CITY AND REGIONAL PLANNING 7230

Cost Minimization and the Cost Function

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February 25, 2015

Contents

1 Introduction

This note discusses the comparative statics of the cost minimization problem, and uses those results as a motivation to consider other aspects of comparative statics and of cost minimization.

2 Cost Minimization

We focus on the behavior of a producer attempting to produce output at least cost. This could be a private-sector firm or a public-sector agency. The important features of the analysis are that the producer has a target output, which it takes as parametric; and it is a price-taker in all input (factor) markets.

Specifically, consider a price-taking producer in a 1-output 3-input world. We work with the 3-input case purely for ease of exposition: the results in the N-input case are precisely analogous. The output is denoted by q, inputs are z_1, z_2 and z_3; and technology is given the continuous, concave, monotonic production function f, so that $q = f(z_1, z_2, z_3)$. The firm faces input prices r_1, r_2 and r_3 respectively. The producer wants to produce a given quantity $\bar{q}$ of output at least input cost. We have three choice variables, the z's; and four parameters: the three input prices (r's), and the target production level $\bar{q}$.

2.1 Definition of the cost function

The Lagrangian for the cost minimization problem is:

$$\mathcal{L} = r_1 z_1 + r_2 z_2 + r_3 z_3 + \mu(\bar{q} - f(z_1, z_2, z_3))$$

where μ is the Lagrange multiplier. The first-order conditions (FOCs) for a minimum are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial z_1} &= 0: \quad r_1 - \mu f_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial z_2} &= 0: \quad r_2 - \mu f_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial z_3} &= 0: \quad r_3 - \mu f_3 = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= 0: \quad \bar{q} - f(z_1, z_2, z_3) = 0
\end{aligned}$$

where $f_i \equiv \partial f / \partial z_i$ is the marginal product of input i. Assuming that the Implicit Function Theorem holds, we can solve the FOCs for the choice functions: the solutions depend on the problem parameters, and we denote them by $z_1^*(r_1, r_2, r_3, \bar{q})$, $z_2^*(r_1, r_2, r_3, \bar{q})$, $z_3^*(r_1, r_2, r_3, \bar{q})$ and $\mu^*(r_1, r_2, r_3, \bar{q})$. The functions $z_i^*(r_1, r_2, r_3, \bar{q})$, and are the *cost-minimizing input demand functions*: they tell us how much of the inputs the firm would use at prices r_1, r_2, r_3 if it wanted to produce the quantity $\bar{q}$ at least cost.

The second-order condition (SOC) for a (constrained) minimum is that the border-preserving principal minors of order $k \geq 2$ of the bordered Hessian,

$$
\begin{aligned}
H &= \begin{bmatrix} \partial^2\mathcal{L}/\partial z_1^2 & \partial^2\mathcal{L}/\partial z_1\partial z_2 & \partial^2\mathcal{L}/\partial z_1\partial z_3 & \partial^2\mathcal{L}/\partial z_1\partial\mu \\ \partial^2\mathcal{L}/\partial z_2\partial z_1 & \partial^2\mathcal{L}/\partial z_2^2 & \partial^2\mathcal{L}/\partial z_2\partial z_3 & \partial^2\mathcal{L}/\partial z_2\partial\mu \\ \partial^2\mathcal{L}/\partial z_3\partial z_1 & \partial^2\mathcal{L}/\partial z_3\partial z_2 & \partial^2\mathcal{L}/\partial z_3^2 & \partial^2\mathcal{L}/\partial z_3\partial\mu \\ \partial^2\mathcal{L}/\partial\mu\partial z_1 & \partial^2\mathcal{L}/\partial\mu\partial z_2 & \partial^2\mathcal{L}/\partial\mu\partial z_3 & \partial^2\mathcal{L}/\partial\mu^2 \end{bmatrix} \\
&= \begin{bmatrix} -\mu\ f_{11} & -\mu\ f_{12} & -\mu\ f_{13} & -f_1 \\ -\mu\ f_{21} & -\mu\ f_{22} & -\mu\ f_{23} & -f_2 \\ -\mu\ f_{31} & -\mu\ f_{32} & -\mu\ f_{33} & -f_3 \\ -f_1 & -f_2 & -f_3 & 0 \end{bmatrix}
\end{aligned}
$$

are negative. (In terms of the static optimization lecture note, this is an $R = 1$ constraint problem, so we are interested in square matrices of dimension $\geq S = 1 + 2R = 3$).

The (total) *cost function* is the indirect objective function for this problem:

$$C^*(r_1, r_2, r_3, \bar{q}) \equiv r_1 z_1^*(r_1, r_2, r_3, \bar{q}) + r_2 z_2^*(r_1, r_2, r_3, \bar{q}) + r_3 z_3^*(r_1, r_2, r_3, \bar{q}).$$

$C^*(r_1, r_2, r_3, \bar{q})$ tells us the minimum expenditure (cost) needed at the given factor prices to produce $\bar{q}$. From now on, we'll write (r_1, r_2, r_3) as r, and the cost function as $C^*(r, \bar{q})$.

2.2 Properties of the cost function

- **Shephard's Lemma**:

$$\frac{\partial C^*}{\partial r_i} = z_i^*(r, \bar{q})$$

Proof 1: Apply the Envelope Theorem to the cost function.

