Applied Microeconomics: Firm and Household

Lecture 6: Comparative Statics and the Slutsky Equation

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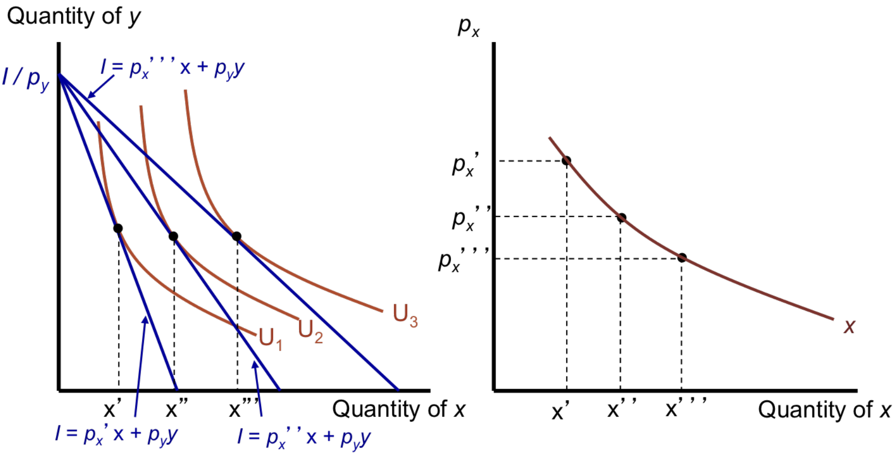
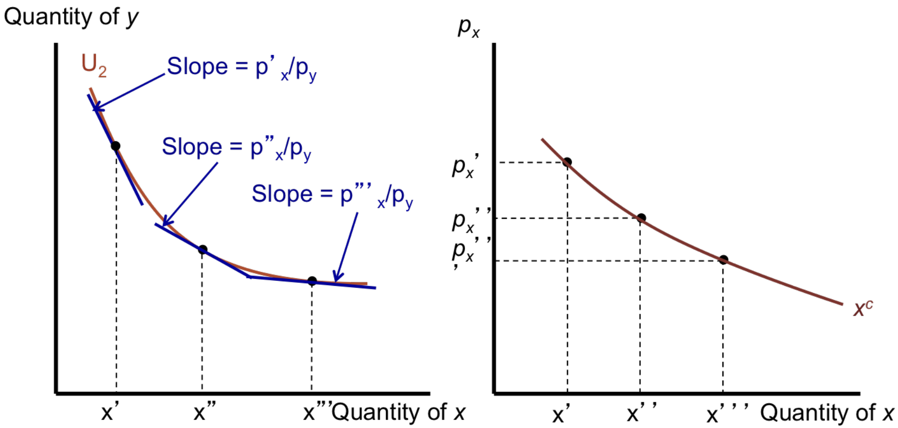
Outline

- Relationship between compensated and uncompensated demand curves
- Comparative statics of the utility maximization model
    - Mathematical analysis
    - Graphical analysis
- Comparative statics of the expenditure minimization model
    - Derivation of the Slutsky Equation (Income and substitution effects)

Construction of an uncompensated demand curve

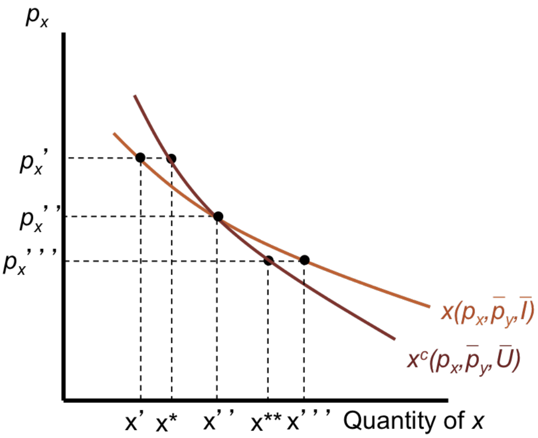


Construction of a compensated demand curve



Here we hold **real income** constant
(Why is this the same as holding utility constant?)

