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AFRE 802
Statistical Methods for Agricultural, Food, & Resource Economists

Continuous random variables & their probability distributions – Part 3 of 3 (WMS Ch. 4.6, 4.8, 4.10, 4.12)

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GAME PLAN

Review

No in-class exercise today (but expect one on Thursday)

Probability distributions for continuous RVs (cont'd)

- Finish normal distribution
- Gamma distribution and two special cases (exponential and Chi-squared)
- Tchebysheff's inequality in the context of continuous RVs

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Review: uniform distribution

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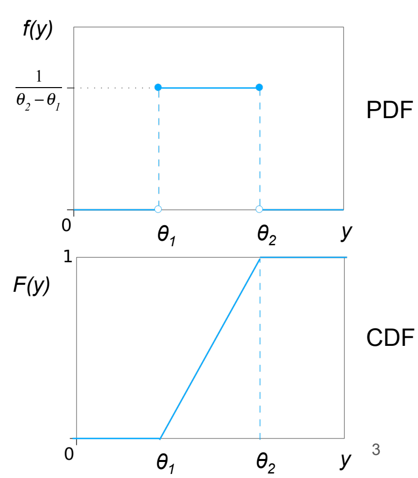
- Equal probability over range w/ non-zero probability
- Also if # of events that occur in time interval (0, t) ~ Poisson, then if exactly one such event occurred in the interval (0, t), then the actual time of occurrence is ~ uniform (0, t)

$$\textbf{PDF:}\ f(y) = \begin{cases} \dfrac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\textbf{CDF:}\ F(y) = \begin{cases} 0, & y < \theta_1 \\ \dfrac{y - \theta_1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 1, & y > \theta_2 \end{cases}$$

where $\theta_1 < \theta_2$

$$E(Y) = \frac{\theta_1 + \theta_2}{2} \qquad V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}$$



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Review: normal distribution

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- Familiar bell-shaped curve; very common; symmetric
- No closed form solution for CDF but can convert any normal RV to a standard normal RV (next slide), then look up probabilities in Table 4

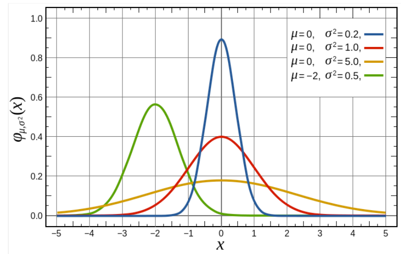
$$\boxed{\begin{array}{l}\textbf{PDF}: f(y)=\dfrac{1}{\sigma\sqrt{2\pi}}e^{-(y-\mu)^2/(2\sigma^2)}, \\ -\infty \le y \le \infty \\ Y \sim N(\mu,\sigma^2)\end{array}}$$

$$\boxed{E(Y)=\mu} \qquad \boxed{V(Y)=\sigma^2}$$

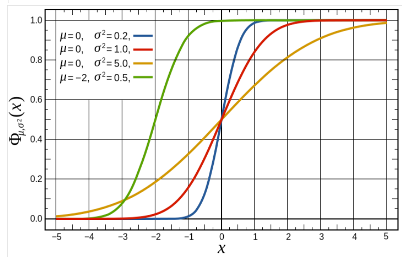
$$\boxed{\begin{array}{l}\text{Standard normal RV:} \\ Z \sim N(0,1)\end{array}}$$

CDF of standard normal: $\Phi(.)$

PDF of standard normal: $\phi(.)$



PDF



CDF

Can convert any normal RV to standard normal

- If RV $Y \sim N(\mu, \sigma^2)$ ("Y is distributed as normal w/ mean μ and variance σ^2")
- Convert to standard normal (Z) using:

$$\boxed{Z=\frac{Y-\mu}{\sigma}, \quad Z \sim N(0,1) \ = \text{standard normal}}$$

- Once converted to standard normal → use Table 4
- EX) The achievement scores for a college entrance exam are normally distributed with mean 75 and standard deviation 10. *What fraction of those scores lies between 80 and 90?*

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Common continuous probability distributions