Proof 2: Suppose z^* is a cost-minimizing input bundle for $\bar{q}$ at input prices r^*. Now let r be any input price vector and consider the function

$$g(r) = C^*(r, \bar{q}) - r\ z^*$$

This compares the minimum cost of producing $\bar{q}$ at prices r (ie $C^*(r, q)$) with the cost if we use the bundle z^*. Since C^* is the minimum production cost, $g(r) \le 0$. Moreover, when $r = r^*$, then $g(r) = 0$. In other words, $g(r)$ has a maximum (of zero) at $r = r^*$. Thus at r^* the derivatives of g are zero, so

$$\frac{\partial g(r)}{\partial r_i} = 0 : \quad \frac{\partial C^*}{\partial r_i} - z_i^*, \quad \forall i$$

which is Shephard's Lemma.

Shephard's Lemma is widely used in both theoretical and empirical contexts. In empirical contexts it can be used as a way to generate additional estimating equations in the unknown parameters of a cost function, thereby improving your degrees of freedom in the estimation process.

- Interpretation of the Lagrange Multiplier: we have:

$$\frac{\partial C^*}{\partial \bar{q}} = \mu^*(r, \bar{q})$$

— that is, the Lagrange multiplier μ is the additional cost incurred per unit change (increase) in the target output level, or the *marginal cost of production*. Proof: apply the Envelope Theorem.

- $C^*(r, 0) = 0$. Proof: obvious, since if you don't produce any output you won't buy any inputs, so the cost is zero.

It's worth pointing out that this result relies on our being able to select the levels of *all* inputs (the long run). In the short run (when the levels of one or more inputs is fixed) it is no longer true. See also section 2.3 below.

- $C^*(tr, \bar{q}) = tC^*(r, \bar{q})$. The cost function is linear homogeneous (homogeneous of degree 1) in the vector of all input prices.

Proof: Suppose all input prices increase by a factor t. Write down the new Lagrangian, and note that the FOCs are exactly the same as before $(r_i/r_j = f_i/f_j)$. Hence the input demand functions don't change. But it costs t times as much to purchase them, so (minimum) total costs increase by a factor of t.

One important use of this property is to check empirical specifications of the cost function. If the right-hand side of your regression equation isn't linear homogeneous in input prices, then you're not estimating a cost function at all.

- $z^*(tr, \bar{q}) = z^*(r, \bar{q})$. The input demands are homogeneous of degree zero in input prices.

Proof: The derivative of a function that is homogeneous of degree k is itself homogeneous of degree $k-1$. The cost function is homogeneous of degree 1 in r, and the input demand functions are the r-derivatives of the cost function, by Shephard's Lemma. So they are homogeneous of degree zero in r.

This property is also useful in checking the "legality" of empirical specifications of a cost function.

- $C^*(r, \bar{q})$ is concave in r.

  This is a bit harder than the other properties. What do we want to prove? Let r^1 and r^2 be (different) input price vectors, and, for $0 \leq t \leq 1$, let $r^t = t\, r^1 + (1-t)\, r^2$. (Note that the superscripts are indices, not powers). Then we want to show that:

  $$t\, C^*(r^1, \bar{q}) + (1-t)\, C^*(r^2, \bar{q}) \leq C^*(r^t, \bar{q}).$$

  Proof: Suppose that z^1, z^2 and z^t are all feasible input bundles for the same output $\bar{q}$ and that:

  - z^1 minimizes cost at r^1, so that $C^*(r^1, \bar{q}) = z^1 r^1$
  - z^2 minimizes cost at r^2, so that $C^*(r^2, \bar{q}) = z^2 r^2$
  - z^t minimizes cost at r^t, so that $C^*(r^t, \bar{q}) = z^t r^t$

    Now, since z^1 minimizes cost at r^1 we have:

    $$r^1 z^1 \leq r^1 z^t$$

    and since z^2 minimizes cost at r^2 we have:

    $$r^2 z^2 \leq r^2 z^t$$

    Now multiply the first of these inequalities by (the positive constant) t and the second by $1-t$ (so the direction of the inequalities is preserved) and add them together. We get:

    $$\begin{aligned} t r^1 z^1 + (1-t) r^2 z^2 &\leq z^t (t r^1 + (1-t) r^2) \\ &= z^t r^t \end{aligned}$$

    and so:

    $$t\, C^*(r^1, \bar{q}) + (1-t)\, C^*(r^2, \bar{q}) \leq C^*(r^t, \bar{q})$$

    which is what we wanted to show.

2.3 Short-run vs long-run cost functions

Our setup has assumed that the producer is able to select *all* the input quantities required to produce the parametric output ($\bar{q}$). This characterizes the producer's *long run*. In the *short run* the producer is constrained to employ parametric quantities of some of the inputs. This can most often arise for contractual reasons: for example, the producer signs a lease on a plot of land, and for the duration of the lease he or she must pay rent on the land, even if no output is produced. These constraints give rise to the *fixed factors* (the factors whose levels are constrained); those factors whose levels may be freely chosen are the *variable factors*. In the long run, by definition, all factors are variable.

