Apec 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 9: Functional Forms for Demand Systems

I. Introduction

The theorems regarding preferences, utility functions and demand behavior covered in previous lectures are a useful framework for analyzing data on demand. In particular:

1. Proposed functional forms for econometric analysis can be evaluated in terms of whether they are consistent with these theorems.

2. Estimates based on reasonable functional forms can be used to test theorems about demand behavior. If the theory is rejected by the data there are 2 possibilities, either people do not really behave as the theory predicts, or the functional form used is too restrictive.

3. Once you have a functional form that gives sensible results, you can proceed to use it to estimate demand behavior of interest, such as price elasticities and income (wealth) elasticities.

This lecture presents the most commonly used functional forms for estimating demand behavior.

II. Linear Expenditure System

Suppose you start out with a set of general (Walrasian) demand functions, one equation for each of the L goods (I use i instead of ℓ to index since we need more than 1 index):

$$x_i = x_i(\mathbf{p}, w) \quad \text{for } i = 1, 2, \ldots L$$

For each equation there are L+1 first derivatives, so for the system of L equations there are $L^2 + L$ first derivatives. We could **impose standard theoretical constraints**, namely Walras' Law, homogeneity of degree zero and Slutsky symmetry. This would **reduce the number of independent derivatives by more than one half** to $(L^2+L)/2 - 1$.

To see this in a demand system, assume a **linear model for expenditures** on each item, which implies that all second derivatives for these expenditure equations = 0:

$$p_i x_i = \beta_i w + \sum_{j=1}^{L} \beta_{ij} p_j$$

Stone showed (1954) that the **only form** of these expenditure equations **that satisfies all three sets of restrictions** is the ***linear expenditure system***:

$$p_i x_i = p_i \gamma_i + \beta_i \left(w - \sum_{k=1}^{L} p_k \gamma_k\right) \quad \text{with } \sum_{k=1}^{L} \beta_k = 1$$

The associated expenditure function for this system of demand equations is:

$$e(\mathbf{p}, u) = \sum_{k=1}^{L} p_k\gamma_k + u\prod_{k=1}^{L} p_k^{\beta_k}$$

which will be concave in prices as long as all β_i's are ≥ 0 and $w \geq \sum_{k=1}^{L} p_k\gamma_k$ (if these conditions do not hold when parameters are estimated then this functional form is not credible because it implies that the expenditure function is not concave, which is inconsistent with the basic economic assumptions of rationality).

Sometimes the γ's are interpreted as minimal quantities that are essential for survival, which implies that $\sum_{k=1}^{L} p_k\gamma_k$ is the minimal expenditure that is necessary for survival. Yet there is no theoretical reason that the γ's have to be > 0.

Perhaps the **greatest attraction of the linear expenditure system** is that it is based on **only 2L parameters** (L γ's and L β's) and in fact only L-1 β's are independent of each other since they sum to 1. Recall that the theory itself allows up to $L^2/2 - L/2 - 1$ independent parameters (first derivatives) after imposing the three sets of restrictions, so what is really going on here is that **the (linear) functional form** of the linear expenditure system **imposes *additional* restrictions**.

In fact, Deaton and Muellbauer (1980) point out that inferior goods are possible only if $\beta_i < 0$, but this violates the concavity of the expenditure function and also implies a positive own-price elasticity. In addition, it does not allow for any goods to be compliments; **all goods must be substitutes**. This suggests that the linear expenditure system is probably not very useful, except in cases where goods are grouped into such broad categories that we would not expect any inferior goods or compliments.

III. Flexible Functional Forms of the 1970s

New approaches developed in the 1970s specify the (direct) utility function, or the indirect utility function, or the expenditure function, using a flexible functional form with many parameters. These typically allow almost any values not only for first derivatives but also for second derivatives of the function. A prominent example of this is the "indirect translog model" of Christiansen, Jorgenson and Lau (1975), which specifies a quadratic (linear terms plus squared terms plus interaction terms) indirect utility function:

$$U = \alpha_0 + \sum_{k=1}^{L} \alpha_k \log(p_k/w) + (1/2) \sum_{k=1}^{L} \sum_{j=1}^{L} \beta_{kj} \log(p_k/w) \log(p_j/w)$$

This is automatically homogenous of degree zero in prices and wealth. It can be argued that it is a second order Taylor approximation to any indirect utility function. The authors applied this to U.S. macro data from 1929 to 1972 and rejected the basic theoretical axioms of demand theory.

