ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 5: Classical Demand Theory: Utility Maximization and Expenditure Minimization
(MWG, Ch. 3, pp.50-63)

I. Introduction

This lecture describes the fundamental "problem" (behavior) of a consumer who wants to maximize his or her utility subject to a wealth constraint. Throughout this lecture we assume that the consumer has a preference relation $\succsim$ that is:

1. Rational
2. Continuous
3. Locally nonsatiated

The first two assumptions imply that these preferences can be represented by a utility function u(x). Assume also that the consumption set is X = $\mathbb{R}^L_+$ (all possible bundles of L commodities with nonnegative values).

Finally, this lecture ends by examining the consumer's expenditure minimization problem: that is, how to attain a certain level of utility at the lowest possible cost. We will see that this is very closely related to utility maximization.

II. The Consumer's Utility Maximization Problem

The consumer's "problem" (goal) is to choose the "best" consumption bundle(s) given prices p >> 0 and w > 0. We call this the **utility maximization problem (UMP)**:

$$\max_{x \geq 0} \; u(x) \qquad \text{subject to } p \cdot x \leq w$$

Thus the UMP is to choose the consumption bundle(s) from all the possibilities in the Walrasian budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ to maximize his or her utility level.

The first useful result is the following:

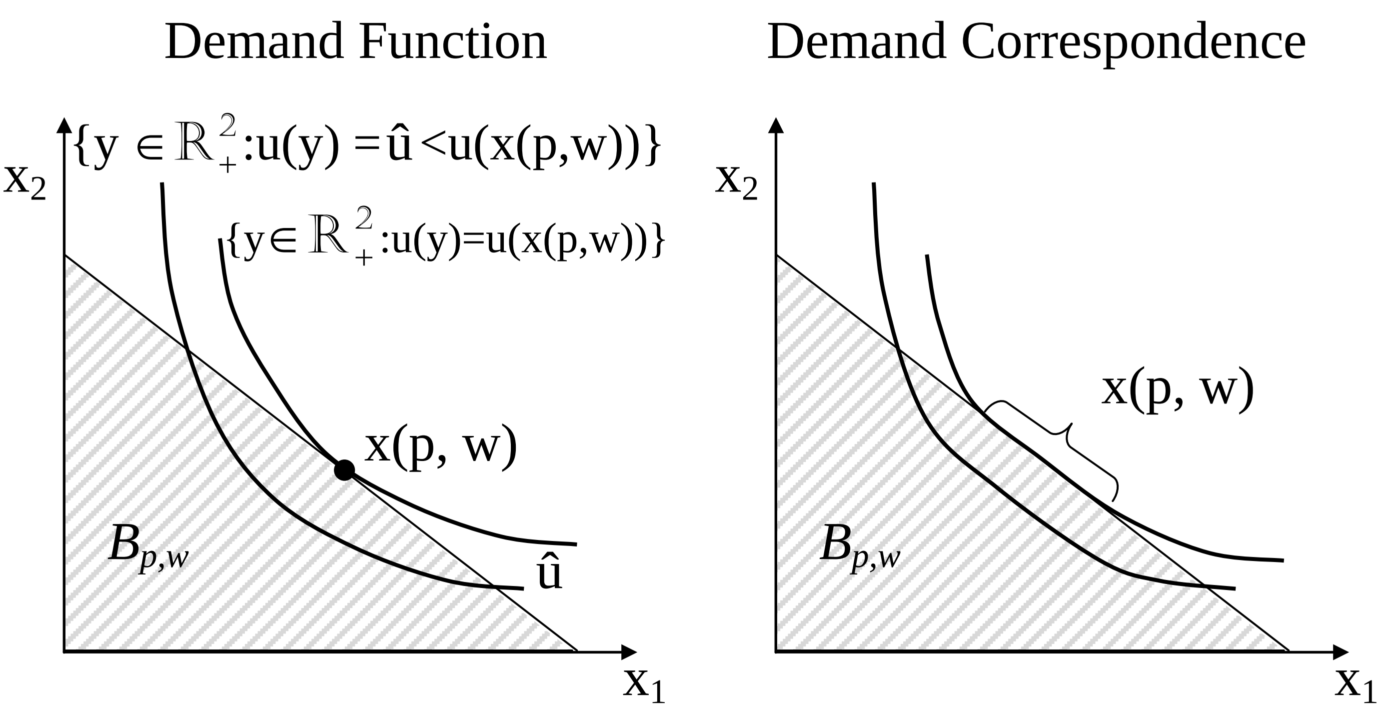
Proposition 3.D.1: If p >> 0 and u( ) is continuous, then the utility maximization problem has a solution.

Proof: If p >> 0 then the budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is bounded (for any $\ell = 1, 2, \ldots L$, we have $x_\ell \leq w/p_\ell$ for all $x \in B_{p,w}$) and closed (it "contains its borders", or more formally all limit points are within the set for sequences that have all members in the set). A set that is closed and bounded is compact, and as explained in the mathematical appendix (section M.F) of Mas Colell et al., any continuous function always has a maximum value on any compact set.

Note: As seen below, there may be more than 1 solution.

III. The Walrasian Demand Correspondence/Function

Definition: The bundle or bundles of commodities, denoted by x, that are the solution to the consumer's UMP are called the **Walrasian** (or **ordinary** or **market** or **Marshallian**) **demand correspondence**. It is denoted by $x(p, w) \in \mathbb{R}^L_+$. The following two figures give two examples for the case of two goods (L = 2):



The first figure has a unique solution, so it is a demand function (note that preferences are strictly quasiconcave). The second figure has multiple solutions, so it is a demand correspondence (and the preferences that produce it are quasiconcave, but **not *strictly*** quasiconcave).

The properties of the demand correspondence/function x(p, w) are given in the following proposition:

Proposition 3.D.2: Suppose that u( ) is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined over the consumption set $X \in \mathbb{R}^{L}_{+}$. Then the **Walrasian demand correspondence x(p, w) has the following properties**:

1. Homogeneity of degree zero in (p, w): $x(\alpha p, \alpha w) = x(p, w)$ for any $p >> 0$, $w > 0$ and $\alpha > 0$.

2. Walras' law, $p{\cdot}x = w$, holds for all $x \in x(p, w)$.

3. Convexity/uniqueness: If $\succsim$ is convex, so that u( ) is quasiconcave, then x(p, w) is a convex set. Moreover, if $\succsim$ is strictly convex, so that u( ) is strictly quasi-concave, then x(p, w) consists of a single element.

Note: Homogeneity & Walras Law were ***assumed*** in Lect. 3.

