COST MINIMIZATION & REVENUE MAXIMIZATION

APEC 8002, Fall 2016 **Instructor: Terry Hurley**

INTRODUCTION (MWG 5.C-D; Varian 4-5; Chambers 2-3; Cornes 5.2,5.6)

We begin our analysis of how to produce and how much to produce by looking at the cost minimization problem. For this problem, we will assume outputs and inputs are clearly delineated. Two more assumptions we will make are producers minimize cost and take input prices as exogenous (competitive input markets). The advantage of the cost minimization problem over the profit maximization problem we will take up later is that it does not assume output markets are competitive. Our ultimate goal is the derivation of conditional input demand and cost functions. As part of this derivation, we will explore what can be said about the properties of these objects that can help guide empirical research. Throughout this exploration, we will assume existence, which will simplify our arguments and not require us to do math beyond the prerequisites of this class.

Once we complete our detailed development of costs, we will quickly turn to summarizing the revenue maximization problem and the key objects and results that emerge from it. The revenue maximization problem with a single output is trivial, which is why it receives such little attention in many microeconomic text books that focus on single output production processes. However, when the prospect of multiple outputs is raised, the revenue maximization problem becomes just as interesting as the cost minimization problem, with slightly different strengths and weakness: we must assume competitive output markets, but no longer need to assume competitive input markets.

COST MINIMIZATION

The cost minimization problem can be framed generally as

CM1 $$\mathbf{Z}(\mathbf{r}, \mathbf{q}) = \{\mathbf{z} \in \mathbf{IRS}(\mathbf{q}) : \mathbf{r} \cdot \mathbf{z}' \geq \mathbf{r} \cdot \mathbf{z} \text{ for all } \mathbf{z}' \in \mathbf{IRS}(\mathbf{q})\}$$

where $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ is referred to as the *Conditional Input Demand* and is analogous to *Hicksian Demand* in the consumer problem. Intuitively, equation CM1 says we want the cheapest way to produce the vector of outputs **q** given strictly positive input prices **r**. Note that **Z**(**r**, **q**) can in general represent a set of cost minimizing vectors of inputs, though we will often want to look at a single vector contained in **Z**(**r**, **q**), which we will refer to as **z**(**r**, **q**) to simplify our exposition.

Several properties follow immediately from this definition. First note that if we multiply **r** by any $\alpha > 0$, $\mathbf{Z}(\mathbf{r}, \mathbf{q}) = \mathbf{Z}(\alpha\mathbf{r}, \mathbf{q})$ because $\alpha\mathbf{r} \cdot \mathbf{z}' \geq \alpha\mathbf{r} \cdot \mathbf{z}$ is no different from $\mathbf{r} \cdot \mathbf{z}' \geq \mathbf{r} \cdot \mathbf{z}$. Therefore, the conditional input demand is homogenous of degree zero in prices. Intuitively, this result says that how we produce depends on relative, not absolute, prices. Next, take any $\mathbf{r}^0$ and $\mathbf{r}^1$, let $\mathbf{z}^0 = \mathbf{z}(\mathbf{r}^0, \mathbf{q})$ and $\mathbf{z}^1 = \mathbf{z}(\mathbf{r}^1, \mathbf{q})$, and note that

$$\mathbf{r}^0\cdot \mathbf{z}^0 \leq \mathbf{r}^0\cdot \mathbf{z}^1 \text{ and } \mathbf{r}^1\cdot \mathbf{z}^1 \leq \mathbf{r}^1\cdot \mathbf{z}^0$$

by the definition of cost minimization in equation CM1. Summing then implies

$$\mathbf{r}^0\cdot \mathbf{z}^0 + \mathbf{r}^1\cdot \mathbf{z}^1 \leq \mathbf{r}^0\cdot \mathbf{z}^1 + \mathbf{r}^1\cdot \mathbf{z}^0 \text{ or } (\mathbf{r}^1 - \mathbf{r}^0)\cdot(\mathbf{z}^1 - \mathbf{z}^0) \leq 0.$$

An important implication of this result is that the conditional input demand is non-increasing in own prices (demand curves will never be upward sloping). To see this more clearly, suppose we compare the conditional input demands for commodity n where $r_n^1 > r_n^0$ and $r_{n'}^1 = r_{n'}^0$ for all $n' \neq n$. This equation then implies $(r_n^1 - r_n^0)\cdot(z_n^1 - z_n^0) \leq 0$. We know $r_n^1 - r_n^0 > 0$ by assumption, so $z_n^1 \leq z_n^0$ for this equation to hold, which means the demand for z_n cannot increase with an increase in price. The implications of this equation go well beyond this however and provide concrete hypotheses to test the validity of our cost minimization assumption.

Recall from our discussion of distance functions that if inputs are weakly freely disposable, then $\mathbf{IRS}(\mathbf{q}) = \{\mathbf{z}: D_I(\mathbf{q}, \mathbf{z}) \geq 1\}$. Therefore, another way to frame the cost minimization problem is

CM1’ $\min_{\mathbf{z}\geq 0} \mathbf{r} \cdot \mathbf{z}$ subject to $D_I(\mathbf{q}, \mathbf{z}) \geq 1$.

If our input requirement set is convex and input distance function is differentiable, we can use Lagrange methods to solve this problem:

CM2 $L = \mathbf{r} \cdot \mathbf{z} + \gamma\big(1 - D_I(\mathbf{q}, \mathbf{z})\big)$,

with the first-order conditions

CM3 $\frac{\partial L}{\partial z_n} = r_n - \gamma^* \frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n} \geq 0, \frac{\partial L}{\partial z_n} z_n{}^* = 0$, and $z_n{}^* \geq 0$ for $n = 1,\ldots, N$,

CM4 $\frac{\partial L}{\partial \gamma} = 1 - D_I(\mathbf{q}, \mathbf{z}^*) \leq 0, \frac{\partial L}{\partial \gamma} \gamma^* = 0$, and $\gamma^* \geq 0$

where the * is used to denote values that satisfy equations CM3 and CM4. Assuming weak free disposal of inputs implies $D_I(\mathbf{q}, \mathbf{z}^*) = 1$. To see why, suppose this is not the case such that $D_I(\mathbf{q}, \mathbf{z}^*) > 1$. Weak free disposal of inputs and the constraint $D_I(\mathbf{q}, \mathbf{z}) \geq 1$ imply $\mathbf{z}^* \in \mathbf{IRS}(\mathbf{q})$. Note that $\frac{\mathbf{z}^*}{D_I(\mathbf{q},\mathbf{z}^*)} \in \mathbf{IRS}(\mathbf{q})$ by definition of the input distance function. Now for $\mathbf{z}^*$ to minimize costs, $\mathbf{r} \cdot \mathbf{z} \geq \mathbf{r} \cdot \mathbf{z}^*$ for all $\mathbf{z} \in \mathbf{IRS}(\mathbf{q})$ implying $\mathbf{r} \cdot \frac{\mathbf{z}^*}{D_I(\mathbf{q},\mathbf{z}^*)} \geq \mathbf{r} \cdot \mathbf{z}^*$. Rearranging then yields $\mathbf{r} \cdot \mathbf{z}^* \left(\frac{1}{D_I(\mathbf{q},\mathbf{z}^*)} - 1\right) \geq 0$, which can only be true if $D_I(\mathbf{q}, \mathbf{z}^*) \leq 1$ — a contradiction.

