Applied Microeconomics: Firm and Household

Course Review

Jason Kerwin

Department of Applied Economics
University of Minnesota

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Math Review

Look over the Math Review slides

Cobb-Douglas utility

A Cobb-Douglas (CD) utility function is given by

- $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, for $\alpha > 0$ and $\beta > 0$

The CD indifference curve is:

- $x_1^{\alpha} x_2^{\beta} = u_0$
- $x_2 = x_1^{\alpha/\beta} u_0^{1/\beta}$

The marginal utilities are:

- $u_1 = \alpha x_1^{\alpha-1} x_2^{\beta}$ and $u_2 = \beta x_1^{\alpha} x_2^{\beta-1}$

The MRS is:

- $MRS = \frac{u_1}{u_2} = \frac{\alpha x_2}{\beta x_1}$

Perfect substitutes

Suppose an individual is buying food for a party. She wants enough food for her guests and considers two hot dogs to be equivalent to one hamburger. These preferences can be represented as

- $u(x_1, x_2) = x_1 + 2x_2$, where x_1 is hot dogs, x_2 is hamburger

In general form, the perfect substitute preferences is represented as:

- $u(x_1, x_2) = \alpha x_1 + \beta x_2$

The indifference curve is:

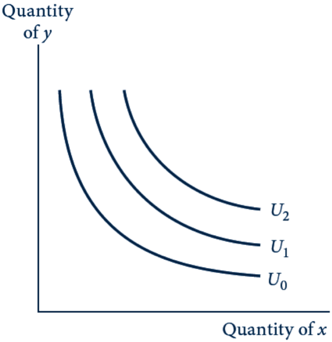
- $x_2 = u_0/\beta - \frac{\alpha}{\beta} x_1$

The marginal utilities and MRS are:

- $u_1 = \alpha$ and $u_2 = \beta$
- $MRS = \frac{\alpha}{\beta}$

Important special classes of preferences

The indifference curves are convex to the origin (diminishing MRS) for Cobb-Douglas but not for perfect substitutes



(a) Cobb-Douglas (b) Perfect substitutes

Perfect complements (Leontief utility)

Suppose an individual consumes a hamburger patty with two slices of bread. If she has 3 patties and 9 slices of bread, then the last 3 slices are worthless. Similarly, if any extra patties without two slices of bread are worthless too.

- $u(x_1, x_2) = min\{2x_1, x_2\}$, where x_1 are patties, x_2 are slices of bread

In general, perfect complements (Leontief) preferences is represented as:

- $u(x_1, x_2) = min\{\alpha x_1, \beta x_2\}$

The marginal utilities and MRS are:

- $u_1 = \alpha$ and $u_2 = 0$ if $\alpha x_1 < \beta x_2$
  - $MRS = \frac{\alpha}{0} = \infty$
- $u_1 = 0$ and $u_2 = \beta$ if $\alpha x_1 > \beta x_2$
  - $MRS = \frac{0}{\beta} = 0$

Constant Elasticity of Substitution (CES)

The CES utility function is given by

- $u(x_1, x_2) = \frac{x_1^\delta}{\delta} + \frac{x_2^\delta}{\delta}$, for $\delta \neq 0$
- $u(x_1, x_2) = ln(x_1) + ln(x_2)$, for $\delta = 0$

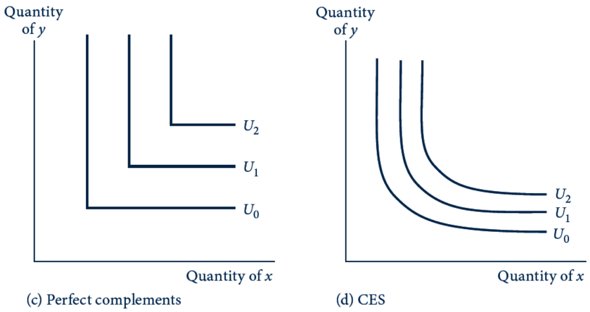
- As $\delta \to 0$, the CES approximates Cobb-Douglas (why?)
- As $\delta \to 1$, the CES approximates perfect substitutes
- As $\delta \to -\infty$, the CES approximates perfect complements

The marginal utilities and MRS are:

- $u_1 = x_1^{\delta-1}$ and $u_2 = x_2^{\delta-1}$
- $MRS = \frac{u_1}{u_2} = \frac{x_2^{1-\delta}}{x_1^{1-\delta}}$

What Leeontief and CES indifferennce curves look like

Indifference curves are convex, implying that individuals prefer balance their consumption



Homothetic Preferences

A utility function exhibits **homothetic preferences** if MRS only depends on the ratio of the amounts of the two goods, not the total.

Properties of homothetic functions:

- Indifference curves exhibit the same curvature at the same ratio of goods
  - i.e., slopes depend on x_1/x_2 not on how far out is the indifference curve
- MRS is constant along any ray line, i.e., MRS is a function of x_1/x_2

Quasi-Linear preferences

An individual has quasi-linear preferences if they can be represented by a utility function of the form

- $u(x_1, x_2) = v(x_1) + x_2$
  - Quasi-linear preferences are linear in x_2
  - These preferences are often used to analyze goods which constitute a small part of an individual's income.
  - Consider x_2 as general consumption (a.k.a. income)

The MRS is equal to

- $MRS = \frac{v'(x_1)}{1} = v'(x_1)$
  - MRS depends only on x_1
  - These preferences are not homothetic.

Utility maximization: FOCs

- $\underset{x_1, x_2}{\text{Max}} \quad L = u(x_1, x_2) + \lambda(M - p_1 x_1 - p_2 x_2)$

The necessary condition for an interior optimum is that the first partials of the Lagrangian equal zero.

1. $\frac{\partial L}{\partial x_1} = L_1 = u_1 - \lambda p_1 = 0$
2. $\frac{\partial L}{\partial x_2} = L_2 = u_2 - \lambda p_2 = 0$
3. $\frac{\partial L}{\partial \lambda} = L_\lambda = M - p_1 x_1 - p_2 x_2 = 0$

From FOC 1 and 2 we can derive the following equilibrium condition

- $\frac{u_1}{u_2} = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2}$
- $MRS_{12} = \frac{p_1}{p_2}$

The first order conditions imply that at the optimum the MRS of the two goods equals their price ratio.

Utility maximization: Graphical interpretation

- $MRS_{12} = \frac{u_1}{u_2} = \frac{p_1}{p_2}$

The condition for an optimum states that a utility-maximizing individual consumes goods x_1 and x_2 at levels where the ratio of prices of the goods equals to the ratio of marginal utilities of the goods (MRS).

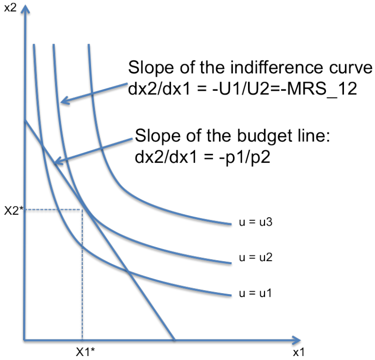


Figure: Graph of utility maximization

Utility maximization: Second order sufficient condition

Using the second partials we can form the bordered Hessian:

- $H_b = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{21} & u_{22} \end{vmatrix}$

Check if $H_{b2} < 0$ and $H_{b3} > 0$ (SOC for a maximum). So long as the SOC holds and $H_b \neq 0$, we can solve for the system of FOC equations to obtain

- $x_1 = x_1^*(p_1, p_2, M)$
- $x_2 = x_2^*(p_1, p_2, M)$
- $\lambda = \lambda^*(p_1, p_2, M)$

x_1^* and x_2^* are the uncompensated demand functions. Also commonly referred as *Marshallian demand functions* or *Money-income-held-constant demand functions*.

Roy's identity

Given an arbitrary indirect utility function we can derive the utility-maximizing Marshallian demand functions using **Roy's Identity**:

- $x_i^* = -\frac{\partial u^* / \partial p_i}{\partial u^* / \partial M}$

A very important result that has strong implications for applied work.

Comes from the **Implicit Function Theorem** – a powerful tool for microeconomic analysis in general.

Expenditure minimization

Consider a problem in which consumer is assumed to minimize the expenditure needed to achieve a given utility level, u^0.

- $\underset{x_1, x_2}{Min}\ E = p_1 x_1 + p_2 x_2$

subject to

- $u^0 = u(x_1, x_2)$

The Lagrangian is formed as

- $\underset{x_1, x_2, \lambda}{Min}\ L = p_1 x_1 + p_2 x_2 + \lambda(u^0 - u(x_1, x_2))$

the FOCs are

1. $L_1 = p_1 - \lambda u_1 = 0$
2. $L_2 = p_2 - \lambda u_2 = 0$
3. $L_\lambda = u^0 - u(x_1, x_2) = 0$

Expenditure minimization: Graphical interpretation

- $MRS_{12} = \frac{u_1}{u_2} = \frac{p_1}{p_2}$

The condition for an optimum states that an expenditure-minimizing individual consumes goods x_1 and x_2 at levels where the ratio of the prices of the goods equals to the ratio of their marginal utilities (MRS).

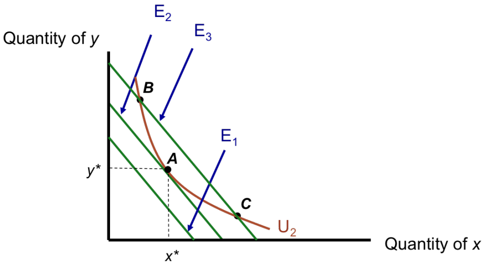


Figure: Graph of expenditure minimization

Marshallian vs. Hicksian demand functions

- Marshallian demands are money-income-constant demand functions. Whereas, Hicksian demands are real-income-constant (i.e. utility constant) demand functions.
- Suppose that $u^*(x_1^*, x_2^*)$ is the maximum utility achieved from a utility maximization problem. Since the tangency condition for expenditure minimization is the same as the tangency condition for utility maximization, an expenditure minimizing consumer achieves the same outcome (x_1^*, x_2^*) by setting $u^0 = u^*$ with the same prices.
- Even though the same outcome is achieved in both utility maximization and expenditure minimization problems, the comparative statics are *not* the same! (We will show this in our next lecture)
- While the partial derivatives of Marshallian demands w.r.t. prices represent combined substitution and income effects, the partial derivatives of Hicksian demands w.r.t. prices represent the *pure substitution effect*.

