ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 2: Consumer Choice (MWG, Ch. 2, pp.17-28)

I. Introduction

The discussion of preference and choice in Lecture 1 was rather abstract. In particular the objects of choice could be almost anything, and the decision maker could be almost any kind of person.

In this lecture we narrow this down. The objects of choice are "bundles" of goods and services, and the decision maker is a consumer. The consumer has an amount of wealth to spend, and all items in the bundles have a price.

We begin by describing the commodities, and then move on to the consumption set (all feasible choices within the budget constraint), followed by a discussion of demand functions.

Notice that we do not yet introduce the concept of a utility function! That will be done next week.

II. Commodities

Commodities are simply goods and services that the consumer values. We assume that there a finite number of possible commodities, denoted by L. We use ℓ to index these commodities, so that $\ell = 1, 2, \ldots L$.

A **commodity vector**, or **commodity bundle**, is denoted by x. It is a "list" of the individual commodities in a specific bundle. It is complete in that it indicates how much of all L commodities are in the bundle. If some commodity is not in the bundle then we assign it a value of zero. Thus the vector x can be explicitly described as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{bmatrix}$$

Although we will usually think of x as amounts consumed during a specific time period, it could be made more general to represent consumption over several time periods. For example, for two time periods we could have consumption of good 1 during time period 1 as one element of the vector x and consumption of good 1 during time period 2 as another element of the vector x.

III. The Consumption Set

Before talking about budget constraints, it is useful to point out that there are "physical" limits to the set of all possible vectors (all possible values of the vector x). For example, for most commodities it is not possible to consume a negative amount.

Formally, we can define the **consumption set** as a subset of the "commodity space", denoted by $\mathbb{R}^L$, whose elements are the consumption bundles that the consumer could possibly consume, subject to physical constraints.

Mas Colell et al. give four examples of physical constraints on pp.18-19:

1. It is not possible to consume more than 24 hours of leisure time in a given day.

2. For some goods, it is not possible to purchase "parts" of a good. That is the number of goods must be an integer. What examples can you think of ?

3. For some pairs of goods, you can only purchase one of the pair, not both. Their example is eating bread in New York City at a given point in time and eating bread in Washington D.C. at the same point in time.

4. For some necessities, such as food, you cannot survive if you eat a total amount that provides insufficient calories for survival.

The consumption sets for these 4 examples are shown in diagrams on p.19 of Mas Colell et al.

These examples are physical constraints, but “institutional” constraints are also possible. For example a law could be passed saying that it is illegal for someone to work more than 16 hours per day. This means that one is “forced” to consume at least 8 hours of leisure every day (assuming any time that is not work time is leisure time). A diagram shows this on p.20 of Mas Colell et al.

For most economic models, it is reasonable to assume that the **consumption set**, which can be **denoted by X**, consists of collections of commodity vectors x for which **all elements are nonnegative**:

$$X = \mathbb{R}^L_+ = \{ x \in \mathbb{R}^L : x_\ell \geq 0 \text{ for } \ell = 1, 2, \ldots, L\}$$

One feature of the consumption set $X = \mathbb{R}^L_+$ is that it is **convex**. Intuitively, a set is convex if, for any two elements in the set, a “weighted average” of the set is also an element in the same set. More formally, if two consumption bundles, x and x′, are in a convex set (in this case, are in $\mathbb{R}^L_+$), then a new bundle, denoted by x'' and defined as $x'' = \alpha x + (1 - \alpha)x'$, where $\alpha \in [0, 1]$, is also a member of that set (is also in $\mathbb{R}^L_+$).

[Don't forget that x, x' and x'' are all vectors. In particular each element of x'', denoted by x_ℓ'' is defined as $x_\ell'' = \alpha x_\ell + (1 - \alpha)x_\ell'$.]

Much of the theory that we will study in this class depends on convexity of consumption sets, but in some cases this convexity is not needed.

IV. Budget Constraints and the Walrasian Budget Set

In addition to physical constraints, consumers are also constrained by how much they can afford to purchase. To define this constraint formally, we need to make **two assumptions**:

1. The L commodities (goods and services) are all traded in a market at given prices, denoted by the vector p.

2. These prices cannot be influenced by any individual consumer; that is, all consumers are **price takers**.

The price vector, like the commodity vector, is a column vector with L elements in it:

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

Technically, some of these prices could be negative. An example would be a "bad" such as pollution; you would have to pay people to consume it. However, unless otherwise noted, for simplicity we will **assume that $\mathbf{p} \gg \mathbf{0}$**, that is $p_\ell > 0$ for all ℓ.