IV. The Search for a Representative Consumer: PIGL & PIGLOG Indirect Utility Functions and Expenditure Functions (Deaton & Muellbauer, appendix, AER, 1980)

Recall from Lecture 8 that an important issue in microeconomics is whether rational behavior at the individual or household level leads to aggregate data that are “rational” and/or can be thought of as a function of the behavior of a “representative consumer”.

Muellbauer (1975, 1976) ***defined*** **a representative consumer** in the following sense: **a representative consumer exists if the aggregate budget share** of a good i (aggregated over H households),

$$\overline{b}_i = \sum_{h=1}^{H} p_i x_{ih} / \sum_{h=1}^{H} w_h = \sum_{h=1}^{H} w_h b_{ih} / \sum_{h=1}^{H} w_h,$$

is a function of prices and a scalar w_0 that is a function of prices and the distribution of w_h. That is, the distribution of w_h is summarized in a scalar index that affects aggregate budget shares for **all** goods: $\overline{b}_i = b_i(w_0(w_1, \ldots w_H; \mathbf{p}), \mathbf{p})$. This w_0 can be thought of as the “representative” w for the economy as a whole. Note that simple studies of demand based on aggregate data simply used $w_0 = \overline{w}$.

Muellbauer showed that **for such a $\mathbf{w_0}$ to exist all the budget share equations at the individual/household level must have the form**:

$$b_{ih}(w_h, \mathbf{p}) = v_h(\mathbf{p}, w_h)A_i(\mathbf{p}) + B_i(\mathbf{p}) + C_{ih}(\mathbf{p})$$

with $\sum_{i=1}^{L} A_i(\mathbf{p}) = 0$, $\sum_{i=1}^{L} C_{ih}(\mathbf{p}) = 0$, $\sum_{h=1}^{H} C_{ih}(\mathbf{p}) = 0$ & $\sum_{i=1}^{L} B_i(\mathbf{p}) = 1$.

If we **further assume that $\mathbf{w_0}$ depends only on the distribution of $\mathbf{w_h}$**, and not on prices, this implies, and is implied by, the following functional form for $v_h(\mathbf{p}, w_h)$:

$$v_h(\mathbf{p}, w_h) = [1 - (w_h/k_h)^{-\alpha}]^{1/\alpha}$$

where α is a constant and **$\mathbf{k_h}$ is a household specific constant** that is not a function of w_h or of $\mathbf{p}$. Budget shares generated by this assumption are said to have the **PIGL** (price-independent generalized linear) form.

Finally, if $\alpha = 0$, then:

$$v_h(\mathbf{p}, w_h) = \log(w_h/k_h)$$

This special case is known as **PIGLOG** form.

The corresponding expenditure functions are:

$$[e(\mathbf{p}, u_h)/k_h]^\alpha = (1 - u_h)[a(\mathbf{p})]^\alpha + u_h[b(\mathbf{p})]^\alpha \quad \text{(PIGL)}$$

$$\log[e(\mathbf{p}, u_h)/k_h] = (1 - u_h)\log[a(\mathbf{p})] + u_h\log[b(\mathbf{p})] \quad \text{(PIGLOG)}$$

where $a(\mathbf{p})$ and $b(\mathbf{p})$ are linearly homogenous concave functions [$b(\mathbf{p})$ is **not** a budget share]. Usually, $0 \le u \le 1$.

There are some **very useful properties of the PIGL and PIGLOG functional forms**. First, differentiating the equation for the PIGL expenditure function by p_i to get "Hicksian" demands, and then substituting w_h for $e(\mathbf{p}, u_h)$ gives the following Walrasian demand budget share equation:

$$b_{ih} = (1\text{-}u_h)[a(\mathbf{p})/w_h]^\alpha \frac{\partial \log[a(\mathbf{p})]}{\partial \log(p_i)} + u_h[b(\mathbf{p})/w_h]^\alpha \frac{\partial \log[b(\mathbf{p})]}{\partial \log(p_i)}$$

with $u_h = (w_h^\alpha - a(\mathbf{p})^\alpha)/(b(\mathbf{p})^\alpha - a(\mathbf{p})^\alpha)$. The budget share equation is even simpler for the PIGLOG form:

$$b_{ih} = (1\text{-}u_h)\frac{\partial \log[a(\mathbf{p})]}{\partial \log(p_i)} + u_h \frac{\partial \log[b(\mathbf{p})]}{\partial \log(p_i)}$$

with $u_h = [\log(w_h) - \log(a(\mathbf{p}))]/[\log(b(\mathbf{p})) - \log(a(\mathbf{p}))]$.