Shephard's lemma (the envelope theorem)

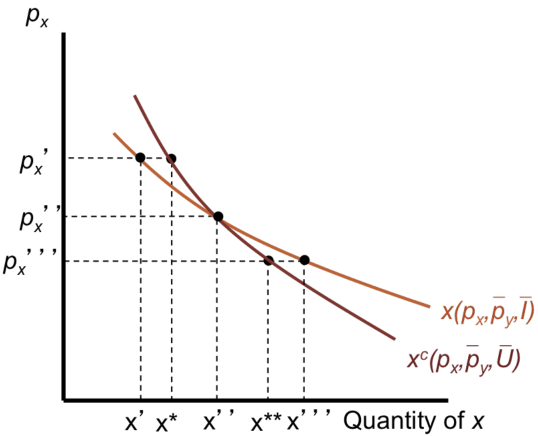
If an expenditure function is differentiable in p, then the Hicksian demand functions can be derived by:

- $x_i^h(p, u) = \frac{\partial E^*(p,u)}{\partial p_i}$

Shephard's Lemma is very useful in applied work.

Note that because expenditure function is linearly homogeneous in prices, its first derivative with respect to prices is homogeneous of degree zero. This is just as we would expect for the Hicksian demand function.

Relationship between compensated and uncompensated demand curves



($\bar{I}$ is **nominal income**)

Comparative statics

To perform comparative statics first we rewrite the FOCs as:

1. $u_1(x_1^*(p_1, p_2, M), x_2^*(p_1, p_2, M)) - \lambda^*(p_1, p_2, M)p_1 \equiv 0$
2. $u_2(x_1^*(p_1, p_2, M), x_2^*(p_1, p_2, M)) - \lambda^*(p_1, p_2, M)p_2 \equiv 0$
3. $M - p_1 x_1^*(p_1, p_2, M) - p_2 x_2^*(p_1, p_2, M) \equiv 0$

Note that the FOCs are now expressed in terms of identities. Because the relation holds for all values of p and M,

- x^* is the optimal decision rule. Therefore, as p or M changes, the utility-maximizing individual adjusts by changing her x^* such that the $FOC = 0$ for any values of p and M.

Comparative statics: income effects

To derive $\frac{\partial x_i^*}{\partial M}$, $i = 1, 2$ we differentiate the identities with respect to M.

- $u_{11}\frac{\partial x_1^*}{\partial M} + u_{12}\frac{\partial x_2^*}{\partial M} - p_1\frac{\partial \lambda^*}{\partial M} \equiv 0$
- $u_{21}\frac{\partial x_1^*}{\partial M} + u_{22}\frac{\partial x_2^*}{\partial M} - p_2\frac{\partial \lambda^*}{\partial M} \equiv 0$
- $1 - p_1\frac{\partial x_1^*}{\partial M} - p_2\frac{\partial x_2^*}{\partial M} \equiv 0$

We can write this system of three equations in matrix notation as

- $\begin{pmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial M} \\ \frac{\partial x_2^*}{\partial M} \\ \frac{\partial \lambda^*}{\partial M} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Comparative statics: income effects

Using Cramer's rule,

- $$\frac{\partial x_1^*}{\partial M} = \frac{\begin{vmatrix} 0 & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ -1 & -p_2 & 0 \end{vmatrix}}{|H|} = -\frac{H_{31}}{|H|} = -\frac{(-p_2 u_{12} + p_1 u_{22})}{|H|} \gtrless 0$$

- $$\frac{\partial x_2^*}{\partial M} = \frac{\begin{vmatrix} u_{11} & 0 & -p_1 \\ u_{21} & 0 & -p_2 \\ -p_1 & -1 & 0 \end{vmatrix}}{|H|} = -\frac{H_{32}}{|H|} = -\frac{(p_2 u_{11} - p_1 u_{21})}{|H|} \gtrless 0$$

- $$\frac{\partial \lambda^*}{\partial M} = \frac{\begin{vmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ -p_1 & -p_2 & -1 \end{vmatrix}}{|H|} = -\frac{H_{33}}{|H|} = -\frac{(u_{11} u_{22} - u_{12}^2)}{|H|} \gtrless 0$$

Comparative statics: price effects

Using Cramer's rule,

- $\dfrac{\partial x_1^*}{\partial p_1} = \dfrac{\begin{vmatrix} \lambda^* & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ x_1^* & -p_2 & 0 \end{vmatrix}}{|H|} = \dfrac{\lambda^* H_{11}}{|H|} + \dfrac{x_1^* H_{31}}{|H|} \gtrless 0$

- $\dfrac{\partial x_2^*}{\partial p_1} = \dfrac{\begin{vmatrix} u_{11} & \lambda^* & -p_1 \\ u_{21} & 0 & -p_2 \\ -p_1 & x_1^* & 0 \end{vmatrix}}{|H|} = \dfrac{\lambda^* H_{12}}{|H|} + \dfrac{x_1^* H_{32}}{|H|} \gtrless 0$

- $\dfrac{\partial \lambda^*}{\partial p_1} = \dfrac{\begin{vmatrix} u_{11} & u_{12} & \lambda_1^* \\ u_{21} & u_{22} & 0 \\ -p_1 & -p_2 & x_1^* \end{vmatrix}}{|H|} = \dfrac{\lambda^* H_{13}}{|H|} + \dfrac{x_1^* H_{33}}{|H|} \gtrless 0$

Slutsky equation

The effect of an increase in price on the demand function is summarized in the Slutsky Equation:

- $\frac{\partial x_i^*}{\partial p_j} = \frac{\partial x_i}{\partial p_j}|_{u=u^0} - x_j^* \frac{\partial x_i^*}{\partial M}$

The equation shows that the price response of a utility-maximizing consumer can be split into:

- i) a pure substitution effect (holding the consumer on the original indifference curve), and
- ii) a pure income effect (holding the prices constant).

Comparative statics: price effects on Hicksian demand

Using Cramer's rule,

- $$\frac{\partial x_1^h}{\partial p_1} = \frac{\begin{vmatrix} -1 & -\lambda^h u_{12} & -u_1 \\ 0 & -\lambda^h u_{22} & -u_2 \\ 0 & -u_2 & 0 \end{vmatrix}}{|H^h|} = -\frac{H_{11}^h}{|H^h|} = -\frac{-u_2^2}{|H^h|} < 0,\ SOC: |H^h| < 0$$
- $$\frac{\partial x_2^h}{\partial p_1} = \frac{\begin{vmatrix} -\lambda^h u_{11} & -1 & -u_1 \\ -\lambda^h u_{21} & 0 & -u_2 \\ -u_1 & 0 & 0 \end{vmatrix}}{|H^h|} = -\frac{H_{12}^h}{|H^h|} = -\frac{u_1 u_2}{|H^h|} > 0,\ SOC: |H^h| < 0$$

The comparative statics results state that Hicksian demand curves are *always* downward sloping. Also, the cross-price effect in the two-good case is positive, indicating that the goods have to be substitutes (note that this does not have to hold in the n-good case if $n > 2$).

Comparative statics: price effects on Hicksian demand

Using Cramer's rule,

- $$\frac{\partial x_1^h}{\partial p_1} = \frac{\begin{vmatrix} -1 & -\lambda^h u_{12} & -u_1 \\ 0 & -\lambda^h u_{22} & -u_2 \\ 0 & -u_2 & 0 \end{vmatrix}}{|H^h|} = -\frac{H_{11}^h}{|H^h|} = -\frac{-u_2^2}{|H^h|} < 0,\ SOC : |H^h| < 0$$
- $$\frac{\partial x_2^h}{\partial p_1} = \frac{\begin{vmatrix} -\lambda^h u_{11} & -1 & -u_1 \\ -\lambda^h u_{21} & 0 & -u_2 \\ -u_1 & 0 & 0 \end{vmatrix}}{|H^h|} = -\frac{H_{12}^h}{|H^h|} = -\frac{u_1 u_2}{|H^h|} > 0,\ SOC : |H^h| < 0$$

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Useful Elasticity Formulas: homogeneity

The Marshallian demand, $x_1^*(p_1, p_2, M)$, is HOD 0 in prices and income, by Euler's theorem:

- $\frac{\partial x_1^*}{\partial p_1} p_1 + \frac{\partial x_1^*}{\partial p_2} p_2 + \frac{\partial x_1^*}{\partial M} M = 0$, divide both sides by x_1^*
- $\epsilon_{11}^* + \epsilon_{12}^* + \epsilon_{1M}^* = 0$
- $\epsilon_{ii}^* + \epsilon_{ij}^* + ... + \epsilon_{in}^* + \epsilon_{1M}^* = 0$, for the case of n goods.

That is, the sum of own-price, cross-price and the income elasticities of the Marshallian demand is equal to zero.

Useful Elasticity Formulas: homogeneity

The Hicksian demand $x_1^h(p_1, p_2, u^0)$ is HOD 0 in prices, by Euler's theorem

- $\frac{\partial x_1^h}{\partial p_1} p_1 + \frac{\partial x_1^h}{\partial p_2} p_2 = 0$, divide both sides by x_1^h
- $\epsilon_{11}^* + \epsilon_{12}^* = 0$
- $\epsilon_{ii}^h + \epsilon_{ij}^h + ... + \epsilon_{in}^h = 0$, for the case of n goods.

That is, the sum of the own- and cross-price elasticities of the Hicksian demand is equal to zero.

Useful Elasticity Formulas: Engel aggregation

Engel's law: As income increases the share of income spent on food decreases.

- Income elasticity of demand for food < 1

Engel's law implies that the income elasticity of all nonfood items must be > 1.

We can establish the formal relationship between income elasticities by differentiating the budget constraint, $p_1 x_1^* + p_2 x_2^* = M$, with respect to income.

- $p_1 \frac{\partial x_1^*}{\partial M} + p_2 \frac{\partial x_2^*}{\partial M} \equiv 1$

Useful Elasticity Formulas: Engel aggregation

- $p_1 \frac{\partial x_1^*}{\partial M} + p_2 \frac{\partial x_2^*}{\partial M} \equiv 1$

Multiplying the first term by $\frac{x_1^* M}{x_1^* M} = 1$ and the second term by $\frac{x_2^* M}{x_2^* M} = 1$ we obtain

- $\frac{p_1 x_1^*}{M} \frac{\partial x_1^*}{\partial M} \frac{M}{x_1^*} + \frac{p_2 x_2^*}{M} \frac{\partial x_2^*}{M} \frac{M}{x_2^*} \equiv 1$
- $s_1 \epsilon_{1M}^* + s_2 \epsilon_{2M}^* = 1$

For the *n*-good case

- $s_1 \epsilon_{1M}^* + s_2 \epsilon_{2M}^* + \ldots + s_n \epsilon_{nM}^* = 1$

That is, the weighted sum of the income elasticities of all goods is equal to 1, where the weights are the shares of income spent on each good.

