Applied Economics 8004
Applied Microeconomic Analysis
II. Public Goods

1 Public Goods and Efficiency

Consider a 2-person economy with 2 goods, x_j (a private good) and y (a public good). Each person has a utility function, $U_j = U_j(x_j, y)$. There is no subscript on y because everyone consumes the same amount of it. Suppose person j has an endowment of the private good given by ω_j. Part or all of this quantity is consumed as x_j and the rest is contributed to the production of y. (See the figure below.) Let $z_j = \omega_j - x_j$ denote j's contribution to the public good. The technology for producing y is given by $y = g(z)$, where $z = \sum_j z_j$. Note that neither person can choose y. It is chosen jointly.

A Pareto-optimal allocation can be found as the solution to the social planner's problem

$$\begin{aligned}&\max_{x_1,x_2,y} W(U_1(x_1,y),U_2(x_2,y))\\ &\text{s.t.} \quad y \leq g(z),\end{aligned}$$

where $W(U_1, U_2)$ is a quasi-concave (in the U_j) social welfare function satisfying $W_j > 0$. The Lagrangian function for this problem is

$$\mathcal{L}(x_1, x_2, y, \lambda) = W(U_1(x_1, y), U_2(x_2, y)) + \lambda\big(g((\omega_1 - x_1) + (\omega_2 - x_2)) - y\big).$$

Assuming an interior solution, the FONCs may be written

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial y} + \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial y} - \lambda = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_1} - \lambda g'(z) = 0, \quad \text{and} \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial x_2} - \lambda g'(z) = 0. \tag{3}$$

Equations (2) and (3) together imply

$$\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_1} = \frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial x_2} \tag{4}$$

and

$$\lambda = W_j \frac{\partial U_j}{\partial x_j}\frac{1}{g'}.$$

Using (1), dividing both sides by $(\partial W/\partial U_1) \cdot (\partial U_1/\partial x_1)$, we have

$$\left(\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial y} \Big/ \frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_1}\right) + \left(\frac{\partial W}{\partial U_2}\frac{\partial U_2}{\partial y} \Big/ \frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_1}\right) = \frac{1}{g'}\left(\frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_1} \Big/ \frac{\partial W}{\partial U_1}\frac{\partial U_1}{\partial x_1}\right).$$

But the large fraction on the right side is 1, and by (4) the denominator of the second term on the left is equal to $(\partial W/\partial U_2)(\partial U_2/\partial x_2)$. Thus, we have the following condition, known as the *Samuelson condition*:

$$\sum_j \left(\frac{\partial U_j}{\partial y} \Big/ \frac{\partial U_j}{\partial x_j}\right) = \frac{1}{g'}. \tag{5}$$

The fraction $1/g'$ can be interpreted as the marginal rate of transformation between y and x, so we can write $\mathrm{MRS}_1 + \mathrm{MRS}_2 = \mathrm{MRT}$. Any Pareto-optimal allocation in a pure public-goods economy (with interior consumption bundles, monotone preferences, and differentiable utility functions, etc.) will satisfy this condition. It is *necessary* for an interior outcome to be PO, Note that Samuelson's condition applies for economies with any number of public goods, in which case the marginal rate of technical substitution is the ratio of marginal products of the respective production technologies for any two public goods.

In the case of private goods, recall that Pareto optimality required that the MRS_j were equal. In a production economy, these must also equal MRT.

Example 1. Consider an economy with one private good x and one pure public good y. The endowment of x is ω and there are J identical agents, each with utility function

$$U_j(x_j, y) = \gamma \ln y + \ln x_j.$$

Because agents are identical, we have $\omega_j = \omega/J$. Assume that the technology for producing y is the identity function: $y = \sum_j z_j$. Letting the social welfare function be $W = \sum_j U_j$, a Pareto-optimal allocation solves

$$\max_{x_1,\ldots x_J} \quad \sum_j \big(\gamma \ln y + \ln x_j\big)$$
$$\text{s.t.} \quad y + \sum x_j = \omega,$$

with Lagrangian function

$$\mathcal{L}_i(x_j, z_j, \lambda; y) = J\gamma \ln y + \sum_j \ln x_j + \lambda\big(\omega - y - \sum_j x_j\big).$$

The FONCs for this problem, where we are now *guaranteed* an interior solution (neither x_j nor

Var	$J=5$	$J=50$	$J=500$
ω	50	500	5000
$\hat{x}_j$	10/3	10/3	10/3
$\hat{z}_j$	20/3	20/3	20/3
$\hat{y}$	100/3	1000/3	10,000/3

z_j will be zero) by the form of the U_j, are

$$\frac{\partial \mathcal{L}_j}{\partial y} = \frac{J\gamma}{y} - \lambda = 0 \quad \text{and}$$
$$\frac{\partial \mathcal{L}_j}{\partial x_j} = \frac{1}{x_j} - \lambda = 0 \quad \forall j.$$

These expressions give us $\sum_j x_j = y/\gamma$ which, combined with the budget constraint $y + \sum_j x_j = \omega$, gives us $(\omega - y) = y/\gamma$ or, rearranging,

$$\hat{y} = \frac{\gamma\omega}{1+\gamma}. \tag{6}$$

But we can use the FONCs again, together with (6), to get

$$\frac{\gamma\omega}{1+\gamma} = x_j J\gamma.$$

This can be manipulated to yield

$$\hat{x}_j = \frac{\omega}{J(1+\gamma)}.$$

A Pareto-optimal allocation is characterized by the vector $(\hat{x}_1, \ldots, \hat{x}_J, \hat{y})$.

A useful insight into how this problem works in economies of different sizes can be gained by fixing the various parameter values and letting J rise. Let

$$\omega_j = 10 \quad \text{and} \quad \gamma = 2.$$

Now let J rise from 5 to 50 to 500. The results for individual consumption of x and contribution to the public good and the total amount of the public good are in the nearby table. Individual contributions remain fixed and the level of y scales upward linearly with J.

2 Quasi-linear utilities and uniqueness of y^*

From micro principles we are familiar with the notion of summing individual demand functions horizontally to obtain aggregate demand. So long as utility functions are quasi-linear, the optimal level of a public good can be determined by an aggregate MRS curve that is obtained by summing

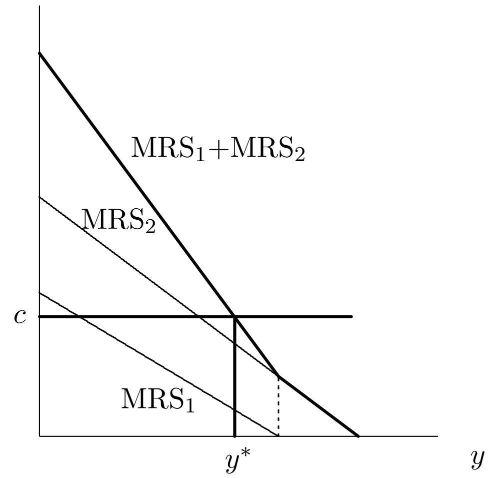

Figure 1: Optimal level of y is where aggregate MRS equals c.

vertically. Recall that a utility function is quasi-linear if it takes the form

$$U_j(x_j, y) = x_j + f_j(y),$$

where it is assumed that $f' > 0$ and $f'' < 0$. If we put x_j on the horizontal axis and y on the vertical axis, the indifference curves for such a consumer will be horizontally parallel. That is, at a given y every indifference curve will have the same slope for any x_j. The implication for demand behavior is that for any set of prices, the consumer's preferred level of the public good will not depend on income.