The convexity assumption assures that equations CM3 and CM4 are both necessary and sufficient for finding the optimum, so we do not need to look at the second order conditions. Convexity also gets us some additional properties.

First, $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ will be a convex set. For $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ to be convex, $\mathbf{z}^1$ and $\mathbf{z}^2 \in \mathbf{Z}(\mathbf{r}, \mathbf{q})$ implies $\mathbf{z}^3 = \alpha\, \mathbf{z}^1 + (1 - \alpha)\, \mathbf{z}^2 \in \mathbf{Z}(\mathbf{r}, \mathbf{q})$ for all $\alpha \in [0, 1]$. By the definition of the conditional factor demand, $\mathbf{r} \cdot \mathbf{z}^1 \leq \mathbf{r} \cdot \mathbf{z}^2$ and $\mathbf{r} \cdot \mathbf{z}^2 \leq \mathbf{r} \cdot \mathbf{z}^1$ such that $\mathbf{r} \cdot \mathbf{z}^1 = \mathbf{r} \cdot \mathbf{z}^2$. This implies that $\mathbf{r} \cdot \mathbf{z}^3 = \mathbf{r} \cdot (\alpha\, \mathbf{z}^1 + (1 - \alpha)\, \mathbf{z}^2) = \mathbf{r} \cdot \mathbf{z}^1 = \mathbf{r} \cdot \mathbf{z}^2$. Convexity of the input requirement set implies $\mathbf{z}^3 \in \mathbf{IRS}(\mathbf{q})$. Therefore, since $\mathbf{r} \cdot \mathbf{z}^1 = \mathbf{r} \cdot \mathbf{z}^2 \leq \mathbf{r} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbf{IRS}(\mathbf{q})$, $\mathbf{r} \cdot \mathbf{z}^3 \leq \mathbf{r} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbf{IRS}(\mathbf{q})$ implying

$\mathbf{z}^3 \in \mathbf{Z}(\mathbf{r}, \mathbf{q})$. Similar arguments can be used to show that if $\mathbf{IRS}(\mathbf{q})$ is strictly convex, $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ will be a singleton set for all $\mathbf{r}$ and $\mathbf{q}$.

Second, if $\mathbf{z}(\mathbf{r}, \mathbf{q})$ is differentiable at $\mathbf{r}^0$, then the input substitution matrix defined by

CM5
$$\mathbf{D}_{\mathbf{r}}\mathbf{z}(\mathbf{r}^0, \mathbf{q}) = \begin{bmatrix} \frac{dz_1(\mathbf{r}^0,\mathbf{q})}{dr_1} & \cdots & \frac{dz_1(\mathbf{r}^0,\mathbf{q})}{dr_N} \\ \vdots & \ddots & \vdots \\ \frac{dz_N(\mathbf{r}^0,\mathbf{q})}{dr_1} & \cdots & \frac{dz_N(\mathbf{r}^0,\mathbf{q})}{dr_N} \end{bmatrix}$$

will be a symmetric and negative semi-definite with $\mathbf{D}_{\mathbf{r}}\mathbf{z}(\mathbf{r}^0, \mathbf{q}) \cdot \mathbf{r}^0 = \mathbf{0}_N$. The symmetry and negative semi-definiteness of $\mathbf{D}_{\mathbf{r}}\mathbf{z}(\mathbf{r}^0, \mathbf{q})$ follow from weak free disposal of inputs and the differentiability and concavity of $D_I(\mathbf{q}, \mathbf{z})$ in $\mathbf{z}$. $\mathbf{D}_{\mathbf{r}}\mathbf{z}(\mathbf{r}^0, \mathbf{q}) \cdot \mathbf{r}^0 = \mathbf{0}_N$ follows from differentiability and the homogeneity of degree zero of $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ in $\mathbf{r}$.

Another point to make is that it is optimal to equate the *MRTS* to the price ratio for all inputs used in positive quantities. That is, for any $z_n{}^* > 0$ and $z_{n'}{}^* > 0$, equation CM3' implies $\frac{r_{n'}}{r_n} = \frac{\frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_{n'}}}{\frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n}}$. Alternatively, if $z_n{}^* > 0$ and $z_{n'}{}^* = 0$, equation CM3' implies $\frac{r_{n'}}{r_n} \geq \frac{\frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_{n'}}}{\frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n}}$. That is, the *MRTS* will never exceed the ratio of input prices at a minimum.

The *Cost Function* is defined as

CM6
$$C(\mathbf{r}, \mathbf{q}) = \mathbf{r} \cdot \mathbf{z}^* = \mathbf{r} \cdot \mathbf{z}(\mathbf{r}, \mathbf{q}).$$

It represents the minimal cost of production given prices and output. Several results follow from this definition. First, if we multiply $\mathbf{r}$ by $\alpha > 0$, we get $C(\alpha\mathbf{r}, \mathbf{q}) = \alpha\mathbf{r} \cdot z(\alpha\mathbf{r}, \mathbf{q}) = \alpha C(\mathbf{r}, \mathbf{q})$ — the cost function is homogeneous of degree one in input prices. Intuitively, if we double all input prices, we must double our cost because optimal input use only depends on relative prices.

We can also show $C(\mathbf{r}, \mathbf{q})$ is a concave function of $\mathbf{r}$. Note that $C(\mathbf{r}, \mathbf{q})$ is a concave function of r if

$$C(\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}) \geq \alpha C(\mathbf{r}^1, \mathbf{q}) + (1 - \alpha)\, C(\mathbf{r}^2, \mathbf{q}) \text{ for all } \alpha \in [0,1].$$

By definition of the cost function,

$$C(\mathbf{r}^1, \mathbf{q}) = \mathbf{r}^1 \cdot \mathbf{z}(\mathbf{r}^1, \mathbf{q}),$$

$$C(\mathbf{r}^2, \mathbf{q}) = \mathbf{r}^2 \cdot \mathbf{z}(\mathbf{r}^2, \mathbf{q}), \text{ and}$$

$$C(\alpha\, \mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}) = (\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2) \cdot \mathbf{z}(\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}).$$

By definition of the conditional input demand,

$$\alpha\mathbf{r}^1 \cdot \mathbf{z}(\alpha\mathbf{r}^1, \mathbf{q}) \leq \alpha\mathbf{r}^1 \cdot \mathbf{z}(\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}) \text{ and}$$

$$(1 - \alpha)\mathbf{r}^2 \cdot \mathbf{z}((1 - \alpha)\mathbf{r}^2, \mathbf{q}) \leq (1 - \alpha)\mathbf{r}^2 \cdot \mathbf{z}(\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}).$$