Relationship between compensated and uncompensated demand curves



($\bar{I}$ is **nominal income**)

Review: The utility maximization model

Consider the following utility maximization problem,

- $\underset{x_1, x_2, \lambda}{\text{Max}} \quad L = u(x_1, x_2) + \lambda(M - p_1 x_1 - p_2 x_2)$

The FOCs are,

1. $\frac{\partial L}{\partial x_1} = L_1 = u_1 - \lambda p_1 = 0$
2. $\frac{\partial L}{\partial x_2} = L_2 = u_2 - \lambda p_2 = 0$
3. $\frac{\partial L}{\partial \lambda} = L_\lambda = M - p_1 x_1 - p_2 x_2 = 0$

Assuming the SOSC holds, the optimum functions in implicit form are

- $x_1 = x_1^*(p_1, p_2, M)$
- $x_2 = x_2^*(p_1, p_2, M)$
- $\lambda = \lambda^*(p_1, p_2, M)$

Comparative statics

To motivate our next steps of the analysis of demand theory let's remember the quote by Paul Samuelson

"*Meaningful theorems in economics consist not in laying out equilibrium conditions which are rarely observable and therefore empirically sterile, but in deriving predictions that the direction of change of some decision variable in response to a change in some observable parameter must be in some particular direction.*"

Once we obtain the optimal solutions we can analyze how a utility-maximizing individual's decisions change under changing market conditions (comparative statics).

Comparative statics

In particular, we seek to establish a relationship between the rates of change of consumption with respect to price changes when money income (nominal income) is held constant and the corresponding rate of change when real income, or utility, held constant.

- i.e., between $\frac{\partial x_i^*}{\partial p_j}$ and $\frac{\partial x_i^h}{\partial p_j}|_{u=u^0}$

Next we will derive this relationship, which is known as the **Slutsky equation.**

Comparative statics

As a first step, we seek to derive the following:

1. How does a utility-maximizing individual adjust her demand for good *i* in response to a change in the price of that good? (own-price effect)
   - $\frac{\partial x_i^*}{\partial p_i}$=?
2. How does a utility-maximizing individual adjust her demand for good *i* in response to a change in the price of good *j*? (cross-price effect)
   - $\frac{\partial x_i^*}{\partial p_j}$=?
3. How does a utility-maximizing individual adjust her demand for good *i* in response to a change in her income? (income effect)
   - $\frac{\partial x_i^*}{\partial M}$=?

Comparative statics

To perform comparative statics first we rewrite the FOCs as:

1. $u_1(x_1^*(p_1, p_2, M), x_2^*(p_1, p_2, M)) - \lambda^*(p_1, p_2, M)p_1 \equiv 0$
2. $u_2(x_1^*(p_1, p_2, M), x_2^*(p_1, p_2, M)) - \lambda^*(p_1, p_2, M)p_2 \equiv 0$
3. $M - p_1 x_1^*(p_1, p_2, M) - p_2 x_2^*(p_1, p_2, M) \equiv 0$

Note that the FOCs are now expressed in terms of identities. Because the relation holds for all values of p and M,

- x^* is the optimal decision rule. Therefore, as p or M changes, the utility-maximizing individual adjusts by changing her x^* such that the $FOC = 0$ for any values of p and M.

Comparative statics: income effects

To derive $\frac{\partial x_i^*}{\partial M}$, $i = 1, 2$ we differentiate the identities with respect to M.

- $u_{11}\frac{\partial x_1^*}{\partial M} + u_{12}\frac{\partial x_2^*}{\partial M} - p_1\frac{\partial \lambda^*}{\partial M} \equiv 0$
- $u_{21}\frac{\partial x_1^*}{\partial M} + u_{22}\frac{\partial x_2^*}{\partial M} - p_2\frac{\partial \lambda^*}{\partial M} \equiv 0$
- $1 - p_1\frac{\partial x_1^*}{\partial M} - p_2\frac{\partial x_2^*}{\partial M} \equiv 0$

We can write this system of three equations in matrix notation as

- $\begin{pmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial M} \\ \frac{\partial x_2^*}{\partial M} \\ \frac{\partial \lambda^*}{\partial M} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Comparative statics: income effects

Using Cramer's rule,

- $$\frac{\partial x_1^*}{\partial M} = \frac{\begin{vmatrix} 0 & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ -1 & -p_2 & 0 \end{vmatrix}}{|H|} = -\frac{H_{31}}{|H|} = -\frac{(-p_2 u_{12} + p_1 u_{22})}{|H|} \gtrless 0$$
- $$\frac{\partial x_2^*}{\partial M} = \frac{\begin{vmatrix} u_{11} & 0 & -p_1 \\ u_{21} & 0 & -p_2 \\ -p_1 & -1 & 0 \end{vmatrix}}{|H|} = -\frac{H_{32}}{|H|} = -\frac{(p_2 u_{11} - p_1 u_{21})}{|H|} \gtrless 0$$
- $$\frac{\partial \lambda^*}{\partial M} = \frac{\begin{vmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & 0 \\ -p_1 & -p_2 & -1 \end{vmatrix}}{|H|} = -\frac{H_{33}}{|H|} = -\frac{(u_{11} u_{22} - u_{12}^2)}{|H|} \gtrless 0$$

