Applied Microeconomics: Firm and Household

Lecture 9: Production Functions

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Outline

Production Functions

- 1-output 1-input production functions
  - Technical aspects (TP, AP, MP, y^*, ϵ)
  - The three stages of production
  - Examples: Linear, power and quadratic production functions
- 1-output 2-input production functions
  - Factor interdependence
  - Isoquants and the rate of technical substitution (RTS)
  - Isoclines and ridgelines
  - Elasticity of Substitution
  - The Function Coefficient, Returns to Scale, and Homogeneity
  - Example: Cobb-Douglas production function

Production functions

A production function is a mathematical description of the technical production possibilities of a firm. It gives the **maximum physical output** of a firm **for each level of inputs.** We begin with single output/single input production functions. Formally:

- $y = f(x_1|x_2, ..., x_N)$

where

- y is physical output
- x_1 is the variable input, i.e., fertilizer, labor
- $x_2, ..., x_N$ are fixed inputs, i.e., capital, machinery

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For convenience we will drop fixed input arguments and denote production function as:

- $y = f(x)$

Technical aspects of production functions

We use the following concepts to characterize production functions.

- Total Product (TP) – sometimes Total Physical Product (TPP)
  - $y = f(x)$
- Average Product (AP) – sometimes Average Physical Product (APP)
  - $AP = \frac{TP}{x} = \frac{f(x)}{x}$

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- Maximum output
  - $y^* = f(x^*)$ where x^* solves $f'(x) = 0$ with $f''(x) < 0$
- Elasticity of production (factor elasticity)
  - $\epsilon_{y,x} = \frac{\partial y}{\partial x}\frac{x}{y} = \underbrace{\frac{\partial f(x)}{\partial x}}_{MP} / \underbrace{\frac{f(x)}{x}}_{AP} = \frac{MP}{AP}$

MP and $\epsilon_{y,x}$

$$MP = \frac{\partial f(x)}{\partial x} = f'(x)$$

MP is the slope of the total product function evaluated at a particular level of the variable factor. It gives the exact rate of change in total product for an infinitesimal change in the factor.

Question: Is it rational for a firm to produce at a level where $MP < 0$?

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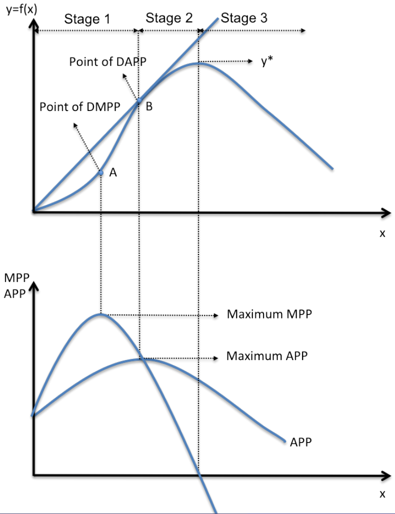
- $\epsilon_{y,x} = \frac{\%\text{change in output}}{\%\text{change in input}} = \frac{\partial y/y}{\partial x/x} = \frac{\partial y}{\partial x}\frac{x}{y} = \frac{MP}{AP}$

$\epsilon_{y,x}$ is the percentage change in output in response to a percent change in the variable input **holding all other inputs fixed.** Note that this is a unit free measure.

The three stages of production



Stage 1:

- $MP > AP$
- $\epsilon_{y,x} > 1$

Stage 2:

- $0 \leqslant MP \leqslant AP$
- $0 \leqslant \epsilon_{y,x} \leqslant 1$

Stage 3:

- $MP < 0$
- $\epsilon_{y,x} < 0$

Law of diminishing marginal returns

Law of diminishing marginal returns: states that as the quantity of an input increases, the marginal product of each additional unit of input at some point will be less than the marginal product of the previously added unit of input. In other words, at some point there will be decreasing MP.

- $\frac{\partial MP}{\partial x} < 0$.

Note that DMP starts at the inflection point of the production function, point A, which also corresponds to the maximum MP.

At point B, the ray has the greatest possible slope among all possible ray lines that is tangent to (or intersecting) TP, therefore AP is at its maximum at point B (the start of the diminishing average physical product, DAP).

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Question: Why does MP always have to intersect AP at its maximum?

