AFRE 802
Statistical Methods for Agricultural, Food, & Resource Economists

Continuous random variables & their probability distributions – Part 2 of 3 (WMS Ch. 4.4 & 4.5)

September 28, 2017

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GAME PLAN

-Hand back in-class exercise & Ch. 3 HW

-Collect Integration HW (Ch. 4 HW won't be due until next Thursday at the earliest)

-Review

-Graded in-class exercise

Probability distributions for continuous RVs (cont'd)

-Specific continuous probability distributions:

1. Uniform
2. Normal

-Next class: gamma distribution & discuss Tchebysheff's inequality in the context of continuous RVs

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Review

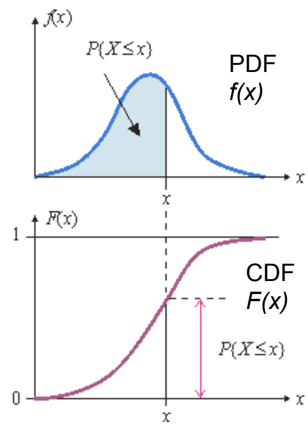
Cumulative distribution function (CDF):
$F(y) = P(Y \le y)$ for $-\infty \le y \le \infty$

Properties: $F(-\infty) = 0$, $F(\infty) = 1$,
$F(y)$ is a nondecreasing function of y

Probability density function (PDF):

$$f(y) = \frac{dF(y)}{dy} = F'(y)$$

Properties: $f(y) \ge 0$ for all y; $\int_{-\infty}^{\infty} f(y)\,dy = 1$



Relationships b / w CDFs & PDFs: $f(y) = \frac{dF(y)}{dy} = F'(y)$; $F(y) = \int_{-\infty}^{y} f(t)\,dt$

Review

How do we find P(a≤Y≤b) using the PDF? CDF?

Review

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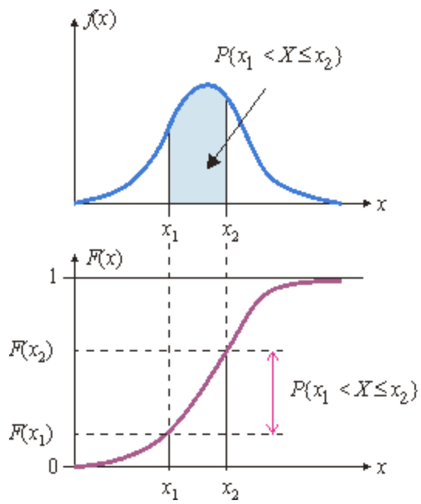
$$P(a \le Y \le b) = \int_a^b f(y)\,dy = F(b) - F(a)$$



For a continuous RV, what is the relationship among :
$P(a \le Y \le b)$
$P(a \le Y < b)$
$P(a < Y \le b)$
$P(a < Y < b)$?

Review: E(Y) and V(Y) for continuous RVs

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$$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy$$

Expected value of a continuous RV

Same rules for expected values & variances apply to both discrete & continuous RVs

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f(y)\,dy$$

Expected value of a function of a continuous RV

$$V(Y) = E\left[(Y-\mu)^2\right] = E(Y^2) - \mu^2 = E(Y^2) - \left[E(Y)\right]^2$$

Variance of an RV

$$V(Y) = \int_{-\infty}^{\infty} (y-\mu)^2 f(y)\,dy = \int_{-\infty}^{\infty} y^2 f(y)\,dy - \left[E(Y)\right]^2$$

Variance of a continuous RV

$E(Y^2)$

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$E(Y)$ and $V(Y)$ for continuous RV - example

In example #2 last class, we found $f(y)=(3/8)y^2$ for $0 \le y \le 2$, and $f(y)=0$ elsewhere. If random variable Y has this continuous PDF, find $E(Y)$ and $V(Y)$.

What did you get?

$$E(Y)=\int_{-\infty}^{\infty} yf(y)\,dy$$

$$V(Y)=E(Y^2)-[E(Y)]^2$$

$$E(Y^2)=\int_{-\infty}^{\infty} y^2 f(y)\,dy$$

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Graded in-class exercise

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Common continuous probability distributions

1. Uniform
2. Normal
3. Gamma and special cases
    1. Exponential
    2. Chi-square
4. Beta (not covered in this class)

*What examples did you find for the **uniform** and **normal** distribution?*

For next Tuesday, please try to find applications for gamma, exponential, or chi-square distribution.

