DUALITY

APEC 8002, Fall 2016 **Instructor: Terry Hurley**

INTRODUCTION

As we have moved through production, cost minimization, revenue maximization, and finally profit maximization, you might have felt déjà vu. We seemed to keep repeating ourselves with a few changes in notation and simply flipping things around. You felt this way because all these objects are very closely related. Duality formalizes these relationships and the conditions needed for these relationships to hold. Essentially it asks questions like:

- If I have estimated a cost function, can I figure out what the underlying production possibilities set is? Or can I use this information to find the profit maximizing output?
- If I have estimated conditional input demands, can I find the input demands without knowing the underlying production possibilities set?
- If I have estimated the revenue function, can I find the output distance function?

The answer to these questions is generally yes, provided the production possibilities set is nonempty, closed, convex, and satisfies free disposal. This is particularly good news for applied economists because we do not always have concrete information on inputs or outputs even though we may have pretty good information on prices, costs, and profits. In this section, we will look a little closer at these various relationships and summarize them.

PROFIT, COST, & REVENUE FUNCTIONS

The first relationship we will explore is the relationship between the profit and cost functions. Recall that our conditional input demand and cost function are defined as

D1$^{\text{cm}}$ $\mathbf{Z}(\mathbf{r}, \mathbf{q}) = \{\mathbf{z} \in \mathbf{IRS}(\mathbf{q}) \colon \mathbf{r} \cdot \mathbf{z}' \geq \mathbf{r} \cdot \mathbf{z} \text{ for all } \mathbf{z}' \in \mathbf{IRS}(\mathbf{q})\}$, and

D2$^{\text{cm}}$ $C(\mathbf{r}, \mathbf{q}) = \mathbf{r} \cdot \mathbf{z}(\mathbf{r}, \mathbf{q})$.

We defined our supply, input demand, and profit function as

D1$^{\text{pm}}$ $\mathbf{Y}(\mathbf{p}, \mathbf{r}) = \{(\mathbf{q}, -\mathbf{z}) \in \mathbf{PPS} \colon \mathbf{p} \cdot \mathbf{q} - \mathbf{r} \cdot \mathbf{z} \geq \mathbf{p} \cdot \mathbf{q}' - \mathbf{r} \cdot \mathbf{z}' \text{ for all } (\mathbf{q}', -\mathbf{z}') \in \mathbf{PPS}\}$

and

D2$^{\mathbf{pm}}$ $\pi(\mathbf{p},\mathbf{r}) = \mathbf{p}\cdot\mathbf{q}(\mathbf{p},\mathbf{r}) - \mathbf{r}\cdot\mathbf{z}(\mathbf{p},\mathbf{r})$.

Consider an alternative definition of our supply and profit function that uses the cost function

D3 $\mathbf{Q}^{a}(\mathbf{p},\mathbf{r}) = \{\mathbf{q} \text{ feasible}: \mathbf{p}\cdot\mathbf{q} - C(\mathbf{r},\mathbf{q}) \geq \mathbf{p}\cdot\mathbf{q}' - C(\mathbf{r},\mathbf{q}') \text{ for all feasible } \mathbf{q}'\}$

and

D4 $\pi^{a}(\mathbf{p},\mathbf{r}) = p\cdot q^{a}(\mathbf{p},\mathbf{r}) - C(\mathbf{r},\mathbf{q}^{a}(\mathbf{p},\mathbf{r}))$

where **q** feasible implies $\mathbf{q} \in \{\mathbf{q} \geq 0: (\mathbf{q}, -\boldsymbol{z}) \in \mathbf{PPS} \text{ for all } \mathbf{z} \geq 0\}$. That is, **q** being feasible means that there is some **z** that is capable of producing it. Equations D1^{cm}, D2^{cm}, D1^{pm}, D3, and D4 imply that for any (**q**, -**z**) $\in \mathbf{Y}(\mathbf{p},\mathbf{r})$, $\mathbf{q} \in \mathbf{Q}^{a}(\mathbf{p},\mathbf{r})$. Suppose this were not the case, which implies that there exists ($\mathbf{q}^0$, -$\mathbf{z}^0$) $\in \mathbf{Y}(\mathbf{p},\mathbf{r})$ where $\mathbf{q}^0 \notin \mathbf{Q}^{a}(\mathbf{p},\mathbf{r})$. By equation D1^{pm},

D5 $(\mathbf{q}^0, -\mathbf{z}^0) \in \mathbf{PPS}$ and $\mathbf{p}\cdot\mathbf{q}^0 - \mathbf{r}\cdot\mathbf{z}^0 \geq \mathbf{p}\cdot\mathbf{q}' - \mathbf{r}\cdot\mathbf{z}'$ for all $(\mathbf{q}', -\mathbf{z}') \in \mathbf{PPS}$.