Formally, the problem of the producer in the short run is similar to the one we're studying here, except that some of the inputs (the fixed factors) become parameters rather than choice variables. Suppose for example that in the 3-input case, input 1 (z_1) is land, and that the producer has signed a lease which requires him or her to pay a rent of r_1 on $\bar{z}_1$ units (acres) for some period of time. As long as this agreement is in force, the producer's problem is to choose levels of the variable inputs (z_2, and z_3) to minimize total costs. The problem can be expressed by the Lagrangian

$$\mathcal{L}' = r_1\bar{z}_1 + r_2 z_2 + r_3 z_3 + \mu'(\bar{q} - f(\bar{z}_1, z_2, z_3))$$

where the (augmented) choice variables are $z_2,\ z_3$ and μ'. The short-run (total) cost function (sometimes called the restricted cost function) is the indirect objective function for this problem:

$$C_s^*(r, \bar{q}, \bar{z}_1) = r_1\bar{z}_1 + r_2 z_2^* + r_3 z_3^*$$

and note that it has an additional parameter, the fixed factor. We can define the short-run marginal cost function as the output-derivative of $C_s^*(r, \bar{q}, \bar{z}_1)$: note that it too will depend on the level of the fixed factor. The Lagrange multiplier for this problem now measures the short-run marginal cost. Some of the properties applicable to the (long-run) cost function will need to be modified when we are considering the short run: for example, it is no longer true (as observed above) that $C_s^*(r, 0, \bar{z}_1) = 0$: if you produce no output, then your cost is still $r_1\bar{z}_1$ as long as you're operating in the short run.

For the remainder of these notes, we focus on the long-run (unrestricted) cost function, when all inputs are variable (freely selectable).

3 Comparative statics of the producer

We turn now to the comparative statics of the long-run cost-minimization problem. If you refer to the lecture note introducing the idea of comparative statics, you'll see that we follow its steps exactly.

3.1 The (general) 3-input case

We know that the choice functions $z_1^*(r,\bar{q})$, $z_2^*(r,\bar{q})$, $z_3^*(r,\bar{q})$ and $\mu^*(r,\bar{q})$ satisfy the FOCs as identities; thus we may write:

$$\begin{aligned} r_1 - \mu^*\ f_1(z_1^*, z_2^*, z_3^*) &\equiv 0 \\ r_2 - \mu^*\ f_2(z_1^*, z_2^*, z_3^*) &\equiv 0 \\ r_3 - \mu^*\ f_3(z_1^*, z_2^*, z_3^*) &\equiv 0 \\ \bar{q} - f(z_1^*, z_2^*, z_3^*) &\equiv 0 \end{aligned}$$

Since these are identities (they hold for any values of the parameters r_1, r_2, r_3 and $\bar{q}$) we may differentiate both sides of them with respect to any parameter. Let's pick r_1 and differentiate. We obtain:

$$\begin{aligned} 1 - \frac{\partial \mu^*}{\partial r_1} f_1 - \mu^* f_{11}\frac{\partial z_1^*}{\partial r_1} - \mu^* f_{12}\frac{\partial z_2^*}{\partial r_1} - \mu^* f_{13}\frac{\partial z_3^*}{\partial r_1} &\equiv 0 \\ -\frac{\partial \mu^*}{\partial r_1} f_2 - \mu^* f_{21}\frac{\partial z_1^*}{\partial r_1} - \mu^* f_{22}\frac{\partial z_2^*}{\partial r_1} - \mu^* f_{23}\frac{\partial z3}{\partial r_1} &\equiv 0 \\ -\frac{\partial \mu^*}{\partial r_1} f_3 - \mu^* f_{31}\frac{\partial z_1^*}{\partial r_1} - \mu^* f_{32}\frac{\partial z_2^*}{\partial r_1} - \mu^* f_{33}\frac{\partial z_3^*}{\partial r_1} &\equiv 0 \\ - f_1\frac{\partial z_1^*}{\partial r_1} - f_2\frac{\partial z_2^*}{\partial r_1} - f_3\frac{\partial z_3^*}{\partial r_1} &\equiv 0 \end{aligned}$$

Since we're interested in the slopes (derivatives) of the choice functions with respect to our chosen parameter (here, r_1), let's isolate them by re-writing this set of equations as a single matrix equation. We obtain:

$$\begin{bmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_2 & -f_3 & 0 \end{bmatrix} \begin{bmatrix} \partial z_1^*/\partial r_1 \\ \partial z_2^*/\partial r_1 \\ \partial z_3^*/\partial r_1 \\ \partial \mu^*/\partial r_1 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

We can now use Cramer's Rule to solve for the quantities of interest, the derivatives of the choice functions. For example, we have:

$$\frac{\partial z_1^*}{\partial r_1} = \frac{\begin{vmatrix} -1 & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ 0 & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ 0 & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ 0 & -f_2 & -f_3 & 0 \end{vmatrix}}{\begin{vmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_2 & -f_3 & 0 \end{vmatrix}}$$

so that:

$$\frac{\partial z_1^*}{\partial r_1} = \frac{\begin{vmatrix} -1 & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ 0 & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ 0 & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ 0 & -f_2 & -f_3 & 0 \end{vmatrix}}{|H|}$$

where in the numerator we have inserted the right-hand side of (1) into column 1 the matrix on the left. (This is Cramer's Rule). The problem now is to see what we can say about this expression.

First, the denominator. We recognize it as the bordered Hessian for the problem. Its sign is therefore known, by the SOC for the problem, and since this is a minimization problem with one constraint, its sign is $(-1)^1 < 0$.

That leaves the numerator. We begin by assuming that the second-order conditions hold, and then see if those conditions imply anything about the sign of the numerator. Any results obtained this way are the strongest results: they depend only on the fact that this is an optimization problem with the given algebraic structure. But for precisely this reason it would be unwise to expect too much here.

