PRODUCTION UNDER UNCERTAINTY

APEC 8002, Fall 2016 **Instructor: Terry Hurley**

INTRODUCTION (MWG Chapter 6, Varian 3rd Edition Chapter 11, CQ Chapters 1-5)

Our objective now is to develop an understanding of production under uncertainty. Up until now, everything we have done assumed that the outcomes of a producer's decisions are precisely known — a truly extraordinary assumption that cannot withstand the scrutiny of casual observation: How many farmers can predict the vagaries of the weather they will face after planting their crops?

The primary theory used by economists to understand decisions under uncertainty is expected utility theory (EUT), which you were introduced to earlier this semester. This theory is still predominant even though there is now plenty of research that has identified important short-comings. While alternative theories have been proposed to supplant EUT, none have been very successful, though Prospect Theory in its various forms seems to be gaining ground. Regardless, we will try to frame the problem more generally, while drawing out important implications of the more restrictive EUT.

Before we jump into risk preferences however, we need to think about how we can characterize uncertainty in the context of production. What is important to realize as we work through this section is that much of the theory of production under uncertainty developed in a way that seems very disconnected with the theory of production we have talked about to date. More recently however, Chambers and Quiggin (CQ) have developed the theory so there are clearer connections to what we have already learned. Part of my goal here is to show that we can push these connections even further.

CHARACTERIZATIONS OF UNCERTAINTY

CQ is critical of the typical characterizations of production under uncertainty. To understand the concerns, it is useful to develop a simple example before turning to the formalities. For our example, we will think in the context of a crop farmer. A key factor in the success of crop farming is weather including precipitation and temperature. To keep things simple, suppose that a farmer can have high or low precipitation, and high or low temperatures. This gives us four mutually exclusive weather scenarios: 1) high precipitation and temperatures, 2) high precipitation and low temperatures, 3) low precipitation and high temperatures, and 4) low precipitation and temperatures. These types of mutually exclusive scenarios are referred to as states or states of nature and it is easy to think about how these different states of nature might affect crop production.

While weather may impact crop production, the farmer is not strictly at its mercy. For example, if there is too little rain, the farmer can irrigate. Alternatively, drainage can be used for too much rain. However, both of these activities require extra management effort. Table 1 shows hypothetical yields for these different states, assuming irrigation, drainage, or no extra management. Note that irrigation boosts yields when there is low rainfall (more so when temperatures are high) and has no effect when there is high rainfall. Drainage boosts yields when there is high rainfall, more so when temperatures are low, but is actually detrimental to yields when there is too little rainfall, especially when temperatures are high.

To keep things simple, let us assume the price the farmer receives per bushel is $1. Also let us assume that the cost of irrigation and drainage are the same: $50 per acre. The implications of these assumptions on net returns are shown in Table 2.

Table 1: Hypothetical Average Yields (bushels/acre)

			Management		
State	**Precipitation**	**Temperature**	**None**	**Irrigation**	**Drainage**
1	High	High	150	150	300
2	High	Low	200	200	400
3	Low	High	100	200	60
4	Low	Low	150	250	120

Finally, we will make some assumptions about the probability of these different states. Again, for simplicity we will assume all states are equally probable. With this equal probability assumption, we can construct the distributions of net returns which are illustrated in Figure 1.

Table 2 and Figure 1 provide two different perspectives on the uncertainty faced by a farmer. In Table 2, management choices result in an association between outcomes and states. In Figure 1, management choices provide an association between outcomes and probabilities. CQ refer to the perspective in Table 2 as the state-space or state-contingent approach and the perspective in Figure 1 as the parameterized distribution or outcome-space approach. What is the distinction? Notice that there is always a clear difference between potential consequences of the different types of management in Table 2. This is not the case in Figure 1. In particular, no management and irrigation yield the exact same distribution of net returns such that they are indistinguishable to a decision maker who only looks at the distribution of outcomes. What this means is that the state-space approach can be thought of as a structural form model, while the parameterized distribution approach can be thought of as a reduced form model. Given a state-space model, you can always construct the parameterized distribution model, but given the parameterized distribution model, you cannot necessarily reconstruct the state-space model.

Table 2: Hypothetical Net Returns ($/acre)

			Management		
State	**Precipitation**	**Temperature**	**None**	**Irrigation**	**Drainage**
1	High	High	$150	$100	$250
2	High	Low	$200	$150	$350
3	Low	High	$100	$150	$10
4	Low	Low	$150	$200	$70

There is another reason to be critical. We have assumed that irrigation and drainage both cost $50 per acre. Suppose that this $50 per acre actually represented 5 hours of labor at $10 per hour. The last two columns of Table 1 and 2 show that the effect of this 5 hours of labor on the possible outcomes depends crucially on whether labor is devoted to irrigation or drainage. This would not pose a conceptual problem if we explicitly modeled labor devoted to irrigation and labor devoted to drainage, but this is usually not how things are done. In the parameterized distribution approach, inputs like labor are typically modeled as a single input without concern for the specific activities for which it is used. This implicitly limits the possible tradeoffs a producer can make between states. For example, for the same $50 per acre, a farmer could irrigate all land, drain all land, or irrigate some proportion of land and drain the rest, which yields a much wider range of possible outcomes across states.

NOTATION

Since the parameterized distribution representation can be derived from the state-space representation, we will outline the state-space representation. To do this, we will need to modify some of our earlier notation.

S:	Finite number of mutually exclusive states of the world.
M:	Number of outputs in a given state.
N:	Number of inputs.
$\mathbf{q} \in \mathbb{R}_+^{M\times S}$:	Vector of outputs across all states.
$\mathbf{q}^s \in \mathbb{R}_+^{M}$:	Vector of outputs if state s occurs.
$q_m^{\ s} \geq 0$:	The mth output if state s occurs.
$\mathbf{p} \in \mathbb{R}_{++}^{M\times S}$:	Vector of output prices.
$\mathbf{p}^s \in \mathbb{R}_{++}^{M}$:	Vector of output prices if state s occurs.
$p_m^{\ s} > 0$:	The mth output price if state s occurs.
$\mathbf{z} \in \mathbb{R}_+^{N}$:	Vector of inputs.
$\mathbf{r} \in \mathbb{R}_{++}^{N}$:	Vector of input prices.
$\mathbf{PPS} \subset \mathbb{R}_+^{M\times S}\times\mathbb{R}_-^{N}$:	The production possibilities set where outputs are measured as positive numbers and inputs are measured as negative numbers.
$\mathbf{FOS}(\mathbf{z}) \subset \mathbb{R}_+^{M\times S}$:	The feasible output set.
$\mathbf{IRS}(\mathbf{q}) \subset \mathbb{R}_+^{N}$:	The input requirement set.

The way we will think about this problem is that a producer must choose inputs before knowing the state of the world. While the producer does not know the state of the world, it does know the outputs that can be produced in each state of the world given the inputs. The producer then sees the state of the world and realizes the output vector given that state and the selected inputs.