By equation D3, $\mathbf{q}^0 \notin \mathbf{Q}^{a}(\mathbf{p},\mathbf{r})$ implies $\mathbf{q}^0$ is not feasible, which contradicts $(\mathbf{q}^0, -\mathbf{z}^0) \in \mathbf{PPS}$, or there exist a feasible **q**'' such that

D6 $\mathbf{p}\cdot\mathbf{q}'' - C(\mathbf{r},\mathbf{q}'') > \mathbf{p}\cdot\mathbf{q}^0 - C(\mathbf{r},\mathbf{q}^0)$ or $\mathbf{p}\cdot\mathbf{q}'' - \mathbf{r}\cdot\mathbf{z}''(\mathbf{r},\mathbf{q}'') > \mathbf{p}\cdot\mathbf{q}^0 - \mathbf{r}\cdot\mathbf{z}^0(\mathbf{r},\mathbf{q}^0)$

which contradicts D5.

What do these equations tell us? They tells us

$$\mathbf{z}(\mathbf{p},\mathbf{r}) = \mathbf{z}\big(\mathbf{r},\mathbf{q}(\mathbf{p},\mathbf{r})\big), \text{ and}$$

$$\pi(\mathbf{p},\mathbf{r}) = \mathbf{p}\cdot\mathbf{q}(\mathbf{p},\mathbf{r}) - C\big(\mathbf{r},\mathbf{q}(\mathbf{p},\mathbf{r})\big).$$

More importantly, they tells us that if we have a cost function, we can find our supply and input demands by solving

$$\max_{\mathbf{q}\geq 0} \mathbf{p}\cdot\mathbf{q} - C(\mathbf{r},\mathbf{q}).$$

That is, we do not need to find the **PPS** in order to solve

$$\max_{\mathbf{q}\geq 0,\mathbf{z}\geq 0} \mathbf{p}\cdot\mathbf{q} - \mathbf{r}\cdot\mathbf{z} \text{ subject to } (\mathbf{q},-\mathbf{z}) \in \mathbf{PPS}.$$

We will now explore the relationship between the profit and revenue function using a different sort of argument. This argument will rely on the added assumption of differentiability. Recall that with a differentiable output distance function, our conditional supply will satisfy

D7$^{\mathbf{rm}}$ $\frac{\partial L}{\partial q_m} = p_m - \gamma^{rm*}\frac{\partial D_O(\mathbf{q}^*,\mathbf{z})}{\partial q_m} \leq 0$, $\frac{\partial L}{\partial q_m} q_m{}^* = 0$, and $q_m{}^* \geq 0$ for $m = 1,\ldots, M$,

D8$^{\mathbf{rm}}$ $D_O(\mathbf{q}^*,\mathbf{z}) = 1$ and $\gamma^{rm*} \geq 0$.

With profit maximization, our supply will satisfy

D7$^{\mathbf{pm}}$ $\frac{\partial L}{\partial q_m} = p_m - \gamma^{pm*}\frac{\partial D_O(\mathbf{q}^*,\mathbf{z}^*)}{\partial q_m} \leq 0$, $\frac{\partial L}{\partial q_m} q_m{}^* = 0$, and $q_m{}^* \geq 0$ for $m = 1,\ldots, M$,

$\frac{\partial L}{\partial z_n} = -r_n - \gamma^{pm*}\frac{\partial D_O(\mathbf{q}^*,\mathbf{z}^*)}{\partial z_n} \leq 0$, $\frac{\partial L}{\partial z_n} z_n{}^* = 0$, and $z_n{}^* \geq 0$ for $n = 1,\ldots, N$,

D8$^{\mathbf{pm}}$ $D_O(\mathbf{q}^*,\mathbf{z}^*) = 1$ and $\gamma^{pm*} \geq 0$.

Consider the alternative problem

D9 $\max_{\mathbf{z}\geq 0} \pi = R(\mathbf{p},\mathbf{z}) - \mathbf{r}\cdot\mathbf{z}$,

which yields the first order conditions

D10 $\frac{\partial \pi}{\partial z_n} = \frac{\partial R(\mathbf{p},\mathbf{z}^*)}{\partial z_n} - r_n \leq 0$, $\frac{\partial \pi}{\partial z_n} z_n{}^* = 0$, and $z_n{}^* \geq 0$ for $n = 1,\ldots, N$.

