ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 6: The Relationship between Demand, Indirect Utility & Expenditure Functions, + Integrability (MWG, Ch. 3, pp.67-80)

I. Introduction

This lecture is a continuation of Lecture 5 in that it shows how demand functions and other functions (in particular the indirect utility function and expenditure function) that are "products" of the UMP and EMP are related to each other. It focuses on three relationships, those between:

1. Hicksian demands and the expenditure function

2. Hicksian demands and Walrasian demands

3. Walrasian demands and the indirect utility function

As in Lecture 5, assume that u( ) is a continuous function that represents the locally nonsatiated preference relation $\succsim$, which is defined on the consumption set $X = R^L_+$. Also assume that $p >> 0$. Finally, **assume that $\succsim$ is strictly convex**, so that both Hicksian and Walrsian demands are functions, not correspondences.

Finally, the lecture ends with a discussion of integrability.

II. Hicksian Demand and the Expenditure Function

If we know the Hicksian demand function h(p, u), we can easily construct the expenditure function by inserting these demands into the budget constraint:

$$e(p, u) = p \cdot h(p, u)$$

In fact, we can go in the other direction. That is, if we know the expenditure function e(p, u) we can obtain the Hicksian demands by differentiating the expenditure function by prices. The following proposition explains:

Proposition 3.G.1 (Shepherd's lemma): Suppose that u( ) is continuous and represents a locally nonsatiated and strictly convex preference relation $\succsim$ that is defined on the consumption set $X = R^L_+$. For all p and u, the Hicksian demand h(p, u) vector is the derivative vector of the expenditure function with respect to prices:

$$h(p, u) = \nabla_p e(p, u).$$

That is, the Hicksian demand for good ℓ, $h_\ell(p, u)$, equals $\partial e(p, u)/\partial p_\ell$, for all $\ell = 1, \ldots L$.

This result is important, and Mas-Colell et al. provide three proofs of it on pp.68-69. Here is the second proof, which is based on first-order conditions:

Proof 2: For simplicity, focus on interior solutions, so that $h(p, u) >> 0$, and assume that $h(p, u)$ is differentiable with respect to both p and u. To start, use the chain rule to differentiate $e(p, u) = p \cdot h(p, u)$ with respect to prices:

$$\nabla_p e(p, u) = \nabla_p [\, p \cdot h(p, u)]$$

$$= h(u, p) + [p \cdot D_p h(p, u)]^T$$

A result that was not shown in Lecture 5, but is easy to show (see p.61 of Mas-Colell et al.) is that one of the first order conditions for solving the EMP (assuming interior solutions) is $p = \lambda \nabla u(x^*) = \lambda \nabla u(h(p, u))$, where $\nabla u(\ )$ is the column vector of derivatives of $u(\ )$ with respect to x and λ is a Lagrangian multiplier ($\neq$ the λ for utility maximization). Substitute out p in the above expression:

$$\nabla_p e(p, u) = h(p, u) + \lambda[\nabla u(h(p, u)) \cdot D_p h(p, u)]^T$$

To finish, note that the constraint $u = u(h(p, u))$ holds for all p for the EMP. Differentiating this constraint by p gives $0 = \nabla u(h(p, u)) \cdot D_p h(p, u)$. Thus the second term in the above expression equals 0, so $\nabla_p e(p, u) = h(p, u)$.

Q.E.D

You are encouraged to look at the other two proofs, one of which uses duality theory and the other of which uses the envelope theorem.

The **intuition** behind all 3 proofs is the same. Both Hicksian demands and the expenditure function are defined in terms of optimal values. In particular, for the expenditure function at the optimal values of x, **very small changes in prices** lead to changes in x that move along the budget constraint (**show for L = 2 diagram**) so such changes have **no income effect** (u is held constant), just a price effect, so we can ignore the second term in $\nabla_p e(p, u)$.

Hicksian demand functions have some important properties, especially the properties of their derivatives with respect to prices. The following proposition summarizes the most important properties:

Proposition 3.G.2: Let u( ) be a continuous function that represents a locally nonsatiated and strictly convex preference relation $\succsim$ that is defined on the consumption set $X = R^L_+$. Assume that h(u, p) is continuously differentiable for any (p, u), and denote its L×L derivative matrix with respect to prices by $D_p h(p, u)$. Then:

1. $D_p h(p, u) = D_p^2 e(p, u)$.

2. $D_p h(p, u)$ is a negative semidefinite matrix.