Comparative statics: income effects

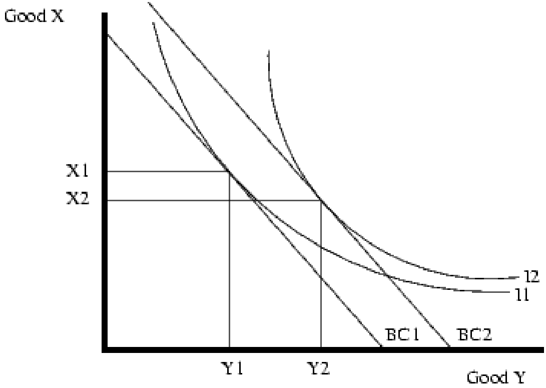
To summarize,

1. $\frac{\partial x_i^*}{\partial M} \gtrless 0$
   - The indeterminate sign implies that the theory does not rule out the possibility of inferior goods.
   - x_1 or x_2 can be an inferior good, but both of them cannot be inferior goods. That is, at least one of them has to be a normal good. Why?
2. $\frac{\partial \lambda^*}{\partial M} \gtrless 0$
   - We know that marginal utility of income, λ^*, must be positive (due to nonsatiation). The comparative statics result state that the marginal utility of income can be *either* increasing *or* decreasing. Matt Rabin says that DMU of income is a results-driven (empirical) assumption — introduced to resolve St. Petersburg paradox).

Graphical analysis: income effects

Definition: A good x_i for which $\frac{\partial x_i}{\partial M} < 0$ over some range of income changes is an **inferior good** in that range. If $\frac{\partial x_i}{\partial M} > 0$ over some range of income variation the good is **normal good** for that range.

Good X is inferior, Good Y is normal



Comparative statics: price effects

Next, we seek to derive $\frac{\partial x_i^*}{\partial p_1}$, $i = 1, 2$. By differentiating the identities with respect to p_1 we obtain:

- $u_{11}\frac{\partial x_1^*}{\partial p_1} + u_{12}\frac{\partial x_2^*}{\partial p_1} - p_1\frac{\partial \lambda^*}{\partial p_1} - \lambda^* \equiv 0$
- $u_{21}\frac{\partial x_1^*}{\partial p_1} + u_{22}\frac{\partial x_2^*}{\partial p_1} - p_2\frac{\partial \lambda^*}{\partial p_1} \equiv 0$
- $-p_1\frac{\partial x_1^*}{\partial p_1} - x_1^* - p_2\frac{\partial x_2^*}{\partial p_1} \equiv 0$

We can write this system of three equations in matrix notation as

- $\begin{pmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^*}{\partial p_1} \\ \frac{\partial x_2^*}{\partial p_1} \\ \frac{\partial \lambda^*}{\partial p_1} \end{pmatrix} \equiv \begin{pmatrix} \lambda^* \\ 0 \\ x_1^* \end{pmatrix}$

Comparative statics: price effects

Using Cramer's rule,

- $\frac{\partial x_1^*}{\partial p_1} = \frac{\begin{vmatrix} \lambda^* & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ x_1^* & -p_2 & 0 \end{vmatrix}}{|H|} = \frac{\lambda^* H_{11}}{|H|} + \frac{x_1^* H_{31}}{|H|} \gtrless 0$
- $\frac{\partial x_2^*}{\partial p_1} = \frac{\begin{vmatrix} u_{11} & \lambda^* & -p_1 \\ u_{21} & 0 & -p_2 \\ -p_1 & x_1^* & 0 \end{vmatrix}}{|H|} = \frac{\lambda^* H_{12}}{|H|} + \frac{x_1^* H_{32}}{|H|} \gtrless 0$
- $\frac{\partial \lambda^*}{\partial p_1} = \frac{\begin{vmatrix} u_{11} & u_{12} & \lambda_1^* \\ u_{21} & u_{22} & 0 \\ -p_1 & -p_2 & x_1^* \end{vmatrix}}{|H|} = \frac{\lambda^* H_{13}}{|H|} + \frac{x_1^* H_{33}}{|H|} \gtrless 0$

Comparative statics: price effects

Interpretations:

- Among all the above cofactors only $H_{11} = -p_2^2 < 0$ has a determinate sign. The rest of the cofactors have indeterminate signs. That is, the own and cross price effects of demand, as well as the effect of price changes on the marginal utility of income, are all indeterminate.
- The indeterminate sign of the own-price effect implies that the theory allows the possibility of Giffen good (i.e., upward sloping demand).
- The indeterminate sign of the cross price effect implies that the goods can be either gross substitutes or gross complements.