1. Uniform
2. Normal
3. **Gamma** and special cases
    1. Chi-square
    2. Exponential

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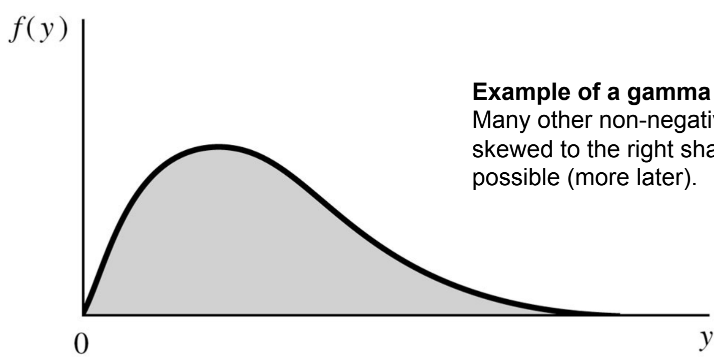
The gamma probability distribution

- Non-negative and skewed (*to the right or left?*)
- *Examples/applications?*



Example of a gamma pdf. Many other non-negative, skewed to the right shapes possible (more later).

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PDF, mean, and variance of gamma distribution

$$f(y) = \begin{cases} \dfrac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & 0 \le y \le \infty \\ 0, & \text{elsewhere} \end{cases}$$

$$where \quad \Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy$$

$$and \quad \alpha > 0,\ \beta > 0$$

Γ(*α*): "gamma function"

$\Gamma(1)=1$

$\Gamma(n)=(n-1)!$ if n is an integer

Note: in general, there is no closed form solution for the CDF for the gamma distribution. (There are closed form solutions for some special cases – e.g., exponential dist.)

What are the parameters of the gamma distribution? α, β

The mean and variance of the gamma distribution:

$$\boxed{E(Y) = \alpha\beta} \qquad \boxed{V(Y) = \alpha\beta^2}$$

Proof is on p. 187 of WMS

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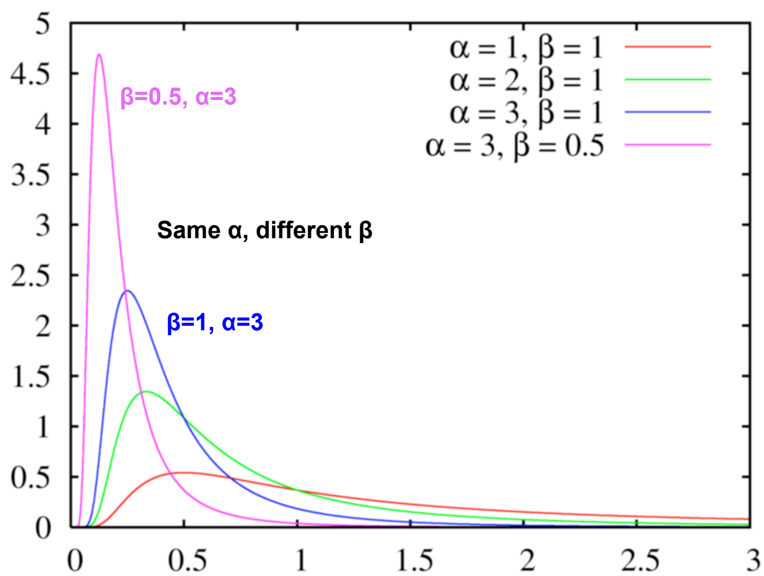
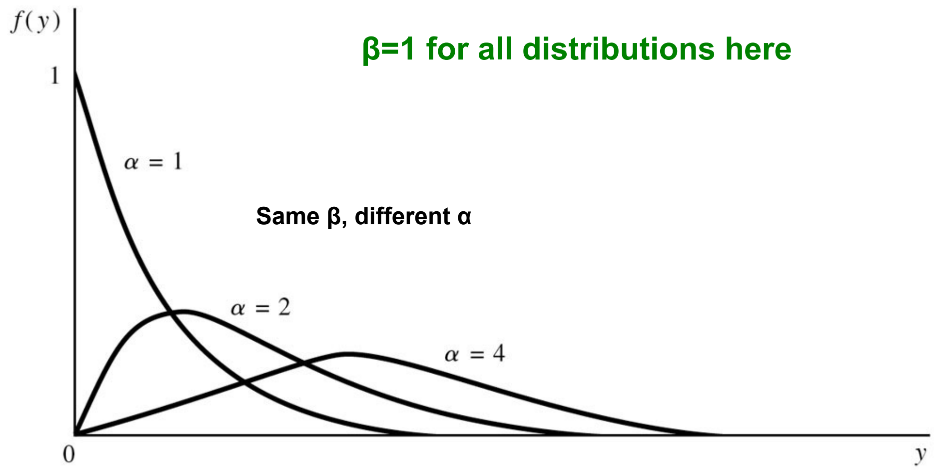
Gamma PDFs with different *β* and *α* values