Useful Elasticity Formulas: Cournot aggregation

Finally, as Cournot once did, we would like to know how a change in a single price might affect the demand for all goods.

To this end, we differentiate the budget constraint, $p_1 x_1^* + p_2 x_2^* = M$, with respect to the price of good 1, p_1, to obtain

- $x_1^* + p_1 \frac{\partial x_1^*}{\partial p_1} + p_2 \frac{\partial x_2^*}{\partial p_1} \equiv 0$

Useful Elasticity Formulas: Cournot aggregation

- $x_1^* + p_1 \frac{\partial x_1^*}{\partial p_1} + p_2 \frac{\partial x_2^*}{\partial p_1} \equiv 0$

Multiplying all terms by $\frac{p_1}{M}$, the second term by $\frac{x_1}{x_1} = 1$ and the third term by $\frac{x_2}{x_2} = 1$ we obtain

- $\frac{p_1 x_1^*}{M} \frac{\partial x_1^*}{\partial p_1} \frac{p_1}{x_1^*} + \frac{p_2 x_2^*}{M} \frac{\partial x_2^*}{\partial p_1} \frac{p_1}{x_2^*} \equiv -\frac{p_1 x_1^*}{M}$
- $s_1 \epsilon_{11}^* + s_2 \epsilon_{21}^* = -s_1$

For the *n* good case

- $s_1 \epsilon_{1j}^* + s_2 \epsilon_{2j}^* + \ldots + s_n \epsilon_{nj}^* = -s_j$

The equation states that the weighted sum of the elasticities of all goods with respect to price of an arbitrary good *j* equals to the negative of the budget spent on good *j*, where the weights are the budget spent on each good.

Equivalent and Compensating Variation

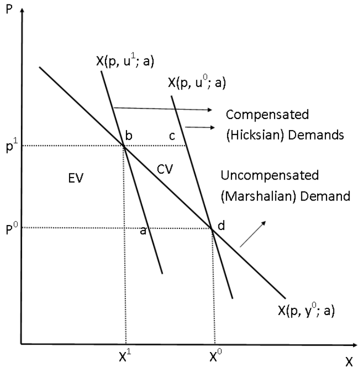


Figure: Welfare measures in case of a price hike

- $EV = P^0P^1ba$
- $CV = P^0P^1cd$
- $CS = P^0P^1bd$

In case of a price increase:

- $|CV| > |CS| > |EV|$

In case of zero income effect

- $|CV| = |CS| = |EV|$

Note that, in case of a price decrease the rankings reverse

- $|EV| > |CS| > |CV|$

Almost Ideal Demand System (AIDS)

Consider the following expenditure function

- $ln(M(p_1,...p_n,u)) = a(p_1,...p_n) + ub(p_1,...p_n)$

where

- $a(p_1,...p_n) = \alpha_0 + \sum_i \alpha_i \ln p_i + \frac{1}{2}\sum_i \sum_j \gamma_{ij} \ln p_i \ln p_j$
- $b(p_1,...p_n) = \beta_0 \prod_j p_j^{\beta_j}$

We impose the following restrictions from economic theory:

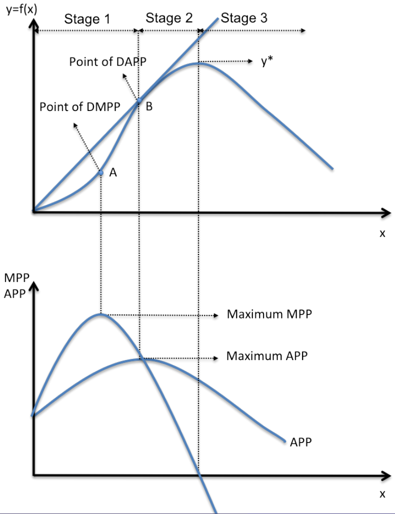
1. $\sum_i \alpha_i = 1 \quad \sum_i \gamma_{ij} = 0 \quad \sum_i \beta_i = 0 \quad (adding - up)$
2. $\sum_j \gamma_{ij} = 0 \quad (homogeneity)$
3. $\gamma_{ij} = \gamma_{ji} \quad (symmetry)$

Technical aspects of production functions

We use the following concepts to characterize production functions.

- Total Product (TP) – sometimes Total Physical Product (TPP)
  - $y = f(x)$
- Average Product (AP) – sometimes Average Physical Product (APP)
  - $AP = \frac{TP}{x} = \frac{f(x)}{x}$
- Marginal Product (MP) – sometimes Marginal Physical Product (MPP)
  - $MP = \frac{\partial TP}{\partial x} = \frac{\partial f(x)}{\partial x} = f'(x)$
- Maximum output
  - $y^* = f(x^*)$ where x^* solves $f'(x) = 0$ with $f''(x) < 0$
- Elasticity of production (factor elasticity)
  - $\epsilon_{y,x} = \frac{\partial y}{\partial x}\frac{x}{y} = \underbrace{\frac{\partial f(x)}{\partial x}}_{MP} / \underbrace{\frac{f(x)}{x}}_{AP} = \frac{MP}{AP}$

The three stages of production



Stage 1:

- $MP > AP$
- $\epsilon_{y,x} > 1$

Stage 2:

- $0 \leqslant MP \leqslant AP$
- $0 \leqslant \epsilon_{y,x} \leqslant 1$

Stage 3:

- $MP < 0$
- $\epsilon_{y,x} < 0$

1-Output 2-Input Production Functions: Technical aspects

We continue our discussion with single output/two input production functions. Formally:

- $y = f(x_1, x_2)$

Suppose, y is wheat output, x_1 is land and x_2 is labor.

AP and MP

We calculate AP and MP for each input holding the other input constant:

- $AP_i = \frac{y}{x_i} = \frac{f(x_1,x_2)}{x_i}, \ i = 1, 2$
- $MP_i = f_i = \frac{\partial y}{\partial x_i} = \frac{\partial f(x_1,x_2)}{\partial x_i}, \ i = 1, 2$

Technical aspects

Total marginal product:

Total marginal product is defined by the total differentiation of the production function:

- $dy = \frac{\partial f(x_1,x_2)}{\partial x_1} dx_1 + \frac{\partial f(x_1,x_2)}{\partial x_2} dx_2$
- $dy = \underbrace{f_1 dx_1}_{dy_1} + \underbrace{f_2 dx_2}_{dy_2}$

where dy_i, $i = 1, 2$ is the marginal product attributable to factor i.

That is, total marginal product is sum of the marginal products that are attributable to each of the factors individually.

Factor interdependence

The technical relationship between two inputs is determined by how the marginal product of one input is affected by the other input. There are three types of technical relationships:

- **Technically complementary:** MP of one input increases as the other input increases.
  - $\frac{\partial^2 y}{\partial x_1 \partial x_2} = \frac{\partial\left(\frac{\partial y}{\partial x_1}\right)}{\partial x_2} = \frac{\partial MP_1}{\partial x_2} = f_{12} = f_{21} > 0$
- **Technically independent:** MP of one input is not affected by changes in the other input.
  - $f_{12} = f_{21} = 0$
- **Technically competitive:** MP of one input decreases as the other input increases.
  - $f_{12} = f_{21} < 0$

Definitions

- **Isoquant:** An isoquant curve represents all input combinations that can produce a given level of output.
- **Marginal rate of technical substitution (MRTS):** is the negative of the slope of the isoquant at an arbitrary point. It gives the rate that one factor must be substituted for the other factor to maintain the same output level.
- **Isoclines:** An isocline is a ray connecting equal MRTS across different isoquants.
- The **ridgelines:** are two special isoclines that define the relevant region of input choices for profitable production. The ridgelines occur at $RTS = 0$ and $RTS = \infty$
- The **elasticity of substitution** is a measure of the degree of substitutability between factors. It is defined as the proportionate rate of the change of input ratio divided by the proportionate rate of change in the MRTS.

Returns to scale

A production function's **returns to scale** is the percentage increase in output when all inputs are increased by the same percentage. A production function can exhibit

- increasing returns to scale
  - i.e., increasing all inputs by one percent increases output by more than one percent.
- constant returns to scale
  - i.e., increasing all inputs by one percent increases output by one percent.
- decreasing returns to scale
  - i.e., increasing all inputs by one percent increases output by less than one percent.

The function coefficient, returns to scale and spacing of isoquants are very closely related.

Profit maximization: output choice

- $MR(q^*) = MC(q^*)$

That is, a profit maximizing monopolist determines its optimal quantity at a level where marginal revenue is equal to its marginal cost.

The SOC of this problem is that

- $\frac{d^2\pi}{dq^2} = \left(\frac{dMR(q)}{q} - \frac{dMC(q)}{q}\right)|_{q=q^*} < 0$

The second order condition of this problem is that the marginal revenue curve has to intersect the marginal cost curve from above.

Profit maximization: input choice

A firms profit maximization problem subject to its production technology can be formalized as:

- $\max_{y,x} \pi(x, y) = py - wx, \quad s.t.\ \ y \leqslant f(x)$.
    - The firm chooses its output and input level. Prices are given.
    - The firm cannot choose output level that is beyond the feasible technology.

Note that, because the firm can choose any level of $y \leqslant f(x)$, so long as $p > 0$ a profit-maximizing firm will always choose $y = f(x)$.

By substituting the binding constraint into the objective function we can transform the problem into a simpler form: an **unconstrained** optimization problem.

- $\max_{x} \pi(x) = pf(x) - wx$

The first order conditions

1. $pf_1(x_1, x_2) = w_1$
2. $pf_2(x_1, x_2) = w_2$

Interpretation: The FOCs say that, for an interior solution, a profit-maximizing firm sets the marginal contribution of each factor to revenues, pf_i, (the factor's **marginal revenue product**) equal to the marginal cost of that factor, w_i.

Implication:

- Profit maximization takes place only at points where the marginal products f_i are positive (regardless of p and w).

Also, note that by dividing (1) and (2) we obtain

- $\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)} = \frac{w_1}{w_2}$

The above equality states that at the optimum a profit-maximizing firm sets the marginal rate of technical substitution, MRTS, equal to the input price ratio.

The second order condition

The SOC for a maximum is that the Hessian has to be *negative semi-definite.* The second partials are:

- $\pi_{11} = pf_{11}$
- $\pi_{22} = pf_{22}$
- $\pi_{12} = pf_{12}$

Hence, the Hessian (why not a bordered Hessian?) is:

- $H(x) = \begin{bmatrix} pf_{11} & pf_{12} \\ pf_{12} & pf_{22} \end{bmatrix}$

- H is negative semidefinite if $f_{11} \leqslant 0$ and $f_{11}f_{22} - f_{12}^2 \geqslant 0$
- From SOC, it must be true that $f_{22} \leqslant 0$
- Note that if the production function is *concave*, the FOC is also sufficient.

