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AFRE 802
Statistical Methods for Agricultural, Food, & Resource Economists

Linear models & estimation by least squares – Part 3 of 3
(WMS Ch. 11.5 & Wooldridge pp. 113-136)

December 5, 2017

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GAME PLAN

- Housekeeping issues:
  - Office hours this week are Wednesday, 11 AM-1 PM
  - Friday optional review session this week will be 4-5 PM
  - I will hold extra office hours next Tuesday (Dec. 12) from 3-5 PM in the Cook Hall basement
- Return take-home graded exercise (see answer key in 2014 final exam on D2L)
- Collect Thursday's additional practice problem
- Distribute new additional practice problem
- Review

- Linear models & estimation by least squares – Part 3 of 3
  - Classical linear model assumptions
  - Inference
    - Hypothesis testing & p-values
    - Confidence intervals

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Review: Total, explained, & residual SS, R^2

Total sum of squares: $$\boxed{SST \equiv \sum_{i=1}^{N}(y_i - \bar{y})^2}$$

Explained sum of squares: $$\boxed{SSE \equiv \sum_{i=1}^{N}(\hat{y}_i - \bar{y})^2}$$

Residual sum of squares: $$\boxed{SSR \equiv \sum_{i=1}^{N}\hat{u}_i^2}$$

$$\boxed{SST = SSE + SSR}$$

Coefficient of determination or R^2:

$$\boxed{R^2 = SSE / SST = 1 - (SSR / SST)}$$

Interpretation?

The proportion of the sample variation in y that is explained by x

Review: Simple linear regression assumptions & implications

SLR.1-SLR.4 ➔ OLS estimators are **unbiased**

SLR.1. Linear in parameters:

SLR.2. Random sampling

****SLR.3. Zero conditional mean (exogeneity):**

$$\boxed{E(u \mid x) = E(u) = 0}$$

SLR.4. Sample variation in x

SLR.5. Homoskedasticity (constant variance):

$$\boxed{V(u \mid x) = V(u) = \sigma^2}$$

→ Formulas for variances of OLS estimators are:

$$\boxed{V(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}} \quad \boxed{V(\hat{\beta}_0) = \frac{\sigma^2 N^{-1}\sum_{i=1}^{N}x_i^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}}$$

→ SLR.1-SLR.5 ➔ OLS is **BLUE** (Gauss-Markov Theorem)

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Unbiased & consistent estimator of $V(u) = \sigma^2$

$$\hat{\sigma}^2 = \frac{1}{N-2}\sum_{i=1}^{N}\hat{u}_i^2 = \frac{SSR}{N-2}$$

Use in formulas to estimate variances and obtain standard errors of our OLS estimators:

$$\hat{V}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\hat{V}(\hat{\beta}_0) = \frac{\hat{\sigma}^2 N^{-1}\sum_{i=1}^{N}x_i^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\hat{\sigma}_{\hat{\beta}_j} = \sqrt{\hat{V}(\hat{\beta}_j)} \text{ for } j = 0,1$$

Review

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Annotated Stata output for simple linear regression so far: N, OLS estimates, SSE, SSR, SST, R^2

```
use "/Users/nicolemason/Documents/AEC802/data/WAGE1_Stata13.dta"

reg wage educ
```

	Source	SS	df	MS	
SSE	Model	1179.73204	1	1179.73204	
SSR	Residual	5980.68225	524	11.4135158	$\hat{\sigma}^2$
SST	Total	7160.41429	525	13.6388844	

Number of obs =	526	
F(1, 524) =	103.36	
Prob > F =	0.0000	
R-squared =	0.1648	
Adj R-squared =	0.1632	
Root MSE =	3.3784	$\hat{\sigma}$

	wage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
$\hat{\beta}_1$	educ	.5413593	.053248	10.17	0.000	.4367534	.6459651
$\hat{\beta}_0$	_cons	-.9048516	.6849678	-1.32	0.187	-2.250472	.4407687

$\hat{\sigma}_{\hat{\beta}_j}$

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What we know about the sampling distributions of the OLS estimators so far

$$y = \beta_0 + \beta_1 x + u$$

OLS estimators for β_0 and β_1:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Expected values (under SLR.1-SLR.4):

$$E(\hat{\beta}_1) = \beta_1 \text{ and } E(\hat{\beta}_0) = \beta_0$$

Sample variances (under SLR.1-SLR.5):

$$\hat{V}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\hat{V}(\hat{\beta}_0) = \frac{\hat{\sigma}^2 N^{-1} \sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

$$\text{where } \hat{\sigma}^2 = \frac{1}{N-2}\sum_{i=1}^{N}\hat{u}_i^2 = \frac{SSR}{N-2}$$

