Applied Microeconomics: Firm and Household

Lecture 10: Profit Maximization

Jason Kerwin

Department of Applied Economics
University of Minnesota

October 11, 2016

Outline

- Profit Maximization
  - Output Choice
  - Input Choice
    - Setting up the optimization problem
    - Deriving and interpreting FOCs and SOC
- Comparative Statics of the Profit Maximization Model
  - Properties of factor demand functions
  - Properties of output supply functions

Profit maximization: output choice

Economic Profit: A firm's economic profits are the difference between its revenues and costs

- $\pi(q) = R(q) - C(q)$

where

- q is output
- $R(q)$ is the revenue function of the firm
- $C(q)$ is the const function of the firm

Firm's problem is to find q^* that maximize $\pi(q^*)$

Profit maximization: output choice

Economic Profit: A firm's economic profits are the difference between its revenues and costs

- $\pi(q) = R(q) - C(q)$

where

- q is output
- $R(q)$ is the revenue function of the firm
- $C(q)$ is the const function of the firm

Firm's problem is to find q^* that maximize $\pi(q^*)$

- $\max\limits_{q} \; \pi = R(q) - C(q)$

The FOC of this problem is

- $\frac{\partial \pi}{\partial q} = \underbrace{R'(q)}_{MR} - \underbrace{C'(q)}_{MC} = 0$
- $MR = MC$

Profit maximization: output choice

- $MR(q^*) = MC(q^*)$

That is, a profit maximizing monopolist determines its optimal quantity at a level where marginal revenue is equal to its marginal cost.

Profit maximization: output choice

- $MR(q^*) = MC(q^*)$

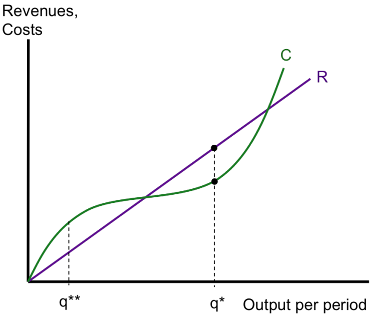
That is, a profit maximizing monopolist determines its optimal quantity at a level where marginal revenue is equal to its marginal cost.

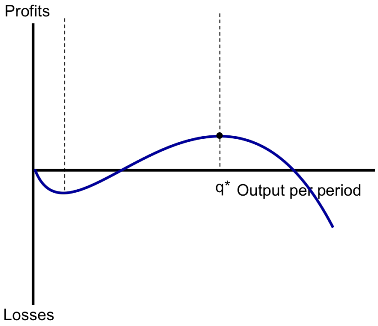
The SOC of this problem is that

- $\frac{d^2\pi}{dq^2} = \left(\frac{dMR(q)}{q} - \frac{dMC(q)}{q}\right)|_{q=q^*} < 0$

The second order condition of this problem is that the marginal revenue curve has to intersect the marginal cost curve from above.

Profit maximization: graphical analysis





Profit maximization: input choice

Assuming perfect competition, we introduce the following notation to set up the problem.

Total revenue is the output price p times total output y

- $TR = py$

Total cost is the input price w times total input x

- $TC = wx$

Profit maximization: input choice

Assuming perfect competition, we introduce the following notation to set up the problem.

Total revenue is the output price p times total output y

- $TR = py$

Total cost is the input price w times total input x

- $TC = wx$

Production technology is represented by a production function:

- $f(x) = \max_{y} \{y : (y, x) \in F\}]$
  - $f(x)$ converts inputs $x = (x_1, ..., x_n)$ into the largest possible output y.
  - F is the feasible set of (y, x). Typically, $y \geqslant 0$ and $x \geqslant 0$ therefore $F \in \mathfrak{R}^2_+$

Next, we formalize the optimization problem.

Profit maximization: input choice

A firms profit maximization problem subject to its production technology can be formalized as:

- $\max_{y,x} \pi(x, y) = py - wx, \quad s.t. \;\; y \leqslant f(x).$
  - The firm chooses its output and input level. Prices are given.
  - The firm cannot choose output level that is beyond the feasible technology.