Comparative statics: properties of factor demands

- $\pi_1 = pf_1(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_1 \equiv 0$
- $\pi_2 = pf_2(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p)) - w_2 \equiv 0$

To obtain $\frac{\partial x^*}{\partial w}$ we differentiate the FOCs with respect to w_1 (using the chain rule):

- $pf_{11}\frac{\partial x_1^*}{\partial w_1} + pf_{12}\frac{\partial x_2^*}{\partial w_1} - 1 \equiv 0$
- $pf_{21}\frac{\partial x_1^*}{\partial w_1} + pf_{22}\frac{\partial x_2^*}{\partial w_1} \equiv 0$

We can write the system of two equations in matrix notation as

- $\begin{pmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Own and cross-price effects

Using Cramer's rule

- $\frac{\partial x_1^*}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12} \\ 0 & pf_{22} \end{vmatrix}}{|H|} = \frac{p\overbrace{f_{22}}^{<0}}{|H|} < 0 \qquad |H| = p^2(f_{11}f_{22} - f_{12}^2) > 0$ by SOC
- $\frac{\partial x_2^*}{\partial w_1} = \frac{\begin{vmatrix} pf_{11} & 1 \\ pf_{21} & 0 \end{vmatrix}}{|H|} = -\frac{pf_{21}}{|H|} \lesseqqgtr 0 \qquad |H| = p^2(f_{11}f_{22} - f_{12}^2) > 0$ by SOC

Similarly, by taking the derivative of FOCs w.r.t w_2 we can find:

- $\frac{\partial x_2^*}{\partial w_2} = \frac{p\overbrace{f_{11}}^{<0}}{|H|} < 0$
- $\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w_1} = -\frac{pf_{12}}{|H|} \lesseqqgtr 0$

Output price effects

Using Cramer's rule

- $\frac{\partial x_1^*}{\partial p} = \frac{\begin{vmatrix} -f_1 & pf_{12} \\ -f_2 & pf_{22} \end{vmatrix}}{|H|} = \frac{p(-f_1 f_{22} + f_2 f_{12})}{|H|} \lesseqqgtr 0$

- $\frac{\partial x_2^*}{\partial p} = \frac{\begin{vmatrix} pf_{11} & -f_1 \\ pf_{21} & -f_2 \end{vmatrix}}{|H|} = \frac{p(-f_2 f_{11} + f_1 f_{12})}{|H|} \lesseqqgtr 0$

The signs of these comparative statics are indeterminate, depending on the sign of f_{12}. That is, for an increase in output prices the model does not predict the profit maximizing firm's response in terms of hiring more or less of the inputs.

Own-price effects on output supply

Next, we seek to analyze $\partial y^*/\partial p$. To this end, we derive the output supply function by substituting the optimal input demand equations in the production function:

- $y^* = f(x_1^*(w_1, w_2, p), x_2^*(w_1, w_2, p))$

By taking the derivative w.r.t p

- $\frac{\partial y^*}{\partial p} = f_1 \frac{\partial x_1^*}{\partial p} + f_2 \frac{\partial x_2^*}{\partial p}$

We can substitute our previous results on $\frac{\partial x_1^*}{\partial p}$ and $\frac{\partial x_2^*}{\partial p}$ to obtain

- $\frac{\partial y^*}{\partial p} = f_1 \frac{-f_1 f_{22} + f_2 f_{12}}{p(f_{11} f_{22} - f_{12}^2)} + f_2 \frac{-f_2 f_{11} + f_1 f_{12}}{p(f_{11} f_{22} - f_{12}^2)}$
- $\frac{\partial y^*}{\partial p} = \frac{\overbrace{-f_1^2 f_{22} + 2f_1 f_2 f_{12} - f_{11} f_{22}^2}^{>0 \Leftrightarrow \textit{ quasiconcavity of } f(.)}}{p(f_{11} f_{22} - f_{12}^2)} > 0$

Homogeneity of demand and supply functions

We find that the FOCs do not change when we change all prices with the same proportion. This implies that the unconditional factor demand functions are **homogeneous of degree zero in prices.**

Formally,

- $x = x^*(tw_1, tw_2, tp) = x^*(w_1, w_2, p)$

Question: Do you find this result economically intuitive?

Question: What can you say about homogeneity of profit function? Of the supply function?

The cost function

The cost function is the *minimum* cost achievable at any given levels of output and factor prices, denoted as:

- $C = C(y, w_1, ...w_n)$

where y is the output level and w_1, ..., w_n denote the prices of factors employed in production of y, i.e., $x_1, ...x_n$.

- Note that the cost function specifies the total cost of producing any given level of output. Thus, output, y, enters as a parameter in the cost function.

Marginal Cost Function: The rate of change of total cost for a change in output.

- $MC(w, y) = \frac{\partial C^*(w_1,...,w_n,y)}{\partial y}$

Average Cost Function: Cost per unit of output.

- $AC(w, y) = \frac{C^*(w_1,...,w_n,y)}{y}$

Cost minimization

- $\underset{x_1, x_2, \lambda}{\text{Min}} \; L = w_1 x_1 + w_2 x_2 + \lambda(y - f(x_1, x_2))$

The FOCs are:

1. $L_1 = \frac{\partial L}{\partial x_1} = w_1 - \lambda f_1 = 0$
2. $L_2 = \frac{\partial L}{\partial x_2} = w_2 - \lambda f_2 = 0$
3. $L_\lambda = \frac{\partial L}{\partial \lambda} = y - f(x_1, x_2) = 0$

From FOCs 1 and 2:

- $\lambda = \frac{w_1}{f_1} = \frac{w_2}{f_2}$

Therefore, at the optimum it must be the case that

- $\frac{w_1}{w_2} = \frac{f_1}{f_2} = RTS$

Cost minimization: graphical interpretation

- $\frac{w_1}{w_2} = \frac{f_1}{f_2}$

The condition for an optimum states that a cost-minimizing firm employs inputs at a level where the ratio of their prices equals to the ratio of their marginal products.

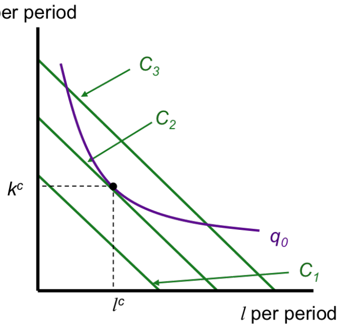


Figure: Graph of cost minimization

Cost minimization: comparative statics

Now we will discuss the comparative statics of this model without a formal derivation (the formal derivation is the same what we did to derive the comparative statics of the expenditure minimization model for consumers).

- How does a cost-minimizing firm adjust its factor demands in response to a change in the price of an input?
  - $\frac{\partial x_1^c}{\partial w_i} =? \;\; i = 1, 2$
- How does a cost-minimizing firm adjust its factor demands in response to a change in the target output?
  - $\frac{\partial x_i^c}{\partial y} =? \;\; i = 1, 2$

Comparative statics: own- and cross-price effect

The own- and cross-price effects can be derived as:

1. $\frac{\partial x_1^c}{\partial w_1} < 0$
    - A cost-minimizing firm uses less of an input when the (own) price of the input increases.
    - The conditional factor demand curve is downward-sloping.
2. $\frac{\partial x_2^c}{\partial w_1} > 0$
    - A cost minimizing firm uses more of the second input when the price of the first input increases.
    - The reason is that when the price of the first input increases a firm uses less of the first input. So, to be able to produce the target output the firm must employ more of the second input.
    - This cross-price effect holds only when there are two inputs. If there are more than two inputs the sign is ambiguous.

Conditional and unconditional demand functions

To formally derive the conditional and unconditional demands, denote the factor demand functions as

- $x_i^c(w, y)$ (conditional) and $x_i^*(p, w)$ (unconditional).

We can establish a relationship between the two demand functions by allowing level of output to vary instead of remaining fixed.

- $x_i^*(p, w) \equiv x_i^c(w, y(p, w))$

Taking the derivative with respect to the price of the input:

- $\frac{\partial x_i^*}{\partial w} = \underbrace{\frac{\partial x_i^c}{\partial w}}_{\text{substitution effect}} + \underbrace{\frac{\partial x_i^c}{\partial y}\frac{\partial y}{\partial w}}_{\text{output effect}}$

Shephard's Lemma

If the cost function is differentiable in w, then the cost-minimizing conditional demand functions are:

- $x_i^c(w, y) = \frac{\partial C(w,y)}{\partial w_i}$

The cost function is linearly homogeneous in input price. Therefore the first derivative of the cost function with respect to each input price is homogeneous of degree zero – as we would expect for the conditional demand function.

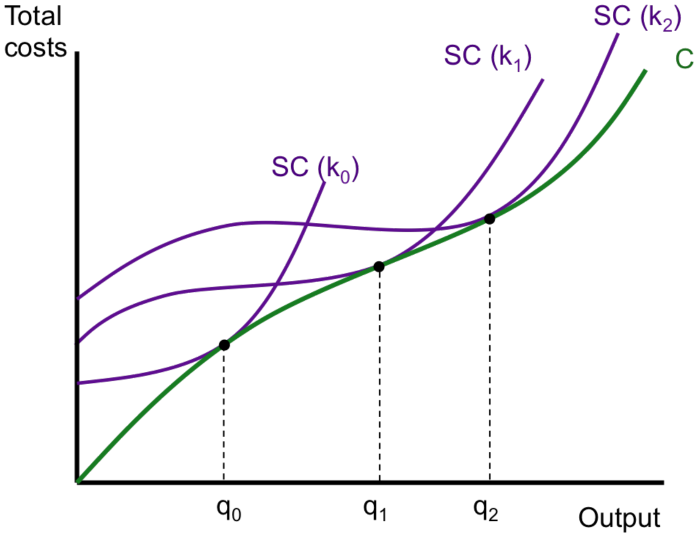
Hotelling's Lemma

If the profit function is differentiable in p and w, the profit maximizing supply and unconditional demand functions are:

- $y(p, w) = \frac{\partial \pi(p,w)}{\partial p}$
- $x_i(p, w) = -\frac{\partial \pi(p,w)}{\partial w}$

Note that because profit function is linearly homogeneous, the first derivative of the profit function is homogeneous of degree zero – as we'd expect for demand and supply functions.