Note that

D11 $R(\mathbf{p},\mathbf{z}) = \mathbf{p}\cdot\mathbf{q}(\mathbf{p},\mathbf{z}) + \gamma^{rm}(\mathbf{p},\mathbf{z})\big(1 - D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})\big)$.

Differentiating with respect to z_n, then implies

D12 $$\frac{\partial R(\mathbf{p},\mathbf{z})}{\partial z_n} = \sum_{m=1}^{M}\left(p_m - \gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial q_m}\right)\frac{\partial q_m(\mathbf{p},\mathbf{z})}{\partial z_n} + \frac{\partial \gamma^{rm}(\mathbf{p},\mathbf{z})}{\partial z_n}\big(1 - D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})\big) - \gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial z_n}.$$

We know $1 - D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z}) = 0$. Differentiating

$$\left(p_m - \gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial q_m}\right) q_m(\mathbf{p},\mathbf{z}) = 0$$

from equation $D7^{rm}$ with respect to z_n then implies

$$\left(p_m - \gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial q_m}\right)\frac{\partial q_m(\mathbf{p},\mathbf{z})}{\partial z_n} = \frac{\partial\left(p_m - \gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial q_m}\right)}{\partial z_n} q_m(\mathbf{p},\mathbf{z}).$$

But equation $D7^{rm}$ then implies $\left(p_m - \gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial q_m}\right)\frac{\partial q_m(\mathbf{p},\mathbf{z})}{\partial z_n} = 0$ because $p_m - \gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial q_m} = 0$ or $q_m(\mathbf{p},\mathbf{z})$=0. Therefore, equation D12 can be written as

$$\frac{\partial R(\mathbf{p},\mathbf{z})}{\partial z_n} = -\gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}),\mathbf{z})}{\partial z_n},$$

which means we can rewrite equation D10 as

D10' $\frac{\partial\pi}{\partial z_n} = -\gamma^{rm}(\mathbf{p},\mathbf{z})\frac{\partial D_O(\mathbf{q}(\mathbf{p},\mathbf{z}^*),\mathbf{z}^*)}{\partial z_n} - r_n \leq 0$, $\frac{\partial\pi}{\partial z_n} z_n{}^*$, and $z_n{}^* \geq 0$ for $n = 1,\ldots, N$.

Comparing equations D7^{rm}, D8^{rm} and D10' to equations D7^{pm} and D8^{pm}, they are identical for $\mathbf{q}(\mathbf{p},\mathbf{z}^*) = \mathbf{q}^*$ after substituting out γ^{rm*} and γ^{pm*}. This means that

$$\mathbf{q}(\mathbf{p},\mathbf{r}) = \mathbf{q}\big(\mathbf{p},\mathbf{z}(\mathbf{p},\mathbf{r})\big), \text{ and}$$

$$\pi(\mathbf{p},\mathbf{r}) = R\big(\mathbf{p},\mathbf{z}(\mathbf{p},\mathbf{r})\big) - \mathbf{r}\cdot\mathbf{z}(\mathbf{p},\mathbf{r}).$$

Therefore, if we have a revenue function, we can find our supply and input demands by solving

$$\max_{\mathbf{z}\geq 0} R(\mathbf{p},\mathbf{z}) - \mathbf{r}\cdot\mathbf{z}.$$

PROFIT, COST, REVENUE & DISTANCE FUNCTIONS

There are a variety of other duality relationships that we should be aware of, though we will not take the time to establish them formally:

(i) $\mathbf{IRS}(\mathbf{q}) = \{\mathbf{z} \geq 0 : \mathbf{r}\cdot\mathbf{z} \geq C(\mathbf{r},\mathbf{q}) \text{ for all } \mathbf{r}\}$;

(ii) $\mathbf{FOS}(\mathbf{z}) = \{\mathbf{q} \geq 0 : R(\mathbf{p},\mathbf{z}) \geq \mathbf{p}\cdot\mathbf{q} \text{ for all } \mathbf{p}\}$;

(iii) $\mathbf{PPS} = \{\mathbf{y} : \pi(\mathbf{p}) \geq \mathbf{p}\cdot\mathbf{y} \text{ for all } \mathbf{p}\}$;

(iv) $D_I(\mathbf{q},\mathbf{z}) = \min_{\mathbf{r}>0}\{\mathbf{r}\cdot\mathbf{z} : C(\mathbf{r},\mathbf{q}) \geq 1\}$;