Let's see how this works out here for $\partial z_1^*/\partial r_1$. Expand the determinant in the numerator along the first column; we see that we need to determine the sign of:

$$-1 \times \begin{vmatrix} -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ -f_2 & -f_3 & 0 \end{vmatrix}.$$

But this is easy: since column 1 has a -1 in the topmost entry and 0's elsewhere, expanding along column 1 amounts to deleting the first row and column of the

Hessian (and then multiplying by -1). This is exactly how we form a 3×3 border-preserving principal minor (of order 2), so the determinant has sign -1 by the SOC, and therefore the sign of the entire numerator (ie $(-1) \times (-1)$) is positive. Then, in terms of signs we have:

$$\frac{\partial z_1^*}{\partial r_1} = \frac{\oplus}{\ominus} = \ominus$$

and we have our first result:

$$\frac{\partial z_1^*}{\partial r_1} < 0$$

— the input demand for z_1 slopes downward in own-price.

The same result obtains for the own-price slope of z_2^* : differentiate the FOCs with respect to r_2 and we get:

$$\begin{aligned}
-\frac{\partial \mu^*}{\partial r_2} f_1 - \mu^* f_{11}\frac{\partial z_1^*}{\partial r_2} - \mu^* f_{12}\frac{\partial z_2^*}{\partial r_2} - \mu^* f_{13}\frac{\partial z_3^*}{\partial r_2} &\equiv 0 \\
1 - \frac{\partial \mu^*}{\partial r_2} f_2 - \mu^* f_{21}\frac{\partial z_1^*}{\partial r_2} - \mu^* f_{22}\frac{\partial z_2^*}{\partial r_2} - \mu^* f_{23}\frac{\partial z_3^*}{\partial r_2} &\equiv 0 \\
-\frac{\partial \mu^*}{\partial r_2} f_3 - \mu^* f_{31}\frac{\partial z_1^*}{\partial r_2} - \mu^* f_{32}\frac{\partial z_2^*}{\partial r_2} - \mu^* f_{33}\frac{\partial z_3^*}{\partial r_2} &\equiv 0 \\
- f_1\frac{\partial z_1^*}{\partial r_2} - f_2\frac{\partial z_2^*}{\partial r_2} - f_3\frac{\partial z_3^*}{\partial r_2} &\equiv 0
\end{aligned}$$

leading to the matrix equation:

$$\begin{bmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_2 & -f_3 & -0 \end{bmatrix} \begin{bmatrix} \partial z_1^*/\partial r_2 \\ \partial z_2^*/\partial r_2 \\ \partial z_3^*/\partial r_2 \\ \partial \mu^*/\partial r_2 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Once again, we see the Hessian on the left, and, solving now for $\partial z_2^*/\partial r_2$ by Cramer's Rule, we get:

$$\frac{\partial z_2^*}{\partial r_2} = \frac{\begin{vmatrix} -\mu^* f_{11} & 0 & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -1 & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & 0 & -\mu^* f_{33} & -f_3 \\ -f_1 & 0 & -f_3 & -0 \end{vmatrix}}{\begin{vmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_2 & -f_3 & -0 \end{vmatrix}}$$

that is:

$$\frac{\partial z_2^*}{\partial r_2} = \frac{\begin{vmatrix} -\mu^* f_{11} & 0 & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -1 & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & 0 & -\mu^* f_{33} & -f_3 \\ -f_1 & 0 & -f_3 & -0 \end{vmatrix}}{|H|}$$

The denominator (= the bordered Hessian) is negative as before; expanding the numerator along the second column we obtain:

$$-1 \times \begin{vmatrix} -\mu^* f_{11} & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{31} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_3 & -0 \end{vmatrix}$$

which is also a border-preserving principal minor of order 2 (we are deleting row and column 2, and multiplying by -1), hence negative, and thus:

$$\frac{\partial z_2^*}{\partial r_2} < 0$$

The same result holds for any input in the N-input problem:

$$\frac{\partial z_i^*}{\partial r_i} < 0$$

the input-demand functions slope down in own price.

What about the *cross-price* slopes, for example, $\partial z_2^*/\partial r_1$? From our first matrix equation:

$$\begin{bmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_2 & -f_3 & 0 \end{bmatrix} \begin{bmatrix} \partial z_1^*/\partial r_1 \\ \partial z_2^*/\partial r_1 \\ \partial z_3^*/\partial r_1 \\ \partial \mu^*/\partial r_1 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and we can solve for $\partial z_2^*/\partial r_1$ using Cramer's Rule, giving:

$$\frac{\partial z_2^*}{\partial r_1} = \frac{\begin{vmatrix} -\mu^* f_{11} & -1 & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & 0 & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & 0 & -\mu^* f_{33} & -f_3 \\ -f_1 & 0 & -f_3 & 0 \end{vmatrix}}{\begin{vmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & -\mu^* f_{32} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_2 & -f_3 & -0 \end{vmatrix}}$$

that is:

$$\frac{\partial z_2^*}{\partial r_1} = \frac{\begin{vmatrix} -\mu^* f_{11} & -1 & -\mu^* f_{13} & -f_1 \\ -\mu^* f_{21} & 0 & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & 0 & -\mu^* f_{33} & -f_3 \\ -f_1 & 0 & -f_3 & 0 \end{vmatrix}}{|H|}$$

As before, we know that the sign of the denominator is negative; and, expanding along the second column of the numerator, we find

$$1 \times \begin{vmatrix} -\mu^* f_{21} & -\mu^* f_{23} & -f_2 \\ -\mu^* f_{31} & -\mu^* f_{33} & -f_3 \\ -f_1 & -f_3 & 0 \end{vmatrix}$$

What say we say about this? The answer — without additional assumptions — is *nothing*: because we formed it by deleting row 1 and column *2*, the determinant of the numerator isn't a border-preserving principal minor of the bordered Hessian and hence nothing is implied about its sign from the SOCs. This illustrates an important feature of comparative statics results: *they are often indeterminate* — that is, we often find that we get no results at all (for particular choice variables and parameters). Them's the breaks.

