Applied Economics 8004

Applied Microeconomic Analysis

I. General Economic Equilibrium

> The reader who now tackles this book should bear the following in mind. What he learns is at the heart of modern economic theory. Almost all of the economic literature, theoretical and applied, turns around models in which the nature of "the equilibrium" is discussed and analyzed. –W. Hildenbrand and A.P. Kirman, *Equilibrium Analysis* (1988, p. 49)

1 Introduction

One can gain a considerable level of intuition, from the partial-equilibrium framework, into the way in which markets work. That intuition can sometimes be misleading, though. The direct effect of a price change in one market is only part of the story. There are also indirect effects as the price change causes demand curves in other markets to shift around, possibly feeding back into the first market in unexpected ways. In this brief overview of the theory of general economic equilibrium, we will examine the basic setup of a general-equilibrium economy and aim at an understanding of three main results: the existence of an equilibrium and the first and second theorems of welfare economics.

People have thought about general equilibrium for a long time, at least since Walras published *Elements of Pure Political Economy* in 1874. The "Walrasian" economy and the general-equilibrium approach have two primary features. First, all markets are interrelated and the economy is closed. Second, the many complex elements of the economy are reduced to a set of fundamentals: people or agents, preferences, production technologies, and endowments of goods. In its usual form, markets are assumed to be perfectly competitive. Walras's program is extremely ambitious. It seeks to describe the performance of the entire economy—the behavior of all consumers and producers as well as the outcome in the market for every good.

The general-equilibrium model is sufficiently powerful to describe, if somewhat abstractly, both production and consumption outcomes. Though production is interesting, we won't have time to cover everything in detail. The production economy will be taken up only if we have time at the end, and then only briefly to see how things change. The Arrow-Debreu model can also be amended to account for uncertainty, but we will leave that aside entirely. Realize that the Arrow-Debreu framework is the foundation of one of the workhorse models in macro, the dynamic-stochastic general-equilibrium (DSGE) model.

The examples that appear here have only two consumers, where the assumption that they take prices as given is untenable. The math all scales up to cases with enough consumers that markets can reasonably be taken to be competitive.

2 The Exchange Economy

2.1 Notation

There are n goods, indexed by an i superscript, and m consumers, indexed by a j subscript. An individual is completely described by three elements, which together we will call her *characteristic*. The first is the *commodity space* $X = \mathbb{R}^n_+$, the same for each agent. Commodity bundles are elements $x = (x^1, \ldots x^n) \in X$. The second element is $\omega_j \in X$, j's *initial endowment*. This is the bundle of goods with which the agent begins his or her participation in exchange. The aggregate endowment, which describes the entire resource base of the economy, is $\Omega = \sum_j \omega_j \in \mathbb{R}^n$. This bundle is not to be confused with $\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^{mn}_+$, which gives each consumer's endowment bundle. The third element is preferences $\succcurlyeq_j$ over X, described below. We will often suppress X, which is usually assumed to be $\mathbb{R}^n_+$. A consumer is then fully characterized by the pair $(\omega_j, \succcurlyeq_j)$.

A *price vector* is $p = (p^1, \ldots, p^n) \in \mathbb{R}^n_+$. For convenience we may sometimes assume that prices are strictly positive: $p \in \mathbb{R}^n_{++}$. Suppose each consumer treats prices parametrically, which means markets are competitive. Then j's *budget set* is $\mathcal{B}_j(p, \omega_j) = \{x \in X \mid px \le p\omega_j\}$.

2.2 Preferences

Each consumer's preferences are given by the binary relation $\succcurlyeq_j \subset X \times X$. For any x and y we write $x \succcurlyeq_j y$ if j weakly prefers x to y. Indifference is the symmetric part of $\succcurlyeq_j$, denoted $\sim_j$, and strict preference is the asymmetric part of $\succcurlyeq_j$, denoted $\succ_j$., with $\sim_j \cup \succ_j = \succcurlyeq_j$. Preferences are variously assumed to satisfy one or more of the following conditions.

Definition 1. The preference relation $\succcurlyeq_j$ is **reflexive** if $\forall x \in X$, $x \succcurlyeq_j x$.

Definition 2. The preference relation $\succcurlyeq_j$ is **complete** if $\forall x, y \in X$ with $x \neq y$, $x \succcurlyeq_j y$ or $y \succcurlyeq_j x$.

Definition 3. The preference relation $\succcurlyeq_j$ is **transitive** if $\forall x, y, z \in X$, $[x \succcurlyeq_j y \text{ and } y \succcurlyeq_j z] \Rightarrow x \succcurlyeq_j z$.

Definition 4. The preference relation $\succcurlyeq_j$ is **continuous** if $\forall y \in X$, the upper contour set $\{x \in X \mid x \succcurlyeq_j y\}$ and the lower contour set $\{x \in X \mid y \succcurlyeq_j x\}$ are closed in X.

Different authors make use of different conventions and definitions, but Hildenbrand and Kirman (1988, p. 60) define a *preference relation* as any binary relation satisfying reflexivity, completeness, transitivity, and continuity.

Several other conditions on preferences are common in the literature. Many will be used here. The next four have to do with the way preferences behave as a consumer obtains more of one or all goods. Local nonsatiation rules out bands of indifference, but not upward-sloping indifference curves.

Definition 5. The preference relation $\succcurlyeq_j$ satisfies **local nonsatiation** (LNS) if $\forall x \in X$, $\forall \epsilon > 0$, there exists $y \in B(x, \epsilon)$, the open ball around x, with $y \succ_j x$.

Monotonicity admits bands of indifference and also horizontal or vertical portions of indifference curves, but rules out upward-sloping indifference curves.

Definition 6. The preference relation $\succsim_j$ is **monotonic** if $\forall x, y \in X$, $x \geq y$ implies that $x \succsim_j y$.

The following rules out bands of indifference but still admits horizontal or vertical portions of indifference curves.

Definition 7. The preference relation $\succsim_j$ is **monotone increasing** if $\forall x, y \in X$, $x \gg y$ implies that $x \succ_j y$.

Strong monotonicity rules out both bands of indifference and vertical and horizontal portions of indifference curves.

Definition 8. The preference relation $\succsim_j$ is **strongly monotonic** if, $\forall x, y \in X$, $x \geq y$ and $x \neq y$ together imply that $x \succ_j y$.

Convexity of preferences plays an important role in the analysis of consumer behavior. The following three definitions are increasingly stringent.

Definition 9. The preference relation $\succsim_j$ is **weakly convex** if $\forall x, y \in X$, $\forall t \in [0, 1]$, $x \succsim_j y$ implies that $tx + (1 - t)y \succsim_j y$.

Definition 10. The preference relation $\succsim_j$ is **convex** if $\forall x, y \in X$, $\forall t \in (0, 1)$, $x \succ_j y$ implies that $tx + (1 - t)y \succ_j y$.

Definition 11. The preference relation $\succsim_j$ is **strictly convex** if $\forall x, y \in X$, $\forall t \in (0, 1)$, $x \sim_j y$ implies that $tx + (1 - t)y \succ_j y$.

Weakly convex preferences admit thick indifference curves. Convex preferences rule out thick indifference curves but allow flat sections. Strictly convex preferences rule out both thick indifference curves and flat sections. All three conditions rule out indifference curves that bulge outward from the origin.

It is sometimes convenient to work with utility functions rather than the underlying, more primitive information contained in $\succsim_j$.

Definition 12. A **utility function** for $\succsim_j$ is a continuous function $U_j : X \to \mathbb{R}$ such that, for all $x, y \in X$, $U_j(x) \geq U_j(y)$ if and only if $x \succsim_j y$.

The following result, Theorem 2.1 in Hildenbrand and Kirman, provides conditions guaranteeing the existence of a utility function that represents $\succsim_j$.

Theorem 1. Suppose the preference relation $\succsim_j$ is reflexive, complete, transitive, and continuous. Then there exists a utility function for $\succsim_j$.

An exchange economy, consisting of consumers, preferences, and endowments, is denoted

$$\mathcal{E} = (m, (\succsim_j, \omega_j)_{j=1}^m).$$

When preferences admit of representation by a utility function, the $\succsim_j$ will sometimes be replaced by U_j.

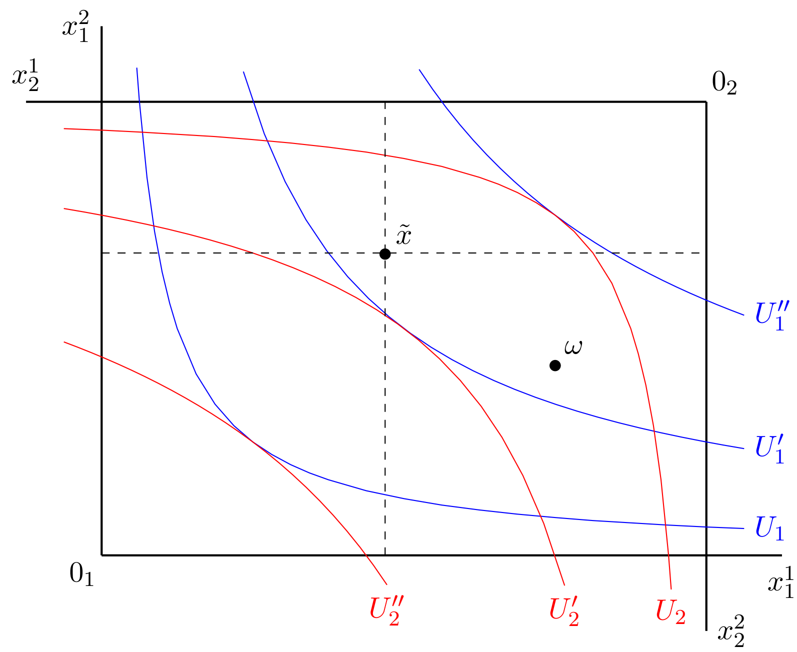

Figure 1: Indifference curves, endowment, and allocation in the Edgeworth box.

Remark 1. As a counterexample to Theorem 1, consider *lexicographic* preferences, according to which for $x, y \in X$ we say that

$$x \succcurlyeq_j y \text{ if } \begin{cases} x^1 > y^1 & \text{or} \\ x^1 = y^1 & \text{and} \quad x^2 > y^2. \end{cases}$$

Upper contour sets are not closed and so these preferences are not continuous. There does not exist a real-valued utility function that represents lexicographic preferences.

Remark 2. We will assume throughout that preferences are continuous.

Remark 3. We will sometimes assume that there is *no free disposal*. This means that the entire aggregate endowment will be consumed: nothing is thrown away.

2.3 The Edgeworth box

We can represent consumer 1's preferences in the Edgeworth box using indifference curves, but consumer 2's indifference curves are in relation to his origin, the northeast corner of the box labeled 0_2. Thus, they are concave rather than convex (relative to 0_1) in Figure 1.

Definition 13. An **allocation** $x = (x_1, \ldots, x_m) \in \mathbb{R}^{mn}$ is an assignment of a consumption bundle to each consumer. The allocation x is **feasible** if there is enough of each good to go around, or if $\sum_j x_j^i \le \sum_j \omega_j^i$ for each good i.

A feasible allocation can be also illustrated in the Edgeworth box. Consumer 1's consumption of good 1 is measured from left to right along the bottom of the box and her consumption of good 2 is measured from bottom to top along the left side of the box. Consumer 2's consumption of good 1 is measured from right to left along the top of the box and his consumption of good 2 is measured from top to bottom along the right side of the box. See allocation $\tilde{x}$ in Figure 1.

2.4 Prices and demand

Pareto optimality has nothing to do with prices. Neither does the core. But prices are at the heart of an equilibrium. Given a price vector and the budget set defined earlier, the consumer's U-max problem is written $\max_{x\in\mathcal{B}_j(p,\omega_j)} U_j(x)$. If preferences are strictly convex, this problem has a unique solution. If they are not, the solution function is a *correspondence*, a mapping that can assign more than one value to a given element of the domain.

Definition 14. For consumer j with preferences $\succsim_j$ and endowment ω_j, the **demand correspondence** is the solution

$$\varphi_j(p,\omega_j) = \{x \in \mathcal{B}_j(p,\omega_j) \mid x \succsim_j y \;\forall\; y \in \mathcal{B}_j(p,\omega_j)\}.$$

Demands are homogeneous of degree 0 (hod 0) in prices. That is, if $t > 0$, for all j and for all $x \in X$ we have $\varphi_j(tp,\omega_j) = \varphi_j(p,\omega_j)$. If demand is homogeneous of degree zero, then any vector of prices $\hat{p}$ can be normalized by dividing each element by a given numerical value. One sometimes encounters normalization on $\hat{p}^1$, so that $p^1 = 1$ and for $i > 1$, $p^i = \hat{p}^i/\hat{p}^1$.

A more common normalization in the theory of general equilibrium is normalization to the unit simplex. This is accomplished by dividing each element of $\hat{p}$ by the sum $\sum_i \hat{p}^i$. For n goods, the relevant simplex, denoted S^{n-1} because it is of one less dimension, is

$$S^{n-1} = \{p \in \mathbb{R}^n_+ \mid \textstyle\sum_i p^i = 1\}.$$

If $n = 2$, the simplex is a line segment connecting the points $(0,1)$ and $(1,0)$. If $n = 3$, the simplex is a triangle with vertices at $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$.

Demand functions are typically depicted in price-quantity diagrams, but they can also be characterized as curves or graphs in multi-demensional quantity space. Suppose preferences for j are strictly convex. Given the budget set $\mathcal{B}_j(p,\omega_j)$, we know that the optimal consumption bundle will be unique. If preferences are also differentiable, it will be determined by the tangency of an indifference curve to the budget line.[1] As the price vector changes, rotating through the endowment vector ω_j (which is always, by definition, on the budget line), the set of tangencies traces out a curve in X. This curve is called the consumer's *offer curve*. It describes vectors of the two goods demanded for any possible price vector, again given ω_j.

[1] Strict convexity does not rule out kinks in indifference curves, so preferences can be strictly convex without being differentiable.

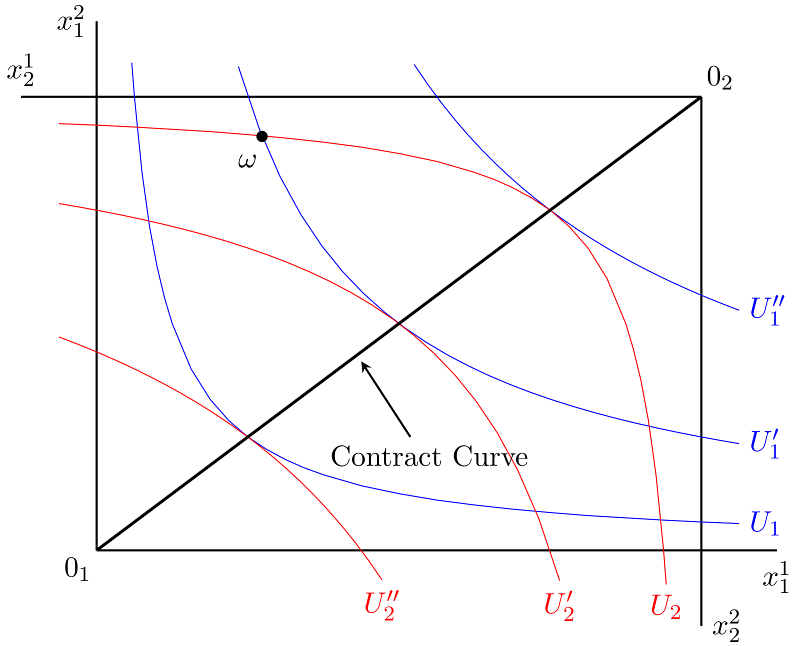

Figure 2: Contract curve for two symmetric Cobb-Douglas consumers.

3 Pareto optimality and the contract curve

It would be nice to have a clear notion of "best" for the entire economy. Which outcomes are desirable? Well, it turns out that the closest to a notion of optimality that general-equilibrium theory takes us is not all that close. It is *Pareto optimality* and it is a weak notion indeed. A given outcome of the economy is Pareto optimal (PO) if no agent can be made better off without someone else being made worse off. To see why this is a weak condition, consider a two-person exchange economy in which both agents have strictly monotonic preferences. That is, more is always better. If one person gets the entire endowment ω and the other person gets nothing at all, we would say POness has been achieved. In order to give the poor person even a little bit, the rich person would be made worse off. Pareto is silent on distributional issues. It does not have anything to do with equity or with who should get what.

We need some definitions, of course.

Definition 15. The allocation x' **Pareto dominates** x if for every j, $x'_j \succcurlyeq_j x_j$ and for at least one j, $x'_j \succ_j x_j$.

Definition 16. The feasible allocation x is **weakly Pareto optimal** (WPO) if there does not exist another feasible allocation x' such that for each j, $x'_j \succ_j x_j$.

Definition 17. The feasible allocation x is **strongly Pareto optimal** (SPO) if there does not exist another allocation x' such that (i) x' is feasible and (ii) x' Pareto dominates x.

The two definitions differ if, for example, one consumer has a band of indifference.