It turns out that this PIGL class of functional forms encompasses many other functional forms (sometimes with an additional assumption or two). First, LES is a special case; in fact, LES is a special case of the Gorman form (see Lecture 8, it is also called Gorman polar form), which is a special case of PIGL if α =1 and k_h = 1. Second, adding a mild restriction to the translog functional form, that $\sum_{j=1}^{L} \sum_{k=1}^{L} \beta_{kj} = 0$, transforms it into a PIGLOG functional form.

V. Almost Ideal Demand System (AIDS)

This functional form was proposed by Deaton and Muellbauer in 1980, and their paper is on your reading list (in 2011 it was chosen as one of the top 20 papers of all papers published in the first 100 years of the *American Economic Review*). It is a flexible functional form that has the added advantage that it can be interpreted in terms of economic models of consumer behavior when estimated either with aggregated (macroeconomic) or disaggregated (household survey) data. This model is generally considered to be the best functional form to use estimate static systems of demand, especially for household level data. No dramatic advances have been made since 1980 on functional forms for demand systems, although some refinements have been made (see below).

The **AIDS model** takes as its **starting point** the following **(PIGLOG) expenditure function**:

$$\log[e(\mathbf{p}, u)] = (1-u)\log[a(\mathbf{p})] + u\log[b(\mathbf{p})]$$

It is a harmless normalization to require u to be ≥ 0 and ≤ 1, so this can be thought of as a weighted average of two functions of prices, with greater weight put on the second one for “richer” people.

The **next step** is to specify the (flexible) functional forms of log[a(**p**)] and log[b(**p**)]:

$$\log[a(\mathbf{p})] = \alpha_0 + \sum_{k=1}^{L} \alpha_k \log(p_k) + (1/2)\sum_{k=1}^{L}\sum_{j=1}^{L} \gamma_{kj}^{*}\log(p_k)\log(p_j)$$

$$\log[b(\mathbf{p})] = \log[a(\mathbf{p})] + \beta_0 \prod_{k=1}^{L} p_k^{\beta_k}$$

These functional forms are sufficiently flexible in the sense that they can reproduce **any** arbitrary set of the first and second derivatives of the expenditure function (at any single point): $\partial e(\mathbf{p}, u)/\partial p_i$, $\partial e(\mathbf{p}, u)/\partial u$, $\partial^2 e(\mathbf{p}, u)/\partial p_i \partial p_j$, $\partial^2 e(\mathbf{p}, u)/\partial u \partial p_i$ and $\partial^2 e(\mathbf{p}, u)/\partial u^2$.

Substitute in these expressions for log[a(**p**)] and log[b(**p**)] to get the AIDS expenditure function:

$$\log[e(\mathbf{p}, u)] = \alpha_0 + \sum_{k=1}^{L} \alpha_k \log(p_k)$$
$$+ (1/2) \sum_{k=1}^{L} \sum_{j=1}^{L} \gamma_{kj}^* \log(p_k)\log(p_j) + u\beta_0 \prod_{k=1}^{L} p_k^{\beta_k}$$

For this expenditure function to be homogenous of degree one in prices the following conditions must hold:

$$\sum_{k=1}^{L} \alpha_k = 1, \quad \sum_{j=1}^{L} \gamma_{kj}^* = 0, \quad \sum_{k=1}^{L} \gamma_{kj}^* = 0, \quad \sum_{k=1}^{L} \beta_k = 0.$$

