Applied Microeconomics: Firm and Household

Lecture 12: Cost and Profit Functions, Duality, and the Short vs. the Long Run

Jason Kerwin

Department of Applied Economics
University of Minnesota

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Outline

- Comparative statics of the cost minimization model
- Substitution and output effects
- The dual approach to producer theory
  - Profit functions & Hotelling's Lemma
  - Cost functions & Shephard's Lemma
- Short-run vs. Long-run production decisions
  - Le Chatelier's principle

Cost minimization

In the previous lecture we discussed the cost minimization model:

- $\underset{x_1, x_2, \lambda}{\text{Min}} \; L = w_1 x_1 + w_2 x_2 + \lambda(y - f(x_1, x_2))$

The FOCs are:

1. $L_1 = \frac{\partial L}{\partial x_1} = w_1 - \lambda f_1 = 0$
2. $L_2 = \frac{\partial L}{\partial x_2} = w_2 - \lambda f_2 = 0$
3. $L_\lambda = \frac{\partial L}{\partial \lambda} = y - f(x_1, x_2) = 0$

Assuming the SOCs hold, the solutions for the system of equations of FOCs are:

- $x_1 = x_1^c(w_1, w_2, y)$
- $x_2 = x_2^c(w_1, w_2, y)$
- $\lambda = \lambda^c(w_1, w_2, y)$

Cost minimization: comparative statics

Now we will discuss the comparative statics of this model without a formal derivation (the formal derivation is the same what we did to derive the comparative statics of the expenditure minimization model for consumers).

- How does a cost-minimizing firm adjust its factor demands in response to a change in the price of an input?
  - $\frac{\partial x_1^c}{\partial w_i} =? \;\; i = 1, 2$
- How does a cost-minimizing firm adjust its factor demands in response to a change in the target output?
  - $\frac{\partial x_i^c}{\partial y} =? \;\; i = 1, 2$

Comparative statics: own- and cross-price effect

The own- and cross-price effects can be derived as:

1. $\frac{\partial x_1^c}{\partial w_1} < 0$
   - A cost-minimizing firm uses less of an input when the (own) price of the input increases.
   - The conditional factor demand curve is downward-sloping.
2. $\frac{\partial x_2^c}{\partial w_1} > 0$
   - A cost minimizing firm uses more of the second input when the price of the first input increases.
   - The reason is that when the price of the first input increases a firm uses less of the first input. So, to be able to produce the target output the firm must employ more of the second input.
   - This cross-price effect holds only when there are two inputs. If there are more than two inputs the sign is ambiguous.

Comparative statics: output effect

The effects of changes in output on factor demands can be derived as:

1. $\frac{\partial x_1^c}{\partial y} \gtrless 0$
2. $\frac{\partial x_2^c}{\partial y} \gtrless 0$
   - A cost minimizing firm's response due to an increase in parametric output is ambiguous. This result is intuitive since the firm may be able to increase production by increasing only one input.

Substitution and output effects in unconditional factor demand

Now we examine the difference in sensitivity to input prices between two different demanded functions: conditional and unconditional.

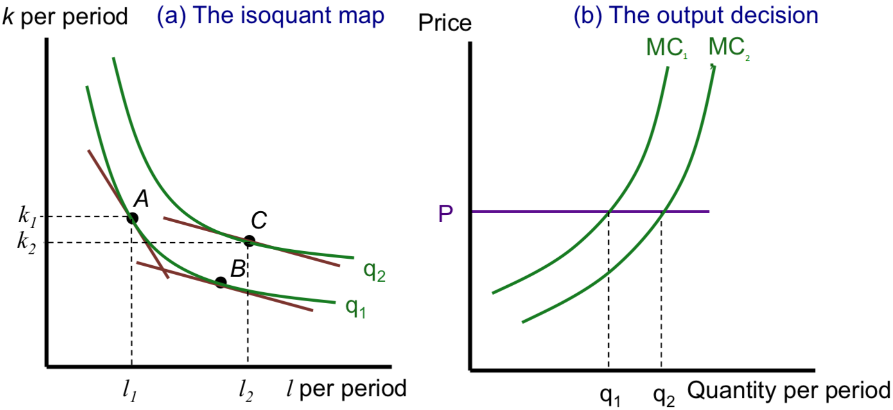
Own-price effects: In our discussion of profit maximization model we derived that the own-price effect for an unconditional demand function is negative:

- $\frac{\partial x_1^*}{\partial w_1} < 0$.

This effect can be decomposed into a substitution effect and an output effect:

- **Substitution effect:** Change in quantity demanded of x_1 due to substitution with other inputs such that the resulting input bundle achieves the same output level.
- **Output effect:** Change in quantity of x_1 demanded due to overall change in marginal cost.

Substitution and output effects



Conditional and unconditional demand functions

To formally derive the conditional and unconditional demands, denote the factor demand functions as

- $x_i^c(w, y)$ (conditional) and $x_i^*(p, w)$ (unconditional).

We can establish a relationship between the two demand functions by allowing level of output to vary instead of remaining fixed.

- $x_i^*(p, w) \equiv x_i^c(w, y(p, w))$

Taking the derivative with respect to the price of the input:

- $\frac{\partial x_i^*}{\partial w} = \underbrace{\frac{\partial x_i^c}{\partial w}}_{\text{substitution effect}} + \underbrace{\frac{\partial x_i^c}{\partial y}\frac{\partial y}{\partial w}}_{\text{output effect}}$

The cost function

The cost function is obtained by substituting the conditional demand functions into the total cost equation

- $C(w, y) = w_1 x_1^c(w_1, w_2, y) + w_2 x_2^c(w_1, w_2, y)$

The cost function $C^*(w, y)$ gives the minimum cost for a firm to achieve an arbitrary output level y for any set of given factor prices.

Note that, just like the conditional demand functions, the cost function is also a function of factor prices and the target output (model parameters.)

Properties of the cost function

1. Nondecreasing in input prices w.
   - if $w^1 \geqslant w^2$ then $C(w^1, y) \geqslant C(w^2, y)$
2. Nondecreasing in output level y.
   - if $y^1 \geqslant y^2$ then $C(w, y^1) \geqslant C(w, y^2)$
3. $C(w, y)$ is concave in input prices w.
4. $C(w, y)$ is linearly homogeneous in input prices
   - $C(tw, y) = tC(w, y)$
5. $C(w, 0) = 0$
   - There are no fixed costs associated with the indirect cost function. So, we are taking a long-run view in which all relevant inputs are variable.

Shephard's Lemma

If the cost function is differentiable in w, then the cost-minimizing conditional demand functions are:

- $x_i^c(w, y) = \frac{\partial C(w,y)}{\partial w_i}$

The cost function is linearly homogeneous in input price. Therefore the first derivative of the cost function with respect to each input price is homogeneous of degree zero – as we would expect for the conditional demand function.

A closer look at Shephard's Lemma

We can rewrite the indirect cost function as

- $C(w, y) = \sum_i^N w_i x_i^c(w, y) + \lambda(\underbrace{y - f(x^c(w, y))}_{foc=0})$

Note that the second term is zero because the constraint holds with equality at the optimum. Taking the derivative with respect to w_i

- $\frac{\partial C(w,y)}{\partial w_i} = x_i^c(w, y) + w\frac{\partial x^c(w,y)}{\partial w_i} - \lambda f_x \frac{\partial x^c(w,y)}{\partial w_i}$
- $\frac{\partial C(w,y)}{\partial w_i} = x_i^c(w, y) + \underbrace{(w - \lambda f_x)}_{foc=0}\frac{\partial x^c(w,y)}{\partial w_i}$
- $\frac{\partial C(w,y)}{\partial w_i} = x_i^c(w, y)$

The result establishes Shephard's lemma.