$\hat{\sigma}$ is the **standard error** of the regression

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The sampling distributions of the OLS estimators

- **By CLT**, Under assumptions SLR.1-SLR.5, the OLS estimators are **asymptotically** (i.e., as $N \to \infty$) **normally distributed** with the means & variances on the previous slides
- If we **add one more assumption**, then we can obtain the sampling distribution of the OLS estimators in **finite samples**

SLR.6. Normality: The population error, ***u*, is independent of *x*** and is **normally distributed** with $E(u)=0$ and $V(u)=\sigma^2$, i.e.: $u \sim Normal(0, \sigma^2)$

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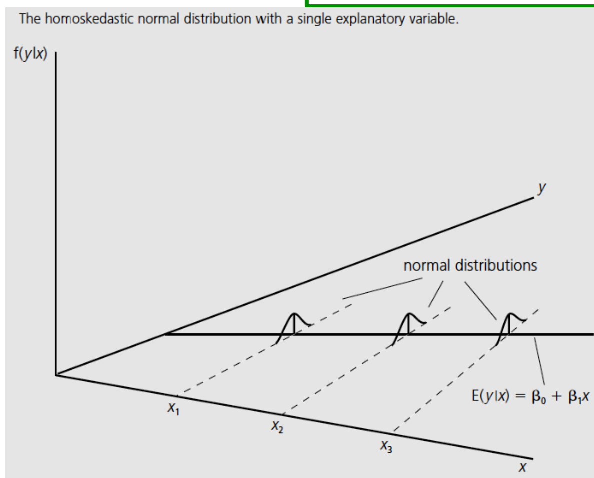
SLR.1-SLR.6 = "classical linear model assumptions"

- CLM = Gauss-Markov + SLR.6 (normality)
- CLM assumptions imply $\boxed{y \mid x \sim Normal(\beta_0 + \beta_1 x,\ \sigma^2)}$



Source: Wooldridge (2003)

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$$y = \beta_0 + \beta_1 x + u$$

The sampling distributions of the OLS estimators under the CLM assumptions (SLR.1-SLR.6):

$\boxed{\hat{\beta}_j \sim Normal\left(\beta_j,\ V(\hat{\beta}_j)\right)}$ where $V(\hat{\beta}_1) = \dfrac{\sigma^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$,

$$V(\hat{\beta}_0) = \frac{\sigma^2 N^{-1} \sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad \hat{\sigma}^2 = \frac{1}{N-2}\sum_{i=1}^{N}\hat{u}_i^2 = \frac{SSR}{N-2}$$

If we **know σ^2,** then we can standardize beta-hat$_j$ to a **Z-statistic**; otherwise, we can **estimate σ^2** and compute a **T-statistic** – i.e.:

$$\boxed{Z = \frac{\hat{\beta}_j - \beta_j}{\sqrt{V(\hat{\beta}_j)}} \sim Normal(0,\ 1)} \qquad \boxed{T = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\hat{V}(\hat{\beta}_j)}} \sim t \text{ with } N-2 \text{ d.f.}}$$

$\sigma_{\hat{\beta}_j}$ $\qquad$ $\hat{\sigma}_{\hat{\beta}_j}$

Testing hypotheses about β_0 or β_1

$$y = \beta_0 + \beta_1 x + u$$

1. State the **null & alternative hypotheses:** e.g., $H_0 : \beta_j = 0, H_1 : \beta_j \neq 0$
2. Define an appropriate **test statistic:** $\hat{\beta}_j$
3. Determine the **distribution of the test statistic under the null** hypothesis $\hat{\beta}_j \sim Normal\left(0,\ V(\hat{\beta}_j)\right)$
4. **Standardize the test statistic** to something with known/tabled probabilities for its sampling distribution (e.g., *Z, t, chi-square, F*)
   $$\boxed{Z = \frac{\hat{\beta}_j - 0}{\sigma_{\hat{\beta}_j}} \sim Normal(0,\ 1)} \qquad \boxed{T = \frac{\hat{\beta}_j - 0}{\hat{\sigma}_{\hat{\beta}_j}} \sim t \text{ with } N-2 \text{ d.f.}}$$
5. Choose a **significance level** (***α***, the *P*(Type I error)=*P*(reject the null when it is true), typically 0.01, 0.05, or 0.10) & a **rejection region** **OR** compute the **p-value** for the test statistic.
6. **Reject the null hypothesis if** the standardized statistic lies **in the rejection region (or if p-value≤*α*)**; fail to reject otherwise

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Example #1: Testing hypotheses about β_0 or β_1

```
. reg bwght cigs
```

Source	SS	df	MS
Model	13060.4194	1	13060.4194
Residual	561551.3	1386	405.159668
Total	574611.72	1387	414.283864

Number of obs	=	1388
F(1, 1386)	=	32.24
Prob > F	=	0.0000
R-squared	=	0.0227
Adj R-squared	=	0.0220
Root MSE	=	20.129

bwght	Coef.	Std. Err.
cigs	-.5137721	.0904909
_cons	119.7719	.5723407

Test the following hypotheses at the a = 0.05 level. Also find the p-values.