The amount of money that the consumer has to purchase commodities is denoted by w ("wealth"). Thus the only consumption bundles that the consumer can afford to purchase are those that satisfy the budget constraint:

$$p \cdot x = p_1x_1 + p_2x_2 + \ldots + p_Lx_L \leq w$$

[In general, in Mas Colell et al. the inner product of two column vectors is denoted by "·"; the transpose for the first column vector is not shown.]

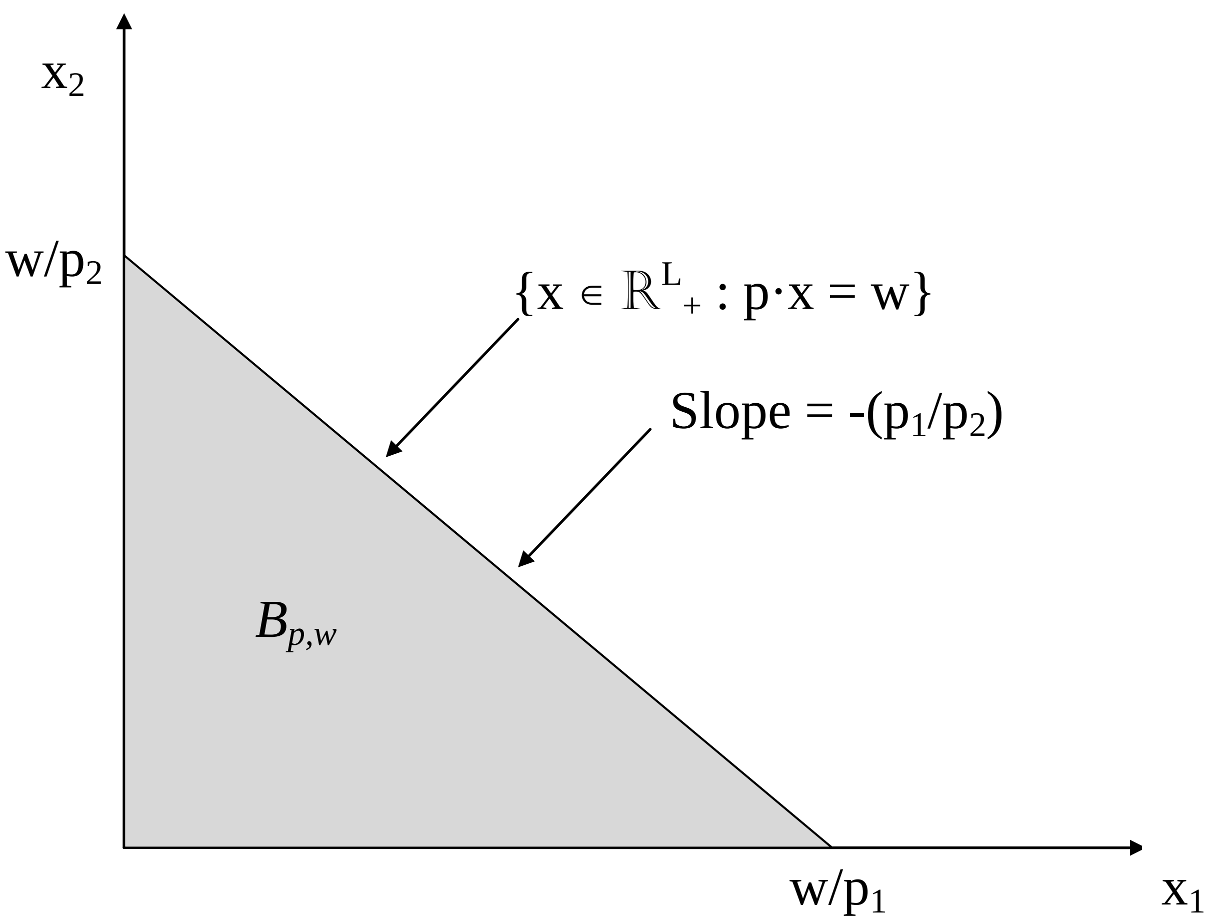
A formal definition of the consumption bundles that the consumer can afford is:

Definition: The **Walrasian budget set** (also known as the **competitive budget set**), which can be denoted as $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$, is the set of **all feasible consumption bundles** for the consumer who faces prices p and has a wealth of w.