Note that, because the firm can choose any level of $y \leqslant f(x)$, so long as $p > 0$ a profit-maximizing firm will always choose $y = f(x)$.

Profit maximization: input choice

A firms profit maximization problem subject to its production technology can be formalized as:

- $\max_{y,x} \pi(x, y) = py - wx, \quad s.t. \ \ y \leqslant f(x).$
    - The firm chooses its output and input level. Prices are given.
    - The firm cannot choose output level that is beyond the feasible technology.

Note that, because the firm can choose any level of $y \leqslant f(x)$, so long as $p > 0$ a profit-maximizing firm will always choose $y = f(x)$.

By substituting the binding constraint into the objective function we can transform the problem into a simpler form: an **unconstrained** optimization problem.

- $\max_{x} \pi(x) = pf(x) - wx$

A side note on technical efficiency

The production function $y = f(x)$ is also called **production possibility frontier, PPF**. A firm is called **technically efficient** if it operates on its PPF. That is, it is impossible for the firm to be able to produce more with the same mix of inputs.

On the flip side, a firm is called **technically inefficient** if $y < f(x)$. That is, the firm is not utilizing the technology at its full capacity. With the available technology firm can increase y with the same level of x.

Note that, because a profit maximizing firm always chooses $y = f(x)$, a profit-maximizing firm is always technically efficient. On the contrary, a technically inefficient firm cannot be a profit maximizer.

Profit maximization: input choice

Consider the following profit maximization problem:

- $\max_{x} \pi(x) = pf(x) - wx$

Denote the solutions to this optimization problem as

- $x^*(p, w)$ and $y^*(p, w) = f(x^*(p, w))$

where $x^*(p, w)$ are the profit-maximizing firm's unconditional input demand functions, and $y^*(p, w)$ is its supply function. It's important to note that the optimal quantities are functions of model parameters. We want to do the following:

- Derive $y^*(p, w)$ and $x^*(p, w)$
- Investigate the properties of $y^*(p, w)$ and $x^*(p, w)$
- Analyze how a profit-maximizing firm's optimal decisons change under varying market conditions (comparative statics).

First order conditions

Without loss of generality, let's focus on the 2-input case, $i = 1, 2$.

- $\max\limits_{x_1, x_2} \pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2$

The FOCs for profit maximization are:

1. $\frac{\partial \pi}{\partial x_1} = \pi_1 = \underbrace{pf_1(x_1, x_2)}_{MRP_1} - w_1 = 0$
   - $pf_1(x_1, x_2) = w_1$

First order conditions

Without loss of generality, let's focus on the 2-input case, $i = 1, 2$.

- $\max_{x_1, x_2} \pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2$

The FOCs for profit maximization are:

1. $\frac{\partial \pi}{\partial x_1} = \pi_1 = \underbrace{pf_1(x_1, x_2)}_{MRP_1} - w_1 = 0$
   - $pf_1(x_1, x_2) = w_1$
2. $\frac{\partial \pi}{\partial x_2} = \pi_2 = \underbrace{pf_2(x_1, x_2)}_{MRP_2} - w_2 = 0$
   - $pf_2(x_1, x_2) = w_2$

The first order conditions

1. $pf_1(x_1, x_2) = w_1$
2. $pf_2(x_1, x_2) = w_2$

Interpretation: The FOCs say that, for an interior solution, a profit-maximizing firm sets the marginal contribution of each factor to revenues, pf_i, (the factor's **marginal revenue product**) equal to the marginal cost of that factor, w_i.

Implication:

- Profit maximization takes place only at points where the marginal products f_i are positive (regardless of p and w).

The first order conditions

1. $pf_1(x_1, x_2) = w_1$
2. $pf_2(x_1, x_2) = w_2$

Interpretation: The FOCs say that, for an interior solution, a profit-maximizing firm sets the marginal contribution of each factor to revenues, pf_i, (the factor's **marginal revenue product**) equal to the marginal cost of that factor, w_i.

