ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 4: Classical Demand Theory: Preferences and Utility (MWG, Ch. 3, pp.40-50)

I. Introduction

The next 4 lectures will focus on "classical demand theory", that is demand theory that is based on utility functions. This is Chapter 3 of Mas-Colell et al.

This lecture examines the relationship between preference relations (the $\succsim$ relation) and utility functions. In general, preference relations have somewhat less restrictive assumptions than do utility functions, and this lecture shows what additional assumptions are needed for preference relations to generate utility functions. The main additional assumption is that a preference relation must be continuous to be represented by a utility function.

II. Basic Properties of Preference Relations

Let's begin by reviewing the two properties that make preferences rational. For most (but not all) of the lecture assume that goods are nonnegative: the consumption set $X \subset \mathbb{R}^L_+$ (For MWG $\subset$ is "weak" inclusion, i.e. $\subseteq$.)

Definition: The preference relation $\succsim$ is **rational** if it has the following two properties:

1. **Completeness**: For all x, y $\in$ X, either x $\succsim$ y or y $\succsim$ x (or both).

2. **Transitivity**: For all x, y, z $\in$ X, if x $\succsim$ y and y $\succsim$ z, then x $\succsim$ z.

We now introduce two other classes of assumptions: desirability assumptions and convexity assumptions.

Desirability Assumptions

Desirability assumptions are based on the plausible notion that larger amounts of commodities are preferred over smaller amounts. In general, we will assume that larger amounts are feasible in the sense that if a smaller amount is a member of the consumption set X than any larger amount is also a member of the consumption set X.

We will examine two desirability assumptions: monotonicity and (local) nonsatiation.

Definition: The preference relation $\succsim$ on X is **monotone** if x $\in$ X and y >> x imply that y $\succ$ x. The relation is **strongly monotone** if y $\geq$ x and y $\neq$ x imply that y $\succ$ x.

(Note: y >> x means that $y_\ell > x_\ell$ for all ℓ = 1, 2, … L.)

Note: For most applications we assume that each commodity is a “good”, as opposed to being a “bad”. If we have something “bad”, such as garbage or waste, we can define a good that is simply the inverse of the “bad” commodity. So this set up can be modified to include “bads” as well as “goods”.

In words, the **monotone** assumption says that if there is a commodity bundle y that contains **greater amounts of all commodities** than the commodity bundle x, then the consumer strictly prefers y to x. In contrast, the **strongly monotone** assumption assumes only that y has **greater amounts of at least one commodity** (and never a lesser amount of any commodity) relative to x.

Question: For strong monotonicity, what in the definition guarantees that there is at least more of one commodity?

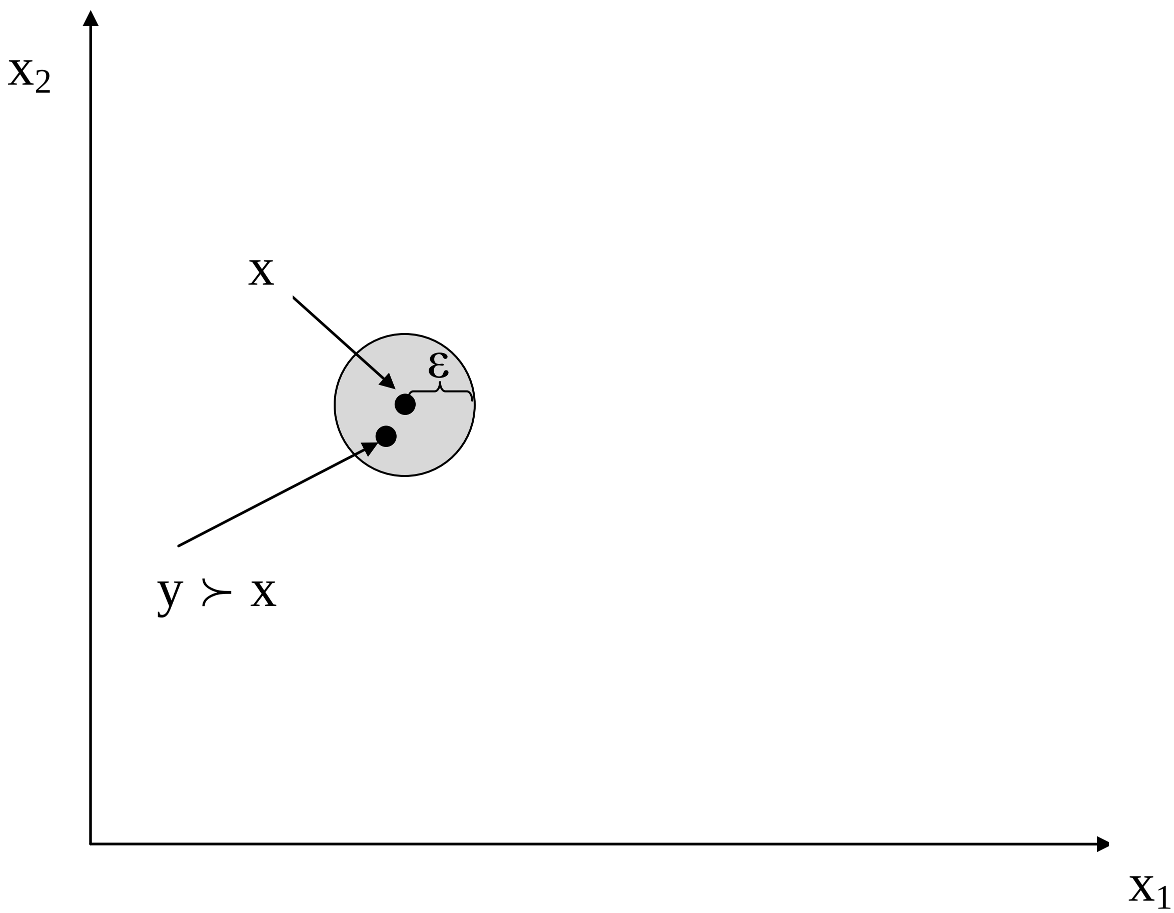
Another question: If a preference relation $\succsim$ is monotone, is it also strongly monotone? If a preference relation $\succsim$ is strongly monotone, is it also monotone?

