AFRE 802
Statistical Methods for Agricultural, Food, & Resource Economists

Tchebysheff's Inequality & Integration (CW Ch. 14.1-14.4)

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GAME PLAN

- Housekeeping issues:
  - Review session 8:30-9:30 AM tomorrow
  - Ch. 3 HW is due on Tuesday
- Review
- No graded in-class exercise today
- Finish discussion of discrete random variables
  - Tchebysheff's Inequality
- Integration
  - Reading for this section is C&W Ch. 14 (on D2L)
- Next week: continuous random variables

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Review

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We have discussed 5 specific, common discrete probability distributions:

1. **Bernoulli** (1 trial, only 2 outcomes: S (Y=1) or F (Y=0); *p* is probability of S, *q=1-p* is probability of F)
2. **Binomial** (n independent Bernoulli trials, *Y* is # of Ss)
3. **Geometric** (series of independent Bernoulli trials, *Y* is the # of the trial on which the 1st S occurs)
4. **Negative binomial** (series of independent Bernoulli trials, *Y* is the # of the trial on which the rth S occurs)
5. **Poisson** (*Y* is the # of times an event occurs in a given interval, and *λ* is the average value of *Y)*

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Review

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Table 1 Discrete Distributions

Distribution	Probability Function	Mean	Variance
Bernoulli	$p(y) = p^y(1-p)^{1-y}$; $y = 0, 1$	p	$p(1-p)$
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$; $y = 0, 1, \ldots, n$	np	$np(1-p)$
Geometric	$p(y) = p(1-p)^{y-1}$; $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$; $y = 0, 1, 2, \ldots$	λ	λ
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$; $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

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Review

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Poisson approximation to the binomial distribution

- Poisson distribution can be derived as the limit of a binomial distribution as the number of trials (n) → ∞
- Because of this relationship, Poisson probabilities can be used to approximate binomial probabilities when:
  - The # of trials **(*n*) is large**, and
  - The probability of success **(*p*) is small**

  Such that:
  - ***λ=np* roughly < 7** (for our purposes; rules of thumb vary)
  - Recall that E(Y)=λ for Poisson and E(Y)=np for binomial (hence λ=np)

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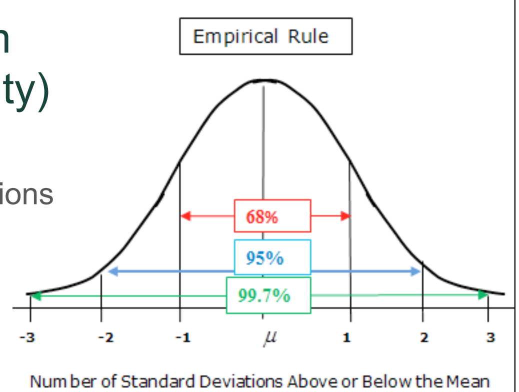
Tchebysheff's Theorem (Chebyshev's Inequality)

- Recall the "**empirical rule**": useful for probability distributions that are roughly bell-shaped → can determine approx. probability of being in $\mu \pm k$



- But many distributions are NOT bell-shaped
- **Tchebysheff's Theorem**: can use for any probability distribution to determine the lower bound for probability of being in $\mu \pm k\sigma$

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Tchebysheff's Inequality

For **any** RV, *Y,* with with mean μ & variance σ^2:

$$P\left[(\mu - k\sigma) < Y < (\mu + k\sigma)\right] \geq 1 - \frac{1}{k^2}$$

or

$$P\left[Y \leq (\mu - k\sigma) \text{ OR } Y \geq (\mu + k\sigma)\right] \leq \frac{1}{k^2}$$

for any constant $k > 0$

The probability of being less than k standard deviations from the mean is at least $1-1/k^2$

The probability of being k or more standard deviations from the mean is no more than $1/k^2$

Tchebysheff's Theorem - example

The number of customers per day at a sales counter, *Y*, has been observed for a long period of time and found to have mean 20 and standard deviation 2. The probability distribution of *Y* is not known. What can be said about the probability that Y will be greater than 16 but less than 24? (Hint: find k by determining # of standard deviations 16 and 24 are from their means, then use the formula on the previous slide.)