$H_0 : \beta_{cigs} = 0$ vs. $H_1 : \beta_{cigs} \neq 0$

and

$H_0 : \beta_{cigs} = 0$ vs. $H_1 : \beta_{cigs} < 0$

$$\boxed{T = \frac{\hat{\beta}_j - 0}{\hat{\sigma}_{\hat{\beta}_j}} \sim t \text{ with } N-2 \text{ d.f.}}$$

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T-stats and p-values in Stata output

```
. reg bwght cigs

      Source |       SS       df       MS              Number of obs =    1388
-------------+------------------------------           F(  1,  1386) =   32.24
       Model |  13060.4194     1  13060.4194           Prob > F      =  0.0000
    Residual |    561551.3  1386  405.159668           R-squared     =  0.0227
-------------+------------------------------           Adj R-squared =  0.0220
       Total |   574611.72  1387  414.283864           Root MSE      =  20.129

------------------------------------------------
       bwght |      Coef.   Std. Err.      t    P>|t|
-------------+----------------------------------
        cigs |  -.5137721   .0904909    -5.68   0.000
       _cons |   119.7719   .5723407   209.27   0.000
------------------------------------------------
```

The p-values reported by Stata are for $H_0 : \beta_j = 0$ vs. $H_1 : \beta_j \neq 0$

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Example #2: Testing hypotheses about β_0 or β_1

$$\log(crime) = \beta_0 + \beta_1 \log(enroll) + u$$

$$\widehat{\log(crime)} = -6.63 + 1.27 \log(enroll)$$
$$(1.03) \quad (0.11)$$
$$n = 97, R^2 = .585.$$

[Aside on interpreting results in log-log models]

crime is the annual number of crimes on college campuses and *enroll* is student enrollment. The numbers in parentheses are standard errors.

Use the regression output above to test the following hypotheses at the α=0.05 level. Also find the associated p-values.

$H_0 : \beta_1 = 1$ vs. $H_1 : \beta_1 \neq 1$
and
$H_0 : \beta_1 = 1$ vs. $H_1 : \beta_1 > 1$

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Summary of Functional Forms Involving Logarithms $y = \beta_0 + \beta_1 x + u$

Model	Dependent Variable	Independent Variable	Interpretation of β_1
level-level	y	x	$\boxed{\beta_1 = \frac{\Delta y}{\Delta x}}$ $\Delta y = \beta_1 \Delta x$
level-log	y	$\log(x)$	$\boxed{\frac{\beta_1}{100} = \frac{\Delta y}{\%\Delta x}}$ $\Delta y = (\beta_1/100)\%\Delta x$
log-level	$\log(y)$	x	$\boxed{100\beta_1 = \frac{\%\Delta y}{\Delta x}}$ $\%\Delta y = (100\beta_1)\Delta x$
log-log	$\log(y)$	$\log(x)$	$\boxed{\beta_1 = \frac{\%\Delta y}{\%\Delta x}}$ $\%\Delta y = \beta_1 \%\Delta x$

Source: Wooldridge (2003)

[back]

Confidence intervals for β_0 or β_1

Recall from earlier in the course:

Two-sided, large-sample (1-α)% confidence interval for θ: $\hat{\theta} \pm z_{\alpha/2}\sigma_{\hat{\theta}}$

Two-sided, small-sample (1-α)% confidence interval for μ: $\bar{Y} \pm t_{\alpha/2}\hat{\sigma}_{\bar{Y}}$,

($N - 1$ d.f. for $t_{\alpha/2}$)

Two-sided, finite sample (1-α)% confidence interval for β_j (in the case of simple linear regression):

$$\boxed{\begin{array}{c}\hat{\beta}_j \pm t_{\alpha/2}\hat{\sigma}_{\hat{\beta}_j} \\ (N - 2 \text{ d.f. for } t_{\alpha/2})\end{array}}$$