The **other** common **desirability assumption** is **local nonsatiation**, which can be defined as follows:

Definition: The preference relation $\succsim$ on X is **locally nonsatiated** if for every $x \in X$ and every $\varepsilon > 0$ there is $y \in X$ such that $\| y - x \| \leq \varepsilon$ and $y \succ x$.

Note: $\| y - x \| = [\sum_{\ell=1}^{L} (y_\ell - x_\ell)^2]^{1/2}$.

Locally nonsatiated preferences are shown for the case where L = 2 in the following figure:



In words, local nonsatiation means that **for any point x** one can define a set of points very close to it (within the distance ε) and within that set ("in the neighborhood of x") **there is a point y** that is **strictly preferred to x**.

Note that, as drawn, the commodity bundle y has ***less*** of both items than the bundle x. This is OK for the purposes of this definition, and it implies that local nonsatiation is weaker than monotonicity because the latter requires the preferred bundle to have more (or the same), but never less, of each commodity. Yet local nonsatiation is stronger in the sense that it posits the *existence* of a preferred bundle.

Question: Suppose the commodities over the set $X = \mathbb{R}^L_+$ are all “bads”, so that the most preferred point in this set is x = 0 (do not consume any good). Is local nonsatiation satisfied for this point x = 0? More generally, is local nonsatiation satisfied for the set $X = \mathbb{R}^L_+$? For $X = R^L_{++}$?

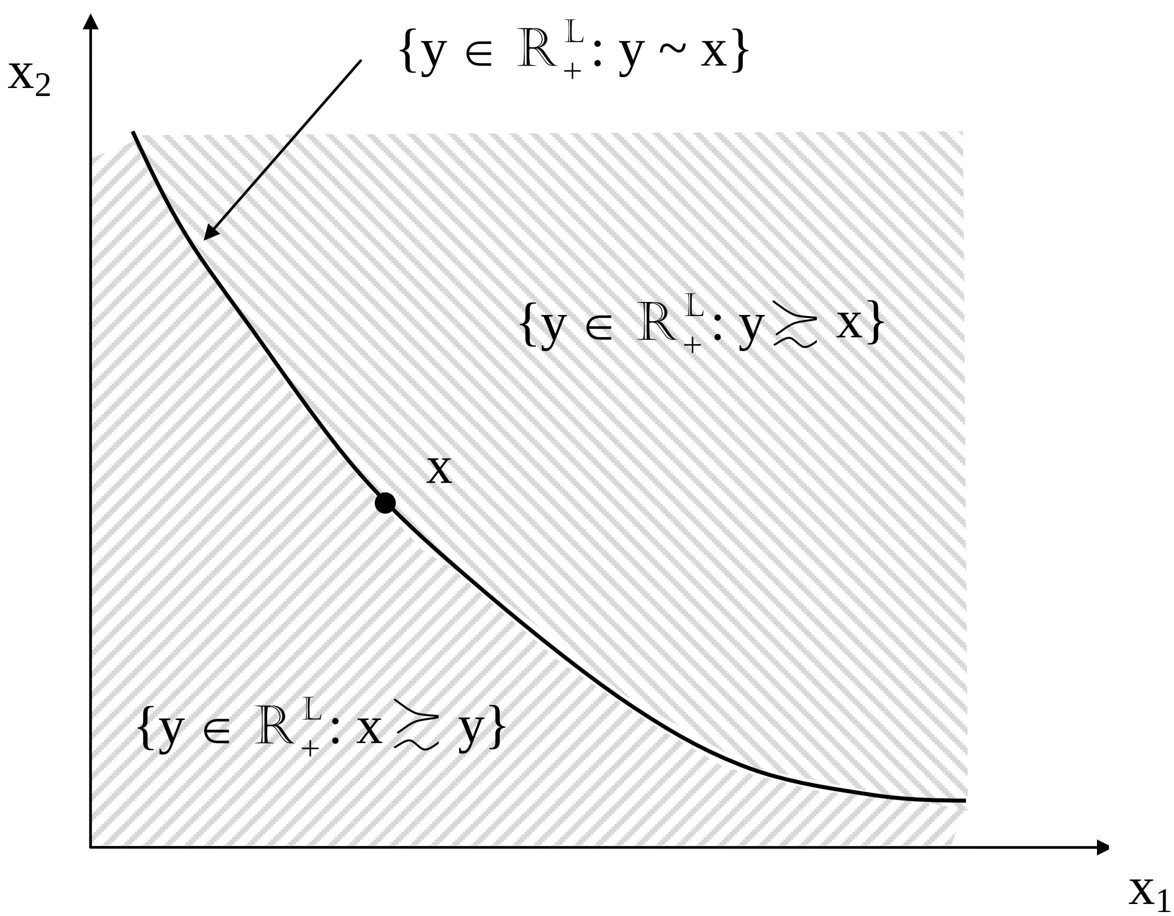
Finally, it is **useful to define the indifference set**, as well as the **upper and lower contour sets**, for a point x. These definitions are as follows:

