ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 7: Welfare Evaluation and the Strong Axiom (MWG, Ch. 3, pp.80-92)

I. Introduction

So far in this class we have discussed consumer and choice behavior in terms of **describing** that **behavior**. This is often called **positive economics**, which consists of studying economic phenomena only to understand those phenomena, not to make policy recommendations (or any other kind of recommendations). Yet demand theory also provides guidance on policy decisions, which is often referred to as **normative economics**. More specifically, demand theory is often used to conduct **welfare analysis**, which examines the impact of different economic (and other) policies on consumers' welfare, as measured by utility (more precisely, by the money required to reach a certain level of utility, via the expenditure function).

This lecture provides an introduction to welfare analysis, focusing on the impact of a change in prices (perhaps brought about by a change in taxes). Apec 8004 will cover this in more detail. Near the end of the lecture, we finish our discussion of demand theory with a brief discussion of the strong axiom of revealed preference.

II. Money Metric Welfare, Equivalent Variation and Compensating Variation

As in previous lectures, we assume that the consumer has a rational, continuous and locally nonsatiated (but not necessarily concave) preference relation $\succsim$. We also assume, when needed, that the consumer's expenditure and indirect utility functions are differentiable.

How can we measure the welfare impact of a change in prices on the consumer's utility using a "metric" that, unlike utility, can be observed? To see how this can be done, let p^0 be the initial vector of prices and w be the consumer's wealth. We want to understand how to measure the impact on the consumer's welfare when prices change from p^0 to another set of prices, p^1.

Let's assume that we know the consumer's preferences. As discussed in Lecture 6, under the assumptions needed for integrability we can obtain the preference relation $\succsim$ if we know the consumer's Walrasian demand function x(p, w), which is something that we can estimate with household level data.

To assess whether the consumer's utility has increased or decreased, the **most convenient relationship to use** is the **indirect utility function v(p, w)**: By definition, the consumer is worse off if $v(p^1, w) - v(p^0, w) < 0$.

We can use the expenditure function to **express this change in utility in money terms**. Indeed, a **money metric indirect utility function** does exactly this. However, it is **always defined with respect to some "reference" level of prices**, which we can denote by $\bar{p}$, which we assume is >> 0. Use the indirect utility function to express the expenditure function as follows:

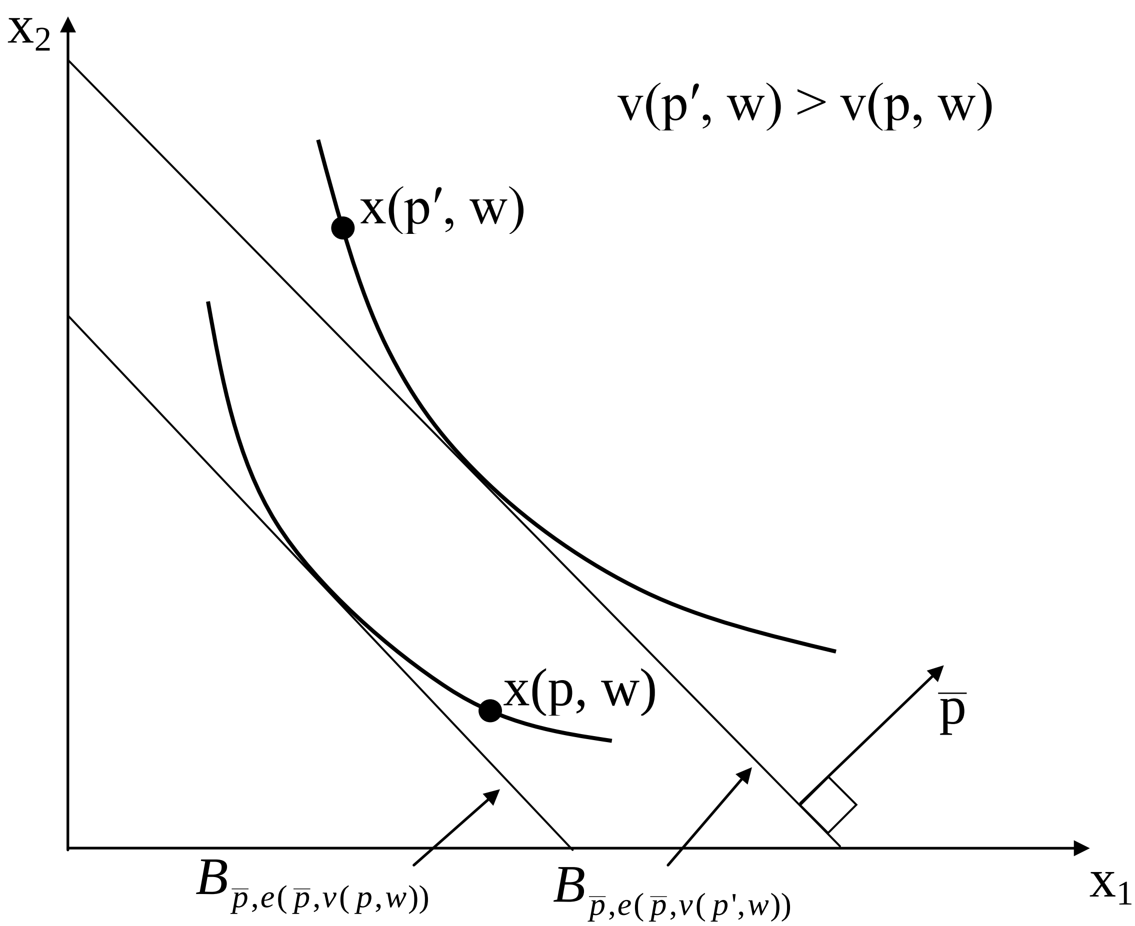
$$e(\bar{p}, u) = e(\bar{p}, v(p, w))$$

Taking $\bar{p}$ as fixed, the function $e(\bar{p}, v(p, w))$ transforms the utility level that can be obtained for the price-wealth pair (p, w) and converts it into a dollar amount that corresponds to prices $\bar{p}$. (Recall from Proposition 3.E.2 that the expenditure function is strictly increasing in u.)