STATE-CONTINGENT PRODUCTION POSSIBILITY SETS

Much of the theory and empirics of production under uncertainty developed under the notion of a stochastic production function, which can generally be written as $q^s = f(\mathbf{z}, s)$ where $M = 1$. Specific examples commonly used in earlier work include $q^s = f(\mathbf{z})e^s$ where s is a continuous random variable such that $E(s) = 0$, $q^s = f(\mathbf{z})s$ where s is a continuous random variable such that $E(s) = 1$, and $q^s = f(\mathbf{z}) + s$ where s is a continuous random variable such that $E(s) = 0$. However, these specifications have tended to be replaced in more recent work by $q^s = f(\mathbf{z}) + h(\mathbf{z})s$ where s is a continuous random variable such that $E(s) = 0$ and $E(s^2) = 1$. This production function is actually a generalization that encompasses the other three. It was introduced by Just and Pope in their influential 1979 *Journal of Econometrics* paper "Stochastic Specification of Production Functions and Economic Implications." The introduction of this Just-Pope production function was motivated by the restrictive nature of the alternatives in terms of characterizing risk. For example, with $q^s = f(\mathbf{z})e^s$ and $q^s = f(\mathbf{z})s$, increasing an input increases both the mean and variance of output under the typical assumption that $\frac{\partial f(\mathbf{z})}{\partial z_n} > 0$. For $q^s = f(\mathbf{z}) + s$, increasing an input increases the mean output, but has no effect on the variance. With $q^s = f(\mathbf{z}) + h(\mathbf{z})s$, an input can increase or decrease the mean output and increase or decrease the variance of output. Still, in the context of the state-space model specified above, the Just-Pope specification or the even more general specification $q^s = f(\mathbf{z}, s)$ can still imply overly restrictive assumptions in terms of characterizing risky production. To get a sense of what types of restrictions may be implied by different assumptions on the form of the production possibilities sets, we will review some alternatives.

Output-Cubicle Technologies

The implications of the stochastic production function alluded to above result in what CQ refer to as an *output-cubicle technology*. These implications are illustrated in Figure 2 assuming $M = 1$ and $S = 2$. In Figure 2, the state-space output vector (q^1, q^2) represents the most output that can be achieved in states 1 and 2. Assuming free disposal of outputs then implies the feasible output set can be represented by the shaded region. An important limitation of this type of specification is that there is no way for a producer to tradeoff output across states, which lacks intuitive appeal. In our initial example with irrigation and drainage, if a farmer can choose how much labor to put into irrigation and how much into drainage, he/she can tradeoff increased yields between states with low precipitation and states with high precipitation contrary to what is inferred by Figure 2.

Generally speaking, a production possibilities set will be output-cubicle if $\mathbf{FOS}(\mathbf{z}) = \times_{s=1}^{S} \mathbf{FOS}^{s}(\mathbf{z})$ where $\mathbf{FOS}^{s}(\mathbf{z}) \subset \mathbb{R}_{+}^{M}$ is the feasible output set in state s given inputs $\mathbf{z}$. Alternatively, the input requirement set will be $\mathbf{IRS}(\mathbf{q}) = \cap_{s=1}^{S} \mathbf{IRS}^{s}(\mathbf{q}^{s})$ where $\mathbf{IRS}^{s}(\mathbf{q}^{s}) \subset \mathbb{R}_{+}^{N}$ is the input requirement set in state s given outputs $\mathbf{q}^{s}$. Intuitively, you have to find input vectors that are capable of producing at least the desired level of output in a state for all states.

State Allocable Input Technologies

Generating output tradeoffs across states as is illustrated in Figure 3 can be accomplished by having some or all inputs being state allocable. A state allocable input is an input that can be divided across activities that are only productive when certain states occur. For example, when we assumed labor could be devoted either to irrigation or drainage activities. Assuming all inputs are state allocable, $\mathbf{FOS}(\mathbf{z}) = \{\times_{s=1}^{S} \mathbf{FOS}^{s}(\mathbf{z}^{s}) : \sum_{s=1}^{S} \mathbf{z}^{s} \leq \mathbf{z}\}$ where $\mathbf{z}^{s}$ is a vector of inputs allocable to activities that produce more output in state s. Alternatively, $\mathbf{IRS}(\mathbf{q}) = \sum_{s=1}^{S} \mathbf{IRS}^{s}(\mathbf{q}^{s})$.

Homothetic Production Technologies

Homothetic production can occur on the output side, the input side, or both. On the output side, the feasible output set for an output homothetic production function can be written as $\mathbf{FOS}(\mathbf{z}) = g(\mathbf{z})\mathbf{G}^{0}$ where $g(\mathbf{z})$ is a real value function of inputs such that $g(\mathbf{z}) > 0$ and $\mathbf{G}^{0} \subseteq \mathbb{R}_{+}^{M\times S}$ is an output set for some fixed reference bundle of inputs. With these assumptions, the output distance function can be written as $D_{O}(\mathbf{q},\mathbf{z}) = \frac{\widehat{D}_{O}(\mathbf{q})}{g(\mathbf{z})}$ where $\widehat{D}_{O}(\mathbf{q})$ is the output distance function for $\mathbf{G}^{0}$. The primary implication of this type of production technology is that $MRT = \left|\frac{dq_{m'}}{dq_{m}}\right| = \frac{\frac{\partial \widehat{D}_{O}(q)}{\partial q_{m}}}{\frac{\partial \widehat{D}_{O}(q)}{\partial q_{m'}}}$, which doesn't depend on the level of inputs employed (the marginal rate of transformation is constant along a ray through the origin).

Input homothetic functions can be written as $\mathbf{IRS}(\mathbf{q}) = h(\mathbf{q})\mathbf{H}^0$ where $h(\mathbf{q})$ is a real value function of inputs such that $h(\mathbf{q}) > 0$ and $\mathbf{H}^0 \subseteq \mathbb{R}_+^N$ is an input set for some fixed reference bundle of state-contingent outputs. With these assumptions, the input distance function can be written as $D_I(\mathbf{q}, \mathbf{z}) = \frac{\widehat{D}_I(\mathbf{z})}{h(\mathbf{q})}$ where $\widehat{D}_I(\mathbf{z})$ is the input distance function for $\mathbf{H}^0$. Similarly, the primary implication of this assumption is that $MRTS = \left|\frac{dz_{n'}}{dz_n}\right| = \frac{\frac{\partial \widehat{D}_I(z)}{\partial z_n}}{\frac{\partial \widehat{D}_I(z)}{\partial z_{n'}}}$, so the marginal rate of technical substitution does not depend on the level of output produced.

If the production possibility set is both output and input homothetic, then it is referred to as completely homothetic or inversely homothetic.

There are other more or less restrictive types of technologies that have been defined, but in the interest of time, we will not go into them here.

COST & REVENUE FUNCTIONS UNDER UNCERTAINTY

Now that we have formalized production possibility sets under uncertainty, it is time to return to our notions of the cost and revenue functions to see what changes we might make to accommodate this new production possibilities set. Let us start with the cost minimization problem and cost function. If we maintain our conventional assumptions (the **PPS** is nonempty, closed, and satisfies weak free disposal of inputs; and competitive pricing of inputs), then hopefully it is clear that there are really no fundamental changes to the unconditional input demands and the cost function. The only real difference is that the vector of outputs that condition our input demands and cost function includes outputs in different states of the world. Therefore, all our standard properties for the conditional input demands and the cost function will still hold. Great news!