Select any level of y and compute j's marginal rate of substitution:

$$\text{MRS}_j = \frac{\partial U_j/\partial y}{\partial U_j/\partial x_j} = f_j'(y).$$

You can see that the marginal rate of substitution does not depend upon x_j, the level of the private good being consumed. Let's make one more simplifying assumption: the level of y that can be produced from the private good is a constant c times the aggregate contribution: $y = c((\omega_1 - x_1) + (\omega_2 - x_2)$. Now we may write (5) as

$$f_1'(y) + f_2'(y) = c.$$

What's more, because utilities are quasi-linear in y, the vertical sum of MRS curves, as they depend upon y, can be used to determine the optimal level of y. This argument is illustrated in Figure 1, where the fact that MRS curves slope downward is guaranteed by the assumption that $f_j'' < 0$.

The reason this diagram is a reliable guide to finding y^* only if utilities are quasi-linear is that otherwise each MRS curve depends upon x_j as well as upon y. Therefore the Samuelson condition may be represented as the vertical sum of MRS curves only if the MRS is independent of x_j, which it is precisely when utilities are quasi-linear.

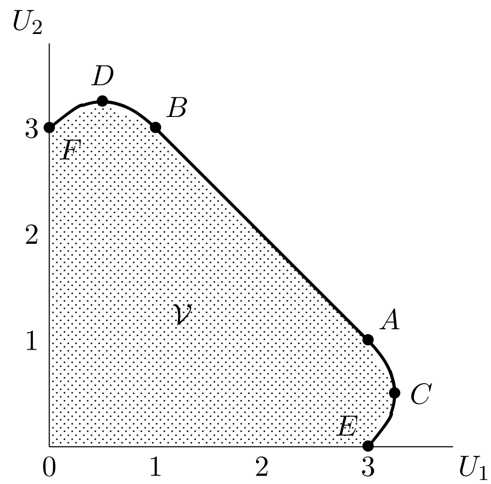

Figure 2: Utility-possibilities set.

3 Utility possibilities and Pareto optimality

The following example, which I adapted from a set of lecture notes by Ted Bergstrom at UC-San Diego, gives us some practice with computing Pareto-optimal allocations in a 2-person economy with a public good.

Example 2. Suppose j's quasi-linear utility function is $U_j(x_j, y) = x_j + \sqrt{y}$. The public good can be produced from x at a cost of one unit of private goods per unit of public good and the aggregate endowment is $\omega = 3$. The set of feasible allocations, then, is $\{(x_1, x_2, y) \in \mathbb{R}^3_+ \mid x_1 + x_2 + y \leq 3\}$. The sum of utilities is

$$U_1(x_1, y) + U_2(x_2, y) = x_1 + x_2 + 2\sqrt{y}.$$

Denote $x = x_1 + x_2$.

To find the set of PO allocations, we can start by maximizing the sum of utilities, $x + 2\sqrt{y}$, subject to $x + y \leq 3$. This will give us only some of the PO allocations, for reasons that will become clear in a moment. The solution is $y = 1$ and $x = 2$. Any allocation $(x_1, x_2, 1) \geq 0$ with $x_1 + x_2 = 2$ a Pareto optimum. The set of all such allocations yields one part of the utility-possibilities frontier: the line segment from A at $(3, 1)$ to B at $(1, 3)$ in Figure 2. At A the allocation is $(2, 0, 1)$. Consumer 1 gets all of x and thus gets utility $U_1(2, 1) = 2 + \sqrt{1} = 3$. Consumer 2 gets none of x and thus gets utility $U_2 = 0 + \sqrt{1} = 1$. The inverse is true at B, which reflects the allocation $(0, 2, 1)$. The utility pairs along the segment AB are achieved by changing the division of x between the two consumers, always keeping $y = 1$.

3.1 Maximizing $\sum_j U_j$ is sufficient but not necessary for PO

The quasi-linear example allows us to see that the set of PO allocations at which both consumers receive a positive amount of the public good will be a straight line in utility space with slope -1.

To see this, note that there is a unique PO level y^* of the public good (in Example 2, $y^* = 1$) at which both consumers receive positive amounts of the private good. Along the utility possibility frontier we have $U_1 + U_2 = x_1 + x_2 + f_1(y^*) + f_2(y^*)$. After paying for the public good, the amount of income that is left to be divided among the consumers for private consumption is $3p_y y^*$. So it must be that $U_1 + U_2 = 3p_y y^* + f_1(y^*) + f_2(y^*)$. The right side of this expression is a constant, so the portion of the utility-possibility frontier that is achievable with $x_j \geq 0$ for both consumers must be a straight line with slope -1.

There are other Pareto-optimal allocations, though, at which $y < 1$. The first point to note about such allocations is that *they do not maximize the unweighted sum of utilities.* But let's consider two ways of identifying PO allocations that do not maximize $\sum_j U_j$. One is to notice that in order to give consumer 2, say, utility greater than $U_2 = 3$, as at B, we need to set $x_1 = 0$. Indeed, consumer 1 already obtains $x_1 = 0$ at B itself. In order to make 2 even better off than at B, we'll definitely need to keep $x_1 = 0$. Now 2's utility will be determined by the amount of x that is used to produce y. That is, allocations will be of the form $(0, x_2, y)$ with $x_2 + y = 3$. The curved portion of the frontier from B to D is determined by $U_2(x_2, 3 - x_2) = x_2 + \sqrt{3 - x_2}$ with x_2 varying from 2 to 11/4.

At D itself, 2's utility achieves its maximum. That outcome is interior for 2 (but not for 1), and we can find it by solving $\max_{x_2} x_2 + \sqrt{3 - x_2}$, with FONC $1 - 1/2\sqrt{3 - x_2} = 0$. The solution is $x_2 = 11/4$, which means $y = 1/4$. The allocation is $(0, 11/4, 1/4)$. Consumer 2 receives utility of $\sqrt{1/4} = 1/2$. The same derivation gives the allocation $(11/4, 0, 1/4)$ that leads to utility outcome C, where $U_2 = 1/2$.

The other way to identify the PO allocations that yields portions BD and AC of the frontier is to use Lagrangian techniques. A social planner could select allocations to solve

$$\begin{aligned} \max_{x_1, x_2} \quad & U_1(x_1, y) + \alpha U_2(x_2, y) \\ \text{s.t.} \quad & x_1 + x_2 + y = 3, \end{aligned}$$

where $\alpha \geq 0$. The Lagrangian for this problem is

$$\mathcal{L}(x_1, x_2, y) = x_1 + \sqrt{y} + \alpha\,(x_2 + \sqrt{y}) + \lambda(3 - y - x_2 - x_2). \tag{7}$$

If $\alpha = 1$ we will find the line segment AB, representing all allocations with $y = 1$. These, again, maximize the unweighted sum of utilities. The FONCs associated with (7) are

$$\begin{aligned} \mathcal{L}_1 &= 1 - \lambda \leq 0, & x_1(1 - \lambda) = 0 \\ \mathcal{L}_2 &= \alpha - \lambda \leq 0, & x_2(\alpha - \lambda) = 0 \\ \mathcal{L}_y &= \frac{3}{2\sqrt{y}} - \lambda = 0, & \end{aligned}$$

where I ignore complementary slackness on y because I know it will be nonzero.