3.2 The 2-input case

The two-input case turns out to be a bit special. Let's focus first on the own-price slopes. (We already know the result here, but we will obtain it a bit differently). Differentiating the FOCs with respect to r_1 gives (eliminate the third row and column from equations (1)):

$$\begin{bmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{bmatrix} \begin{bmatrix} \partial z_1^*/\partial r_1 \\ \partial z_2^*/\partial r_1 \\ \partial \mu^*/\partial r_1 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

so that:

$$\frac{\partial z_1^*}{\partial r_1} = \frac{\begin{vmatrix} -1 & -\mu^* f_{12} & -f_1 \\ 0 & -\mu^* f_{22} & -f_2 \\ 0 & -f_2 & 0 \end{vmatrix}}{\begin{vmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix}}$$

The denominator is negative by the SOC for a minimization problem. The numerator has us looking at:

$$-1 \times \begin{vmatrix} -\mu^* f_{22} & -f_2 \\ -f_2 & 0 \end{vmatrix}$$

What can we say about this? The SOC instructs us to look at border-preserving principal minors of size greater than $S = 1 + 2R = 3$, and this is a 2×2 matrix: hence SOC conditions imply *nothing* about its sign.

But all is not lost. We can actually *evaluate* this determinant, and when we do, we find that it is:

$$-1 \times [-(f_2)^2]$$

which is > 0. So even in this case we can conclude that:

$$\frac{\partial z_1^*}{\partial r_1} = \frac{\oplus}{\ominus} = \ominus \qquad \text{ie } < 0$$

The important thing is that looking for minors isn't the only way to sign the numerator: sometimes brute force works too.

In the two-input case only, we *may* be able to get an additional result. The cross-price slope is:

$$\frac{\partial z_2^*}{\partial r_1} = \frac{\begin{vmatrix} -\mu^* f_{11} & -1 & -f_1 \\ -\mu^* f_{21} & 0 & -f_2 \\ -f_1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -f_2 \\ -f_1 & -f_2 & -0 \end{vmatrix}}$$

or:

$$\frac{\partial z_2^*}{\partial r_1} = \frac{\begin{vmatrix} -\mu^* f_{11} & -1 & -f_1 \\ -\mu^* f_{21} & 0 & -f_2 \\ -f_1 & 0 & 0 \end{vmatrix}}{|H|}$$

As always, the sign of the denominator is known, negative, by the SOCs. What about the numerator? Expanding along the second column, we need to examine:

$$1 \times \begin{vmatrix} -\mu^* f_{21} & -f_2 \\ -f_1 & 0 \end{vmatrix}$$

The determinant isn't a border-preserving principal minor of order 2 or larger (it's too small); but actually computing the determinant shows that it is equal to:

$$-f_1 f_2.$$

Still no help — unlike the previous case, the sign isn't determined by the mathematics. So let's think about this in terms of the *economics*. The quantity f_i is the marginal product of input i: the way a change in z_i changes output, all other inputs held constant. Suppose we are prepared to assume $f_i > 0$, that is, the marginal product of an input is always positive. What this says is that if you add more of either input to the input mix, then you always manage to increase output. Note that this is *not* always true: for example let the product (q) be flowers, and let the inputs be water (z_1) and seeds (z_2). At some point, continuing to pour water on the seeds will drown them; at this point f_1 goes negative. But *if* we are prepared to assume positive marginal products, then in the numerator $-f_1 f_2 < 0$, and

$$\frac{\partial z_2^*}{\partial r_1} > 0$$

Note that this result is not obtained, even with positive marginal products, in the 3+-input case. But it requires additional assumptions, and these may or may not be justified.

4 Further results

4.1 Instant comparative statics setup

It took a certain amount of effort to obtain the matrix equation for comparative statics: we needed to plug the optimal choice functions into the FOCs and then differentiate repeatedly using the composite function rule. Is there a faster way?

Look again at the basic comparative statics setup (here, from the 2-input case, for simplicity):

$$\begin{bmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{bmatrix} \begin{bmatrix} \partial z_1^*/\partial r_1 \\ \partial z_2^*/\partial r_1 \\ \partial \mu^*/\partial r_1 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

On the left we have the (bordered) Hessian, pre-multiplying the vector of derivatives of the (augmented) choice functions with respect to our selected parameter. As we have stressed several times, this will always be true. So the only real task is to obtain the vector on the right. If you look carefully at the derivations, you will see that we begin with the system of identities obtained by inserting the optimal choice functions into the FOCs. If the original problem was:

$$\max / \min_x \ \mathcal{L} = f(x;\alpha) + \lambda(b - g(x;\alpha))$$

then the i-th identity will be:

$$\frac{\partial \mathcal{L}}{\partial x_i} = f_i(x^*(\alpha);\alpha) - \lambda^*(\alpha) g(x^*(\alpha);\alpha)) \equiv 0$$

When we differentiate this with respect to our chosen parameter we will first of all be using the composite-function rule to assemble terms multiplying $\partial x_i^*/\partial \alpha_j$ and $\partial \lambda_i^*/\partial \alpha_j$. These will end up in the Hessian. The terms we're focussing on now will be those terms that do *not* involve the choice functions. These will be terms of the form:

$$\frac{\partial^2 \mathcal{L}}{\partial x_i \partial \alpha_j}$$

ie the second derivatives of the (original) Lagrangian, which will in turn be the α_j-derivatives of the (ordinary, not the identity-form) FOCs. Then, when we form our matrix equation for comparative statics, equation (2), we take theese terms over to the right-hand side, where they get a minus sign. This means that we can derive an equation like (2) via the following steps:

1. Set up the Lagrangian, obtain the FOCs and the (bordered) Hessian, H.
2. Pick a parameter and differentiate each of the (ordinary, ie not the identity-form) FOCs with respect to this parameter. The result is a vector, call it θ.
3. The basic equation for comparative statics is

   $$Hy = -\theta$$

   where y is the vector of the partial derivatives of the (augmented) choice functions with respect to the chosen parameter, and H (the [bordered] Hessian) is evaluated at the optimal choices.