The profit function

Recall the profit maximization problem:

- $\max_{x_1, x_2} \pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2$

The FOCs are:

- $\frac{\partial \pi}{\partial x_i} = \pi_i = pf_i(x_1, x_2) - w_i = 0, \quad i = 1, 2$

Assuming the SOSC holds, the optimum functions in implicit form are:

- $x_i = x_i^*(p, w_1, w_2), \quad i = 1, 2$ and $y = y^*(p, w_1, w_2)$

The indirect profit function can be obtained as:

- $\pi(p, w) = pf(x^*(p, w)) - wx^*(p, w)$

Definition: The firm's profit function shows its maximal profits as a function of the prices that the firm faces:

- $\pi(p, w) = \underset{x}{\text{Max}} \{pf(x) - wx\}$

Properties of the profit function

1. Nondecreasing in output price p.
   - if $p^1 \geqslant p^2$ then $\pi(p^1, w) \geqslant \pi(p^2, w)$
2. Nonincreasing in input prices w.
   - if $w^1 \geqslant w^2$ then $\pi(p, w^1) \leqslant \pi(p, w^2)$
3. $\pi(p, w)$ is convex in prices, p and w.
4. $\pi(p, w)$ is linearly homogeneous in prices
   - $\pi(tp, tw) = t\pi(p, w)$

Hotelling's Lemma

If the profit function is differentiable in p and w, the profit maximizing supply and unconditional demand functions are:

- $y(p, w) = \frac{\partial \pi(p,w)}{\partial p}$
- $x_i(p, w) = -\frac{\partial \pi(p,w)}{\partial w}$

Note that because profit function is linearly homogeneous, the first derivative of the profit function is homogeneous of degree zero – as we'd expect for demand and supply functions.

A closer look at Hotelling's Lemma

The indirect profit function is:

- $\pi(p, w) = pf(x^*(p, w)) - wx^*(p, w)$

Taking the derivative with respect to the output price, p:

- $\frac{\partial \pi(p,w)}{\partial p} = f(x^*(p, w)) + pf_x \frac{\partial x^*(p,w)}{\partial p} - w \frac{\partial x^*(p,w)}{\partial p}$
- $\frac{\partial \pi(p,w)}{\partial p} = \underbrace{f(x^*(p, w))}_{y(p,w)} + \underbrace{(pf_x - w)}_{foc=0} \frac{\partial x^*(p,w)}{\partial p}$
- $\frac{\partial \pi(p,w)}{\partial p} = y(p, w)$

Hotelling's lemma: the first derivative of the indirect profit function with respect to the price of output yields the supply function.

Exercise: Following similar steps, show that the first derivative of the indirect profit function w.r.t. each input price yields the negative of the respective unconditional demand function, i.e., $\frac{\partial \pi(p,w)}{\partial w} = -x^*$.

Cost function: Cobb-Douglas example

Consider the following Cobb-Douglas indirect cost function

- $C(w, y) = Ay^{\alpha} w_1^{\beta_1} w_2^{\beta_2}, \quad \alpha, \beta_1, \beta_2 > 0$

We want to do the following:

1. Derive the conditional demand functions
2. Derive the own- and cross-price elasticities of factor demands

Exercise: Show that this is a legitimate cost function.

