha)L''$$

Now we can **see what assumptions are needed for** the **existence of an extended expected utility function**:

Propostion 6.E.1: Extended Expected Utility Theorem. Suppose that the preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and extended independence axioms. Then we can assign a utility function $u_s(\ \ )$ for money in every state s such that, for any $L = (F_1, F_2, \ldots F_S)$ and any $L' = (F_1', F_2', \ldots F_S')$, we have:

$$L \succsim L' \text{ if and only if } \sum_{s=1}^{S}\Big(\int_0^\infty u_s(x_s)\, dF_s(x_s)\Big) \geq \sum_{s=1}^{S}\Big(\int_0^\infty u_s(x_s)\, dF_s'(x_s)\Big)$$

The proof is given at the bottom of p.203 of Mas Colell et al.

Thus utility is additively separable across states.

IV. Subjective Probability Theory

What if the probabilities were not objectively known to everyone, but instead each person has their own "guess" as to what all the π's are. Assuming that the π's sum to 1, it may be that people behave rationally in the sense that their behavior is consistent with expected utility theory based on their subjective probabilities. In theory, observed behavior could even allow us to derive their expected probabilities. This leads to **subjective probability theory**.

As before, start with a set of possible states {1, 2, … S}. **We do not know the probabilities $\pi_1, \pi_2, \ldots \pi_S$; the goal is to derive them based on observed behavior**.

Let $x = (x_1, x_2, \ldots x_S) \in \mathbb{R}_+^S$ be the associated monetary payoffs. To be even more general, assume that these payoffs are not certain but each has its own distribution function, denoted by $(F_1, F_2, \ldots F_S)$.

Suppose there is a rational **preference relation** $\succsim$ on the space of lotteries $\mathscr{L}$ that **satisfies continuity and extended independence**. By Proposition 6.E.1 we know that those preferences are consistent with a set of Bernoulli utility functions that use $\sum_{s=1}^{S} u_s(x_s)$ to evaluate lotteries.

But these $u_s(\ )$ functions do not yet allow us to derive the subjective probabilities. For any $(\pi_1, \pi_2, \ldots \pi_S) >> 0$ we could define $\tilde{u}_s(\ ) = (1/\pi_s)u_s(\ )$ and then we could evaluate a given $(x_1, x_2, \ldots x_S)$ by $\sum_{s=1}^{S} \pi_s \tilde{u}_s(x_s)$. The **problem** is that **we cannot distinguish probabilities from the $u_s(\ )$ functions**.

Example. Suppose that a gamble that gives \$1 in state 1 and \$0 in state 2 is preferred to a gamble that gives \$0 in state 1 and \$1 in state 2. This implies that the decision maker thinks that state 1 is more likely than state 2 **as long as the state does not influence the value of money.**

Next, define state s preferences $\succsim_s$ on state s lotteries as:

$$F_s \succsim_s F_s' \text{ if } \int_0^\infty u_s(x_s)\, dF_s(x_s) \geq \int_0^\infty u_s(x_s)\, dF_s'(x_s)$$

Definition 6.F.1: The state preferences $(\succsim_1, \succsim_2, \ldots \succsim_S)$ on state lotteries are **state uniform** if $\succsim_s = \succsim_{s'}$ for any s and s′.

State uniformity implies that two **state-specific utility functions**, $u_s(\ )$ and $u_{s'}(\ )$ can **differ only by a linear transformation**. That is, **there is a u( ) such that**:

$$u_s(\ ) = \pi_s u(\ ) + \beta_s \text{ for all } s = 1, 2, \ldots S.$$

These π_s's will be the subjective probabilities. To sum up:

Proposition 6.F.1: (Subjective Expected Probability Theorem). Suppose that the preference relation $\succsim$ on $\mathscr{L}$ satisfies the continuity and extended independence axioms. **Assume** also that the derived state preferences are **state uniform**. Then there are probabilities $(\pi_1, \pi_2, \ldots \pi_S)$ and a utility function u( ) such that for any $(x_1, x_2, \ldots x_S)$ and $(x_1', x_2', \ldots x_S')$ we have:

$$(x_1, x_2, \ldots x_S) \succsim (x_1', x_2', \ldots x_S') \text{ if and only if } \sum_{s=1}^{S} \pi_s u(x_s) \geq \sum_{s=1}^{S} \pi_s u(x_s')$$

These probabilities are unique, as is u( ) (plus linear trans).

Some people dispute this. A well-known example is the Ellsberg paradox. There are 2 urns, each of which has 100 balls. Balls are either black or white. In Urn 1 there are 49 white balls and 51 black balls. Nothing is known about how many white balls and black balls are in Urn 2.

Each urn is like a lottery. Suppose there is a payoff of $1000 if a **black** ball is chosen. Which urn/lottery do you prefer, Urn 1 or Urn 2?

Suppose there is a payoff of $1000 if a **white** ball is chosen. Which urn/lottery do you prefer, Urn 1 or Urn 2?

Some people prefer Urn 1 in both cases, which can be depicted as “uncertainty aversion”.

Material that will NOT be on the exam:

1. Any proofs in any lectures
2. Strong axiom of revealed preference (Lecture 7)
3. Quasilinear preferences (Lecture 4)
4. Lexicographic preferences
5. Integrability (Lexture 6)
6. You do not need to memorize the functional forms of demand systems (Lecture 9)
7. State-dependent utility and subjective probability theory (in Lecture 12)
8. Lecture 13.