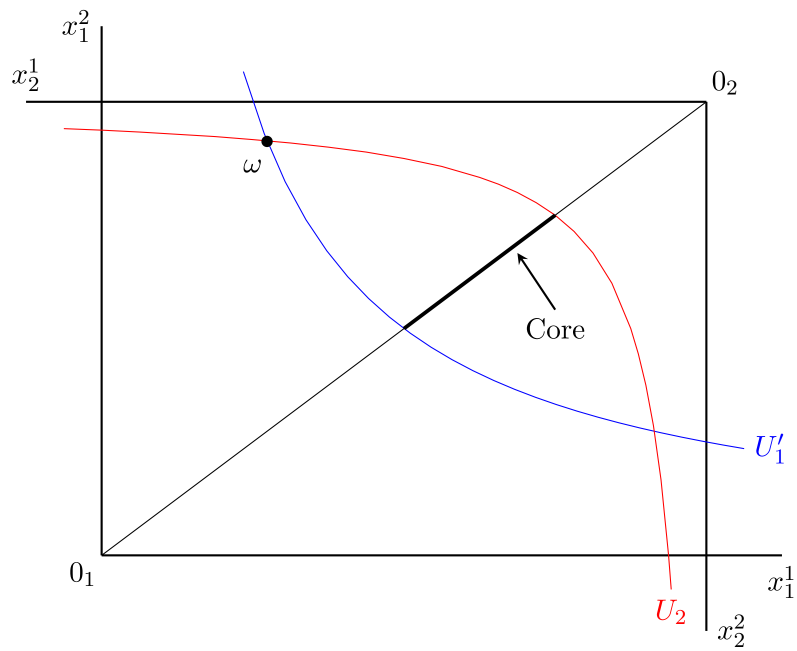

Figure 3: The core for two symmetric Cobb-Douglas consumers.

Given an exchange economy, the *contract curve* is the set of allocations in the Edgeworth box that are Pareto optimal. In a two-person, two-good exchange economy, if utility functions are differentiable and if demanded quantities are always strictly positive, an allocation is on the contract curve if it is the joint tangency of two indifference curves in the Edgeworth box. In this special differentiable case, with two goods, an allocation is Pareto optimal if for each consumer j, the marginal rate of substitution of good 1 for good 2, $\mathrm{MRS}_j^{1,2}$, is equal. Figure 2 illustrates the contract curve as the set of tangencies between, in this case, symmetric Cobb-Douglas indifference curves (that is, $U_j = x_j^1 x_j^2$) for both consumers. Notice that from the endowment, ω, any allocation within the lens-shaped region bounded by the two indifference curves through ω represents a Pareto improvement relative to ω. Any allocation not on the contract curve is Pareto dominated; we say that it is inefficient.

Note that Mas-Colell, Whinston, and Green use different and somewhat idiosyncratic terminology here. For them, the object that I (and many others, including Hindenbrand and Kirman) call the contract curve is called the "Pareto set." The object that they call the contract curve, depicted as the bold segment in Figure 3, I (and many others) call the *core*. For a two-person economy, an allocation $\tilde{x}$ is in the core if (i) $\tilde{x}$ is Pareto optimal and (ii) for each consumer j, $U_j(\tilde{x}_j) \geq U_j(\omega_j)$. For an m-person economy, the core must also ensure that no coalition, acting on its own, could secure a better outcome for each of its members than at the candidate allocation in question. The connection between the core of an economy and the set of equilibrium allocations is an important part of the theory of general equilibrium that we will touch on if time permits.

4 Optimality: an example

In this section we pause to compute the set of Pareto-optimal allocations for a 2×2 example exchange economy. Consumer 1 has strictly convex preferences given by the utility function $U_1 : X \longrightarrow \mathbb{R}$ with $U_1(x_1^1, x_1^2) = x_1^1 x_1^2$. Her endowment is $\omega_1 = (12, 4)$. Consumer 2 has utility function $U_2(x_2^1, x_2^2) = 3 \ln x_2^1 + x_2^2$ and endowment of $\omega_2 = (4, 8)$.

Let us now compute the set of PO allocations. Mathematically, one may solve for the contract curve in more than one way. I think the following method is the most straightforward: maximize consumer 1's utility subject to (i) an arbitrary mininum level of $\bar{U}_2$ and (ii) the resource constraints. The problem is given as

$$\begin{aligned} &\max_{x_1^1, x_1^2} \; x_1^1 x_1^2 \\ \text{s.t.} \quad & \bar{U}_2 \leq 3 \ln x_2^1 + x_2^2 \\ & x_2^1 = 16 - x_1^1 \\ & x_2^2 = 12 - x_1^2. \end{aligned}$$

The resulting Lagrangian function, after inserting the last two constraints into the first, is

$$\mathcal{L}(x, \lambda; \omega) = x_1^1 x_1^2 + \lambda \left(\bar{U}_2 - 3 \ln(16 - x_1^1) - (12 - x_1^2) \right),$$

with interior FONCs

$$\mathcal{L}_1 = x_1^2 + \frac{3\lambda}{16 - x_1^1} = 0 \qquad \text{and} \qquad \mathcal{L}_2 = x_1^1 + \lambda = 0.$$

These may be solved to obtain an expression for *the interior part* of the contract curve:

$$x_1^2(x_1^1) = \frac{3x_1^1}{16 - x_1^1}. \tag{1}$$

Be careful here. One of the features of quasi-linear utility, which is what our consumer 2 has, is that indifference curves can meet one of the axes with finite slope. (Don't rely always on the Cobb-Douglas function for your intuition!) In this case, at low levels of utility 2's indifference curves meet the horizontal axis in the range of consumption represented by the box. (In fact, one can show that for x_2^1 sufficiently high, any indifference curve eventually meets the horizontal axis at finite slope.) Set $x_1^2 = 12$ in (1) and solve to find $x_1^1 = 12.8$. The contract curve meets the top of the box at $(12.8, 12)$ relative to 0_1 or, equivalently, $(3.2, 0)$ relative to 0_2.

Figure 4 illustrates. There, we see several indifference curves for each consumer and their tangencies along the bold black curve, from equation (1), that characterizes the interior portion of the contract curve. The line segment along the rightmost portion of the top of the box represents Pareto-optimal allocations at which indifference curves are not tangent.

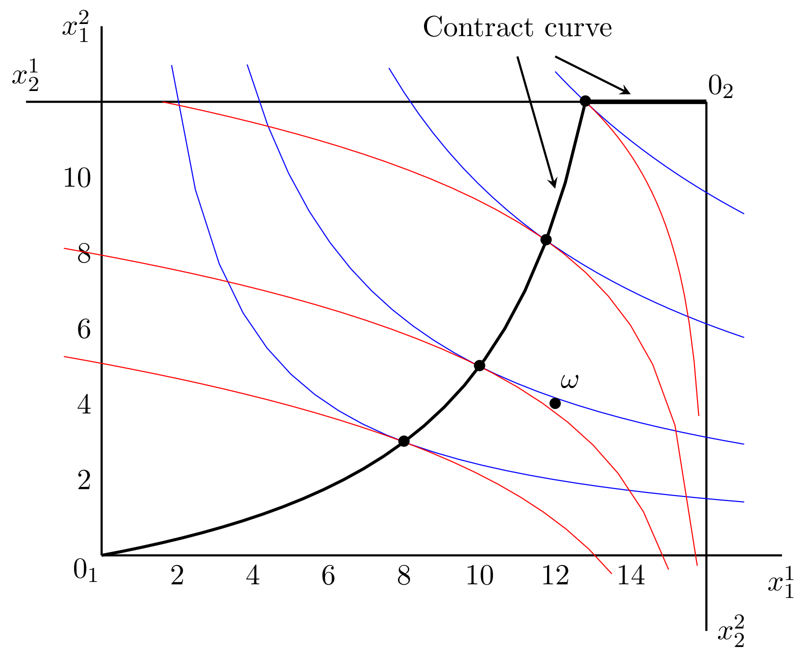

Figure 4: Contract curve in two parts.

5 Equilibrium and Walras's law

The definition of a *Walrasian equilibrium* may be stated in different ways. In this first version it is a price vector and an allocation at which consumers optimize and markets clear.

Definition 18 (Walrasian equilibrium I)**.** Given an exchange economy consisting of a set of agents $J = \{1, \dots m\}$, their endowment vectors ω_j and their preferences $\succsim_j$, $j = 1, \dots m$, a **Walrasian equilibrium** is a pair $(p^*, x^*) \in \mathbb{R}^{n+nm}$ such that

i.) the allocation is feasible ($\sum_j \varphi_j^i(p^*, \omega_j) = \sum_j \omega_j^i$ for each good i), and
ii.) x_j^* is optimal for each agent j (if $x_j' \succ_j x_j^*$, then $p^* x_j' > p^* \omega_j$).

Because preferences are continuous, demands are also continuous. For each consumer, for each good, we can define *excess demand* for good i as the difference between the consumer's demanded quantity for the good and her endowment of the good:

$$z_j^i(p, \omega_j) = \varphi_j^i(p, \omega_j) - \omega_j^i.$$

Aggregate excess demand for good i is the sum of z_j^i over all j:

$$z^i(p, \omega) = \sum_j z_j^i(p, \omega_j),$$

where $\omega^i = \sum_j \omega_j^i$ is the total endowment of good i and $\omega = (\omega^1, \dots \omega^n)$ is the vector of aggregate endowment. The vector of excess demands is denoted $z(p, \omega) = (z^1(p, \omega), \dots, z^n(p, \omega))$.

Because ω is given exogeneously, the function $z(p, \omega)$ will usually be expressed as $z(p)$, with dependence on ω understood. The value of excess demand *at any price vector* must be zero. This fundamental fact is given the hallowed designation of a law. We state it in two forms, depending upon whether each consumer's budget is required to be satisfied as an equality. That is, whether free disposal is possible. (The weak form can apply if preferences are monotonic but neither strongly monotonic nor LNS.)

Walras's Law (strong form). Suppose that, for each j, $\succcurlyeq_j$ satisfies local nonsatiation. For any price vector p, it must be true that $p \cdot z(p) = 0$.

Proof. Let p denote the vector of prices and $z(p)$ the vector of excess demands. Now multiply p and z together:

$$\begin{aligned} p \cdot z(p) &= p\Big[\sum_j \varphi_j(p, \omega_j) - \sum_j \omega_j\Big] \\ &= \sum_j \big[p \cdot \varphi_j(p, \omega_j) - p \cdot \omega_j\big] = 0 \end{aligned}$$

The last equality must hold because the term in square backets is simply the budget constraint for j. By local nonsatiation it has to be zero. □

Walras's Law (weak form). Suppose that, for each j, $\succcurlyeq_j$ is monotonic. For any price vector p, it must be true that $p \cdot z(p) \leq 0$.

An easy consequence of Walras's law is this. If $p^i > 0$ for any good i, then $z^i(p) = 0$, or the law would be violated.

An alternative definition, more in the spirit of the mathematical elegance of the theory, is the following. The optimality of the allocation associated with the equilibrium price is implicit here.

Definition 19 (Walrasian equilibrium II)**.** The price vector p^* is an equilibrium price vector if $z(p^*) \leq 0$.

A second easy consequence of Walras's law is that if p^* is the price associated with a Walrasian equilibrium and $z^i(p^*) < 0$, then $p^{j*} = 0$. The only way a good can be in excess supply at an equilibrium is if it is free.

6 Equilibrium: the example continued

In this section we continue the 2×2 example introduced earlier, but now the goal is to derive offer curves and find the equilibrium prices and allocation. Recall that consumer 1 has utility $U_1(x_1^1, x_1^2) = x_1^1 x_1^2$ and endowment $\omega_1 = (12, 4)$. We may write her optimization problem as $\max_{x_1^1, x_1^2} x_1^1 x_1^2$ subject to $p^1 x_1^1 + p^2 x_1^2 = 12p^1 + 4p^2$. The Lagrangian is

$$\mathcal{L}(x_1^1, x_1^2, \lambda; p^1, p^2, \omega_1) = x_1^1 x_1^2 + \lambda(12p^1 + 4p^2 - p^1 x_1^1 - p^2 x_1^2).$$

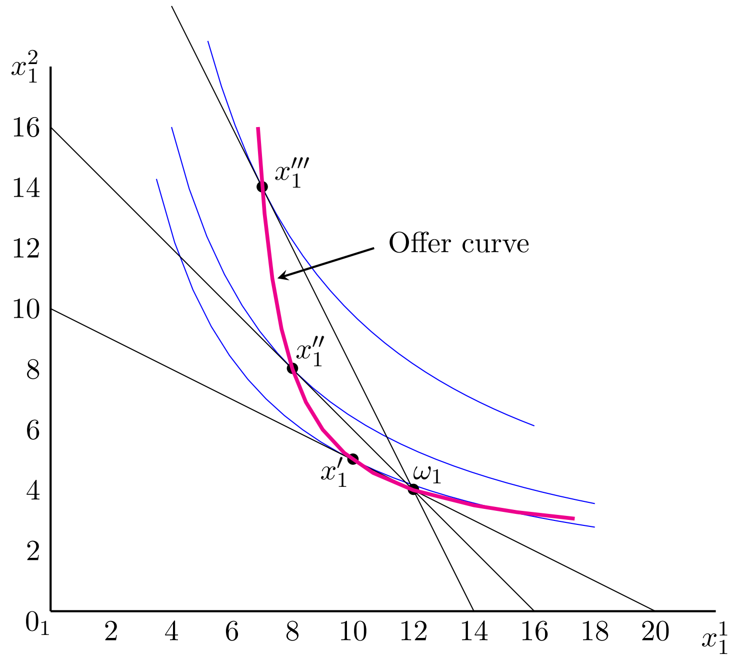

Figure 5: Offer curve for consumer 1.

The FONCs for an interior solution to this problem are

$$
\begin{aligned}
\mathcal{L}_1 &= x_1^2 - \lambda p^1 = 0 \\
\mathcal{L}_2 &= x_1^1 - \lambda p^2 = 0 \\
\mathcal{L}_\lambda &= 12p^1 + 4p^2 - p^1 x_1^1 - p^2 x_1^2 = 0.
\end{aligned}
$$

Solving the first two equations for λ and setting the results equal to each other, we obtain

$$x_1^2 = x_1^1(p^1/p^2). \tag{2}$$

Plug this into the constraint to get

$$p^1 x_1^1 + p^2 x_1^1 \frac{p^1}{p^2} = 12p^1 + 4p^2.$$

Solve for x_1^1 to get

$$x_1^1(p^1, p^2, \omega_1) = 6 + \frac{2p^2}{p^1}. \tag{3}$$

From equation (2) we can now obtain

$$x_1^2(p^1, p^2, \omega_1) = 2 + \frac{6p^1}{p^2}. \tag{4}$$

Because these demand functions, as expected, are hod 0 in prices (notice that there is no income

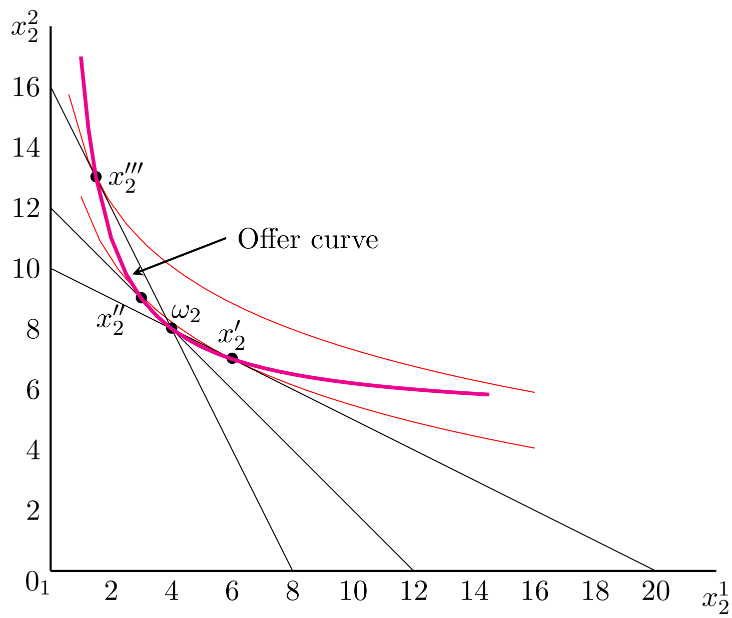

Figure 6: Offer curve for consumer 2.

variable here; price times ω_j is income and we've plugged in numbers for the ω_j^i), we can "normalize" prices in any way we choose. This does not change demand. Therefore, divide each price by the sum of the prices, so that the resulting price vector sums to 1. Consider now three price vectors: $p' = (1/3, 2/3)$, $p'' = (1/2, 1/2)$, and $p''' = (2/3, 1/3)$. The slopes of the resulting budget lines, all of which must pass through $\omega_1 = (12, 4)$, are $-1/2$, -1, and -2 respectively. From equations (3) and (4), we find that the resulting demanded bundles are

$$x_1' = (10, 5), \quad x_1'' = (8, 8), \quad \text{and} \quad x_1''' = (7, 14).$$

Figure 5, with a set of tangent indifference curves in blue and the offer curve shown in pink, illustrates.