Thus, for a given price level $\bar{p}$, $e(\bar{p}, v(p, w))$ is an indirect utility function that "converts" utility into the amount of money needed for a consumer to attain the utility that he or she could obtain with wealth w and facing prices p.

This conversion of utility into a money amount is **very useful for welfare analysis**. For example, if prices rise but wealth does not, utility declines, and the money metric indirect utility function shows that this loss of welfare is equivalent to a decrease in wealth of a certain amount, given prices $\bar{p}$. Note that, **in general, such conversions will differ for different reference prices (different $\bar{p}$)**.

The following diagram shows this visually:



Prices have changed from p to p′.

Question: In the diagram, how did each price change?

For a given wealth w, the new prices p′ allow the consumer to attain a higher level of utility than he or she had obtained under the old prices p ($v(p', w) > v(p, w)$. Consider the reference prices $\bar{p}$. This increase in welfare is equivalent to an increase in w from $e(\bar{p}, v(p, w))$ to $e(\bar{p}, v(p', w))$.

Equivalent Variation and Compensating Variation

A money metric indirect utility function can be defined for any price vector $\bar{p} >> 0$. What price vector should we use for $\bar{p}$? Two "natural" choices are p^0 and p^1. Choosing p^0 yields a measure of the welfare change that is called the **equivalent variation** (EV):

$$EV(p^0, p^1, w) \equiv e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w$$

The equivalent variation compares the new level of utility attained (u^1) to the previous level of utility attained (u^0), when both levels of utility are evaluated at the **old** prices. If the change in prices leads to an increase (decrease) in utility, **EV shows how much wealth would have had to increase (decrease) under the OLD prices to lead to the NEW utility attained under the new prices** (and the initial wealth). Notice that $EV < 0$ if utility decreases, and > 0 if utility increases, after prices change.

Finally, note that EV can also be expressed using the indirect utility function $v(p, w)$: $v(p^0, w + EV) = u^1$. **Why?**

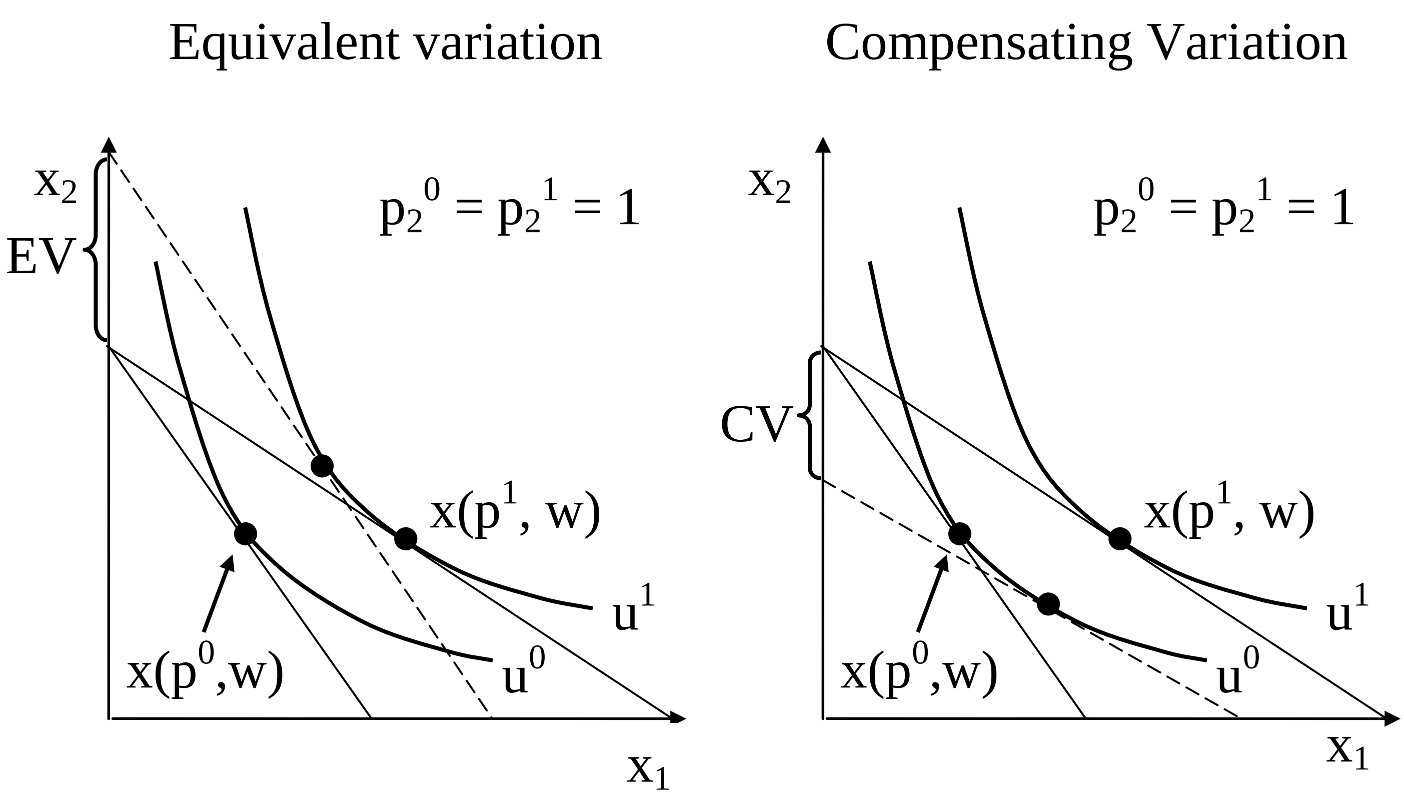
Choosing p^1 yields a measure of the welfare change that is called the **compensating variation** (CV):