Cost function: Cobb-Douglas example

We can obtain the conditional demand functions by applying Shephard's Lemma:

- $\frac{\partial C(w,y)}{\partial w_1} = x_1 = A\beta_1 y^{\alpha} w_1^{\beta_1 - 1} w_2^{\beta_2}$
- $\frac{\partial C(w,y)}{\partial w_2} = x_2 = A\beta_2 y^{\alpha} w_1^{\beta_1} w_2^{\beta_2 - 1}$

To obtain the own- and cross-price elasticities of the conditional demand functions:

- $\epsilon_{11} = \frac{\partial x_1}{\partial w_1}\frac{w_1}{x_1} = A\beta_1(\beta_1 - 1)y^{\alpha} w_1^{\beta_1 - 2} w_2^{\beta_2}\frac{w_1}{x_1}$
- $\epsilon_{11} = \frac{(\beta_1 - 1)}{x_1}\underbrace{A\beta_1 y^{\alpha} w_1^{\beta_1 - 1} w_2^{\beta_2}}_{x_1}$
- $\epsilon_{11} = (\beta_1 - 1)\frac{x_1}{x_1} = \beta_1 - 1$

Exercise: Derive that $\epsilon_{22} = \beta_2 - 1$, $\epsilon_{12} = \beta_2$, and $\epsilon_{21} = \beta_1$

Short-run vs long-run costs

In the short run

- Economic actors have only limited flexibility in their actions

Specifically, we assume:

- That capital input is held constant at x_2^0
- The firm is free to vary only its labor input, x_1

The production function becomes

- $q = f(x_1, x_2^0)$

Short-Run total costs

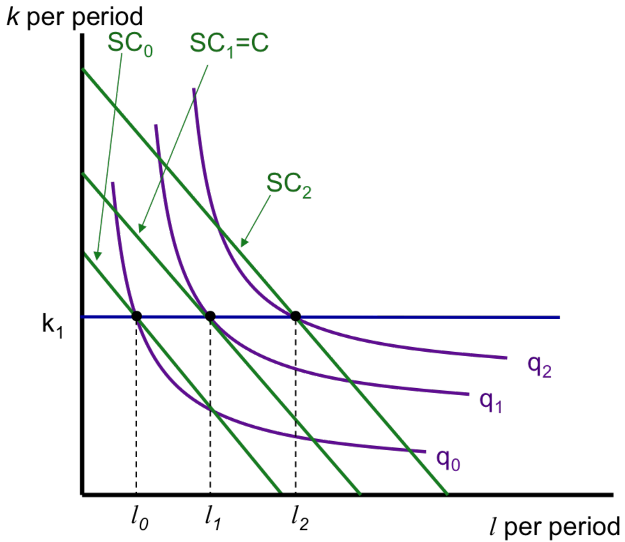
The short-run total cost for the firm is:

- SC = $w_1 x_1 + w_2 x_2^0$

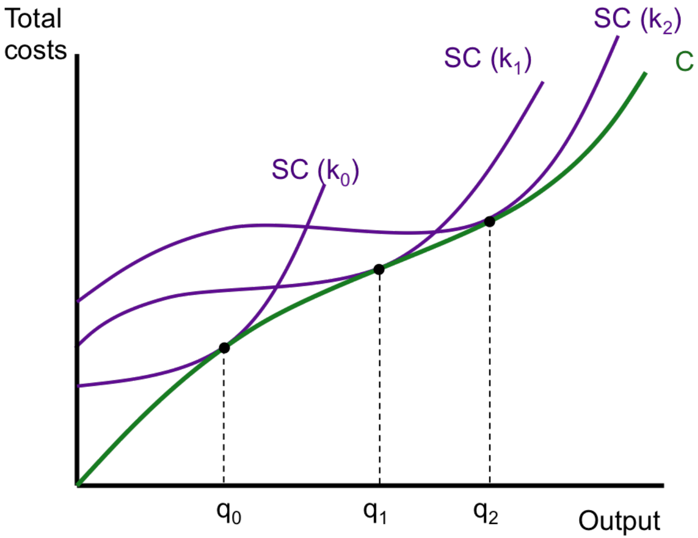
There are two types of short run costs

- Short-run fixed costs: $w_2 x_2^0$
- Short-run variable costs: $w_1 x_1$

Non-optimality of short-run costs



The long-run total cost curve



Short-run vs long-run demand and supply elasticities

The distinction between long-run and short-run costs creates a difference between the short-run and the long-run demand and supply elasticities. We want to analyze these differences.