The fact that $\partial\log(e(\mathbf{p}, u))/\partial\log(p_i) = b_i$ (this is easy to show) implies that the budget shares b_i take the form:

$$b_i = \alpha_i + \sum_{j=1}^{L} \gamma_{ij}\log(p_j) + \beta_i u\beta_0 \prod_{k=1}^{L} p_k^{\beta_k}$$
$$= \alpha_i + \sum_{j=1}^{L} \gamma_{ij}\log(p_j) + \beta_i \log(w/P)$$

where $\gamma_{ij} = (1/2)(\gamma_{ij}^* + \gamma_{ji}^*)$ and
$$\log(P) = \alpha_0 + \sum_{k=1}^{L} \alpha_k \log(p_k) + (1/2) \sum_{j=1}^{L} \sum_{k=1}^{L} \gamma_{kj}\log(p_k)\log(p_j).$$

(think of P, which equals [a(**p**)], as a price index)

A nice feature of the AIDS functional form is that the **standard restrictions of demand theory are simple expressions of the parameters**. Namely:

Walras' Law implies: $\sum_{k=1}^{L} \alpha_k = 1$, $\sum_{k=1}^{L} \gamma_{kj} = 0$, $\sum_{k=1}^{L} \beta_k = 0$.

Homogeneity implies: $\sum_{k=1}^{L} \gamma_{jk} = 0$.

Slutsky symmetry implies: $\gamma_{ij} = \gamma_{ji}$.

As in many functional forms, the unrestricted AIDS model automatically satisfies Walras' Law, so that "law" cannot be tested. Yet homogeneity and symmetry can be tested.

Finally, the negative definiteness of the Slutsky matrix is tested using the L×L matrix C, the elements of which are defined as:

$$c_{ij} = \gamma_{ij} + \beta_i\beta_j \log(w/P) - b_i\delta_{ij} + b_i b_j$$

where $\delta_{ij} = 1$ if i = j and equals zero otherwise.

The budget share (b_i) equation on the previous page has a very simple interpretation, where w/P can be thought of as real wealth (P is a price index). Price effects work through the γ's and changes in real income work through the β's. **These budget share equations can be estimated one by one using OLS**.

All of the above was at the level of the household. But from the 1950's to the 1970's economists used aggregate data. If we assume that this flexible functional form is "true" for households, what does this tell us about the "behavior" of aggregate data?

Another nice characteristic of the AIDS models is that it has a straightforward way to be applied to aggregate data. Using Muellbauer's above results, one can show that **exact aggregation is possible if**:

$$b_{ih} = \alpha_i + \sum_{j=1}^{L} \gamma_{ij}\log(p_j) + \beta_i \log(w_h/k_h P)$$

where h indicates household h and k_h can be interpreted as household composition (size). Basically, k_h is "effective" household size so that w_h/k_h is "per capita" wealth.

Summing up over all H households gives the **budget shares for the economy as a whole** (denoted by $\bar{b}_i$):

$$\bar{b}_i = \left(\sum_{h=1}^{H} p_i x_{ih}\right) \Big/ \left(\sum_{h=1}^{H} w_h\right) = \left(\sum_{h=1}^{H} w_h b_{ih}\right) \Big/ \left(\sum_{h=1}^{H} w_h\right)$$

$$= \alpha_i + \sum_{j=1}^{L} \gamma_{ij}\log(p_j) - \beta_i \log(P) + \beta_i \left[\sum_{h=1}^{H} w_h \log(w_h/k_h) \Big/ \left(\sum_{h=1}^{H} w_h\right)\right]$$

Define the aggregate index of effective household size, denoted by k, as the value of k that satisfies:

$$\log(\overline{w}/k) = \sum_{h=1}^{H} w_h \log(w_h/k_h)/(\sum_{h=1}^{H} w_h)$$

the expression for the **aggregate budget share** becomes:

$$\overline{b}_i = \alpha_i + \sum_{j=1}^{L} \gamma_{ij} \log(p_j) + \beta_i \log(\overline{w}/(kP))$$

This is exactly the form of the budget share for the individual household shown above. It is a first-order approximation to any arbitrary demand system, whether consistent with utility maximization or not.

Note finally that the **parameter k does two things**. First, it **accounts for** aggregate (effective) **household size**. Second, if (effective) household size were the same for all house-holds (e.g. $k_h = 1$ for all households), **the expression for log($\overline{w}$/k) measures inequality** in w over households (Theil inequality index). Thus aggregate budget shares can vary if the distribution of income varies.