Implication:

- Profit maximization takes place only at points where the marginal products f_i are positive (regardless of p and w).

Also, note that by dividing (1) and (2) we obtain

- $\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)} = \frac{w_1}{w_2}$

The above equality states that at the optimum a profit-maximizing firm sets the marginal rate of technical substitution, MRTS, equal to the input price ratio.

The second order condition

The SOC for a maximum is that the Hessian has to be *negative semi-definite.* The second partials are:

- $\pi_{11} = pf_{11}$
- $\pi_{22} = pf_{22}$
- $\pi_{12} = pf_{12}$

Hence, the Hessian (why not a bordered Hessian?) is:

- $H(x) = \begin{bmatrix} pf_{11} & pf_{12} \\ pf_{12} & pf_{22} \end{bmatrix}$

- H is negative semidefinite if $f_{11} \leqslant 0$ and $f_{11}f_{22} - f_{12}^2 \geqslant 0$
- From SOC, it must be true that $f_{22} \leqslant 0$
- Note that if the production function is *concave*, the FOC is also sufficient.

The second order condition

1. $f_{11} \leqslant 0$ and $f_{22} \leqslant 0$
2. $f_{11} f_{22} - f_{12}^2 \geqslant 0$

The conditions in 1 state that a profit maximizing firm employs inputs at a level where technology exhibits *diminishing marginal returns* in each input.

Q: Why can't operating at increasing marginal returns maximize profits?

The second order condition

1. $f_{11} \leqslant 0$ and $f_{22} \leqslant 0$
2. $f_{11} f_{22} - f_{12}^2 \geqslant 0$

The conditions in 1 state that a profit maximizing firm employs inputs at a level where technology exhibits *diminishing marginal returns* in each input.

Q: Why can't operating at increasing marginal returns maximize profits?

If hiring the first unit of input was profitable at the first place, without diminishing marginal returns the firm would hire that input without bound.

The second condition arises from the fact that factors can be dependent (hiring more of one factor affects the productivity of the other). The condition requires that the decrease in the own marginal productivity of an input from an additional unit cannot be overcompensated by productivity increases coming from cross effects.

Review: The profit maximization model

Consider the following profit maximization problem:

- $\max_{x_1, x_2} \pi = pf(x_1, x_2) - w_1 x_1 - w_2 x_2$

The FOCs are,

1. $\frac{\partial \pi}{\partial x_1} = \pi_1 = pf_1(x_1, x_2) - w_1 = 0$
2. $\frac{\partial \pi}{\partial x_2} = \pi_2 = pf_2(x_1, x_2) - w_2 = 0$

Assuming the SOSC holds, the optimum functions in implicit form are:

- $x_1 = x_1^*(p, w_1, w_2)$
- $x_2 = x_2^*(p, w_1, w_2)$
- $y = y^*(p, w_1, w_2)$

Comparative statics

We seek analyze how a profit-maximizing firm's decisions change under changing market conditions (comparative statics)

To perform comparative statics we rewrite the FOCs as:

- $\pi_1 = pf_1(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_1 \equiv 0$
- $\pi_2 = pf_2(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_2 \equiv 0$
    - Once again, x^* can also be interpreted as the optimal decision rule. It tells us how the profit maximizing firm adjusts (by changing its x^* such that the FOC holds) as w or p changes.

Comparative statics

Now we want to derive the following:

1. How does a profit-maximizing firm adjust its factor demand in response to a change in the price of an input?
   - $\frac{\partial x^*}{\partial w}$=?
2. How does a profit-maximizing firm adjust its factor demand in response to a change in the price of its output?
   - $\frac{\partial x^*}{\partial p}$=?
3. How does a profit-maximizing firm adjust its output supply in response to a change in the price of output?
   - $\frac{\partial y^*}{\partial p}$=?