- ***β* is sometimes called the "scale" parameter**, and *α* the "shape" parameter



Gamma PDFs with different β and α values

- β is sometimes called the "scale" parameter, and ***α* the "shape" parameter**



Example #1 – gamma distribution

Four-week summer rainfall totals in a section of the Midwest United States have approximately a gamma distribution with $\alpha=1.6$ and $\beta=2.0$. Find the mean and variance of the four-week summer rainfall totals.

$$\boxed{E(Y) = \alpha\beta}$$

$$\boxed{V(Y) = \alpha\beta^2}$$

Example #2 – gamma distribution

The response times of a tablet used for CAPI have approximately a gamma distribution with mean 4 seconds and variance 8 seconds. Write the probability density function (PDF) for the response times.

$$\boxed{f(y)=\begin{cases}\dfrac{y^{\alpha-1}e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & 0\le y\le\infty\\ 0, & \text{elsewhere}\end{cases}}$$

$$\boxed{E(Y)=\alpha\beta}$$

$$\boxed{V(Y)=\alpha\beta^2}$$

$\Gamma(n)=(n-1)!$ if n is an integer

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Common continuous probability distributions

1. Uniform
2. Normal
3. Gamma and special cases
   1. **Chi-square** (*Examples/applications?*)
   2. Exponential

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2 important special cases of the gamma distribution

$$\boxed{1.\ \textbf{Chi-square}\ (\chi^2)\ \text{distribution: gamma w/ } \alpha = \frac{v}{2},\ \beta = 2}$$

*What are the parameters of the **chi-square** distribution?*

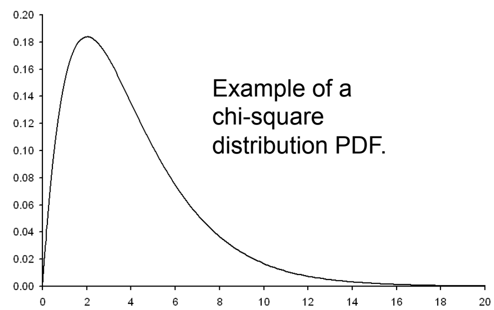
v, also called the number of "degrees of freedom" of the χ^2 distribution

*What are the mean and variance of the **chi-square** distribution?*

$$\boxed{E(Y) = \alpha\beta = \frac{v}{2} \cdot 2 = v}$$

$$\boxed{V(Y) = \alpha\beta^2 = \frac{v}{2} \cdot 2^2 = 2v}$$

No closed form solution for CDF but probabilities tabulated in Table 6. We'll use these later in the course.



Example of a chi-square distribution PDF.

Common continuous probability distributions

1. Uniform
2. Normal
3. Gamma and special cases
   1. Chi-square
   2. **Exponential** (*Examples/applications?*)

2 important special cases of the gamma distribution

2. **Exponential** distribution: gamma w/ $\alpha = 1, \beta > 0$

$$f(y) = \begin{cases} \dfrac{e^{-y/\beta}}{\beta}, & 0 \le y \le \infty \\ 0, & \text{elsewhere} \end{cases}$$

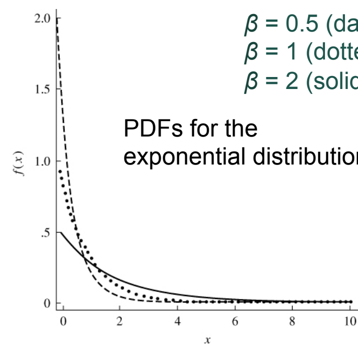
$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y/\beta}, & y \ge 0 \end{cases}$$

What are the parameters of the ***exponential*** *distribution?* β

What are the mean and variance of the ***exponential*** *distribution?*

$$E(Y) = \alpha\beta = 1\beta = \beta$$

$$V(Y) = \alpha\beta^2 = 1\beta^2 = \beta^2$$



β = 0.5 (dashed)
β = 1 (dotted)
β = 2 (solid)

PDFs for the exponential distribution

Example – exponential distribution (and improper definite integrals)

4.97 A manufacturing plant uses a specific bulk product. The amount of product used in one day can be modeled by an exponential distribution with $\beta = 4$ (measurements in tons). Find the probability that the plant will use more than 4 tons on a given day.

$$f(y) = \begin{cases} \dfrac{e^{-y/\beta}}{\beta}, & 0 \le y \le \infty \\ 0, & \text{elsewhere} \end{cases}$$