The "consumer's problem" can be stated as: Given prices p and wealth w, choose a consumption bundle x from $B_{p,w}$.

From now on we will always assume that $w > 0$.

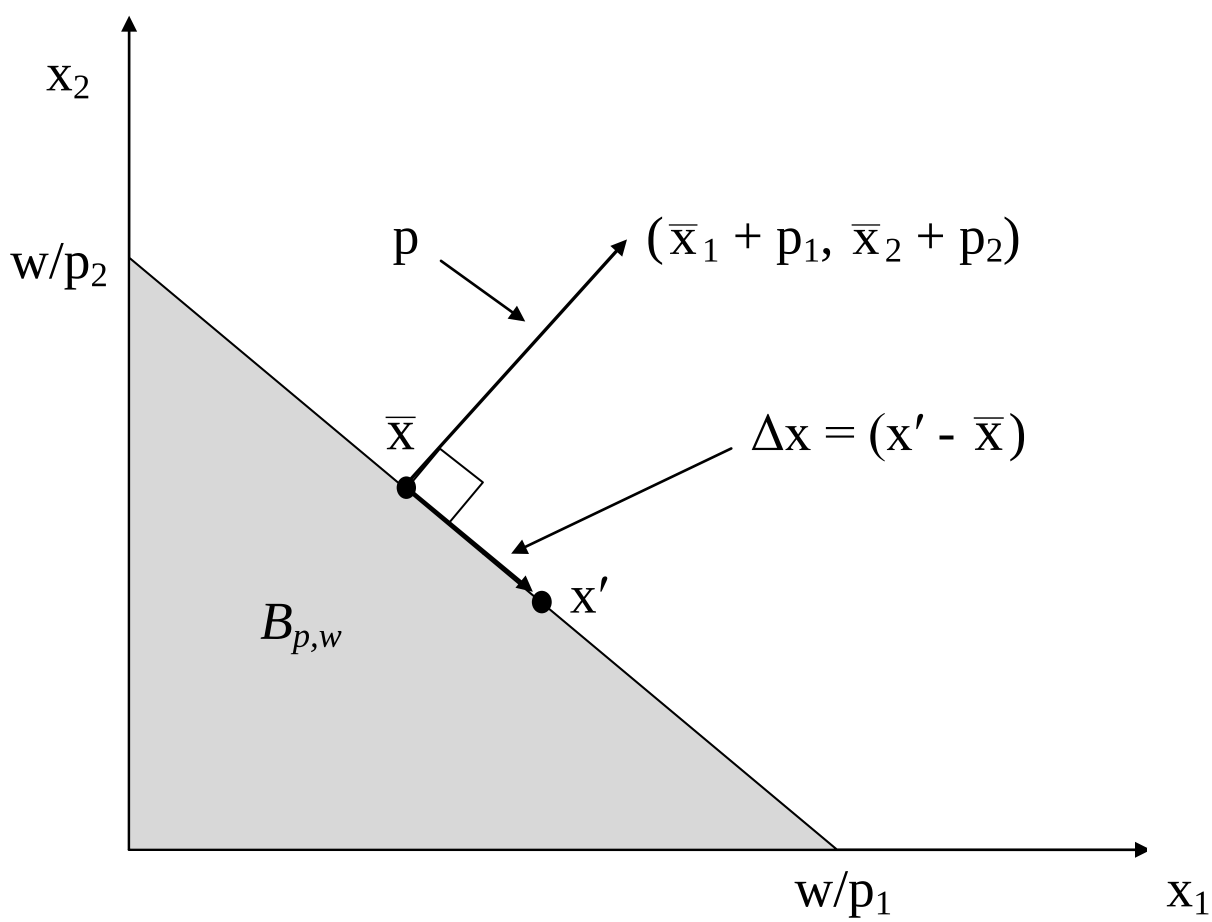
The following graph shows a Walrasian budget set:



The (sub)set $B_{p,w} = \{x \in \mathbb{R}^L : p \cdot x = w\}$ is called the **budget hyperplane** (note that it has =, not ≤). (If L = 2, as in the above diagram, it is a **budget line**.) It shows the "rate of exchange" between goods given their prices.

Question: What happens to the diagram if p_2 decreases?