$$CV(p^0, p^1, w) \equiv e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0)$$

The compensating variation compares the new level of utility attained (u^1) to the previous level of utility attained (u^0), when both levels of utility are evaluated at the **new** prices. If the change in prices increases (decreases) in utility, **CV shows how much wealth would have had to increase (decrease) under the NEW prices to remain at the OLD level of utility**. Note that, as with EV, CV < 0 if utility decreases, and > 0 if utility increases, after prices change. (The amount needed to "compensate" is -CV.)

Finally, note that CV can also be expressed using the indirect utility function v(p, w): $v(p^1, w - CV) = u^0$.

The following diagram expresses both EV and CV:



In these diagrams, the price of good 2 does not change; only the price of good 1 changes.

Question: Does the price of good 1 increase or decrease?

Welfare Evaluation and Hicksian Demand Curves

When only one price changes, Equivalent Variation (EV) and Compensating Variation (CV) can also be shown in terms of area "under" (more specifically, to the left of) the Hicksian demand curve.

Assume that the price of good 1 changes, while no other prices change: $p_1^0 \neq p_1^1$ but $p_\ell^0 = p_\ell^1$ for $\ell \neq 1$. Since w is unchanged, we have $w = e(p^0, u^0) = e(p^1, u^1)$. Recall (Shepard's lemma) that $h_1(p, w) = \partial e(p, u)/\partial p_1$. Putting this together implies:

$$EV(p^0, p^1, w) = e(p^0, u^1) - w$$

$$= e(p^0, u^1) - e(p^1, u^1)$$

$$= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$$

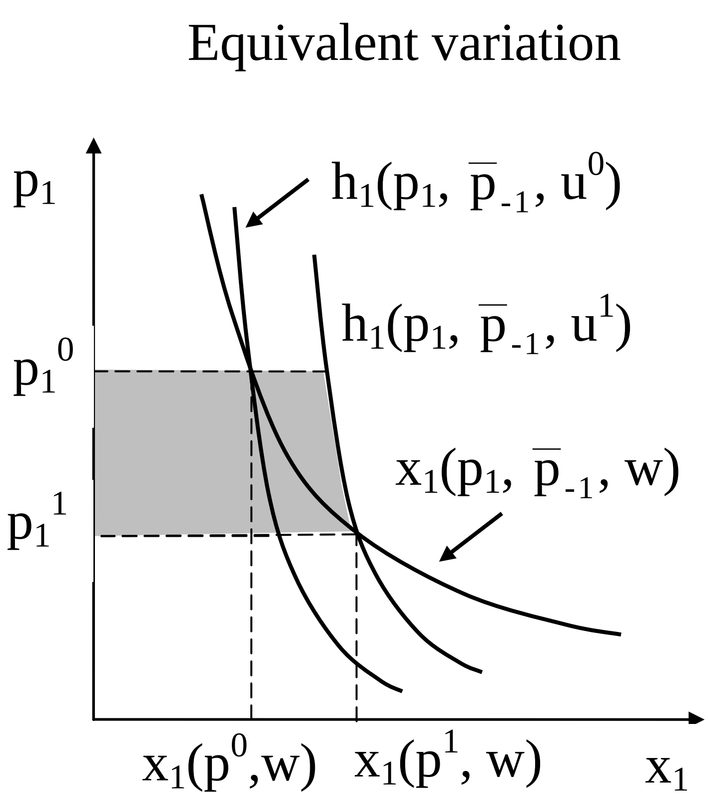
where $\bar{p}_{-1} = (\bar{p}_2, \bar{p}_3, \ldots \bar{p}_L)$. Thus the change in consumer welfare measured by equivalent variation can be represented by the area between p_1^0 and p_1^1 to the left of the Hicksian demand curve for good 1 (at the utility level u^1).