3. $D_p h(p, u)$ is a symmetric matrix.

4. $D_p h(p, u)p = 0$

Proof: The first property follows immediately from the result in Proposition 3.G.1 that $\nabla_p e(p, u) = h(p, u)$, and then differentiating both sides of this by p (which yields a matrix, so we use the D_p notation). The second and third properties follow from the fact that e(p, u) is a twice continuously differentiable concave function; it has a symmetric and negative semidefinite Hessian (matrix of second derivatives). See Section M.C. in the math appendix in Mas-Colell et al. for details. Finally, the fourth property can be shown by recalling that h(p, u) is homogenous of degree 0 in p, which implies that $h(\alpha p, u) - h(p, u) = 0$. Differentiating this expression w.r.t. α yields $D_p h(p, u) \times (\partial \alpha p / \partial \alpha) = D_p h(p, u) p = 0$.

The **result that $D_p h(p, u)$ is a negative semidefinite matrix** may not seem to be of much interest, but in fact it **expresses the compensated law of (Hicksian) demand** (see Lecture 5). In particular, this property implies that $dp \cdot D_p h(p, u) dp \leq 0$ for all dp. Holding u constant, note that we can express $dh(p, u) = D_p h(p, u) dp$ for a small change in p. But this implies that $dp \cdot dh(p, u) \leq 0$, which is exactly the compensated law of demand.

What about the **symmetry** of $D_p h(p, u)$? It implies that, for any two goods ℓ and k, $\partial h_\ell(p, u)/\partial p_k = \partial h_k(p, u)/\partial p_\ell$. This is surprising and has **no clear economic intuition or interpretation**. Mathematically, it reflects the property that cross derivatives of twice continuously differentiable functions are equal to each other (the order of differentiation doesn't matter). This can be tested (as explained further below), so if data are inconsistent with this either people are not rational, or our model is still too simplistic.

Finally, we end with some useful definitions:

1. Two goods ℓ and k are **substitutes** at the price-wealth pair (p, w) if $\partial h_\ell(p, u)/\partial p_k \geq 0$. That is, if the price of one of the goods increases the consumer switches to the other good.

2. Two goods ℓ and k are **compliments** at the price-wealth pair (p, w) if $\partial h_\ell(p, u)/\partial p_k \leq 0$.

The result that $D_p h(p, u)p = 0$, along with the result that $\partial h_\ell(p, u)/\partial p_\ell \leq 0$ (compensated law of demand), implies that **at least one substitute must exist for any good**. This is quite intuitive, if the price of one good rises you will tend to buy less of it, but if utility is to be maintained there must be some other good that you buy more of.

III. The Hicksian and Walrasian Demand Functions

Hicksian demand functions cannot be estimated directly because they are functions of utility, which is very hard, if not impossible, to observe. Then how can you test the validity of demand theory by, for example checking whether $D_p h(p, u)$ is negative definite or symmetric? (Recall proposition 3.G.2 above.) In fact, it is possible to **estimate** $D_p h(p, u)$ and other aspects of Hicksian demand functions **indirectly** because they are **closely related to Walrasian demand functions**, which are functions of observable variables (p and w). **This section explains this relationship**. The key result is the **Slutsky equation**:

Proposition 3.G.3: Let u( ) be a continuous function that represents a locally nonsatiated and strictly convex preference relation $\succsim$ that is defined on the consumption set $X = R^L_+$. Then, for all (p, w) and u = v(p, w), the following equation, called the **Slutsky equation**, holds:

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p, w) \quad \text{for all } \ell, k$$

This can be written in matrix form as:

$$D_p h(p, u) = D_p x(p, w) + D_w x(p, w) x(p, w)^T$$

where $D_p h(p, u)$ and $D_p x(p, w)$ are L×L matrices and $D_w x(p, w)$ and x(p, w) are L×1 column vectors.