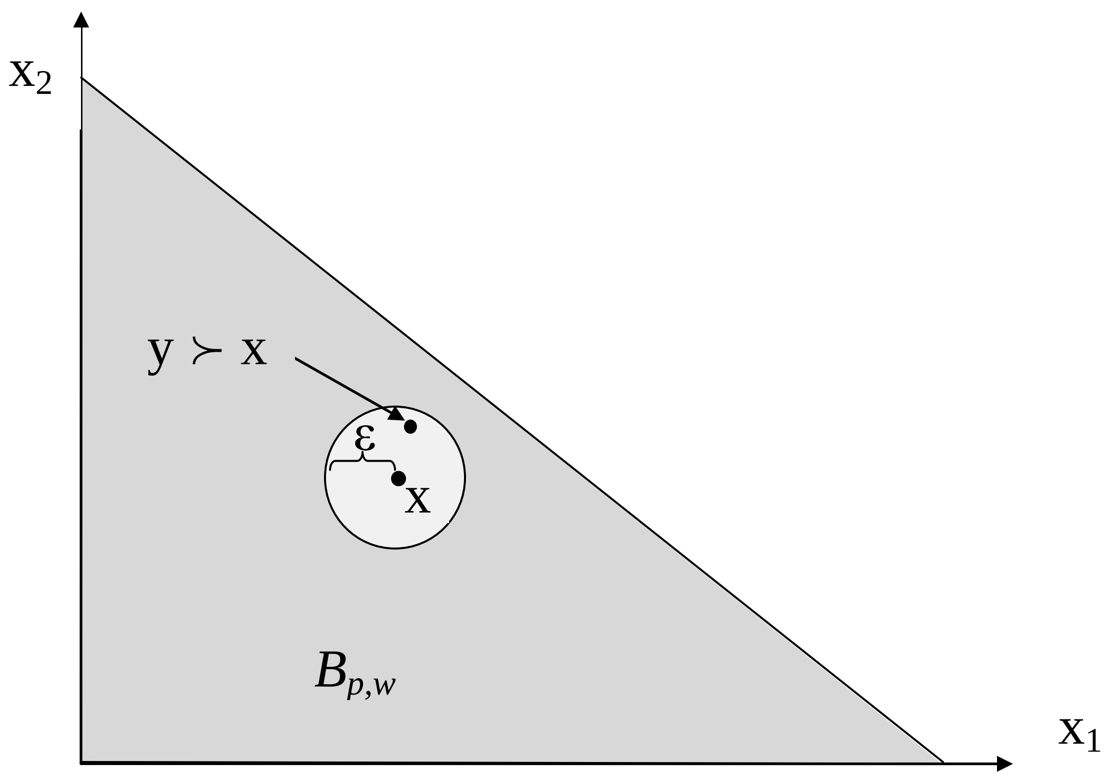
Proof: Let's examine each of the 3 properties separately.

1. **Homogeneity**. Consider the set of feasible consumption bundles, denoted by $\{x \in \mathbb{R}^{L}_{+}: p{\cdot}x \le w\}$. If prices are changed to αp and wealth changes to αw, **the feasible consumption bundle is unchanged**. That is:

$$\{x \in \mathbb{R}^{L}_{+}: \alpha p{\cdot}x \le \alpha w\} = \{x \in \mathbb{R}^{L}_{+}: p{\cdot}x \le w\}$$

Since the (constrained) choice problem has not changed, the choices made should not change, so the demand correspondence must be the same: $x(p, w) = x(\alpha p, \alpha w)$. Note that this property does not require *any* assumptions about the utility function u( ).

2. **Walras' Law.** This follows from the nonsatiated preferences assumption. Suppose Walras' law does not hold; that is, $p \cdot x < w$ for some $x \in x(p, w)$. Then by local nonsatiation there must be another consumption bundle y sufficiently close to x that satisfies both $\| y - x \| < \varepsilon$ and $y \succ x$. We can choose ε sufficiently small so that $p \cdot y < w$. But then x cannot be part of $x(p, w)$ because there is another affordable bundle that is strictly preferred to x. Thus Walras' law must hold. This is seen in a diagram:



3. **Convexity/Uniqueness.** Recall that u( ) is quasi-concave. Assume that there are two bundles, x and x′, with $x \neq x'$, both of which are in x(p, w), i.e. in the demand correspondence. To show convexity we need to show that, for any $0 \leq \alpha \leq 1$, $x'' = \alpha x + (1 - \alpha)x'$ is also a member of x(p, w). Note first that it must be that u(x) = u(x′). Call this utility level u*. Quasiconcavity implies that $u(x'') \geq u^*$. Also, since $p \cdot x \leq w$ and $p \cdot x' \leq w$ then:

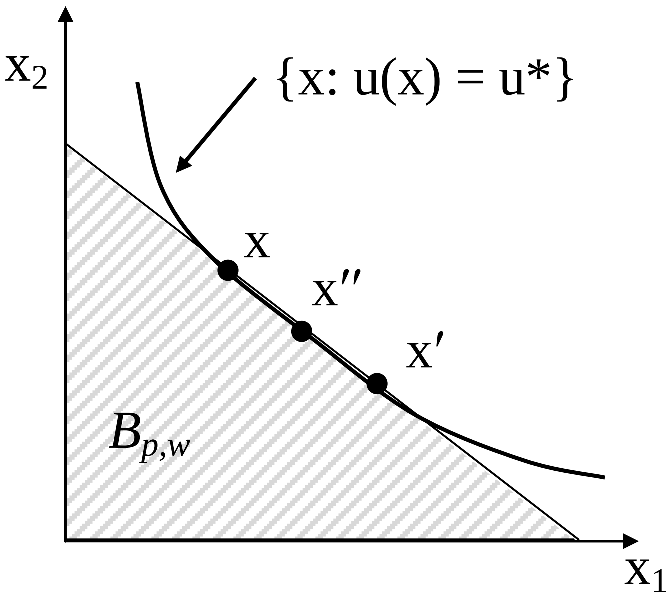
$$p \cdot x'' = p[\alpha x + (1 - \alpha)x'] \leq w$$

Therefore x″ is a feasible choice in the UMP. Since x″ is feasible and $u(x'') \geq u^*$ then $x'' \in x(p, w)$. Thus x(p, w) is a convex set. This is shown in the diagram on the left at the top of the next page.