Comparative statics: properties of factor demands

- $\pi_1 = pf_1(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_1 \equiv 0$
- $\pi_2 = pf_2(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_2 \equiv 0$

To obtain $\frac{\partial x^*}{\partial w}$ we differentiate the FOCs with respect to w_1 (using the chain rule):

- $pf_{11} \frac{\partial x_1^*}{\partial w_1} + pf_{12} \frac{\partial x_2^*}{\partial w_1} - 1 \equiv 0$
- $pf_{21} \frac{\partial x_1^*}{\partial w_1} + pf_{22} \frac{\partial x_2^*}{\partial w_1} \equiv 0$

Comparative statics: properties of factor demands

- $\pi_1 = pf_1(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_1 \equiv 0$
- $\pi_2 = pf_2(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_2 \equiv 0$

To obtain $\frac{\partial x^*}{\partial w}$ we differentiate the FOCs with respect to w_1 (using the chain rule):

- $pf_{11}\frac{\partial x_1^*}{\partial w_1} + pf_{12}\frac{\partial x_2^*}{\partial w_1} - 1 \equiv 0$
- $pf_{21}\frac{\partial x_1^*}{\partial w_1} + pf_{22}\frac{\partial x_2^*}{\partial w_1} \equiv 0$

We can write the system of two equations in matrix notation as

- $\begin{pmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Own and cross-price effects

Using Cramer's rule

- $\frac{\partial x_1^*}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12} \\ 0 & pf_{22} \end{vmatrix}}{|H|} = \frac{p\overbrace{f_{22}}^{<0}}{|H|} < 0 \quad |H| = p^2(f_{11}f_{22} - f_{12}^2) > 0$ by SOC

Own and cross-price effects

Using Cramer's rule

- $\frac{\partial x_1^*}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12} \\ 0 & pf_{22} \end{vmatrix}}{|H|} = \frac{p \overbrace{f_{22}}^{<0}}{|H|} < 0 \qquad |H| = p^2(f_{11}f_{22} - f_{12}^2) > 0 \text{ by SOC}$

- $\frac{\partial x_2^*}{\partial w_1} = \frac{\begin{vmatrix} pf_{11} & 1 \\ pf_{21} & 0 \end{vmatrix}}{|H|} = -\frac{pf_{21}}{|H|} \lesseqgtr 0 \qquad |H| = p^2(f_{11}f_{22} - f_{12}^2) > 0 \text{ by SOC}$

Own and cross-price effects

Using Cramer's rule

- $\frac{\partial x_1^*}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12} \\ 0 & pf_{22} \end{vmatrix}}{|H|} = \frac{p \overbrace{f_{22}}^{<0}}{|H|} < 0 \qquad |H| = p^2(f_{11}f_{22} - f_{12}^2) > 0 \text{ by SOC}$
- $\frac{\partial x_2^*}{\partial w_1} = \frac{\begin{vmatrix} pf_{11} & 1 \\ pf_{21} & 0 \end{vmatrix}}{|H|} = -\frac{pf_{21}}{|H|} \lesseqqgtr 0 \qquad |H| = p^2(f_{11}f_{22} - f_{12}^2) > 0 \text{ by SOC}$

Similarly, by taking the derivative of FOCs w.r.t w_2 we can find:

- $\frac{\partial x_2^*}{\partial w_2} = \frac{p \overbrace{f_{11}}^{<0}}{|H|} < 0$
- $\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w_1} = -\frac{pf_{12}}{|H|} \lesseqqgtr 0$

Own-price effects of factor demands

- $\frac{\partial x_1^*}{\partial w_1} \equiv \frac{\overbrace{f_{22}}^{<0}}{\underbrace{p(f_{11}f_{22} - f_{12}^2)}_{>0}} < 0$

Own-price effects of factor demands

- $\frac{\partial x_1^*}{\partial w_1} \equiv \frac{\overbrace{f_{22}}^{<0}}{\underbrace{p(f_{11}f_{22} - f_{12}^2)}_{>0}} < 0$
- $\frac{\partial x_2^*}{\partial w_2} \equiv \frac{\overbrace{f_{11}}^{<0}}{\underbrace{p(f_{11}f_{22} - f_{12}^2)}_{>0}} < 0$

The profit maximization model implies that own-price effects on factor demands are negative. That is, the factor demand curves must be downward-sloping.