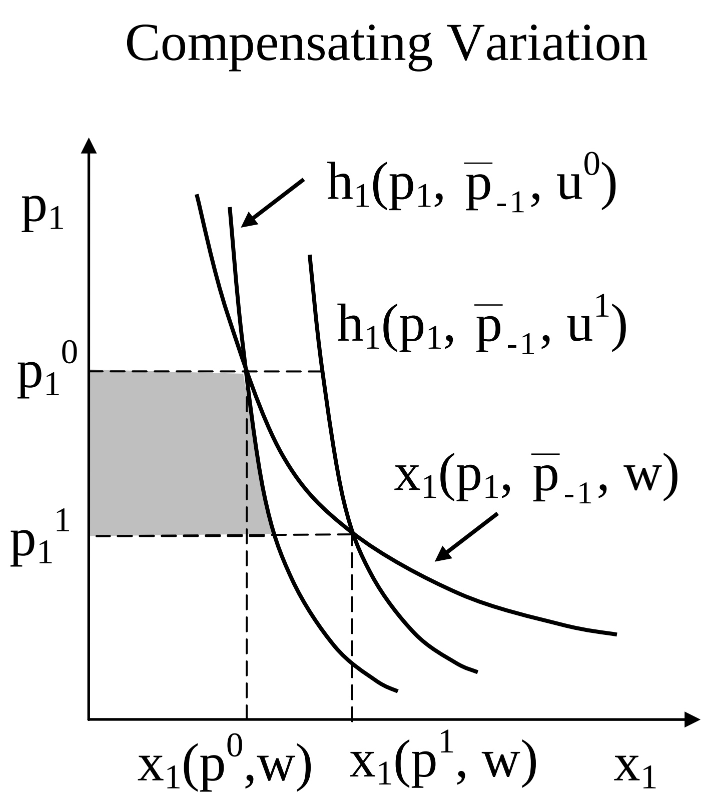
[Note: EV equals this area if $p_1^0 > p_1^1$, i.e. utility increases, and it equals the negative of this area if $p_1^0 < p_1^1$, i.e. utility decreases.]

Following the same method, compensating variation is:

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0)dp_1$$

The following figures show these areas:





So, why is equivalent variation a larger area (larger amount of money) than compensating variation? The first thing to note is that good 1 is a normal good. It turns out that EV > CV when the good for which the price changes is a normal good, and EV < CV when it is an inferior good. If the demand for the good does not change when wealth rises (holding prices constant) then EV = CV. This is the case because for such a good we have:

$$h_1(p_1, \bar{p}_{-1}, u^0) = x_1(p_1, \bar{p}_{-1}, w) = h_1(p_1, \bar{p}_{-1}, u^1)$$

That is, all three curves in the above diagram are identical. In this case, and only this case, the welfare change is the area to the left of the Walrasian (Marshallian) demand curve.

III. Example: Deadweight Loss from a Commodity Tax

At last we get to apply the theory to a policy question, which is: To raise a specific amount of money, is it better for the government to tax commodities or to tax wealth? For simplicity, consider the case where there is a tax on only one commodity, which we will call commodity 1. The "pre-tax" vector of prices was p^0, and the new prices after the tax are p^1, where $p_\ell^{\ 1} = p_\ell^{\ 0}$ for $\ell \neq 1$ and $p_1^{\ 1} = p_1^{\ 0} + t$, where t is the tax rate on good 1. The total revenue raised, denoted by T, will be $T = tx_1(p^1, w)$.

The alternative tax policy is to take T directly from the consumer's wealth, a "lump-sum" tax.

Recall that equivalent variation (EV) is the cost, in money terms, of the loss in utility that the consumer experiences under the new, higher prices (p^1), evaluated under the old prices. This cost is a negative number. The lump-sum tax is also a loss, that is –T. The question thus becomes: which loss is greater? Another way to express this is to say that **if**:

$$– T > e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w$$

$$\text{which implies } w – T > e(p^0, u^1)$$

then the consumer is better off under lump-sum taxation.

If this is the case (which will be shown below), then the difference in these two expressions is the **deadweight loss of commodity taxation**:

$$\text{Deadweight loss} = w – T - e(p^0, u^1)$$

We can use the consumer's Hicksian demand curve to see whether the deadweight loss is positive. First, express the lump-sum tax as: $T = tx_1(p^1, w) = th_1(p^1, u^1)$.

Question: Why is Hicksian demand at u^1 instead of u^0?

We can then express the deadweight loss as:

$$w - T - e(p^0, u^1)$$

$$= e(p^1, u^1) - e(p^0, u^1) - th_1(p^1, u^1)$$

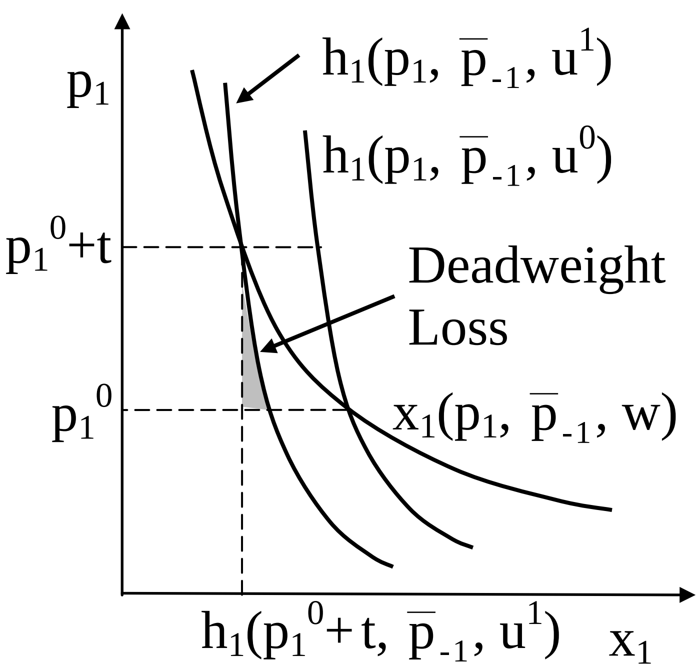
$$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1 - th_1(p_1^0 + t, \bar{p}_{-1}, u^1)$$