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Tchebysheff's Theorem (cont'd)

k	$P[(\mu-k\sigma)<Y<(\mu+k\sigma)] \geq 1-1/k^2$	$P[Y\leq(\mu-k\sigma) \text{ OR } Y\geq(\mu+k\sigma)] \leq 1/k^2$
1	0	1
2	0.750	0.250
3	0.889	0.111
4	0.938	0.063
5	0.960	0.040
6	0.972	0.028
7	0.980	0.020
8	0.984	0.016
9	0.988	0.012
10	0.990	0.010
	Etc.	

- *Which of these is **upper bound** (max.) vs. **lower bound** (min.) of a probability?*
- *Lower bound (min.) probability of being less than 4 standard deviations from the mean for any distribution?*
- *Upper bound (max.) probability of being 3 or more standard deviations from the mean for any distribution?*

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Example - Tchebysheff's Theorem

The U.S. mint produces dimes with an average diameter of .5 inch and standard deviation .01. Using Tchebysheff's theorem, find a lower bound for the number of coins in a lot of 400 coins that are expected to have a diameter between .48 and .52.

We'll come back and do this example at the end of class if we finish the rest of the material early

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Integration: hopefully this is review from college/high school!

- Notation

Integral sign

"integrate w.r.t. x"

$$\int \underbrace{f(x)}_{\text{Integrand}} dx$$

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Integration: the reverse of differentiation

$$\boxed{\frac{dF(x)}{dx} = f(x) \qquad \Rightarrow \qquad \int f(x)\,dx = F(x) + c}$$

Why do we need to add a constant ("+ c")?

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Indefinite vs. definite integrals

What is the difference between indefinite and definite integrals?

$$\int f(x)\,dx \qquad\qquad \int_a^b f(x)\,dx$$

Indefinite Definite

Indefinite integral is a function; definite integral is a number.

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Rules of integration

Rule 1. The power rule

What is $\int x^n\,dx$ *? Remember that we need it to satisfy* $\frac{dF(x)}{dx} = f(x) \quad \Rightarrow \quad \int f(x)\,dx = F(x) + c$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)}$$

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Rules of integration

Rule 1. The power rule - examples

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

1. $\int x^2\,dx = ?$

2. $\int \frac{1}{x^4}\,dx = ?$

3. $\int x^{3/2}\,dx = ?$

Confirm your answer by checking that the derivative of the integral equals the integrand.

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Rules of integration

Rule 2. The exponential rule

What is $\int e^x\,dx$ *? Remember that we need it to satisfy* $\frac{dF(x)}{dx} = f(x) \quad \Rightarrow \quad \int f(x)\,dx = F(x) + c$

$$\boxed{\int e^x\,dx = e^x + c}$$

$$\boxed{\text{More generally, } \int f'(x)e^{f(x)}\,dx = e^{f(x)} + c}$$

EX) $\int 2xe^{x^2}\,dx = ?$

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Rules of integration

Rule 3. The logarithmic rule

What is $\int \frac{1}{x} dx$ *? Remember that we need it to satisfy* $\frac{dF(x)}{dx} = f(x) \quad \Rightarrow \quad \int f(x)\,dx = F(x) + c$

$$\int \frac{1}{x} dx = \ln x + c \quad (x > 0)$$

$$\int \frac{1}{x} dx = \ln |x| + c \quad (x \neq 0)$$

More generally,

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c \quad (f(x) > 0)$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c \quad (f(x) \neq 0)$$