The three stages of production

Highlights:

- Stage 1: Between 0 and maximum AP.
  - AP is always increasing.
  - MP is always above AP, $\epsilon_{y,x} > 1$
  - Economically optimal *only* if there is some constraint forcing firm to produce within this region.

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  - This is the economically rational region.

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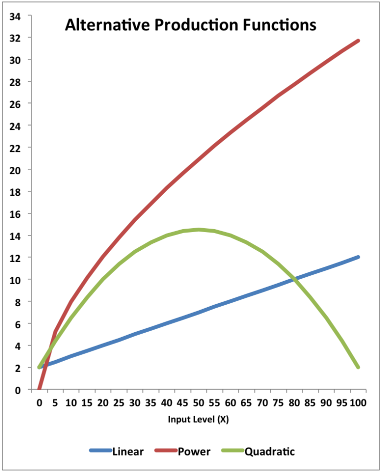
- Stage 3: Beyond maximum TP
  - MP is below zero, $\epsilon_{y,x} < 0$
  - Economically irrational stage.



To fix ideas we will investigate some commonly used production functions in empirical studies. Namely:

- Linear function
  - $y = \alpha + \beta x$
- Power function (Cobb-Douglas)
  - $y = \alpha x^{\beta}$
- Quadratic function
  - $y = \alpha + \beta_1 x + \beta_2 x^2$

Linear production function

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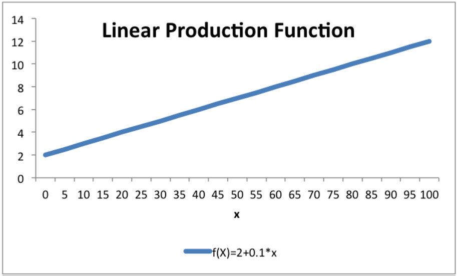
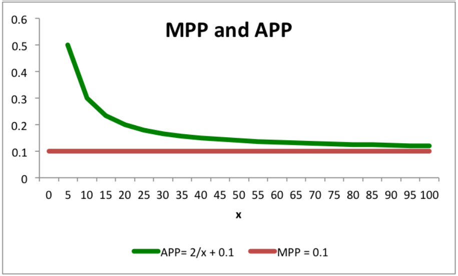
- y^* does not exist, ($MP > 0, \ \forall x$).
- $\epsilon_{y,x} = \frac{MP}{AP} = \frac{\beta x}{\alpha+\beta x}$
  - $\epsilon_{y,x} = 1$ if $\alpha = 0$,
  - $0 < \epsilon_{y,x} < 1$ if $\alpha > 0$,
  - $\epsilon_{y,x} > 1$ if $\alpha < 0$ and $|\beta x| > |\alpha| \ \forall x$





Properties:

- Has no maximum
  - Always increasing
- No diminishing returns
  - MP is constant
- AP is always decreasing
- at the limit AP is constant

Power production function

A **power function** production function is given as

$$y = \alpha x^{\beta}, \qquad \alpha > 0, \text{ and } 0 < \beta < 1$$

- $AP = \frac{y}{x} = \frac{\alpha x^{\beta}}{x} = \alpha x^{\beta - 1}$
- $MP = \frac{\partial y}{\partial x} = \frac{\partial(\alpha x^{\beta})}{\partial x} = \beta \alpha x^{\beta - 1}$
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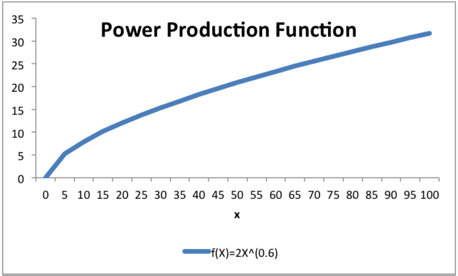
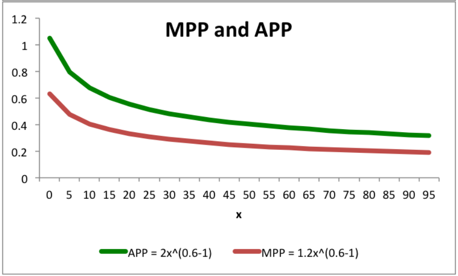
- y^* does not exist ($MP > 0,\ \forall x$).
- $\epsilon_{y,x} = \frac{MPP}{APP} = \frac{\beta \alpha x^{\beta - 1}}{\alpha x^{\beta - 1}} = \beta$