Now let's suppose $\alpha > 1$. Specifically, set $\alpha = 2$, which means the planner favors consumer 2

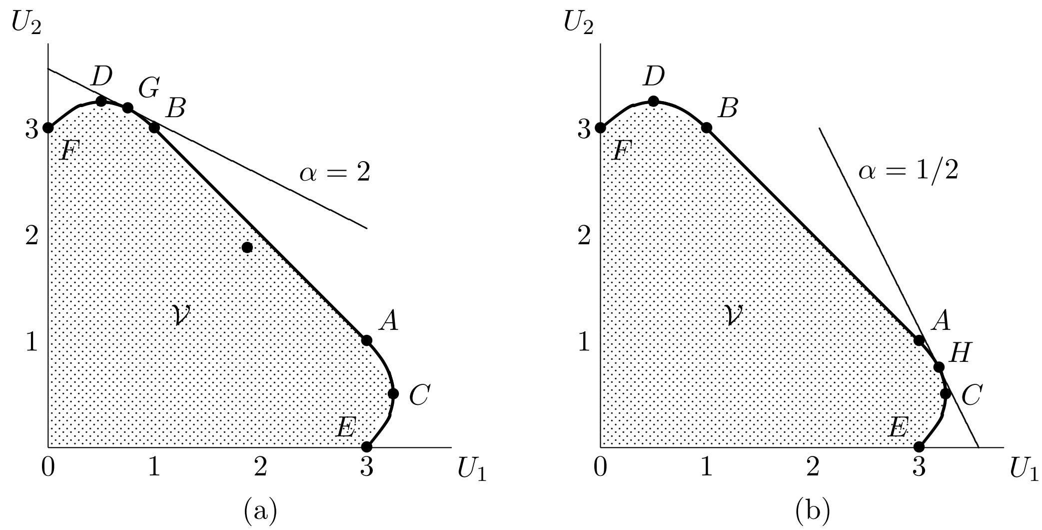

Figure 3: Pareto-optimal utility pairs that do not satisfy the Samuelson condition.

for some reason. We should be able to identify one of the points along the curve from B to D. In fact, we already know that $x_1 = 0$ here, so the first FONC is a strict inequality: $1 < \lambda$. We can solve the last two to get

$$2 = \frac{3}{2\sqrt{y}} \quad \text{or} \quad \sqrt{y} = \frac{3}{4}, \quad \text{or} \quad y = \left(\frac{3}{4}\right)^2.$$

Then

$$x_2 = 3 - \left(\frac{3}{4}\right)^2.$$

The corresponding utility outcome is denoted G in Figure 3(a). The line through G is the linear social indifference curve associated with the social welfare function $U_1 + 2U_2$. But notice something: the slope of the utility frontier cannot be -1 here. In fact, the sum of MRS_j's is

$$\sum_j \frac{\partial U_j/\partial y}{\partial U_j/\partial x_j} = \frac{1}{\sqrt{(3/4)^2}} \neq 1.$$

The outcome obtained when $\alpha = 1/2$, labeled H, is shown in Figure 3(b). In summary, keep these two facts in mind:

1. The Samuelson is *necessary* but not *sufficient* for an interior allocation to be PO. That is, there can be non-PO allocations that satisfy Samuelson.
2. Maximizing the unweighted sum of utilities is *sufficient* but not *necessary* for an allocation to be PO. With one exception all of the PO allocations may be identified using (7), by varying the α parameter. If $\alpha = 0$, we obtain utility outcome C. The one exception to my statement is outcome D, for which we would need to set $\alpha = \infty$. That doesn't make sense, so to obtain D we have to maximize not (7) but rather $U_2(x_2, y)$.

The complete set of PO utility outcomes is the frontier from point C to point D. But there are other boundary points to $\mathcal{V}$, between C and E and between D and F. These outcomes are not PO. Let y vary from 0.5 down to 0, computing both utility levels along the way to get the non-PO curves.

4 Voluntary-contribution equilibrium

Now it is time to take the social planner out of the picture and investigate what people in the model do when given their volition. The primary question is, will the Pareto-optimal allocation obtain in this case? Remember that the first welfare theorem guarantees that a competitive equilibrium—in which people do whatever they want—will be PO. The analogous result does not hold for public goods.

Suppose now that each person j seeks to maximize his or her utility, taking others' contributions $z_{-j} = (z_1, \ldots, z_{j-1}, z_{j+1} \ldots, z_J)$ as given. Person j solves

$$\begin{aligned} \max_{x_j, z_j} \quad & U_j(x_j, y) \\ \text{s.t.} \quad & y \le g\Big(z_j + \sum_{k \neq j} z_k\Big), \end{aligned}$$

with Lagrangian function

$$\mathcal{L}_i(x_j, z_j, \lambda; y) = U_j(x_j, y) + \lambda\Big(g\big(z_j + \sum_{k \neq j} z_k\big) - y\Big). \tag{8}$$

An equilibrium for this problem is called a voluntary-contribution equilibrium (VCE).

Definition 1. *A **voluntary-contribution equilibrium** for a public-goods exchange economy is a vector $\{x_j^*, y^*\}_{j=1}^J$ such that for each j, x_j^* solves program (8) taking x_{-j}^* as given and $y^* = g(\sum_j (\omega_j - x_j^*))$.*

The FONCs for j's problem, again assuming an interior solution, are

$$\begin{aligned} \frac{\partial \mathcal{L}_j}{\partial y} &= \frac{\partial U_j}{\partial y} - \lambda = 0 \quad \text{and} \\ \frac{\partial \mathcal{L}_j}{\partial x_j} &= \frac{\partial U_j}{\partial x_j} - \lambda g'(z) = 0. \end{aligned}$$

These expressions can be combined to give us

$$\frac{\partial U_j / \partial y}{\partial U_j / \partial x_j} = \frac{1}{g'(z)}.$$

This is not the Samuelson condition, so we know right away that the VCE is not necessarily Pareto optimal. The problem is that each person ignores the benefits his or her contribution of z_j bestows upon others, and sets his or her private MRS_j equal to the marginal cost of the public good. In general this leads to underprovision of a public good in a VCE.

Example 1 continued. To find the VCE in the Cobb-Douglas example, for each person we solve

$$\begin{aligned}\max_{x_j,z_j} \quad & \gamma \ln y + \ln x_j \\ \text{s.t.} \quad & x_j = \omega_j - z_j \quad \text{and} \\ & y = z_j + \sum_{k \neq j} z_k.\end{aligned}$$

Incorporating the constraints into the objective function, and again noting that we must have an interior solution, we have

$$\max_{z_j \leq \omega_j} U_j = \gamma \ln\left(z_j + \sum_{k \neq j} z_k\right) + \ln\left(\frac{w}{J} - z_j\right).$$

The FONC for this problem is

$$\frac{\partial U_j}{\partial z_j} = \frac{\gamma}{z_j + \sum_{k \neq j} z_k} - \frac{1}{\omega_j - z_j} = 0.$$

or

$$\gamma\big(\omega_j - z_j\big) = z_j + \sum_{k \neq j} z_k. \tag{9}$$

This may be rearranged to yield

$$\gamma \omega_j = \big(1 + \gamma\big) z_j + \sum_{k \neq j} z_j.$$

Summing over the j,

$$\begin{aligned}\gamma \omega &= (1 + \gamma) \sum_j z_j + \sum_j \sum_{k \neq j} z_k \\ &= (J + \gamma) \sum_j z_j.\end{aligned}$$

But $\sum_j z_j = y$, so

$$y^* = \frac{\gamma \omega}{J + \gamma}$$

From the constraint we know that $x_j^* = \omega_j - z_j^*$ and, using (9),

$$\begin{aligned}x_j^* &= \omega_j - z_j^* \\ &= \frac{\sum_j z_j^*}{\gamma} = \frac{y^*}{\gamma} \\ &= \frac{\omega}{J + \gamma}.\end{aligned}$$

Var	$J = 5$	$J = 50$	$J = 500$
ω	50	500	5000
x_j^*	50/7	500/52	5000/502
y^*	100/7	1000/52	10,000/502

It is now easy to compare the two solutions. In particular, we have

$$\hat{y} = \frac{\gamma\omega}{1+\gamma} > \frac{\gamma\omega}{J+\gamma} = y^*.$$

Thus, in the voluntary-contribution (Nash) equilibrium, we see that the public good is indeed underprovided. Furthermore, the difference between $\hat{y}$ and y^* grows as J grows. In a private-goods economy, on the other hand, as the economy grows we expect to get closer and closer to the PO allocation. The trajectory of contributions and public good is in the table. Now instead of growing linearly in J, the level of the public good approaches 20. The contribution of each individual approaches zero.

Example 2 continued.