- The **indifference set** for the commodity bundle x is the set of all commodity bundles for which the consumer is indifferent when comparing them to x. More formally, it is defined as $\{y \in X: y \sim x\}$.
- The **upper contour set** for the commodity bundle x is the set of all commodity bundles that are “at least as good as” x: $\{y \in X: y \succsim x\}$.
- The **lower contour set** for the commodity bundle x is the set of all commodity bundles that x is “at least as good as”: $\{y \in X: x \succsim y\}$.

Question: For each of these sets, is x a member of the set?

Another question: Suppose that there is an area for which preferences are "thick" in the sense that, for a bundle x within the indifference set, small movements in any direction do not lead to any commodity bundles that are strictly preferred to x. [Draw a picture to make this scenario clearer.] Does this x satisfy local nonsatiation?

The following diagram shows the indifference set, the upper contour set and the lower contour set:

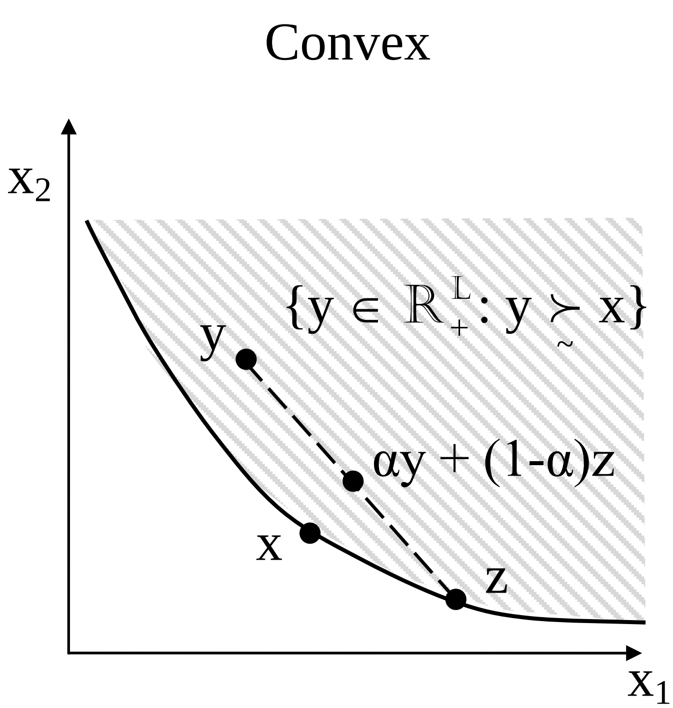


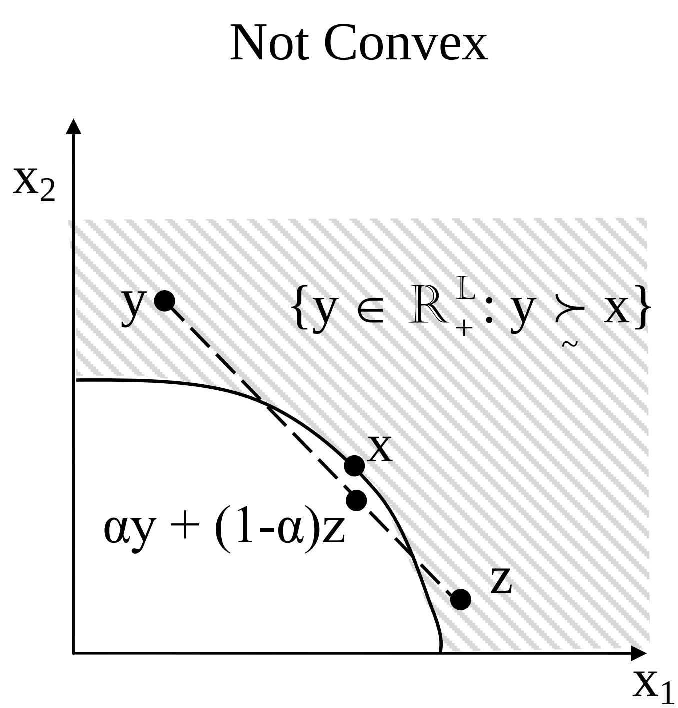
Convexity Assumptions

Assumptions regarding the convexity of the preference relation $\succsim$ are concerned with trade-offs that the consumer is willing to make among different goods. Let's start by defining convexity:

Definition: The preference relation $\succsim$ on X is **convex** if for every $x \in X$, the upper contour set $\{y \in X: y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