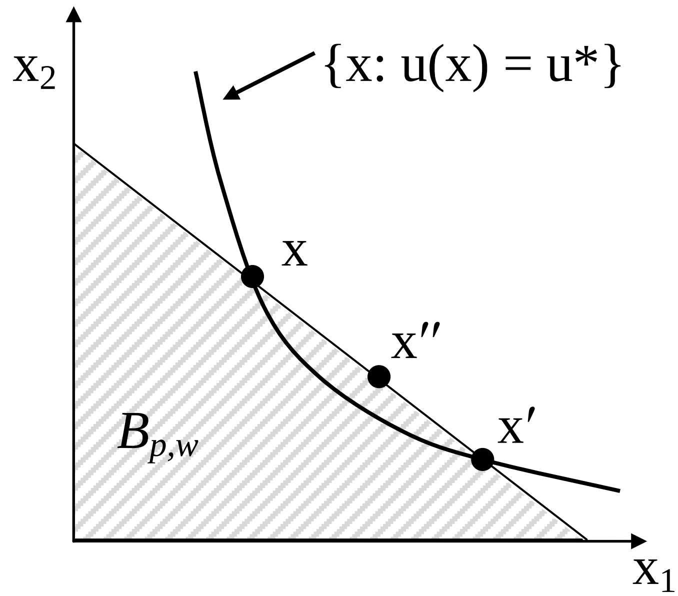
The last thing to show is that strict quasiconcavity of u( ) implies that there is a unique solution to x(p, w). This is also a "proof by contradiction". Assume again that there are two bundles, x and x′, with $x \neq x'$, both of which are in x(p, w). Define x″ as above. We already know that x″ is a feasible choice. But we saw (Lecture 4, p.18) that strict quasiconcavity implies that $u(x'') > u^*$ for $\alpha \in (0, 1)$. Thus x and x′ **cannot** be elements of x(p, w). So the solution to x(p, w) must be unique.   **Q.E.D.**

Uniqueness is shown in the diagram on the right at the top of the next page:

Convexity



Uniqueness (strict convexity)



More Properties of the Optimal Consumption Bundle

If the utility function u( ) is continuously differentiable, an optimal consumption bundle $x^* \in x(p, w)$ can be characterized using first order conditions. The **Kuhn-Tucker (necessary) conditions** (see Section M.K in the Mathematical Appendix of MWG) demonstrate that if $x^* \in x(p, w)$ is a solution to the UMP, then a **Lagrange multiplier** $\lambda \geq 0$ exists such that, for all $\ell = 1, 2, \ldots L$:

$$\partial u(x^*)/\partial x_\ell \leq \lambda p_\ell, \quad \text{and} \quad \partial u(x^*)/\partial x_\ell = \lambda p_\ell \text{ if } x_\ell^* > 0$$

This can be expressed for all L commodities in vector notation by defining $\nabla u(x) = [\partial u(x)/\partial x_1, \ldots \partial u(x)/\partial x_L]^T$, that is the **gradient vector** for u( ) at point x:

$$\nabla u(x^*) \leq \lambda p \quad \text{and} \quad x^* \cdot [\nabla u(x^*) - \lambda p] = 0$$

If x^* is an interior solution, $x_\ell^* > 0$ for all ℓ ($x^* >> 0$), so:

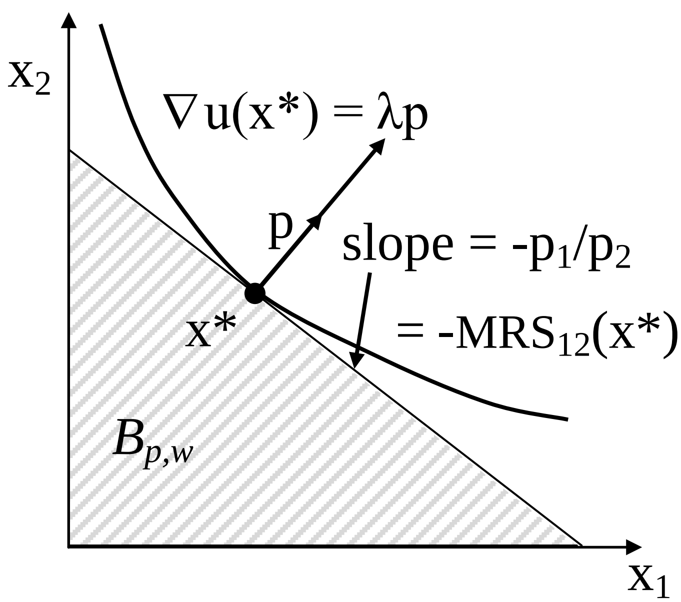
$$\nabla u(x^*) = \lambda p$$

An interior solution implies, for any two goods ℓ and k, that:

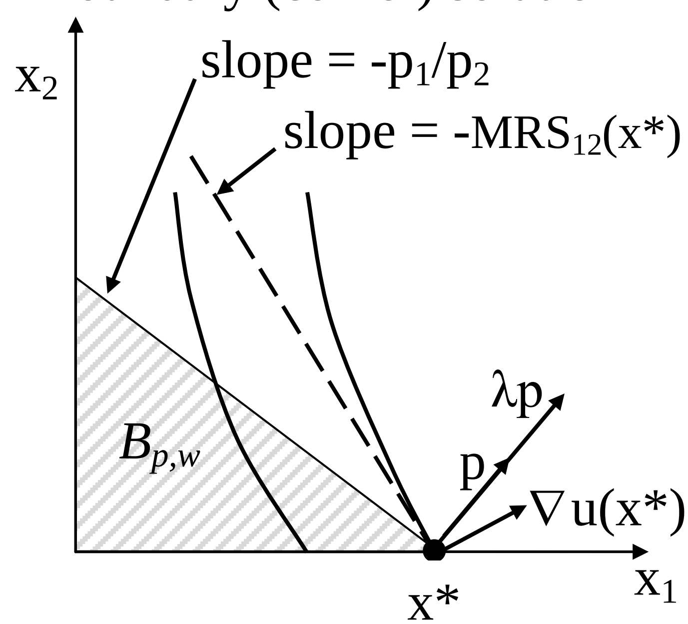
$$MRS_{\ell k}(x^*) = \frac{\partial u(x^*)/\partial x_\ell}{\partial u(x^*)/\partial x_k} = p_\ell/p_k$$

This is the **marginal rate of substitution** of good ℓ relative to good k at the optimal consumption bundle x^*. Intuitively, it shows the relative "value" of the two goods in terms of their contribution to the consumer's utility. That is, it shows how much good ℓ the consumer requires to obtain the same increase in utility from a 1 unit increase in good k. The slope of the indifference curve in the diagram below at the utility maximizing point is –MRS.