Summing then implies

$$\alpha\mathbf{r}^1 \cdot \mathbf{z}(\alpha\mathbf{r}^1, \mathbf{q}) + (1 - \alpha)\mathbf{r}^2 \cdot \mathbf{z}((1 - \alpha)\mathbf{r}^2, \mathbf{q}) \leq (\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2) \cdot \mathbf{z}(\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}).$$

Finally, since the conditional input demand is homogeneous of degree zero in prices, we get the desired result

$$\begin{aligned}\alpha C(\mathbf{r}^1, \mathbf{q}) + (1 - \alpha)\, C(\mathbf{r}^2, \mathbf{q}) &= \alpha\mathbf{r}^1 \cdot \mathbf{z}(\mathbf{r}^1, \mathbf{q}) + (1 - \alpha)\mathbf{r}^2 \cdot \mathbf{z}(\mathbf{r}^2, \mathbf{q}) \\ &\leq (\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2) \cdot \mathbf{z}(\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}) = C(\alpha\mathbf{r}^1 + (1 - \alpha)\mathbf{r}^2, \mathbf{q}).\end{aligned}$$

Equation CM3 implies $r_n z_n{}^* = \gamma^* \frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n} z_n{}^*$ for all n. Summing then implies

$$C(\mathbf{r},\mathbf{q}) = \mathbf{r}\cdot\mathbf{z}^* = \gamma^* \sum_{n=1}^{N} \frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n} z_n{}^*.$$

Recall however that $D_I(\mathbf{q},\mathbf{z})$ is homogenous of degree one in inputs such that

$$\sum_{n=1}^{N} \frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n} z_n{}^* = D_I(\mathbf{q},\mathbf{z}^*) = 1$$

or $C(\mathbf{r},\mathbf{q}) = \gamma^*$ — the Lagrange multiplier at the optimum is equal to the minimum cost of production.

If $\mathbf{Z}(\mathbf{r}^0, \mathbf{q})$ is a singleton set for $\mathbf{r}^0$, then $\frac{\partial c(\mathbf{r}^0,\mathbf{q})}{\partial r_n} = z_n(\mathbf{r}^0,\mathbf{q})$ for all n, which is known as *Shephard's lemma*. This result can be established by noting that

$$C(\mathbf{r}^0,\mathbf{q}) = \mathbf{r}^0 \cdot z(\mathbf{r}^0,\mathbf{q}) + \gamma(\mathbf{r}^0,\mathbf{q})\left(1 - D_I\big(\mathbf{q},\mathbf{z}(\mathbf{r}^0,\mathbf{q})\big)\right).$$

Differentiating with respect to r_n then yields

$$\frac{\partial C(\mathbf{r}^0,\mathbf{q})}{\partial r_n} = z_n(\mathbf{r}^0,\mathbf{q}) + \frac{\partial \gamma(\mathbf{r}^0,\mathbf{q})}{\partial r_n}\left(1 - D_I\big(\mathbf{q},\mathbf{z}(\mathbf{r}^0,\mathbf{q})\big)\right)$$
$$+ \sum_{n'=1}^{N} \frac{\partial z_{n'}(\mathbf{r}^0,\mathbf{q})}{\partial r_n}\left(r_{n'}{}^0 - \gamma(\mathbf{r}^0,\mathbf{q}) \frac{\partial D_I\big(\mathbf{q},\mathbf{z}(\mathbf{r}^0,\mathbf{q})\big)}{\partial z_{n'}}\right).$$

We know the second term on the right-hand side is 0 because production will be efficient with weak free disposal of inputs. We also know the third term is zero because if we differentiate $\frac{\partial L}{\partial z_{n'}} z_{n'}{}^* = 0$ from equation CM3 with respect to r_n and rearrange, we get

$$\frac{\partial\left(r_{n'} - \gamma(\mathbf{r}^0,\mathbf{q})\frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_{n'}}\right)}{\partial r_n} z_{n'}(r^0,q) = -\frac{\partial z_{n'}(\mathbf{r}^0,\mathbf{q})}{\partial r_n}\left(r_{n'} - \gamma(\mathbf{r}^0,\mathbf{q})\frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_{n'}}\right).$$

If $z_{n'}(\mathbf{r}^0, \mathbf{q}) = 0$, we know the left-hand side of this equation is zero by assumption, so the right-hand side must also be zero. If $z_{n'}(\mathbf{r}^0, \mathbf{q}) > 0$ we know, $r_{n'} - \gamma(\mathbf{r}^0, \mathbf{q}) \frac{\partial D_I(\mathbf{q}, \mathbf{z}^*)}{\partial z_{n'}} = 0$ by equation CM3. Therefore, $\frac{\partial z_{n'}(\mathbf{r}^0, \mathbf{q})}{\partial r_n} \left(r_{n'}{}^0 - \gamma(\mathbf{r}^0, \mathbf{q}) \frac{\partial D_I\left(\mathbf{q}, \mathbf{z}(\mathbf{r}^0, \mathbf{q})\right)}{\partial z_{n'}} \right) = 0$ for $n' = 1, \ldots, N$.

For convex $IRS(q)$ and twice differentiable $C(r^0, q)$ at r^0,

CM7
$$\mathbf{D_r}^2\mathbf{C}(\mathbf{r}^0, \mathbf{q}) = \begin{bmatrix} \frac{d^2C(\mathbf{r}^0, \mathbf{q})}{dr_1{}^2} & \cdots & \frac{d^2C(\mathbf{r}^0, \mathbf{q})}{dr_1 dr_N} \\ \vdots & \ddots & \vdots \\ \frac{d^2C(\mathbf{r}^0, \mathbf{q})}{dr_N dr_1} & \cdots & \frac{d^2C(\mathbf{r}^0, \mathbf{q})}{dr_N{}^2} \end{bmatrix}$$

is a symmetric and negative semi-definite matrix with $\mathbf{D_r}^2\mathbf{C}(\mathbf{r}^0, \mathbf{q}) \cdot \mathbf{r}^0 = \mathbf{0}_N$, which follows immediately from $\frac{\partial C(\mathbf{r}^0, \mathbf{q})}{\partial r_n} = z_n(\mathbf{r}^0, \mathbf{q})$ for all n and $\mathbf{D_r z}(\mathbf{r}^0, \mathbf{q})$ being symmetric and negative semi-definite with $\mathbf{D_r z}(\mathbf{r}^0, \mathbf{q}) \cdot \mathbf{r}^0 = \mathbf{0}_N$.