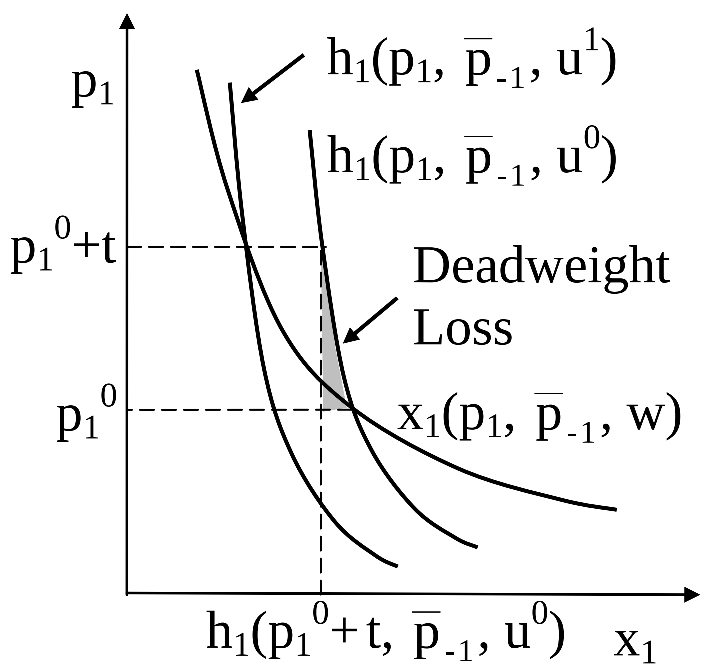
$$= \int_{p_1^0}^{p_1^0+t} [h_1(p_1, \bar{p}_{-1}, u^1) - h_1(p_1^0 + t, \bar{p}_{-1}, u^1)]\, dp_1$$

where $\bar{p}_{-1} = (\bar{p}_2, \bar{p}_3, \ldots \bar{p}_L)$.

Since Hicksian demands are nonincreasing in p, this integral cannot be negative. If $h_1(p, u)$ is strictly decreasing in p_1, it is positive.

The figure on the left shows the area of deadweight loss:





The deadweight loss is the triangle-shaped shaded area under the Hicksian demand curve. The **one on the left** is for the Hicksian demand based, using **equivalent valuation** (EV). The intuition is that the consumer's loss of utility, evaluated in money terms, is the shaded area **plus** the box immediately to the left. The gain in government revenue is that box to the left, so the difference is the shaded area.

The **right figure** on p.11 uses **compensating variation**, and it can be derived as follows. Suppose that after taxing the consumer the government provides a lump-sum transfer to keep him or her at the initial utility level u^0. In this case, the Hicksian demand to examine is that for the new prices but at the old utility level, so government revenue would be $th_1(p_1^1, u^0)$. We compare this to CV:

$$e(p^1, u^0) - w - T$$

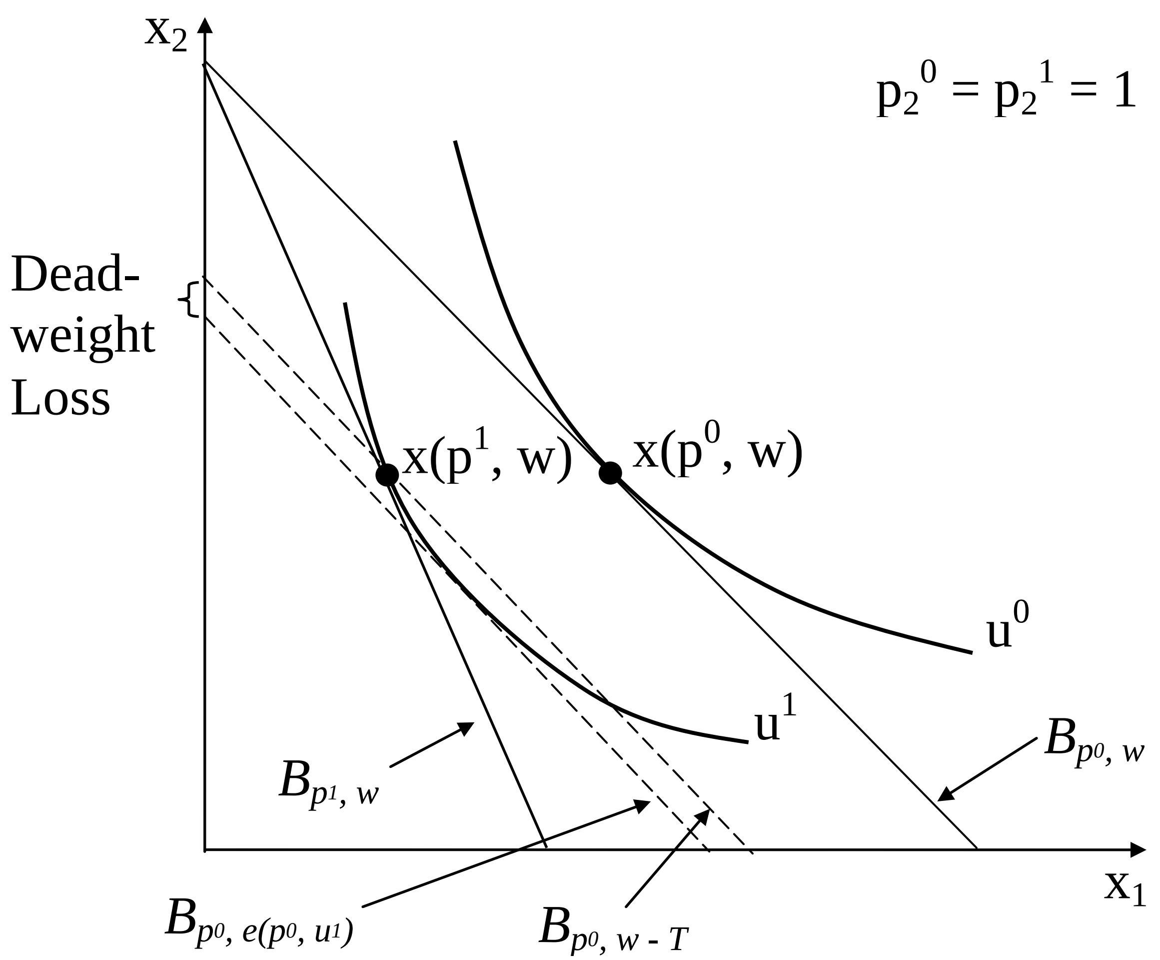
$$= e(p^1, u^0) - e(p^0, u^0) - th_1(p_1^1, u^0)$$