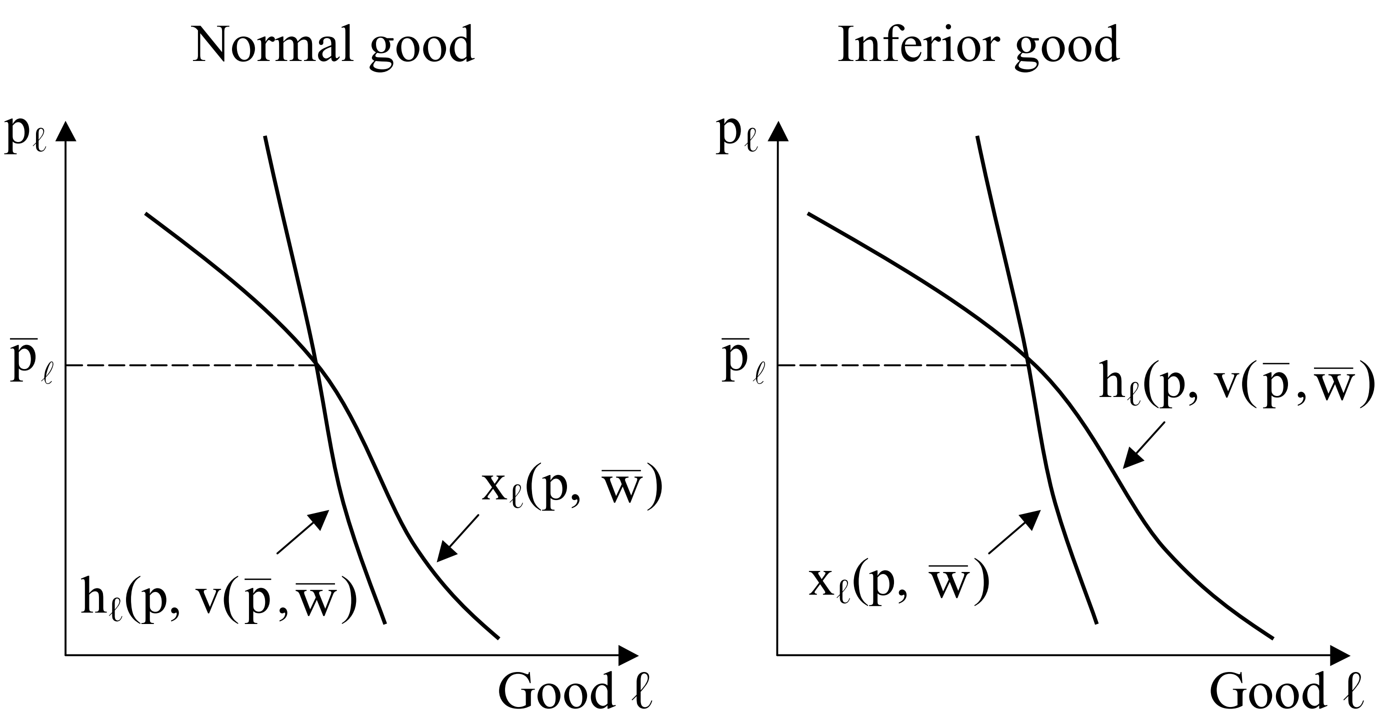
Proof: Consider a consumer facing the price-wealth pair $(\bar{p}, \bar{w})$ and attaining the utility level $\bar{u}$. The consumer's wealth must satisfy $\bar{w} = e(\bar{p}, \bar{u})$. For any price-wealth pair (p, w), the associated Hicksian and Walrasian demands for any good ℓ must be equal (see Lecture 5), so $h_\ell(p, u) = x_\ell(p, e(p, u))$. Differentiate this equality with respect to p_k and evaluate it at $(\bar{p}, \bar{u})$:

$$\frac{\partial h_\ell(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_\ell(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_\ell(\bar{p},e(\bar{p},\bar{u}))}{\partial w}\frac{\partial e(\bar{p},\bar{u})}{\partial p_k}$$

$$= \frac{\partial x_\ell(\bar{p},\bar{w})}{\partial p_k} + \frac{\partial x_\ell(\bar{p},\bar{w})}{\partial w} h_k(\bar{p}, \bar{u})$$

$$= \frac{\partial x_\ell(\bar{p},\bar{w})}{\partial p_k} + \frac{\partial x_\ell(\bar{p},\bar{w})}{\partial w} x_k(\bar{p},\bar{w}) \quad \textbf{Q.E.D.}$$

Hicksian and Walrasian demand curves can be compared in a simple diagram (holding other prices fixed):



For a **normal good** ($\partial x_\ell(\bar{p}, \bar{w})/\partial w > 0$), the impact of a change in price on the Walrasian demand is larger (in absolute value) because the **wealth effect reinforces the price effect**. For example, an increase in the price will reduce demand both because the relative price has increased and because real wealth has fallen. In contrast, Hicksian demand is less sensitive to price because, to maintain the same utility level, price changes are accompanied by wealth changes in the same direction.