(v) $D_O(\mathbf{q},\mathbf{z}) = \max_{\mathbf{p}>0}\{\mathbf{p}\cdot\mathbf{q} : R(\mathbf{p},\mathbf{z}) \leq 1\}$;

(vi) $D_I(\mathbf{q},\mathbf{z}) = \min_{\mathbf{p}>0,\mathbf{r}>0}\{\mathbf{r}\cdot\mathbf{z} : \mathbf{p}\cdot\mathbf{q} - \pi(\mathbf{p},\mathbf{r}) \geq 1\}$; and

(vii) $D_O(\mathbf{q},\mathbf{z}) = \max_{\mathbf{p}>0,\mathbf{r}>0}\{\mathbf{p}\cdot\mathbf{q} : \mathbf{r}\cdot\mathbf{z} + \pi(\mathbf{p},\mathbf{r}) \leq 1\}$.

Figure D1 provides a visual illustration. These results can be formally established by appealing to our assumptions that feasible output and input requirement sets are nonempty, closed, convex, and satisfy weak free disposal. If we wanted to be truly rigorous, min in (iv) and (vi) should be replaced with inf and max in (v) and (vii) should be replaced with sup.

The duality relationships in (i) – (iii) tell us that if we have bothered to estimate a cost function, revenue function, or profit function, then we can construct the underlying input requirement, feasible output, or production possibilities sets, though to do so could get quite tedious. They also tell us that these functions embody anything we might want to know about the technical relationship between inputs and outputs. They are of importance to theorist, but of more limited utility for empiricists. The duality relationships in (iv) – (vii) give empiricists a straightforward way to recover the input or output distance function describing the technical relations of interest given some cost, revenue, or profit function they may have estimated.

To convince ourselves that all of this actually works, recall the input distance function $D_I(\mathbf{q},\mathbf{z}) = \frac{\sqrt{z_1 z_2}}{q_1{}^2+q_2{}^2}$ which we used to derive the cost function $C(\mathbf{r},\mathbf{q}) = 2\sqrt{r_1 r_2}(q_1{}^2 + q_2{}^2)$. Let us solve the problem

$$\min_{\mathbf{r}>0} r_1 z_1 + r_2 z_2 \text{subject to } 2\sqrt{r_1 r_2}(q_1{}^2 + q_2{}^2) \geq 1.$$

The Lagrangian and first-order conditions are

$$L = r_1 z_1 + r_2 z_2 + \gamma\left(1 - 2\sqrt{r_1 r_2}(q_1{}^2 + q_2{}^2)\right),$$

$$\frac{\partial L}{\partial r_1} = z_1 - \gamma^* \sqrt{\frac{r_2}{r_1{}^*}}(q_1{}^2 + q_2{}^2) \geq 0, \frac{\partial L}{\partial r_1} r_1{}^* = 0, \text{ and } r_1{}^* > 0,$$

$$\frac{\partial L}{\partial r_2} = z_2 - \gamma^* \sqrt{\frac{r_1{}^*}{r_2{}^*}}(q_1{}^2 + q_2{}^2) \geq 0, \frac{\partial L}{\partial r_2} r_2{}^* = 0, \text{ and } r_2{}^* > 0, \text{ and}$$

$$\frac{\partial L}{\partial \gamma} = 1 - 2\sqrt{r_1{}^* r_2{}^*}(q_1{}^2 + q_2{}^2) \leq 0, \frac{\partial L}{\partial \gamma} \gamma^* = 0, \text{ and } \gamma^* \geq 0.$$

For an interior solution, the middle two equations imply $r_2{}^* = r_1{}^* \frac{z_1}{z_2}$. Substituting into the last equation then yields $r_1{}^* = \sqrt{\frac{z_2}{z_1}} \frac{1}{2(q_1{}^2+q_2{}^2)}$ such that $r_2{}^* = \sqrt{\frac{z_1}{z_2}} \frac{1}{2(q_1{}^2+q_2{}^2)}$. Therefore,

$$D_I(q_1, q_2, z_1, z_2) = \sqrt{\frac{z_2}{z_1}} \frac{1}{2(q_1{}^2 + q_2{}^2)} z_1 + \sqrt{\frac{z_1}{z_2}} \frac{1}{2(q_1{}^2 + q_2{}^2)} z_2 = \frac{\sqrt{z_1 z_2}}{(q_1{}^2 + q_2{}^2)}$$

as expected.

Figure D1: Duality relationships