The following diagrams should make this point clearer:





Convexity plays a central role in economic analysis of consumer behavior. The **intuition** for it is that there are **diminishing marginal rates of substitution** between different goods. For any two commodities in x, the consumer's willingness to trade the first for the second will decline as the amount of the first is reduced.

An even more intuitive (and less precise) interpretation is that consumers prefer variety. For a given budget set the consumer prefers bundles of goods with many different goods over bundles consisting of one or a small number of goods. Most people would agree that this seems to be a reasonable assumption, but you can always come up with a specific scenario where it may not be reasonable. For example, given a choice of watching two 2-hour films, you probably prefer one or the other rather than watching one for one hour and the other for another hour.

Sometimes we will need a somewhat stronger assumption:

Definition: The preference relation $\succsim$ on X is **strictly convex** if for every $x \in X$, we have that $y \succsim x$, $z \succsim x$, and $y \neq z$ implies that $\alpha y + (1 - \alpha)z \succ x$ for any $\alpha \in (0, 1)$.

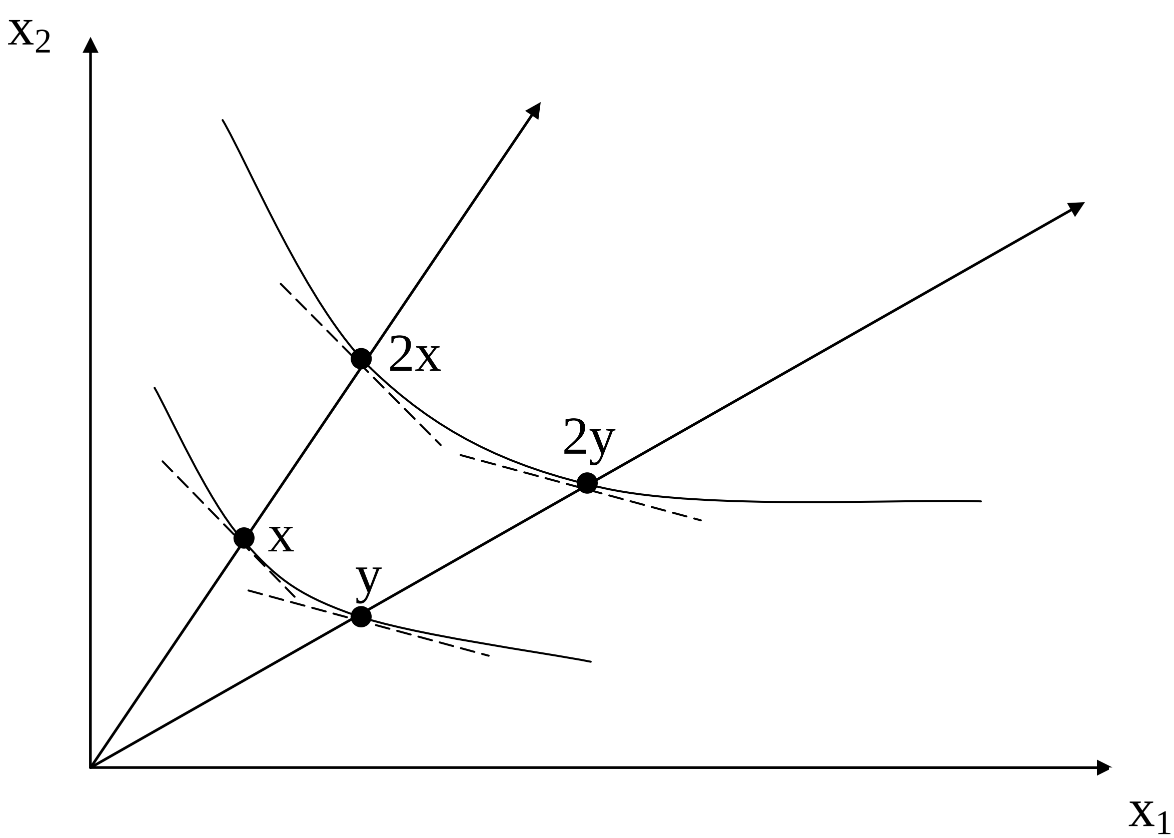
The convex diagram on the previous page is also strictly convex. It would be convex, but not strictly convex, if some or all of the indifference set containing the point x were a straight line.