Let's try it out on the cost-minimizing problem in the 2-input case.

1. The Lagrangian is:

   $$\mathcal{L} = r_1 z_1 + r_2 z_2 + \mu(\bar{q} - f(z_1, z_2))$$

   and the FOCs are:

   $$\begin{aligned} r_1 - \mu f_1 &= 0 \\ r_2 - \mu f_2 &= 0 \\ \bar{q} - f(z_1, z_2) &= 0 \end{aligned}$$

while the bordered Hessian is:

$$H = \begin{bmatrix} -\mu f_{11} & -\mu f_{12} & -f_1 \\ -\mu f_{21} & -\mu f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{bmatrix}$$

2. Pick a parameter, say r_1. Differentiate the FOCs with respect to that parameter. The result is:

$$\theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and we have, for this choice of parameter:

$$y = \begin{bmatrix} \partial z_1^* / \partial r_1 \\ \partial z_2^* / \partial r_1 \\ \partial \mu_1^* / \partial r_1 \end{bmatrix}$$

3. Assemble the basic equation, evaluating H at the optimal choices:

$$\begin{aligned} Hy &= -\theta \\ \begin{bmatrix} -\mu^* f_{11} & -\mu^* f_{12} & -f_1 \\ -\mu^* f_{21} & -\mu^* f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{bmatrix} \begin{bmatrix} \partial z_1^* / \partial r_1 \\ \partial z_2^* / \partial r_1 \\ \partial \mu_1^* / \partial r_1 \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

which is precisely the result we obtained by going through the steps of forming the FOCs as identities, differentiating them with respect to a parameter and assembling the results.

This can make setting up the framework for comparative statics investigations almost routine. If course, it doesn't help in actually obtaining the results: that you will need to do for yourself.

4.2 More comparative statics results

Here is another way to get comparatve statics results, which may also allow you to generate some additional relations.

With the same notation as before, consider choosing $z_{1,} z_2, z_3, \mu, r_1, r_2, r_3$ and $\bar{q}$ (this is: all the augmented choice variables in the original Lagrangian, followed by all the problem parameters, ie everything in sight) to:

$$\begin{array}{ll} \text{maximize} & C^*(r_1, r_2, r_3, \bar{q}) - (r_1 z_1 + r_2 z_2 + r_3 z_3) \\ \text{s.t.} & f(z_1, z_2, z_3) = \bar{q} \end{array}$$

This is the so-called *primal-dual problem*. Before proceeding, let's see how to put it together in general. First, if the original problem was a maximization problem, the new problem will be a minimization problem, and vice-versa. Second, the new objective function is the difference between the indirect objective function of the original problem (here $C^*(r_1, r_2, r_3, \bar{q})$) and the original objective function (here, $r_1 z_1 + r_2 z_2 + r_3 z_3$). Finally, the constraint is the constraint of the original optimization problem.

What do we know about the primal-dual problem?

Well, for one thing, we know its *solution*. Since $C^*(r_1, r_2, r_3, \bar{q})$ is the indirect objective function of the original problem and therefore tells us the *minimum* cost needed to produce output $\bar{q}$, it follows that, as long as $f(z_1, z_2, z_3) = \bar{q}$, we must have $C^*(r_1, r_2, r_3, \bar{q}) \leq r_1 z_1 + r_2 z_2 + r_3 z_3$. Therefore the new objective function is ≤ 0. Moreover, the new objective function will be zero precisely when $z_1 = z_1^*$, $z_2 = z_2^*$, $z_3 = z_3^*$ and $\mu = \mu^*$. Finally, all this holds *for any values of* r_1, r_2, r_3 and $\bar{q}$. In other words, as long as $f(z_1, z_2, z_3) = \bar{q}$, the primal-dual problem has a maximum (of zero) at $z_1 = z_1^*$, $z_2 = z_2^*$, $z_3 = z_3^*$ and $\mu = \mu^*$, where the starred values are precisely the solutions of the *original* problem. So the same functions solve both the original problem and the primal-dual problem we've just constructed.