Interior Solution



Boundary (corner) solution



The expression for $MRS_{\ell k}(x^*)$ also shows that, given an interior solution, utility maximization requires $MRS_{\ell k}(x^*)$ to be equal to p_ℓ/p_k. If not, the consumer could increase utility by purchasing more of good ℓ if $MRS_{\ell k}(x^*) > p_\ell/p_k$ or more of good k if $MRS_{\ell k}(x^*) < p_\ell/p_k$.

More precisely, suppose $MRS_{\ell k}(x^*) > p_\ell/p_k$. Then $(\partial u(x)/\partial x_\ell)dx_\ell > (\partial u(x)/\partial x_k)(p_\ell/p_k)dx_\ell$ and thus $(\partial u(x)/\partial x_\ell)dx_\ell - (\partial u(x)/\partial x_k)(p_\ell/p_k)dx_\ell > 0$. So reducing consumption of good k by $(p_\ell/p_k)dx_\ell$ (which equals dx_k) and spending the money saved on good ℓ will raise utility.

What if x* is not an interior solution? Then the utility maximizing bundle is at the boundary of the consumption set, so one or more commodities are not consumed. Suppose that commodity ℓ is not consumed. This implies that $MRS_{\ell k}(x^*) \leq p_\ell/p_k$ where the value for x_ℓ in x^* is 0, and that $\partial u(x)/\partial x_\ell \leq \lambda p_\ell$. This is shown for the case of two goods in the diagram on the left of the previous page.

So what does λ (the **Lagrange multiplier**) in the UMP represent? It is the marginal (shadow) value of relaxing the wealth constraint, which means that it **is the marginal utility of an increase in w** (for the optimal bundle of x). To see why, consider an interior solution ($x(p, w) >> 0$). The change in utility from a marginal increase in w is:

$$\partial u/\partial w = \nabla u(x(p, w)) \cdot D_w(x(p, w))$$

where $D_w(x(p, w))$ denotes the column vector with the elements $\partial x_1(p, w)/\partial w$, $\partial x_2(p, w)/\partial w$, … $\partial x_L(p, w)/\partial w$. Recall that an interior solution implies that $\nabla u(x^*) = \lambda p$. This implies that:

$$\partial u/\partial w = \lambda p \cdot D_w(x(p, w)) = \lambda$$

where $p \cdot D_w(x(p, w)) = 1$, as shown by the Engel aggregation condition in Lecture 2.

Note: The first order conditions $\nabla u(x^*) \leq \lambda p$ and $x^* \cdot [\nabla u(x^*) - \lambda p] = 0$ are **necessary** for x* to be a solution to the UMP. They are **sufficient** if u( ) is quasiconcave and monotone and $\nabla u(x) \neq 0$ for all $x \in \mathbb{R}^L_+$. See MGW, p.55.

Example: Deriving a Demand Function from a Utility Function

Let's consider a situation with only 2 goods, and assume the following utility function:

$$u(x_1, x_2) = kx_1^{\alpha}x_2^{1-\alpha}, \qquad \alpha \in (0, 1) \text{ and } k > 0.$$

It is convenient to maximize the log of u( ), which gives the same behavior. The Lagrangean is:

$$\mathscr{L} = \ln(k) + \alpha\ln(x_1) + (1-\alpha)\ln(x_2) + \lambda[w - p_1x_1 - p_2x_2]$$

The F.O.C. are:

$$\partial\mathscr{L}/\partial x_1 = \alpha/x_1 - \lambda p_1 = 0, \text{ which implies } \alpha/x_1 = \lambda p_1$$

$$\partial\mathscr{L}/\partial x_2 = (1-\alpha)/x_2 - \lambda p_2 = 0, \text{ which implies } (1-\alpha)/x_2 = \lambda p_2$$

Combining these two F.O.C. gives:

$(x_1 p_1)/\alpha = (x_2 p_2)/(1-\alpha)$, which implies $p_1 x_1 = [\alpha/(1-\alpha)]p_2 x_2$

Using the budget constraint $w = p_1 x_1 + p_2 x_2$ yields:

$$w = [\alpha/(1-\alpha)]p_2 x_2 + p_2 x_2 = [1/(1-\alpha)]p_2 x_2$$

Thus $x_2 = (w/p_2)(1-\alpha)$. You can also show that $x_1 = (w/p_1)\alpha$.

This functional form for utility is rather restrictive. For example, it implies that the budget share for good 1 $(p_1 x_1/w)$ equals α, and that the budget share for good 2 equals $(1-\alpha)$. Thus the fraction of total wealth spent on each good is unaffected by wealth and by prices.

Note: It is very useful if the demand function x(p, w) is both continuous and differentiable. See Appendix A of Chapter 3 of Mas-Colell (pp. 92-95) for the technical assumptions needed regarding preferences for x(p, w) to be continuous and differentiable.

IV. The Indirect Utility Function and Its Properties

We can take the demands x(p, w) that are the solutions to the UMP and plug them into the utility function to get an expression for utility that is a function of p and w, as opposed to a function of x:

$$u(x_1, x_2, \ldots x_L) = u(x_1(p, w), x_2(p, w), \ldots x_L(p, w)) = v(p, w)$$

The expression v(p, w) is called the **indirect utility function**. Note that, unlike the utility function, the indirect utility function **assumes maximizing behavior**. The following proposition explains what properties it has:

Proposition 3.D.3: Let u( ) be a continuous utility function that represents a locally nonsatiated preference relation $\succsim$ that is defined on the consumption set $X = \mathbb{R}^L_+$. The corresponding indirect utility function, v(p, w), is:

1. Homogenous of degree zero (in prices and wealth).

2. Strictly increasing in w and nonincreasing in p_ℓ for any ℓ

3. Quasiconvex: i.e. the set $\{(p, w): v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

4. Continuous in p and w.

Proof: Let's check these properties one by one:

1. Homogeneity of degree zero follows from the result that demand functions are homogenous of degree zero. If demands are unchanged then utility is unchanged.

2. Strictly increasing in w follows because an increase in w increases the consumption set, and local nonsatiation implies that there is a consumption bundle in the neighborhood of the previous optimal demand that is strictly preferred the previous optimal demand. Nonincreasing in p follows from the fact that an increase in any price decreases the consumption set, and thus cannot increase utility.