Deaton and Muellbauer estimated this using aggregate British data from 1954 to 1976, for 8 types of goods. Note that they did not use any household survey data to get an estimate of k, so they basically ignored it in their analysis.

They rejected homogeneity for 4 types of goods (food, clothing, housing and transport). When they estimated their model in differenced form ($\Delta b_i = \sum_{j=1}^{L} \gamma_{ij} \Delta\log(p_j) + \beta_i \Delta\log(w/P)$) and add a constant (which implies a time trend), then homogeneity is rejected only for food and transport.

VI. The Quadratic AIDS (QUAIDS) Functional Form

An useful generalization of the AIDS functional form for demand systems is the Quadratic AIDS (QUAIDS) model of Banks, Blundell and Lewbel (1996, 1997).

First, consider the AIDS demand system discussed above. The AIDS expenditure function has the following form:

$$\log[e(\mathbf{p}, u)] = \log[a(\mathbf{p})] + u[b(\mathbf{p})]$$

where

$$\log[a(\mathbf{p})] = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log(p_i) + (1/2) \sum_{i=1}^{L} \sum_{j=1}^{L} \gamma_{ij} \log(p_i)\log(p_j)$$

$$b(\mathbf{p}) = \prod_{i=1}^{L} p_i^{\beta_i}$$

(Note: D. & M. notation is $\log[e(\mathbf{p}, u)] = (1-u)\log[a(\mathbf{p})] + u\log[b(\mathbf{p})]$ and $\log[b(\mathbf{p})] = \log[a(\mathbf{p})] + \beta_0 \prod_{i=1}^{L} p_i^{\beta_i}$.

Question: Why can Banks et al. drop β_0 from $\log[b(\mathbf{p})]$?)

We can derive the indirect utility function by solving for u and noting that $e(\mathbf{p}, u)$ simply equals total wealth:

$$u = \frac{\log(w) - \log[a(\mathbf{p})]}{b(\mathbf{p})}$$

We can use a variant of Shepherd's lemma, namely that $\partial\log[e(\mathbf{p}, u)]/\partial\log(p_i) = b_i$, to get the budget share of good i given by the AIDS system:

$$b_i = \frac{\partial \log[a(\mathbf{p})]}{\partial \log(p_i)} + u\frac{\partial b(\mathbf{p})}{\partial \log(p_i)}$$

$$= \frac{\partial \log[a(\mathbf{p})]}{\partial \log(p_i)} + \frac{\log(w) - \log[a(\mathbf{p})]}{b(\mathbf{p})}\frac{\partial b(\mathbf{p})}{\partial \log(p_i)}$$

$$= \frac{\partial \log[a(\mathbf{p})]}{\partial \log(p_i)} + \frac{\log(w^r)}{b(\mathbf{p})}\frac{\partial b(\mathbf{p})}{\partial \log(p_i)}$$

$$= \frac{\partial \log[a(\mathbf{p})]}{\partial \log(p_i)} + \log(w^r)\frac{\partial \log[b(\mathbf{p})]}{\partial \log(p_i)}$$

where Banks et al. define $w^r = w/a(\mathbf{p})$ as real (price adjusted) wealth (I have changed their notation; they use x, I use w^r).

Working out these derivatives for the functional forms given above for $a(\mathbf{p})$ and $b(\mathbf{p})$ yields (see page 10):

$$b_i = \alpha_i + \sum_{j=1}^{L} \gamma_{ij}\log(p_j) + \beta_i\log[w/a(\mathbf{p})]$$

which is what we had for the AIDS model.

The **important thing** to note here is that $\mathbf{w_i}$ **is linear in** terms of (the log of) **real wealth** (expenditure). Banks et al. present evidence from the U.K. to show that this assumption is restrictive and thus doubtful for at least some categories of goods. That is, it **may make sense to add a quadratic** (squared) **term** to the budget share equation.