Notice that the endowment point $(12, 4)$ is itself on the offer curve. This will be true except in unusual cases with endowments on the boundary of the box or with peculiar preferences. A good practice problem is to determine the price pair at which ω_1 is itself the demanded bundle.

We now add consumer 2, who you'll recall has utility function $U_2(x_2^1, x_2^2) = 3 \ln x_2^1 + x_2^2$ and endowment $\omega_2 = (4, 8)$. His demand functions are

$$x_2^1(p^1, p^2, \omega_2) = \frac{3p^2}{p^1} \quad \text{and} \quad x_2^2(p^1, p^2, \omega_2) = 5 + \frac{4p^1}{p^2}.$$

Again consider the three price vectors we used earlier: $p' = (1/3, 2/3)$, $p'' = (1/2, 1/2)$, and $p''' = (2/3, 1/3)$. These prices generate budget lines that pass through $\omega_2 = (4, 8)$, and have slope

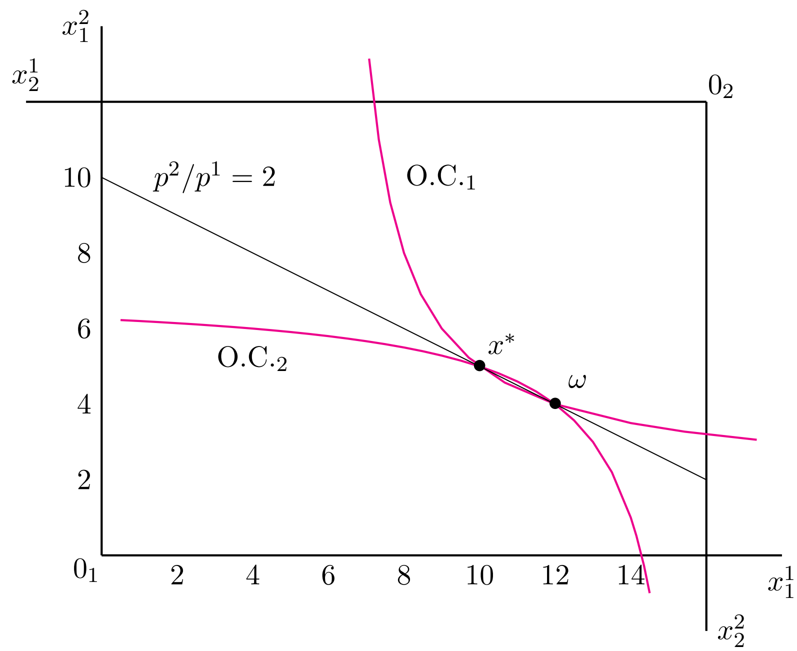

Figure 7: Offer curves cross at the equilibrium allocation.

$-1/2$, -1, and -2 respectively. The resulting demanded bundles are

$$x_2' = (6,7), \quad x_2'' = (3,9), \quad \text{and} \quad x_2''' = (1.5, 13).$$

Figure 6, again with the offer curve in pink, illustrates consumer 2's optimal behavior. Notice that the consumer's utility is almost identical for the first two price vectors, $p' = (1/3, 2/3)$ and $p'' = (1/2, 1/2)$. This explains why there are only two indifference curves in the figure.

Now we want to solve for the equilibrium price and allocation. Putting the two offer curves together in the Edgeworth box, we find the equilibrium allocation at the point where they cross. To compute the equilibrium, we begin by solving for the equilibrium prices, remembering that all we care about is the relative prices. Consumer 1's demand for x^1, as a function of prices only, is $x_1^1(p) = 6 + 2(p^2/p^1)$. Consumer 2's demand for x^1 is $x_2^1(p) = (3p^2/p^1)$. We know that at an equilibrium, $x_1^1 + x_2^1 = 16 = \omega^1$. Here is the algebra:

$$\left(6 + 2\frac{p^2}{p^1}\right) + \left(\frac{3p^2}{p^1}\right) = 16,$$

or

$$5\frac{p^2}{p^1} = 10, \quad \text{or} \quad \frac{p^2}{p^1} = 2.$$

The equilibrium solution, depicting offer curves, the endowment point, the allocation (x_1^*, x_2^*) where the offer curves cross, and the price line, appears in Figure 7. Because we care only about relative prices, any set of values for p^2 and p^1 will work here, so long as their ratio is 2.[2]

The equilibrium allocation is found by plugging these prices into the demand functions as follows.

$$x_1^{1*} = 6 + 2 \cdot 2 = 10, \qquad x_1^{2*} = 2 + 6 \cdot \frac{1}{2} = 5,$$
$$x_2^{1*} = 3 \cdot 2 = 6, \qquad x_2^{2*} = 5 + 4 \cdot \frac{1}{2} = 7.$$

As you can easily see, the quantities of both goods sum to the endowed values. Figure 8 again shows the equilibrium allocation $x^* = (x_1^*, x_2^*)$ and the price line through ω and x^*. Both the endowment point and the equilibrium allocation always lie on the budget line determining the equilibrium prices. But here, instead of showing the two offer curves we instead have indifference curves at x^*, one for each consumer. These are not the same as the offer curves! Offer curves cross at x^*. Figure 8 shows that indifference curves are tangent at x^*. It also shows the contract curve derived earlier.

7 Walras's Law illustrated

We can also explore and illustrate Walras's law using this example. For convenience, demands for the two consumers are repeated here:

$$\begin{aligned} &\text{Consumer 1}: \quad x_1^1(p^1, p^2, \omega_1) = 6 + \frac{2p^2}{p^1}, \quad x_1^2(p^1, p^2, \omega_1) = 2 + \frac{6p^1}{p^2} \\ &\text{Consumer 2}: \quad x_2^1(p^1, p^2, \omega_2) = \frac{3p^2}{p^1}, \qquad x_2^2(p^1, p^2, \omega_2) = 5 + \frac{4p^1}{p^2}. \end{aligned}$$

Excess demands, suppressing the ω_j as arguments, are

$$\begin{aligned} z^1(p) &= x_1^1(p) + x_2^1(p) - \omega^1 = -10 + \frac{5p^2}{p^1} \quad \text{and} \\ z^1(p) &= x_1^2(p) + x_2^2(p) - \omega^2 = -5 + \frac{10p^1}{p^2}. \end{aligned}$$

Consider the three price vectors and one more: $p^* = (1/3, 2/3)$, $p'' = (1/2, 1/2)$, $p''' = (2/3, 1/3)$, and $p'''' = (1/5, 4/5)$. Excess demand quantities are

$$\begin{aligned} p^* &: \quad z^1 = -10 + 10 = 0, \qquad z^2 = -5 + 5 = 0, \\ p'' &: \quad z^1 = -10 + 5 = -5, \qquad z^2 = -5 + 10 = 5, \\ p''' &: \quad z^1 = -10 + 2.5 = -7.5, \quad z^2 = -5 + 20 = 15, \\ p'''' &: \quad z^1 = -10 + 20 = 10, \qquad z^2 = -5 + 2.5 = -2.5. \end{aligned}$$

[2] If, for example, we want the prices to sum to 1, we can set $p^{1*} = 1/3$, so that $p^{2*} = 2/3$.

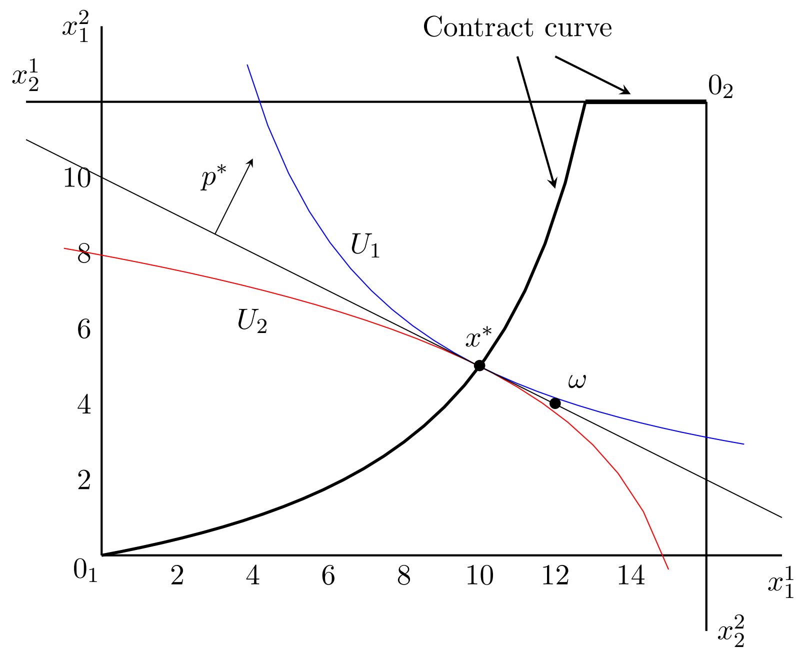

Figure 8: Indifference curves are tangent at the equilibrium allocation.

Perform the vector multiplication to confirm that Walras's law works every time.

$$
\begin{aligned}
p^* \cdot z(p^*) &= (1/3, 2/3) \cdot (0, 0) = 0, \\
p'' \cdot z(p'') &= (1/2, 1/2) \cdot (-5, 5) = -2.5 + 2.5 = 0, \\
p''' \cdot z(p''') &= (2/3, 1/3) \cdot (-7.5, 15) = -5 + 5 = 0, \\
p'''' \cdot z(p'''') &= (1/5, 4/5) \cdot (10, -2.5) = 2 - 2 = 0.
\end{aligned}
$$

The price vectors and excess demands are illustrated in Figure 9. Notice that each pair of arrows of the same color is at right angles in the figure, as is required for any two vectors whose dot product is $x \cdot y = 0$. Notice also that for the price vectors below p^* along the simplex, that is for p'' and p''' where p^1 is greater than its equilibrium level, excess demand is negative for x^1 and positive for x^2. At p'''', which has p^1 less than its equilibrium level, excess demand is positive for x^1 and negative for x^2.

8 The First Theorem of Welfare Economics

The fact that x^* lies along the contract curve, that indifference curves are tangent there, is no accident. Indeed, it's an illustration for our example of the First Welfare Theorem. The theorem states that any Walrasian equilibrium must be Pareto optimal. The proof of this result is instructive and surprisingly, not to say encouragingly, straightforward. Harder work is coming soon.

Theorem 2 (First Theorem of Welfare Economics)**.** Suppose (p^*, x^*) is a Walrasian equilibrium

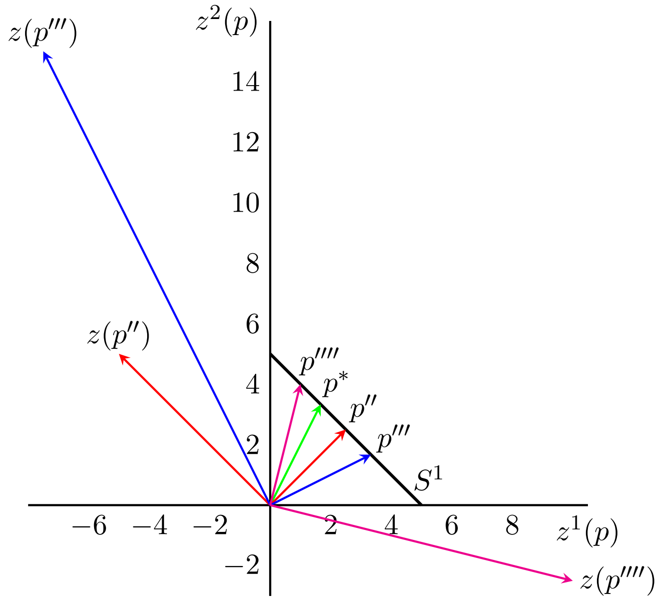

Figure 9: Walras's law means that price and excess demand are orthogonal at every price vector. Equilibrium price is p^*, where excess demand is zero.

for an exchange economy. If preferences are locally nonsatiated, then x^* is strongly Pareto optimal.

Proof. By way of contradiction, suppose that x^* is not Pareto optimal. Then there is another feasible allocation, say x, that is weakly preferred to x^* by all consumers and strictly preferred by one, say k. At an equilibrium we have $p^* \sum_j x_j^* = p^* \sum_j \omega_j$. We know that

$$p^* x_j \geq p^* x_j^* \quad \text{for } j = 1, \ldots m. \tag{5}$$

(To be slightly pedantic, otherwise suppose not: $p^* x_j < p^* x_j^*$ for some j. Then by LNS, for any $\epsilon > 0$ there is x_j' with $\|x_j' - x_j\| < \epsilon$ and $x_j' \succ x_j \succcurlyeq x_j^*$. Thus $p^* x_j' < p^* x_j + \|p^*\|\epsilon$ and for ϵ sufficiently small we have $p^* x_j' \leq p^* x_j^*$ and, $x_j' \succ x_j^*$, contradicting that x_j^* was optimal at p^*. We conclude that $p^* x_j \geq p^* x_j^*$.)

By the definition of equilibrium we also have that

$$p^* x_k > p^* x_k^*. \tag{6}$$

Combine (5) and (6) and sum over j to get

$$p^* \sum_j x_j > p^* \sum_j x_j^*. \tag{7}$$

Because x is feasible, we have $\sum_j \omega_j \geq \sum_j x_j$ and so

$$p^* \sum_j \omega_j \geq p^* \sum_j x_j. \tag{8}$$

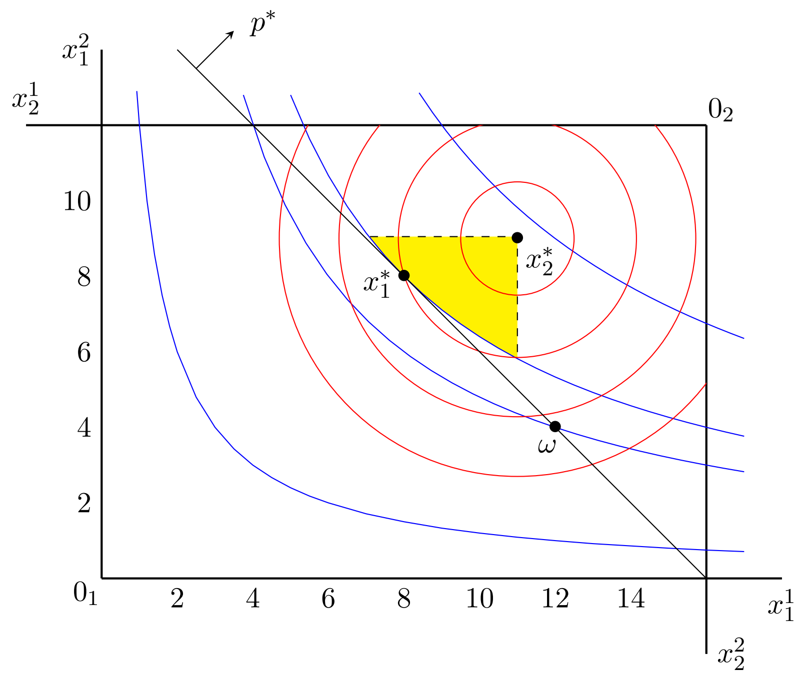

Figure 10: First welfare theorem doesn't hold with satiated preferences.

Because $p^* \sum_j x_j^* = p^* \sum_j \omega_j$, equations (7) and (8) can be combined to obtain

$$p^* \sum\nolimits_j x_j > p^* \sum\nolimits_j x_j^* = p^* \sum\nolimits_j \omega_j \geq p^* \sum\nolimits_j x_j,$$

a contradiction. We conclude that x^* is Pareto optimal. □

8.1 Anomaly 1: first welfare theorem doesn't apply

Consider what goes wrong in the first welfare theorem if one of the consumers has a satiation point or "bliss point," a bundle at which utility reaches a global maximum. Such a consumer may not spend all of her income and we cannot expect that excess demand will be zero. Suppose that consumer 1 has Cobb-Douglas utility $U_1(x_1^1, x_1^2) = x_1^1 x_1^2$. Nothing unusual here. Her endowment is $\omega_1 = (12, 4)$. But consumer 2, with endowment $\omega_2 = (4, 8)$, has very strange preferences. They violate every variant of montotonicity and also, at one particular bundle, local nonsatiation. These preferences are quadratic, given by

$$U_2(x_2^1, x_2^2) = -(x_2^1 - 5)^2 - (x_2^2 - 3)^2.$$

Consumer 2's preferences are satiated at the bundle $x_2^* = (5, 3)$. Any other bundle leads to lower utility. The utility surface is a parabaloid opening downward, with a maximum at x_2^*. Indifference curves are circles centered at x_2^*. The farther away this person gets from his bliss point, the lower his utility.

The situation is depicted in the Edgeworth box shown in Figure 10. Let us consider whether the combination of $p^* = (1/2, 1/2)$ and the allocation given by $x_1^* = (8, 8)$ and $x_2^* = (5, 3)$ is an equilibrium. Yes, it is. Consumer 1 optimizes, and so does consumer 2. Also, there is enough of each good to go around: $x_1^{1*} + x_2^{1*} = 13 < \omega^1$ and $x_1^{2*} + x_2^{*2} = 11 < \omega^2$. Some of each good is being thrown away at this equilibrium allocation, so the assumption of no free disposal is violated.