We can also find the VCE from our quasi-linear example. Suppose the initial endowment is divided equally, so that $\omega_j = 3/2$. Let $z_j = 3/2 - x_j$. Consumer j takes z_{-j} as given and solves

$$\max_{z_j} \ \omega_j - z_j + \sqrt{z_j + z_{-j}},$$

which gives a FONC of $\sqrt{z_j + z_{-j}} = 1/2$. The best-response functions are

$$z_j(z_{-j}) = \frac{1}{4} - z_{-j}.$$

Solve these simultaneously to obtain $z_1 = z_2$. Combine with the FONC to get $z_j^* = 1/8$ and thus $y^* = 1/4$. Figure 4 shows that the VCE utility outcome with $U_j = 3/2 + \sqrt{1/4}$, labeled Z, is not PO.

Another way to find this outcome is simply to set either agent's MRS equal to MRT, which is 1. Given the symmetry of the problem, we can find the equilibrium level of y in this way: $\text{MRS}_j = 1/2\sqrt{y} = 1$. This also gives $y^* = 1/4$.

5 Lindahl equilibrium

The Nash equilibrium in the VCE is one form of a *mechanism*, although it is a particularly simple and ineffective one. There are others, one of which is the *Lindahl equilibrium*. Though this has its own unsatisfying properties, it does yield a PO allocation.

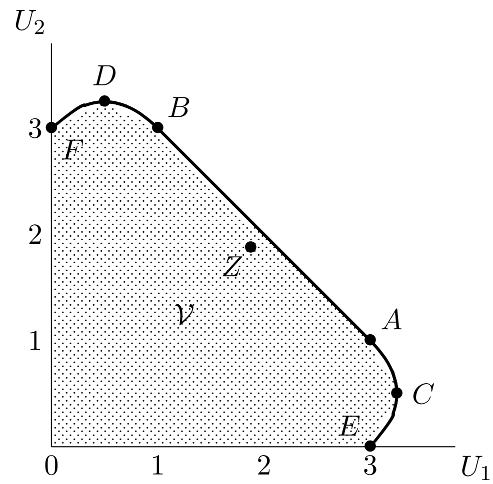

Figure 4: The voluntary-contribution equilibrium is not PO.

We shall require a separate price of y for each person j. Let this price be denoted p_j. The interpretation is that if, for example, 3 units of y are provided, person j must pay $3p_j$. Let the price of x be normalized at 1. Given prices, agent j solves

$$\begin{aligned} \max_{x_j} \quad & U_j(x_j, y) \\ \text{s.t.} \quad & x_j + p_j y \leq \omega_j. \end{aligned}$$

The Lagrangian function for this problem is

$$\mathcal{L}_j(x_j, y, \lambda) = U_j(x_j, y) + \lambda\big((\omega_j - x_j - p_j y\big).$$

The FONCs (interior solution assumed yet again) are

$$\begin{aligned} \frac{\partial \mathcal{L}_j}{\partial y} &= \frac{\partial U_j}{\partial y} - \lambda p_j = 0 \quad \text{and} \\ \frac{\partial \mathcal{L}_j}{\partial x_j} &= \frac{\partial U_j}{\partial x_j} - \lambda = 0. \end{aligned}$$

These can be combined to give us

$$\frac{\partial U_j/\partial y}{\partial U_j/\partial x_j} = p_j,$$

and, summing over the j, we find

$$\sum_j \frac{\partial U_j/\partial y}{\partial U_j/\partial x_j} = \sum_j p_j. \tag{10}$$

Now, suppose that a competitive firm produces y according to the technology $y = g(z)$, seeking to maximize profit using the price $p = \sum_j p_j$ for its output. This firm will solve

$$\begin{aligned} \max_{y,z} \quad & \pi = py - z \\ \text{s.t.} \quad & y \leq g(z). \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(y, \lambda; p) = py - z + \lambda\big(g(z) - y\big),$$

with FONCs

$$\begin{aligned} \frac{\partial \mathcal{L}_j}{\partial y} &= \sum_j p_j - \lambda = 0 \quad \text{and} \\ \frac{\partial \mathcal{L}_j}{\partial z} &= -1 + \lambda g' = 0. \end{aligned}$$

These may be combined to obtain

$$\sum_j p_j = 1/g'. \tag{11}$$

Combining (10) and (11), we find that

$$\sum_j \frac{\partial U_j/\partial y}{\partial U_j/\partial x_j} = 1/g',$$

which is precisely the Samuelson condition. A Lindahl equilibrium is Pareto optimal.

But there are problems. The main one is the assumption of personalized prices. This derivation requires the assumption of perfect competition. Does this make sense when each person knows that he or she is the only person in his or her "market" for y? Probably not. But if people recognize that their behavior affects p_j, the entire problem unravels. Suppose, for example, that in order to make a decision regarding the provision of a public good, the local authorities issue a survey asking for $U_j(x_j, y)$ and each person's preferred level of y. If the scheme proposes issuing a tax given by $t_j = p_j y$, so that each person pays his or her share, people will have an incentive to lie. Assuming that everyone else tells the truth, by claiming that their demand for y is lower than it is, each person can gain.

Incentives are the problem. The Clarke-Groves mechanism is one way to inject the proper incentives into a public-goods provision problem.

6 The simplified Vickrey-Clarke-Groves mechanism

The Vickrey-Clarke-Groves (VCG) mechanism is a scheme for eliciting people's true wishes regarding a public project. The mechanism applies more widely than just to public goods, but that is the main use to which it is put. It is attractive in that it can be used to induce truth telling and it also selects the optimal level of the public good. It doesn't balance the budget, though, and so it is not Pareto optimal overall.

Here we will take a first look at the VCG mechanism through a particularly simple example in which a pre-specified public project is under consideration. In a separate set of notes we will consider the more general problem in which the mechanism itself does the work of selecting the project too. Keep in mind that throughout, we're assuming that individual utility functions for the public good and all private goods are quasi-linear.

6.1 Vickrey auctions

Suppose an object is to be auctioned among J bidders. Each bidder knows his or her *valuation* of the object, given by v_j. Bidders submit sealed bids, where w_j denotes the value of j's bid. The highest bidder gets the object and pays the second-highest bid. Everyone knows the rules before the auction begins, but j does not know the other bidders' valuations, v_{-j}. The attractive feature of this auction is that no one can gain by lying. Truth-telling is a *dominant strategy* in the sense that no matter what everyone else does, the payoff to j from bidding $w_j = v_j$ is always at least as great as the payoff from any other bid.

To see this, we consider two possible cases: either (1) j is the highest bidder when $w_j = v_j$ or (2) j is not the highest bidder when $w_j = v_j$. (This argument does not require j to know the other bidders' bids.) In case (1), agent j gets the object at the second-highest bid. Should the agent have bid $w_j \neq v_j$? No. By bidding lower, there is no gain if j remains the highest bidder. If w_j becomes small enough so that w_j is no longer the highest bid, then j loses the increment in value equal to $v_j - v_2$, where v_2 is the second-highest bid. Bidding higher does not help either. Bidder j remains the high bidder and still pays the second-highest bid amount. In case (2), there is a bid greater than v_j. If j bids less than v_j, nothing changes for him or her. If j bids greater than v_j, he or she may end up getting the object, but at a loss, since the price will be the formerly highest bid, which is greater than v_j.

6.1.1 VCG

The VCG mechanism requires a crucial assumption: all preferences are quasi-linear. The following covers the simplest possible decision for the mechanism, but it allows us to see why telling the truth is optimal for participants. Suppose a government must decide whether to build a *given* public project. By "given" I mean that the project has been designed and the plans made. The decision is thus binary: either build or don't build. The plans consist of a blueprint for the project itself, together with a taxation scheme that will cover the cost of the project. (This payment scheme may be an equal poll tax for each citizen.) It is important to note that the scheme, as proposed, contains provisions for funding the project itself.