Properties:

- Has no maximum
  - $\frac{\partial y}{\partial x} > 0, \ \forall x$
- $MP < AP, \ \forall x$
- $\frac{\partial MP}{\partial x} < 0, \ \forall x$
  - diminishing returns
- $\frac{\partial AP}{\partial x} < 0, \ \forall x$
- $\epsilon_{y,x}$ is constant

Quadratic production function

A **quadratic function** is a second-order approximation to an unknown production function (due to the square term). The sign of the intercept is important in representing the stages of production.

$$y = \alpha + \beta_1 x + \beta_2 x^2, \qquad \beta_1 > 0, \text{ and } \beta_2 < 0$$

- $AP = \frac{y}{x} = \frac{\alpha}{x} + \beta_1 + \beta_2 x$
- $MP = \frac{\partial y}{\partial x} = \beta_1 + 2\beta_2 x$
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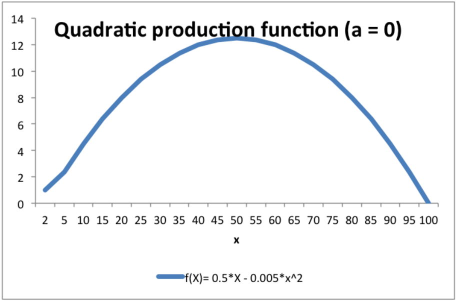
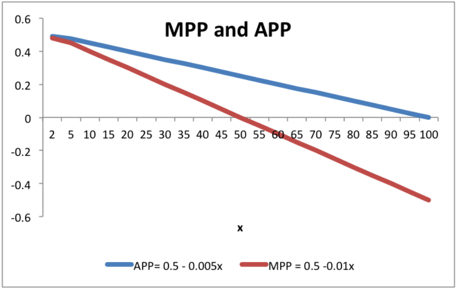
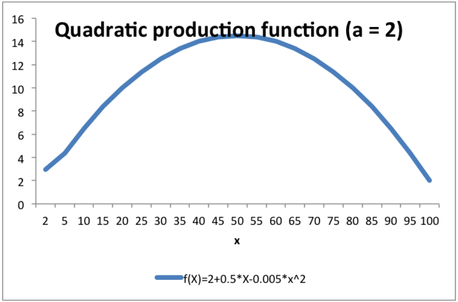
- $y^* = f(x^*)$ where $x^* = -\frac{\beta_1}{2\beta_2}$
- $\epsilon_{y,x} = \frac{MP}{AP} = \frac{(\beta_1 + 2\beta_2 x)x}{\alpha + \beta_1 x + \beta_2 x^2} = \frac{MP * x}{TP}$

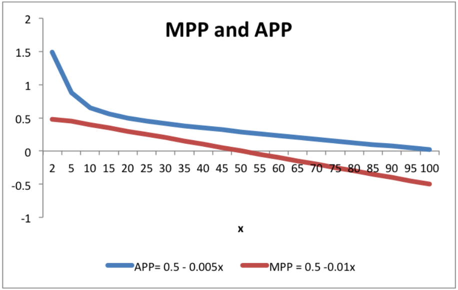




Properties:

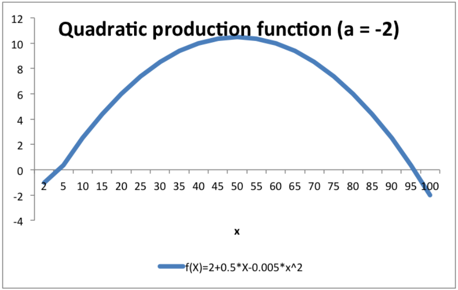
- Starts at stage 2
  - $\frac{\partial AP}{\partial x} > 0, \ \forall x$
- $MP < AP, \ \forall x$
- $\frac{\partial MP}{\partial x} < 0, \ \forall x$
  - diminishing returns
- $\exists y^*$
  - Stage 3 starts after y^*
- $0 < \epsilon_{y,x} < 1$ in stage 2
- $\epsilon_{y,x} < 0$ in stage 3