If output is strongly freely disposable, $C(\mathbf{r}, \mathbf{q}^1) \geq C(\mathbf{r}, \mathbf{q}^0)$ for all $\mathbf{q}^1 \geq \mathbf{q}^0$, which is analogous to the *Weak Axiom of Cost Minimization* discussed in Varian for example. Suppose this were not the case such that $C(\mathbf{r}, \mathbf{q}^1) < C(\mathbf{r}, \mathbf{q}^0)$ for some $\mathbf{q}^1 \geq \mathbf{q}^0$. By the definition of the cost function, $C(\mathbf{r}, \mathbf{q}^1) = \mathbf{r} \cdot \mathbf{z}(\mathbf{r}, \mathbf{q}^1)$. By definition of conditional input demand,

$$\mathbf{z}(\mathbf{r}, \mathbf{q}^1) \in \{\mathbf{z} \in \mathbf{IRS}(\mathbf{q}^1) \colon \mathbf{r} \cdot \mathbf{z}' \geq \mathbf{r} \cdot \mathbf{z} \text{ for all } \mathbf{z}' \in \mathbf{IRS}(\mathbf{q}^1)\}.$$

By definition of the input requirement set, $\left(\mathbf{q}^1, -\mathbf{z}(\mathbf{r}, \mathbf{q}^1)\right) \in \mathbf{PPS}$. By the definition of the feasible output set and strong free disposal of output, $\left(\mathbf{q}^0, -\mathbf{z}(\mathbf{r}, \mathbf{q}^1)\right) \in \mathbf{PPS}$ since $\mathbf{q}^1 \geq \mathbf{q}^0$. By the definition of the input requirement set, $\mathbf{z}(\mathbf{r}, \mathbf{q}^1) \in \mathbf{IRS}(\mathbf{q}^0)$. By the definition of the cost function, $\mathbf{r} \cdot \mathbf{z}(\mathbf{r}, \mathbf{q}^1) \geq \mathbf{r} \cdot \mathbf{z}(\mathbf{r}, \mathbf{q}^0)$ implying the contradiction $C(\mathbf{r}, \mathbf{q}^1) \geq C(\mathbf{r}, \mathbf{q}^0)$.

Similar arguments can be used to show that $C(\mathbf{r}, \alpha\mathbf{q}) \geq C(\mathbf{r}, \mathbf{q})$ for all $\alpha \geq 1$ if output is weakly freely disposable. Intuitively, if we have a cost minimizing vector of inputs, we cannot increase output and lower our costs at the same time. This results also tells us that the *Marginal Cost of Production* (i.e., $\frac{\partial C(\mathbf{r},\mathbf{q})}{\partial q_m}$ for differentiable costs with strong free disposal and $\frac{\partial C(\mathbf{r},\alpha\mathbf{q})}{\partial \alpha}$ for differentiable costs with weak free disposal) is non-negative.

With constant returns to scale, $\mathbf{Z}(\mathbf{r},\mathbf{q})$ and $C(\mathbf{r},\mathbf{q})$ are homogeneous of degree one in $\mathbf{q}$. Note that $\mathbf{Z}(\mathbf{r},\mathbf{q})$ is homogeneous of degree one in $\mathbf{q}$ such that $\mathbf{Z}(\mathbf{r},\alpha\mathbf{q}) = \alpha\mathbf{Z}(\mathbf{r},\mathbf{q})$ for all $\alpha > 0$. Therefore, what we need to show is that $\mathbf{z}^0 \in \mathbf{Z}(\mathbf{r},\alpha^0\mathbf{q})$ if and only if $\mathbf{z}^0 \in \alpha^0\mathbf{Z}(\mathbf{r},\mathbf{q})$. Suppose this is not the case such that there exists a $\alpha^0 > 0$ and a $\mathbf{z}^0$ where (i) $\mathbf{z}^0 \in \mathbf{Z}(\mathbf{r},\alpha^0\mathbf{q})$ and $\mathbf{z}^0 \notin \alpha^0\mathbf{Z}(\mathbf{r},\mathbf{q})$ or (ii) $\mathbf{z}^0 \notin \mathbf{Z}(\mathbf{r},\alpha^0\mathbf{q})$ and $\mathbf{z}^0 \in \alpha^0\mathbf{Z}(\mathbf{r},\mathbf{q})$. Consider (i) first. By the definition of the conditional input demand $\mathbf{z}^0 \in \mathbf{Z}(\mathbf{r},\alpha^0\mathbf{q})$ implies $\mathbf{r}\cdot\alpha^0\mathbf{z} \geq \mathbf{r}\cdot\mathbf{z}^0$ for all $\alpha^0\mathbf{z} \in \mathbf{IRS}(\alpha^0\mathbf{q})$. By constant returns to scale $\alpha^0\mathbf{z} \in \mathbf{IRS}(\alpha^0\mathbf{q})$ implies $\mathbf{z} \in \mathbf{IRS}(\mathbf{q})$, such that $\mathbf{r}\cdot\mathbf{z} \geq \mathbf{r}\cdot\frac{\mathbf{z}^0}{\alpha^0}$ for all $\mathbf{z} \in \mathbf{IRS}(\mathbf{q})$ implies $\frac{\mathbf{z}^0}{\alpha^0} \in \mathbf{Z}(\mathbf{r},\mathbf{q})$ for all $\frac{\mathbf{z}^0}{\alpha^0} \in \mathbf{IRS}(\mathbf{q})$ by the definition of the conditional input demand. But $\frac{\mathbf{z}^0}{\alpha^0} \in \mathbf{Z}(\mathbf{r},\mathbf{q})$ then yields the contradiction $\mathbf{z}^0 \in \alpha^0\mathbf{Z}(\mathbf{r},\mathbf{q})$. For (ii), these arguments can simply be reversed to show $\mathbf{z}^0 \in \alpha^0\mathbf{Z}(\mathbf{r},\mathbf{q})$ yields the contradiction that $\mathbf{z}^0 \in \mathbf{Z}(\mathbf{r},\alpha^0\mathbf{q})$. With $\mathbf{Z}(\mathbf{r},\mathbf{q})$ homogeneous of degree one in $\mathbf{q}$, $C(\mathbf{r},\alpha\mathbf{q}) = \mathbf{r}\cdot\mathbf{z}(\mathbf{r},\alpha\mathbf{q}) = \alpha\mathbf{r}\cdot\mathbf{z}(\mathbf{r},\mathbf{q}) = \alpha C(\mathbf{r},\mathbf{q})$ establishes that $C(\mathbf{r},\mathbf{q})$ is also homogeneous of degree one in $\mathbf{q}$.

The previous discussion introduced a lot of important properties for the conditional input demand and the cost function that would be useful to summarize in one place.