For inferior goods ($\partial x_\ell(\bar{p}, \bar{w})/\partial w < 0$), the wealth effect for Walrasian demand is the opposite direction of price effects.

Proposition 3.G.3 implies that the **matrix of the price derivatives of Hicksian demand can be written as**:

$$S(p, w) = \begin{bmatrix} s_{11}(p, w) & \cdots & s_{1L}(p, w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p, w) & \cdots & s_{LL}(p, w) \end{bmatrix}$$

where $s_{\ell k}(p, w) = \partial x_\ell(p, w)/\partial p_k + [\partial x_\ell(p,w)/\partial w]x_k(p, w)$. This is called the **Slutsky substitution matrix**. It can be calculated since $x(p, w)$ is **a function of observed variables** and so it can be estimated. If the assumptions in Proposition 3.G.2 are true, then $S(p, w)$ is negative semi-definite, symmetric, and satisfies $S(p, w)p = 0$.

The fact that $D_p h(p, u) = D_p x(p, w) + D_w x(p, w)x(p, w)^T$ ($= S(p, w)$) **allows one to compare the preference-based approach to consumer demand to the choice-based approach** based on the weak axiom of revealed preference. Recall from Lecture 3 that the weak axiom (plus homogeneity of degree zero and Walras' law) implies that S(p, w) is negative semidefinite, and that $S(p, w)p = 0$. However, for $L \geq 3$ the **choice-based approach does not imply symmetry of S(p, w)**. Thus "weaker" assumptions lead to "weaker" results. It turns out that it is impossible to find preferences that imply that demand is rational when S(p, w) is not symmetric.

IV. Walrasian Demand and the Indirect Utility Function

In Section II we saw how the expenditure function from the EMP, e(p, u), can be differentiated to give Hicksian demand functions. This **suggests** that **an analogous result** holds for the UMP: if we differentiate the function that we are trying to optimize we will get the Walrasian demand function. This is almost true, but not quite.

One reason why this cannot be true is that utility is an ordinal concept; any monotonic transformation of the utility function yields the same choices even though the utility function, and any derivative of the utility function, gives a different value. Also, if we cannot observe utility directly it is hard to imagine how we can observe its derivative. **However**, it turns out that **we can "normalize" the derivative** by the marginal utility of wealth and thus **obtain Walrasian demand** as the ratio of two derivatives of the utility function. This is **Roy's identity** and it is stated in the following proposition:

Proposition 3.G.4: Let u( ) be a continuous function that represents a locally nonsatiated and strictly convex preference relation $\succsim$ that is defined on the consumption set $X = R^L_+$. Assume also that the indirect utility function $v(\bar{p}, \bar{w})$ is differentiable at $(\bar{p}, \bar{w}) >> 0$. Then:

$$x(\bar{p}, \bar{w}) = - \frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w})$$

That is, for every $\ell = 1, 2, \ldots L$, we have:

$$x_\ell(\bar{p}, \bar{w}) = - \frac{\partial v(\bar{p}, \bar{w})/\partial p_\ell}{\partial v(\bar{p}, \bar{w})/\partial w}$$

MWG give three proofs on pp.74-75. Here is one of them.

Proof 2: Assume that x(p, w) is differentiable and that $x(\bar{p}, \bar{w}) >> 0$ (i.e. an interior solution). Recalling that $v(\bar{p}, \bar{w}) = u(x(\bar{p}, \bar{w}))$, and using the chain rule gives:

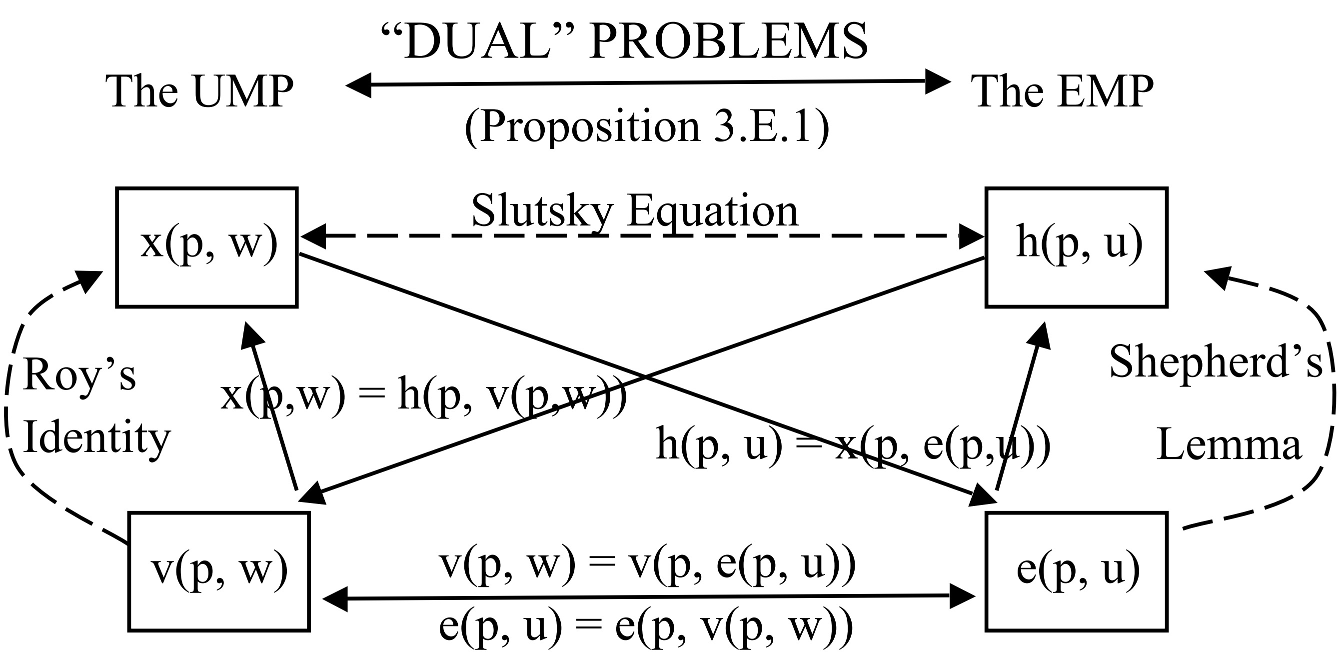
$$\frac{\partial v(\bar{p}, \bar{w})}{\partial p_\ell} = \sum_{k=1}^{L} \frac{\partial u(x(\bar{p}, \bar{w}))}{\partial x_k} \frac{\partial x_k(\bar{p}, \bar{w})}{\partial p_\ell}$$

Use the UMP F.O.C. to substitute out $\partial u(x(\bar{p}, \bar{w}))/\partial x_k$:

$$\frac{\partial v(\bar{p},\bar{w})}{\partial p_\ell} = \sum_{k=1}^{L} \lambda p_k \frac{\partial x_k(\bar{p},\bar{w})}{\partial p_\ell} = -\lambda x_\ell(\bar{p}, \bar{w})$$

where the last equality uses the Cournot aggregation condition (see Lecture 2). **Q.E.D.**

The following diagram summarizes the main results thus far (including results from Lecture 5):



The solid arrows are derivations from Lecture 5, and the dashed arrows from Lecture 6.

Starting with the UMP, we can derive Walrasian demands x(p, w). Starting with the EMP, we can derive Hicksian demands h(p, u). Plugging Walrasian demands into the utility function gives the indirect utility function v(p, w) [no arrow for this]. Plugging Hicksian demands into the budget constraint gives the expenditure function e(p, u) [no arrow for this]. The expenditure function e(p, u) and the indirect utility function v(p, w) are "inversions" of each other. We can get Hicksian demands from Walrasian demands by plugging in the expenditure function, and we can go in the other direction by plugging in the indirect utility function. Finally, by differentiation we can obtain Walrasian demands from the indirect utility function, and Hicksian demands from the expenditure function.

V. Integrability

Thus far, we have shown that if rational, locally non-satiated preferences generate a continuously differentiable demand function x(p, w), then that demand function must be homogenous of degree zero (in p and w), must satisfy Walras' law, and must have a substitution matrix S(p, w) that is symmetric and negative semidefinite.

Can we go in the other direction? That is, **if we have a demand function** that is homogenous of degree zero, satisfies Walras' law, and has a symmetric and negative semidefinite substitution matrix S(p, w), **can we find rational preferences that are consistent with it?** The answer is **yes**. This is called the **integrability question**.

Why is this an interesting question? There are (at least) two theoretical reasons and two empirical reasons:

1. It shows that these consequences of rational preferences (homogeneity, Walras' law and a symmetric and negative semidefinite substitution matrix) are the "complete" properties of those preferences, in the sense that those properties allow us to go back from the demand function to the preferences from which they come.