Another characteristic of the budget hyperplane is that it is orthogonal (perpendicular) to the price vector p. This is shown in the following diagram:



Let $\bar{x}$ be any point on the budget hyperplane. Draw the price vector p from this point. It is orthogonal to any vector lying within the budget hyperplane. This is the case because for all points on the hyperplane we have $p \cdot x = w$. Thus $p \cdot \bar{x} = w$, and for any x' on the hyperplane $p \cdot x' = w$, which implies that $p \cdot (x' - \bar{x}) = 0$. Thus $p \cdot \Delta x = 0$ for any vector Δx within the hyperplane.

A final property of the **Walrasion budget set** $\boldsymbol{B_{p,w}}$ is that it **is convex**, which means that if any bundles x and x' are both elements of $B_{p,w}$, then $x'' = \alpha x + (1-\alpha)\, x'$ is also in $B_{p,w}$ (where $0 \leq \alpha \leq 1$). This is easy to show. First, since both x and $x' \in \mathbb{R}^L_+$ (both are nonnegative), then $x'' \in \mathbb{R}^L_+$. Second, the assumption that both x and x' are in $B_{p,w}$ means that $p \cdot x \leq w$ and $p \cdot x' \leq w$, which in turn implies that $\alpha(p \cdot x) + (1-\alpha)p \cdot x' \leq \alpha w + (1-\alpha)w = w$. Finally, $p \cdot x'' = p \cdot [\alpha x + (1-\alpha)\, x'] = \alpha(p \cdot x) + (1-\alpha)p \cdot x' \leq w$, so x'' is in $B_{p,w}$.

Final note: There are other types of budget sets that consumers may face in the real world that are not Walrasion, due to (for example) taxes and overtime payments that lead to kinks in the budget set. These may not be convex; an example is given on p.22 of Mas Colell et al.

V. Demand Functions

We can define a consumer's **Walrasian demand correspondence**, $x(p, w)$, as a relationship that assigns a set of chosen consumption bundles for every possible value of p and w. In theory, it is possible that, for a given (p, w) pair, there is more than one consumption bundle. However, we will usually assume that there is a single unique consumption bundle for each (p, w) pair, and so we can call this relationship the **Walrasian demand function** (also called the Marshallian demand function).

For the rest of this lecture (and the next lecture) we will make two assumptions about the Walrasian demand correspondence: it is **homogenous of degree zero** and it **satisfies Walras' law**. Let's define these two concepts:

Definition: The Walrasian demand correspondence $x(p, w)$ is **homogenous of degree zero** if, for any $\alpha > 0$, $x(\alpha p, \alpha w) = x(p, w)$ for any values of p and w.

The intuition here is that the consumer does not suffer from "money illusion"; if prices and wealth increase or decrease by the same proportion then there is no reason to change his or her consumption bundle. More formally, such a change in prices and wealth does not change the consumer's feasible consumption bundles (that is $B_{p,w} = B_{\alpha p,\alpha w}$) and thus his or her choice should not change.

Definition: The Walrasian demand correspondence **satisfies Walras' law** if, for every $p >> 0$ and $w > 0$, $p \cdot x(p, w) = w$.

This simply states that **the consumer spends all the wealth that he or she has**, which is reasonable to assume for most people (most people do not reach a "satiation point"). It is sometimes called the "adding up restriction". This assumption should be interpreted flexibly. In particular, for a consumer who spends money over several time periods during his or her life all, we are assuming is that all money (resources) are spent by the end of his or her lifetime, not in every time period. Thus "saving" in an earlier time period to spend in a later one is perfectly consistent with Walras' law.

See the bottom of p.23 of Mas-Colell et al. for a fairly easy exercise that checks whether Walras' law holds for a specific type of a demand function.

We will see in later lectures that both homogeneity of degree zero and Walras' law hold under a wide variety of circumstances for demand functions that are derived from utility maximization. **For the rest of this lecture we will focus on what can be learned from demand functions that satisfy these two conditions**, without asking exactly where the demand functions come from.