If $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ is the conditional input demand and $C(\mathbf{r}, \mathbf{q})$ is the cost function derived from an input requirement set that is nonempty, closed, and satisfies weak free disposal,

(i) $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ is homogeneous of degree zero and $C(\mathbf{r}, \mathbf{q})$ is homogeneous of degree one in $\mathbf{r}$;

(ii) $(\mathbf{r}^1 - \mathbf{r}^0)\cdot(\mathbf{z}(\mathbf{r}^1, \mathbf{q}) - \mathbf{z}(\mathbf{r}^0, \mathbf{q})) \leq 0$ for all $\mathbf{r}^1$ and $\mathbf{r}^0$;

(iii) $C(\mathbf{r}, \mathbf{q})$ is a concave and continuous function of $\mathbf{r}$;

(iv) $C(\mathbf{r},\mathbf{q}^1) \geq C(\mathbf{r},\mathbf{q}^0)$ for $\mathbf{q}^1 \geq \mathbf{q}^0$ if $\mathbf{FOS}(\mathbf{z})$ satisfies strong free disposal and $C(\mathbf{r},\alpha\mathbf{q}) \geq C(\mathbf{r},\mathbf{q})$ for $\alpha \geq 1$ if $\mathbf{FOS}(\mathbf{z})$ satisfies weak free disposal;

(v) $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ is convex/singleton set for all $\mathbf{r}$ if $\mathbf{IRS}(\mathbf{q})$ is convex/strictly convex;

(vi) $C(\mathbf{r}, \mathbf{q})$ is differentiable with respect to $\mathbf{r}$ at $\mathbf{r}^0$ and $\frac{\partial C(\mathbf{r}^0,\mathbf{q})}{\partial r_n} = z_n(\mathbf{r}^0, \mathbf{q})$ for all n if $\mathbf{Z}(\mathbf{r}^0, \mathbf{q})$ is a singleton set;

(vii) $\mathbf{D_r}^2\mathbf{C}(\mathbf{r}^0, \mathbf{q}) = \mathbf{D_r z}(\mathbf{r}^0, \mathbf{q})$ are symmetric and negative semi-definite matrices with $\mathbf{D_r z}(\mathbf{r}^0, \mathbf{q}) \cdot \mathbf{r}^0 = \mathbf{0}_N$ if $\mathbf{z}(\mathbf{r}^0, \mathbf{q})$ is differentiable at $\mathbf{r}^0$; and

(viii) $\mathbf{Z}(\mathbf{r}, \mathbf{q})$ and $C(\mathbf{r}, \mathbf{q})$ are homogeneous of degree one in $\mathbf{q}$ if *PPS* exhibits constant returns to scale.

Note that these properties are generalizations of properties (i), (ii), and (iv) – (ix) listed in MWG on page 141. We will come back to property (iii) in MWG when we talk about duality.

These properties are important for several reasons. First, they provide specific hypotheses that we can test to see if observed behavior is consistent with our various assumptions. If some of these hypotheses can be rejected, but others cannot, we can get a sense of which assumptions are being violated. Second, if we are interested in estimating a firm's cost function or conditional input demand, they provide a wealth of guidance in terms of choosing functional forms and appropriate parameter restrictions that can increase the efficiency of our estimation.

To demonstrate how all this works, let us return to the two-output and two-input technology we introduced in our discussion about the production possibilities sets that gave us $D_I(\mathbf{q}, \mathbf{z}) = \frac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2}$ (assuming $q_1 > 0$ or $q_2 > 0$). The Lagrangian and first order conditions for the cost minimization problem are

CM2$^{\text{exp}}$ $$L = r_1 z_1 + r_2 z_2 + \gamma\left(1 - \frac{\sqrt{z_1 z_2}}{q_1^2 + q_2^2}\right),$$

CM3$^{\text{exp}}$ $$\frac{\partial L}{\partial z_1} = r_1 - \gamma^* \frac{\sqrt{z_2^*}}{2(q_1^2 + q_2^2)\sqrt{z_1^*}} \geq 0,\ \frac{\partial L}{\partial z_1} z_1^* = 0, \text{ and } z_1^* \geq 0,$$

$$\frac{\partial L}{\partial z_2} = r_2 - \gamma^* \frac{\sqrt{z_1^*}}{2(q_1^2 + q_2^2)\sqrt{z_2^*}} \geq 0,\ \frac{\partial L}{\partial z_2} z_2^* = 0, \text{ and } z_2^* \geq 0,$$

CM4$^{\text{exp}}$ $\frac{\partial L}{\partial \gamma} = 1 - \frac{\sqrt{z_1{}^* z_2{}^*}}{q_1{}^2 + q_2{}^2} \leq 0, \frac{\partial L}{\partial \gamma}\gamma^* = 0$, and $\gamma^* \geq 0$.

For an interior solution, equation **CM3$^{\text{exp}}$** tells us that $\frac{r_1}{r_2} = \frac{z_2{}^*}{z_1{}^*}$ or $z_2{}^* = z_1{}^* \frac{r_1}{r_2}$. Can you show that we must have an interior solution? We also know $\frac{\sqrt{z_1{}^* z_2{}^*}}{q_1{}^2 + q_2{}^2} = 1$. Substitution and a little algebra then implies $z_1(\mathbf{r}, \mathbf{q}) = \sqrt{\frac{r_2}{r_1}}(q_1{}^2 + q_2{}^2)$, such that $z_2(\mathbf{r}, \mathbf{q}) = \sqrt{\frac{r_1}{r_2}}(q_1{}^2 + q_2{}^2)$ and $C(\mathbf{r}, \mathbf{q}) = 2\sqrt{r_1 r_2}(q_1{}^2 + q_2{}^2)$.

LONG-RUN VS SHORT-RUN COST MINIMIZATION

Our earlier discussion of production talked about long-run and short-run production. This notion is designed to capture the fact that some inputs can be more freely reallocated than others depending on time. We will now review some familiar relations between production in the long-run and the short-run.

Let $\mathbf{z} = (\mathbf{z}^v, \mathbf{z}^f)$ where $\mathbf{z}^v$ is a $1\times N^v$ vector of variable inputs and $\mathbf{z}^f$ is a $1\times N^f$ vector of fixed inputs such that $N^v + N^f = N$. We can define a short-run input distance function as

CM8 $D_I(\mathbf{q}, \mathbf{z}^v, \mathbf{z}^f) = \max_\delta \left\{\delta > 0: \left(q, -\frac{\mathbf{z}^v}{\delta}\right) \in \mathbf{PPS}(\mathbf{z}_f)\right\} = \max_\delta \left\{\delta > 0: \frac{\mathbf{z}^v}{\delta} \in \mathbf{IRS}(\mathbf{q}, \mathbf{z}^f)\right\}$.

Before going any further, it is worth thinking a little about our assumptions. Assumptions that hold for our production possibilities in the long-run need not hold in the short-run. For example, $\mathbf{IRS}(\mathbf{q}, \mathbf{z}^f)$ does not have to satisfy weak free disposal in $\mathbf{z}^v$ just because $\mathbf{IRS}(\mathbf{q})$ satisfies weak free disposal in $\mathbf{z}$; though, $\mathbf{IRS}(\mathbf{q}, \mathbf{z}^f)$ will satisfy strong free disposal in $\mathbf{z}^v$ if $\mathbf{IRS}(\mathbf{q})$ satisfies strong free disposal in $\mathbf{z}$. With that said, we will assume that both the long-run and short-run input requirement sets satisfy A1, A2, A3.IW, A4.IS and $D_I(\mathbf{q}, \mathbf{z})$, and $D_I(\mathbf{q}, \mathbf{z}^v, \mathbf{z}^f)$ are differentiable, so we know a solution is unique and we can use Lagrange methods.