The decision will be made using input from citizens. Suppose citizen j places a *net* value on the project—the design and the payment plan—at v_j. As before, let w_j denote j's bid or expressed

Case	$v_j + \sum_{k \neq j} w_k$	$\sum_{k \neq j} w_k$	$t_j(w)$	Payoff to j
I	≥ 0	≥ 0	0	v_j
II	≥ 0	< 0	$\sum_{k \neq j} w_k$	$v_j + \sum_{k \neq j} w_k$
III	< 0	≥ 0	$-\sum_{k \neq j} w_k$	$-\sum_{k \neq j} w_k$
IV	< 0	< 0	0	0

Table 1: Clarke-Groves outcomes, incentives for j

value of the project. The search here is for a rule for turning individual bids into a collective decision about whether to build the project. Let x denote the binary decision variable, where $x = 1$ if the project is built and $x = 0$ if it is not built. The VCG mechanism consists of two parts. The first is a binary decision rule that takes the form

$$x(w) = \begin{cases} 1 & \text{if } \sum_j w_j \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

The second part is a transfer scheme (apart from the taxation scheme that will actually fund the project) given by

$$t_j(w) = \begin{cases} - \mid \sum_{k \neq j} w_k \mid & \text{if } (\sum_{k \neq j} w_k)(\sum_j w_j) < 0; \\ 0 & \text{otherwise.} \end{cases}$$

If $t_j < 0$, it is a tax.

This transfer scheme affects only *pivotal* people, where j is pivotal if the decision would have been different if not for his or her bid. In either direction. This is stated formally by the condition in the first part of the $t_j(w)$ expression. The product of sums is different precisely when j's bid changes the outcome from "build" to "not build" or vice versus.

Now, the claim is that everyone should report the truth in this case. Why would this be true? The idea is the same as in the Vickrey auction. The payment t_j does not depend directly upon j's bid. The key is to cut the tie between w_j and the transfer amounts. To see why the claim is true, suppose person j considers lying and reporting $w_j \neq v_j$. Table 1 shows what happens if j tells the truth. There are 4 possibilities.

In the first 2 cases, the project is built when j reports truthfully. To see that lying never pays, we compare the payoff to honesty and the payoff to lying. Let this difference be denoted ΔU_j = payoff to honesty – payoff to lying. In case I, there are two possibilities.

1. Suppose person j reports a value greater than the true valuation: $w_j > v_j$. Nothing will change: $\Delta U_j = v_j - v_j = 0$.

2. Now suppose j reports $w_j < v_j$. If w_j is close to v_j, it may still be the case that nothing changes. However, if w_j is sufficiently small, it may be that $w_j + \sum_{k \neq j} w_k < 0$. In this case, the project will not be built. Person j does not receive net benefit v_j. Moreover, his or her transfer is now $\sum_{k \neq j} w_k < 0$. The difference is $\Delta U_j = v_j + \sum_{k \neq j} w_j > 0$. The payoff to honesty is strictly greater than the payoff to lying.

The same argument can be pursued for each case. The lie may not matter (as in 1) or it may change the outcome. In either case, lying never helps and it sometimes hurts. Because the decision w_j must be made without knowledge of the w_k, nobody can tell which case they're in before the fact. Thus, telling the truth is best.

This scheme is nice because telling the truth is a (weakly) dominant strategy. It also has problems, however. The main one is that the outcome may not be Pareto optimal. There is no guarantee that $\sum_j t_j(w) = 0$, which is required for POness. In fact, it has been shown that there is no mechanism in which truth-telling is a dominant strategy and that always yields a Pareto-optimal allocation.

In fact, the VCG mechanism is more general and more powerful than this simple binary problem would suggest. In the lingo of the mechanism-design literature, the VCG mechanism is truth revealing and it chooses the optimal level of the public good. It is not efficient, though, because it can raise revenue through the VCG taxes. These taxes are not necessary to fund the public good and so an inefficiency creeps in.

7 The general VCG mechanism

This section borrows heavily from a set of lecture notes by Ted Bergstrom, for his Economics 230B course at UC-Santa Barbara. There are n people in an economy consisting of two goods, a private good x_j and a public good y. Person j's utility function is

$$U_j(x_j, y) = x_j + F_j(y), \tag{12}$$

and her endowment of the private good is ω_j. The production of y units of the public good requires a total amount $C(y)$ of private goods. The units on $C(y)$ are in bushels or pounds, not dollars. Assume that F_j is strictly concave and C is strictly convex. Both are differentiable. We'll also assume allocations are interior, so that $x_j > 0$. This means there is a unique Pareto-optimal quantity of the public good, which maximizes

$$\sum_j F_j(y) - C(y). \tag{13}$$

You should be able to convince yourself that the maximum of (13) is identical to the solution to

$$\max_{\{x_j\}} \quad \sum\nolimits_j (x_j + F_j(y))$$
$$\text{s.t.} \quad \sum\nolimits_j x_j + C(y) = \sum\nolimits_j \omega_j.$$

A planner wishes to select the optimal level of y, together with a suitable way of paying for it. To this end, consumers are asked to reveal their functions F_j to the government. Let M_j (possibly different from F_j) be the function that consumer j reports, and let $M = (M_1, \ldots, M_n)$. Given M, the planner chooses $y(M)$ that would be Pareto optimal if everyone were telling the truth about their utilities. That is, the government chooses $y(M)$ so that:

$$\sum\nolimits_j M_j(y(M)) - C(y(M)) \geq \sum\nolimits_j M_j(y) - C(y), \qquad \text{all } y \geq 0.$$

The planner then assigns taxes $T_j(M)$ according to

$$T_j(M) = C(y(M)) - \sum_{i \neq j} M_i(y(M)) - R_j(M), \tag{14}$$

where $R_j(M)$ is some function that may depend on the functions M_i reported by others, but not on M_j.

For a vector M of reported function, j's private consumption is

$$x_j(M) = \omega_j - T_j(M). \tag{15}$$

Inserting (14) and (15) into (12) we may write j's utility as

$$x_j(M) + F_j(y(M)) = \omega_j + \sum_{i \neq j} M_i(y(M)) + F_j(y(M)) - C(y(M)) + R_j(M). \tag{16}$$

Notice that $\omega_j + R_j(M)$ is independent of M_j, so that the only way in which j's reported function M_j can affect her utility is through the dependence of $y(M)$ on M_j. This means that, from (16), given any vector of reported functions by others, consumer j's best choice of M_j is the one that causes the government to choose $y(M)$ to maximize

$$\sum_{i \neq j} M_i(y) + F_j(y) - C(y). \tag{17}$$

But recall that the planner's goal is to maximize $\sum_{j=1}^n M_j(y) - C(y)$. *This is the heart of the insight of Clarke and Groves:* precisely when the consumer reports her true F_j function, the planner chooses y to maximize j's preferred objective function in (17). Reporting a function other than the true F_j can never help, and it might hurt. Honest revelation is a weakly dominant strategy. And notice that we haven't said what the R_j functions look like, other than that they are constant in M_j.

When everyone reports her true F_j, the planner chooses y to maximize

$$\sum_j F_j(y) - C(y).$$

This leads to the PO level of public goods, which is nice. We would also like the outcome of the scheme to satisfy at least the first or, better yet, the second of these properties:

1. *Feasibility:* the total taxes collected are at least as large as the cost of producing y, or

$$\sum_j T_j(M) \geq C(y(M)); \tag{18}$$

2. *Ex ante optimality:* the total taxes collected exactly equal the cost of producing y, or

$$\sum_j T_j(M) = C(y(M)). \tag{19}$$

Getting this right depends on choosing the $R_j(M)$ functions carefully. The bad news is that it turns out to be impossible to find $R_j(M)$ functions that that are independent of M_j for each j and such that (19) is satisfied.