$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y/\beta}, & y \ge 0 \end{cases}$$

Distribution	Probability Function	Mean	Variance
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$ $Z = \frac{Y-\mu}{\sigma}$, $Z \sim N(0,1)$ = standard normal	μ	σ^2
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y > 0$	v	$2v$

Review

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Tchebysheff's Inequality

For **any** RV, *Y*, with with mean μ & variance σ^2:

$$P\left[(\mu - k\sigma) < Y < (\mu + k\sigma)\right] \ge 1 - \frac{1}{k^2}$$

or

$$P\left[Y \le (\mu - k\sigma) \text{ OR } Y \ge (\mu + k\sigma)\right] \le \frac{1}{k^2}$$

for any constant $k > 0$

Interpretation?

The probability of being less than k standard deviations from the mean is at least 1-1/k^2

Interpretation?

The probability of being k or more standard deviations from the mean is no more than 1/k^2

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Tchebysheff's Theorem – Example w/ continuous RV

Recall for gamma RV:

$\boxed{E(Y)=\alpha\beta}$ $\boxed{V(Y)=\alpha\beta^2}$

Suppose that experience has shown that the length of time *Y* (in minutes) required to conduct a periodic maintenance check on a maize grinding mill follows a gamma distribution with α=3.1 and β=2. A new maintenance worker takes 22.5 minutes to check the machine. What is the upper bound on the probability of being this far or more above or below the mean?

$$P\left[(\mu-k\sigma)<Y<(\mu+k\sigma)\right]\geq 1-\frac{1}{k^2} \quad \text{or} \quad P\left[Y\leq(\mu-k\sigma)\text{ OR Y}\geq(\mu+k\sigma)\right]\leq\frac{1}{k^2}$$

for any constant $k>0$

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Homework:

- WMS Ch. 4 (part 3 of 3)
  - Gamma & related distributions: 4.88, 4.89. 4.91
  - Tchebysheff's theorem: 4.146, 4.147
- Ch. 4 HW due Thurs. (or Tues. if we don't finish Ch. 4 today)

Next class:

- Multivariate probability distributions (Part 1 of 3)

Reading for next class:

- WMS Ch. 5 (sections 5.1-5.3)

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In-class exercises #1: normal distribution

4.63 A company that manufactures and bottles apple juice uses a machine that automatically fills 16-ounce bottles. There is some variation, however, in the amounts of liquid dispensed into the bottles that are filled. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation 1 ounce.

a Use Table 4, Appendix 3, to determine the proportion of bottles that will have more than 17 ounces dispensed into them.

In-class exercises #2: normal distribution

4.68 The grade point averages (GPAs) of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation .8. What fraction of the students will possess a GPA in excess of 3.0?

a Answer the question, using Table 4, Appendix 3.

In-class exercises #3: exponential distribution

4.176 If Y has an exponential distribution with mean β, find (as a function of β) the median of Y.

$$\boxed{f(y)=\begin{cases}\dfrac{e^{-y/\beta}}{\beta}, & 0\le y\le\infty\\ 0, & \text{elsewhere}\end{cases}}$$

$$\boxed{F(y)=\begin{cases}0, & y<0\\ 1-e^{-y/\beta}, & y\ge 0\end{cases}}$$

$$\boxed{E(Y)=\beta}\quad \boxed{V(Y)=\beta^2}$$

In-class exercises #4: gamma & chi-square dist.

4.96 Suppose that a random variable Y has a probability density function given by

$$f(y)=\begin{cases}ky^3e^{-y/2}, & y>0,\\ 0, & \text{elsewhere.}\end{cases}$$

a Find the value of k that makes $f(y)$ a density function.
b Does Y have a χ^2 distribution? If so, how many degrees of freedom?
c What are the mean and standard deviation of Y?

Gamma

$$\boxed{f(y)=\begin{cases}\dfrac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}, & 0\le y\le\infty\\ 0, & \text{elsewhere}\end{cases}}$$

$\Gamma(n)=(n-1)!$ if n is an integer

$$\boxed{E(Y)=\alpha\beta=\frac{\nu}{2}\cdot 2=\nu}$$

$$\boxed{V(Y)=\alpha\beta^2=\frac{\nu}{2}\cdot 2^2=2\nu}$$

Extra slide

Gamma PDFs with different β and α values

- β is sometimes called the “scale” parameter, and α the “shape” parameter