EX) $\int \frac{2x+1}{x^2+x-5} dx = ?$

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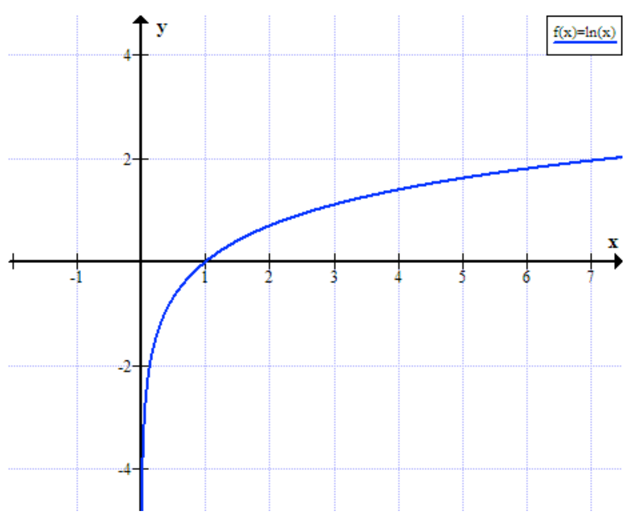
ln(x) does not exist for negative numbers



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Rules of integration

Rule 4. The integral is a linear operator

$$\boxed{\begin{array}{l}\int\left[af(x)+bg(x)\right]dx = a\int f(x)\,dx + b\int g(x)\,dx \\ \text{for constants } a \text{ and } b, \text{ and functions } f(.) \text{ and } g(.)\end{array}}$$

EX) $\int\left(x^3+2x+4\right)dx = ?$

Rules of integration

Rule 5. The substitution rule

$$\boxed{\int f(u)\frac{du}{dx}dx = \int f(u)\,du = F(u)+c}$$

EX) $\int\left(x^3+2x+1\right)^{-2}(3x^2+2)\,dx = ?$

EX) $\int xe^{x^2}\,dx = ?$

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Rules of integration

Rule 6. Integration by parts

$$\boxed{\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx}$$

Tip: In general you want to make your g(x) the part that is difficult to integrate.

EX) $\int xe^x\,dx = ?$

EX) $\int x^3 \ln x\,dx = ?$

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Definite integrals

$\int f(x)\,dx$ — Indefinite

$\int_a^b f(x)\,dx$ — Definite

Recall that an **indefinite integral is a function**; a **definite integral is a number.**

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Definite integrals

$$\boxed{\int_a^b f(x)\,dx = F(x)+c\Big]_a^b = F(b)-F(a)}$$

What happened to the c's?

EX) $\int_1^5 3x^2\,dx = ?$

EX) $\int_a^b ke^x\,dx = ?$

EX) $\int_a^x f(x)\,dx = ?$

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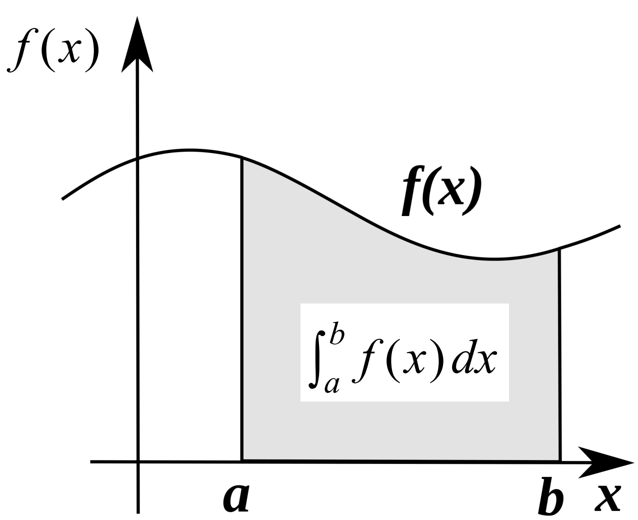
A definite integral as an area under a curve



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Properties of definite integrals

$$1.\ \int_a^b \left[\alpha f(x) + \beta g(x)\right] dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

$$2.\ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$3.\ \int_a^a f(x)\,dx = 0$$

4. If $a < b < c < d$

$$\int_a^d f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

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See handout for more practice problems

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Homework:

- Chiang & Wainwright (CW), Ch. 14:
  - Exercise 14.2: 1, 2, 3, 4
  - Exercise 14.3: 1, 2
- **CW Ch. 14 HW will be due next Thursday, Sep. 28
- Reminder: Ch. 3 HW due Tuesday, Sep. 26

Next class:

- Continuous random variables (Part 1 of 3)

Reading for next class:

- WMS Ch. 4 (sections 4.1-4.3)

Application for next class:

- Look up application in your field of uniform or normal distribution