Comparative Statics and Slutsky Equation

Interpretations:

- The above comparative statics results are known as **Slutsky Equations**. Note that, the second term of each equations is the same (multiplied by x_1) as the relevant comparative static results with respect to money income. Thus these terms represent the income effect of a price change on demand. (Next we will show that the first terms in these equations are the pure substitution effects.)
- Therefore, the Slutsky equations show that the response of a utility-maximizing consumer to a price change can be split into two parts. The substitution effect (holding income constant) and the income effect (holding prices constant).

Slutsky equation

The effect of an increase in price on the demand function is summarized in the Slutsky Equation:

- $\frac{\partial x_i^*}{\partial p_j} = \frac{\partial x_i}{\partial p_j}|_{u=u^0} - x_j^* \frac{\partial x_i^*}{\partial M}$

The equation shows that the price response of a utility-maximizing consumer can be split into:

- i) a pure substitution effect (holding the consumer on the original indifference curve), and
- ii) a pure income effect (holding the prices constant).

Graphical analysis: own-price effects for normal goods

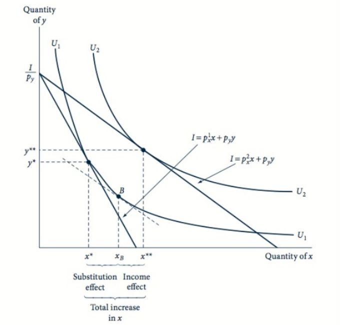


Figure: A decrease in P_{x1}

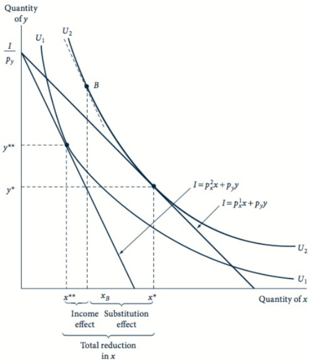
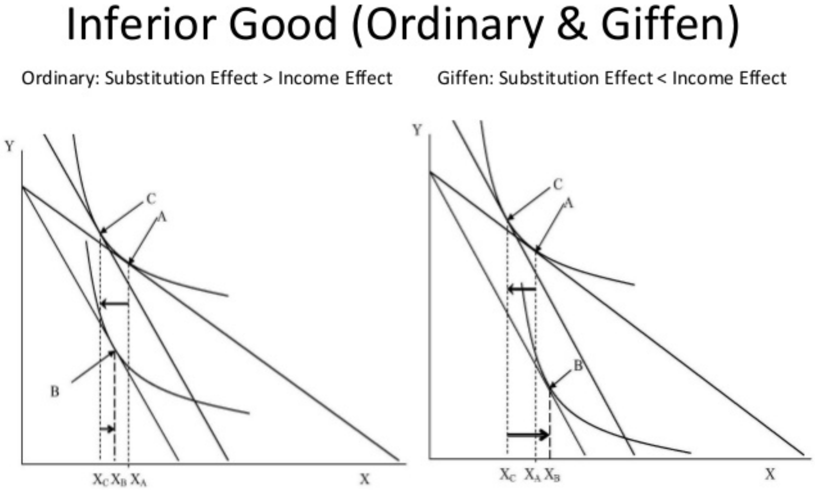


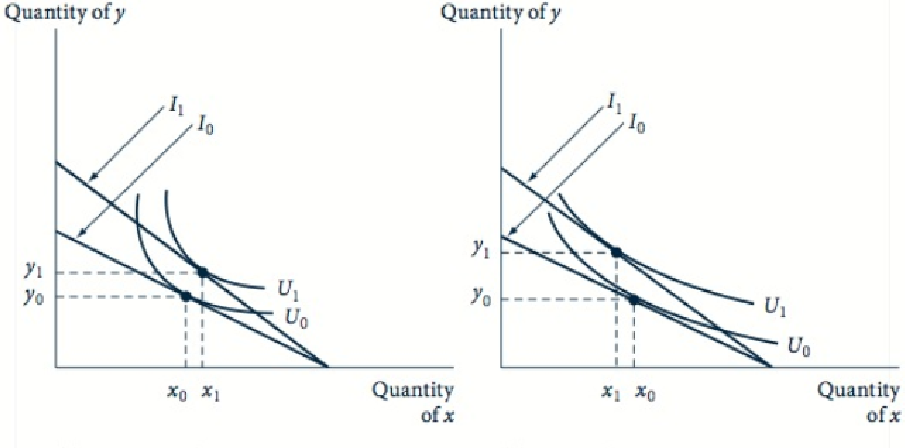
Figure: An increase in P_{x1}

Graphical analysis: own-price effects for inferior goods



Cross-price effects

Definition: Goods *i* and *j* are *gross complements* if an increase in the price of *j* reduces the quantity demanded of good *i*, i.e., $\frac{\partial x_i}{\partial p_j} < 0$. If the opposite is true, i.e., $\frac{\partial x_i}{\partial p_i} > 0$, then the goods are *gross substitutes*.