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The uniform probability distribution

Intuiting the PDF for the uniform distribution

- Suppose the CATA bus arrives at a particular stop between 8:00 and 8:10 AM, and that it is equally likely to arrive at any time in that interval
- Let *Y* be the length of time (in minutes) a person has to wait for the bus if they arrive at the bus stop at exactly 8:00:00 AM.
- *What is the interval over which f(y)>0?*
- *What does the PDF of this RV look like and why? (e.g., is f(y) bell-shaped? declining? flat line?)*

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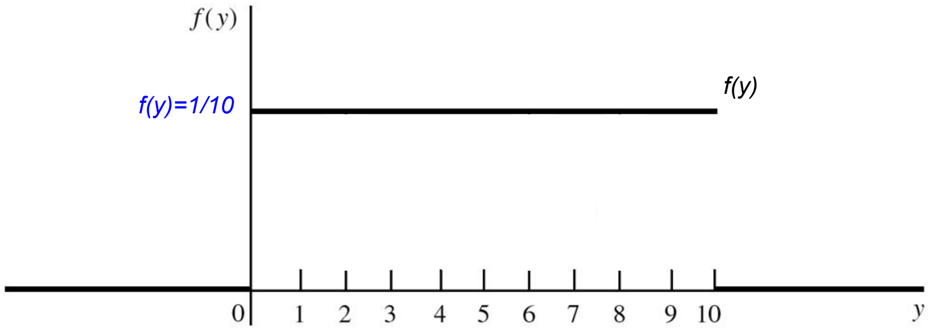
So *f(y)* is constant ("uniform") over the range with positive probability.

What does f(y) equal here? (Hint: recall what the area under f(y) from -∞ to +∞ has to equal to be a valid PDF.)



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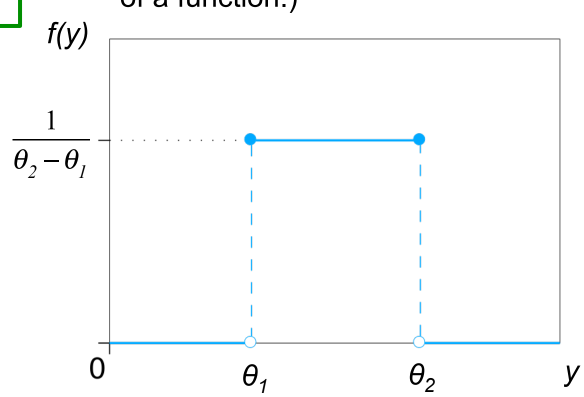
PDF for a uniform continuous RV

$$f(y) = \begin{cases} \dfrac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0, & \text{elsewhere} \end{cases}$$

where $\theta_1 < \theta_2$

What are the parameters of a uniform distribution?
(**Parameters** = the constants that determine the specific form of a function.)

If the bus arrives between 10:15 and 10:34 AM, and X is the number of minutes someone has to wait if they arrive at the bus stop at 10:15, what is the PDF of this uniform RV, X?



PDF & CDF of uniform distribution

Find the formula for the CDF for an RV with a uniform distribution over (θ_1, θ_2).

Recall $F(y) = \int_{-\infty}^{y} f(t)\, dt$

$$f(y) = \begin{cases} \dfrac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0, & \text{elsewhere} \end{cases}$$

where $\theta_1 < \theta_2$

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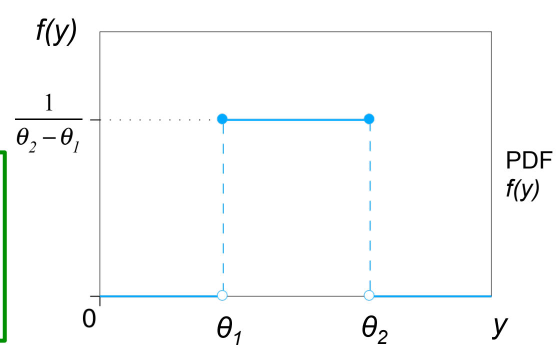
PDF & CDF of uniform distribution

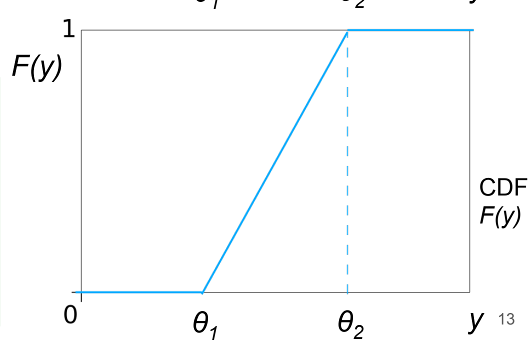
$$f(y) = \begin{cases} \dfrac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0, & \text{elsewhere} \end{cases}$$

where $\theta_1 < \theta_2$

$$F(y) = \begin{cases} 0, & y < \theta_1 \\ \dfrac{y - \theta_1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 1, & y > \theta_2 \end{cases}$$

where $\theta_1 < \theta_2$





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Uniform distribution – link to Poisson & example

Suppose the number of events that occur in a time interval (0, *t*) ~ Poisson. If exactly one such event has occurred in the interval (0, *t*), then the actual time of occurrence is ~ uniform (0, t)

- "~" means "distributed as"

EX) Suppose that the arrival of customers at a checkout counter follows a Poisson distribution. During a given 30-minute period, one customer arrived at the counter. *What is the probability that the customer arrived during the last 5 minutes of the 30-minute period?* (Hint: The actual time of arrival, *Y*, follows a uniform distribution. First determine the interval over which f(y)>0.)