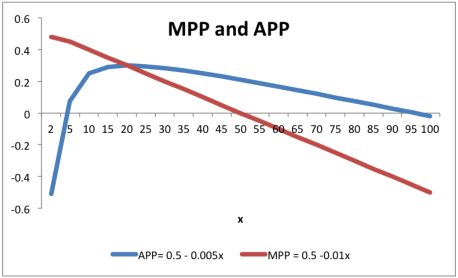




Properties:

- Exhibits same properties as in the case of no intercept.
- The main difference is that the intercept defines whether the input is **essential** (required for non-zero output).
- if $x = 0 \rightarrow y = 0$ then x is essential.
- Here, *x* is *not* essential.





Negative intercept has two main differences from other cases:

1. Displays all three stages of production
   - AP has a maximum that delineates stages 1 and 2.
2. The input is not only essential but it has a minimum threshold that must be used to generate positive output.

1-Output 2-Input Production Functions: Technical aspects

We continue our discussion with single output/two input production functions. Formally:

- $y = f(x_1, x_2)$

Suppose, y is wheat output, x_1 is land and x_2 is labor.

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AP and MP

We calculate AP and MP for each input holding the other input constant:

- $AP_i = \frac{y}{x_i} = \frac{f(x_1,x_2)}{x_i},\ i = 1,2$
- $MP_i = f_i = \frac{\partial y}{\partial x_i} = \frac{\partial f(x_1,x_2)}{\partial x_i},\ i = 1,2$

Technical aspects

Total marginal product:

Total marginal product is defined by the total differentiation of the production function:

- $dy = \frac{\partial f(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial f(x_1, x_2)}{\partial x_2} dx_2$
- $dy = \underbrace{f_1 dx_1}_{dy_1} + \underbrace{f_2 dx_2}_{dy_2}$

where $dy_i,\ i = 1, 2$ is the marginal product attributable to factor i.

That is, total marginal product is sum of the marginal products that are attributable to each of the factors individually.

Factor interdependence

The technical relationship between two inputs is determined by how the marginal product of one input is affected by the other input. There are three types of technical relationships:

- **Technically complementary:** MP of one input increases as the other input increases.
  - $\frac{\partial^2 y}{\partial x_1 \partial x_2} = \frac{\partial\left(\frac{\partial y}{\partial x_1}\right)}{\partial x_2} = \frac{\partial MP_1}{\partial x_2} = f_{12} = f_{21} > 0$

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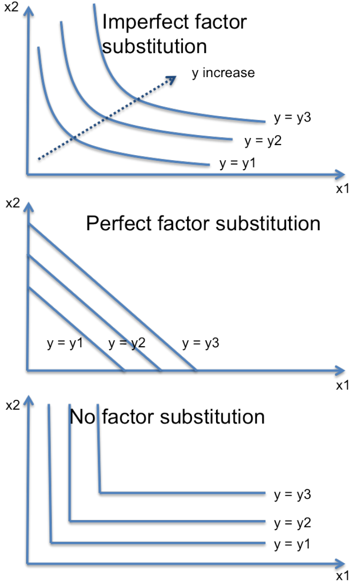
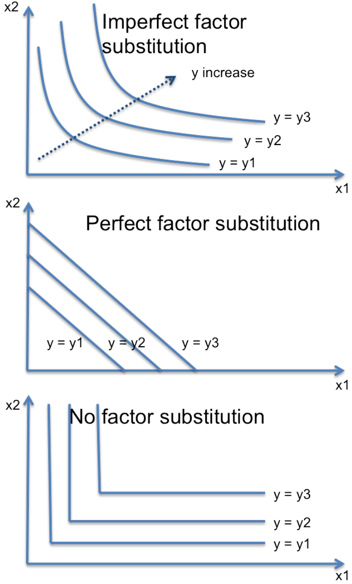
- **Technically independent:** MP of one input is not affected by changes in the other input.
  - $f_{12} = f_{21} = 0$
- **Technically competitive:** MP of one input decreases as the other input increases.
  - $f_{12} = f_{21} < 0$