Homothetic and Quasilinear Preferences

Sometimes it is useful to "simplify" a preference relation by taking a particular indifference set and "expanding" it over the entire space X. Two common examples of this are homothetic preferences and quasilinear preferences.

Definition: A monotone preference relation $\succsim$ on $X = \mathbb{R}_+^L$ is **homothetic** if all indifference sets are related by proportional expansion along rays from the origin. That is, if $x \sim y$ then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.

This is illustrated in the following diagram (for $\alpha = 2$):



Homothetic preferences have the property that, for any ray drawn from the origin, the slopes of all indifference sets at the points crossed by that ray are equal. This means that **as the budget set expands** (w increases), if prices are unchanged the **consumer increases the purchases of all goods by the same proportion that w increases**.

Question: Does this seem realistic?

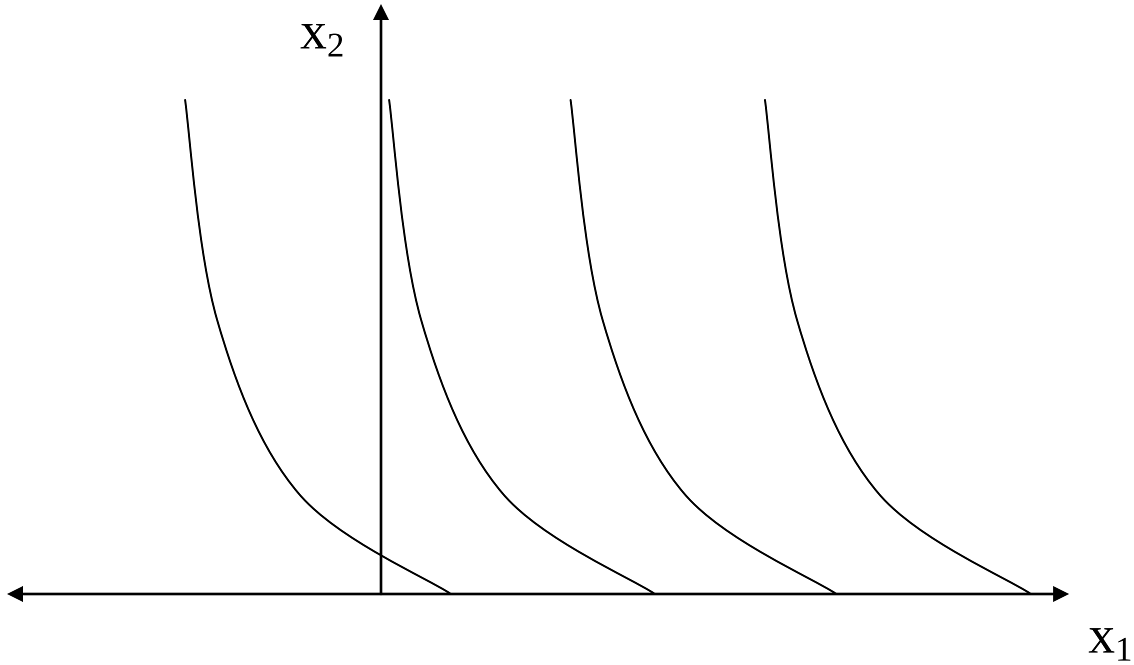
Now let's turn to quasilinear preferences

Definition: A preference relation $\succsim$ on $X = (-\infty, \infty)\times\mathbb{R}_{+}^{L-1}$ is **quasilinear** with respect to commodity 1 (the *numeraire* commodity) if:

1. All indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots 0)$ and any $\alpha \in \mathbb{R}$.

2. Good 1 is desirable. That is, $x + \alpha e_1 \succ x$ for all x and all $\alpha > 0$.

An unusual characteristic of quasilinear preferences is that there is no lower bound on the numeraire commodity; it can take negative values. This is rather odd, but it is a consequence of the definition. Here is a graph of them:



III. Preference and Utility

While the preference relation $\succsim$ is a very general and perhaps reasonably credible starting point for economic analysis, for applied/empirical economic research it is more convenient, and easier to understand, if we start with a utility function. Even for theoretical research utility functions are very useful because there is a much larger set of mathematical tools that one can use on utility functions.