Cross-price effects on factor demands

For cross-price effects we found that:

- $\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w_1} \equiv -\frac{\overbrace{f_{12}}^{\gtreqless 0}}{\underbrace{p(f_{11}f_{22} - f_{12}^2)}_{>0}} \gtreqless 0$

The profit maximization model implies a reciprocity relation between cross-price effects. That is, the cross-price effects are equal (the symmetry condition). However, the model does not predict the sign of the cross price effects.

Comparative statics: output price effect

Next, we derive comparative statics for changes in output prices, $\frac{\partial x^*}{\partial p}$. This time we differentiate the FOCs wrt p:

- $\pi_1 = pf_1(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_1 \equiv 0$
- $\pi_2 = pf_2(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_2 \equiv 0$

Note that differentiation involves both the chain rule and the multiplication rule

- $f_1 + pf_{11}\frac{\partial x_1^*}{\partial p} + pf_{12}\frac{\partial x_2^*}{\partial p} \equiv 0$
- $f_2 + pf_{21}\frac{\partial x_1^*}{\partial p} + pf_{22}\frac{\partial x_2^*}{\partial p} \equiv 0$

Comparative statics: output price effect

Next, we derive comparative statics for changes in output prices, $\frac{\partial x^*}{\partial p}$. This time we differentiate the FOCs wrt p:

- $\pi_1 = pf_1(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_1 \equiv 0$
- $\pi_2 = pf_2(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_2 \equiv 0$

Note that differentiation involves both the chain rule and the multiplication rule

- $f_1 + pf_{11}\frac{\partial x_1^*}{\partial p} + pf_{12}\frac{\partial x_2^*}{\partial p} \equiv 0$
- $f_2 + pf_{21}\frac{\partial x_1^*}{\partial p} + pf_{22}\frac{\partial x_2^*}{\partial p} \equiv 0$

In matrix form,

- $\begin{pmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial p} \\ \frac{\partial x_2^*}{\partial p} \end{pmatrix} \equiv \begin{pmatrix} -f_1 \\ -f_2 \end{pmatrix}$

Output price effects

Using Cramer's rule

- $\frac{\partial x_1^*}{\partial p} = \frac{\begin{vmatrix} -f_1 & pf_{12} \\ -f_2 & pf_{22} \end{vmatrix}}{|H|} = \frac{p(-f_1 f_{22} + f_2 f_{12})}{|H|} \lesseqqgtr 0$

Output price effects

Using Cramer's rule

- $\frac{\partial x_1^*}{\partial p} = \frac{\begin{vmatrix} -f_1 & pf_{12} \\ -f_2 & pf_{22} \end{vmatrix}}{|H|} = \frac{p(-f_1 f_{22} + f_2 f_{12})}{|H|} \lesseqqgtr 0$

- $\frac{\partial x_2^*}{\partial p} = \frac{\begin{vmatrix} pf_{11} & -f_1 \\ pf_{21} & -f_2 \end{vmatrix}}{|H|} = \frac{p(-f_2 f_{11} + f_1 f_{12})}{|H|} \lesseqqgtr 0$

The signs of these comparative statics are indeterminate, depending on the sign of f_{12}. That is, for an increase in output prices the model does not predict the profit maximizing firm's response in terms of hiring more or less of the inputs.

Output price effects

However, if

- Inputs are complementary or independent, $f_{12} > 0$ or $f_{12} = 0$, then $\frac{\partial x^*}{\partial p} > 0$.
    - For an increase in output price a profit maximizing firm unambiguously hires more of the each factor if factors are technically complementary or independent.