The good news is that Clarke and Groves and Loeb managed to find specific $R_j(M)$ functions that at least guarantee (18) is satisfied. The insight works as follows. Suppose that for each j, the planner sets a "target share" $\theta_j \geq 0$, with $\sum_j \theta_j = 1$. The planner then tries to fix the $R_j(M)$ so that $T_j(M) \geq \theta_j C(y(M))$ for every j. If this attempt is successful then of course we'll have $\sum_j T_j(M) \geq C(y(M))$ and y is feasible. Subtract $\theta_j C(y(M))$ from both sides of (14) to obtain

$$T_j(M) - \theta_j C(y(M)) = (1-\theta_j)C(y(M)) - \sum_{i \neq j} M_i(y(M)) - R_j(M), \tag{20}$$

Thus, the planner could set $T_j(M) = \theta_j C(y(M))$ if and only if it the right side of (20) is zero, which would mean that

$$R_j(M) = (1-\theta_j)C(y(M)) - \sum_{i \neq j} M_i(y(M)).$$

Unfortunately this choice of $R_j(M)$ doesn't work, because it can depend on M_j via the optimal $y(M)$ function, where M_j appears.

The tax can be made to work, though, and this is the second insight in the Clarke-Groves scheme. Suppose the planner chooses the R_j according to

$$R_j(M) = \min_y \left[(1-\theta_j)C(y) - \sum_{i \neq j} M_i(y)\right]. \tag{21}$$

Ah, this $R_j(M)$ function depends on all the M_i for $i \neq j$, but it is independent of M_j. So the independence of the tax from j's report is restored. To see this, insert (21) into the right side of

(20) to get

$$(1-\theta_j)C(y(M)) - \sum\nolimits_{i\neq j} M_i(y(M)) - \min_y \Big((1-\theta_j)C(y) - \sum\nolimits_{i\neq j} M_i(y)\Big) \geq 0.$$

This means the left side of (20) is also nonnegative. Therefore, when taxes are set according to the R_j functions in (21) we also have enough resources to provide y:

$$\begin{aligned} T_j(M) &\geq \theta_j C(y(M)) \quad \forall j, \quad \text{and so} \\ \sum\nolimits_j T_j(M) &\geq C(y(M)). \end{aligned}$$

The Clarke-Groves mechanism works as claimed.

Let's recap. The mechanism, now usually called "Vickrey-Clarke-Groves" to recognize its roots in Vickrey's second-price auction, has some excellent properties:

1. Truth telling is a dominant strategy;
2. The Pareto-optimal level of the public good is selected; and
3. There is enough money to provide the public good.

We say that VCG is "*ex post* efficient." But there is one very discouraging property too:

4. There is no way to guarantee a balanced budget.

VCG is not "*ex ante* efficient" and it cannot be made so. If $\sum_j T_j(M) > C(y)$, then some of the contributions are wasted. And any method of returning the surplus to consumers will cause the incentive properties to unravel. You should always remember another important qualification to the entire discussion. That is, VCG doesn't work at all unless we're willing to assume that utility functions are quasi-linear.

One final note: In a dismal and negative theorem from 1979, Green and Laffont showed that any truth-revealing mechanism for the provision of public goods must be of the VCG class. Because VCG mechanisms cannot guarantee Pareto optimality, this discouraging result tells us that one must choose between *ex post* optimality on one hand, and incentive compatibility on the other. You can have one, but you can't have both. At least not in general.

An exception is when the planner is himself inserted into the game. This is the trick discussed in Mas-Colell on pp. 876–882 and also exploited in a very clever 2008 paper by Juan-Pablo Montero. We don't have time to get into these matters.

8 Bergstrom-Blume-Varian

The current literature on public goods features two strands that we'll have to bypass, at least mostly. One is on mechanism design, where the question is how to build a mechanism that provides incentives for people to reveal their true valuation. The VCG mechanism is the leading example, but not the only one. The other strand is BBV, a literature stemming directly from a 1986 *JPubE* paper by Berstrom, Blume, and Varian.[1] They provided a model of "crowding out," the idea that government provision of a public good can supplant its private provision. The innovation of the model is a careful consideration of the effects of wealth redistributions on equilibrium contribution levels.

BBV consider a model with n consumers of one public good and one private good. Let x_i be i's consumption of the private good and g_i be i's contribution of the public good, with $G = \sum_i g_i$. Consumer i is endowed with wealth $w_i = x_i + g_i$ and has utility function $u_i(x_i, G)$. At a Nash equilibrium, the vector g^* maximizes each consumer's utility subject to the wealth constraint and the constraint that $G^*_{-i} = G * -g_i$ is taken as given. BBV provide the following theorem.

Theorem 1 (Bergstrom, Blume, and Varian). *Assume that consumers have convex preferences and that contributions are originally in a Nash equilibrium. Consider a redistribution of income among contributing consumers such that no consumer loses more income than his original contribution. After the redistribution there is a new Nash equilibrium in which every consumer changes the amount of his gtft by precisely the change in his income. In this new equilibrium, each consumer consumes the same amount of the public good and the private good that he did before the redistribution.*

This result was a big surprise in 1986. Their existence results (Theorems 2 and 3) weren't surprising, but this result was:

Theorem 4 (Bergstrom, Blume, and Varian). *In a Nash equilibrium:*

i. *Any change in the wealth distribution that leaves unchanged the aggregate wealth of current contributors will either increase or leave unchanged the equilibrium supply of public good.*

ii. *Any change in the wealth distribution that increases the aggregate wealth of current contributors will necessarily increase the equilibrium supply of the public good.*

iii. *If a redistribution of income among current contributors increases the equilibrium supply of the public good, then the set of contributing consumers after the redistribution must be a proper subset of the original set of contributors.*

iv. *Any simple transfer of income from another consumer to a currently contributing consumer will either increase or leave constant the equilibrium supply of the public good.*

Perhaps the best way to get to the bottom of this result is through an example.

Example 3. Suppose two people inhabit a BBV-style economy with one private good x and one public good G. Each consumer's wealth w_i is divided between consumption and contribution, so

[1] Bergstrom, Theodore, Lawrence Blume, and Hal Varian, "On the Private Provision of Public Goods," *Journal of Public Economics*, 29 (1986), 25–49.

that, as above, $w_i = x_i + g_i$ and $G = g_1 + g_2$. Utility functions are given by

$$U_1(x_1, G) = \ln x_1 + \ln G \qquad \text{and} \qquad U_2(x_2, G) = \ln x_2 + (\ln G)/2.$$

Ignoring the $G \geq g_1$ constraint, consumer 2's optimization problem is

$$\max_{x_2, G} \ \ln x_2 + (\ln G)/2$$
$$\text{s.t. } x_2 + G = w_2 + g_1.$$

This yields FONCs that may be reduced to

$$x_2 = 2G,$$

or, inserting this into the budget constraint, $w_2 + g_1 = 3G$. Rearrange another step to get 2's unconstrained demand for G of

$$G = \frac{w_2 + g_1}{3}.$$

Following BBV's notation, $f_i(w)$ is i's demand function for the public good, which can also be written as $G = \max\{f_i(w_i + G_{-i}), G_{-i}\}$. For consumer 2 this becomes

$$G = \max\{(w_2 + g_1)/3, g_1\}$$

or, subtracting g_1 from both sides, we get 2's reaction function

$$g_2(g_1) = \max\big\{(w_2 + g_1)/3 - g_1, 0\big\}.$$

Repeat the exercise for consumer 1 to get

$$g_1(g_2) = \max\{(w_1 + g_2)/2 - g_2, 0\}.$$

Suppose that initially $\bar{w}_1 = 2.5$ and $\bar{w}_2 = 3.5$. If we solve the best-response functions simultaneously we get $g_1^* = 1$, $g_2^* = 0.5$, and so $G^* = 1.5$. Also, $x_1^* = 1.5$ and $x_2^* = 3$, as Figure 5 illustrates. This is the interior case with both people contributing to the public good. The tangencies must be along the same horizontal line, at G^*, which remains at 1.5 so long as $w_1 \in [1.5, 3]$, with a constant aggregate endowment of $w = 6$.

Now suppose we take one unit of w from consumer 2 and give it to consumer 1. This results in $w_1' = 3.5$ and $w_2' = 2.5$. The resulting equilibrium contributions are $g_1' = 1.75$ and $g_2' = 0$, so that $G' = 1.75 > G^*$. Strange. By shifting w from 2 to 1, consumer 2 quits contributing altogether but consumer 1 contributes so much more that the level of the public good goes up.