But the equilibrium is not Pareto optimal. The reason is that we can make consumer 1 better off, by leaving 2 at x_2^* and shifting x_1 to any point in the yellow shaded region, without reducing 2's utility and without violating the feasibility condition. Any x_1 in that region, together with x_2^*, is both Pareto superior to x^* and also feasible given ω. So the allocation in question is a Walrasian equilibrium but it is not Pareto optimal.

Keep in mind some important facts about this example. First, the example economy has many equilibria, but none of the equilibrium allocations is PO. The first welfare theorem does not fail though. Rather, the example shows how the conclusion of a theorem may not be obtained if one of the required assumptions is not satisfied.

9 The Second Theorem of Welfare Economics

The second welfare theorem turns the first on its head. It says that any PO allocation can be the basis for a Walrasian equilibrium for some vector of prices and possibly after a suitable redistribution of the endowment. The proof of the second welfare theorem relies upon some version of a *separating-hyperplane theorem*. The following version is about the simplest. It is applicable in $\mathbb{R}^n$; other versions apply to more general topological spaces. For any set $A \subset \mathbb{R}^n$, its *closure*, denoted $\mathrm{cl}(A)$ is the set together with all its boundary points: $\mathrm{cl}(A) = A \cup \partial A$.[3]

Theorem 3 (Separating-hyperplane theorem)**.** Suppose that $A \subset \mathbb{R}^n$ is open and convex and consider a vector $x \in \mathbb{R}^n$ with $x \notin A$. There exists $p \neq 0$ such that $p \cdot a \geq p \cdot x$ for all $a \in \mathrm{cl}(A)$.

The second fundamental theorem of welfare economics relies crucially upon the idea of a separating hyperplane.

Theorem 4 (Second Welfare Theorem)**.** Consider an exchange economy $\mathcal{E} = \{m, (\succcurlyeq_j, \omega_j)_{j=1}^m\}$ in which $\Omega = \sum_j \omega_j \gg 0$ and in which, for each consumer j, preferences are (i) continuous; (ii) convex; and (iii) strongly monotonic. If $x^* \in \mathbb{R}^n_{++}$ is Pareto optimal then there exists a price vector $p^* \in \mathbb{R}^n_{++}$ and an associated endowment vector $\tilde{\omega}$ with $\sum_j \tilde{\omega}_j = \sum_j \omega_j$ such that (p^*, x^*) is a Walrasian equilibrium for the economy $\tilde{\mathcal{E}} = \{m, (\succcurlyeq_j, \tilde{\omega}_j)_{j=1}^m\}$.

Proof. We begin by assuming that the initial endowment is redistributed (miraculously, by some exogenous power) so that $\tilde{\omega}_j = x_j^*$ for each j. The proof proceeds in three steps:

1. Define the aggregate strict upper set A relative to x^*. Convexity of preferences guarantees that A is convex. It is also open;
2. Invoke the separating-hyperplane theorem: $\exists\, p^*$ with $p^* \cdot a \geq p^* \cdot \Omega$ for all $a \in A$. Show that $p^* \gg 0$;

[3]Given a set $A \subset \mathbb{R}^n$, the vector $x \in \mathbb{R}^n$ is an element of the boundary of A, denoted ∂A, if for any $\epsilon > 0$, the ball $B(x, \epsilon) = \{y \in \mathbb{R}^n \mid \|y - x\| < \epsilon\}$ contains both elements of A and elements of A^c.

3. Show that (p^*, x^*) is a W.E. for $\tilde{\mathcal{E}}$.

Step 1. Define A_j as the strict upper contour set for j relative to x_j^*:

$$A_j = \{a_j \in \mathbb{R}^n_+ \mid a_j \succ_j x_j^*\}.$$

Because $\succsim_j$ is convex, A_j is convex. It is also nonempty and open, and clearly $x_j^* \notin A_j$. Define

$$A = \sum_{j=1}^{m} A_j = \left\{a \in \mathbb{R}^n \mid \exists\, a_1 \in A_1, \ldots, a_m \in A_m \text{ with } a = \sum_{j=1}^{m} a_j\right\}. \tag{9}$$

This is the set of aggregate endowment vectors that could be distributed among the agents so that each is strictly better off than at x_j^*. Being the sum of nonempty, open, and convex sets, A is nonempty, open, and convex.

Step 2. By the separating hyperplane theorem there is a price vector $p^* \neq 0$ with $p^* \cdot a \geq p^* \cdot \Omega$ for all $a \in \text{cl}(A)$. Furthermore, we claim that $p^* \gg 0$. Let $e_k = (0, \ldots, 0, 1, 0, \ldots 0)$ be the unit vector in $\mathbb{R}^n$ and let $\mathbf{1}$ be the vector of all ones. By strong monotonicity, the bundle $\Omega + e_k \in A$ because the increment of extra good k can be distributed across people to make everyone better off than at x_j^*. This means $p^* \cdot (\Omega + e_k) \geq p^* \cdot \Omega$, or $p^* \cdot e_k \geq 0$ for any k. Thus, $p^* \geq 0$ in all its elements. To see that, indeed, $p^* \gg 0$, consider $\Omega + e_k - \varepsilon\mathbf{1}$ for some small $\varepsilon > 0$. By continuity of preferences and strong monotonicity, and because $\Omega \gg 0$, using the same argument as before, we know there is ε small enough so that $p^* \cdot (\Omega + e_k - \varepsilon\mathbf{1}) \geq p^* \cdot \Omega$ or, equivalently, $p^* \cdot e_k \geq \varepsilon p^* \cdot \mathbf{1} > 0$. Therefore $p^* \gg 0$.

Step 3. The claim is that this price vector supports x^* as a Walrasian equilibrium. By feasibility, $x^*(= \tilde{\omega})$ clears markets. All that remains is to show that x_j^* is optimal for every j. Fix prices at p^*. Consider consumer j and suppose, by way of contradiction, that there is $x'_j \in \mathbb{R}^n_+$ such that $p^* x'_j \leq p^* x_j^*$ (x'_j is affordable) and $x'_j \succsim_j x_j^*$. By continuity of preferences, we know that there is $\lambda \in (0, 1)$ and sufficiently close to 1 so that $\lambda x'_j \succsim_j x_j^*$. This means that $\lambda x'_j \in A_j$. But then, by the separating hyperplane theorem result, we have that

$$p^* \cdot (\Omega - (x_j^* - \lambda x'_j)) \geq p^* \cdot \Omega, \tag{10}$$

which may be simplified as

$$p^* \cdot \lambda x'_j \geq p^* \cdot x_j^*.$$

Because $p^* \gg 0$ and $x_j^* \geq 0$ and $\lambda < 1$, we have that $p^* \cdot x'_j > p^* \cdot x_j^*$, contradicting that x'_j is affordable. We conclude that x_j^* must be optimal for each j and so (p^*, x^*) is a Walrasian equilibrium for $\tilde{\mathcal{E}}$. This completes the proof. $\square$

The red bits in the proof are to indicate where we used each of the four key conditions.

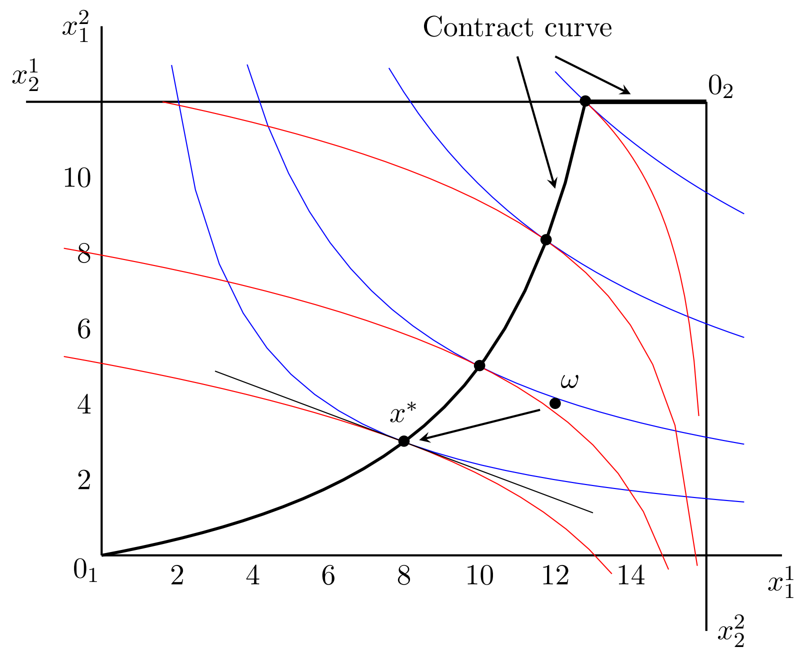

Figure 11: Endowment redistributed to PO allocation x^*.

10 Illustrating WTII in the example

We may use the example to explore how the Second Welfare Theorem works, and to gain insights into the proof. Consider the allocation $x^* = (8, 3, 8, 9)$ in Figure 11, which is *not* the equilibrium allocation derived earlier. Because x^* is Pareto optimal and also satisfies $x_j^* \gg 0$, WTII ensures that there is a set of prices for which it is the equilibrium allocation, given a suitable redistribution of resources. In our proof we first assumed that the initial endowment vector Ω is redistributed so that x^* itself is the new translated endowment. An arrow depicts this translation. Because x^* is optimal, as expected we see that the two indifference curves through that point are tangent. A line segment drawn carefully through x^* suggests there must be a price vector, determining the slope of that line, that would make x_j^* the optimal choice for each j. The suggestion is true, as the proof confirms.

First disassemble the box, placing the two bundles $x_1^* = (8, 3)$ and $x_2^* = (8, 9)$ in the same set of coordinate axes. Figure 12 illustrates. There, $\Omega = (16, 12)$ is shown as the vector sum of the x_j^*, where both are now in relation to the same origin. Confirm that, for both consumers, the slope of the indifference curve through x_j^* is the same. If this were not true, the allocation would not be PO.) Lines of that slope pass through each of the x_j^*. A line of the same slope through Ω is the separating hyperplane of Step 2 of the proof, with p^* depicted as orthogonal to the line. The A_j sets are simply strict upper contour sets relative to the x_j^*. The lower, southwest boundary of each open A_j is the dashed indifference curve labeled ∂A_j. It is clear that the A_j are open and strictly convex, as required.

The dashed curve labeled ∂A is the boundary of the summation set given in equation (9) in the

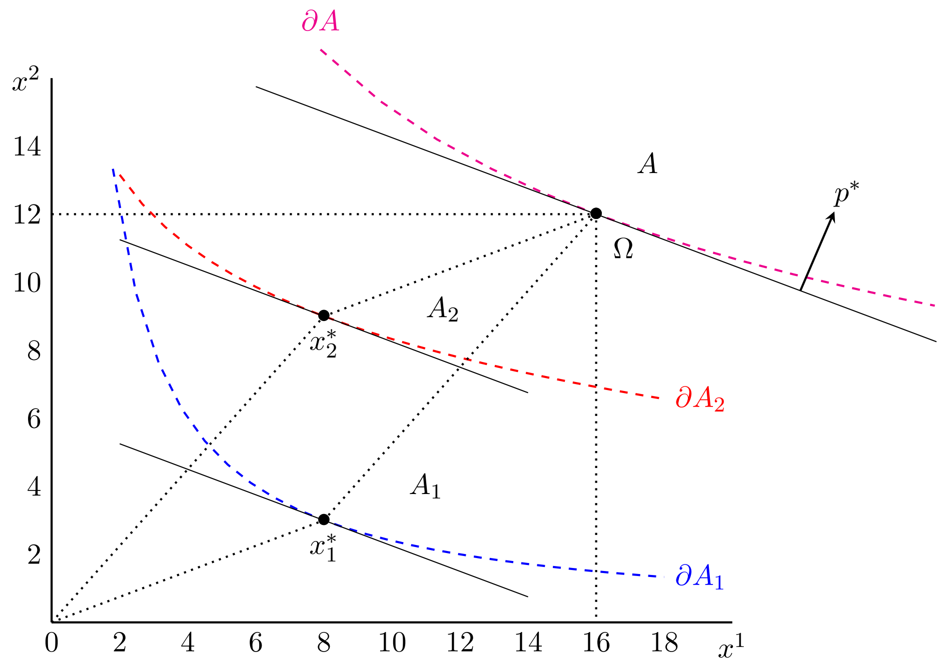

Figure 12: A_j and A sets relative to Pareto optimal allocation x^*.

proof, and it is the main object of Step 1 there. This set is also open and strictly convex. The ∂A curve is a bit subtle conceptually, and deriving it for a specific example can be a chore analytically. The same device, a convex set obtained from all possible vector summations of other convex sets, is used in a number of places in economics. In welfare economics the boundary is called a Scitovsky indifference curve.[4] Because it is slightly technical, a detailed explanation of the derivation of ∂A appears in Section 10.1. I'm tempted to call that section optional, but I believe a close study of it will help in understanding the main WTII proof and so I recommend it.

The last bit of Step 2 of our proof was to establish that $p \gg 0$. The argument was to show that we can add one unit of some good to Ω, and then move back a tiny amount from that bundle and remain in the set A. Pick any good, say good 1, and consider Figure 13, in which we have a separating line with $p^{*1} < 0$. (We could have drawn this line horizontally and the argument would still work.) The movement right from Ω to $\Omega + e_1$ is exaggerated (greater than 1), to be clear. The movement southwest from there to $\Omega + e_1 - \varepsilon\mathbf{1}$ shows that, with strongly monotone preferences, the last bundle must also be in A. But we have a bundle $\Omega + e_1 - \varepsilon\mathbf{1}$ that is both in A and below the separating line, violating the conclusion of the hyperplane theorem. A zero or negative price is ruled out.

The final step of the proof was to show that x_j^* must be the optimal choice for each j at prices p^*. The strategy is to assume otherwise: for some j we can find a strictly preferred bundle, x'_j, that is affordable at prices p^*. Take $j = 1$ and consider x'_1 with $x'_1 \succ_j x^*_1$. We want to show that x'_1 cannot be affordable. The challenge is that our separating hyperplane argument applies not to

[4]Tibor Scitovsky, a well-known Hungarian economist from the last century, really enjoyed adding sets together in clever ways.

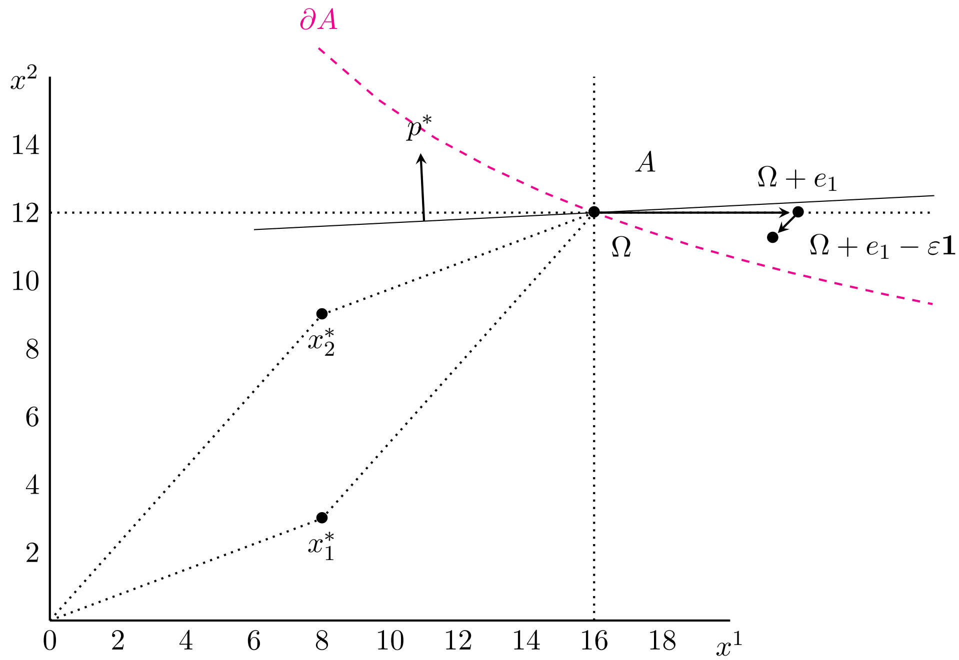

Figure 13: Separating hyperplane must have $p^* \gg 0$.