$$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1 - th_1(p_1^0 + t, \bar{p}_{-1}, u^0)$$

$$= \int_{p_1^0}^{p_1^0+t} [h_1(p_1, \bar{p}_{-1}, u^0) - h_1(p_1^0 + t, \bar{p}_{-1}, u^0)]\, dp_1$$

Again, the expression in brackets is strictly positive as long as $h_1(p, u)$ is strictly increasing in p_1. This calculation of deadweight loss is shown on the right of the above diagram.

Here is a depiction of deadweight loss, in "commodity space" for the case of two goods (L = 2). The price for good 2 is fixed while the price of good 1 increases.



Note: Things get more complicated when trying to compare three different vectors of prices. See p.86 of MWG.

IV. Welfare Analysis with Partial Information

Even if the consumer's expenditure function is unknown, it may still be possible to say something about the welfare impact of a change in prices if all we know are the initial prices, p^0, the new prices, p^1, and consumer demand under the initial prices, $x^0 = x(p^0, w)$. This is explained in the following proposition:

Proposition 3.I.1: Assume that the consumer has a locally nonsatiated preference relation $\succsim$. If $(p^1 - p^0)\cdot x^0 < 0$, then the consumer is strictly better off under the price-wealth pair (p^1, w) than under the price-wealth pair (p^0, w).

Proof: This follows directly from revealed preference. Since $p^0\cdot x^0 = w$ by Walras' law, if $(p^1 - p^0)\cdot x^0 < 0$ then $p^1\cdot x^0 < w$. But if this is so then x^0 is still affordable under prices p^1 **and** is in the interior of the budget set $B_{p^1, w}$. By local nonsatiation, there must be a consumption bundle in $B_{p^1, w}$ that the consumer strictly prefers to x^0. **Q.E.D.**

The "comparison" in this proposition (of $p^1\cdot x^0$ with $p^0\cdot x^0$) can be viewed as a first-order approximation to the true welfare change. This can be seen by taking a first-order Taylor expansion of $e(p, u)$ around the initial prices p^0:

$$e(p^1, u^0) = e(p^0, u^0) + (p^1 - p^0)\cdot\nabla_p e(p^0, u^0) + o(\|p^1 - p^0\|)$$

where $o(\|p^1 - p^0\|)$ is the remainder term.

The concavity of e(p, u) in p implies that this remainder term is nonpositive (see theorem M.C.2 on p.933 of Mas-Colell et al.). Thus the above expression implies:

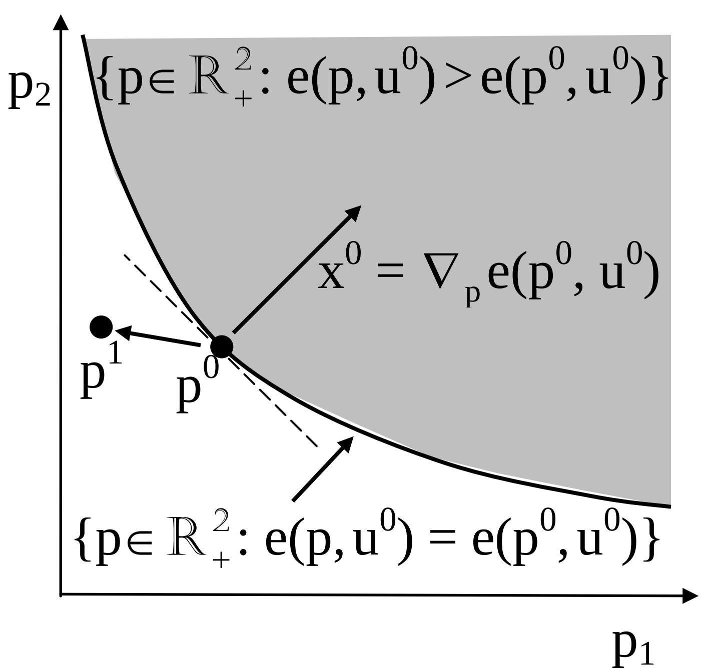
$$e(p^1, u^0) \le e(p^0, u^0) + (p^1 - p^0)\cdot\nabla_p e(p^0, u^0)$$