The long-run total cost curve



Market demand

Suppose an individual spends her income, I, on two goods, x and y. Her demand for x is denoted as

- Quantity of x demanded = $x(p_x, p_y, I)$

Definition: The market demand function for X is the sum of each individual's demand for x. Formally,

- $X(p_x, p_y, I) = \sum_i^M x_i(p_x, p_y, I_i)$

where subscript i reflects each individual in the market.

- If all individual demand functions are downward-sloping, the market demand function is also downward-sloping.

Elasticities of market demand

- Own-price elasticity of demand
  - Sometimes just "price elasticity of demand"
  - $\epsilon_{Q,p} = \frac{\partial Q_D(p,p',I)}{\partial p}\frac{p}{Q_d}$
  - $\epsilon_{Q,p}$ measures percentage change in the quantity demanded in response to a one-percent change in the good's price.
  - if $\epsilon_{Q,p} \begin{cases} < -1 & \text{demand is elastic;} \\ (-1,0) & \text{demand is inelastic.} \end{cases}$
- Cross-price elasticity of demand
  - $\epsilon_{Q,p'} = \frac{\partial Q_D(p,p',I)}{\partial p'}\frac{p'}{Q_d}$
  - $\epsilon_{Q,p'}$ measures percentage change in the quantity demanded in response to a one-percent change in a related good's price.
- Income elasticity of demand
  - $\epsilon_{Q,I} = \frac{\partial Q_D(p,p',I)}{\partial I}\frac{I}{Q_d}$
  - $\epsilon_{Q,I}$ measures percentage change in the quantity demanded in response to a one-percent change in income.

Perfect competition: The behavior of a single firm

Because price is determined by the market, in perfect competition a firm decides how much to produce. The profit maximization problem of a firm is:

- $\max_{q} \ \pi = pq - C(q)$

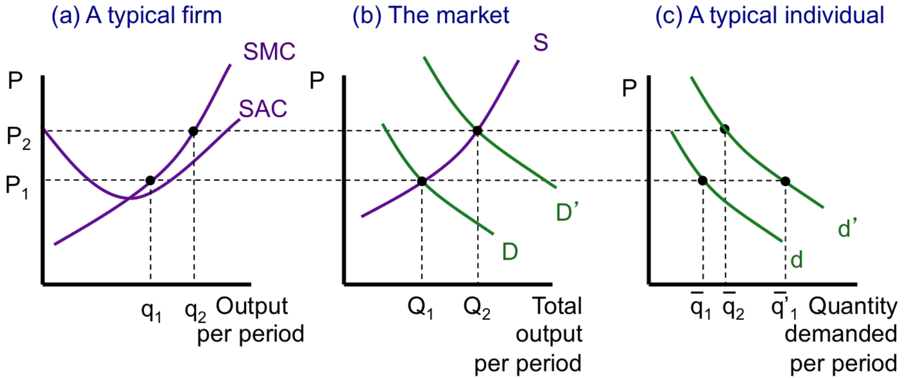
where p is price, q is the quantity the firm produces, and C(q) is total cost.

The FOC of this problem is:

- $\frac{\partial \pi}{\partial q} = p - \underbrace{\frac{\partial C(q)}{\partial q}}_{MC(q)} = 0$
- $p = MC(q)$

Under perfect competition a profit-maximizing firm produces at a level where the cost of producing the last unit of output is equal to the market price.

Short-run equilibrium: shift in an individual demand



Comparative statics of market equilibrium

Denote the market demand and supply functions as

- $D(p, \alpha)$ and $S(p, \beta)$

where α and β are demand and supply shifters respectively. The equilibrium condition is

- $D(p, \alpha) = S(p, \beta)$

Suppose we want to know the impact of a demand shifter on market price, i.e., $\frac{\partial p}{\partial \alpha}$. We start by taking the α derivative of demand and supply:

- $\frac{\partial D}{\partial \alpha} = D_p \frac{\partial p}{\partial \alpha} + D_\alpha$
- $\frac{\partial S}{\partial \alpha} = S_p \frac{\partial p}{\partial \alpha}$

Comparative statics of market equilibrium

At the equilibrium $\frac{\partial D}{\partial \alpha} = \frac{\partial S}{\partial \alpha}$ (why?)

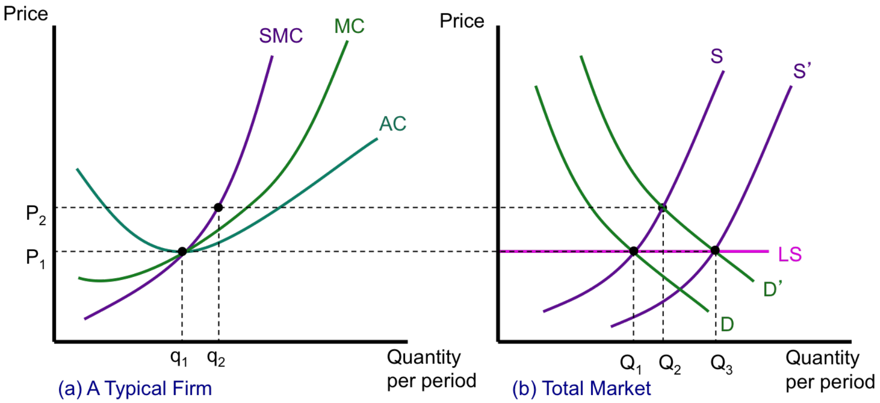
- $D_p \frac{\partial p}{\partial \alpha} + D_\alpha = S_p \frac{\partial p}{\partial \alpha}$
- $\frac{\partial p}{\partial \alpha} = \frac{D_\alpha}{S_p - D_p} \gtrless 0$

we can rewrite the comparative static result in elasticity form as

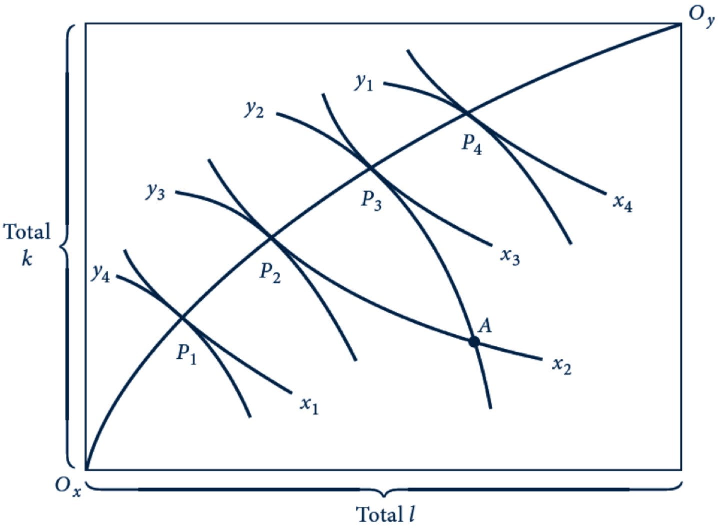
- $\epsilon_{p,\alpha} = \frac{\epsilon_{D,\alpha}}{\epsilon_{S,p} - \epsilon_{D,p}}$

That is, the proportional response of the market price to a one-percent increase in a demand shifter (e.g. income) is equal to the proportional change in demand weighted by the summation of the price elasticities of demand and supply.

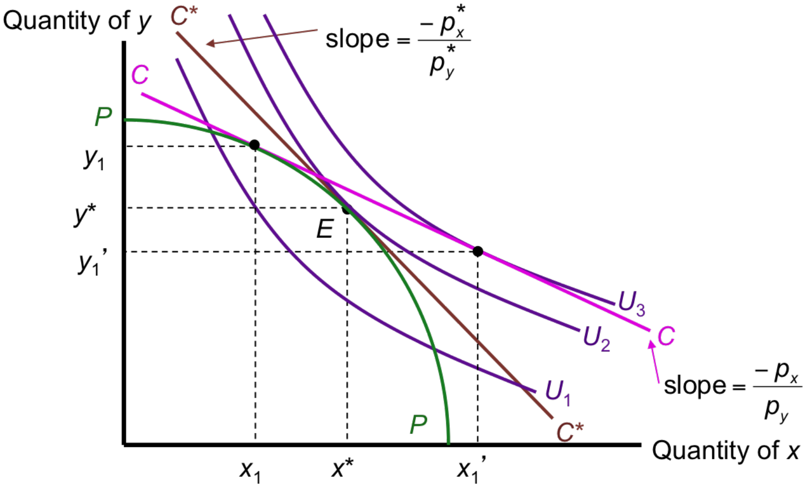
Long run equilibrium: constant-cost industry



Edgeworth box — Efficient allocations



Determination of equilibrium prices



Walrasian equilibrium

- $\underset{x^i}{\text{Max}} \quad u^i(x^i)$ such that $px^i = p\bar{x}^i$

The solution to this problem is the consumer demand function:

- $x^i(p, p\bar{x}^i)$

Definition: (p^*, x^*) is a **Walrasian equilibirum** if

- $\sum_i^m x^i(p^*, p^*\bar{x}^i) \leqslant \sum_i^m \bar{x}^i$

That is, p^* is a Walrasian equilibrium if there is no good for which there is positive excess demand.

Q: Will there always exist a price vector where all markets clear?

Walras' law

Walras' law: For any price vector p, we have $pz(p) \equiv 0$; i.e., the value of excess demand is identically zero.

Proof: By the definition of excess demand

- $pz(p) = p\left[\sum_i^m (x^i(p, p\bar{x}^i) - \bar{x}^i)\right]$
- $pz(p) = \sum_i^m (px^i(p, p\bar{x}^i) - p\bar{x}^i) = 0$

because $x^i(p, p\bar{x}^i)$ must satisfy budget constraint $px^i = p\bar{x}^i \;\; \forall i$ ■

Walras' law simply says that if each individual satisfies her budget constraint the value of the individual excess demand is zero.

Then, the value of the sum of the excess demands must also be zero.

The first theorem of welfare economics

The first theorem of welfare economics says that if (x, p) is a Walrasian equilibrium, then x is Pareto efficient.