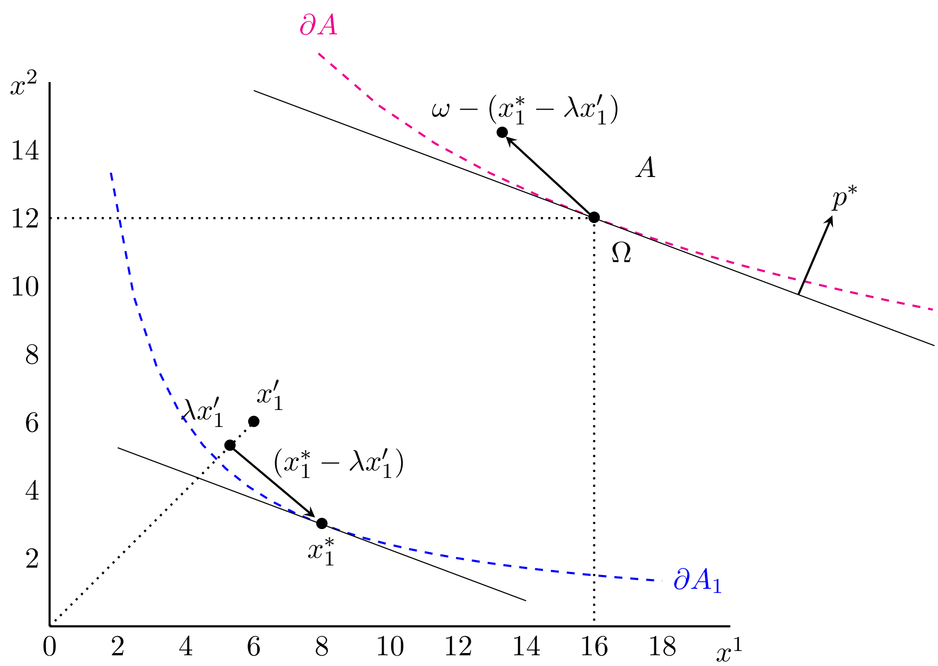

Figure 14: x_j' strictly preferred to x_j^* must be unafforadable at prices p^*.

A_1 but to A. Consider the situation depicted in Figure 14. Continuity of preferences means that for any bundle strictly preferred to x_1^*, there is another strictly preferred bundle that lies slightly to the southwest. So choose λ sufficiently close to 1 that $\lambda x_1'$ is also strictly preferred to x_1^*. The idea is to show that $\lambda x_1'$ is at least as expensive as x_1^* and so x_1' must be strictly more expensive.

We need to turn the properties of 1's situation into an argument that applies to p^* and A, which is the only argument relevant to the hyperplane theorem. To this end, translate the vector $(x_1^* - \lambda x_1')$ up to Ω. The direction of the vector arrow is reversed because it is being subtracted from Ω. Because $p \cdot \Omega$ appears on both sides of (10) and so can be cancelled, the transition from a property regarding Ω and A has become an argument regarding A_1 and 1's resource constraint. Mathematically, the $p^* \cdot \Omega$ term drops out on both sides of (10). Conceptually, this argument shows that x_1' must be unaffordable to consumer 1. The bundle $\lambda x_1'$ is at least as expensive as x_1^*, so x_1' must be strictly more expensive, contradicting that it was affordable.

10.1 A mathematical derivation of ∂A

The construction of A from the A_j is not exactly a straightforward operation, but it is useful. The key to this idea is that each point on the boundary ∂A is the sum of a matched pair of elements, one each from ∂A_1 and ∂A_2, at which the two slopes are equal. The slope of ∂A at the resulting summation point must be the same.

For our example, this construction may be carried out as follows. Select an arbitrary value of x^1. Denote this value y, to emphasize that the entire calculation starts here, and compute the slope of one indifference curve. We choose 2's indifference curve, where the slope is given by

$$\mathrm{MRS}_2 = \frac{\partial U_2/\partial x_2^1}{\partial U_2/\partial x_2^2} = \frac{3}{y}. \tag{11}$$

It is convenient, but will not usually be true, that here the slope of 2's indifference curve does not depend on x^2. This is the essential feature of quasi-linear preferences. Utility at $x_2^* = (8, 9)$, the known point on this indifference curve, is $U_2(x_2^*) = 3 \ln 8 + 9 \approx 15.2383$. Thus, for a given y we have $x_2^2(y) = 15.2383 - 3 \ln y$ and so

$$x_2^2(y) = (y, 15.2383 - 3 \ln y)\,.$$

Now compute the vector x_1 at which 1's indifference curve has the same slope. We can compute the slope of 1's indifference curve as

$$\mathrm{MRS}_1 = \frac{\partial U_1/\partial x_1^1}{\partial U_1/\partial x_1^2} = \frac{x_1^2}{x_1^1}. \tag{12}$$

Consumer 1's utility at x_1^* is $U_1 = 8 \cdot 3 = 24$, so it must also be true that $x_1^1 \cdot x_1^2 = 24$. Thus, $x_1^2 = 24/x_1^1$. In order for the two indifference curves to have the same slope, combining (11) and (12) we must have

$$x_1^2 = \frac{3x_1^1}{y} = \frac{24}{x_1^1},$$

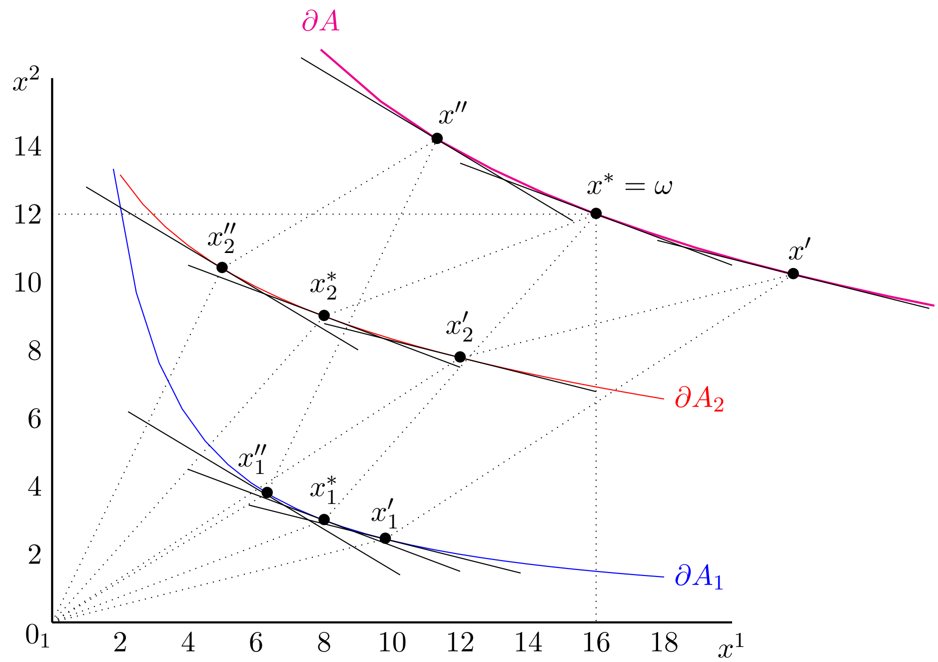

Figure 15: Construction of A by vector summation of the A_j.

which may be rearranged to yield

$$x_1^1(y) = \sqrt{8y},$$

and so we get

$$x_1^2(y) = \frac{24}{\sqrt{8y}} = 48\sqrt{\frac{2}{y}}.$$

The resulting vector on 1's indifference curve at which the slope equals that along 2's indifference curve with $x_2^1 = y$ is

$$x_1(y) = \left(\sqrt{8y}, 48\sqrt{\frac{2}{y}}\right).$$

We now have the pair of bundles, $x_1(y)$ and $x_2(y)$, that guarantee the equality of slopes of indifference curves for any starting point y. The corresponding point along the Scitovsky curve, ∂A, is the sum

$$x(y) = \left(\left(y + \sqrt{8y}\right), \left(15.2383 - 3\ln y + 48\sqrt{\frac{2}{y}}\right)\right).$$

This derivation is depicted in Figure 15. There are three sets of dots. The triple (x_1^*, x_2^*, x^*) is familiar. The double-primed set (x_1'', x_2'', x'') is derived by starting with $y = 5$ and the single-primed set (x_1', x_2', x') is derived by starting with $y = 12$. You can confirm that the slope of the two indifference curves is equal in each set. The slope of ∂A must, by construction, agree.

11 Existence I: strict convexity

The question of whether an exchange economy of the sort described here actually has an equilibrium is quite important. Why should we be interested in the Walrasian scheme as a reasonable description of the economy, and of its equilibrating properties, if we cannot rely upon there being an equilibrium at all?

Walras devised a cumbersome iterative method to show that an equilibrium exists. This method is not very useful and so we will not spend time with it. In a flurry of papers in the early 1950's, several economists (notably Arrow and Debreu 1954 and McKenzie 1954) worked out the modern argument for the existence of an equilibrium in a general-equilibrium economy. The formulation and argument of Arrow and Debreu have become one of the standard existence proofs. We'll turn to that result later; it involves a "private-ownership" economy that includes production. Here we treat the exchange economy.

The first existence theorem, for the easier case in which preferences are strictly convex, will rely upon a powerful topological result that guarantees that a function mapping a set into itself must have a fixed point.

Theorem 5 (Brouwer's fixed-point theorem)**.** If $f : S^{n-1} \longrightarrow S^{n-1}$ is a continuous function, then there is some $x \in S^{n-1}$ such that $x = f(x)$.

Proof. For $n \geq 3$ Brouwer's theorem is a deep topological result. If $n = 2$, though, the theorem can be proved using only the intermediate-value theorem. The domain and range of f can be associated with the unit interval $[0, 1]$. For any $x \in [0, 1]$ we must have $f(x) \in [0, 1]$. Consider the function $g(x) = f(x) - x$. A fixed point of f is any x at which $g(x) = 0$. We know that $g(0) \geq 0$ because $f(0) \in [0, 1]$. We also know that $g(1) \leq 0$ because $f(1) \in [0, 1]$. Because f is continuous, the intermediate-value theorem guarantees that there is some $x \in [0, 1]$ such that $g(x) = f(x) - x = 0$. This completes the proof. $\square$

The proof of existence in this section is based upon the assumption that preferences are strictly convex. This is a significant simplification because it means that demands are functions rather than correspondences.[5] The fact the demands are functions in turn means that the argument can be carried by Brouwer's theorem. Note that continuity of preferences implies that, if demands are functions then they are also continuous.

Theorem 6 (Existence of equilibrium I: strict convexity)**.** Suppose excess demand $z : S^{n-1} \longrightarrow \mathbb{R}^n$ is continuous and that $p \cdot z(p) \equiv 0$ for any $p \in S^{n-1}$. There is $p^* \in S^{n-1}$ such that $z(p^*) \leq 0$.

Proof. The proof is adapted from Luenberger (1995). Define the function $\tilde{z}^i(p) = \max[0, z^i(p)]$. This function is strictly positive only if $z_j(p) > 0$. Define the following mapping on the simplex S^{n-1}:

$$h^k(p) = \frac{p^k + \tilde{z}^k(p)}{1 + \sum_i \tilde{z}^i(p)}, \quad k = 1, 2, \ldots n. \tag{13}$$

[5] In his great *Theory of Value*, Debreu writes: "The assumption of weak convexity and even convexity are intuitively justified; it is not so for the assumption of strict convexity.

Imagine that a Walrasian "auctioneer" is at work behind the scene, calling out a price vector and, upon discovering the excess demand that results, adjusting the price to move the outcome toward an equilibrium. On this fictional interpretation (there is no passage of time here), the function in (13) is a price-adjustment function. The auctioneer uses it to raise the price of goods in excess demand and to reduce the price of goods in excess supply. The $\max[\cdot, \cdot]$ operator ensures that prices never go negative: $h^k(p) \geq 0$. Note that the denominator in (13) is nonzero by Walras's law (otherwise all goods would be in excess supply, violating the assumption of local nonsatiation). Note too that $\sum_j h^j(p) = 1$, so the vector-valued function h maps S^{n-1} into itself.

Because $z(p)$ is continuous, $h(p)$ is continuous. By Brouwer's fixed-point theorem it has a fixed point p^* at which

$$p^{*k} = \frac{p^{*k} + \tilde{z}^k(p^*)}{1 + \sum_i \tilde{z}^i(p^*)}, \quad k = 1, 2, \ldots n. \tag{14}$$

It remains only to show that this p^* is a Walrasian equilibrium. Multiply both sides of (14) by the dominator and cancel p^{*k} to obtain

$$p^{*k} \sum_{i=1}^{n} \tilde{z}^i(p^*) = \tilde{z}^k(p^*), \quad k = 1, 2, \ldots n.$$

Multiply the kth equation by $z^k(p^*)$ to get

$$p^{*k} z^k(p^*) \sum_{i=1}^{n} \tilde{z}^i(p^*) = z^k(p^*)\tilde{z}^k(p^*), \quad k = 1, 2, \ldots n.$$

Sum the resulting n expressions to get

$$\left(\sum_{k=1}^{n} p^{*k} z^k(p^*)\right) \sum_{i=1}^{n} \tilde{z}^i(p^*) = \sum_{k=1}^{n} z^k(p^*)\tilde{z}^k(p^*). \tag{15}$$

By Walras's law, $\sum_{k=1}^{n} p^{*k} z^k(p^*) = 0$, so (15) becomes

$$\sum_{k=1}^{n} z^k(p^*)\tilde{z}^k(p^*) = 0. \tag{16}$$

For each k, $z^k(p^*) \cdot \tilde{z}^k(p^*)$ can never be negative because the $\max[\cdot, \cdot]$ operator guarantees that if $z^k(p^*) < 0$, we must have $\tilde{z}^k(p^*) = 0$. Thus, the kth term is nonzero if and only if it positive. But it can't be positive and still have equation (16) satisfied. We conclude that $z^k(p^*) \leq 0$ for $k = 1, 2, \ldots, n$. This completes the proof. $\square$

The theorem is illustrated in Figure 16 (adapted from Luenberger's Figure 7.5). In Figure 16(a), the equilibrium is at p', where $z(p') = 0$. At p, where p^1/p^2 is higher than the equilibrium level, excess demand for good 1 is negative and for good 2 positive. The signs of excess demands are reversed at p''. As required by Walras's law, in each case the vector of excess demand is orthogonal to the associated price vector in S^1. In Figure 16(b), we have the boundary outcome. At the equilibrium price vector, p'', the price of good 1 is zero and excess demand points straight to the

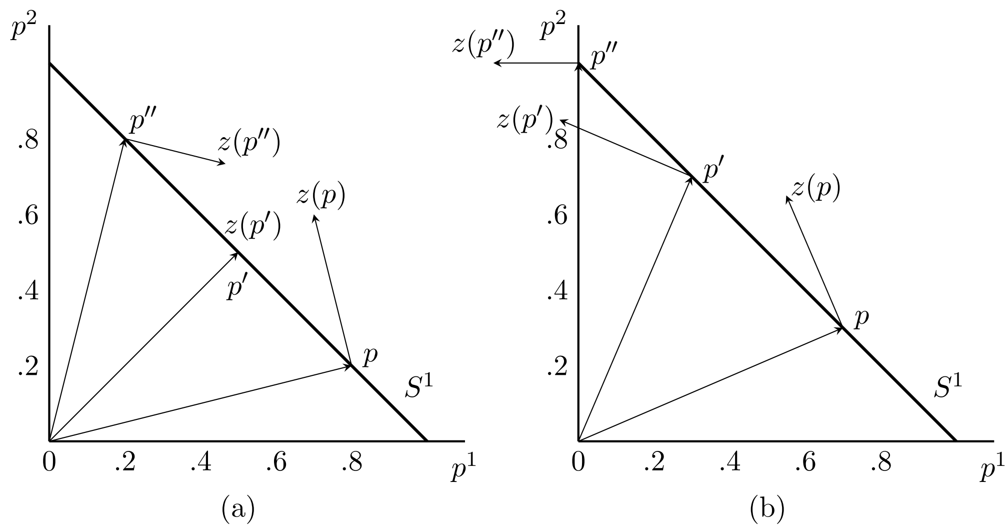

Figure 16: Equilibrium price and excess demand. (a) Equilibrium at p'. (b) equilibrium at boundary price p'' with $p^1 = 0$.

left. This means that excess demand for good 2 is zero and for good 1 negative. In either case, we have a price vector at which $z(p) \leq 0$.

12 Existence II: convexity

If one is unwilling to assume that preferences are strictly convex, an assumption that Debreu found objectionable, the previous proof simply does not go through. Without that assumption, demands may not be functions and so Brouwer's theorem is no help. Another drawback to the setup of Theorem 6 is that it assumes Walras's law. One might, quite reasonably, prefer to derive that result on the way to proving existence, rather than assuming it as a necessary condition of the result. The proof in this section removes both of these objections. We assume convexity rather than strict convexity. This means that demands are not necessarily functions. If a consumer's indifference curves have flat spots, at a given price vector she might be indifferent between a set of affordable bundles. Thus, demand is a multi-valued function or correspondence.

A different fixed-point result does the heavy lifting in this case. First, though, we need to define the notion of continuity for correspondences.

Definition 20. The correspondence $\varphi : S \Longrightarrow T$ is **upper hemi-continuous (u.h.c.) at** $x \in S$ if for every open set $\mathcal{O}$ containing $\varphi(x)$ there exists a neighborhood Y of x such that $\varphi(x') \subset \mathcal{O}$ for every $x' \in Y$. φ is **upper hemi-continuous** (u.h.c.) if it is u.h.c. at every $x \in S$.

If both S and T lie in the reals of some dimension, and T is compact, then φ is u.h.c. if the graph $G_\varphi = \{(x, y) \mid y \in \varphi(x)\}$ is a closed subset of $S \times T$.

A companion property, lower hemi-continuity, will not be used in these notes.