Output price effects

However, if

- Inputs are complementary or independent, $f_{12} > 0$ or $f_{12} = 0$, then $\frac{\partial x^*}{\partial p} > 0$.
  - For an increase in output price a profit maximizing firm unambiguously hires more of the each factor if factors are technically complementary or independent.
- Inputs are competitive, $f_{12} < 0$, then the sign of $\frac{\partial x^*}{\partial p}$ is ambiguous.
  - For example, for an increase in output price a profit maximizing firm might employ less of some of the inputs more of the others. However, as we will see shortly, it would never decrease all of the inputs.

Own-price effects on output supply

Next, we seek to analyze $\partial y^*/\partial p$. To this end, we derive the output supply function by substituting the optimal input demand equations in the production function:

- $y^* = f(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p))$

By taking the derivative w.r.t p

- $\frac{\partial y^*}{\partial p} = f_1 \frac{\partial x_1^*}{\partial p} + f_2 \frac{\partial x_2^*}{\partial p}$

Own-price effects on output supply

Next, we seek to analyze $\partial y^* / \partial p$. To this end, we derive the output supply function by substituting the optimal input demand equations in the production function:

- $y^* = f(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p))$

By taking the derivative w.r.t p

- $\frac{\partial y^*}{\partial p} = f_1 \frac{\partial x_1^*}{\partial p} + f_2 \frac{\partial x_2^*}{\partial p}$

We can substitute our previous results on $\frac{\partial x_1^*}{\partial p}$ and $\frac{\partial x_2^*}{\partial p}$ to obtain

- $\frac{\partial y^*}{\partial p} = f_1 \frac{-f_1 f_{22} + f_2 f_{12}}{p(f_{11} f_{22} - f_{12}^2)} + f_2 \frac{-f_2 f_{11} + f_1 f_{12}}{p(f_{11} f_{22} - f_{12}^2)}$
- $\frac{\partial y^*}{\partial p} = \frac{\overbrace{-f_1^2 f_{22} + 2 f_1 f_2 f_{12} - f_{11} f_{22}^2}^{>0 \Leftrightarrow \textit{quasiconcavity of } f(.)}}{p(f_{11} f_{22} - f_{12}^2)} > 0$

Own-price effect of output supply

- $\frac{\partial y^*}{\partial p} > 0$

A profit maximizing firm increases its production as the output price increases, (upward sloping supply curve).

Note that this result also explains why in the case of competitive inputs a profit maximizing firm increases at least one of the inputs as a response to an increase in output price: an increase in production is not feasible if all the inputs are decreased.

Homogeneity of demand and supply functions

We derived demand functions implicit form as:

- $x = x^*(w_1, w_2, p)$

From its definition the homogeneity of demand functions requires:

- $x = x^*(tw_1, tw_2, tp) = t^k x^*(w_1, w_2, p)$

In this implicit form we cannot directly check the homogeneity of demand functions. However, because demand functions are the solutions to FOCs, we can analyze how FOCs change as we change the prices.

Homogeneity of demand and supply functions

The FOCs are:

1. $pf_1(x_1, x_2) - w_1 = 0$
2. $pf_2(x_1, x_2) - w_2 = 0$

Homogeneity of demand and supply functions

The FOCs are:

1. $pf_1(x_1, x_2) - w_1 = 0$
2. $pf_2(x_1, x_2) - w_2 = 0$

Let's first focus on FOC 1, and multiply all prices by t:

- $tpf_1(x_1, x_2) - tw_1 = 0$
- $t(pf_1(x_1, x_2) - w_1) = 0$
- $pf_1(x_1, x_2) - w_1 = 0$

That is, FOC 1 is the same even when we scale all prices by t. Performing the same steps for FOC 2, we will find the same result:

- $tpf_2(x_1, x_2) - tw_2 = pf_2(x_1, x_2) - w_2 = 0$

Homogeneity of demand and supply functions

We find that the FOCs do not change when we change all prices with the same proportion. This implies that the unconditional factor demand functions are **homogeneous of degree zero in prices.**

Formally,

- $x = x^*(tw_1, tw_2, tp) = x^*(w_1, w_2, p)$

Question: Do you find this result economically intuitive?

Question: What can you say about homogeneity of profit function? Of the supply function?