Let's set up a short-run profit maximization problem where $x_2 = x_2^0$:

- $\max_{x_1} \pi = pf(x_1, x_2^0) - w_1 x_1 - w_2 x_2^0$

Note that firm is choosing only x_1 because $x_2 = x_2^0$.

The FOC for this profit maximization problem is:

- $\pi_1 = \frac{\partial \pi}{\partial x_1} = pf_1(x_1, x_2^0) - w_1 = 0$

The SOSC is:

- $\pi_{11} = \frac{\partial \pi_1}{\partial x_1} = pf_{11}(x_1, x_2^0) < 0$

Short-run demand function

The solution to the problem is the *conditional factor demand* (conditional on the level of the fixed input) denoted as:

- $x_1 = x_1^{sr}(w_1, x_2^0, p)$

where *sr* denotes short run. Because the total cost of the second input is fixed at $w_2 x_2^0$, only the level of the fixed input x_2^0 is an argument of the factor demand function.

Question: What can we say about the relative elasticities of the short-run vs. long-run factor demands?

Le Chatelier's principle

Suppose the fixed factor is allowed to vary such that it is equal to its long-run equilibrium level:

- $x_2^0 \equiv x_2^*(w_1, w_2, p)$

We can write the long-run factor demand as:

- $x_1^* \equiv x_1^{sr}(w_1, p, x_2^*(w_1, w_2, p))$

Differentiating both sides with respect to each input price gives:

1. $\frac{\partial x_1^*}{\partial w_1} \equiv \frac{\partial x_1^{sr}}{\partial w_1} + \frac{\partial x_1^{sr}}{\partial x_2^0}\frac{\partial x_2^*}{\partial w_1}$
2. $\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_1^{sr}}{\partial x_2^0}\frac{\partial x_2^*}{\partial w_2}$

That the sign of the term $\frac{\partial x_1^{sr}}{\partial x_2^0}\frac{\partial x_2^*}{\partial w_1}$ in (1) determines the difference between LR and SR responses.
Q:Why is this term always negative?

Le Chatelier principle

For a formal derivation, note that by using the second equation and reciprocity ($\frac{\partial x_1}{\partial w_2} = \frac{\partial x_2}{\partial w_1}$), we can solve for $\frac{\partial x_1^{sr}}{\partial x_2^0}$ as

- $\frac{\partial x_1^{sr}}{\partial x_2^0} = \frac{\partial x_2^*}{\partial w_1} / \frac{\partial x_2^*}{\partial w_2}$

Substituting this term back into the equation of $\frac{\partial x_1^{lr}}{\partial w_1}$ we obtain

- $\frac{\partial x_1^*}{\partial w_1} \equiv \frac{\partial x_1^{sr}}{\partial w_1} + \left(\frac{\partial x_2^*}{\partial w_1} / \frac{\partial x_2^*}{\partial w_2}\right) \frac{\partial x_2^*}{\partial w_1}$
- $\underbrace{\frac{\partial x_1^*}{\partial w_1}}_{-} \equiv \underbrace{\frac{\partial x_1^{sr}}{\partial w_1}}_{-} + \underbrace{\frac{(\partial x_2^* / \partial w_1)^2}{\partial x_2^* / \partial w_2}}_{-}$

The numerator is positive. The denominator is the slope of the factor demand, hence negative.

This equation states that long-run factor demand curves are more negative (hence more elastic) than the short-run factor demand curve.

Short run vs. long run response

The intuition behind “Le Chatelier’s Principle” is that in the short run firm’s response to a change in price is limited because it cannot substitute one factor to another. In the long run, however, the ability to substitute between factors provides the firm with the option of substituting out of the more expensive factor.

Next Class

Please Read NS Chapter 12: Partial Equilibrium