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Uniform distribution: expected value & variance

- Recall the waiting for the bus example but now suppose the bus arrives between 8:10 and 8:30 AM, and *Y* is the number of minutes you have to wait for the bus if you arrive at the bus stop at 8:10 AM exactly.
- *What time do you expect the bus to arrive (on average), and what is E(Y)?*
- *What does this imply about E(Y) in general for a uniform RV?*

$$\boxed{E(Y) = \frac{\theta_1 + \theta_2}{2}}$$

$$\boxed{V(Y) = \frac{(\theta_2 - \theta_1)^2}{12}}$$

What is V(Y) in the example above?

If we have time at the end of class, we will prove E(Y).

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Summary: uniform distribution

Distribution	Probability Function (PDF)	Mean	Variance
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$

Common continuous probability distributions

1. Uniform
2. **Normal**
3. Gamma and special cases
    1. Exponential
    2. Chi-square
4. Beta (not covered in this class)

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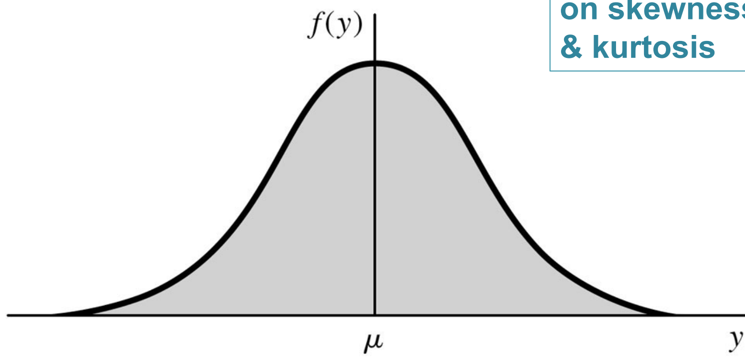
The normal probability distribution

- Most common & widely used continuous probability distribution
- Symmetric, bell-shaped

Quick aside on skewness & kurtosis



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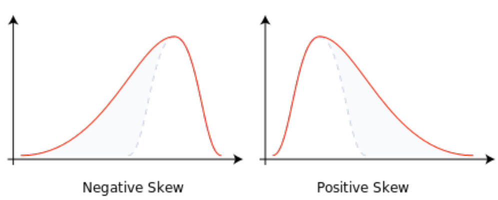
Aside on **skewness** & **kurtosis**

- Skewness (lack of symmetry):

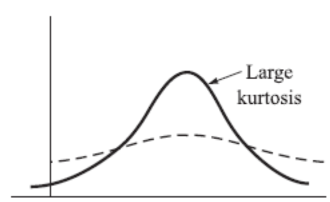
$$E\left[\frac{(Y-\mu)^3}{\sigma^3}\right]$$



Skewed to the left   Skewed to the right

- Kurtosis (peakedness):

$$E\left[\frac{(Y-\mu)^4}{\sigma^4}\right]$$



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Normal distribution (cont'd)

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PDF, mean, & variance of the normal distribution

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}, \qquad -\infty \le y \le \infty$$

What are the parameters of the normal distribution?

$\mu, \ \sigma$

The mean and variance of the normal distribution:

$E(Y) = \mu$ $\qquad$ $V(Y) = \sigma^2$

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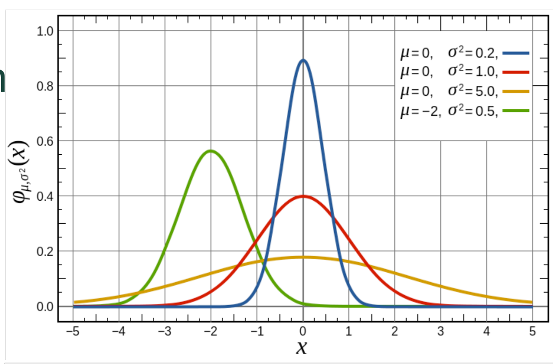
Normal distribution examples

$\varphi_{\mu,\sigma^2}(\,.\,)$

or $\phi_{\mu,\sigma^2}(\,.\,)$

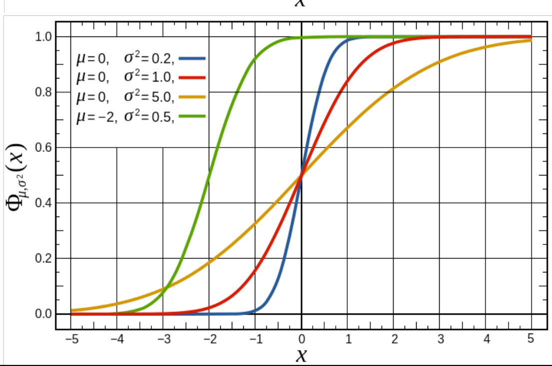
denotes a normal PDF with mean μ and variance σ^2