The short-run cost minimization problem can be written as

CM1$^{\text{sr}}$ $\min_{\mathbf{z}^v \geq 0} \mathbf{r}^v \cdot \mathbf{z}^v + \mathbf{r}^f \cdot \mathbf{z}^f$ subject to $D_I(\mathbf{q}, \mathbf{z}^v, \mathbf{z}^f) \geq 1$

where $\mathbf{r}^v$ is a $1 \times N^v$ vector of variable input prices and $\mathbf{r}^f$ is a $1 \times N^f$ vector of fixed input prices. The Lagrangian and first order condition for this problem are

CM2$^{\text{sr}}$ $L = \mathbf{r}^v \cdot \mathbf{z}^v + \mathbf{r}^f \cdot \mathbf{z}^f + \gamma\left(1 - D_I(\mathbf{q}, \mathbf{z}^v, \mathbf{z}^f)\right)$,

CM3$^{\text{sr}}$ $\frac{\partial L}{\partial z_n^v} = r_n^v - \gamma^* \frac{\partial D_I(\mathbf{q}, \mathbf{z}^{v*}, \mathbf{z}^f)}{\partial z_n^v} \geq 0$, $\frac{\partial L}{\partial z_n^v} z_n^{v*} = 0$, and $z_n^{v*} \geq 0$ for $n = 1, \ldots, N^v$,

CM4$^{\text{sr}}$ $\frac{\partial L}{\partial \gamma} = 1 - D_I(\mathbf{q}, \mathbf{z}^{v*}, \mathbf{z}^f) \leq 0$, $\frac{\partial L}{\partial \gamma} \gamma^* = 0$, and $\gamma^* \geq 0$.

What is important to recognize in equations CM3$^{\text{sr}}$ and CM4$^{\text{sr}}$ is the absence of $\mathbf{r}^f$, which tells us our solution does not depend on of the fixed input prices. However, $\mathbf{z}^f$ does appear in these equations, so our solution will depend on fixed inputs. Therefore, the short-run conditional input demand depends on the prices of variable inputs, output, and fixed inputs: $\mathbf{z}^v(\mathbf{r}^v, \mathbf{q}, \mathbf{z}^f)$. The short-run cost function is $C(\mathbf{r}, \mathbf{q}, \mathbf{z}^f) = \mathbf{r}^v \cdot \mathbf{z}^v(\mathbf{r}^v, \mathbf{q}, \mathbf{z}^f) + \mathbf{r}^f \cdot \mathbf{z}^f$, which is often broken up into variable and fixed costs: $C(\mathbf{r}, \mathbf{q}, \mathbf{z}^f) = C^v(\mathbf{r}^v, \mathbf{q}, \mathbf{z}^f) + C^f(\mathbf{r}^f, \mathbf{z}^f)$ where $C^v(\mathbf{r}^v, \mathbf{q}, \mathbf{z}^f) = \mathbf{r}^v \cdot \mathbf{z}^v(\mathbf{r}^v, \mathbf{q}, \mathbf{z}^f)$ are the variable costs and $C^f(\mathbf{r}^f, \mathbf{z}^f) = \mathbf{r}^f \cdot \mathbf{z}^f$ are the fixed costs. Note the dependency of variable costs on variable input prices, output, and fixed inputs and the dependency of fixed costs only on fixed inputs and their prices. Also, note that the short-run conditional input demand and variable cost function will satisfy properties (i) – (ix) stated above.

The long-run cost minimization problem can be written as

CM1$^{\text{lr}}$ $\min_{\mathbf{z} \geq 0} \mathbf{r}^v \cdot \mathbf{z}^v + \mathbf{r}^f \cdot \mathbf{z}^f$ subject to $D_I(\mathbf{q}, \mathbf{z}) \geq 1$.

The Lagrangian and first order conditions for this problem are

CM2lr $L = \mathbf{r}^v \cdot \mathbf{z}^v + \mathbf{r}^f \cdot \mathbf{z}^f + \gamma\big(1 - D_I(\mathbf{q},\mathbf{z})\big)$,

CM3lr $\frac{\partial L}{\partial z_n^v} = r_n^v - \gamma^* \frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n^v} \geq 0$, $\frac{\partial L}{\partial z_n^v} z_n^{v*} = 0$, and $z_n^{v*} \geq 0$ for $n = 1,\ldots, N^v$,

$\frac{\partial L}{\partial z_n^f} = r_n^f - \gamma^* \frac{\partial D_I(\mathbf{q},\mathbf{z}^*)}{\partial z_n^f} \geq 0$, $\frac{\partial L}{\partial z_n^f} z_n^{f*} = 0$, and $z_n^{f*} \geq 0$ for $n = 1,\ldots, N^f$,

CM4lr $\frac{\partial L}{\partial \gamma} = 1 - D_I(\mathbf{q},\mathbf{z}^*) \leq 0$, $\frac{\partial L}{\partial \gamma}\gamma^* = 0$, and $\gamma^* \geq 0$.

The solution to this problem is the long-run conditional input demand $\mathbf{z}(\mathbf{r},\mathbf{q}) = \big(\mathbf{z}^v(\mathbf{r},\mathbf{q}), \mathbf{z}^f(\mathbf{r},\mathbf{q})\big)$, which depends on output and all input prices unlike the short-run conditional input demand. The long-run cost function is $C(\mathbf{r},\mathbf{q}) = \mathbf{r}\cdot\mathbf{z}(\mathbf{r},\mathbf{q}) = \mathbf{r}^v \cdot \mathbf{z}^v(\mathbf{r},\mathbf{q}) + \mathbf{r}^f \cdot \mathbf{z}^f(\mathbf{r},\mathbf{q})$ where all costs are now variable. Like the short-run conditional input demand and variable cost function, the long-run conditional input demand and cost function will satisfy properties (i) – (viii) stated above.

Comparing the solutions for the short-run and long-run problems, we know $D_I\left(\mathbf{q}, \mathbf{z}^v\left(\mathbf{r}^v, \mathbf{q}, \mathbf{z}^f(\mathbf{r},\mathbf{q})\right), \mathbf{z}^f(\mathbf{r},\mathbf{q})\right) = 1$ and $D_I\big(\mathbf{q},\mathbf{z}(\mathbf{r},\mathbf{q})\big) = 1$ by weak free disposal of inputs. From the definition of the input distance function, we know that

$$\left(\mathbf{q}, -\mathbf{z}^v\left(\mathbf{r}^v, \mathbf{q}, \mathbf{z}^f(\mathbf{r},\mathbf{q})\right), -\mathbf{z}^f(\mathbf{r},\mathbf{q})\right) \in PPS \text{ and } \left(\mathbf{q}, -\mathbf{z}^v(\mathbf{r}^v,\mathbf{q}), -\mathbf{z}^f(\mathbf{r},\mathbf{q})\right) \in \mathbf{PPS}.$$

By definition of cost minimization,

$$C(\mathbf{r},\mathbf{q}) \geq C\left(\mathbf{r},\mathbf{q},\mathbf{z}^f(\mathbf{r},\mathbf{q})\right) \text{ and } C\left(\mathbf{r},\mathbf{q},\mathbf{z}^f(\mathbf{r},\mathbf{q})\right) \geq C(\mathbf{r},\mathbf{q})$$

such that $C(\mathbf{r},\mathbf{q}) = C\left(\mathbf{r},\mathbf{q},\mathbf{z}^f(\mathbf{r},\mathbf{q})\right)$. Furthermore, since $(\mathbf{q}, -\mathbf{z}^v(\mathbf{r}^v,\mathbf{q},\mathbf{z}^f), -\mathbf{z}^f) \in \mathbf{PPS}$ for all $\mathbf{z}^f$,