We could also solve the revenue maximization problem with our standard assumptions (the **PPS** is nonempty, closed, and satisfies weak free disposal of outputs; and competitive pricing of outputs) just as before. If we did, we would see that our conditional supplies and revenue function will also satisfy all the properties we have already derived. ***But does this make sense?*** Actually, it doesn't make much sense because our outputs and therefore, the revenues we realize will depend on the state. Doing the standard problem amounts to simply adding up revenues across all states and then maximizing this aggregate revenue. This is analogous to maximizing expected revenue assuming all states are equally likely, which will not make a lot of sense if states are not equally likely or if producers care about risk. The bottom line is that the revenue maximization problem is not as sensible in the case of uncertainty.

Revenue Cost Function

Given the standard revenue maximization problem doesn't make much sense under uncertainty, is there an alternative way to think about the problem that is more sensible and useful? The answer to this question is yes, but it requires making a change to how we think about the cost function rather than the revenue function.

Suppose we know both input and output prices are competitive and our other typical assumptions hold (the **PPS** is nonempty, closed, and satisfies weak free disposal of inputs and outputs). An alternative way to think about the cost minimization problem is in terms of producing revenues of at least $R_s > 0$ rather than outputs $\mathbf{q}^s$ in each state s. We can write this problem as

PU1 $\min_{\mathbf{q}\geq 0} C(\mathbf{r},\mathbf{q})$ subject to $\mathbf{p}^s \cdot \mathbf{q}^s \geq R_s$ for $s = 1,\ldots, S$.

The Lagrangian is

PU2 $L = C(\mathbf{r},\mathbf{q}) + \sum_{s=1}^{S} \lambda^s (R_s - \mathbf{p}^s \cdot \mathbf{q}^s)$

which (assuming differentiability) has the first order conditions

PU3 $\frac{\partial L}{\partial q_m^s} = \frac{\partial C(\mathbf{r},\mathbf{q}^*)}{\partial q_m^s} - \lambda^{s*} p_m^s \geq 0, \frac{\partial L}{\partial q_m^s} q_m^{s\,*} = 0$, and $q_m^{s\,*} \geq 0$ for $m = 1,\ldots, M$ and $s = 1,\ldots, S$;

PU4 $\frac{\partial L}{\partial \gamma} = R_s - \mathbf{p}^s \cdot \mathbf{q}^{s*} \leq 0, \frac{\partial L}{\partial \lambda^s} \lambda^{s*} = 0$, and $\lambda^s \geq 0$ for $s = 1,\ldots, S$.

The solution to this problem is a set of state-contingent supplies that are conditional on prices and the desired state-contingent revenues: $\mathbf{Q}(\mathbf{p}, \mathbf{r}, \mathbf{R})$ where $\mathbf{R} \in \mathbb{R}_{++}^S$ is a vector of state-contingent revenues. The *revenue cost function* is then $C(\mathbf{p}, \mathbf{r}, \mathbf{R}) = C(\mathbf{r}, \mathbf{q}(\mathbf{p}, \mathbf{r}, \mathbf{R}))$.

Before thinking about the properties of $\mathbf{Q}(\mathbf{p}, \mathbf{r}, \mathbf{R})$, and $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ it is useful to see what the first-order conditions tell us about the solution. Equation PU3 implies $\frac{\frac{\partial C(\mathbf{r},\mathbf{q}^*)}{\partial q_{m'}^{s'}}}{\frac{\partial C(\mathbf{r},\mathbf{q}^*)}{\partial q_m^s}} = \frac{\lambda^{s'*} p_{m'}^{s'}}{\lambda^{s*} p_m^s}$ for positive outputs. If $s = s'$, the Lagrangian multipliers cancel each other out and we simply have the ratio of marginal costs (or the marginal rate of transformation) equaling the price ratios. With $s \neq s'$, the ratio of marginal costs will not in general equal the price ratio because there is a tradeoff in revenues across states that may or may not be equal depending on the chosen $\mathbf{R}$. Equation PU4 simply says we must be producing at least the desirable level of revenue in each state.

Some properties of $\mathbf{Q}(\mathbf{p}, \mathbf{r}, \mathbf{R})$ and $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ that follow if the input requirement set is nonempty, closed, and satisfies weak free disposal of inputs are

(i) $\mathbf{Q}(\mathbf{p}, \mathbf{r}, \mathbf{R})$ and $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ are homogeneous of degree zero and one in $\mathbf{r}$;

(ii) $\mathbf{Q}(\mathbf{p}, \mathbf{r}, \mathbf{R})$ and $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ are homogeneous of degree zero in $\mathbf{p}^s$ and R_s for all s;

(iii) $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ is concave, continuous, and non-decreasing in $\mathbf{r}$;

(iv) $C(\mathbf{p}, \mathbf{r}, \mathbf{R}^1) \geq C(\mathbf{p}, \mathbf{r}, \mathbf{R}^0)$ for $\mathbf{R}^1 \geq \mathbf{R}^0$ if **FOS**($\mathbf{z}$) satisfies strong free disposal of output;

(v) $z_n(\mathbf{p}, \mathbf{r}^0, \mathbf{R}) = \frac{\partial C(\mathbf{p},\mathbf{r}^0,\mathbf{R})}{\partial r_n}$ for all n if **FOS**($\mathbf{z}$) satisfies strong free disposal of output and $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ is differentiable at $\mathbf{r} = \mathbf{r}^0$;

(vi) $q_m^s(\mathbf{p}^0, \mathbf{r}, \mathbf{R}^0) = -\frac{\frac{\partial C(\mathbf{p}^0,\mathbf{r},\mathbf{R}^0)}{\partial p_m^s}}{\frac{\partial C(\mathbf{p}^0,\mathbf{r},\mathbf{R}^0)}{\partial R_s}}$ for all s and m if $\mathbf{FOS}(\mathbf{z})$ satisfies strong free disposal of output and $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ is differentiable at $\mathbf{p} = \mathbf{p}^0$ and $\mathbf{R} = \mathbf{R}^0$;

(vii) $C(\mathbf{p}, \mathbf{r}, \mathbf{R})$ is convex in $\mathbf{R}$ if $\mathbf{FOS}(\mathbf{z})$ is convex; and

(viii) $C(\mathbf{p}^0, \mathbf{r}, \mathbf{R}) \geq C(\mathbf{p}^1, \mathbf{r}, \mathbf{R})$ for $\mathbf{p}^1 \geq \mathbf{p}^0$.