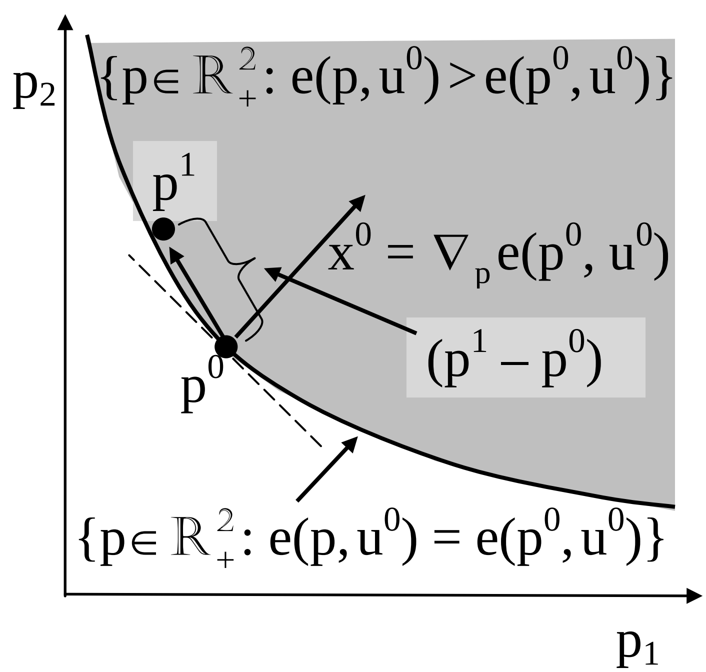
$$\le e(p^0, u^0) + (p^1 - p^0)\cdot h(p^0, u^0)$$

$$\le e(p^0, u^0) + (p^1 - p^0)\cdot x^0$$

where the second line uses Shepard's lemma. Thus if $(p^1 - p^0)\cdot x^0 < 0$ it follows that $e(p^1, u^0) < e(p^0, u^0)$, so the consumer is better off after the price change from p^0 to p^1 (he or she can attain the utility u^0 at lower cost and use the savings to move to a very nearby bundle that offers higher utility, which is available by the nonsatiation assumption).

This is shown in the diagram on the left:





What if $(p^1 - p^0)\cdot x^0 > 0$? In general, in this case it is not possible to make a general conclusion about the impact on welfare. **However**, if the price change is small enough then it is the case that the consumer is worse off:

Proposition 3.I.2: Assume that the consumer has a differentiable expenditure function. If $(p^1 - p^0)\cdot x^0 > 0$, then there is a sufficiently small $\overline{\alpha} \in (0, 1)$ such that for all $\alpha < \overline{\alpha}$, we have $e((1-\alpha)p^0 + \alpha p^1, u^0) > w$, and so the consumer is strictly better off under the price-wealth pair (p^0, w) than under $((1-\alpha)p^0 + \alpha p^1, w)$.

This is shown in the right side of the diagram above.

V. Approximating Welfare Using Walrasian Demand

Some economists, especially in relatively early studies, used Walrasian, rather than Hicksian, demand curves to approximate welfare changes. In this section we examine the conditions under which this is reasonable.

To start, we can define the **area variation** (AV) welfare measure to be the area to the left of the Walrasian demand curve for a given change in prices:

$$AV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x_1(p_1, \overline{p}_{-1}, w)\, dp_1$$

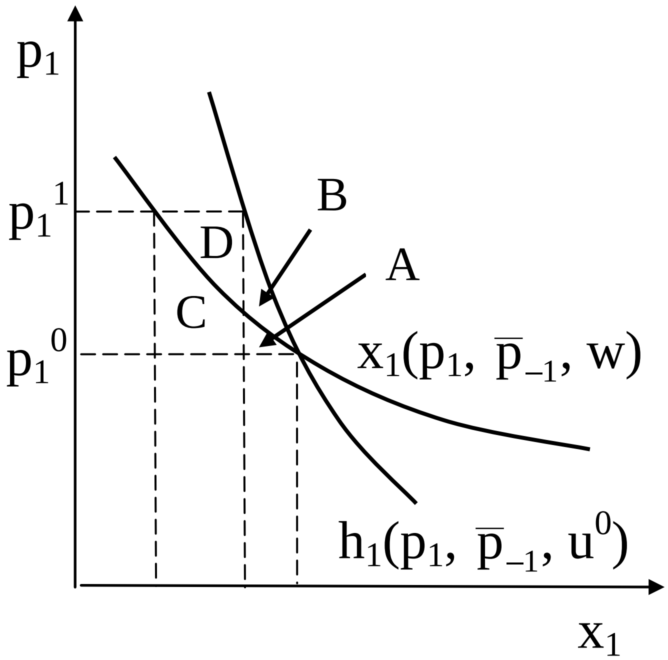
The first thing to note is that if wealth effects did not exist, then the Hicksian and Walrasian demand curves are the same, which implies that AV = EV = CV.