We also assume for the rest of this lecture **that x(p, w) has a single bundle** for each possible (p, w) pair. That is, we are working with demand functions, not demand correspondences. We will also assume that x(p, w) is differentiable. This allows us to express x(p, w) as a vector of demand functions for individual commodities:

$$x(p, w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ \vdots \\ x_L(p,w) \end{bmatrix}$$

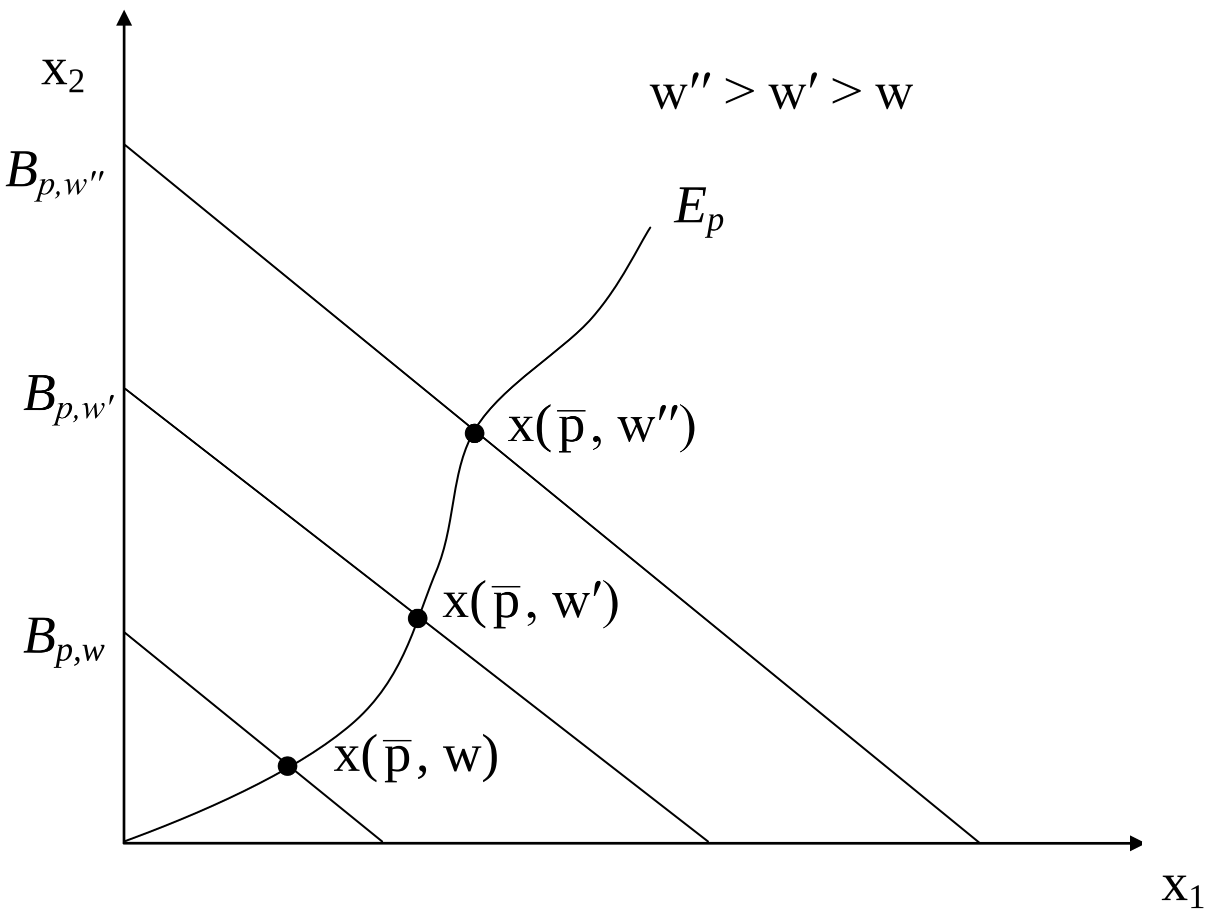
Note: This approach of discussing demand functions without reference to any underlying preferences or utility function can be interpreted as an application of the "choice rules" approach discussed in Lecture 1.

Comparative Statics

Many aspects of human behavior that economists (and others) would like to understand can be modeled as changes in demand functions in response to changes in prices (p) or wealth (w).

First we will consider wealth effects (often called income effects). Let's fix prices at $\bar{p}$ and see how the consumer's demand for each good changes as wealth increases. For the entire vector of goods, x, this can be depicted as the consumer's **Engel function**, denoted by $x(\bar{p}, w)$. The changes in demand for all L goods as w increases is called the **wealth expansion path** (also known as the income expansion path), and the derivative of the Engel function for good ℓ, $\partial x_\ell(\bar{p}, w)/\partial w$ is called the **wealth effect** (also known as the income effect).