Now let's study the primal-dual problem formally. The Lagrangian is:

$$\mathcal{L}^* = C^*(r_1, r_2, r_3, \bar{q}) - [r_1 z_1 + r_2 z_2 + r_3 z_3] + \mu(\bar{q} - f(z_1, z_2, z_3))$$

and, treating the z's, r's, $\bar{q}$ and the Lagrange multiplier μ as choice variables, the

FOCs are:

$$\frac{\partial \mathcal{L}^*}{\partial z_1} = 0: \quad -r_1 - \mu f_1 = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}^*}{\partial z_2} = 0: \quad -r_2 - \mu f_2 = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}^*}{\partial z_3} = 0: \quad -r_3 - \mu f_3 = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}^*}{\partial r_1} = 0: \quad \frac{\partial C^*}{\partial r_1} - z_1 = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}^*}{\partial r_2} = 0: \quad \frac{\partial C^*}{\partial r_2} - z_2 = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}^*}{\partial r_3} = 0: \quad \frac{\partial C^*}{\partial r_3} - z_3 = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}^*}{\partial \bar{q}} = 0: \quad \frac{\partial C^*}{\partial \bar{q}} + \mu = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}^*}{\partial \mu} = 0: \quad \bar{q} - f(z_1, z_2) = 0 \tag{10}$$

If the conditions of the implicit function theorem are satisfied we can solve the FOCs to obtain (among other things) the *original* solution functions z_1^*, z_2^*, z_3^* and μ^*. The SOC for this problem is the 8×8 bordered Hessian which we'll obtain by differentiating the FOCs in the following order:

$$z_1,\ z_2, z_3,\ r_1,\ r_2,\ r_3, \bar{q},\ \mu$$

(Note that this is: original choice variables, original parameters, Lagrange multipliers; we want to keep the derivatives with respect to any Lagrange multipliers in the last columns, in order to be able to visualize the border-preserving minors, otherwise the order is arbitrary, but convenient). We obtain:

$$H = \begin{vmatrix} -\mu f_{11} & -\mu f_{12} & -\mu f_{13} & -1 & 0 & 0 & 0 & -f_1 \\ -\mu f_{21} & -\mu f_{22} & -\mu f_{23} & 0 & -1 & 0 & 0 & -f_2 \\ -\mu f_{31} & -\mu f_{32} & -\mu f_{33} & 0 & 0 & -1 & 0 & -f_3 \\ -1 & 0 & 0 & \partial^2 C^*/\partial r_1^2 & \partial^2 C^*/\partial r_1 \partial r_2 & \partial^2 C^*/\partial r_1 \partial r_3 & \partial^2 C^*/\partial r_1 \partial \bar{q} & 0 \\ 0 & -1 & 0 & \partial^2 C^*/\partial r_2 \partial r_1 & \partial^2 C^*/\partial r_2^2 & \partial^2 C^*/\partial r_2 \partial r_3 & \partial^2 C^*/\partial r_2 \partial \bar{q} & 0 \\ 0 & 0 & -1 & \partial^2 C^*/\partial r_3 \partial r_1 & \partial^2 C^*/\partial r_{31} \partial r_2 & \partial^2 C^*/\partial r_3^2 & \partial^2 C^*/\partial r_3 \partial \bar{q} & 0 \\ 0 & 0 & 0 & \partial^2 C^*/\partial \bar{q} \partial r_1 & \partial^2 C^*/\partial \bar{q} \partial r_2 & \partial^2 C^*/\partial \bar{q} \partial r_3 & \partial^2 C^*/\partial \bar{q}^2 & 1 \\ -f_1 & -f_2 & -f_3 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

As you differentiate the FOCs to get the bordered Hessian, it is important to keep the functional dependencies in mind. For example, in the fourth row, when we are differentiating $(\partial C^*/\partial r_1) - z_1$, the variable z_1 is *not* a function of r_1, r_2 or $\bar{q}$: the *function* z_1^* is, but z_1^* does not appear in the fourth FOC.

We now want to relate as many of the terms in the bordered Hessian as possible to what we are eventually interested in, namely the derivatives of the choice functions. Equations (6) – (9) tell us how to do this: you will recognize them as a statement of the Envelope Theorem for this problem. For example, equation (6) tells us that whenever we see a partial derivative of C^* with respect to r_1 we may replace that with z_1 — or more precisely, by z_1^*, since the Envelope Theorem holds only at the optimal values of the choice functions. When we have a term like $\partial^2 C^*/\partial r_1 \partial r_2$, we re-write it as:

$$\frac{\partial^2 C^*}{\partial r_1 \partial r_2} = \frac{\partial}{\partial r_2}\frac{\partial C^*}{\partial r_1} = \frac{\partial}{\partial r_2} z_1^* = \frac{\partial z_1^*}{\partial r_2}$$

Doing this throughout the Hessian (and using all the four Envelope Theorem results) gives us

$$H = \begin{vmatrix} -\mu f_{11} & -\mu f_{12} & -\mu f_{13} & -1 & 0 & 0 & 0 & -f_1 \\ -\mu f_{21} & -\mu f_{22} & -\mu f_{23} & 0 & -1 & 0 & 0 & -f_2 \\ -\mu f_{31} & -\mu f_{32} & -\mu f_{33} & 0 & 0 & -1 & 0 & -f_3 \\ -1 & 0 & 0 & \partial^2 z_1^*/\partial r_1 & \partial^2 z_1^*/\partial r_2 & \partial^2 z_1^*/\partial r_3 & \partial^2 z_1^*/\partial \bar{q} & 0 \\ 0 & -1 & 0 & \partial^2 z_2^*/\partial r_1 & \partial^2 z_2^*/\partial r_2 & \partial^2 z_2^*/\partial r_3 & \partial^2 z_2^*/\partial \bar{q} & 0 \\ 0 & 0 & -1 & \partial^2 z_3^*/\partial r_1 & \partial^2 z_3^*/\partial r_2 & \partial^2 z_3^*/\partial r_3 & \partial^2 z_3^*/\partial \bar{q} & 0 \\ 0 & 0 & 0 & \partial \mu^*/\partial r_1 & \partial \mu^*/\partial r_2 & \partial \mu^*/\partial r_3 & \partial \mu^*/\partial \bar{q} & 1 \\ -f_1 & -f_2 & -f_3 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

The interesting results will of course be those involving starred quantities, ie the choice functions of the primal-dual problem *which are also the choice functions of the original problem.* (The primal-dual problem is just a device to help us study the original problem).