With this revenue cost function, we can start to explore the notion of technological risk. One way to characterize technological risk is to ask the question: What is the greatest certain revenue that can be obtained for the same cost as some stochastic revenue? The answer to this question is referred to as the *certainty equivalent revenue*:

$$ce^R(\mathbf{p}, \mathbf{r}, \mathbf{R}) = \max_e\{e \in \mathbb{R}: C(\mathbf{p}, \mathbf{r}, e\mathbf{1}^S) \leq C(\mathbf{p}, \mathbf{r}, \mathbf{R})\}$$

where $\mathbf{1}^S$ is a vector of S ones. Technically, the maximum may not exist, so if we want to be truly rigorous we should replace max with sup. This certainty equivalent revenue is similar to the certainty equivalent wealth you learned about when discussing choice under uncertainty earlier this semester. There is however an important distinction. The certainty equivalent wealth is a characterization of risk attitudes, while the certainty equivalent revenue is a characterization of technological risk.

Up to this point, our formalities have been devoid of any discussion of probabilities or other measures of the likelihood of states. However, if we introduce probabilities, we can further characterize technological risk. Let $\phi_s \geq 0$ be the probability of state s such that $\sum_{s=1}^S \phi_s = 1$. The expected revenue is then $\bar{R}(\boldsymbol{\phi}, \mathbf{R}) = \sum_{s=1}^S \phi_s R_s$ where $\boldsymbol{\phi}$ is the vector of state contingent probabilities. This allows us to define the *production-risk premium* as $arp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R}) = \bar{R}(\boldsymbol{\phi}, \mathbf{R}) - ce^R(\mathbf{p}, \mathbf{r}, \mathbf{R})$. Note that the production-risk premium is written explicitly as a function of the probabilities used to calculate the mean revenue because, quite frankly, not everyone agrees on probabilities, but more on this later. The production-risk premium is similar to the risk premium you talked about when looking at choice under uncertainty, but the production-risk premium is a measure of technological risk, while the risk premium is a measure of risk preferences. A technology is said to be *inherently risky* at $\mathbf{R}$ if $arp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R}) > 0$ and *not inherently risky* at $\mathbf{R}$ if $arp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R}) < 0$. Intuitively, under strong free disposal of output, a technology is inherently risky at $\mathbf{R}$ if it costs more to produce a certain revenue vector than the stochastic revenue vector with the same mean. That is, if removing uncertainty is costly.

There is another way of characterizing technological risk called the *relative production-risk premium*, which is written as $rrp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R}) = \frac{\bar{R}(\boldsymbol{\phi}, \mathbf{R})}{ce^R(\mathbf{p}, \mathbf{r}, \mathbf{R})}$. With the relative production-risk premium, a technology is inherently risky if $rrp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R}) \geq 1$.

When we first started talking about production, we explored different ways of characterizing various tradeoffs. With our revenue cost function, it is useful to characterize the tradeoff in costs for revenues in different states. To do this, we can differentiate the isocost function defined by $C(\mathbf{p}, \mathbf{r}, \mathbf{R}) = \bar{C}$ with respect to the revenues in states s and s', which yields $\left|\frac{dR_s}{dR_{s'}}\right| = \frac{\frac{\partial C(\mathbf{p},\mathbf{r},\mathbf{R})}{\partial R_{s'}}}{\frac{\partial C(\mathbf{p},\mathbf{r},\mathbf{R})}{\partial R_s}}$ after taking the absolute value. This expression is essentially the marginal rate of transformation between state-contingent revenues and represents the absolute value of the slope of an isocost curve.

Note that it is possible to show a number of useful properties of these certainty equivalents and risk premiums:

- $ce^R(\alpha\mathbf{p}, \mathbf{r}, \alpha\mathbf{R}) = \alpha ce^R(\mathbf{p}, \mathbf{r}, \mathbf{R})$ for $\alpha > 0$.
- $arp^P(\boldsymbol{\phi}, \alpha\mathbf{p}, \mathbf{r}, \alpha\mathbf{R}) = \alpha arp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R})$ for $\alpha > 0$.
- $rrp^P(\boldsymbol{\phi}, \alpha\mathbf{p}, \mathbf{r}, \alpha\mathbf{R}) = rrp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R})$ for $\alpha > 0$.
- If $arp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R} - \alpha\mathbf{1}^S) = arp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R})$ for any $\alpha \in \mathbb{R}$, $\left|\frac{dR_s}{dR_{s'}}\right|$ will be constant along rays that are parallel to the equal revenue vector, which is referred to as *constant absolute riskiness*.
- If $rrp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \alpha\mathbf{R}) = rrp^P(\boldsymbol{\phi}, \mathbf{p}, \mathbf{r}, \mathbf{R})$ for $\alpha > 0$, $\left|\frac{dR_s}{dR_{s'}}\right|$ will be constant along rays through the origin, which is referred to as *constant relative riskiness*.

Below we will develop similar concepts in more detail, that should help you better understand these concepts.

RISK & UNCERTAINTY PREFERENCES

The revenue cost function allows us to characterize the riskiness of producing different combinations of revenue across states. This is certainly of interest, but not really where we hope to end up. Ultimately, we would like to know what to produce in different states and how to produce it. To answer these questions, we need to know how the producer values outputs and inputs in alternative states. That is, we need to say something about the producer's preferences across states. Under our competitive market assumptions with production under certainty, we assumed the producer's goal was to maximize profit. We could continue with this assumption, but face some difficulties. Note that we can define the profit in state s using our revenue cost function as $\pi_s(\mathbf{p}, \mathbf{r}, \mathbf{R}) = R_s - C(\mathbf{p}, \mathbf{r}, \mathbf{R})$, so the first difficulty we face is that we no longer have a single profit function to consider. Instead, we have $s = 1,\ldots, S$ different profit functions. We could simply sum these profits up, but as mention above this would imply that the producer doesn't care about the likelihood of alternative states or risk, which is certainly contrary to casual observation. So what we need is some way to aggregate all these profit functions up into a single measure of value to the producer. The predominant way of aggregating up is EUT. With EUT, we can use the probabilities we just defined to write the producer's expected utility of profit as

PU5 $$EU(\boldsymbol{\pi}) = \sum_{s=1}^{S} \phi_s u(\pi_s)$$

where $\boldsymbol{\pi}$ is a vector of state-contingent profits and $u(\cdot)$ is an increasing and differentiable real value function such that $u'(\cdot) > 0$. Recall that we can use $u(\cdot)$ to characterize risk preferences: $u''(\cdot) = 0$ implies risk neutral, $u''(\cdot) < 0$ implies risk averse, and $u''(\cdot) > 0$ implies risk seeking/loving preferences.

While EUT is certainly a convenient and widely used framework for analyzing production decisions under risk, it is also very restrictive in terms of the assumptions it imposes on preferences and there are many examples of where it fails to adequately characterize behavior. Furthermore, its development has really been disconnected from classical consumer and producer theory. Therefore, we also want to consider more general characterizations of preferences that have clearer connections to classical theory.