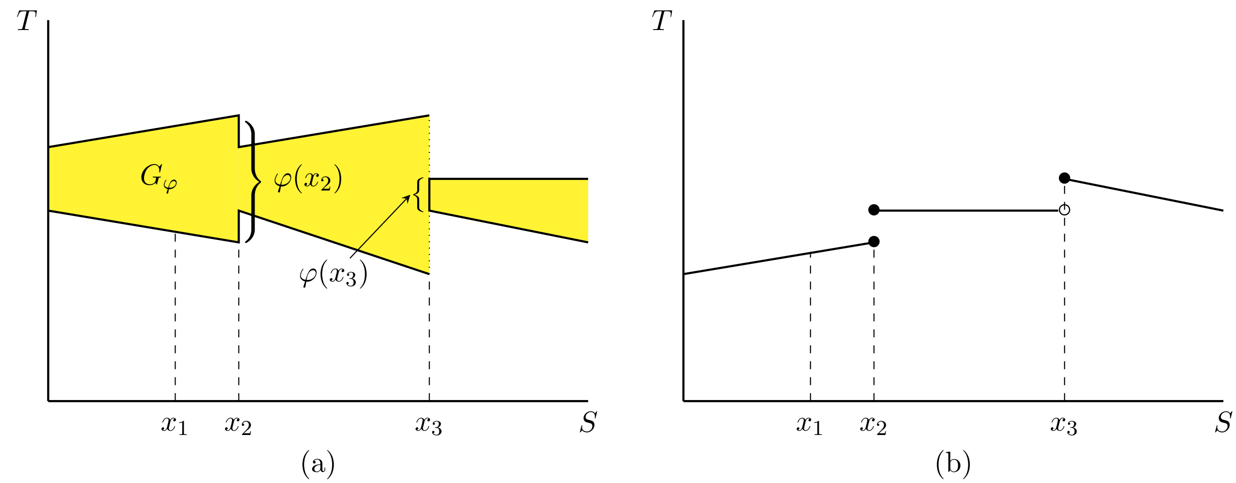

Figure 17: The correspondence φ is upper hemi-continuous at x_2 and lower hemi-continuous at x_3, but not the other way around.

Definition 21. The correspondence $\varphi : S \Longrightarrow T$ is **lower hemi-continuous (l.h.c.) at** $x \in S$ if for every open set $\mathcal{O}$ in T with $\varphi(x) \cap \mathcal{O} \neq \emptyset$ there exists a neighborhood Y of x such that $\varphi(x') \cap \mathcal{O} \neq \emptyset$ for every $x' \in Y$. φ is **lower hemi-continuous** (l.h.c.) if it is l.h.c. at every $x \in S$.

Figure 17 illustrates. There, in both panels the correspondence φ is u.h.c. (but not l.h.c.) at x_2. It is l.h.c. (but not u.h.c.) at x_3. Is it both u.h.c. and l.h.c. at x_1. The correspondence in panel (b) is almost a function, except at x_2 where it takes two values. (A continuous function on a convex domain is both u.h.c. and l.h.c.)

An alternative definition, based on convergent sequences, complements the open-set definition and might be more transparent to some.

Definition 22. The correspondence $\varphi : S \Longrightarrow T$ is **upper hemi-continuous (u.h.c.) at** $x \in S$ if for every sequence $\{x^i\}_{i=1}^{\infty}$ converging to $x \in S$ and every sequence $\{y^i\}_{i=1}^{\infty}$ with $y^i \in \varphi(x^i)$ there exists a converging subsequence of $\{y^i\}_{i=1}^{\infty}$ whose limit belongs to $\varphi(x)$.

Notice how this definition emphasizes the fact that for each x, $\varphi(x)$ be closed and bounded. In Figure 18, $\varphi(x)$ is not closed for $x \in (x', x'')$. A sequence of pairs (x^i, y^i) converges to (x^0, y^0), but $y^0 \notin \varphi(x^0)$. The correspondence is not u.h.c. We state also the definition of l.h.c. using convergent sequences.

Definition 23. The correspondence $\varphi : S \Longrightarrow T$ is **lower hemi-continuous (l.h.c.) at** $x \in S$ if for every sequence $\{x^i\}_{i=1}^{\infty}$ converging to $x \in S$ and every $y \in \varphi(x)$, there exists a sequence $\{y^i\}_{i=1}^{\infty}$ converging to y with $y^i \in \varphi(x^i)$ for each i.

Finally, we round out the discussion by putting the two notions of hemi-continuity together.

Definition 24. The correspondence $\varphi : S \Longrightarrow T$ is **continuous** if it is both lower and upper hemi-continuous.

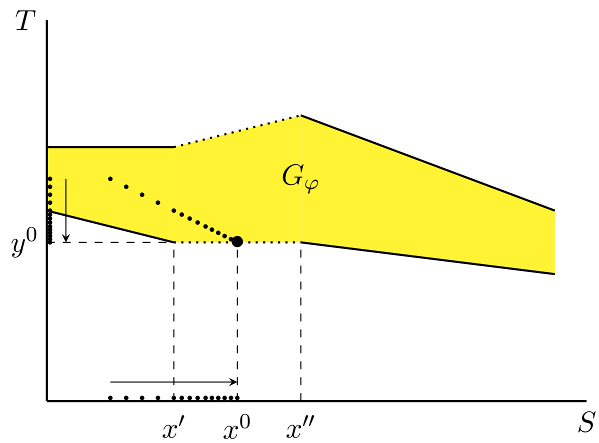

Figure 18: The correspondence φ is not upper hemi-continuous at any $x \in (x', x'')$ because it is not compact-valued there.

For T a compact and convex subset of $\mathbb{R}^n$, the following is the counterpart to Brouwer that applies to correspondences.

Theorem 7 (Kakutani's fixed-point theorem)**.** Let T be a non-empty, compact, and convex subset of $\mathbb{R}^n$ and φ a convex-valued correspondence of T into T. If φ is upper hemi-continuous then there exists a fixed point $\hat{x}$ for φ. That is, $\hat{x} \in \varphi(\hat{x})$.

Kakutani obtained this result in 1941. Its proof is also quite deep and even the two-dimensional version is complicated. Note that Nash's 1950 proof of the existence of an equilibrium in a non-cooperative game used the Kakutani fixed-point theorem.

In addition to Kakutani's theorem, we'll also use Berge's theorem (this is Theorem M.K.6 in Mas-Colell, and I use their notation). The problem is that of maximizing a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ on the nonempty constraint set $C(q)$, where $q = (q_1, \dots q_s)$. If $x(q) \subset C(q)$ is the set of solutions given q, and $v(q)$ is the associated maximum value, then Berge's theorem gives conditions under which $x(q)$ is upper hemi-continuous and the value function is continuous.

Theorem 8 (Berge's theorem)**.** Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function and the constraint correspondence $C : Q \longrightarrow \mathbb{R}^n$ is continuous with $C(q)$ compact and non-empty for all $q \in Q$. Consider the maximization problem $\max_{x \in C(q)} f(x)$. The solution correspondence is upper hemi-continuous and the value function is continuous.

In our application, the function f corresponds to utility and the constraint set $C(q)$ corresponds to a consumer's budget set.

Because they are continuous, we know that consumer i's preferences may be represented by a utility function $u_i : X \to \mathbb{R}$. The following notation is used to denote an abstract exchange economy.

Definition 25. Given a set of goods $I = \{1, \dots, n\}$, a set of consumers $J = \{1, \dots m\}$ and their common commodity space $X = \mathbb{R}^n_+$ and utilities $\{U_j(x_j)\}_{j=1}^m$, and the set of prices S^{n-1}, an **abstract economy** is $\mathcal{E} = (\{X\}_{j=1}^m, \{U_j(x_j)\}_{j=1}^m, S^{n-1})$.

Theorem 9 (Existence of equilibrium II: convexity)**.** Suppose that each consumer's preferences are continuous, strongly monotonic, and convex. Suppose also that, for each consumer i, $\omega_i \gg 0$. Then there exists a Walrasian equilibrium (p^*, x^*) for $\mathcal{E}$.

The proof is developed along the lines of Arrow and Debreu, but we have simplified things somewhat by ruling out production.[6] The innovation in the proof is the addition of an actor called the "auctioneer," whose role is to select a price vector. Consumers will select a demanded bundle given prices and the auctioneer will select a price vector given demands. In equilibrium the selections are self-reinforcing, in precisely the manner in which a Nash equilibrium exhibits self-reinforcing strategies.

In our proof of Theorem 6 the mathematical device was an adjustment function that mapped prices in the simplex while (i) ensuring that the adjustment never produces a negative price and (ii) normalizing to ensure that the prices also sum to one. Here the mathematical device is a mapping, a correspondence that takes as its argument a price vector and an allocation, and maps it into another such pair. Consumers play the role of mapping prices into x according to their preferences, and we must show that their aggregate demand has the properties required by Kakutani's theorem. The auctioneer plays the role of mapping allocations into p, and we must show that her choice correspondence also has the properties required by Kakutani's theorem. The domain and range of the joint correspondence are the same compact, convex-valued set. Thus, it must have a fixed point. The last step of the proof is to show that the fixed point must be a Walrasian equilibrium for the underlying economy.

Proof. The **first step** of the proof is to invoke Berge's theorem to show that each consumer's demand correspondence $\varphi_j(p, \omega)$ is upper hemi-continuous. A difficulty is that the budget correspondence $\mathcal{B}_j(p, \omega_j)$ is not compact valued if any $p^i = 0$, or if the price vector lies on the boundary ∂S^{n-1} of the simplex. Define the compact set

$$T = \left\{ x \in \mathbb{R}^n_+ \mid x \leq 2\Omega \right\},$$

which is a cube whose ith side is twice the aggregate endowment of good i. Now for each j consider the demand correspondence

$$\varphi_j(p, \omega_j) = \arg \max_{x \in \mathcal{B}(p,\omega_j) \cap T} U_j(x).$$

By Berge's theorem this correspondence is non-empty valued and u.h.c. for each j. Moreover, because preferences are convex φ_j is convex-valued. Note that the assumption that $\omega_j \gg 0$ is crucial here: if $\omega_j^i = 0$ for some i, the budget correspondence itself is not continuous at $p^i = 0$.

Define the aggregate demand correspondence for consumers by

$$\varphi(p, \omega) = \textstyle\sum_j \varphi_j(p, \omega_j) = \Big\{ x \mid \exists\, x_1 \in \varphi_1(p, \omega_1), \ldots, x_m \in \varphi_m(p, \omega_m) \text{ with } \textstyle\sum_j x_j = x \Big\}.$$

Because each $\varphi_j(p, \omega_j)$ is non-empty, convex-valued, and upper hemi-continuous in prices, the correspondence $\varphi : S^{n-1} \Longrightarrow T$ is also non-empty, convex-valued, and upper hemi-continuous in

[6] I have adapted the proof from Jonathan Levin's 2006 lecture notes.

prices.

The **second step** of the proof is to introduce the auctioneer, whose payoff correspondence $\varphi_A : T \Longrightarrow S^{n-1}$ is defined by

$$\varphi_A(x) = \arg \max_{p \in S^{n-1}} p(x - \omega).$$

The auctioneer chooses a price vector to maximize the value of aggregate excess demand given the allocation x. Note that φ_A is also non-empty, convex-valued, and upper hemi-continuous.

The **third step** is to invoke Kakutani. Define $\Phi : S^{n-1} \times T \Longrightarrow S^{n-1} \times T$ by

$$\Phi(p, x) = (\varphi_A(x), \varphi(p)) .$$

The Cartesian product of non-empty, convex-valued, u.h.c. correspondences is also non-empty, convex-valued, and u.h.c. Thus Kakutani's fixed-point theorem applies and we may conclude that there exists a fixed point $(p^*, x^*) \in \Phi(p^*, x^*)$.

The **fourth step** is to show that (p^*, x^*) is a Walrasian equilibrium for $\mathcal{E}$. Because $x^* \in \varphi(p^*)$, it must be true that for each j, $x_j^* \in \arg\max_{x \in \mathcal{B}(p^*,\omega_j) \cap T} U_j(x)$. We must show that $x_j^* \in \arg\max_{x \in \mathcal{B}(p^*,\omega_j)} U_j(x)$. That is, the optimum cannot be in $\mathcal{B}(p^*, \omega_j) \backslash T$. To see that it cannot, note that because $p^* \in \varphi_A(x^*)$, for all $p \in S^{n-1}$ we must have

$$0 \geq p^*(x^* - \omega) \geq p(x^* - \omega),$$

or otherwise the auctioneer would have chosen p rather than p^*. The first inequality is just Walras's law and the second implies that $x^* \leq \omega$ and thus that $x_j^* < 2\omega$. This means that $x_j^* \in \arg\max_{x \in \mathcal{B}(p^*,\omega_j)} U_j(x)$. By way of contradiction, suppose not: there is $x_j' \in \mathcal{B}(p^*, \omega_j)$ with $U_j(x_j') > U_j(x_j^*)$. By continuity, there is a λ sufficiently small so that $\lambda x_j' + (1 - \lambda)x_j^* \in \mathcal{B}(p) \cap T$. By monotonicity and convexity of preferences, $U_j(\lambda x_j' + (1 - \lambda)x_j^*) > U_j(x_j^*)$, contradicting that $U_j(x_j') > U_j(x_j^*)$. We conclude that $x_j^* \in \arg\max_{x \in \mathcal{B}(p^*,\omega_j)} U_j(x)$.

Finally, we show that markets clear: $x^* = \omega$. By Walras's law, we know that $p^* x^* = p^* \omega$. If $x^{*i} < \omega^i$ for some good i, we must have $p^{*i} = 0$ because p^* is the auctioneer's optimum. Replace x_1^{*i} by $x_1^{*i} - (x^{*i} - \omega^i)$ to restore market clearing. $\square$

12.1 Anomaly 2: no equilibrium exists

Now let's turn to another example. Consumer 2 has utility $U_2(x_2^1, x_2^2) = \max[x_2^1, x_2^2]$. His endowment is $\omega_2 = (4, 12)$. What is this consumer's offer curve? It is illustrated in Figure 19. At prices $p = (1/2, 1/2)$, the consumer is indifferent between bundles $(16, 0)$ and $(0, 16)$. As soon as p^1 becomes slightly larger than $1/2$, though, as at p', so that the budget line through ω_2 is a little steeper than at p, the demanded bundle will be where the new budget line intercepts the vertical axis. Thus, part of the offer curve is the pink bolded segment that rises above $(0, 16)$. A price with p^1 slightly lower than $1/2$, as at p'', puts the optimal bundle on the horizontal axis. The other part of the offer curve is the pink bolded segment that shoots upward from $(0, 16)$.

The important thing to notice is that the offer curve is not continuous. Nonconvex preferences can lead to discontinuous demands, which in turn can lead to a failure for the economy to have an

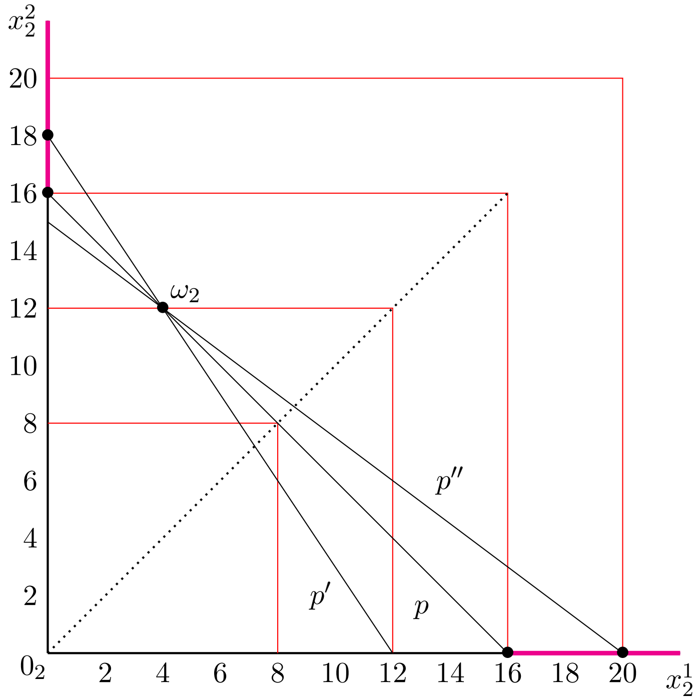

Figure 19: Discontinuous offer curve for $U_2 = \max[x_2^1, x_2^2]$. At prices $p^1 = p^2$, consumer 2 is indifferent between bundles $(16, 0)$ and $(0, 16)$.

equilibrium.

Now add Consumer 1 to this economy. She is the same consumer we saw in the previous example, with utility given by $U_1(x_1^1, x_1^2) = x_1^1 x_1^2$ and an endowment of $\omega_1 = (12, 4)$. This consumer's offer curve is depicted in Figure 5. When we put these two consumers together in an Edgeworth box, we obtain Figure 20. The aggregate endowment is now $\Omega = (16, 16)$ and one sees that there is no equilibrium for this economy. The offer curves never meet.