To do this rigorously, they start with the following general functional form for budget shares, which allows not only for a linear term in w^r (with a flexible coefficient) but also for another term in w^r that has some general form $g(w^r)$ (of which quadratic is only one example):

$$b_i = A_i(\mathbf{p}) + B_i(\mathbf{p})\log(w^r) + C_i(\mathbf{p})g(w^r)$$

where $w^r = w/a(\mathbf{p})$ and A_i, B_i and C_i are scalars that are functions of **p**. Comparing this to the AIDS budget share equation above we see that $A_i(\mathbf{p})$ and $\alpha_i + \sum_{j=1}^{L} \gamma_{ij}\log(p_j)$ are roughly comparable, but $B_i(\mathbf{p})$ is much more flexible than the constant β_i, and finally the term $C_i(\mathbf{p})g(w^r)$ has no comparable part in the AIDS model.

Banks et al. show that this general type of budget share is **consistent with utility maximization** (and an exactly aggregable functional form) **only for the following two possibilities**:

1. $C_i(\mathbf{p})$ has the functional form $d(\mathbf{p})B_i(\mathbf{p})$, where $d(\mathbf{p})$ is independent of the good in question (no i subscript), or

2. The indirect utility function has the form:

$$\log V = \left[\left(\frac{\log(w) - \log[a(\mathbf{p})]}{b(\mathbf{p})} \right)^{-1} + \lambda(\mathbf{p}) \right]^{-1}$$

where $\lambda(\mathbf{p})$ is homogenous of degree zero.

Banks et al. show that the **first possibility**, that $C_i(\mathbf{p})$ has the functional form $d(\mathbf{p})B_i(\mathbf{p})$, is **inconsistent with the UK data**, so **we will focus on the second possibility**, namely the generalized indirect utility function given above.

Note that if $\lambda(\mathbf{p}) = 0$, we have the indirect utility function of the AIDS model (see p.15; although on that page we have u instead of log of u, this is just a monotonic transformation).

For the general case where $\lambda(\mathbf{p}) \neq 0$, we can derive the budget share equation again, this time using Roy's identity:

$$b_i = \frac{\partial \log[a(\mathbf{p})]}{\partial \log(p_i)} + \frac{\partial \log[b(\mathbf{p})]}{\partial \log(p_i)} \log(w^r) + \frac{\partial \lambda(\mathbf{p})}{\partial \log(p_i)} \frac{1}{b(\mathbf{p})} [\log(w^r)]^2$$

The authors show (Corollary 1) that for demand systems to have "rank 3" (roughly speaking, more than just linear in expenditures), then the functional form of $g(w^r)$ in the general form of the budget share given above must be $(\log(w^r))^2$. Of course, this implies that the budget share is quadratic in (log) real (and nominal) expenditures (w^r).

Finally, the authors also show (Corollary 2) that one cannot have both $B_i(\mathbf{p})$ and $C_i(\mathbf{p})$ be independent of prices. In other words, one or both must be a function of prices. This has an important implication for estimation: the budget share equation will not be linear in parameters, so **nonlinear estimation will be needed**.

In order not to make the new demand system overly complicated and different from AIDS, Banks et al. use the same functional forms for a(**p**) and b(**p**) given above. What remains to be done is to specify the $\lambda(\mathbf{p})$ equation. They use the following flexible functional form:

$$\lambda(\mathbf{p}) = \sum_{i=1}^{L} \lambda_i \log(p_i), \quad \text{where } \sum_{i=1}^{L} \lambda_i = 0$$

(The λ's must sum to zero to ensure that this function is homogenous of degree zero in prices).

Putting all of this together yields the following budget share equation for good i:

$$b_i = \alpha_i + \sum_{j=1}^{L} \gamma_{ij} \log(p_j) + \beta_i \log\left[\frac{w}{a(\mathbf{p})}\right] + \frac{\lambda_i}{b(\mathbf{p})}\left[\log\left[\frac{w}{a(\mathbf{p})}\right]\right]^2$$

Note that these budget share equations are *nonlinear* in parameters, so it cannot be estimated by OLS. You need to use a nonlinear estimator, such as nonlinear least squares.

Price and income elasticities for this functional form are given on p.534 of the 1997 article.

Banks et al. applied this to 17 years worth of household survey data from the U.K. An interesting point is that they imposed homogeneity. The reason for this is that, at least for the AIDS functional form, symmetry and Walras law (adding up) imply homogeneity (see top of page 11), so if Walras law holds a test for symmetry is a joint test for symmetry and homogeneity.

After imposing homogeneity, they show that the estimated results are consistent with symmetry and (almost) with a negative semidefinite Slutsky matrix.