2. It clarifies the relationship between choice-based demand theory and preference-based demand theory. We saw that demand derived from choice-based theory does not require a symmetric substitution matrix, and when that matrix is not symmetric the demand cannot be derived from any rational preferences. Here we will see that any demand that satisfies the weak axiom of revealed preference (and is homogenous of degree zero and satisfies Walras' law) is consistent with rational preferences **if and only if** the substitution matrix S(p, w) is symmetric.

3. To analyze the welfare effects of price changes (and other changes or constraints) we need knowledge of consumers' preferences; thus the "yes" answer to the integrability question means that we can use demand relationships that we estimate to do welfare analysis. This will be discussed in detail in Apec 8004.

4. When conducting empirical research on consumer demand, we may want to choose a functional form for the demand equations that can be linked to a rational preference relation. The integrability results allow us to check whether our functional form satisfies this requirement without us having to "solve for" the utility function from our demand equations.

The task of recovering the preference relation $\succsim$ from the demand functions x(p, w) can be divided into two parts:

1. Solve for the expenditure function e(p, u) from the demand functions x(p, w).

2. "Recover" preferences from the expenditure function e(p, u).

We will now examine these steps, starting with the second and then turning to the first.

Recovering Preferences from the Expenditure Function

Assume that a consumer has (locally nonsatiated) preferences that imply a continuous utility function u( ). We have seen (Lecture 5, Proposition 3.E.2) that e(p, u), the associated expenditure function, is strictly increasing in u, is continuous in p and u, nondecreasing, homogenous of degree one, and concave in p. We will also assume that demand is single valued (a function, not a correspondence), which implies that $e(p, u)$ is differentiable.

So, **given the expenditure function $e(p, u)$, how can we recover a preference relation** (that is, a utility function) **that generates it?** To start, we must find, for each utility level u, an "at least as good as" set of commodity bundles, denoted by $V_u \subset R^L_+$, that is defined in terms of the expenditure function. This can be done by using $e(p, u)$ to calculate the minimum expenditure required, given (relative) prices p, to purchase a bundle in the set V_u.

More precisely, for each u we want to identify a set V_u such that, for all $p >> 0$, we have:

$$e(p, u) = \min_{x \geq 0} p \cdot x \quad \text{subject to } x \in V_u$$

The following proposition shows that this is accomplished by the set $V_u = \{x \in R^L_+ : p \cdot x \geq e(p, u) \text{ for all } p >> 0\}$.