Definitions

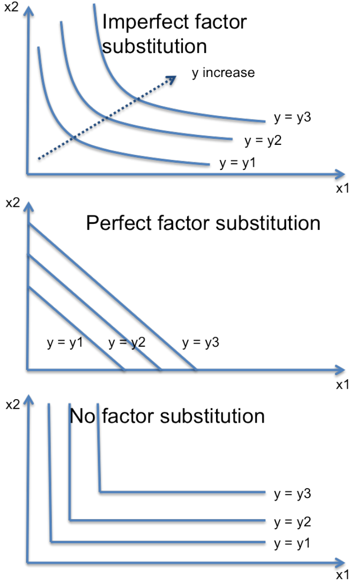
- **Isoquant:** An isoquant curve represents all input combinations that can produce a given level of output.
- **Marginal rate of technical substitution (MRTS):** is the negative of the slope of the isoquant at an arbitrary point. It gives the rate that one factor must be substituted for the other factor to maintain the same output level.
- **Isoclines:** An isocline is a ray connecting equal MRTS across different isoquants.
- The **ridgelines:** are two special isoclines that define the relevant region of input choices for profitable production. The ridgelines occur at $RTS = 0$ and $RTS = \infty$
- The **elasticity of substitution** is a measure of the degree of substitutability between factors. It is defined as the proportionate rate of the change of input ratio divided by the proportionate rate of change in the MRTS.


x2
Imperfect factor substitution
y increase
y = y3
y = y2
y = y1
x1
x2
Perfect factor substitution
y = y1
y = y2
y = y3
x1
x2
No factor substitution
y = y3
y = y2
y = y1
x1



Upper panel:

- Negatively sloped. As you decrease one input it takes increasingly more of the other to hold the output constant.
- Both inputs are essential.



Upper panel:

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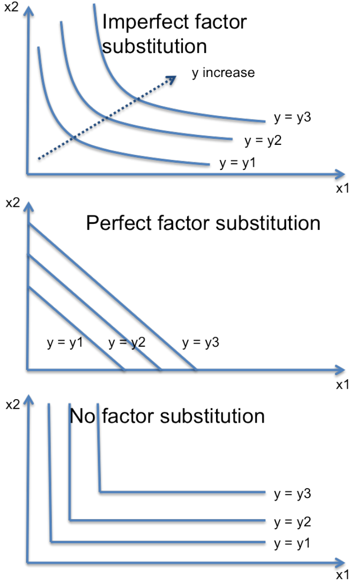
Middle panel:

- Constant MRTS.
- Neither input is essential.



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Lower Panel:

- At the corner, it is impossible to maintain the same amount of output by reducing one of the inputs.
- Both inputs are essential.

Marginal rate of technical substitution (MRTS or RTS)

MRTS is negative one times the slope of the isoquant defined as:

- $MRTS = -\frac{dx_2}{dx_1}$

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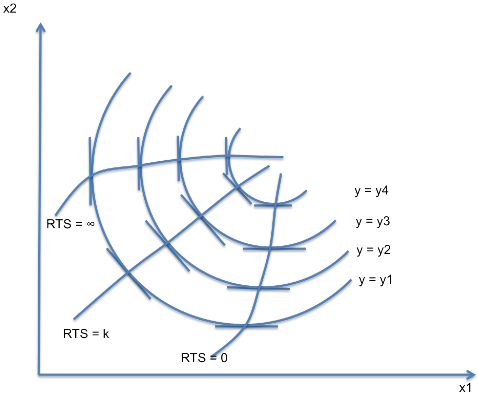
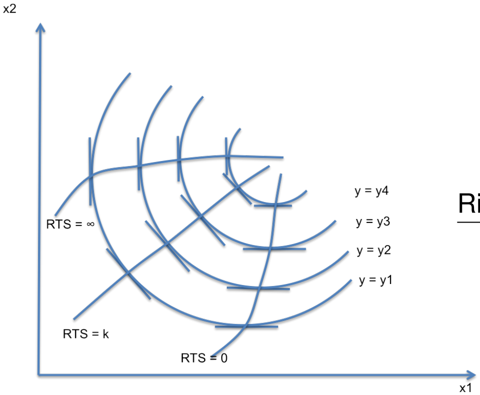
- $dy = f_1 dx_1 + f_2 dx_2$
- $0 = f_1 dx_1 + f_2 dx_2, \qquad$ (holding output constant)
- $-f_2 dx_2 = f_1 dx_1,$
- $-\frac{dx_2}{dx_1} = \frac{f_1}{f_2} = RTS_{12}$

This rather useful result states that the marginal rate of technical substitution is equal to the ratio of the marginal physical products.