Proof: Assume the opposite is true. Let $x^{'}$ be a feasible allocation that all agents prefer to x. Then, from the 2^{nd} property of a Walrasian equilibrium we have:

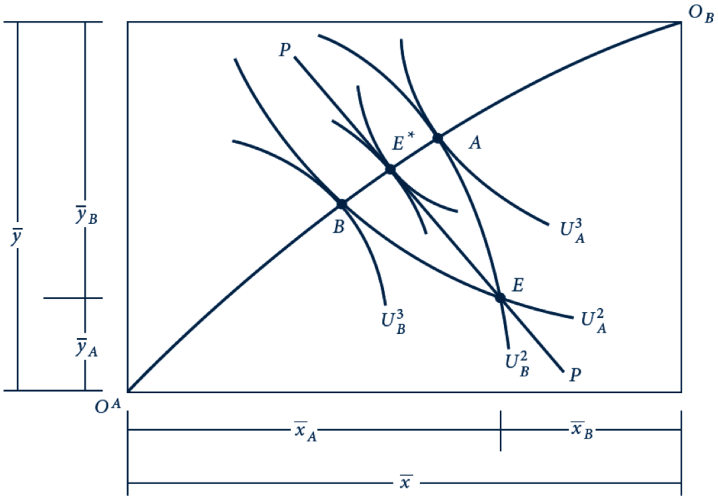
- $px^{'i} > p\bar{x}^i$ for $i = 1, ..., m$, and so
- $p\sum_i^m x^{'i} > \sum_i^m p\bar{x}^i$

However, from the 1^{st} property:

- $\sum_i^m \bar{x}^i = \sum_i^m x^{'i}$, or
- $p\sum_i^m \bar{x}^i = p\sum_i^m x^{'i}$,

which is a contradiction. ■

The first theorem of welfare economics, graphically

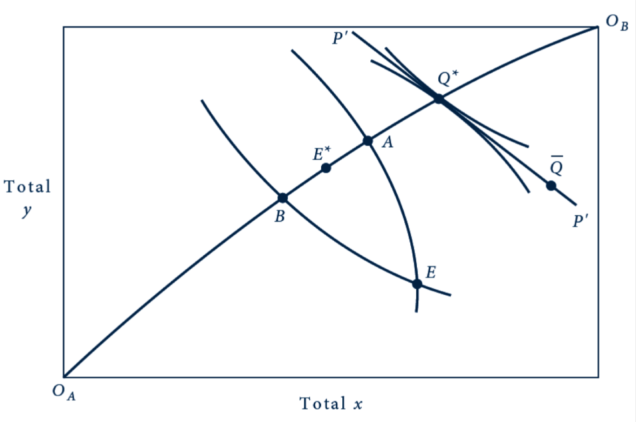


The second theorem of welfare economics

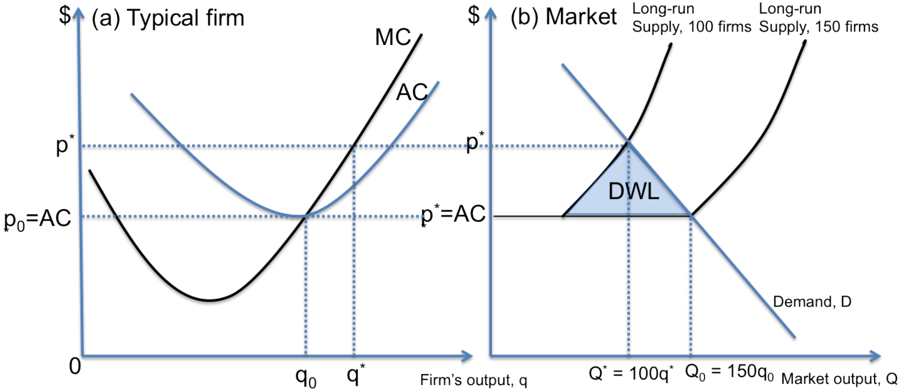
The second theorem of welfare economics:
Suppose that x^* is a Pareto efficient allocation. Suppose further that a competitive equilibrium exists given the initial endowments $\bar{x}^i = x^{*i}$; denote it as $(p^{'}, x^{'})$. Then, in fact $(p^{'}, x^*)$ is a Walrasian equilibrium.

- In words, if a competitive equilibrium exists from a Pareto efficient allocation, then that Pareto efficient allocation is itself a competitive equilibrium.
- Note that in the definition of the theory the initial endowment is allowed to adjust.
- That is, a Pareto optimal allocation is also a Walrasian equilibrium so long as the initial endowments are adjusted accordingly.

The second theorem of welfare economics, graphically



LR equilibrium with an entry restriction



- In the absence of restrictions on entry, the competitive equilibrium is (Q_0, p_0), and there are 150 firms in the market.
- If the government restricts number of firms to 100, the equilibrium is (Q^*, p^*)

Monopoly profit maximization

Suppose $C(Q)$ is the total cost function of monopolist. Then the profit maximization problem of a monopolist can be written as:

- $\max\limits_{Q} \ \pi = p(Q)Q - C(Q)$

The FOC of this problem is:

- $\frac{\partial \pi}{\partial Q} = \underbrace{p(Q) + Qp'(Q)}_{MR} - \underbrace{C'(Q)}_{MC} = 0$
- $MR = MC$

Thus, a profit-maximizing monopolist determines its optimal quantity at a level where marginal revenue is equal to marginal cost.

I won't derive this, but the second-order condition of this problem is that the marginal revenue curve has to intersect the marginal cost curve from above.

Price-cost margin and the Lerner index

Question: How much can a monopoly raise price above MC?

The answer depends on the price elasticity of demand. To observe this rewrite the marginal revenue of a monopolist as:

- $MR = p(Q) + p'(Q)Q$
- $MR = p[1 + \underbrace{p'(Q)\frac{Q}{p}}_{\frac{1}{\epsilon}}]$
- $MR = p(1 + \frac{1}{\epsilon})$
- $\begin{cases} MR \geqslant 0, & \text{if } \epsilon \leqslant -1; \\ MR < 0, & \text{if } -1 < \epsilon < 0. \end{cases}$

A monopolist always produces in the elastic portion of the demand curve.

Price-cost margin and Lerner index

At the optimum,

- $MR = p(1 + \frac{1}{\epsilon}) = MC$
- $p - MC = -\frac{p}{\epsilon}$
- $\frac{p-MC}{p} = -\frac{1}{\epsilon}$

$\frac{p-MC}{p}$ is the **price-cost margin**, also known as the **Lerner Index.**

The price-cost margin of a monopolist depends solely on the price elasticity of demand. The less elastic demand is, the higher is the price-cost margin will be .

At the extreme, if , $\epsilon = -\infty$, the price-cost margin is zero (i.e., $p = MC$).

Conditions for price discrimination

There are three conditions needed for a successful price discrimination:

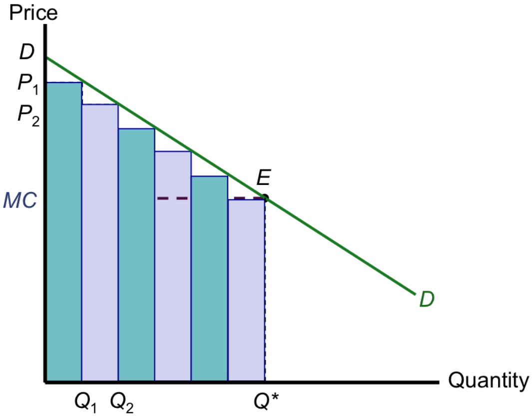
1. A firm must have market power – the ability to set the price above marginal cost.
2. A firm must be able to identify whom to charge a higher price – it must know, or infer, consumers' willingness to pay, and WTP must vary across consumers or units
3. A firm must be able to prevent, or limit, resales by customers who pay the lower price to those who pay the higher price.

First-degree price discrimination

With perfect price discrimination consumers are left with no surplus. There are three alternative but equivalent ways of perfect price discrimination:

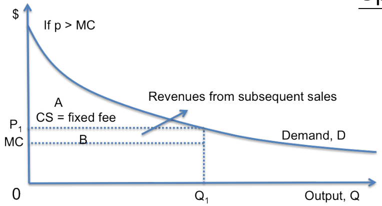
1. Each consumer buys one unit, but each of them pays a different amount.
2. Each consumer buys more than one unit, but pays different amount for each unit.
3. Two-part tariff: Each consumer pays a lump sum fee for the right to purchase plus a uniform per-unit charge for each unit consumed.

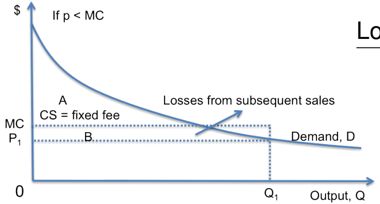
Perfect price discrimination



- The monopoly charges a different price to each buyer. Q_1 units are sold at P_1, and $Q_2 - Q_1$ units at P_2.

Optimal two-part tariffs: finding the optimal price, p^*.





Upper Figure:

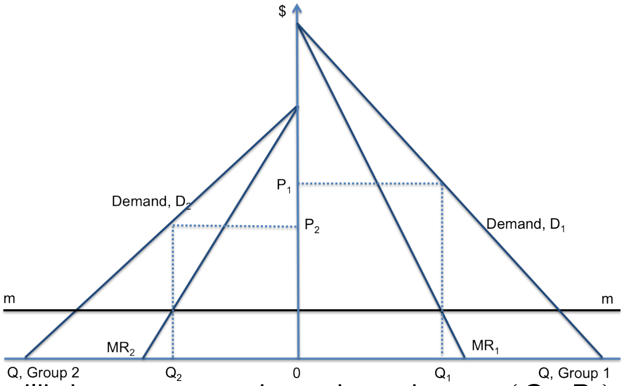
- The monopolist's profit is $A + B$.
  - A = CS captured via fixed fee.
  - B = profits from subsequent sales.
- Note that lowering the price slightly would increase profits.

Lower Figure:

- The monopolist's profit is $A - B$.
  - A = CS captured via fixed fee.
  - B = losses from subsequent sales.
- Note that increasing the price slightly would increase profits.

Thus $p^* = MC$.

Graphical analysis: Third degree price discrimination



- The equilibrium outcomes in each market are (Q_1, P_1) and (Q_2, P_2)
- At each quantity, group 1 is willing to pay a higher amount than group 2. Since group 1's demand schedule is more inelastic, $P_1 > P_2$.

Second degree price discrimination: an example

Suppose there are only two consumers and their valuations are:

Products	Valuations of A	Valuations of B
$Prod_1$	9,000	10,000
$Prod_2$	3,000	2,000
Bundle	12,000	12,000

If the monopolist sells the products separately, it maximizes revenue at

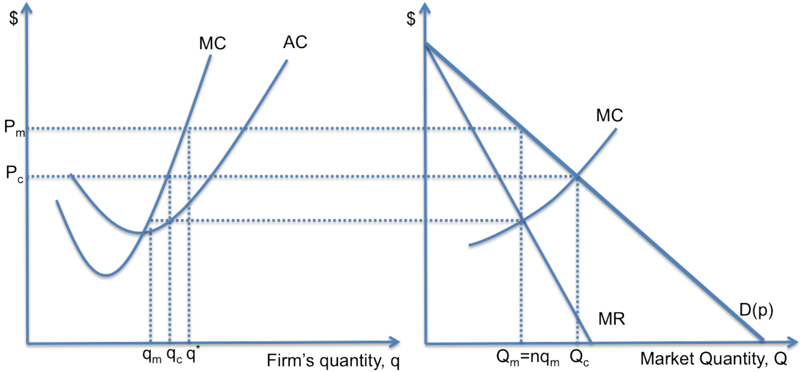
- $Prod_1$ = 9,000, so that $TR_1 = 18,000$
- $Prod_2$ = 2,000, so that $TR_2 = 4,000$
- $TR = 18,000 + 4,000 = \underline{22,000}$.