This figure shows the wealth expansion path (denoted by E_p) for two goods:



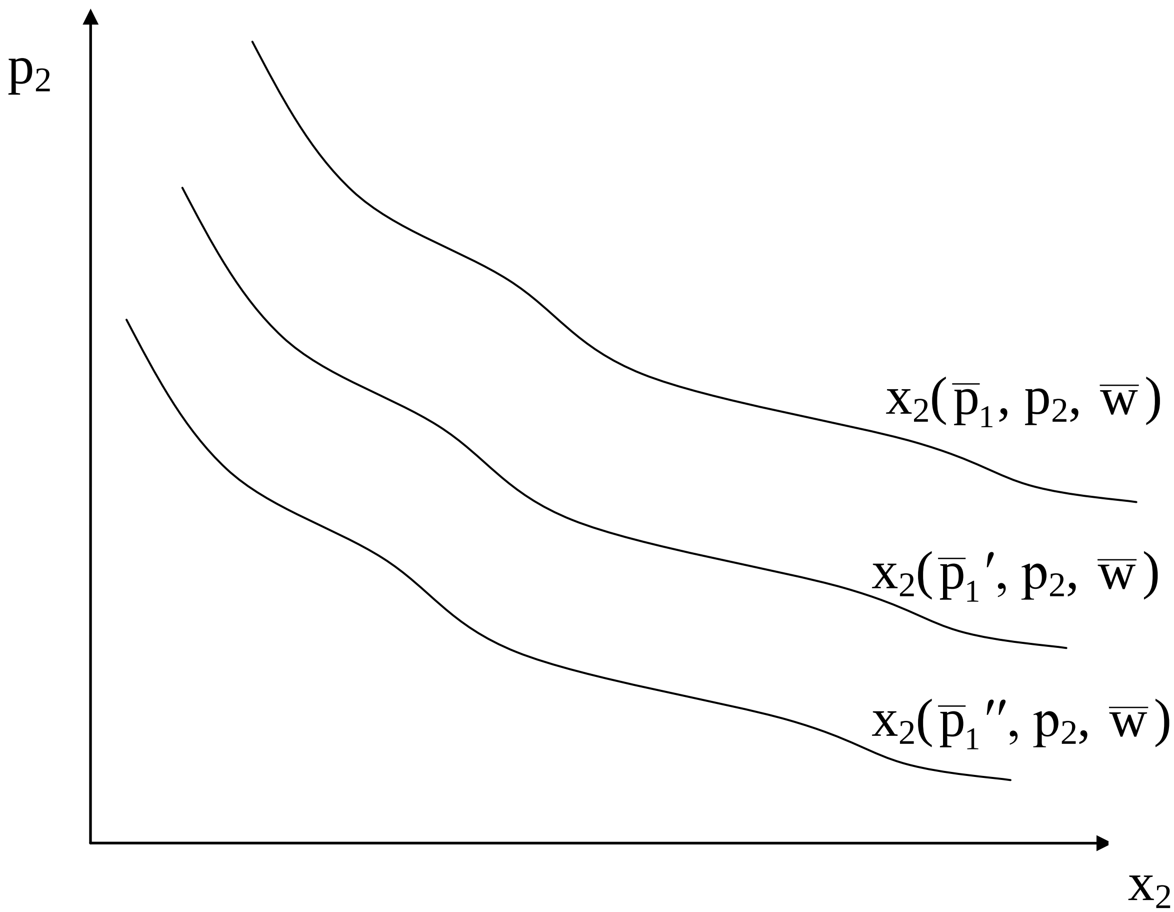
Perhaps the first way to classify commodities according to their wealth effects is whether the consumer increases or decreases consumption of the good as his or her wealth increases. A commodity ℓ is **normal** at the price-wealth pair (p, w) if $\partial x_\ell(p, w)/\partial w \geq 0$. The other possibility is that the commodity ℓ is **inferior** at the price-wealth pair (p, w), which is defined as if $\partial x_\ell(p, w)/\partial w < 0$. If a commodity is **normal** for all possible price-wealth pairs then we say it is a **normal good**. If **every** good is a normal good than we say "**demand is normal**".

Question: Is it also possible for every good to be an inferior good?