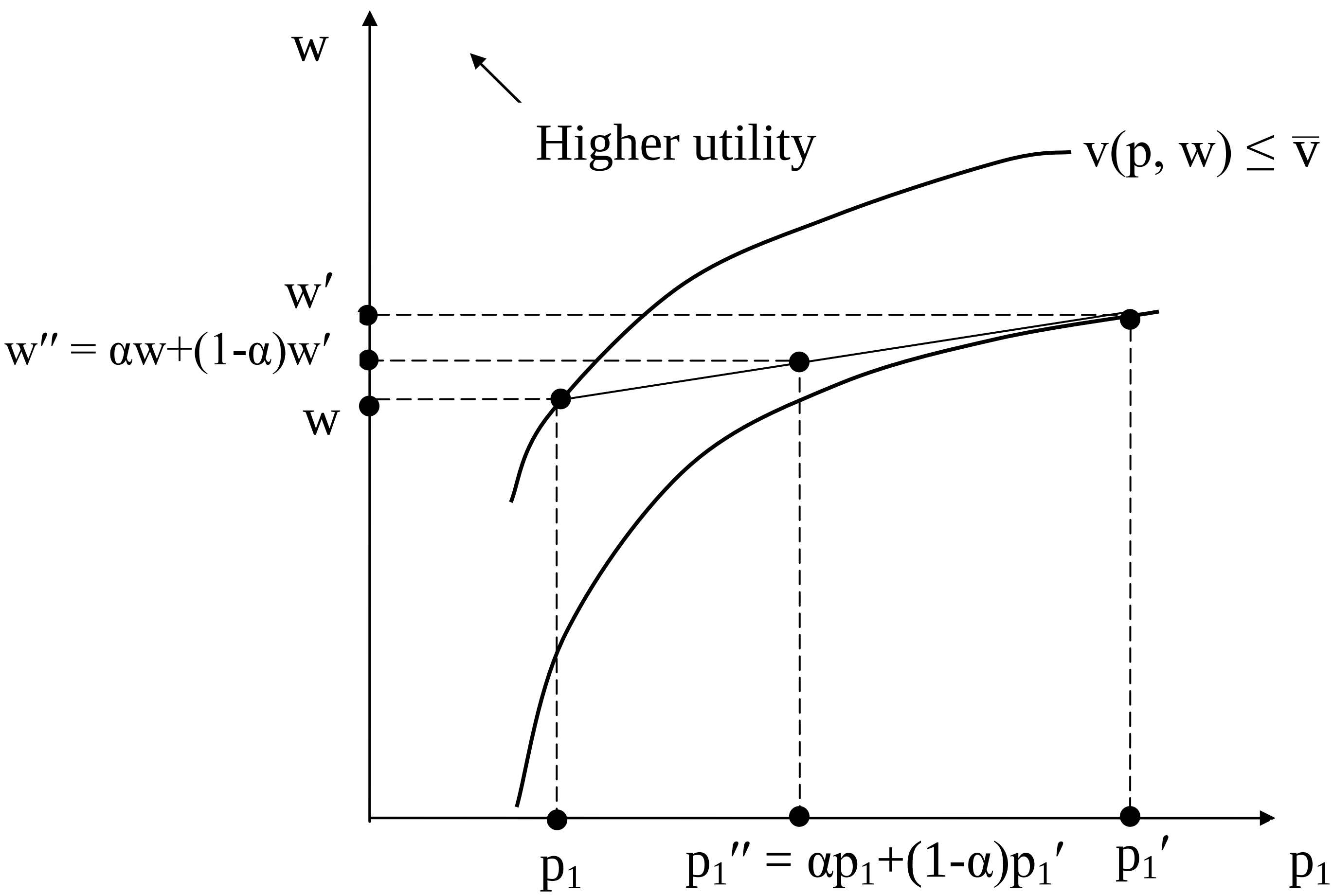
3. To verify that $v(p, w)$ is quasiconvex, start with two price wealth pairs, $v(p, w) \leq \bar{v}$ and $v(p', w') \leq \bar{v}$. Consider any price-wealth pair, denoted by (p'', w''), defined as $(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$ for any $\alpha \in [0, 1]$. We want to show that $v(p'', w'') \leq \bar{v}$. This is seen by showing that for any x for which $p''\cdot x \leq w''$, we must have $u(x) \leq \bar{v}$. To start, note that if $p''\cdot x \leq w''$ then:

   $$\alpha p\cdot x + (1-\alpha)p'\cdot x \leq \alpha w + (1-\alpha)w'$$

   Thus either $p\cdot x \leq w$ or $p'\cdot x \leq w'$ (or both). If $p\cdot x \leq w$, then $u(x) \leq v(p, w) \leq \bar{v}$, which is what we wanted to show. If $p'\cdot x \leq w'$, then $u(x) \leq v(p', w') \leq \bar{v}$, which is again what we wanted to show.

4. For continuity, we will just consider the case where preferences are strictly convex. Given that u( ) is assumed to be continuous and x(p, w) is continuous (shown in Appendix A of Chapter 3 of MWG) then v(p, w) = u(x(p, w)) will also be continuous. **Q.E.D.**

   This diagram shows the intuition for quasiconvexity:



The utility from the weighted average is ≤ the highest of the two “original” utilities.

This diagram is from Rubinstein, *Lecture Notes in Microeconomic Theory*, second edition (p.77).

Example continued: For Cobb-Douglas preferences, the indirect utility function is:

$$u = \alpha\ln(x_1) + (1-\alpha)\ln(x_2)$$

$$= \alpha[\ln(\alpha) + \ln(w) - \ln(p_1)] + (1-\alpha)[\ln(1-\alpha) + \ln(w) - \ln(p_2)]$$

$$= \ln(w) + \alpha\ln(\alpha) + (1-\alpha)\ln(1-\alpha) - \alpha\ln(p_1) - (1-\alpha)\ln(p_2)$$