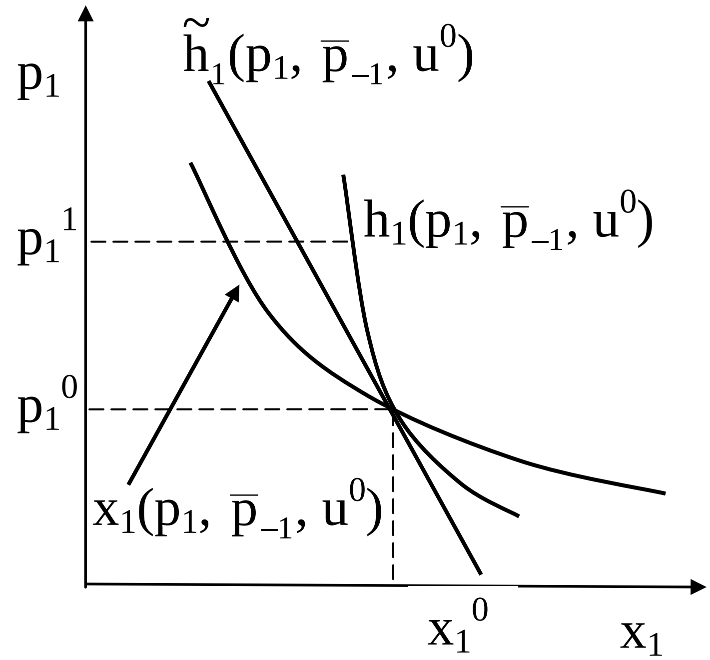
The second thing to note, which can be seen in the diagrams on p.8, is that when the good in question is a normal good, we have EV > AV > CV. Analogously, when the good in question is an inferior good, we have EV < AV < CV.

In general, when wealth effects are “small” AV is a “reasonable” measure of welfare effects.

Also, if the change in prices is small (that is, if $(p_1^1 - p_1^0)$ is small), then the error involved in using AV becomes an increasingly smaller proportion of the total welfare change. This can also be seen by referring to the diagram on p.8.

However, the error could be quite large **as a fraction of the deadweight loss**. This is shown in the diagram on the left of the next page. In it, area B + D is the difference between area variation and true compensating variation. The (inaccurate) deadweight loss calculated using the Walrasian demand curve is A + C, while the true deadweight loss is A + B. The (proportionate) difference will **not** necessary disappear for smaller changes in price.





A final point is that there is another method based on Walrasian demand functions that provides a better approximation of welfare changes than AV does when $(p_1^1 - p_1^0)$ is small. To see how this works, take a first-order Taylor approximation of $h(p, u^0)$ at p^0:

$$\tilde{h}(p, u^0) = h(p^0, u^0) + D_p h(p^0, u^0)(p - p_0)$$

The welfare change can be approximated as:

$$\int_{p_1^1}^{p_1^0} \tilde{h}_1(p_1, \bar{p}_{-1}, u^0)dp_1$$

The intuition for why this is a better approximation than AV is because at p^0 the slope of $\tilde{h}(p, u^0)$ equals the slope of $h(p, u^0)$, since neither has a wealth effect. In contrast, the Walrasian demands have a wealth effect, so the slope is different. This is shown in the second diagram on p.18.

The last step is to show that the integral shown immediately above can be calculated using Walrasian demands. The procedure is based on the fact that $h(p^0, u^0) = x(p^0, w)$, and that $D_p h(p^0, u^0) = S(p^0, w)$. That is:

$$\tilde{h}(p, u^0) = x(p^0, w) + S(p^0, w)(p - p^0)$$

Since the price of only one good is changing, we have:

$$\tilde{h}_1(p_1, \bar{p}_{-1}, w) = x_1(p_1^0, \bar{p}_{-1}, w) + s_{11}(p_1, \bar{p}_{-1}, w)(p_1 - p_1^0)$$

with $s_{11}(p_1, \bar{p}_{-1}, w) = \partial x_1(p^0, w)/\partial p_1 + (\partial x_1(p^0, w)/\partial w)x_1(p^0, w)$.

Note that the "superiority" of this method over AV requires that $(p^1 - p^0)$ be "small".

VI. The Strong Axiom of Revealed Preference

Recall from earlier lectures that rational behavior in the context of the preference-based approach to demand implies that behavior satisfies the weak axiom of revealed preference associated with the choice-based approach, but that the weak axiom did not imply rational preferences. This section shows that a stronger choice-based axiom, called the **strong axiom of revealed preference**, does imply demand behavior that satisfies rational preferences.

Definition: The market demand function x(p, w) satisfies the strong axiom of revealed preference (SA) if, for **any** list of price-wealth pairs:

$$(p^1, w^1), (p^2, w^2), \dots (p^N, w^N)$$

with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ $\forall n \leq N-1$. **If** $\forall n \leq N-1$:

$$p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$$

then $p^N \cdot x(p^1, w^1) > w^N$.

That is, if $x(p^1, w^1)$ is **directly or indirectly revealed to be preferred** to $x(p^N, w^N)$, then $x(p^N, w^N)$ cannot be (directly) revealed to be preferred to $x(p^1, w^1)$. [Thus $x(p^1, w^1)$ cannot be affordable at (p^N, w^N).]

It is easy to show that SA is satisfied for a demand function that is derived from rational preferences. It is harder to show that such a demand function implies SA:

Proposition 3.J.1: If the Walrasian demand function x(p, w) satisfies the strong axiom of revealed preference, then there is a rational preference relation $\succsim$ that "rationalizes" x(p, w), so that for all (p, w), $x(p, w) \succ y$ for every $y \neq x(p, w)$ such that $y \in B_{p,w}$.
The proof is given at the top of p.92 in Mas-Colell et al.