We will sometimes use matrix notation to show the wealth effects for all L goods:

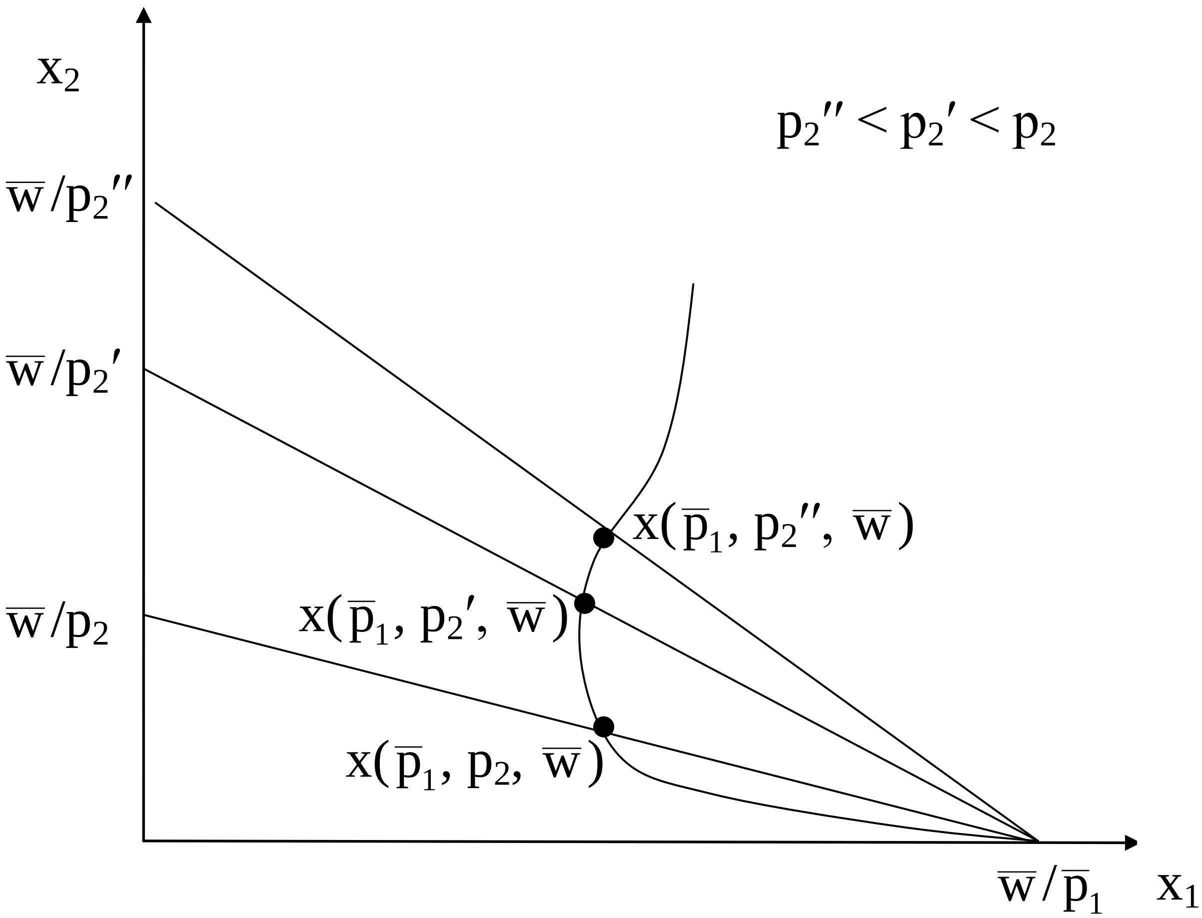
$$D_w x(p, w) = \begin{bmatrix} \dfrac{\partial x_1(p, w)}{\partial w} \\ \dfrac{\partial x_2(p, w)}{\partial w} \\ \vdots \\ \dfrac{\partial x_L(p, w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L$$

Now consider price effects, that is how demand changes as prices vary. To keep things simple, suppose that there are only two commodities. We can show an "Economics 101 demand curve" if we fix w and p_1 at $\overline{w}$ and $\overline{p}_1$, and see how the demand for good 2 changes as p_2 changes:



Question: Is $\overline{p}_1' >$ or $< \overline{p}_1$?

Another useful way to show price effects in a diagram is an **offer curve**, which shows how the demand for two goods changes as the prices change. In the following diagram, w and p_1 are held constant while p_2 changes:



In this diagram, as p_2 decreases the demand for x_2 increases, which is what one would expect. The impact on x_1 is ambiguous, since price and income effects work in the opposite direction (this will discussed more in later lectures). Returning to x_2, it is theoretically possible, but empirically very rare, for the demand for x_2 to decline as p_2 decreases. Such a good is called a **Giffen good**; see p.27 of Mas-Colell et al. to see a diagram showing this.