We can also derive the set of Pareto-optimal allocations. Any allocation along the top or right side of the box is PO because a move in either direction along either axis makes one of the agents worse off. The left and bottom sides are also PO in this case. The Cobb-Douglas consumer receives zero utility if either x_1^i is zero, so the entire coordinate axis system is an indifference curve for 1. But the axes also form an indifference curve for consumer 2. So along the left and bottom we can move any place we want but we cannot make either better off. Figure 21 illustrates. Note that this would not be true if the box weren't square, or if 2's utility function weren't symmetric.

Finally, let's look at a slight variant of this example to get an idea of the peculiar nature of Cobb-Douglas preferences. Suppose consumer 1 has Stone-Geary utility given by $U_1(x_1^1, x_1^2) = (x_1^1 + \gamma^1)(x_1^2 + \gamma^2)$, with $\gamma^1 = \gamma^2 = 2$. These are like Cobb-Douglas preferences, but the "origin" of the indifference curves is at $(-2, -2)$.[7] This means that indifference curves meet the axes with a finite slope. Figure 22 illustrates. The left and bottom sides of the box are no longer PO because

[7] This utility function satisfies the Stone-Geary definition technically, but in demand analysis one would expect the γ^i to be negative. The resulting demand system is the linear expenditure system.

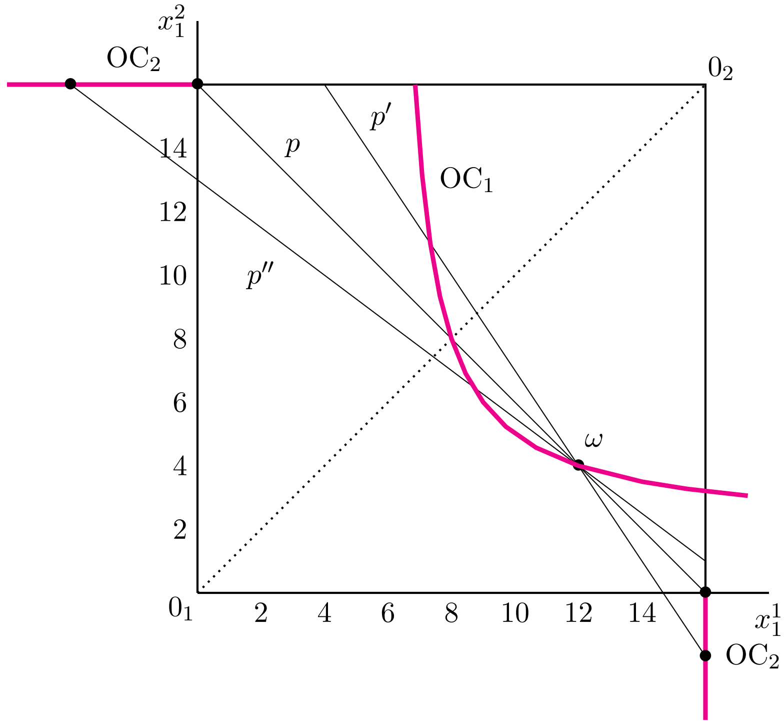

Figure 20: No equilibrium due to 2's nonconvex preferences and discontinuous offer curve.

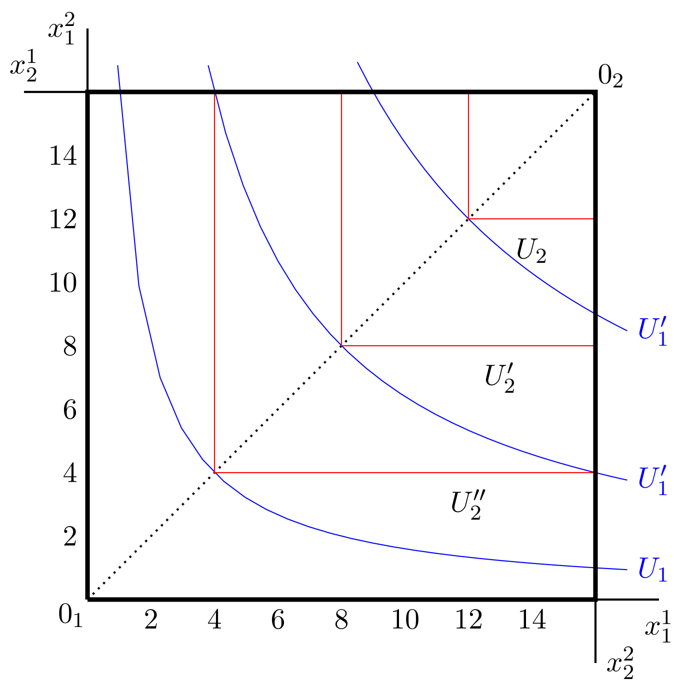

Figure 21: Pareto optimality for nonconvex preferences, $U_1(x_1^1, x_1^2) = x_1^1 x_1^2$ and $U_2(x_2^1, x_2^2) = \max[x_2^1, x_2^2]$. The contract curve is the entire outline of the Edgeworth box.

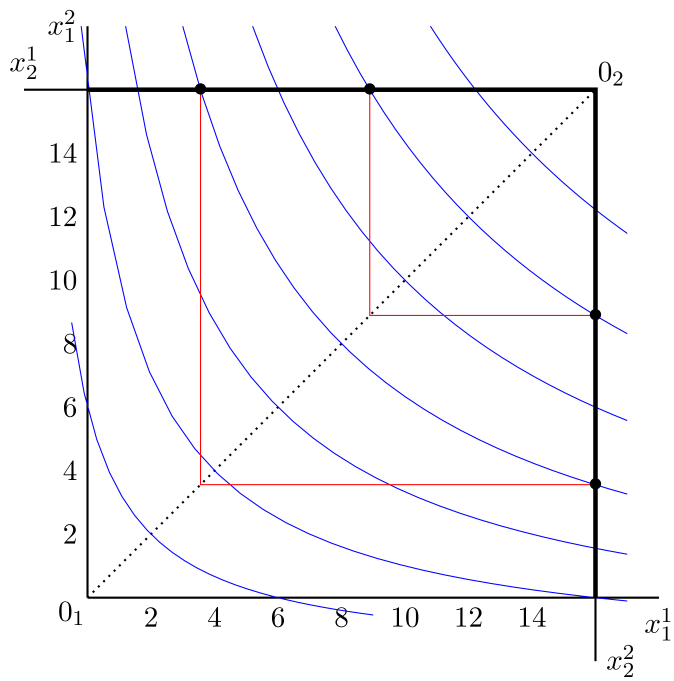

Figure 22: Pareto optimality for nonconvex preferences. $U_1(x_1^1, x_1^2) = (x_1^1 + 2)(x_1^2 + 2)$ and $U_2(x_2^1, x_2^2) = \max[x_2^1, x_2^2]$.

at any point along those sides we can move up or right and make 1 better off without making 2 worse off. The top and right sides are still PO, as with the Cobb-Douglas example.

As with Anomaly 1, the lesson from this example is not that the existence theorems are violated. The proofs are sound. Rather, the example shows that the conclusion of the theorem may not be true if one or more of its assumptions fails to hold.

13 Utility possibilities and social welfare

Another useful construct, the *utility possibilities set*, is the set of all achievable utility combinations for a given collection of utility functions and an aggregate endowment $\omega = \sum_j \omega_j$, with $\omega \in \mathbb{R}^n$. Denote this set $\mathcal{V} = \{(U_1(x_1), \ldots U_n(x_n)) \in \mathbb{R}^J \mid \sum_j x_j^i \leq \sum \omega_j^i \text{ for each } i = 1, \ldots n\}$. The northeast boundary of this set in J-dimensional space is the *utility possibilities frontier*. To any point on this frontier can be associated a unique (if utility functions are strictly quasi-concave) allocation, which will be on the *contract curve* in the space of allocations.

To fix the idea of the utility possibilities set and its associated frontier, consider the following 2-person, 2-good exchange economy. Person 1's utility function is $U_1(x_1^1, x_1^2) = (x_1^1)^{1/3}(x_1^2)^{2/3}$ and person 2's utility function is $U_2(x_2^1, x_2^2) = x_2^1 + x_2^2$. Endowments are $\omega_1 = (6, 3)$ and $\omega_2 = (3, 6)$. First let us determine the contract curve for this economy. The slope of 2's indifference curves is simply -1. We can find the points at which 1's indifference curves have slope -1, which will give us part of the contract curve.

Solve 1's indifference curve for x_1^2 as a function of x_1^1 as follows. First, $(x_1^2)^{2/3} = U/(x_1^1)^{1/3}$. Then

$$x_1^2(x_1^1) = \frac{U^{3/2}}{(x_1^1)^{1/2}},$$

whose derivative is

$$\frac{dx_1^2}{dx_1^1} = -\frac{1}{2}\frac{U^{2/3}}{(x_1^1)^{3/2}}.$$

Set this equal to -1 to obtain

$$\frac{1}{2}\frac{U^{2/3}}{(x_1^1)^{3/2}} = 1,$$

or $x_1^1 = U/2^{2/3}$. Plug this into the expression for x_1^2 to get $x_1^2 = U \cdot 2^{1/3}$. Along any indifference curve for person 1, at the point at which the slope is -1 we know that the ratio between x_1^1 and x_1^2 is

$$\frac{x_1^1}{x_1^2} = \frac{(1/2)^{2/3}}{2^{1/3}} = \frac{1}{2},$$

or $x_1^2 = 2x_1^1$. Thus, the set of tangencies between 1's indifference curves and 2's indifferences curves is along a ray from the origin with slope 2. A portion of this ray (the portion that remains in the box) forms one part of the contract curve. It is a line segment, shown in Figure 23 from the origin to the point $(4.5, 9)$, relative to 0_1.

The remainder of the contract curve is determined by the constraint represented by the box, not by a tangency condition. It runs along the top of the box and forms a line segment from the point $(4.5, 9)$ to the point $(9, 9)$, again relative to 0_1.

To map this contract curve into the utility space, first fix the endpoints of the utility-possibilities frontier by plugging the entire endowment first into U_1 and then into U_2. We get the following points that belong to the frontier in (U_1, U_2)-space: $(9, 0)$ and $(0, 18)$. At the kink in the contract curve, utility levels are

$$U_1(4.5, 9) = 7.14284 \quad \text{and} \quad U_2(4.5, 0) = 4.5.$$

I claim that the frontier to the northwest of the kink (associated with the sloping portion of the contract curve) is a line segment connecting $(0, 18)$ and the kink. This is because of the linear homogeneity of both utility functions.[8]

The portion of the frontier to the southeast of the kink (associated with the border portion of the contract curve) is not linear. We can solve for the relationship between U_1 and U_2 there by noting two facts. One is that $x_2^2 = 0$ and, therefore, $x_1^2 = 9$ on this portion of the contract curve. The other is that $x^1 = 9$ is divided between the two consumers. For allocations on the border

[8] To see that this claim is true, note that this portion of the frontier can be specified as a line, $U_2 = 18 - bU_1$ with intercept 18 and slope given by the rise over the run between $(0, 18)$ and $(7.14284, 4.5)$. Solve for $b = \text{rise/run} = 13.5/7.14284 = 1.890004$. Take any point on the interior portion of the contract curve, say $x' = (2.25, 4.5, 6.75, 4.5)$. Calculate the utility levels at this allocation, getting $U_1(2.25, 4.5) = 3.57152$ and $U_2(6.75, 4.5) = 11.25$. This pair should be on the line given above, and it is. For $U_1 = 3.57152$, we get $U_2 = 18 - (1.89004)(3.57152) = 11.25$.

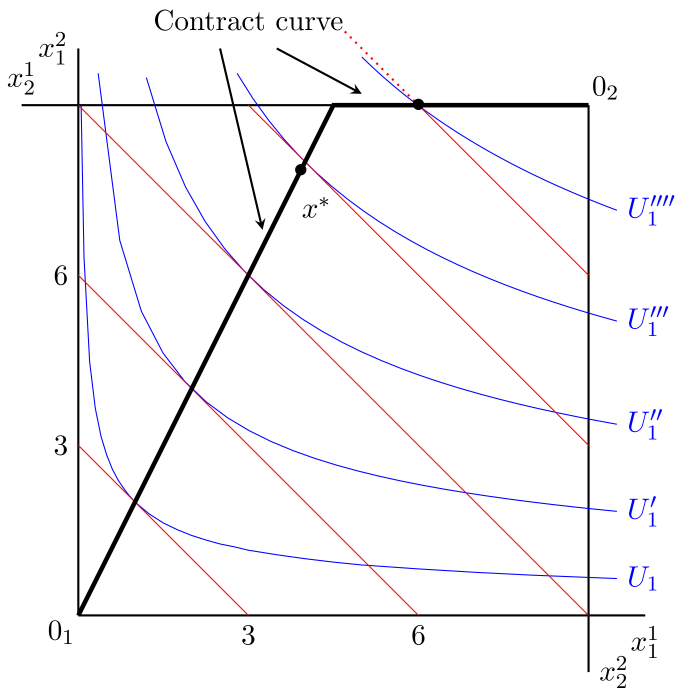

Figure 23: Two-piece contract curve

portion of the contract curve we may write

$$\begin{aligned} U_2 &= x_2^1 + x_2^2 \\ &= (9 - x_1^1) + (9 - x_1^2) \\ &= 9 - x_1^1. \end{aligned}$$

We also know that $U_1 = (x_1^1)^{1/3} \cdot 9^{2/3}$, or

$$x_1^1 = \left(\frac{U_1}{9^{2/3}}\right)^3 = \frac{U_1^2}{81}.$$

The relationship between U_1 and U_2 along the frontier, then, is

$$U_2(U_1) = 9 - \frac{U_1^3}{81}.$$

This curve is only slightly nonlinear below the kink, but it becomes more obviously nonlinear above, where it is shown as a faint dashed curve. The utility-possibility set is

$$\mathcal{V} = \{(U_1, U_2) \in \mathcal{R}^2 \mid U_j \geq 0, \text{ and } (U_1, U_2) \text{ is on or below the frontier}\}.$$

Suppose a social planner wants to select an allocation so as to maximize a social welfare function given by $W = \min\{U_1, U_2\}$. In Figure 24 it is easy to see that the solution must lie on the longer

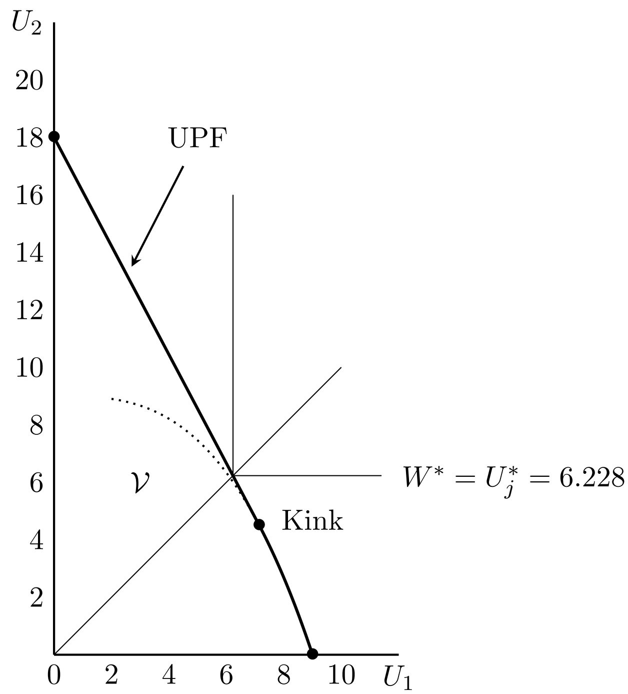

Figure 24: Utility-possibility frontier and Leontief social optimum.

portion of the frontier, above the kink. We can find the optimal utility outcome by setting $U_1 = U_2$ in the expression for the upper line segment, which yields

$$U_1 = 18 - 1.890004U_1.$$

Solve this equation to get $2.890004U_1 = 18$ or

$$U_1 = \frac{18}{2.890004} = 6.22837.$$

(Test that this is correct by plugging $U_1 = 6.22837$ back into the formula for the line. We do indeed get $U_2 = 6.22837$.)

Because we know that $x_1^2 = 2x_1^1$ at this point, we can obtain the underlying allocation by solving for x_1^1 in the expression

$$(x_1^1)^{1/3}(x_1^2)^{2/3} = 6.22837,$$

using the fact that $x_1^2 = 2x_1^1$. This gives us $x_1^1 = 3.92363$ and $x_1^2 = 7.84725$. Taking $\omega^i - x_1^i$, we get 2's allocation at the social optimum: $x_2^1 = 5.07637$ and $x_2^2 = 1.15275$. The socially optimal allocation is then

$$x^* = (3.92363, 7.84725, 5.07637, 1.15279).$$

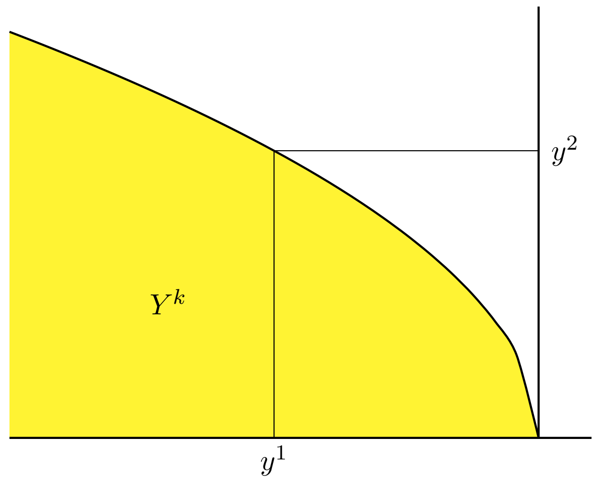

Figure 25: Production set Y^k for a competitive firm.