The classical theory of consumer behavior uses rationality (complete and transitive preference orderings), nonsatiation/monotonicity, and continuity assumptions to derive a real value utility function that characterizes an individual's preferences over vectors of consumption goods. When these preferences are combined with a budget constraint, much can be said about how the individual will behave. In the context of production under uncertainty, we can think of the individual as a producer and the vector of consumption goods of interest as profits in alternative states. If the producer's preferences for profit in alternative states satisfy rationality, nonsatiation/monotonicty, and continuity, then we can derive a real value utility function to characterize these preferences. This utility function will have all of the same properties as the utility function used in consumer theory. Therefore, we will generally let $W(\boldsymbol{\pi})$ represent the producer's utility of profit in different states. To keep our life a little simpler, we will assume that $W(\boldsymbol{\pi})$ is differentiable and preferences are monotonic such that $\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s} \geq 0$ for all s, which means more profit never makes the producer worse off (getting away from nonsatiation helps us deal with the fact the profits can be negative, which is different from consumer theory where we assumed the quantity of goods consumed was non-negative). These assumptions also make EUT a special case where $W(\boldsymbol{\pi}) = \sum_{s=1}^{S} \phi_s u(\pi_s)$. Finally, it will also be convenient to assume preferences are strictly convex, so we are guaranteed to have a unique solution to our problem. Though, it is important to note that some of the theories that have come closest to displacing EUT (e.g., prospect theory and cumulative prospect theory) do not satisfy strict convexity or even convexity for that matter.

Earlier this semester you talked about different ways to characterize and measure individual risk preferences. We will want to be able to do the same here with our more general preferences. We will emphasize the notion of risk aversion because it implies diminishing marginal returns to state-contingent profits and is intuitively appealing. The notion of risk aversion is the preference for certainty. The typical way this preference is expressed is through the comparison of an uncertain outcome to its expected value with certainty:

Definition: A producer is *risk averse* with respect to the probability vector $\boldsymbol{\phi}$ if $W(\bar{\pi}\mathbf{1}^S) \geq W(\boldsymbol{\pi})$ for all $\boldsymbol{\pi}$ where $\bar{\pi} = \sum_{s=1}^{S} \phi_s \pi_s$ is the expected outcome of $\boldsymbol{\pi}$ given $\boldsymbol{\phi}$.

This definition implies that whether or not a producer is risk averse depends on the probabilities of the various states; but where do these probabilities come from? Actually, this definition implies that they are embedded in $W(\boldsymbol{\pi})$, at least when $W(\boldsymbol{\pi})$ is nicely differentiable. To see why, consider Figure 4. The line denoted by $\bar{\pi}^0$ reflects all combinations of $\boldsymbol{\pi}$ that yield the same expected value given $\boldsymbol{\phi}$. Note that the absolute value of the slope of this line will be equal the ratio of probabilities: $\frac{\phi_1}{\phi_2}$. The curve denoted by W is a representative indifference curve. The 45° line represents all combinations of equal profit. W is drawn so it is just tangent to $\bar{\pi}^0$ along the 45° line, which means there is no way to increase utility given this expected profit and probabilities, so these probabilities and preferences are consistent with our definition of risk aversion. To see this, suppose we increase ϕ_1 and decrease ϕ_2 so the expected profit doesn't change. This will rotate $\bar{\pi}^0$ clockwise around (π^0, π^0). Doing this would mean that W would no longer be tangent along $\bar{\pi}^0$ implying $W(\bar{\pi}^0 1^S) < W(\boldsymbol{\pi})$ for some $\boldsymbol{\pi}$ on $\bar{\pi}^0$ violating our definition of risk aversion. We would get an analogous result if we decrease ϕ_1 and increase ϕ_2 so the expected profit doesn't change. What all this means is that for a producer's preferences to be consistent with our definition of risk aversion, $W(\boldsymbol{\pi})$ must satisfy the condition that the slopes of our indifference curves (or marginal rate of substitution) evaluated along the equal profits line must be equal. Note that the marginal rate of substitution between states s^0 and s^1 is generally defined as $\left|\frac{d\pi_{s^1}}{d\pi_{s^0}}\right| = \frac{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_{s^0}}}{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_{s^1}}}$. Therefore, we need to impose the restriction $\frac{\frac{\partial W(\mathbf{1}^S)}{\partial \pi_{s^0}}}{\frac{\partial W(\mathbf{1}^S)}{\partial \pi_{s^1}}} = \frac{\frac{\partial W(\alpha\mathbf{1}^S)}{\partial \pi_{s^0}}}{\frac{\partial W(\alpha\mathbf{1}^S)}{\partial \pi_{s^1}}}$ for all α and all s^0, s^1 = 1,…, S. To satisfy this restriction, we can define the probability of state s as $\phi_s = \frac{\frac{\partial W(\mathbf{1}^S)}{\partial \pi_s}}{\sum_{s'=1}^{S} \frac{\partial W(\mathbf{1}^S)}{\partial \pi_{s'}}}$. For these to be valid probabilities $\phi_s \geq 0$, which is true because we have assumed $\frac{\partial W(\mathbf{1}^S)}{\partial \pi_s} \geq 0$ for all s, and $\sum_{s=1}^{S} \phi_s = 1$, which is true because $\sum_{s=1}^{S} \frac{\frac{\partial W(\mathbf{1}^S)}{\partial \pi_s}}{\sum_{s'=1}^{S} \frac{\partial W(\mathbf{1}^S)}{\partial \pi_{s'}}} = 1$.

Certainty Equivalent & Risk Premiums

Recall that it is possible to measure an individual's degree of risk aversion using the *risk premium*, which is the difference in the expected outcome, $\bar{\pi}(\boldsymbol{\phi}, \boldsymbol{\pi})$, and the *certainty equivalent*, $ce(\boldsymbol{\pi})$ (implicitly defined by $u\big(ce(\boldsymbol{\pi})\big) = \sum_{s=1}^{S} \phi_s u(\pi_s)$ in the expected utility case). This definition still holds with more general preferences with a suitable generalization of the definition of the certainty equivalent:

Definition: $ce(\boldsymbol{\pi}) = \min_c\{c \in \mathbb{R}: W(c\mathbf{1}^S) \geq W(\boldsymbol{\pi})\}$.

This definition tells us that the certainty equivalent is the least amount of certain profit we can give the producer and have it be no worse off than with an uncertain profit (min can be replaced by inf if a minimum doesn't exist). It is worth noting that the certainty equivalent is actually a complete characterization of a producer's preferences, much like the output and input distance functions are a complete characterizations of a producer's production possibilities set. A more formal definition of the *risk premium* is

Definition: $arp(\boldsymbol{\phi}, \boldsymbol{\pi}) = \max_c\big\{c \in \mathbb{R}: W\big((\bar{\pi} - c)\mathbf{1}^S\big) \geq W(\boldsymbol{\pi})\big\}$,

which we will also referred to as the *absolute risk premium*. This definition says that the absolute risk premium is the most we can take away from the average profit in each state and still have the producer be just as well off as if it received the uncertain profit. Another way to approach this question is to ask how much can we proportionally reduce the average profit in each state and still have the producer be just as well off as with an uncertain profit. The answer to this question is the *relative risk premium*, which is defined as

Definition: $rrp(\boldsymbol{\phi}, \boldsymbol{\pi}) = \max_c\left\{c > 0: W\left(\frac{\bar{\pi}\mathbf{1}^S}{c}\right) \geq W(\boldsymbol{\pi})\right\}$.