Sometimes it is convenient to denote the L×L matrix of price effects as $D_p x(p, w)$:

$$D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$$

Implications of Homogeneity and Walras' Law

Both homogeneity and Walras' Law impose restrictions on the price and wealth effects on demand functions.

Let's start with homogeneity. Recall that it implies that $x(\alpha p, \alpha w) - x(p, w) = 0$ for all $\alpha > 0$. The first property can be derived by differentiating this with respect to α and then setting α equal to 1:

Proposition 2.E.1: If the Walrasian demand function x(p, w) is homogenous of degree zero, then for all p and w:

$$\sum_{k=1}^{L} \frac{\partial x_\ell(p,w)}{\partial p_k} p_k + \frac{\partial x_\ell(p,w)}{\partial w} w = 0 \text{ for } \ell = 1, 2, \ldots L$$

This can be written more compactly in matrix notation:

$$D_p x(p, w)p + D_w x(p, w)w = 0$$

Partial intuition: increases in prices include an implicit reduction in real income and this will tend to reduce consumption, but an increase in wealth usually has the opposite effect.

The first expression of Proposition 2.E.1 can be written in elasticity form. First, we need to define elasticities:

$$\varepsilon_{\ell k}(p, w) = \frac{\partial x_\ell(p, w)}{\partial p_k} \frac{p_k}{x_\ell(p, w)} \quad \text{(price elasticity)}$$

$$\varepsilon_{\ell w}(p, w) = \frac{\partial x_\ell(p, w)}{\partial w} \frac{w}{x_\ell(p, w)} \quad \text{(wealth elasticity)}$$

Elasticities show the **percentage change** in the demand for good ℓ in response to a (small) percentage change in the price of good k (which could be good ℓ) or wealth. In particular, $\varepsilon_{\ell k}(p, w) \approx (\Delta x_\ell/x_\ell)/(\Delta p_k/p_k)$ and $\varepsilon_{\ell w}(p, w) \approx (\Delta x_\ell/x_\ell)/(\Delta w/w)$. Elasticities are useful because they have no units (the units cancel out). But note that elasticities could vary over p and w; there is no reason to expect them to be constant as prices and wealth change.

Here is Proposition 2.E.1 in elasticity form:

$$\sum_{k=1}^{L} \varepsilon_{\ell k}(p, w) + \varepsilon_{\ell w}(p, w) = 0 \quad \text{for all } \ell = 1, 2, \ldots L$$

This form reflects the homogeneity assumption: the same percentage increase in prices and wealth will lead to no change in consumption.

Now let's consider two implications of Walras' law on the impact of prices and wealth on demand for x. First, differentiating the expression $p \cdot x(p, w) = w$ with respect to prices gives:

Proposition 2.E.2 (Cournot aggregation condition): If the Walrasian demand function satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p, w)}{\partial p_k} + x_k(p, w) = 0 \quad \text{for } k = 1, 2, \ldots L$$

This can be written in matrix notation as:

$$p \cdot D_p x(p, w) + x(p, w)^T = 0^T$$

[The "T" notation indicates a transpose, which simply means that a column vector becomes a row vector. Also, the "·" notation implies that the vector in front of it, in this case p, is also a row vector.]

Next, differentiating $p \cdot x(p, w) = w$ with respect to w gives:

Proposition 2.E.3 (Engel aggregation condition): If the Walrasian demand function satisfies Walras' law, then for all p and w:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell(p, w)}{\partial w} = 1$$

This can be written in matrix notation as:

$$p \cdot D_w x(p, w) = 1$$

These two propositions are quite intuitive:

1. Total expenditure cannot change in response to a change in prices (you need to cut back on something when the price of good k increases).

2. Total expenditure must change by an amount equal to any change in wealth.

Both of these propositions can be expressed in elasticity terms ($b_\ell(p, w) = p_\ell x_\ell(p, w)/w$: the budget share):

$$\sum_{\ell=1}^{L} b_\ell(p, w)\varepsilon_{\ell k}(p, w) + b_k(p, w) = 0 \quad \text{(Cournot)}$$

$$\sum_{\ell=1}^{L} b_\ell(p, w)\varepsilon_{\ell w}(p, w) = 1 \quad \text{(Engel)}$$