(a) Gross complements

(b) Gross substitutes

Reminder: the expenditure minimization model

The two-good expenditure minimization model is:

- $\underset{x_1, x_2, \lambda}{Min}\ L = p_1 x_1 + p_2 x_2 + \lambda(u^0 - u(x_1, x_2))$

The FOCs are:

1. $L_1 = p_1 - \lambda u_1 = 0$
2. $L_2 = p_2 - \lambda u_2 = 0$
3. $L_\lambda = u^0 - u(x_1, x_2) = 0$

Assuming the SOSC holds, the optimum functions in implicit form are:

- $x_1 = x_1^h(p_1, p_2, u^0)$
- $x_2 = x_2^h(p_1, p_2, u^0)$
- $\lambda = \lambda^h(p_1, p_2, u^0)$

Comparative statics

Just like in the utility maximization model, to derive the comparative statics of the expenditure minimization model we substitute the solutions into the first order conditions:

- $p_1 - \lambda^h(p_1, p_2, u^0)u_1(x_1^h(p_1, p_2, u^0), x_2^h(p_1, p_2, u^0)) \equiv 0$
- $p_2 - \lambda^h(p_1, p_2, u^0)u_2(x_1^h(p_1, p_2, u^0), x_2^h(p_1, p_2, u^0)) \equiv 0$
- $u^0 - u(x_1^h(p_1, p_2, u^0), x_2^h(p_1, p_2, u^0)) \equiv 0$

We want to know the following:

- How does an expenditure-minimizing individual adjust her demand to own- and cross-price changes, i.e. what is $\frac{\partial x_i^h}{\partial p_j} \forall i, j$?

To learn this we differentiate the above identities w.r.t. the parameters of interest.

Comparative statics: price effects on Hicksian demand

To derive $\frac{\partial x_i^h}{\partial p_1}$, $i = 1, 2$ we differentiate our identities with respect to p_1.

- $1 - \lambda^h u_{11} \frac{\partial x_1^h}{\partial p_1} - \lambda^h u_{12} \frac{\partial x_2^h}{\partial p_1} - u_1 \frac{\partial \lambda^h}{\partial p_1} \equiv 0$
- $-\lambda^h u_{21} \frac{\partial x_1^h}{\partial p_1} - \lambda^h u_{22} \frac{\partial x_2^h}{\partial p_1} - u_2 \frac{\partial \lambda^h}{\partial p_1} \equiv 0$
- $-u_1 \frac{\partial x_1^h}{\partial p_1} - u_2 \frac{\partial x_2^h}{\partial p_1} \equiv 0$

We can write this system of three equations in matrix notation as

- $\begin{pmatrix} -\lambda^h u_{11} & -\lambda^h u_{12} & -u_1 \\ -\lambda^h u_{21} & -\lambda^h u_{22} & -u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x_1^h}{\partial p_1} \\ \frac{\partial x_2^h}{\partial p_1} \\ \frac{\partial \lambda^h}{\partial p_1} \end{pmatrix} \equiv \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

Comparative statics: price effects on Hicksian demand

Using Cramer's rule,

- $$\frac{\partial x_1^h}{\partial p_1} = \frac{\begin{vmatrix} -1 & -\lambda^h u_{12} & -u_1 \\ 0 & -\lambda^h u_{22} & -u_2 \\ 0 & -u_2 & 0 \end{vmatrix}}{|H^h|} = -\frac{H_{11}^h}{|H^h|} = -\frac{-u_2^2}{|H^h|} < 0,\ SOC : |H^h| < 0$$
- $$\frac{\partial x_2^h}{\partial p_1} = \frac{\begin{vmatrix} -\lambda^h u_{11} & -1 & -u_1 \\ -\lambda^h u_{21} & 0 & -u_2 \\ -u_1 & 0 & 0 \end{vmatrix}}{|H^h|} = -\frac{H_{12}^h}{|H^h|} = -\frac{u_1 u_2}{|H^h|} > 0,\ SOC : |H^h| < 0$$

The comparative statics results state that Hicksian demand curves are *always* downward sloping. Also, the cross-price effect in the two-good case is positive, indicating that the goods have to be substitutes (note that this does not have to hold in the n-good case if $n > 2$).