Example #1: Confidence intervals for β_0 or β_1

```
. reg bwght cigs

      Source |       SS       df       MS              Number of obs =    1388
-------------+------------------------------           F(  1,  1386) =   32.24
       Model |  13060.4194     1  13060.4194           Prob > F      =  0.0000
    Residual |    561551.3  1386  405.159668           R-squared     =  0.0227
-------------+------------------------------           Adj R-squared =  0.0220
       Total |   574611.72  1387  414.283864           Root MSE      =  20.129

------------------------------------------------------
       bwght |      Coef.   Std. Err.      t    P>|t|
-------------+----------------------------------------
        cigs |  -.5137721   .0904909    -5.68   0.000
       _cons |   119.7719   .5723407   209.27   0.000
------------------------------------------------------
```

$$\hat{\beta}_j \pm t_{\alpha/2}\hat{\sigma}_{\hat{\beta}_j}$$
$(N-2$ d.f. for $t_{\alpha/2})$

a. Find the 95% (two-sided) confidence interval for β_{cigs}.
Relate this to $H_0: \beta_{cigs} = 0$ vs. $H_1: \beta_{cigs} \neq 0$ at α=0.05.

b. Find the 95% upper confidence interval for β_{cigs}.
Relate this to $H_0: \beta_{cigs} = 0$ vs. $H_1: \beta_{cigs} < 0$ at α=0.05.

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95% confidence intervals in Stata output

```
. reg bwght cigs

      Source |       SS       df       MS              Number of obs =    1388
-------------+------------------------------           F(  1,  1386) =   32.24
       Model |  13060.4194     1  13060.4194           Prob > F      =  0.0000
    Residual |    561551.3  1386  405.159668           R-squared     =  0.0227
-------------+------------------------------           Adj R-squared =  0.0220
       Total |   574611.72  1387  414.283864           Root MSE      =  20.129

------------------------------------------------------------------------------
       bwght |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        cigs |  -.5137721   .0904909    -5.68   0.000    -.6912861   -.3362581
       _cons |   119.7719   .5723407   209.27   0.000     118.6492    120.8946
------------------------------------------------------------------------------
```

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Example #2: Confidence intervals for β_0 or β_1

$$\log(crime) = \beta_0 + \beta_1 \log(enroll) + u$$

$$\widehat{\log(crime)} = \underset{(1.03)}{-6.63} + \underset{(0.11)}{1.27} \log(enroll)$$
$$n = 97,\ R^2 = .585.$$

crime is the annual number of crimes on college campuses and *enroll* is student enrollment. The numbers in parentheses are standard errors.

a. Find the 95% (two-sided) confidence interval for β_1.
   Relate this to H_0: $\beta_1 = 1$ vs. H_1: $\beta_1 \sim= 1$ at $\alpha=0.05$.
b. Find the 95% lower confidence interval for β_1.
   Relate this to H_0: $\beta_1 = 1$ vs. H_1: $\beta_1 > 1$ at $\alpha=0.05$.

$$\boxed{\hat{\beta}_j \pm t_{\alpha/2}\hat{\sigma}_{\hat{\beta}_j} \quad (N-2 \text{ d.f. for } t_{\alpha/2})}$$

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Answers :

$2-sided\ \ 95\%\ CI\ for\ \beta_1:\ \ [1.054,\ 1.486]$

$Lower\ \ 1-sided\ \ 95\%\ CI\ for\ \beta_1:\ \ [1.089,\ \infty)$

Annotated Stata output (see handout)

Homework: Ch. 11 (cont'd)

1. Finish the other parts of Thursday's HW
2. Using the data in WMS 11.3, test H_0: β_1 =0 vs. H_1: β_1 ~= 0, and H_0: β_1 =0 vs. H_1: β_1 < 0, both at the at α=0.05 level. Also find the 95% two-sided and upper CIs, and relate the results to your hypothesis tests above.
3. Using the data in tourism.dta (on D2L) and Stata, regress household tourism expenditure (*tourismexp*) on household income (*income*). Interpret the estimate for β_1, and construct 99% two-sided and lower CIs for β_1. Use the CI results to test H_0: β_1=0.05 vs. H_1: β_1~= 0.05, and H: β_1=0.05 vs. H_1: β_1>0.05 at α=0.01 level.

- Please try to complete all Ch. 11 HW before class on Thursday so that we can go over it then (you won't turn in Ch. 11)

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Game plan for Thursday (last day of class)

- Finish any material on today's slides that we didn't get to
- Go over answers to additional practice problem
- Go over any questions you have on the Ch. 11 HW, past final exams, or other HWs/course material

Final exam details

- Cumulative but with emphasis on Ch. 7-Ch. 11
- Please bring paper, pencil, calculator, and cheat sheets (two 8.5x11" sheets, front and back). Please write last 4 digits of your PID on paper in advance to save time.
- Exam is closed book/notes except for cheat sheets
- Exam is in this room from 12:45-2:45 PM (hard stop) next Thursday, December 14

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