This case confirms the surprising part (*iii.*) of BBV's Theorem 4: the level of the public good went up as a result of a redistribution amongst the initial contributors. After redistribution, the set of contributors ($C' = \{1\}$) is a proper subset of the initial set of contributors ($C = \{1, 2\}$). Figure 6 shows that consumer 1, now much richer than before, optimizes by providing quite a

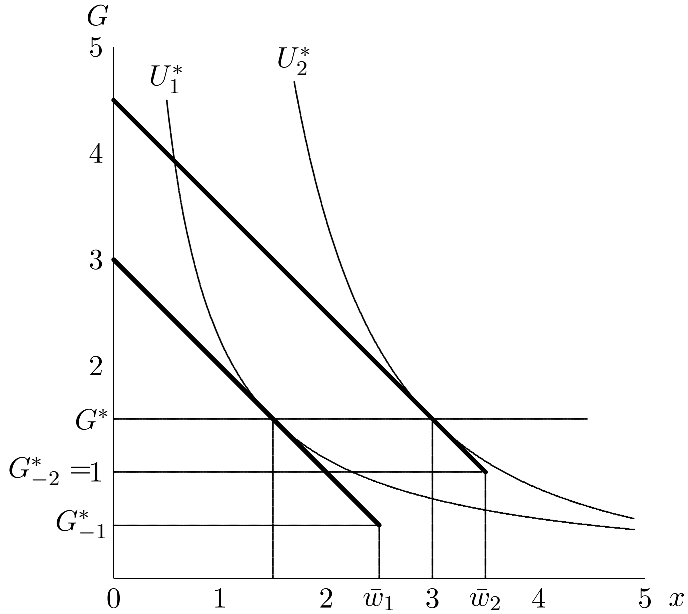

Figure 5: BBV solution, interior case

lot of the public good. Further redistribution to 1, so that $w_1' > 3.5$, causes further increases in the equilibrium G^*. Why? Because BBV assume that both goods are normal for all consumers. An outward shift of 1's budget line, when she is the only contributor, must necessarily lead to an increase in 1's (unilateral) provision of the public good. At w' consumer 2 is worse off than at $\bar{w}$, but does benefit from 1's contribution of G', on which 2 free rides entirely. Note 2's corner solution, with U_2' steeper than the bold budget line at $(x_2', g_2') = (2.5, 0)$.

The BBV paper, and its introduction of the idea of crowding out, still drives a large literature. It's especially popular among experimental economists seeking to understand whether and when public goods (schools, parks, and so on) should be provided by the government and when they shouldn't.

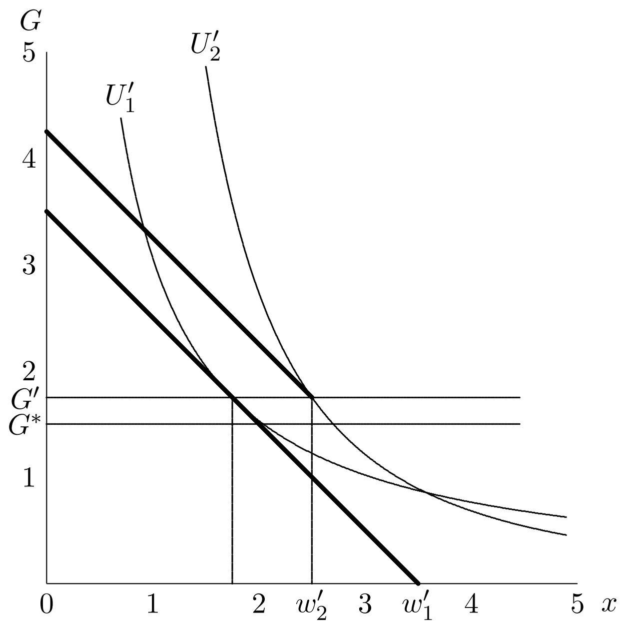

Figure 6: BBV solution, corner case

9 Exercises

1. An economy consists of two consumers, each of whom consumes two goods: a public good q and a private good x_j. Consumer 1's utility function is $U_1(x_1, q) = x_1 q$ and consumer 2's utility function is $U_2(x_2, q) = x_2 + q$. Let $\omega_j = 10$ be j's endowment of good x, of which the amount $z_j \in [0, 10]$ is devoted to the provision of the public good, whose production technology is described by $q = (z_1 + z_2)/2$. The remainder of j's endowment is consumed directly as good x. (That is, $x_j + z_j = \omega_j$.) Determine the Pareto-optimal allocation for this economy. Does this allocation satisfy the Samuelson condition? (*Hint:* You may want to use the fact that with linear utility, we can't be sure that the solution is interior.)

   Solution: This problem is a great illustration of how something that appears to be simple can turn out to be complicated mathematically. There are two sources of complication. One is that 2's utility is linear, which can drive us to corner outcomes. The other is that 1's utility is "increasing returns to scale," which can make the set of possibility utility outcomes nonconvex. To see how strange the problem is, take a look at Figure 7. Each plotted line is the set of utility pairs that can be achieved for a given level of $z = z_1 + z_2$. This set is linear because z determines q and, given q, the tradeoff between U_1 and U_2 linear. The other complication is that the PO solutions that are easiest to find are corner solutions. One may set up the social planner's objective function as follows, substituting the resource constraint for each person and the technology into the respective utility functions:

$$\begin{aligned}\max_{z_1,z_2} W &= U_1 + U_2 \\ &= (10 - z_1)\left(\frac{z_1 + z_2}{2}\right) + (10 - z_2) + \left(\frac{z_1 + z_2}{2}\right).\end{aligned}$$

The FONCs for this problem are

$$\frac{\partial W}{\partial z_1} = 5 - z_1 - z_2/2 + 0.5 = 0 \tag{22}$$

$$\frac{\partial W}{\partial z_2} = 5 - z_1/2 - 1 + 0.5 = 0 \tag{23}$$

If one attempts to solve these expressions for the z_i^*, one finds that the solution involves a negative value for z_2. The second FONC yields $z_1^* = 9$. Plugging this into the first yields $z_2^* = -7$, which is of course infeasible.

Or, we can have social planner's objective function with demands of private goods:

$$\begin{aligned}\max_{x_1,x_2} W &= U_1 + U_2 \\ &= x_1\left(\frac{20 - x_1 - x_2}{2}\right) + x_2 + \left(\frac{20 - x_1 - x_2}{2}\right).\end{aligned}$$

The FONCs for this problem are

$$\begin{aligned}\frac{\partial W}{\partial x_1} &= 19/2 - x_1 - x_2/2 = 0 \\ \frac{\partial W}{\partial x_2} &= 1/2 - x_1/2 = 0.\end{aligned}$$

The second FONC yields $x_1^* = 1$. Plugging this into the first yields $x_2^* = 17$, which is of course infeasible.

Then, we simplify this problem to find an outcome with either U_1 or U_2. These will be PO by definition. By solving $\max U_1$, s.t. $x_2 = 0$ and $q = \frac{z_1+z_2}{2}$, the highest level that U_1 can ever take is achieved when $z_1 = 0$ and $z_2 = 10$ ($x_1 = 10$ and $x_2 = 0$). This gives a utility pair of $(50, 5)$, with $W = U_1 + U_2 = 55$. By solving $\max U_2$, s.t. $x_1 = 0$ and $q = (z_1 + z_2)/2$, the highest level that U_2 can ever take is achieved when $z_1 = 0$, $z_2 = 0$, $x_1 = 0$, and $x_2 = 20$. This gives a utility pair of $(0, 20)$, with $W = U_1 + U_2 = 15$. Both of these outcomes are Pareto optimal.