Isoclines

- An **isocline** is a line connecting a set of points that have the same RTS.
- $RTS_{12} = k$
- There are an infinite number of isoclines



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Ridgelines:

- The **ridgelines** are the two special isoclines that bound the economically rational region.
- $RTS_{12} = 0$
- $RTS_{12} = \infty$

Elasticity of substitution

Defined as the proportionate rate of change of the input ratio divided by the proportionate rate of change of the MRTS. Formally:

- $\sigma = \frac{d(x_2/x_1)}{(x_2/x_1)} \Big/ \frac{d(f_1/f_2)}{(f_1/f_2)} = \frac{f_1 f_2 (f_1 x_1 + f_2 x_2)}{x_1 x_2 (2 f_1 f_2 f_{12} - f_1^2 f_{22} - f_2^2 f_{11})}$

The elasticity of substitution, σ, (with $0 < \sigma < \infty$) is a unitless measure of the substitution possibilities between factors. At the two extremes:

- $\sigma = 0$
  - Factors are independent.
- $\sigma = \infty$
  - Factors are perfect substitutes.

Later in the course we will see that the marginal condition for cost minimization requires that the MRTS be equal to the price ratio. So σ tells us how input ratios adjust when the relative prices of input change.

The function coefficient

Definition: The **function coefficient**, E, measures the proportional change in output from a fixed proportional change in all inputs.

1. The function coefficient is equal to the sum of all the factor elasticities of production (the total elasticity).
   - $E = \sum_i^N \frac{dy}{y} / \frac{dx_i}{x_i} = \sum_i^N \epsilon_i = \sum_i^N \frac{MP_i}{AP_i}$

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Definition: The **function coefficient**, E, measures the proportional change in output from a fixed proportional change in all inputs.

1. The function coefficient is equal to the sum of all the factor elasticities of production (the total elasticity).
   - $E = \sum_i^N \frac{dy}{y} / \frac{dx_i}{x_i} = \sum_i^N \epsilon_i = \sum_i^N \frac{MP_i}{AP_i}$
2. Also, the function coefficient characterizes the returns to scale.
   - A function exhibits **variable proportional returns**, if $E = f(x_1, x_2)$.
   - A function exhibits **constant proportional returns**, if $E = c$. These class of functions are **homogeneous functions**.
   - A function exhibits **constant returns to scale** if $E = 1$.

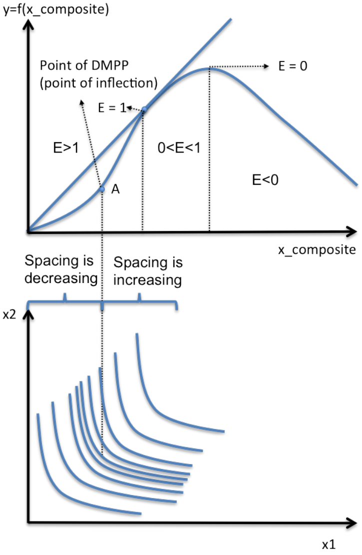
Exercise: (try this at home) Derive the result that $E = \sum_i^N \epsilon_i$

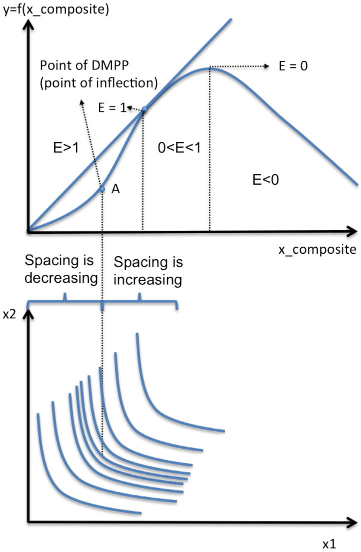
Returns to scale

A production function's **returns to scale** is the percentage increase in output when all inputs are increased by the same percentage. A production function can exhibit

- increasing returns to scale
  - i.e., increasing all inputs by one percent increases output by more than one percent.
- constant returns to scale
  - i.e., increasing all inputs by one percent increases output by one percent.
- decreasing returns to scale
  - i.e., increasing all inputs by one percent increases output by less than one percent.