Note that these definitions imply $arp(\boldsymbol{\phi}, \boldsymbol{\pi}) = \bar{\pi} - ce(\boldsymbol{\pi})$ and $rrp(\boldsymbol{\phi}, \boldsymbol{\pi}) = \frac{\bar{\pi}}{ce(\boldsymbol{\pi})}$

Absolute & Relative Risk Aversion

In consumer theory, we characterize tradeoffs in consumption using the marginal rate of substitution. Here we can do the same thing where $\left|\frac{d\pi_t}{d\pi_s}\right| = \frac{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}}{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_t}}$ is our marginal rate of substitution. Note that for EUT $\left|\frac{d\pi_t}{d\pi_s}\right| = \frac{\phi_s u'(\pi_s)}{\phi_t u'(\pi_t)}$ such that the marginal rate of substitution equals the ratio of probabilities $\left(\left|\frac{d\pi_t}{d\pi_s}\right| = \frac{\phi_s}{\phi_t}\right)$ when $\pi_t = \pi_s$. To continue to explore the implications of risk preferences, we can analyze these implied tradeoffs further. For example, what happens to the marginal rate of substitution if we proportionally increase profits in all states? That is, how does the slope of our indifference curves change as we move along a ray through the origin? To answer this question, we can totally differentiate $\left|\frac{d\pi_t}{d\pi_s}\right| = \frac{\frac{\partial W(\alpha\boldsymbol{\pi})}{\partial \pi_s}}{\frac{\partial W(\alpha\boldsymbol{\pi})}{\partial \pi_t}}$ with respect to α for $\alpha = 1$, which yields

PU6 $$\frac{d\left|\frac{d\pi_t}{d\pi_s}\right|}{d\alpha} = \left|\frac{d\pi_t}{d\pi_s}\right| \left(\frac{\sum_{v=1}^{S} \pi_v \frac{\partial^2 W(\boldsymbol{\pi})}{\partial \pi_s \partial \pi_v}}{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}} - \frac{\sum_{v=1}^{S} \pi_v \frac{\partial^2 W(\boldsymbol{\pi})}{\partial \pi_t \partial \pi_v}}{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_t}} \right).$$

Admittedly, this expression does not look too interpretable, but notice that for EUT it reduces to

PU7 $$\frac{d\left|\frac{d\pi_t}{d\pi_s}\right|}{d\alpha} = \left|\frac{d\pi_t}{d\pi_s}\right| \left(\pi_s \frac{u''(\pi_s)}{u'(\pi_s)} - \pi_t \frac{u''(\pi_t)}{u'(\pi_t)} \right).$$

Recall from earlier this semester that the Arrow-Pratt coefficient of relative risk aversion is defined as $RRA(c) = -c\frac{u''(c)}{u'(c)}$ such that the marginal rate of substitution in equation PU7 will be increasing or decreasing as $RRA(\pi_t) > (<) RRA(\pi_s)$. With constant relative risk aversion (CRRA), $RRA(\pi_t) = RRA(\pi_s)$ such that the marginal rate of substitution will be the same along a ray through the origin, which is a characteristic of homothetic functions. If $\pi_t > \pi_s$, then the marginal rate of substitution will be increasing (decreasing) with increasing (decreasing) relative risk aversion.

Alternatively, what happens to the marginal rate of substitution as we increase profits in all states by the same amount? That is, what happens to the marginal rate of substitution as we move along a ray parallel to the equal profits ray? We can answer this question by totally differentiating $\left|\frac{d\pi_t}{d\pi_s}\right| = \frac{\frac{\partial W\left(\boldsymbol{\pi}+\alpha\mathbf{1}^S\right)}{\partial \pi_s}}{\frac{\partial W\left(\boldsymbol{\pi}+\alpha\mathbf{1}^S\right)}{\partial \pi_t}}$ with respect to α for $\alpha = 0$, which yields

PU8 $$\frac{d\left|\frac{d\pi_t}{d\pi_s}\right|}{d\alpha} = \left|\frac{d\pi_t}{d\pi_s}\right|\left(\frac{\sum_{v=1}^{S}\frac{\partial^2 W(\boldsymbol{\pi})}{\partial \pi_s \partial \pi_v}}{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}} - \frac{\sum_{v=1}^{S}\frac{\partial^2 W(\boldsymbol{\pi})}{\partial \pi_t \partial \pi_v}}{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_t}}\right).$$

For EUT equation PU8 becomes

PU9 $$\frac{d\left|\frac{d\pi_t}{d\pi_s}\right|}{d\alpha} = \left|\frac{d\pi_t}{d\pi_s}\right|\left(ARA(\pi_t) - ARA(\pi_s)\right)$$

where $ARA(c) = -\frac{u''(c)}{u'(c)}$ is the Arrow-Pratt coefficient of absolute risk aversion. Therefore, with constant absolute risk aversion (CARA), the marginal rate of substitution will not change as we increase profits in all states by the same amount. Similarly, if $\pi_t > \pi_s$, the marginal rate of substitution will be increasing (decreasing) with increasing (decreasing) absolute risk aversion.

More generally, CQ defines relative and absolute risk aversion in terms of the risk premiums.

Definition: $W(\boldsymbol{\pi})$ displays

Constant Absolute Risk Aversion (CARA) if, for any π, $arp(\boldsymbol{\phi}, \boldsymbol{\pi} + \alpha\mathbf{1}^S) = arp(\boldsymbol{\phi}, \boldsymbol{\pi})$ for $\alpha \in \mathbb{R}$;

Increasing Absolute Risk Aversion (IARA) if, for any π, $arp(\boldsymbol{\phi}, \boldsymbol{\pi} + \alpha\mathbf{1}^S) \geq arp(\boldsymbol{\phi}, \boldsymbol{\pi})$ for $\alpha > 0$;

Decreasing Absolute Risk Aversion (DARA) if, for any π, $arp(\boldsymbol{\phi}, \boldsymbol{\pi} + \alpha\mathbf{1}^S) \leq arp(\boldsymbol{\phi}, \boldsymbol{\pi})$ for $\alpha > 0$;

Constant Relative Risk Aversion (CRRA) if, for any π, $rrp(\boldsymbol{\phi}, \alpha\boldsymbol{\pi}) = rrp(\boldsymbol{\phi}, \boldsymbol{\pi})$ for $\alpha > 0$;

Increasing Relative Risk Aversion (IRRA) if, for any π, $rrp(\boldsymbol{\phi}, \alpha\boldsymbol{\pi}) \geq rrp(\boldsymbol{\phi}, \boldsymbol{\pi})$ for $\alpha > 1$; and

Decreasing Relative Risk Aversion (DRRA) if, for any π, $rrp(\boldsymbol{\phi}, \alpha\boldsymbol{\pi}) \leq rrp(\boldsymbol{\phi}, \boldsymbol{\pi})$ for $\alpha > 1$.