First, H is symmetric, since it is the matrix of second partials of the Lagrangian. It follows *almost by inspection* that we obtain the *reciprocity relations*:

$$\partial z_i^*/\partial r_j = \partial z_j^*/\partial r_i$$

Note that this is actually an interesting economic result about the cost-minimizing factor demands, and one which isn't at all obvious from the economics itself.

Next, we focus on the border-preserving principal minors of H. Consider the border-preserving principal minor of order 2 (a 3×3 submatrix) obtained by deleting rows and columns 1, 2, 3, 5 and 6:

$$\begin{vmatrix} \partial z_1^*/\partial r_1 & \partial z_1^*/\partial \bar{q} & 0 \\ \partial \mu^*/\partial r_1 & \partial \mu^*/\partial \bar{q} & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

Since this is a 1-constraint maximization problem, its sign is $(-1)^{3-1} = (-1)^2$, ie positive. Then:

$$\begin{aligned} 0 &< \begin{vmatrix} \partial z_1^*/\partial r_1 & \partial z_1^*/\partial \bar{q} & 0 \\ \partial \mu^*/\partial r_1 & \partial \mu^*/\partial \bar{q} & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &= -1 \times \begin{vmatrix} \partial z_1^*/\partial r_1 & \partial z_1^*/\partial \bar{q} \\ 0 & 1 \end{vmatrix} \\ &= -1 \times \frac{\partial z_1^*}{\partial r_1} \end{aligned}$$

(the second line is obtained by expanding the determinant in the first line along the last column; the third line is simple computation) which implies that:

$$\frac{\partial z_1^*}{\partial r_1} < 0$$

A similar result is obtained for $\partial z_2^*/\partial r_2$ by deleting rows 1, 2, 3, 4 and 6 of the bordered Hessian.

In general, what you want to do here is write down all the border preserving principal minors of the bordered Hessian of the primal-dual problem which involve starred quantities. (Obviously a computer algebra package can be a big help here). The signs of each of these will be known, by the SOCs. Then expand out these determinants and see if anything interesting emerges, and if so, whether you can give the results any interesting interpretation. (Often the determinants won't evaluate to something as simple as, say, $\partial z_i^*/\partial r_i$: in that case it can be something of a challenge to explain just what you've found).

4.3 WACM

Suppose I face prices r^1 and use inputs z^1 to produce output q. Suppose you face prices r^2 and use inputs z^2 to produce exactly the same quantity of output. What would it take to show that I am *not* cost-minimizing? Well, I could have used your input bundle z^2 to produce my output. So if it turned out that I could have spent less by using z^2 rather than the z^1 I actually used, then clearly I'm doing something wrong in the cost-minimization department. Thus, I would not be cost minimizing if $r^1z^2 < r^1z^1$. And it should be clear that it's not necessary that you and I produce the same output: if you produced *more* output than I did, the same test would apply. (Assuming the Free Disposal property of production sets).

4.3.1 Consistency with cost-minimization

Now suppose we have a N-unit dataset, where each observation i consists in the triple: output, input price vector, input quantity vector, which we symbolize as (q^i, r^i, z^i). (Typically this will be a cross-section of producers, in order to avoid problems connected with price inflation, which could arise if we had a time-series). Could this dataset have been generated by cost-minimizers? Our starting point is the following formal statement of our intuitive test:

WACM: A dataset satisfies the *weak axiom of cost-minimization* (WACM) if:

$$r^s z^s \leq r^s z^t \quad \text{for all } s, t \text{ such that } q^s \leq q^t$$

So all we need to do to check whether our dataset is consistent with cost minimization is to check this condition. Run through each of the units in the dataset. For each unit (s), pick out all observations t which produce q^s or more. Then check the WACM condition for this s and all the t's. This amounts to asking: at s's input prices (r^s), could s have reduced its costs by purchasing the alternative feasible input bundle z^t (z^t is feasible for q^s since $q^t \geq q^s$)? If the WACM condition holds for all these comparisons, the dataset is consistent with cost-minimizing behavior. Finally, note that nothing here restricts us to scalar outputs: if we take '$\leq$' to mean element-by-element comparisons, then the WACM test will work in multiple-output settings, too. (Especially in the multi-output case, you may want to check that the sets t of 'comparison' producers aren't empty, ie that we're actually doing some comparisons).

4.3.2 Input demand functions slope downwards

We can use the WACM to generate a form of our comparative statics result on the own-price slopes of the input demand functions. If observations s and t are cost minimizers, then we must have:

$$r^s z^s \leq r^s z^t$$

since s is a cost minimizer; and:

$$r^t z^t \leq r^t z^s$$

since t is a cost minimizer. Now re-write the first inequality as:

$$\begin{aligned} r^s z^s - r^s z^t &\leq 0 \\ r^s (z^s - z^t) &\leq 0 \end{aligned}$$

and the second as:

$$\begin{aligned} r^t z^t - r^t z^s &\leq 0 \\ r^t (z^t - z^s) &\leq 0 \\ -r^t (z^s - z^t) &\leq 0. \end{aligned}$$

And now add the two. We get:

$$(r^s - r^t)(z^s - z^t) \leq 0$$

and if we write this as:

$$\Delta r \, \Delta z \leq 0$$

we get a finite-change form of our comparative statics result: changes in input prices (Δr) and input quantities (Δz) run in opposite directions.