The function coefficient, returns to scale and spacing of isoquants are very closely related.


y=f(x_composite)
Point of DMPP
(point of inflection)
E = 1
E = 0
E>1
0<E<1
E<0
A
x_composite
Spacing is
decreasing
Spacing is
increasing
x2
x1



- First, note that this function exhibits variable proportional returns
  - Increasing marginal returns up to point A
  - Increasing average returns up to point B.
- Isoquant spacing and marginal returns are directly related
  - isoquant spacing is $\Downarrow$ ($\Uparrow$) if marginal returns is $\Uparrow$ ($\Downarrow$).
- The function coefficient and average returns are directly related
  - $E > 1$ if average returns $\Uparrow$
  - $E < 1$ if average returns $\Downarrow$.

Returns to scale: constant proportional returns

If a production function exhibits constant proportional returns (i.e. if it is a homogeneous function) then the relationship between the function coefficient, returns to scale and isoquant spacing is straightforward:

- increasing returns to scale if $E > 1$, decreasing isoquant spacing.
- constant returns to scale if $E = 1$, constant isoquant spacing.
- decreasing returns to scale if $E < 1$, increasing isoquant spacing.

Homogeneity of production functions

Homogeneous functions are widely in used economic analyses of both production and consumption. In general a function is **homogeneous of degree k** iff it can be expressed as:

- $f(tx_1, tx_2) = t^k f(x_1, x_2)$

For a homogeneous production function the degree of homogeneity is also equal to the function coefficient.

- $E = k$

That is, the degree of homogeneity of a production function determines its returns to scale.

Cobb-Douglas production function

Returning to our favorite example of a Cobb-Douglas function, we will use the tools we have developed to do the following:

$$f(x_1, x_2) = \alpha x_1^{\beta_1} x_2^{\beta_2}; \quad \alpha, \beta_1, \beta_2 > 0$$

1. Find AP, MP, factor elasticity, and total marginal product
2. What type of technical interdependence exists between the two inputs?
3. Derive the isoquant equation & find its ridgelines
4. Derive the elasticity substitution
5. Calculate the function coefficient
6. Is this production function homogeneous?
7. Comment on returns to scale

AP, MP, ϵ_i

For $\{i, j\} \in \{1, 2\}$

- $AP_i = \frac{f(x_i, x_j)}{x_i} = \frac{\alpha x_i^{\beta_i} x_j^{\beta_j}}{x_i} = \alpha x_i^{\beta_i - 1} x_j^{\beta_j}$
- $MP_i = \frac{\partial f(x_i, x_j)}{\partial x_i} = \beta_i \underbrace{\alpha x_1^{\beta_i - 1} x_j^{\beta_j}}_{y/x_i} = \beta_i \frac{y}{x_i} = \beta_i AP_i$
- $\epsilon_i = \frac{MP_i}{AP_i} = \beta_i$

Total marginal product

- $dy = f_i dx_i + f_j dx_j$
- $dy = \beta_i AP_i dx_i + \beta_j AP_j dx_j$

Factor Interdependence

Taking derivative of MP_1 with respect to x_2 we find:

- $\frac{\partial MP_1}{\partial x_2} = f_{12} = \alpha\beta_1\beta_2 x_1^{\beta_1 - 1} x_2^{\beta_2 - 1} > 0, \;\alpha, \beta_1, \beta_2, \forall x_1, \forall x_2 > 0$

This result implies that

- A C-D production function can only be used to represent technically complementary factors.
- If we believe, for example, that the factors are technically competitive, C-D is not a good choice.