This section explains what additional assumption or assumptions are needed to move from preference relations to utility functions. In fact, we focus on just one assumption; **if preferences are continuous** (in the sense explained below), **they can be represented by a** (continuous) **utility function**.

Here is an example of a **preference relation that cannot be represented by a utility function: lexicographical preferences**. Assume there are only two goods, so that $X = \mathbb{R}^2_+$. Lexicographical preferences are defined as:

$$x \succsim y \text{ if either "} x_1 > y_1 \text{" or "} x_1 = y_1 \text{ and } x_2 \geq y_2 \text{"}$$

That is, whenever two bundles (each of which has only two elements), x and y, are compared, first check whether x has more of good 1 than y. If it does, then x is preferred to y. But if x and y have equal amounts of good 1, then check the amounts of good 2: if $x_2 \geq y_2$ then x is preferred to y. This is called "lexicographical" because it is like looking up a word in a dictionary (lexicon); start with the first letter of the word, then the second letter, and so on.

It is impossible write a utility function that can represent this preference ordering. The intuition here is that utility sets (curves) represent points that are of equal value to the consumer, but for these preferences **every point has a different value (utility)**. A more rigorous argument is given on p.46 of Mas-Colell et al.

The **assumption needed to ensure the existence of a utility function** is that the **preference relation is continuous**, in the following sense:

Definition: The preference relation $\succsim$ on X is **continuous** if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ for which $x^n \succsim y^n$ for all n, and where $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.

Intuitively, this definition of continuity means that preferences cannot have "jumps" or "reversals" when the bundles being compared change slightly (e.g. when x^n and y^n for very large n are very close to x and y).

An equivalent way to state this definition of continuity is to say that, for all x, both the **upper contour set** $\{y \in X: y \succsim x\}$ and the **lower contour set** $\{y \in X: x \succsim y\}$ **are closed** (the limit of any sequence of elements within the set is also within the set). In other words, they both include their boundaries (i.e. the indifference set).

Lexicographical preferences are **not continuous**. To see why, consider two sequences of bundles:

$$x^n = (1/n, 0)$$

$$y^n = (0, 1) \text{ [same for all n]}$$

For all values of n, $x^n \succ y^n$ (since $x^n \succsim y^n$ but it does not hold that $y^n \succsim x^n$). But at the limit the preference reverses, since $\lim_{n\to\infty} x^n = (0, 0)$ and $\lim_{n\to\infty} y^n = (0, 1)$, that is $\lim_{n\to\infty} y^n \succsim \lim_{n\to\infty} x^n$.

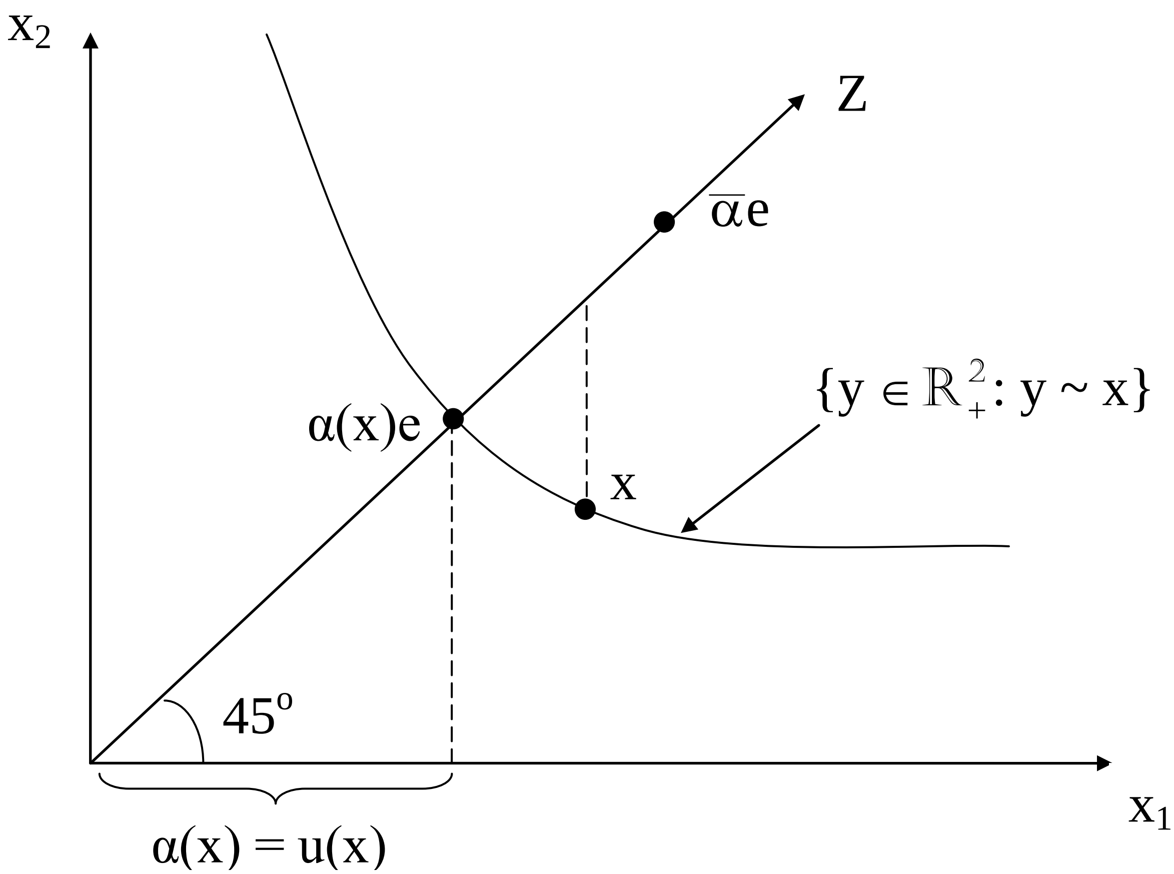
Next, we show that continuity of the preference relation $\succsim$ implies that there exists a (continuous) utility function that represents those preferences:

Proposition 3.C.1: Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function u(x) that represents $\succsim$.

This is an important result, so let's go through the proof for the case where X = $\mathbb{R}^L_+$ *and* the preference relation $\succsim$ is monotone:

Proof: Let Z denote the "diagonal" ray in $\mathbb{R}^L_+$ space (that is, all the points in $\mathbb{R}^L_+$ for which $x_1 = x_2 = \ldots = x_L$). Let e be the L dimensional vector with all elements = 1. Thus $\alpha e \in Z$ for all $\alpha \geq 0$.

(This shown on the diagram on the next page.)



For all vectors $x \in \mathbb{R}^L_+$, monotonicity implies that $x \succsim 0$. In addition, for any $\overline{\alpha}$ such that $\overline{\alpha}e \gg x$, monotonicity implies that $\overline{\alpha}e \succsim x$ (and that $\overline{\alpha}e \succ x$). One can then show that continuity and monotonicity together imply that there is a unique value $\alpha(x) \in [0, \overline{\alpha}]$ such that $\alpha(x)e \sim x$. [This is shown at the top of p.48 of Mas-Colell.]

This function α(x) will be our utility function: u(x) = α(x). To finish, we need to check that this utility function has two properties:

1. It "replicates" the preference relation $\succsim$ (that is, $\alpha(x) \geq \alpha(y) \Leftrightarrow x \succsim y$)

2. It is a continuous function.

The first property is relatively easy to show. Suppose that $\alpha(x) \geq \alpha(y)$. Monotonicity implies that $\alpha(x)e \succsim \alpha(y)e$. Recall that the definition of the $\alpha(\ )$ function implies that $x \sim \alpha(x)e$ and $y \sim \alpha(y)e$; thus $x \succsim y$. **[Why?]** To go in the other direction, suppose that $x \succsim y$. Then we have that $\alpha(x)e \sim x \succsim y \sim \alpha(y)e$ and so by monotonicity $\alpha(x) \geq \alpha(y)$.

To show that $\alpha(\ )$ is continuous is more tedious. See Mas-Colell, pp.48-49. **Q.E.D.**