PDF
f(x)

$\Phi_{\mu,\sigma^2}(\,.\,)$

denotes a normal CDF with mean μ and variance σ^2



CDF
F(x)

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The CDF of the normal distribution & tabulated values

- No closed form solution/formula. Values calculated numerically and tabulated in Appendix 3, Table 4 for the "standard normal" distribution.
- ***Standard normal: $\mu = 0$, $\sigma^2 = 1$**
- Check out **Table 4**. ***Note that the values here are for ***P(Z>z)***.
- Let Z denote a standard normal random variable. *Use Table 4 to find:*
  1. $P(Z > 2)$
  2. $P(-2 \le Z \le 2)$
  3. $P(0 \le Z \le 1.73)$

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Can convert any normal RV to standard normal

- Note: $Y \sim N(\mu, \sigma^2)$ means "Y is distributed as normal w/ mean μ and variance σ^2"
- Convert normal RV (Y) to a standard normal RV (Z) using:

$$\boxed{Z = \frac{Y-\mu}{\sigma}, \quad Z \sim N(0,1) \ = \text{standard normal}}$$

- Once converted to standard normal → use Table 4
- CDF of standard normal: $\Phi(.)$
- PDF of standard normal: $\phi(.)$
- EX) The achievement scores for a college entrance exam are normally distributed with mean 75 and standard deviation 10. *What fraction of those scores lies between 80 and 90?*

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Summary: uniform & normal distributions

Distribution	Probability Function (PDF)	Mean	Variance
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2

$Z = \frac{Y-\mu}{\sigma}$, $Z \sim N(0,1)$ = standard normal

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If have time left …

- Prove E(Y) for uniform distribution
- Try some of additional in-class exercises at the end of the slides

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Homework:

- WMS Ch. 4 (part 2 of 3)
  - Uniform distribution: 4.38, 4.41, 4.42, 4.44, 4.49
  - Normal distribution: 4.59, 4.61, 4.62, 4.64 (a only), 4.71, 4.72
  - Ch. 4 HW will probably be due next Thurs. (or the class after we finish Ch. 4)

Next class:

- Continuous random variables (Part 3 of 3) – continue common probability distributions (gamma and special cases) and revisit Tschebysheff's inequality

Reading for next class:

- WMS Ch. 4 (sections 4.6, 4.8, 4.10, 4.12)

Application for next class:

- Look up application in your field of the gamma, exponential, or chi-square distribution

In-class exercises #1 – *E(Y)* & *V(Y)* for continuous RV

4.21 If, as in Exercise 4.17, Y has density function

$$f(y) = \begin{cases} (3/2)y^2 + y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of Y.

In-class exercises #2 – *E(Y)* & *V(Y)* for continuous RV

4.32 Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} (3/64)y^2(4-y), & 0 \le y \le 4, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the expected value and variance of weekly CPU time.

b The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

c Would you expect the weekly cost to exceed \$600 very often? Why?

In-class exercises #3 & #4 – uniform distribution

4.45 Upon studying low bids for shipping contracts, a microcomputer manufacturing company finds that intrastate contracts have low bids that are uniformly distributed between 20 and 25, in units of thousands of dollars. Find the probability that the low bid on the next intrastate shipping contract

a is below \$22,000.

b is in excess of \$24,000.

4.46 Refer to Exercise 4.45. Find the expected value of low bids on contracts of the type described there.

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In-class exercises #5

4.47 The failure of a circuit board interrupts work that utilizes a computing system until a new board is delivered. The delivery time, Y, is uniformly distributed on the interval one to five days. The cost of a board failure and interruption includes the fixed cost c_0 of a new board and a cost that increases proportionally to Y^2. If C is the cost incurred, $C = c_0 + c_1 Y^2$.

a Find the probability that the delivery time exceeds two days.

b In terms of c_0 and c_1, find the expected cost associated with a single failed circuit board.

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In-class exercises #6

4.63 A company that manufactures and bottles apple juice uses a machine that automatically fills 16-ounce bottles. There is some variation, however, in the amounts of liquid dispensed into the bottles that are filled. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation 1 ounce.

a Use Table 4, Appendix 3, to determine the proportion of bottles that will have more than 17 ounces dispensed into them.

In-class exercises #7

4.68 The grade point averages (GPAs) of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation .8. What fraction of the students will possess a GPA in excess of 3.0?

a Answer the question, using Table 4, Appendix 3.