Isoquant equation and RTS

To derive the isoquant equation we solve the C-D production function for x_2 while keeping the output level fixed:

- $f(x_1, x_2) = \alpha x_1^{\beta_1} x_2^{\beta_2}$
- $x_2^{\beta_2} = \alpha^{-1} x_1^{-\beta_1} y$
- $x_2 = \alpha^{-1/\beta_2} x_1^{-\beta_1/\beta_2} y^{1/\beta_2}$

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Derive the MRTS by using either $-\frac{\partial x_2}{\partial x_1}$ or $\frac{f_1}{f_2}$

- $\frac{\partial x_2}{\partial x_1} = -\frac{\beta_1}{\beta_2} \alpha^{-1/\beta_2} x_1^{-\beta_1/\beta_2 - 1} (\alpha^{1/\beta_2} x_1^{\beta_1/\beta_2} x_2)$
- $MRTS = -\frac{\partial x_2}{\partial x_1} = \frac{\beta_1}{\beta_2} \frac{x_2}{x_1}$

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2nd approach:

- $MRTS = \frac{f_1}{f_2} = \frac{\beta_1 \frac{y}{x_1}}{\beta_2 \frac{y}{x_2}} = \frac{\beta_1}{\beta_2} \frac{x_2}{x_1}$

Ridgelines

The ridgelines that bound the economic region are:

- $RTS_{12} = \frac{\beta_1}{\beta_2}\frac{x_2}{x_1} = 0$
- $x_2 = 0$

and

- $RTS_{12} = \frac{\beta_1}{\beta_2}\frac{x_2}{x_1} = \infty$
- $x_1 = 0$

Elasticity of substitution

We can rewrite σ as:

- $\sigma = \frac{d(x_2/x_1)}{(x_2/x_1)} / \frac{d(f_1/f_2)}{(f_1/f_2)} = \underbrace{\frac{(f_1/f_2)}{(x_2/x_1)}}_{\text{A}} / \underbrace{\frac{d(f_1/f_2)}{d(x_2/x_1)}}_{\text{B}}$

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Previously we found that $f_1/f_2 = \frac{\beta_1}{\beta_2}\frac{x_2}{x_1}$. Therefore,

- $A = \frac{(f_1/f_2)}{(x_2/x_1)} = \frac{\frac{\beta_1}{\beta_2}\frac{x_2}{x_1}}{\frac{x_2}{x_1}} = \frac{\beta_1}{\beta_2}$

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- $B = \frac{d(\frac{\beta_1}{\beta_2}\frac{x_2}{x_1})}{d(\frac{x_2}{x_1})} = \frac{\beta_1}{\beta_2}$
- $\sigma = A/B = 1 \rightarrow$ constant elasticity of substitution.

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Exercise: (fun algebra to on your own!) Confirm the same result using the formula

$$\sigma = \frac{f_1 f_2 (f_1 x_1 + f_2 x_2)}{x_1 x_2 (2 f_1 f_2 f_{12} - f_1^2 f_{22} - f_2^2 f_{11})}$$

Function coefficient

- $E = \sum_1^2 \epsilon_i$
- $\epsilon_1 = \frac{MP_1}{AP_1} = \frac{\alpha\beta_1 x_1^{\beta_1 - 1} x_2^{\beta_2}}{\alpha x_1^{\beta_1 - 1} x_2^{\beta_2}} = \beta_1$
- $\epsilon_2 = \frac{MP_2}{AP_2} = \frac{\alpha\beta_2 x_1^{\beta_1} x_2^{\beta_2 - 1}}{\alpha x_1^{\beta_1} x_2^{\beta_2 - 1}} = \beta_2$
- $E = \sum_1^2 \epsilon_i = \beta_1 + \beta_2$

Homogeneity and returns to scale

Homogeneity:

- $f(tx_1, tx_2) = \alpha(tx_1)^{\beta_1}(tx_2)^{\beta_2}$
- $f(tx_1, tx_2) = t^{\beta_1+\beta_2}\underbrace{\alpha(x_1)^{\beta_1}(x_2)^{\beta_2}}_{f(x_1,x_2)}$
- $f(tx_1, tx_2) = t^{\beta_1+\beta_2}f(x_1, x_2)$
- $k = \beta_1 + \beta_2 = E$

C-D functions are homogeneous of degree $(\beta_1 + \beta_2)$.

Returns to scale:

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- $k = \beta_1 + \beta_2 = E$

C-D functions are homogeneous of degree $(\beta_1 + \beta_2)$.

Returns to scale:

- if $\beta_1 + \beta_2 > 1$ increasing returns to scale
- if $\beta_1 + \beta_2 = 1$ constant returns to scale
- if $\beta_1 + \beta_2 < 1$ decreasing returns to scale