V. The Expenditure Minimization Problem

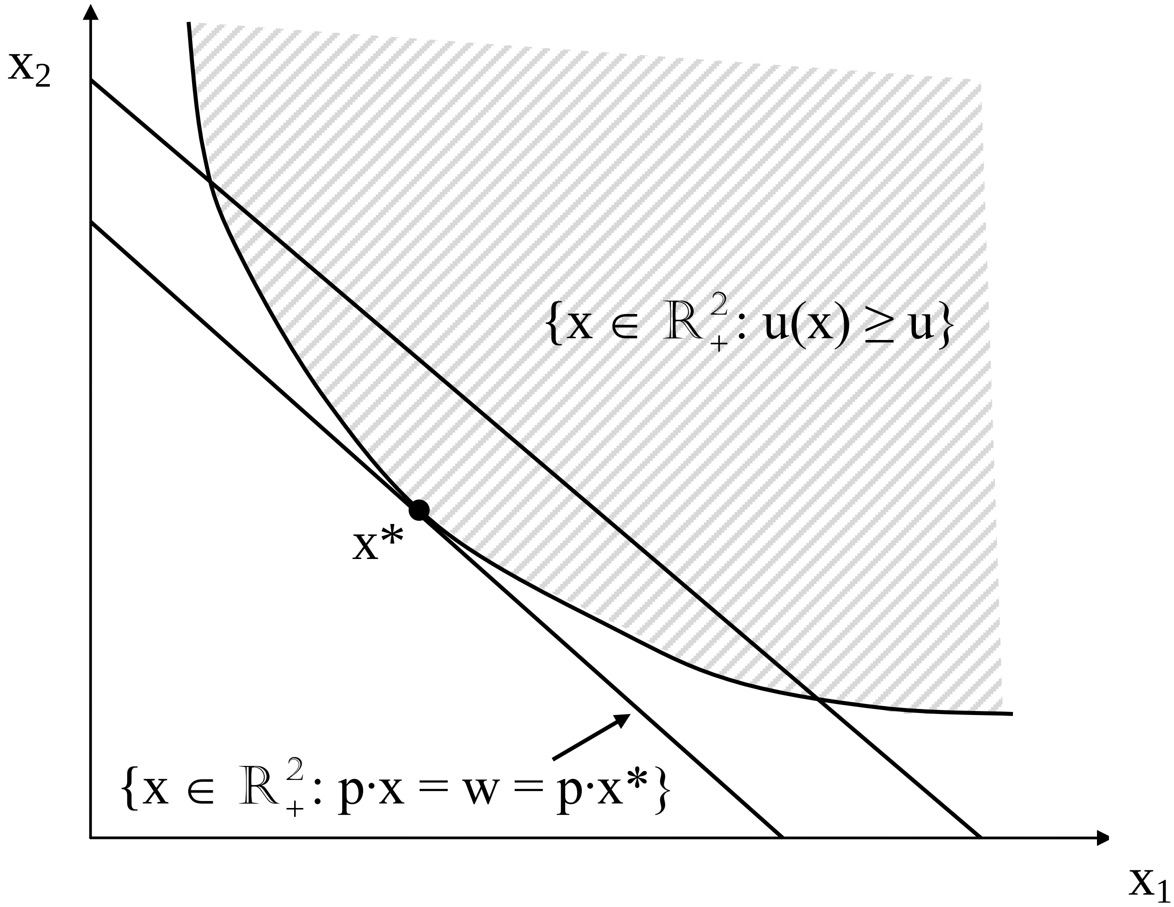
The "expenditure minimization problem" (**EMP**) is the mirror image ("dual") of the utility maximization problem. Instead of choosing a bundle of commodities to attain the highest utility possible given a fixed level of wealth (and given a set of prices), EMP chooses a bundle of commodities to **attain a given level of utility at the lowest possible cost**:

$$\underset{x \geq 0}{\text{Min}}\ p \cdot x \quad \text{subject to } u(x) \geq u$$

Compared to the UMP, the roles of the constraint and the objective are reversed.

For this discussion of EMP, assume that u( ) is continuous and represents a nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Also, assume that $p \gg 0$.

The expenditure function can be depicted in a diagram with two goods (L = 2). In the diagram below, x* is the optimal consumption bundle both for the UMP and the EMP. From the UMP perspective, it is the value of x on the budget line p·x = w that leads to the highest utility level. From the EMP perspective, it is the value of x on the utility curve u that attains that utility at the lowest cost.



Formally, the relationship between the EMP and the UMP can be stated as follows:

Proposition 3.E.1: Let u( ) be a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set X = $\mathbb{R}^L_+$. Assume that p >> 0 (all prices are > 0). Then:

1. If x* is optimal in the UMP when wealth is w (> 0), then x* is optimal in the EMP when the target utility level is u(x*). In addition, the minimized expenditure level for this EMP is exactly equal to w.

2. If x* is optimal in the EMP when the target utility level is u (> u(0)), then x* is optimal in the UMP when the wealth constraint is p·x* (= w). In addition, the maximized utility level for this UMP is exactly u.

Proof: Let's consider the two statements separately:

1. Suppose that x* is **not** optimal in the EMP with the target utility level u(x*). Then there exists an x′ such that u(x′) ≥ u(x*) **and** p·x′ < p·x* ≤ w. **By local nonsatiation**, we can find an x″ very close to x′ such that u(x″) > u(x′) and p·x″ < w. Note that x″ ∈ $B_{p,w}$ and u(x″) > u(x*), **but** this contradicts the assumption that x* is the optimal solution for the UMP. Thus x* must also be optimal for the EMP when the target utility is u(x*), and p·x* is the minimized expenditure level. Finally, since x* solves the UMP when wealth is w, by Walras' law we have p·x* = w.

2. Since u > u(0) it must be that $x^* \neq 0$, and so $p \cdot x^* > 0$. Suppose x^* is **not** optimal for the UMP when wealth is $p \cdot x^*$. Then there exists an x' such that $u(x') > u(x^*)$ **and** $p \cdot x' \leq p \cdot x^*$. Consider a bundle $x'' = \alpha x'$, where $\alpha \in (0, 1)$. **By continuity of u( )**, if α is close enough to 1 then we have $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$. **But** this contradicts the assumption that x^* is the optimal solution for the EMP. Thus x^* must optimal for the UMP when wealth (w) = $p \cdot x^*$, and the maximized utility level is therefore $u(x^*)$. [In proposition 3.E.3 below we will see that if x^* solves the EMP when the required utility level is u, then $u(x^*) = u$.] **Q.E.D.**

As long as $p >> 0$, there is a solution for the EMP under very general conditions. All we really need is for the constraint set to be non-empty; there must be some x for which $u(x) \geq u$ (the target utility level). We will assume this from now on.

The Expenditure Function

The x that minimizes expenditure for a given utility level u and a set of prices p is a function of those variables. This can be expressed as x = x(p, u). If we plug these expressions for the elements of x into the budget constraint (x = $p \cdot x$), we get the **expenditure function**, which is denoted by e(p, u). [In some papers/textbooks it is called the "cost function".] Note that e(p, u) is simply the optimal value of the function that is being minimized in the EMP.