Now, we solve for the UPF for this economy. The frontier is illustrated in the figure, where we denote $z = z_1 + z_2 = q/2$. Each of the individual lines in the diagram represents the possible utility outcomes for a given level of z. To draw these line segments, we don't need to define any social welfare function. However, because we have a public good in the economy, we need to consider the cases of all possible z's. Observe that the blue part of the UPF consists of the ending points on the right side of the line segments. Therefore, points on the blue part represent plans with $x_2 = 0$ and only U_1 is maximized. Given $z \in [0, 20]$, and $x_2 = 0$, utility pairs are $(U_1, U_2) = \left(\frac{z}{2}(20 - z), \frac{z}{2}\right)$. Thus, the blue portion of the UPF is the parabola

$$U_1(U_2(z))) = 50 - 2(U_2 - 5)^2.$$

The upward-sloping portion of the blue curve, from $(0, 0)$ to $(50, 5)$, is not PO.

The red part of UPF consists of interior solutions with different z at which both $x_j > 0$. These outcomes are PO, as can be determined from the following planner specification for a general form of social welfare function:

$$\begin{aligned}
\max_{z_1, z_2} W &= W(U_1, U_2) \\
&= W\left((10 - z_1)\left(\frac{z_1 + z_2}{2}\right), (10 - z_2) + \left(\frac{z_1 + z_2}{2}\right)\right).
\end{aligned}$$

The FONCs for this problem are

$$\begin{aligned}
\frac{\partial W}{\partial z_1} &= \frac{\partial W}{\partial U_1}\left(\frac{10 - 2z_1 - z_2}{2}\right) + \frac{\partial W}{\partial U_2}\frac{1}{2} = 0 \\
\frac{\partial W}{\partial z_2} &= \frac{\partial W}{\partial U_1}\left(\frac{10 - z_1}{2}\right) - \frac{\partial W}{\partial U_2}\frac{1}{2} = 0
\end{aligned}$$

Rearranging these FONCs, we have

$$\begin{aligned}
\frac{\partial W}{\partial U_1}\left(\frac{10 - 2z_1 - z_2}{2}\right) &= -\frac{\partial W}{\partial U_2}\frac{1}{2} \\
\frac{\partial W}{\partial U_1}\left(\frac{10 - z_1}{2}\right) &= \frac{\partial W}{\partial U_2}\frac{1}{2}
\end{aligned}$$

We can cancel $\partial W / \partial U_1$ and $\partial W / \partial U_2$, to get

$$\frac{(10 - 2z_1 - z_2)/2}{(10 - z_1)/2} = -1,$$

which can be rearranged to obtain

$$20 - 3z_1 - z_2 = 0 \qquad \text{or} \qquad z_2 = 20 - 3z_1. \tag{24}$$

Applying (24), we have $U_1 = (10 - z_1)^2$, and $U_2 = 2z_1$. Therefore, the red part of the UPF can be expressed as

$$U_1 = \left(10 - \frac{U_2}{2}\right)^2,$$

where $U_2 \leq 20$. Setting $50 - 2(U_2 - 5)^2 = (10 - \frac{U_2}{2})^2$, we have the transition utility pair between the blue and red parts of UPF: $(U_1, U_2) = (\frac{400}{9}, \frac{20}{3})$.

Now, we have derived the UPF for this economy. We move on to solve for the Pareto optimal allocation for this economy that will satisfy the Samuelson condition with the social welfare function $W = U_1 + U_2$. From (22) and (23), we have shown that there is no interior solution when $z = z_1 + z_2 = 2$. (You can observe this result from the diagram as well. When $z = z_1 + z_2 = 2$, the line is within the utility possibility set. The social welfare function cannot be tangent to the set at $z = 2$). Also, we notice that this purely utilitarian social welfare function can only be tangent to the utility possibility set on the blue part. We know that the blue part is formed by corner solutions with $x_2 = 0$. Let $x_2 = 0$, so $x_1 = 20 - z$. The social welfare function is modified as $W = \frac{z}{2}(20 - z) + \frac{z}{2} = \frac{21z}{2} - \frac{z^2}{2}$. Therefore, the highest possible level of $W = U_1 + U_2$ is achieved when $z = 10.5$. Then, from $x_2 = 0$, we can have $z_1 = 0.5$, and $z_2 = 10$. This gives a utility pair of $(\hat{U_1}, \hat{U_2}) = (49.875, 5.25)$, and you can find this optimal plan in the diagram. This solution satisfies the Samuelson condition, because $\alpha = 1$ ($W = U_1 + \alpha U_2$).

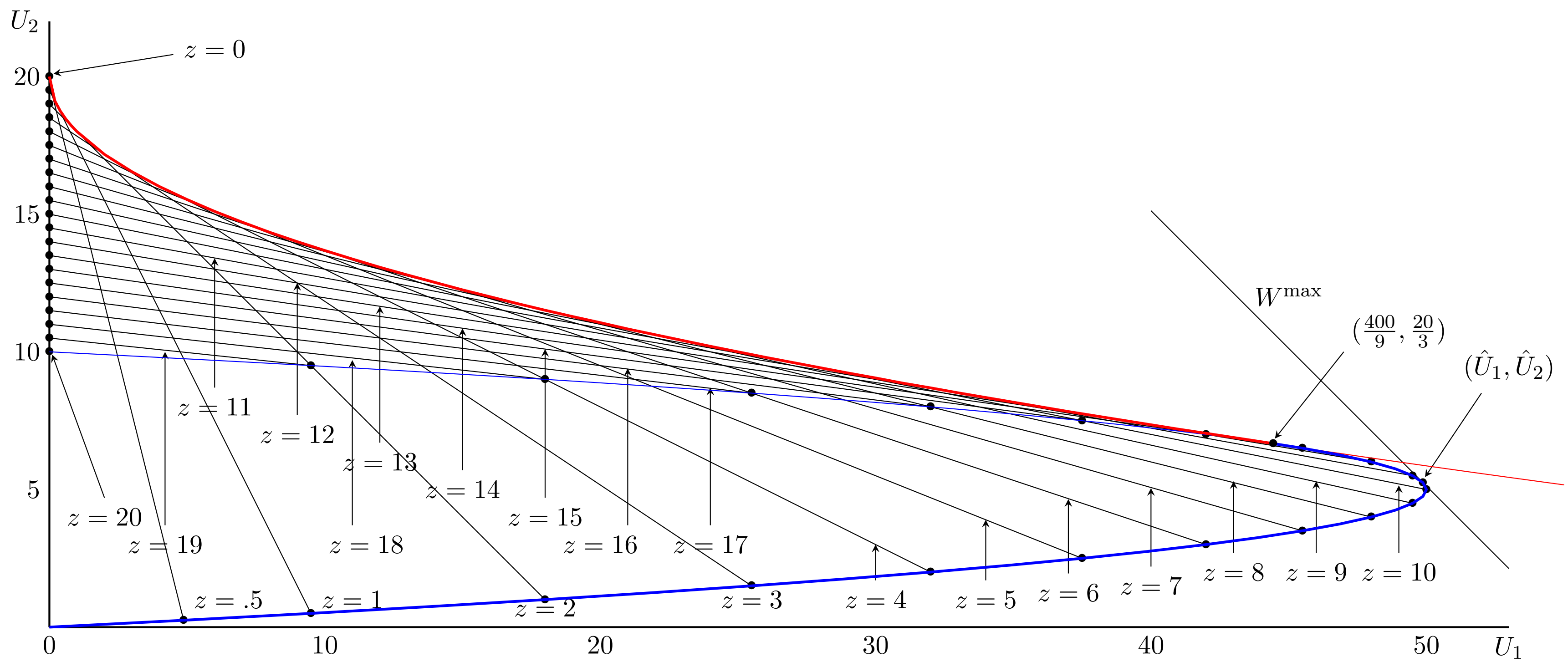

Figure 7: Utility possibilities, with $z = z_1 + z_2$. Samuelson utility pair is at $(\hat{U}_1, \hat{U}_2)$

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