Stochastic Dominance

With EUT, you were introduced to the concept of first- and second-order stochastic dominance (FOSD and SOSD). These concepts provide a partial ordering of different distributions of income/wealth in terms of an individual's preferences. If an individual's utility of income/wealth is increasing (e.g., $u'(\cdot) > 0$) and the distribution *F* first-order stochastically dominates the distribution *G*, then *F* is preferred. Alternatively, if an individual's utility of income/wealth is increasing at a decreasing rate (e.g., $u'(\cdot) > 0$ and $u''(\cdot) < 0$) implying risk aversion and the distribution *F* second-order stochastically dominates the distribution *G*, then *F* is preferable. These notions are actually quite useful and make it possible to answer a lot of questions. A question we might ask now is whether this notion can be extended to more general representations of preferences. The answer to this question is yes through the notion of *generalized Schur-concavity*, which places some additional restrictions on $W(\boldsymbol{\pi})$.

The concept of SOSD developed around a rather specific notion of increasing risk. In particular, the notion of a *mean preserving spread*. Intuitively, a mean preserving spread makes relatively large and relatively small profits more likely without changing the mean. Before providing a more formal definition of a mean preserving spread, we will need some more notation. First, we need to recall the definition of the cumulative distribution for the discrete case:

Definition: The *cumulative distribution function* given $t \in \mathbb{R}$, profits $\boldsymbol{\pi}$, and the probabilities $\boldsymbol{\phi}$ is

$$F(t, \boldsymbol{\pi}, \boldsymbol{\phi}) = \sum_{\{s:\pi_s \leq t\}} \phi_s.$$

Let the subscript $[s]$ indicate the ranking of profits in each state in increasing order such that $\pi_{[s]} \leq \pi_{[s+1]}$ for all s = 1,…, S – 1 and define $s(t, \boldsymbol{\pi})$ = $[s]$ such that $\pi_{[s]} \leq t < \pi_{[s+1]}$, which essentially tells us all of the states that yield a profit lower than t.

Definition: $\boldsymbol{\pi}'$ is a *mean preserving spread* of $\boldsymbol{\pi}$ if $G(t, \boldsymbol{\pi}', \boldsymbol{\phi}) \geq G(t, \boldsymbol{\pi}, \boldsymbol{\phi})$ for all $t \in \mathbb{R}$ and $\sum_{s=1}^{S} \phi_s \pi_s = \sum_{s=1}^{S} \phi_s \pi_s{}'$ where $G(t, \boldsymbol{\pi}, \boldsymbol{\phi}) = \sum_{s=1}^{s(t,\boldsymbol{\pi})} \left(\pi_{[s]} - \pi_{[s-1]}\right) F\left(\pi_{[s-1]}, \boldsymbol{\pi}, \boldsymbol{\phi}\right)$.

Note that these conditions for a mean preserving spread essentially amount to the conditions for SOSD, which we can use to restrict producer preferences, without having to fall back to EUT:

Definition: Given probabilities $\boldsymbol{\phi}$, $W(\boldsymbol{\pi})$ is *generalized Schur-concave* if $G(t, \boldsymbol{\pi}', \boldsymbol{\phi}) \geq G(t, \boldsymbol{\pi}, \boldsymbol{\phi})$ for all $t \in \mathbb{R}$ and $\sum_{s=1}^{S} \phi_s \pi_s = \sum_{s=1}^{S} \phi_s \pi_s{}'$ imply $W(\boldsymbol{\pi}) \geq W(\boldsymbol{\pi}')$.

This is all very exciting given the nice intuition that mean preserving spreads are bad for producers who don't like risk and all the concerns surrounding EUT. However, there is an implication of generalized Schur-concavity that takes us right back to where we started, at least to some degree. Recall that we criticized the outcome-space approach due to its reduced form representation of the production possibilities set. Producers who only care about this reduced form representation are referred to as probabilistically sophisticated:

Definition: A producer's utility of profit function $W(\boldsymbol{\pi})$ exhibits *probabilistic sophistication* with respect to probabilities $\boldsymbol{\phi}$ if for any $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ such that $F(t, \boldsymbol{\pi}, \boldsymbol{\phi}) = F(t, \boldsymbol{\pi}', \boldsymbol{\phi})$ for all t, $W(\boldsymbol{\pi}) = W(\boldsymbol{\pi}')$.

Generalized Schur-concave preferences are probabilistically sophisticated, so producers with these types of preferences only care about the distribution of outcomes, which means that an outcome-space approach is adequate for analyzing uncertain production choices. With that said, if we are willing to dump the assumption that producers do not like mean preserving spreads, we can still rely on our more general notions of risk aversion even though we can no longer exploit stochastic dominance concepts.

Before moving on, there are some important implications of generalized Schur-concave preferences that are worth mentioning: If $\boldsymbol{\pi}'$ is a mean preserving spread of $\boldsymbol{\pi}$ and $W(\boldsymbol{\pi})$ is generalized Schur-concave, then

(i) $\left(\frac{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}}{\phi_s} - \frac{\frac{\partial W(\boldsymbol{\pi})}{\partial \pi_t}}{\phi_t}\right)(\pi_s - \pi_t) \leq 0$ for all s and t if $W(\boldsymbol{\pi})$ is differentiable,

(ii) $\sum_{s=1}^{S} \frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}(\pi_s - \sum_{k=1}^{S} \phi_k \pi_k) \leq 0$ if $W(\boldsymbol{\pi})$ is differentiable,

(iii) $ce(\boldsymbol{\pi})$ is generalized Schur-concave, such that $ce(\boldsymbol{\pi}) \geq ce(\boldsymbol{\pi}')$,

(iv) $arp(\boldsymbol{\phi}, \boldsymbol{\pi})$ is generalized Schur-convex, such that $arp(\boldsymbol{\phi}, \boldsymbol{\pi}) \leq arp(\boldsymbol{\phi}, \boldsymbol{\pi}')$, and

(v) $rrp(\boldsymbol{\phi}, \boldsymbol{\pi})$ is generalized Schur-convex, such that $rrp(\boldsymbol{\phi}, \boldsymbol{\pi}) \leq rrp(\boldsymbol{\phi}, \boldsymbol{\pi}')$.

Figure 1: Cumulative Distributions of Net Returns Assuming Irrigation and Drainage Costs Are $50 Per Acre