The expenditure function has the following properties:

Proposition 3.E.2: Let u( ) be a continuous utility function that represents the (locally nonsatiated) preference relation $\succsim$ defined on the consumption set X = $\mathbb{R}^L_+$. The expenditure function e(p, u) is:

1. Homogenous of degree one in p.

2. Strictly increasing in u and nondecreasing in p_ℓ for all ℓ.

3. Concave in p.

4. Continuous in p and u.

Proof (for the first three properties only):

1. The constraint set ($u(x) \geq u$) is unaffected by a change in prices. As long as **relative** prices do not change, then the expenditure minimizing bundle will not change. If prices change by the factor α, then the cost of obtaining this same bundle becomes $\alpha p \cdot x^*$, so expenditure, and thus the expenditure function, increases by α: $e(\alpha p, u) = \alpha p \cdot x^* = \alpha e(p, u)$.

2. Proofs by contradictions work pretty well, so let's try it here. Suppose that e(p, u) is **not** strictly increasing

in u, and let x' and x'' be optimal consumption bundles for the target utility levels u' and u'', respectively, with $u'' > u'$ and $p \cdot x' \geq p \cdot x'' > 0$. Consider a commodity bundle $\tilde{x} = \alpha x''$, where $\alpha \in (0, 1)$. By continuity of u( ), there exists an α sufficiently close to 1 such that $u(\tilde{x}) > u'$ and $p \cdot x' \geq p \cdot x'' > p \cdot \tilde{x}$. **But** this contradicts the assumption the x' is optimal for the EMP with the required utility level u'.

To show that e(p, u) is nondecreasing in p_ℓ, consider two price vectors, p'' and p', for which $p_\ell'' \geq p_\ell'$ and $p_k'' = p_k'$ for $k \neq \ell$. Let x'' be an optimizing vector for the EMP for prices p''. Then $e(p'', u) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p', u)$. [The last inequality follows because x'' may not be the cost minimizing commodity bundle for p'; i.e. there may be another bundle that costs less.]

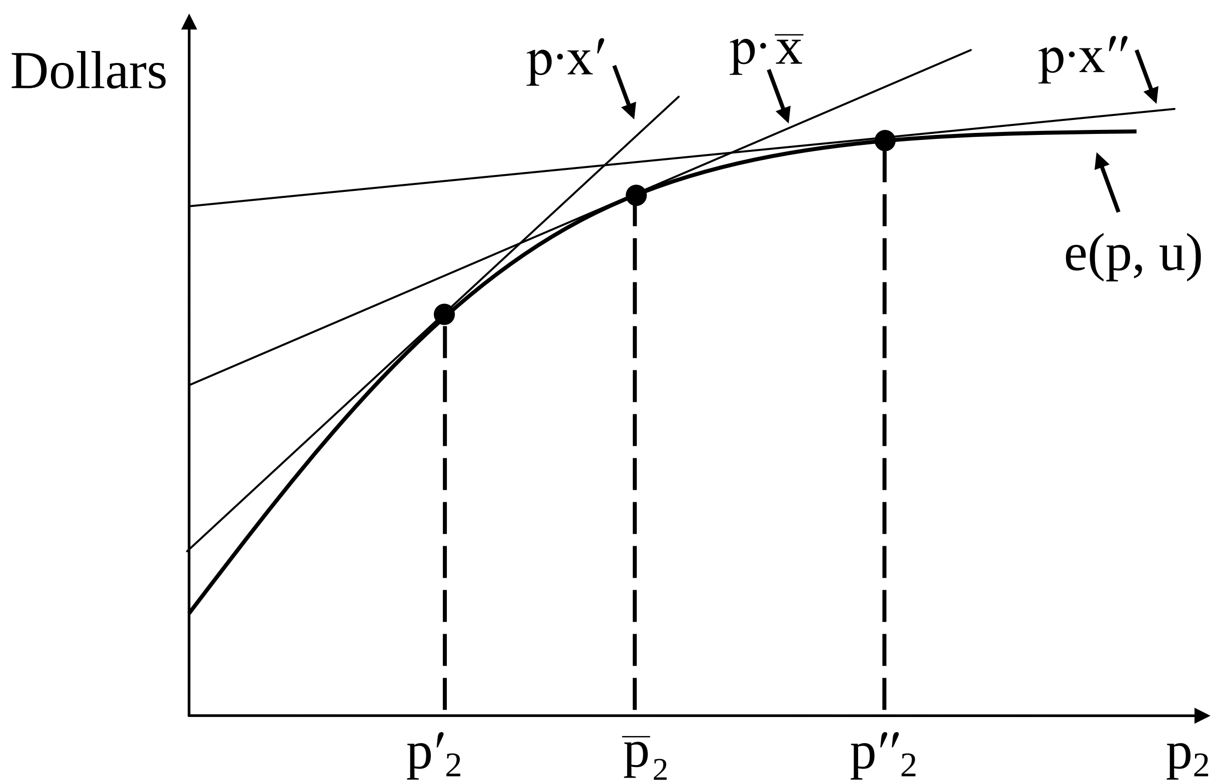
3. For concavity, set a target utility level, $\bar{u}$, and let $p'' = \alpha p + (1-\alpha)p'$ for any $\alpha \in [0, 1]$. Let x'' be the optimal bundle for the EMP with prices p''. Then

$$e(p'', \bar{u}) = p'' \cdot x'' = \alpha p \cdot x'' + (1-\alpha) p' \cdot x''$$

$$\geq \alpha e(p, \bar{u}) + (1-\alpha) e(p', \bar{u}).$$

where the last inequality follows because $u(x'') \geq \bar{u}$ and by the definition of the expenditure function, which implies $p \cdot x'' \geq e(p, \bar{u})$ and $p' \cdot x'' \geq e(p', \bar{u})$.

While this proof of concavity, a very important property, may not be very intuitive, the property itself is intuitive. Suppose that we initially have prices $\bar{p}$ and that $\bar{x}$ is (one of) the optimal consumption vector(s) for the EMP for prices $\bar{p}$. If prices change to p but we don't let the consumer change his or her consumption, then his or her expenditure will be $p \cdot \bar{x}$, which is a linear function of p. If the consumer is allowed to change $\bar{x}$, the value of the expenditure function will be $\leq p \cdot \bar{x}$. This is shown in the following figure, where p_2 varies and all other prices are fixed:



Proposition E.3.1 provides an important connection between the expenditure function and the indirect utility function: For any $p \gg 0$, $w > 0$ and $u > u(0)$, we have:

$$e(p, v(p, w)) = w \quad \text{and} \quad v(p, e(p, u)) = u$$

These relationships imply that, for any fixed price vector $\bar{p}$, the functions $e(\bar{p}, \cdot)$ and $v(\bar{p}, \cdot)$ are inverses of each other. See p.60 of Mas-Colell et al. for further discussion.