CM9 $$g(\mathbf{r},\mathbf{q},\mathbf{z}^f) = C(\mathbf{r},\mathbf{q},\mathbf{z}^f) - C(\mathbf{r},\mathbf{q}) \geq 0$$

for all $\mathbf{z}^f$ and $g\left(\mathbf{r},\mathbf{q},\mathbf{z}^{f^0}\right) = 0$ when $\mathbf{z}^{f^0} = \mathbf{z}^f(\mathbf{r},\mathbf{q})$. Therefore, $g\left(\mathbf{r},\mathbf{q},\mathbf{z}^{f^0}\right)$ is at a minimum. Differentiating with respect to q_m gives us first and second order conditions

CM10 $$\frac{\partial C\left(\mathbf{r},\mathbf{q},\mathbf{z}^{f^0}\right)}{\partial q_m} - \frac{\partial C(\mathbf{r},\mathbf{q})}{\partial q_m} = 0 \text{ and}$$

CM11 $$\frac{\partial^2 C\left(\mathbf{r},\mathbf{q},\mathbf{z}^{f^0}\right)}{\partial {q_m}^2} - \frac{\partial^2 C(\mathbf{r},\mathbf{q})}{\partial {q_m}^2} \geq 0$$

that must be satisfied at a minimum. Equation CM10 says that the short-run and long-run marginal cost of producing q_m are equal at the long-run minimum cost. Equation CM11 says that the short-run marginal cost of producing q_m must intersect the long-run marginal cost of producing q_m from below at the long-run minimum cost. Note that similar arguments can be made to show that

CM10 $$0 \geq \frac{\partial z_n^v\left(\mathbf{r},\mathbf{q},\mathbf{z}^{f^0}\right)}{\partial r_n^v}\frac{r_n^v}{z_n^{v^0}} \geq \frac{\partial z_n^v(\mathbf{r},\mathbf{q})}{\partial r_n^v}\frac{r_n^v}{z_n^{v^0}} \text{ or } \varepsilon_{z_n^v}^{sr} \leq \varepsilon_{z_n^v}^{lr}$$

where $\varepsilon_{z_n^v}^{sr} = \left|\frac{\partial z_n^v\left(\mathbf{r},\mathbf{q},\mathbf{z}^{f^0}\right)}{\partial r_n^v}\frac{r_n^v}{z_n^{v^0}}\right|$ and $\varepsilon_{z_n^v}^{lr} = \left|\frac{\partial z_n^v(\mathbf{r},\mathbf{q})}{\partial r_n^v}\frac{r_n^v}{z_n^{v^0}}\right|$ are the short- and long-run *Conditional Input Demand Own Price Elasticities.*

Returning to our example, suppose z_2 is fixed in the short-run at z_2^0 and let us find the short-run input distance function, conditional factor demand, and cost function. First, the short-run input distance function is defined by

$$D_I(\mathbf{q},\mathbf{z}) \equiv \max_\delta \left\{\delta > 0: \sqrt{\left(\frac{z_1}{\delta}\right) z_2{}^0} \geq q_1{}^2 + q_2{}^2\right\} \text{ or}$$

$$D_I(\mathbf{q},\mathbf{z}) = \frac{z_1 z_2{}^0}{(q_1{}^2+q_2{}^2)^2}.$$

To find the conditional demand for z_1, we know $D_I(\mathbf{q},\mathbf{z}) = 1$ such that $z_1(\mathbf{q}, z_2{}^0) = \frac{(q_1{}^2+q_2{}^2)^2}{z_2{}^0}$ and $C(\mathbf{r},\mathbf{q}, z_2{}^0) = r_1 \frac{(q_1{}^2+q_2{}^2)^2}{z_2{}^0} + r_2 z_2{}^0$ where $C^v(r_1,\mathbf{q}, z_2{}^0) = r_1 \frac{(q_1{}^2+q_2{}^2)^2}{z_2{}^0}$ and $C^f(r_2, z_2{}^0) = r_2 z_2{}^0$ are our variable and fixed costs. Now recall from above that $z_2(\mathbf{r},\mathbf{q}) = \sqrt{\frac{r_1}{r_2}}(q_1{}^2 + q_2{}^2)$ in the long-run. Substitution then implies $z_1(\mathbf{r},\mathbf{q}) = z_1\big(\mathbf{q}, z_2(\mathbf{r},\mathbf{q})\big) = \frac{(q_1{}^2+q_2{}^2)^2}{\sqrt{\frac{r_1}{r_2}}(q_1{}^2+q_2{}^2)} = \sqrt{\frac{r_2}{r_1}}(q_1{}^2 + q_2{}^2)$ and $C(\mathbf{r},\mathbf{q}) = C\big(\mathbf{r},\mathbf{q}, z_2(\mathbf{r},\mathbf{q})\big) = r_1 \frac{(q_1{}^2+q_2{}^2)^2}{\sqrt{\frac{r_1}{r_2}}(q_1{}^2+q_2{}^2)} + r_2\sqrt{\frac{r_1}{r_2}}(q_1{}^2 + q_2{}^2) = 2\sqrt{r_1 r_2}(q_1{}^2 + q_2{}^2)$ as expected.

Another way to get to the long-run cost minimizing z_2 is to minimize $C(r_1, r_2, q_1, q_2, z_2{}^0)$ with respect to $z_2{}^0$:

$$\frac{\partial C(\mathbf{r},\mathbf{q},z_2{}^0)}{\partial z_2{}^0} = -r_1 \frac{(q_1{}^2+q_2{}^2)^2}{z_2{}^{0^2}} + r_2 = 0 \text{ and}$$

$$\frac{\partial^2 C(\mathbf{r},\mathbf{q},z_2{}^0)}{\partial z_2{}^{0^2}} = 2r_1 \frac{(q_1{}^2+q_2{}^2)^2}{z_2{}^{0^3}} \geq 0$$

implying $z_2(\mathbf{r},\mathbf{q}) = \sqrt{\frac{r_1}{r_2}}(q_1{}^2 + q_2{}^2)$ as expected.

COST MINIMIZATION WITH A SINGLE OUTPUT

So far we have framed the cost minimization problem quite generally in terms of many outputs and many inputs. All of the results we have obtained are applicable to the more familiar case of a single output and many inputs. We will now review this case and in the process, show how some well-known properties of average and marginal cost curves can be formally estblished.

In the classic single output and many input model, we are used to defining a production function as $q = f(\mathbf{z})$, which tells us the most output that can be obtained from a given combination of inputs. We often employ a monotonicity/free disposal assumption that says q is feasible for any $\mathbf{z}$ such that $f(\mathbf{z}) \geq q$ and write the cost minimization problem as

CM1'' $\min_{\mathbf{z}\geq 0} \mathbf{r} \cdot \mathbf{z}$ subject to $f(\mathbf{z}) \geq q$.