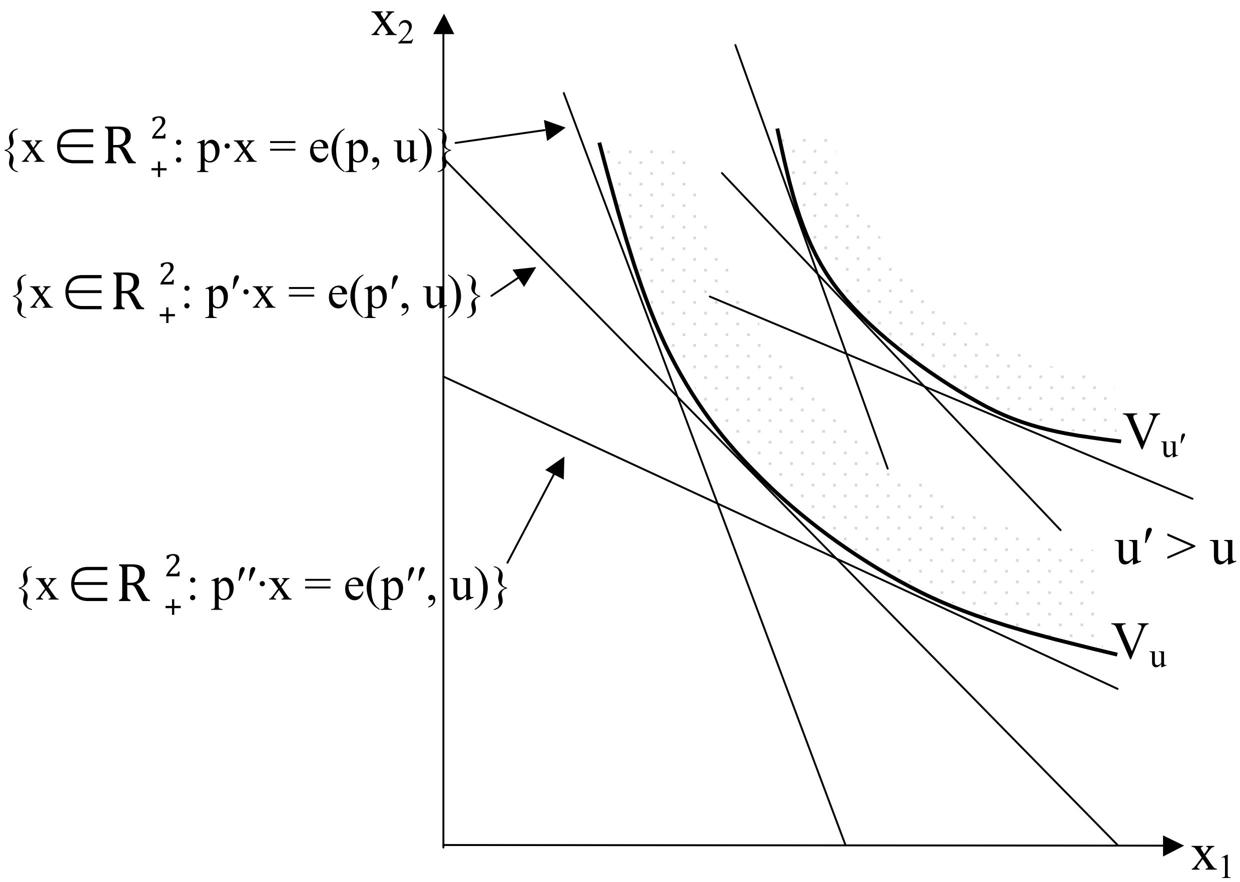
Proposition 3.H.1: Let the expenditure function $e(p, u)$ be strictly increasing in u, and continuous, nondecreasing, homogenous of degree one, concave, and differentiable in p. Then for every utility level u, $e(p, u)$ is the expenditure function associated with the "at least as good as" set:

$$V_u = \{x \in R^L_+ : p \cdot x \geq e(p, u) \text{ for all } p >> 0\}$$

This means that $e(p, u) = \underset{x \geq 0}{\text{Min}} \{p \cdot x : x \in V_u\}$ for all $p >> 0$.

The proof is short but optional; see Mas-Colell et al., p.77.

The following diagram helps with the intuition:



For a given utility level (u or u′), the expenditure function is applied for all possible (relative) prices to create the "lower border" for the set V_u (or $V_{u'}$). This consists of all points where the equality holds for $p \cdot x = e(p, u)$. Note that each set V_u is closed, convex and bounded from below, and if $u' > u$, then $V_{u'}$ contains V_u (but V_u does not contain $V_{u'}$).

Recovery of the Expenditure Function from Demand

Let's start with the case of two commodities ($L = 2$), and normalize $p_2 = 1$. Choose an arbitrary price-wealth point $(p_1^0, 1, w^0)$ and assign a utility of u^0 to the Walrasian demand for that point, $x(p_1^0, 1, w^0)$.

So how can one obtain the value of the expenditure function, denoted by $(p_1, 1, u^0)$ for all prices $p_1 > 0$? Recall that compensated demand is the derivative of the expenditure function with respect to prices; to go from demand to the expenditure function **we need** to integrate, or more specifically **to solve a differential equation** with the independent variable p_1 and the dependent variable e.

For simplicity, write $e(p_1, 1, u^0)$ as $e(p_1)$ and $x_1(p_1, 1, w)$ as $x_1(p_1, w)$. We need to solve the differential equation:

$$\frac{de(p_1)}{dp_1} = x_1(p_1, e(p_1))$$

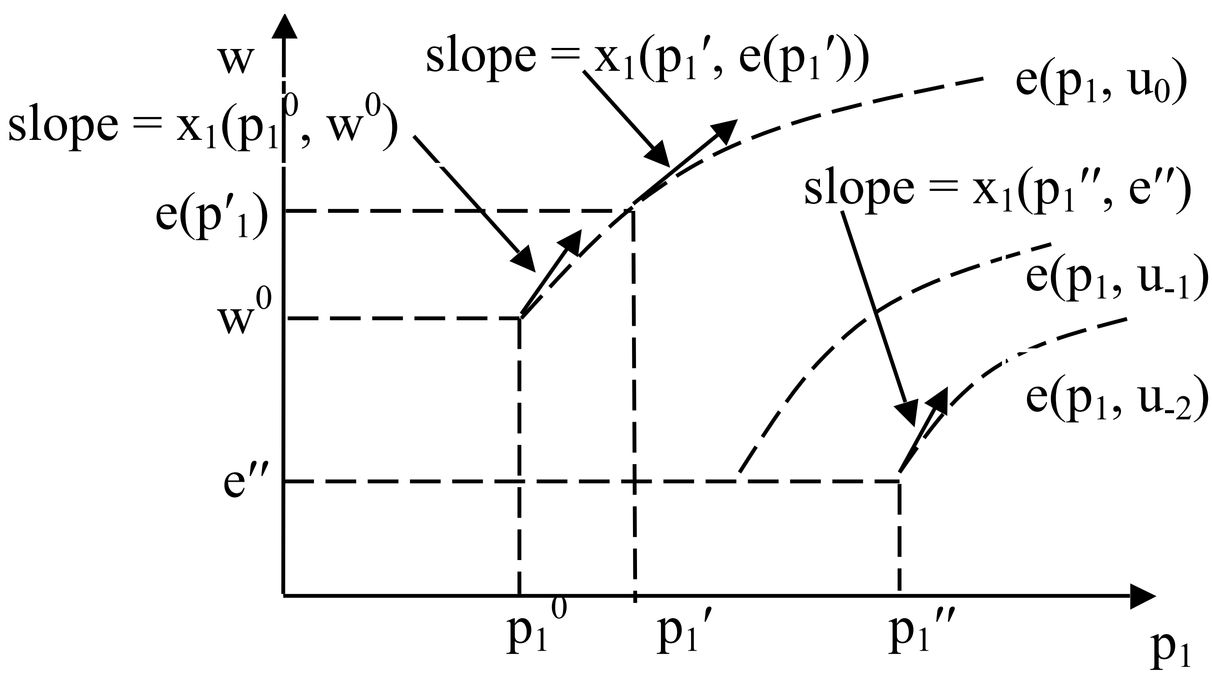
with the initial condition $e(p_1^0) = w^0$.

IF a solution $e(p_1)$ is found for this differential equation that satisfies the initial condition, **it is the expenditure function** for the utility level u^0. If the Hicksian substitution matrix for the Hicksian demands is negative semi-definite then $e(p_1)$ has all the properties of an expenditure function: continuous and nondecreasing in p_1. It is also concave in p_1 since the own price elasticity for a Hicksian demand function is nonpositive:

$$\frac{d^2 e(p_1)}{dp_1^{\,2}} = \frac{\partial x_1(p_1, 1, e(p_1))}{\partial p_1} + \frac{\partial x_1(p_1, 1, e(p_1))}{\partial w} x_1(p_1, 1, e(p_1))$$

$$= s_{11}(p_1, 1, e(p_1)) \le 0$$

Solving the differential equation $de(p_1)/dp_1 = x_1(p_1, e(p_1))$ is a standard ordinary partial differential equation problem; **a few weak regularity conditions guarantee that a solution exists** for any initial condition $(p_1^{\,0}, w^0)$.



Finally, consider the case of L (> 2) commodities. The single differential equation is replaced by a system of L partial differential equations:

$$\frac{\partial e(p)}{\partial p_1} = x_1(p, e(p))$$

$$\vdots$$

$$\frac{\partial e(p)}{\partial p_L} = x_L(p, e(p))$$

The initial conditions are p^0 and $e(p^0) = w^0$. The **existence of a solution** to this system of differential equations is **not guaranteed if L > 2**. If there is a solution, call it e(p), then its Hessian matrix $D_p^2e(p)$ must be symmetric because the Hessian of any twice continuously differentiable function must be symmetric. Differentiating these equations with respect to p gives the matrix of Slutsky equations:

$$D_p^2e(p) = D_px(p, e(p)) + D_wx(p, e(p))x(p, e(p))^T$$

$$= S(p, e(p))$$

Thus a **necessary condition** for the existence of a solution is that the **Slutsky matrix of x(p, w) is symmetric**. In fact, this symmetry is **also a sufficient condition** to find a solution to these differential equations ("recovering" the expenditure function). Finally, the negative definiteness of the Slutsky matrix ensures that the expenditure function that is "recovered" has all the standard properties of an expenditure function. See p.80 of Mas Colell for further discussion.