If it bundles the products, it maximizes revenue at

- Bundle = 12,000, so that $TR = \underline{24,000}$

The monopolist is better of with bundling. Note that it could have charged $11,500$ for the bundle (less than the sum of consumers, valuations of each product), and still be better off $(23,000 > 22,000)$.

Cartels Look Like Monopolies, With an Incentive to Defect



- a cartel can increase the market price and total industry profits by producing at Q_m, P_m.
- each firm has an incentive to cheat, and produce q^*.

Game theory: a formal definition

Game theory is a formal way of analyzing *interactions* among a group of rational agents who behave *strategically*.

Specifically, **game theory** is formalizes each of the following items:

- *group:* In any game there is more than one decision-maker; each decision maker is referred as a *player*.
- *interaction:* What any one individual player does directly affects at least one other player in the group.
- *strategic:* An individual player accounts for this interdependence in deciding what action to take.
- *rational:* While accounting for this interdependence, each player chooses her best action.

The formal structure of games

Every game is played by a set of rules which have to specify four things:

1. **who** is playing – the group of players that strategically interact.
2. **what** they are playing with – the alternative actions or choices, and hence the strategies that each player has available.
3. **when** each player gets to play (in what order)
4. **how much** each player gains (or loses) from choices made in the game.

Types of games

- Simultaneous vs. Sequential
  - whether all players choose their actions at the same time, or whether some players can observe the actions of others before they make their decisions.
- Pure vs. Mixed strategy games
  - whether players can choose only a single action, or if they can randomize across two or more actions with positive probability on each
- Single-period vs. Repeated games
  - whether players play the game just for one period or two or more periods.
- Complete vs. Incomplete information (Bayesian games)
  - whether some players have private information that is not known by others.

Common knowledge

Throughout our analysis we will maintain the important assumption that *the rules of a game are common knowledge*.

Common knowledge means every player knows something, and the fact that they know it is commonly known.

Common knowledge simply means that if any two players in a game were asked a question about who, what, when and how much, they would give the same answer, and also that each player knows that other players will give the same answer.

Common knowledge does not mean that players are equally influential. Common knowledge is also domain-specific: we assume that players are equally well-informed about the rules of the game, but there could be other factors that are part of a game that are not common knowledge. For example, when you buy a used car both you and the seller know that the seller is better-informed about the condition of the car. But the condition of the car itself is not common knowledge.

Normal-form games

Again, consider the prisoner's dilemma game:

		Suspect 2	
		Fink	Silent
Suspect 1	Fink	$u_1 = 1, u_2 = 1$	$u_1 = 3, u_2 = 0$
	Silent	$u_1 = 0, u_2 = 3$	$u_1 = 2, u_2 = 2$

- In this version, there are two players (Suspects 1 and 2)
- Each player has two actions, Fink (F) or Silent (S)
- There are 4 strategy combinations: (F, F), (F, S), (S, F), and (S, S)
- Each strategy combination has a payoff: (1, 1), (0,3), (3, 0), (2, 2)

Notation

We use the following notation for the three components of strategic form games.

- Players are $i = 1, 2, ..., N$
- S_i denotes player *i*'s set of potential strategies.
- s_i, s_i^* or $s_i^{'}$ denotes a specific player *i*'s strategy
- s_{-i} denotes the strategy choices of all other players besides player *i*
- u_i denotes player *i*'s payoff
- $(s_1^*, s_2^*, ..., s_N^*)$ denotes a combination of strategies, one strategy for each player
- $u_i(s_1^*, s_2^*, ..., s_N^*)$ denotes player *i*'s payoff when $(s_1^*, s_2^*, ..., s_N^*)$ is the set of strategies played

Nash equilibrium

Note that for an equilibrium to hold there needs to be a condition to ensure that player i is correct in his conjecture that the other players are going to play s_{-i}. Similarly, the condition should ensure that the other players correct in their conjectures. This takes us to the concept of *Nash equilibrium*.

Definition: The strategy vector $s^* = s_1^*, ..., s_N^*$ is a Nash equilibrium if

- $u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i, s_{-i}^*)$ for all s_i and all i.

That is,

- At the equilibrium each player must be playing a best response against the conjectured strategies of her opponents.
- The conjecture must be correct.
  - No one has an incentive to change their strategy s_i^*. Thus, s^* is stable.

Dominant strategies

Definition: Strategy $s_i^{'}$ **strictly dominates** all other strategies of player i if the payoff to $s_i^{'}$ is strictly greater than the payoff to any other strategy, regardless of which strategy is chosen. Formally:

- $u_i(s_i^{'}, s_{-i}) > u_i(s_i, s_{-i}) \forall s_{-i}$

where s_{-i} is the strategy vector of players other than i.

Consider a game with two players (1 and 2) and two strategies (a and b). If, for example s_1^b is a dominant strategy for player 1, then it must be the case that:

1. $u_1(s_1^b, s_2^a) > u_1(s_1^a, s_2^a)$
2. $u_1(s_1^b, s_2^b) > u_1(s_1^a, s_2^b)$

Definition: A combination of strategies is a **dominant strategy solution** if each player's strategy is a dominant strategy.

Tragedy of commons: formal description

The formal summary of the model is:

- Players: N={herder i, herder j} $i, j = 1, 2$
- Actions: choice of number of sheep - $q_i \in S_i = [0, \infty)$
- Timing: simultaneous
- Payoffs: $\pi_i(q_i, q_j) = q_i v_i(q_i, q_j)$,
  - where $v_i(q_i, q_j) = 120 - (q_i + q_j)$ is value of grazing each sheep

We can solve for the Nash equilibrium using best-response functions:

- The **best-response function**, $b_i(q_j)$ describes herder i's optimal reaction to the output choice of herder j:
  - $b_i(q_j) = \max_{q_i} \ \pi_i(q_i, q_j)$
- Nash equilibrium, (q_i^*, q_j^*) is the solution to:
  - $q_i = b_i(q_j)$ and $q_j = b_j(q_i)$

Best-response functions

Herder 1 faces the following optimization problem:

- $\underset{q_1}{\text{Max}} \ \pi_1(q_1, q_2) = q_1(120 - q_1 - q_2)$

Assuming π_1 strictly concave in q_1 and twice differentiable, the FOC is

- $\pi_1'(q_1, q_2) = 120 - 2q_1 - q_2 = 0$

The best-response function of herder 1 is:

- $q_1 = b_1(q_2) = 60 - \frac{q_2}{2}$

By symmetry, the best-response function of herder 2 is:

- $q_2 = b_2(q_1) = 60 - \frac{q_1}{2}$

Best response functions

1. $q_1 = b_1(q_2) = 60 - \frac{q_2}{2}$
2. $q_2 = b_2(q_1) = 60 - \frac{q_1}{2}$

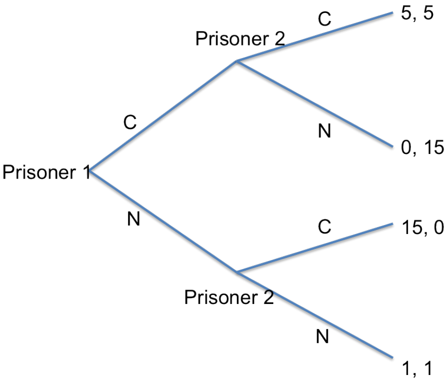
The Nash equilibrium (q_1^*, q_2^*) is the pair of strategies that satisfies 1 and 2:

- $q_1 = 60 - \frac{1}{2}\left(60 - \frac{q_1}{2}\right)$
- $3q_1/4 = 30$
- $q_1^* = 40$, similarly $q_2^* = 40$

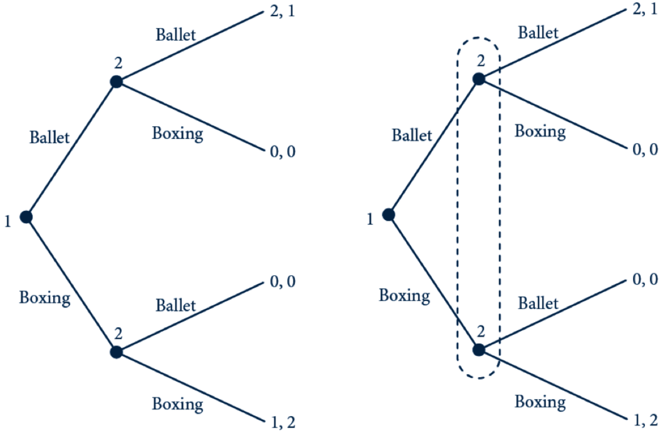
Compare the total herd Nash equilibrium solution (q_1^*, q_2^*) to the social optimum (q_1^s, q_2^s)

The extensive form

Extensive form: A pictorial representation of the game. The main pictorial form is called the *game tree*, which is made up of a root and branches arranged in order.