IV. Further Discussion of Utility Functions

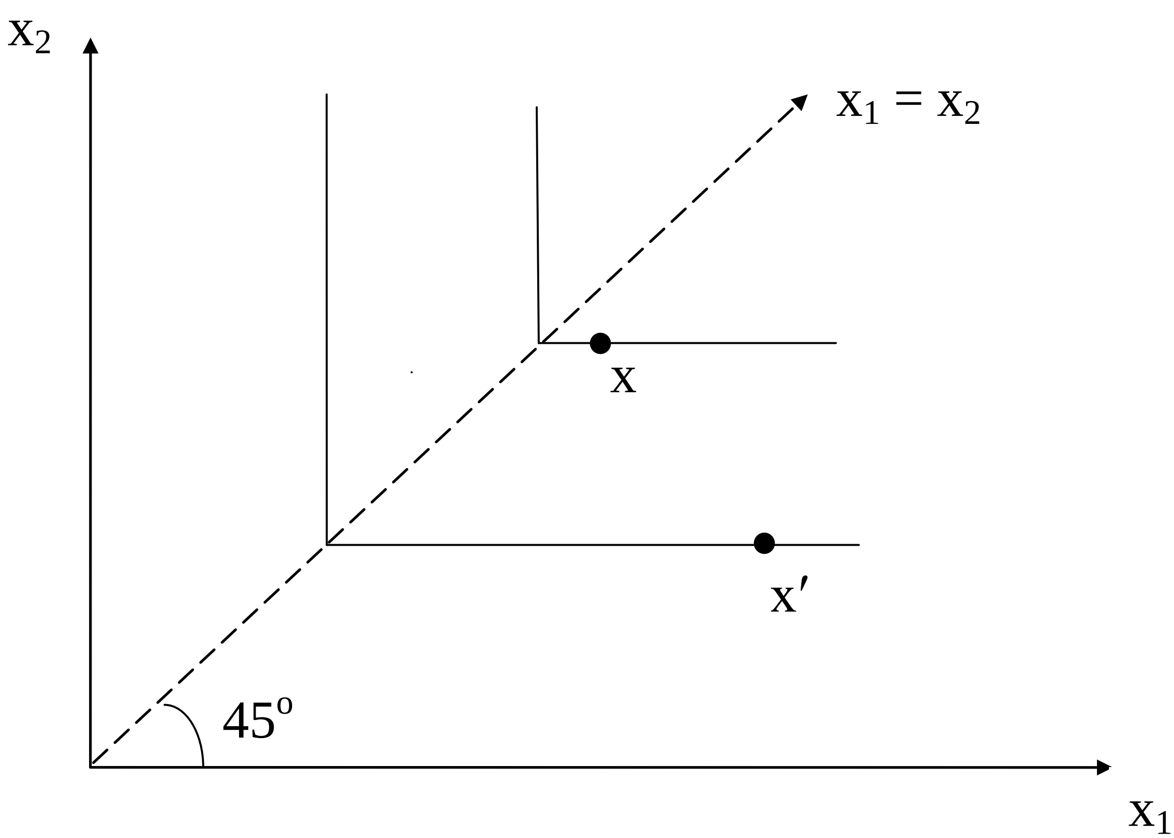
Unless otherwise indicated, for the rest of this class we will assume that consumers' preference relations are continuous, and thus they can be represented by a continuous utility function.

However, any utility function $u(\ )$ that represents a preference relation $\succsim$ is **not unique**. Any strictly increasing transformation of $u(\ )$, such as $v(x) = f(u(x))$, where $f(\ )$ is a strictly increasing function, also represents $\succsim$.

Another point is that although all continuous preference relations can be represented by a continuous utility function, it is also possible to represent those preferences using a utility function that is not continuous; the above statement did not require f( ) to be continuous – it only has to be strictly increasing, which could include "jumps".

It is also very convenient to assume that u( ) is **differentiable**. Yet there are continuous preferences that cannot be represented by a differentiable utility function. The simplest example is **Leontief preferences**, which are shown in the following diagram. They are defined as:

$x \succsim x'$ if an only if $\min\{x_1, x_2, \ldots x_L\} \geq \min\{x_1', x_2', \ldots x_L'\}$



In future lectures we will often assume that u( ) is twice differentiable.

Most restrictions on preferences can be incorporated into the utility functions that represent those preferences by placing restrictions on the utility functions. For example, monotone preferences imply that u(x) > u(y) if x >> y.

Another important example of this is that **convex preferences imply that u( ) is quasiconcave**. (And strictly convex preferences imply strict quasiconcavity.)

Definition: A utility function u( ) is **quasiconcave** if the set $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex for all x.

An equivalent definition is: $u(\alpha x + (1-\alpha)y) \geq \mathrm{Min}\{u(x), u(y)\}$ for any x, y and all $\alpha \in [0, 1]$.

[If the inequality is strict for all $x \neq y$ and all $\alpha \in (0, 1)$ then u( ) is **strictly** quasiconcave.]

Technically speaking, convexity of the preference relation $\succsim$ does not imply that u( ) is concave, which is a stronger property than quasiconcavity. Mas-Colell et al. also state that there can be a convex preference relation $\succsim$ that can't be represented by **any** concave utility function. [They do not say that such preferences also cannot be represented by a quasiconcave utility function, but I am 99% sure of it.]

Two other results regarding preferences and utility are:

1. A continuous $\succsim$ on X = $\mathbb{R}^{L}_{+}$ is homothetic if and only if it corresponds to a utility function u(x) that is homogenous of degree one [$u(\alpha(x)) = \alpha u(x)$ for all $\alpha > 0$].

2. A continuous $\succsim$ on $(-\infty, \infty) \times \mathbb{R}^{L-1}_{+}$ is quasilinear if and only if it corresponds to a utility function u(x) of the form $u(x) = x_1 + \phi(x_2, x_3, \ldots x_L)$.

Finally, note that increasingness (that $u(x) > u(y)$ if $x >> y$) and quasiconcavity are **ordinal** properties of u( ). That is, these properties continue to hold if utility is redefined to be v( ) = f(u( )) for a strictly increasing function f′( ).

In contrast, utility functions that are homogenous of degree one or have the form $u(x) = x_1 + \phi(x_2, x_3, \ldots x_L)$ are **not** ordinal. These properties are called **cardinal** properties.