14 General equilibrium with production

The case of an exchange economy, in which the resources available to the entire economy are handed down exogenously, might seem to be extremely special. Of course in an actual economy productive activities, ruled out in an exchange economy, are quite important. It turns out, though, that from the perspective of the mathematics it is not difficult to add a productive sector and, when one does so, the main elements of the three theorems survive almost intact.

A production economy includes the same n goods and the same m consumers as before. Consumers have the same endowments and the same preferences over $X \in \mathbb{R}^n_+$. But now we add in K firms, indexed by $k = 1, \ldots, K$, each of which owns a production set $Y_k \in \mathbb{R}^n$, defined as all vectors of inputs ($y^l_k < 0$) and outputs ($y^l_k > 0$) that are technically feasible for firm k. By convention, inputs are treated as negative and outputs as positive elements of the *netput vector* y_k, which belongs to Y_k if the set of outputs may be produced by the firm's technology using the set of inputs. A firm can use as inputs any subset of the goods in $\mathbb{R}^n_-$ and it can produce as outputs any subset of $\mathbb{R}^n_+$. But of course a firm's inputs cannot also be that firm's outputs. Figure 25 illustrates a technology with one input y^1 and one output y^2.

The production sets are assumed to satisfy a collection of conditions. The first two together rule out increasing returns to scale.

Assumption 1. For each firm, Y_k is closed and convex in $\mathbb{R}^n$.

Assumption 2. For all firms, $0 \in Y_k$ and $\mathbb{R}^n_- \subset Y_k$.

The last part of the second assumption guarantees that disposal in production is unlimited. It is also useful to rule out the possibility that two or more firms can coordinate their activity to generate infinite joint output. We would not want to allow one firm to produce a pound of steel from a pound of iron, and then another to produce two pounds of iron from the pound of steel. Debreu (1959) specifies an assumption to rule out this possibility.

Assumption 3. If $Y = \sum_k Y_k$, then $Y \cap -Y = \{0\}$.

The firms have a single goal. The vector of input prices is $p \in \mathbb{R}^n_+$. Firms take this entire vector, input and output prices alike, as given, and choose y_k to maximize profit. That is, firm k chooses y^i to solve

$$\max_{y_k \in Y_k} \ \pi_k = p \cdot y_k.$$

The economy as a whole yields total profit of $\pi = \sum_k \pi_k$. Because a general-equilibrium economy is a closed system, this money must go somewhere. We assume a *private-ownership economy*, in which each firm is owned entirely by the consumers. Let $\theta_{jk} \in [0,1]$ be the share of firm k owned by consumer j, with $\theta_{jk} \geq 0$ and $\sum_j \theta_{jk} = 1$ for every k. We will assume that the number of firms producing each output and buying each input is sufficiently large that all output and input markets are perfectly competitive. We will also assume that production technologies are convex sets in $\mathbb{R}^n$. A production economy is fully characterized by

$$\mathcal{E} = \left(m, (U_j, \omega_j, (\theta_{jk})_{k=1}^K)_{j=1}^m, (Y_k)_{k=1}^K\right).$$

For any price vector, a consumer's budget constraint now includes both the value of ω_j and the sum of profit shares in the firms:

$$px_j \leq p\omega_j + \sum\nolimits_k \theta_{jk}\pi_k.$$

Keep in mind that consumers might now sell part of their endowment to the firms (who use them as inputs) as well as to other consumers (who consume them).

I will use the term *outcome* to refer to the vector (x, y) consisting of an allocation, x, together with a *production plan*, $y = (y_1, \ldots, y_K)$. Feasibility of an outcome requires that there is enough of everything to go around.

Definition 26. Given a production economy $\mathcal{E}$, the outcome (x, y) is **feasible** if $\sum_j (x_j - \omega_j) - \sum_k y_k \leq 0$.

Pareto optimality requires only that consumers cannot be made better off. The "preferences" of firms, however one might think of that notion (profits? utility of profits?, is irrelevant.

Definition 27. A feasible outcome (x, y) is **Pareto optimal** if there is no other feasible outcome (x', y') such that $U_j(x'_j) \geq U_j(x_j)$ for all j, with $U_j(x'_j) > U_j(x_j)$ for at least one j.

The definition of an equilibrium is similar to what we've seen, with the difference being that now firms too must optimize.

Definition 28 (Walrasian equilibrium in a production economy)**.** Given a production economy $\mathcal{E} = \Big(m, (U_j, \omega_j, (\theta_{jk})_{k=1}^K)_{j=1}^m, (Y_k)_{k=1}^K\Big)$, a **Walrasian equilibrium** is a vector (p^*, x^*, y^*) such that

i.) the allocation is feasible ($\sum_j \varphi^i_j(p^*, \omega_j) = \sum_j \omega^i_j + \sum_k y^{*i}_k$);
ii.) x^*_j is optimal for each agent j (if $x'_j \succ_j x^*_j$, then $p^* x'_j > p^*\omega_j + \sum_k \theta_{jk}\pi^*_k$); and
iii.) y^*_k is optimal for each firm k ($\pi^*_k \geq p^* y_k$ for all $y_k \in Y_k$)l

14.1 The welfare theorems with production

The technical arguments necessary to prove the first and second welfare theorems change very little when we add production. The statement and proof of the first welfare theorem may be found on p. 549 of Mas-Colell. Notice the similarity to the proof in an exchange economy provided earlier in these notes.

For the second welfare theorem we will first examine the Mas-Colell definition of an equilibrium with transfers.

Definition 29 (Mas-Colell *et al.* 16.D.1)**.** Given an economy $\mathcal{E} = \Big(m, (U_j, \omega_j, (\theta_{jk})_{k=1}^K)_{j=1}^m, (Y_k)_{k=1}^K\Big)$, an outcome (x^*, y^*) and price vector p is a **price quasi-equilibrium with transfers** if there is an assignment of wealth levels $(w_1, \ldots, w_m)$ with $\sum_j w_j = p\omega + \sum_k py_k^*$ such that

i.) For every k, y_k^* maximizes profits in Y_k (that is, $py_k \leq py_k^*$ for all $y_k \in Y_k$;

ii.) For every j, if $U_j(x_j) > U_j(x_j^*)$ then $px_j \geq w_j$; and

iii.) $\sum_j x_j^* = \omega + \sum_k y_k^*$.

We may now state the Mas-Colell version of the second welfare theorem.

Proposition 1 (Mas-colell *et al.* 16.D.1)**.** Consider an economy specified by $\mathcal{E} = (m, (U_j, \omega_j, (\theta_{jk})_{k=1}^K)_{j=1}^m, (Y_k)_{k=1}^K\Big)$, and suppose that every Y_k is convex and every preference relation is convex and locally nonsatiated. Then, for every Pareto-optimal outcome (x^*, y^*), there is a price vector $p \neq 0$ such that (x^*, y^*, p) is a price quasi-equilibrium with transfers.

The proof of this result begins on p. 552 in Mas-Colell. The time we will give the proof depends on how much time is left in the semester.

14.2 The existence theorem with production

The coverage of existence theorems in Mas-Colell is found in their Section 17.C, which spans about 6 pages. We will explore the new aspects of adding production if time permits.

15 Exercises

1. Consider an exchange economy consisting of two consumers and two goods, x^1 and x^2. Consumer 1 has initial endowment $\omega_1 = (3, 6)$ and consumer 2 has initial endowment $\omega_2 = (3, 6)$. The consumers' utility functions are given by $U_1(x_1^1, x_1^2) = x_1^1 + 2x_1^2$ and $U_2(x_2^1, x_2^2) = (x_2^1 x_2^2)^{1/2}$ respectively. *True or False:* The initial endowment is Pareto optimal. In an Edgeworth box (or using mathematical notation), describe the set of Pareto optimal allocations. Find the Walrasian equilibrium for the economy.

   Solution. *False.* Consumer 1's indifference curves are lines with slope $-1/2$. At any allocation, the slope of consumer 2's indifference curve is

$$-\mathrm{MRS}_2 = -\frac{\partial U_2/\partial x_2^1}{\partial U_2/\partial x_2^2} = -\frac{(x_2^2/x_2^1)^{1/2}}{(x_2^1/x_2^2)^{1/2}} = -\frac{x_2^2}{x_2^1}.$$

   At his endowment, the slope of 2's indifference curve is $-x_2^2/x_2^1 = -6/3 = -2$. This is not equal to the slope of 1's indifference curve there, so the endowment is not Pareto optimal.

   We find the set of PO allocations by first setting the slope of 2's indifference curves equal to $-1/2$. This gives $-x_2^2/x_2^1 = -1/2$ or $x_2^1 = 2x_2^2$. The set of tangencies between 1's indifference curves and 2's indifferences curves is along a ray from 2's origin (the upper right corner of the box) with slope 1/2. A portion of this ray (the portion that remains in the box) forms one part of the contract curve. It is a line segment from 2's origin to the point $(6, 3)$, relative to 0_2. The remainder of the contract curve is determined by the constraint represented by the box, not by a tangency condition. It runs along the left side of the box and forms a line segment from the point $(6, 3)$ to the point $(6, 12)$, again relative to 0_2. In Figure 26, the lines of slope $-1/2$ are 1's indifference "curves" and the Cobb-Douglas curves are 2's. The two bold line segments make up the contract curve.

2. Using the utility functions from exercise 1., determine the utility-possibilities set. In a diagram, with the U_i on the axes, depict the utility-possibilities set and its northeast boundary, the utility-possibilities frontier. Find the allocation that will be chosen if the social planner for this economy seeks to maximize the social-welfare function $W = \min\{2U_1, U_2\}$.

3. Consider an exchange economy consisting of two consumers and two goods, x^1 and x^2. Initial endowments are $\omega_1 = (4, 0)$ and $\omega_2 = (0, 4)$ and utility functions are $U_1(x_1^1, x_1^2) = \sqrt{x_1^1 x_1^2}$ and $U_2(x_2^1, x_2^2) = \sqrt{x_2^1 x_1^2}$. Find the set of PO allocations for this economy. Derive the offer curves and the competitive equilibrium price and allocation. Find the price vector and a corresponding (translated) endowment vector that would support the allocation $x_1 = (1, 1)$ and $x_2 = (3, 3)$ as a competitive equilibrium.

4. Consider a 2-person exchange economy with 2 goods. Utility functions are given by $U_1 = \min\{x_1^1, x_1^2\}$ and $U_2 = \min\{x_2^1, x_2^2\}$ respectively. The aggregate endowment for the economy is $\omega = (8, 8)$. How would a social planner who maximizes $W = U_1 U_2$ choose to allocate the resources?

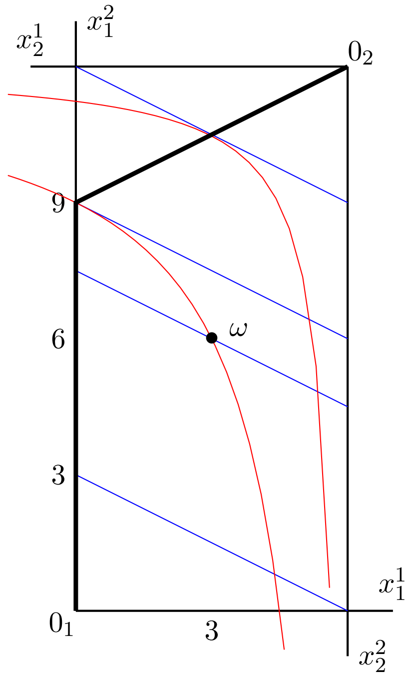

Figure 26: The contract curve

5. Consider a 2-person exchange economy with 2 goods. Utility functions are identical and are given by $U_j = x_j^1 x_j^2$. Derive the contract curve for this economy.

   Solution. The tangency between indifference curves will always be along the diagonal of the Edgeworth box. See Figure 2.

6. Consider the exchange economy from Section 8.1, with utilities of $U_1(x_1^1, x_1^2) = x_1^1 x_1^2$ and $U_2(x_2^1, x_2^2) = -(x_2^1 - 5)^2 - (x_2^2 - 3)^2$ and with endowments of $\omega_1 = (12, 4)$ and $\omega_2 = (4, 8)$

   a. Derive the contract curve for this economy.

   b. Find all competitive equilibria.

   c. Prove that no competitive equilibrium allocation is Pareto optimal.

7. Consider a two-consumer two-good exchange economy with no free disposal. The commodity space is $X = \mathbb{R}_+^2$. Consumer 1 has preferences represented by the utility function

$$U_1\left(x_1^1, x_1^2\right) = 5x_1^1 x_1^2 + 4.$$

   Consumer 2 has lexicographic preferences (with commodity x_2^1 primary). The consumers' endowments are $\omega_1 = (10, 20)$ and $\omega_2 = (10, 0)$.

   a. Construct an Edgeworth box diagram showing (and labeling) the endowment allocation and typical indifference sets and directions of increasing preference for each consumer.

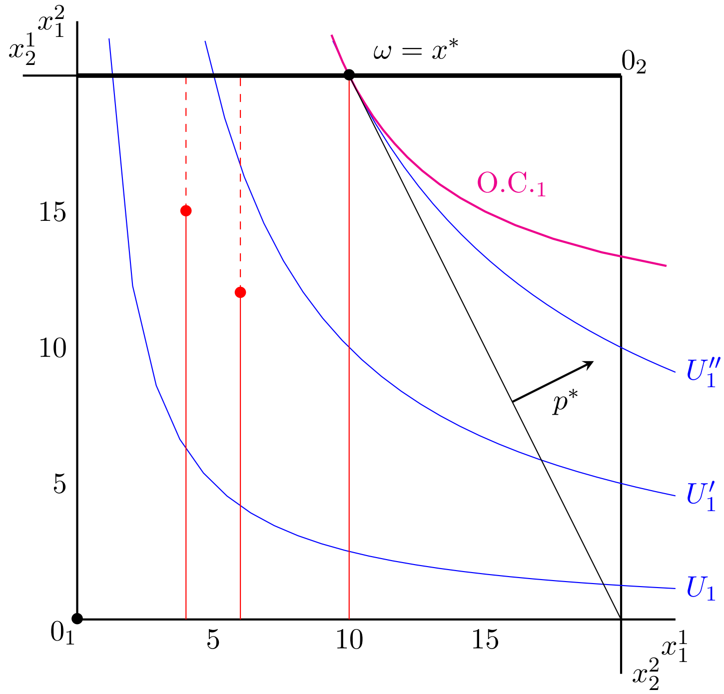

Figure 27: Optimality and equilibrium, exercise 7.

Solution. Consumer 1 has familiar Cobb-Douglas indifference curves, as depicted in Figure 27. Every bundle is its own indifference set for consumer 2 because he is unwilling to trade off even a tiny amount of good 1 for any amount of good 2. Two bundles are shown as red dots, along with the solid red line representing preferred bundles with the same amount of x_2^1 and the dashed line representing less preferred bundles. Consumer 2 has only good 1 in his endowment, so his preference direction is primarily to the West, and secondarily South in the figure.

b. If this economy has Pareto optimal allocations, indicate them clearly on the diagram. If there are none, state that.

Solution. The set of Pareto optimal allocations is the top boundary of the box, together with consumer 1's origin. For any interior allocation, consumer 2 can be made better off with slightly more of good 1, while consumer 1 can give up some good 1 for more of good 2 and be made better off. Therefore, we can always make 2 better off without hurting 1 by moving northwest along 1's indifference curve until we hit the north boundary.

c. If this economy has any competitive Walrasian equilibria, show them clearly on the diagram, indicating price ratios, budget lines, and allocations. If there are none, state so and explain why.

Solution. The initial endowment is the unique Walrasian equilibrium allocation. For any price ratio, p^1/p^2, less than infinity consumer 2 will demand only good 1. To find the equilibrium price ratio we must determine prices at which 1's demanded bundle is also ω_1. Demands are

$$x_1^1(p,\omega_1) = 5 + \frac{10p^2}{p^1} \quad \text{and} \quad x_1^2(p,\omega_1) = 10 + \frac{5p^1}{p^2},$$

which generate the offer curve indicated in pink in the figure. This confirms that the endowment vector is the equilibrium allocation. The equilibrium price ratio will be the slope of 1's indifference curve through ω_1, which is given by

$$-\mathrm{MRS}_1 = -\frac{x_2^1}{x_1^1} = -\frac{20}{10} = -2$$

For any price ratio $p^1/p^2 < 2$, her demand for good 2 will be less than the endowment, 20. For any price ratio greater than 2 the demand for good 2 will be greater than 20. Therefore, for any price ratio not equal to 2 the market will not clear. For $p^1/p^2 = 2$, the market clears and both consumers are optimizing. Therefore, the equilibrium is

$$p^* = \left(\frac{2}{3}, \frac{1}{3}\right) \quad \text{and} \quad x^* = (10, 10, 10, 0)\,.$$

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