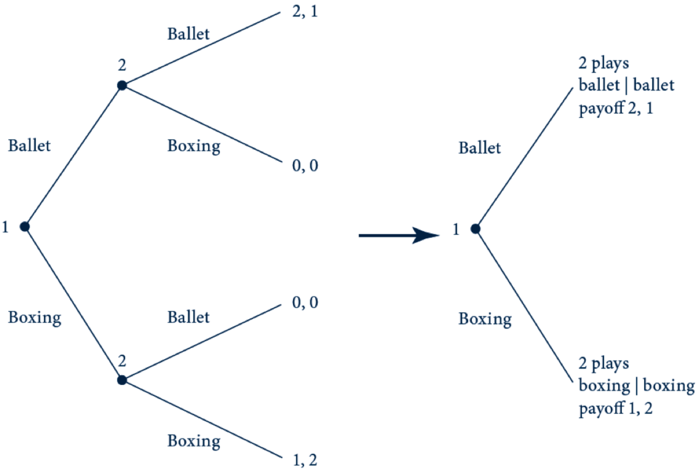
Extensive form for the battle of the sexes



(a) Sequential version

(b) Simultaneous version

Subgame perfect Nash equilibrium of the battle of the sexes

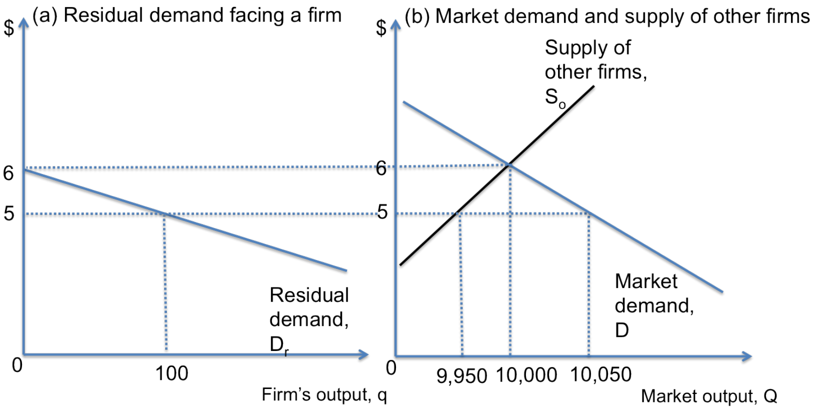


The residual demand curve

The **residual demand curve** is the demand curve facing a particular firm. A firm sells to consumers whose demand is not met by other firms in the market. Formally,

1. $D_r(p) = D(p) - S_o(p)$
    - $D_r(p)$ is the residual demand
    - $D_r(p)$ is the market demand
    - $S_o(p)$ is the supply of all other firms

Derivation of residual demand curve



Classic models of oligopoly

The three best known oligopoly models are:

- Cournot model
- Bertrand model
- Stackelberg model

Models of oligopoly mainly differ by:

- Type of actions: price setting vs. quantity setting firms.
- Order of actions: sequential vs. simultaneous moves
- Type of goods: homogeneous vs. differentiated
- Length of time: single-period vs. multi-period games

The Cournot model of duopoly

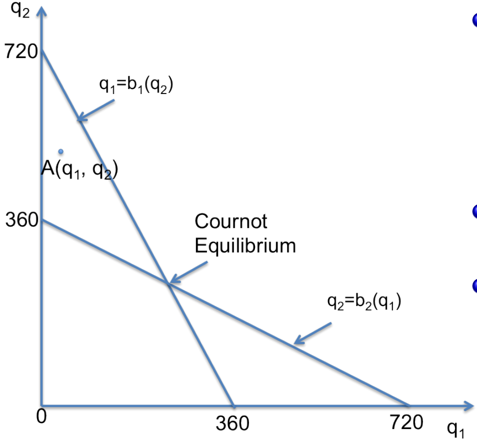
The formal summary of the model is:

- Players: N={firm 1, firm 2}
- Actions: Quantity choice – $q_i \in S_i = [0, \infty)$
- Payoffs: Firm profits – $\pi_i(q_i, q_j)$

We can solve for the Cournot-Nash equilibrium using best-response functions:

- Reminder: The **best response function**, $b_i(q_j)$, describes firm i's optimal reaction to the output choice of firm j. It is given by:
  - $b_i(q_j) = \max_{q_i} \ \pi_i(q_i, q_j)$
- A Cournot-Nash equilibrium is a solution to the system of equations:
  - $q_1^c = b_1(q_2^c)$
  - $q_2^c = b_2(q_1^c)$

Graph of best response functions



- Can point $A(q_1, q_2)$ be an equilibrium?
- From the definition of a best-response function we know that given other firm's production level each firm has an incentive to be on its best-response curve.
- Point A cannot be an equilibrium.
- Equilibrium requires that both firms are simultaneously on their best-response functions. This occurs at the intersection of the curves.

Cournot vs. perfect competition and monopoly

Now let's compare each of the outcomes

- $(Q^c, P^c) = (480, 0.52)$
- $(Q^*, P^*) = (720, 0.28)$
- $(Q^m, P^m) = (360, 0.64)$

Total output of Cournot duopoly is greater than the output of monopolized industry but less than that of a perfectly-competitive industry.

- $Q^* > Q^c > Q^m$

The Cournot-Nash equilibrium price is less than the monopoly price, but greater than the perfectly-competitive price

- $P^* < P^c < P^m$

Cournot-Nash equilibrium with few and many firms

			Firm		Industry	
	Number of Firms	Price	Output	Profit ($)	Output	Profit ($)
Monopoly	1	64	360	129.60	360	129.60
	2	52	240	57.60	480	115.20
	3	46	180	32.40	540	97.20
	4	42.4	144	20.74	576	82.94
	5	40	120	14.40	600	72.00
	6	38.3	102.9	10.58	617.1	63.48
	7	37	90	8.10	630	56.70
	8	36	80	6.40	640	51.20
	9	35.2	72	5.18	648	46.66
	10	34.5	65.5	4.28	654.5	42.84
	15	32.5	48	2.30	675	32.26
	20	31.4	34.3	1.18	685.7	23.51
	50	29.4	14.1	0.20	705.9	9.97
	100	28.7	7.1	0.05	712.9	5.08
	500	28.1	1.4	0.002	718.6	1.03
	1000	28.1	0.7	0.001	719.3	0.52
Competition	∞	28	~ 0	0.00	720	0.00

The Stackelberg (leader-follower) model

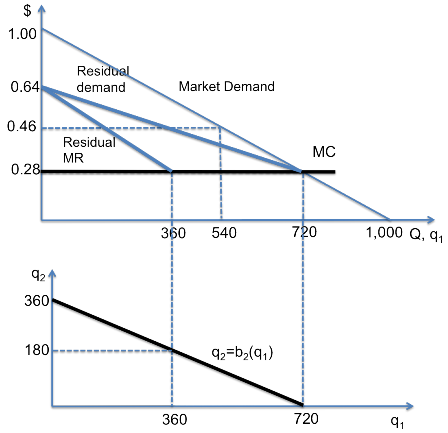
The main assumptions of the Stackelberg model are:

1. There are two identical firms (firm 1 and firm 2)
2. Products are homogeneous
3. Firms choose output
4. Firm 1 chooses its output first, then firm 2 chooses its output knowing the output level of firm 1 (a sequential-move game)
5. Firms compete with each other just once (a single-period game)
6. There is no entry by other firms

The formal summary of the model is:

- Players: N={firm 1, firm 2}
- Actions: Quantity choice – $q_i \in S_i = [0, \infty)$
- Timing: Firm 1 moves first, then firm 2
- Payoffs: Firm profits – $\pi_i(q_1, q_2)$

The Stackelberg equilibrium



- Leader (firm 1) calculates its residual demand provided that follower (firm 2) chooses its output using its Cournot best response, $q_2 = b_2(q_1)$.
- The leader's best response is to behave like a monopoly on its residual demand.
- Thus, at $MC = RMR$, the outcome is $q_1 = 360$, $q_2 = 180$.

Cournot vs. Stackelberg equilbrium

The Stackelberg equilibrium

- $(q_1^s, q_2^s) = (360, 180)$
- $Q^s = 540$
- $P^s = 0.46$
- $\pi_1^s = 64.8$
- $\pi_2^s = 32.4$
- Total Profits: $\pi^s = 97.2$

The Cournot equilibrium

- $(q_1^c, q_2^c) = (240, 240)$
- $Q^c = 480$
- $P^c = 0.52$
- $\pi_1^c = 57.6$
- $\pi_2^c = 57.6$
- Total Profits: $\pi^c = 115.2$

- $Q^s > Q^c$, $P^s < P^c$, $\pi_1^s > \pi_1^c$, $\pi_2^s < \pi_2^c$, , $\pi^s < \pi^c$
- Bottom line: The advantage of moving first and knowing how its rival will behave allows the leader to profit at the follower's expense.

The Bertrand model of oligopoly

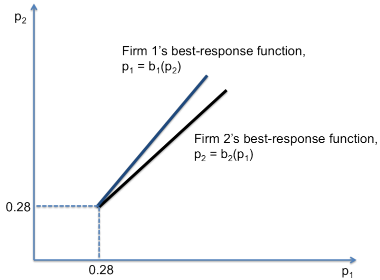
Main assumptions:

1. Products are homogeneous
2. Firms choose price (“firms compete on price”)
3. Firms compete with each other just once and make their pricing decisions simultaneously (a single-period simultaneous-move game)
4. There are two identical firms in the industry and there is no entry by other producers

The formal summary of the model is:

- Players: N={firm 1, firm 2}
- Actions: Price choice – $p_i \in S_i = (0, \infty)$
- Timing: Firms choose their actions simultaneously
- Payoffs: Firm profits – $\pi_i(p_1, p_2)$

Bertrand best-response functions



- MC = 0.28
- Best response functions intersect at MC

- Given any price p_1, Firm 2 will set its price p_2 slightly lower than p_1.
- The same is true for Firm 1. That is, each firm's best strategy is to undercut its rival's price.
- But, neither firm is willing to undercut below MC.
- The only possible Bertrand-Nash equilibrium is $p = MC = 0.28$
- This is efficient outcome, i.e. perfect competition.

Comparison of oligopoly outcomes

Now let's compare all the various oligopoly model outcomes, using the same example we have been using.

- Cournot Duopoly: Each firm makes $57.60
- Stackelberg Model: The leader makes $64.80 while the follower makes $32.40
- Bertrand Model: Neither firm makes positive profits.
- Cartel: If firms can sustain a cartel they make the highest collective profits, totaling = $129.60

Definition: The **profit possibility frontier** shows the highest profit one firm could earn holding the profit of other firm constant.

- summarizes any combination of firms profits in which the sum of profits is maximum (129.60), e.g., firms can split (64.80, 64.80), or one firm can get 0, other gets the rest.

Reasons for the Bertrand paradox

The Bertrand outcome depends on the strong assumptions made by the model. Namely:

1. Any firm can produce as much as it wants at constant marginal cost
2. The products are homogeneous
3. The market lasts for only one period

When at least one of these assumptions is relaxed, the Bertrand outcome is *not* $p = MC$. In reality:

1. firms face capacity constraints
2. products are differentiated
3. markets lasts for many periods (multi-period games)

Top dog vs. puppy dog strategies

Whether competitor's decisions are **strategic substitutes** or **complements** is determined by whether more "aggressive" (increasing quantity, decreasing price) play by one firm in a market lowers or raises competing firms' marginal profitabilities in that market.

- If decisions are strategic substitutes, the best-response functions are downward-sloping
  - In this case, a firm can induce a rival to respond less aggressively by playing more aggressively (a top dog strategy)
- If decisions are strategic complements, the best-response functions are upward-sloping
  - In this case, a firm can induce a rival to respond less aggressively by playing less aggressively (a puppy dog strategy)