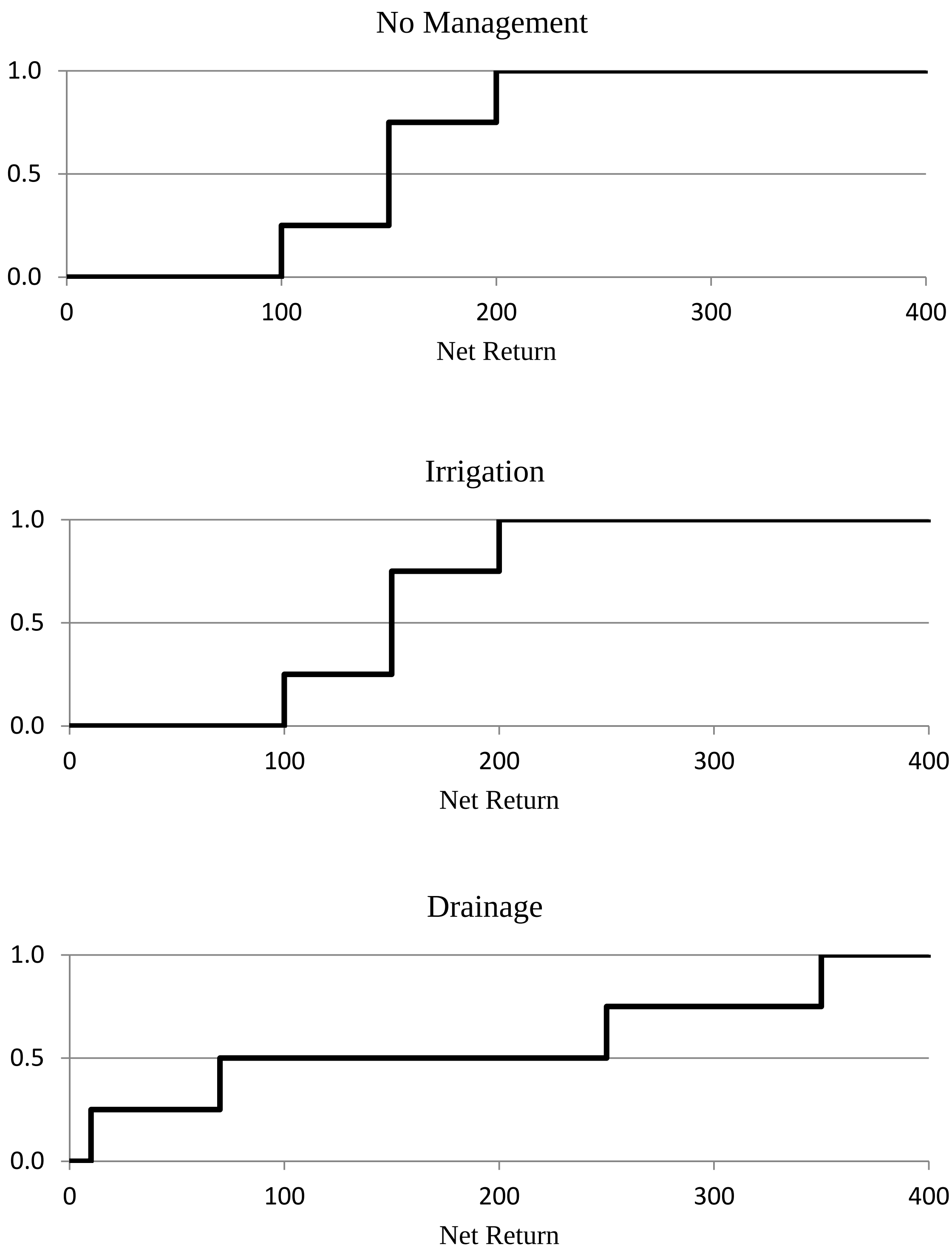

Figure 2: Output Cubicle Technologies

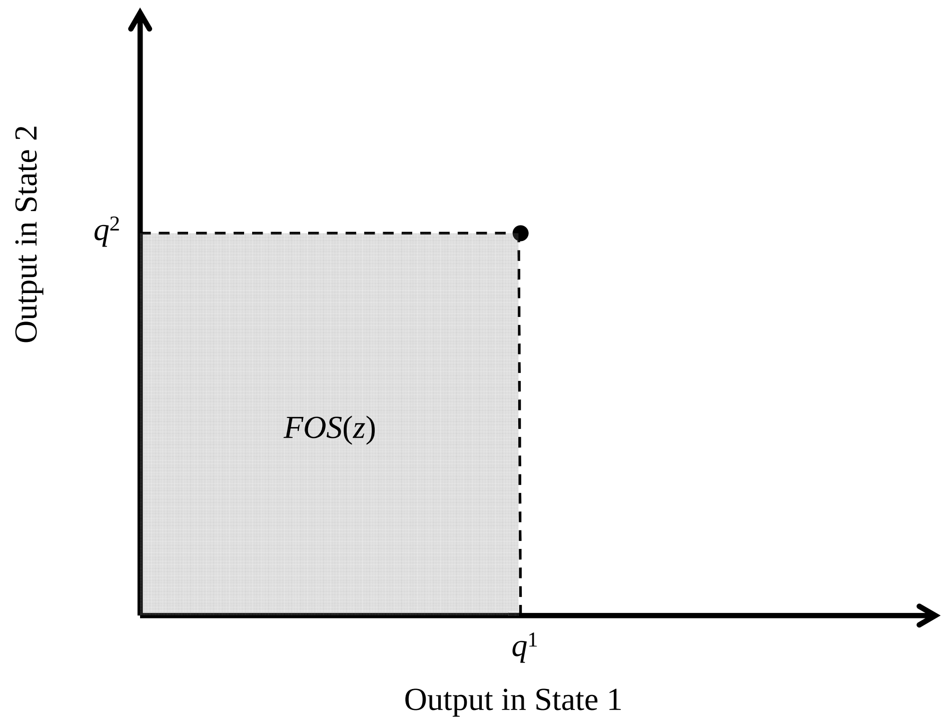

Figure 3: State Allocable Inputs

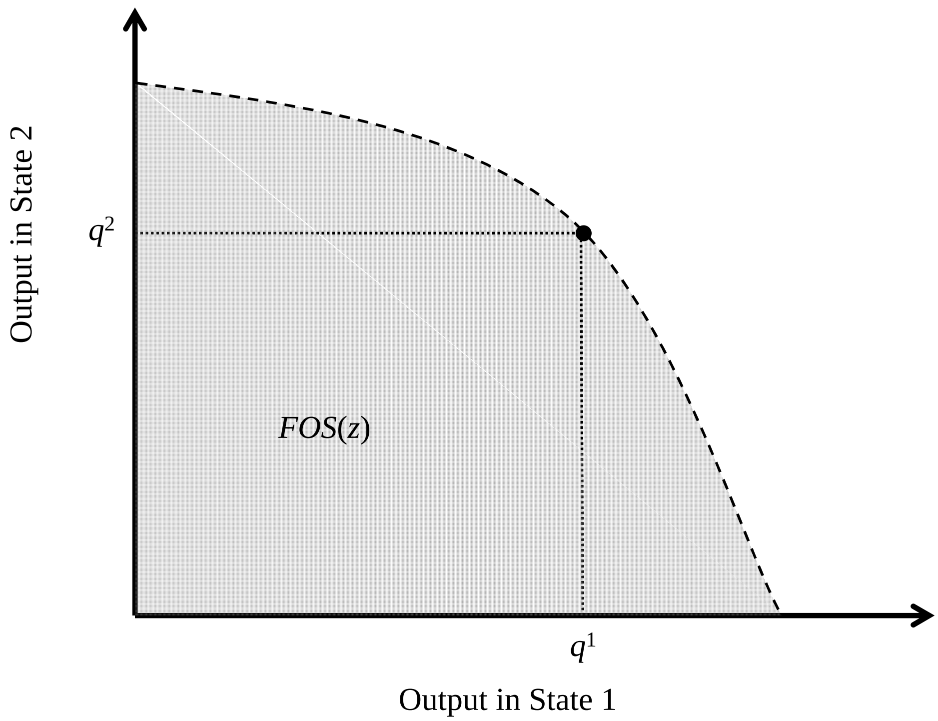

Figure 4