The Hicksian (Compensated) Demand Function

We can denote the optimal commodity vectors for the EMP by $h(p, u) \subset \mathbb{R}^L_+$; h(p, u) is called the Hicksian (or compensated) demand correspondence. If h(p, u) is unique then we call it the Hicksian demand function.

The three basic properties of Hicksian demand are given in Proposition 3.E.3, which is analogous to Proposition 3.D.2 for Walrasian Demand.

Proposition 3.E.3: Let u( ) be a continuous utility function that represents a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. For any $p \gg 0$, the Hicksian demand correspondence h(p, u) has the following properties:

1. Homogenous of degree zero in p: $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$.

2. No excess utility: For any $x \in h(p, u)$, $u(x) = u$.

3. Convexity/uniqueness: If $\succsim$ is convex, then h(p, u) is a convex set; and if $\succsim$ is strictly convex, so that u( ) is strictly quasiconcave, then h(p, u) is unique.

The proof is on p.61 of Mas Colell et al.

Proposition 3.E.1 implies the following relationships between Hicksian and Marshallian demands:

$$h(p, u) = x(p, e(p, u)) \quad \text{and} \quad x(p, w) = h(p, v(p, w))$$

Hicksian Demand and the Compensated Law of Demand

An important property of Hicksian demand is that it satisfies the **compensated law of demand**, which is that demand and price move in opposite directions. This can be summarized in the following proposition:

Proposition 3.E.4: Suppose that u( ) is a continuous utility function that represents a locally nonsatiated preference relation $\succsim$, and that h(p, u) consists of a single element for all $p >> 0$. Then the Hicksian demand function $h(p, u)$ satisfies the compensated law of demand:

$$(p'' - p')\cdot[h(p'', u) - h(p', u)] \le 0$$

Note: If only one price changes, this implies that the change in the Hicksian demand for that good is ≤ 0.

Proof: For any $p >> 0$, the consumption bundle $h(p, u)$ is optimal for the EMP, so it achieves a lower expenditure at prices p than any other bundle offering utility $\ge u$. Thus:

$$p''\cdot h(p'', u) \le p''\cdot h(p', u) \quad \text{and} \quad p'\cdot h(p'', u) \ge p'\cdot h(p', u)$$

These in turn imply:

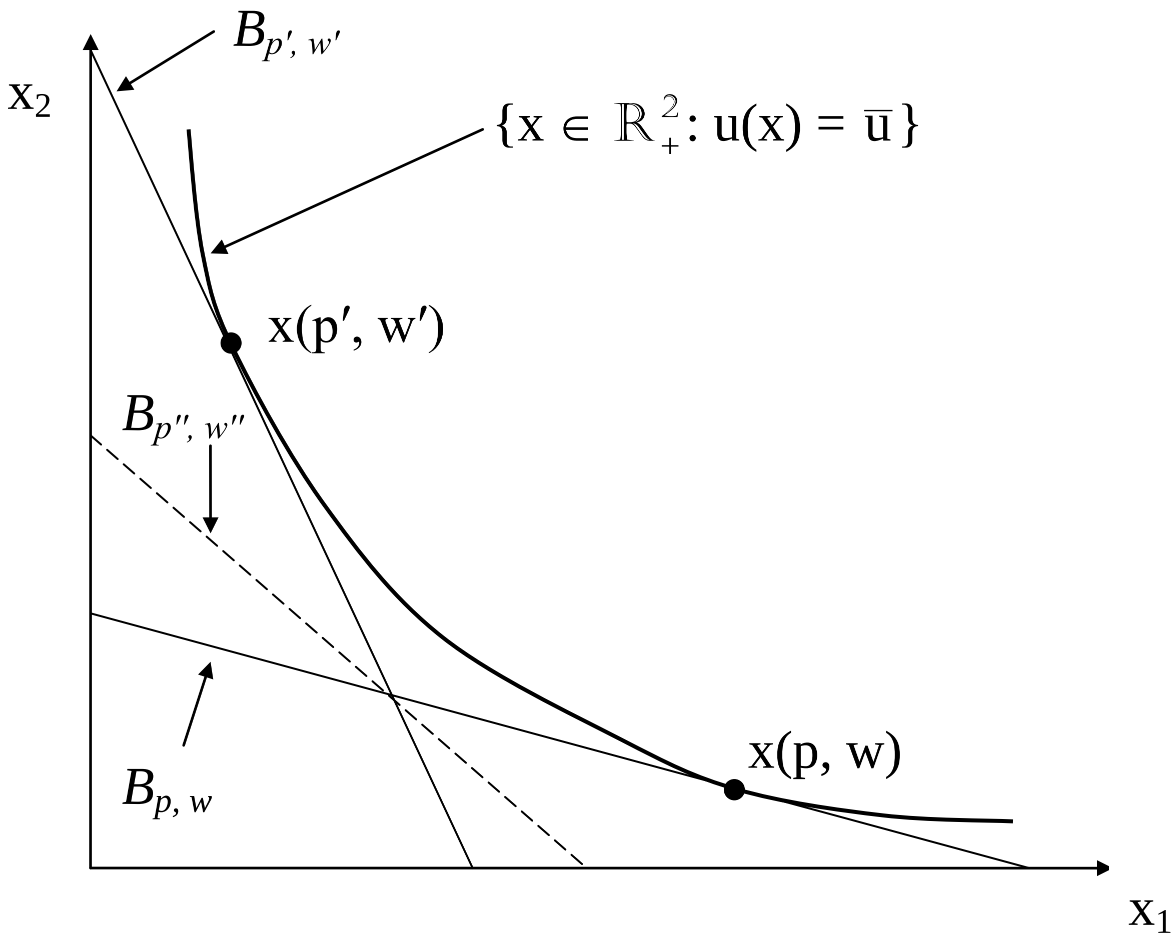
$$p''\cdot[h(p'', u) - h(p', u)] \le 0 \;\; \text{and} \;\; p'\cdot[h(p'', u) - p'\cdot h(p', u)] \ge 0$$

Combining these yields: $(p'' - p')\cdot[h(p'', u) - h(p', u)] \le 0$.

Question: How is this different (or not different) from the compensated law of demand discussed in Lecture 3?

Old diagram for intuition of quasi-convexity of indirect utility function:

This diagram shows the intuition for quasiconvexity:



The budget line corresponding to (p″, w″) is the dashed line. It is a weighted average of the other two budget lines in the diagram. Clearly, the maximum utility that can be attained on the budget line (p″, w″) cannot exceed the maximum amount attained on the other two lines.