Assuming $f(\mathbf{z})$ is concave and differentiable, we can use Lagrangian methods:

CM2'' $L = \mathbf{r} \cdot \mathbf{z} + \lambda\big(q - f(\mathbf{z})\big)$,

with the first-order conditions

CM3'' $\frac{\partial L}{\partial z_n} = r_n - \lambda^* \frac{\partial f(\mathbf{z}^*)}{\partial z_n} \geq 0$, $\frac{\partial L}{\partial z_n} z_n{}^* = 0$, and $z_n{}^* \geq 0$ for $n = 1,\ldots, N$,

CM4'' $\frac{\partial L}{\partial \gamma} = q - f(\mathbf{z}^*) \leq 0$, $\frac{\partial L}{\partial \lambda}\lambda^* = 0$, and $\lambda^* \geq 0$.

As before, the solution is the conditional input demand $\mathbf{z}(\mathbf{r}, q)$, which we can use to get our cost function $C(\mathbf{r}, q) = \mathbf{r} \cdot z(\mathbf{r}, q)$. For any $z_n{}^* > 0$ and $z_{n'}{}^* \geq 0$, equation CM3'' implies $\frac{r_{n'}}{r_n} \geq \frac{\frac{\partial f(\mathbf{z}^*)}{\partial z_{n'}}}{\frac{\partial f(\mathbf{z}^*)}{\partial z_n}}$, which holds with equality when $z_{n'}{}^* > 0$. Above we showed that weak free disposal of inputs implied $D_I\big(\mathbf{q}, \mathbf{z}(\mathbf{r}, \mathbf{q})\big) = 1$. Similar arguments can be used to show that $q = f\big(\mathbf{z}(\mathbf{r}, \mathbf{q})\big)$, such that

$$C(\mathbf{r},q) = \mathbf{r}\cdot\mathbf{z}(\mathbf{r},q) + \lambda(\mathbf{r},q)\left(q - f\big(\mathbf{z}(\mathbf{r},q)\big)\right).$$

Differentiating with respect to q

CM11 $\frac{\partial C(\mathbf{r},q)}{\partial q} = \sum_{n=1}^{N}\frac{\partial z_n(\mathbf{r},q)}{\partial q}\left(r_n - \lambda(\mathbf{r},q)\frac{\partial f(z(\mathbf{r},q))}{\partial q}\right) + \frac{\partial \lambda(\mathbf{r},q)}{\partial q}\left(q - f\big(z(\mathbf{r},q)\big)\right) + \lambda(\mathbf{r},q).$

Differentiating equation CM3'' with respect to q and rearranging implies

CM12

$$\frac{\partial z_n(\mathbf{r},q)}{\partial q}\left(r_n - \lambda(\mathbf{r},q)\frac{\partial f(z(\mathbf{r},q))}{\partial z_n}\right)$$
$$= z_n(\mathbf{r},q)\left(\lambda(\mathbf{r},q)\sum_{n'=1}^{N}\frac{\partial^2 f(z(\mathbf{r},q))}{\partial z_n \partial z_{n'}}\frac{\partial z_{n'}(\mathbf{r},q)}{\partial q} + \frac{\partial \lambda(\mathbf{r},q)}{\partial q}\frac{\partial f(z(\mathbf{r},q))}{\partial z_n}\right) = 0$$

because $r_n - \lambda(\mathbf{r},q)\frac{\partial f(z(\mathbf{r},q))}{\partial z_n} = 0$ or $z_n(\mathbf{r},q) = 0$ by equation CM3''. Therefore, equation CM11 implies $\frac{\partial C(\mathbf{r},q)}{\partial q} = \lambda(\mathbf{r},q)$. That is, the optimal value of the Lagrange multiplier is the marginal cost of production, which also implies $\frac{\partial C(\mathbf{r},q)}{\partial q} = \frac{r_n}{\frac{\partial f(\mathbf{z}^*)}{\partial z_n}}$ for $z_n{}^* > 0$. Intuitively, the marginal cost of production is equal to an input's price divided by its *Marginal Product*.

Recall from above that we showed $C(\mathbf{r},\mathbf{q}) = \gamma(\mathbf{r},\mathbf{q})$ when using the input distance function to solve the problem, which is a different interpretation for the Lagrange multiplier. *While the conditional input demands for both problems are identical, the Lagrange multipliers are not because the constraint is defined differently in each problem. Therefore, it is important to remember that how we frame our constraints to production affects the interpretation of our Lagrange multiplier.*

The average cost of production is defined as

CM13 $AC(\mathbf{r},q) = \frac{C(\mathbf{r},q)}{q}.$

Differentiating with respect to q twice, we get

CM14 $\frac{\partial AC(\mathbf{r},q)}{\partial q} = \frac{\frac{\partial C(\mathbf{r},q)}{\partial q} - AC(\mathbf{r},q)}{q}$ and

CM15 $\frac{\partial^2 AC(\mathbf{r},q)}{\partial q^2} = \frac{\left(\frac{\partial^2 C(\mathbf{r},q)}{\partial q^2} - \frac{\partial AC(\mathbf{r},q)}{\partial q}\right)q - \left(\frac{\partial C(\mathbf{r},q)}{\partial q} - AC(\mathbf{r},q)\right)}{q^2}$.

If average cost is at a minimum q^{min}, equation CM14 must equal 0 and equation CM15 must be non-negative, which implies

CM14' $\frac{\partial C(\mathbf{r},q^{min})}{\partial q} = AC(\mathbf{r}, q^{min})$ and

CM15' $\frac{\partial^2 C(\mathbf{r},q^{min})}{\partial q^2} \geq 0$.

Equations CM14' and CM15' say that at the minimum average cost, marginal cost equals average cost and the marginal cost is non-decreasing, which are conditions that should be familiar from previous coursework that discussed cost curves.

REVENUE MAXIMIZATION

The cost minimization problem lets us say quite a bit about input choices under the assumption of cost minimizing behavior and competitive input markets. We can also say a lot about output choices under the assumption of revenue maximization and competitive output markets. The revenue maximization problem can be framed generally as

RM1 $\mathbf{Q}(\mathbf{p}, \mathbf{z}) = \{\mathbf{q} \in \mathbf{FOS}(\mathbf{z}): \mathbf{p} \cdot \mathbf{q} \geq \mathbf{p} \cdot \mathbf{q}' \text{ for all } \mathbf{q}' \in \mathbf{FOS}(\mathbf{z})\}$

where $\mathbf{Q}(\mathbf{p}, \mathbf{z})$ is referred to as the *Conditional Supply*. With weak free disposal of output, $\mathbf{FOS}(\mathbf{z}) = \{\mathbf{q}: D_O(\mathbf{q}, \mathbf{z}) \leq 1\}$ such that another framing of the revenue maximization problem is

RM1' $\max_{\mathbf{q} \geq 0} \mathbf{p} \cdot \mathbf{q}$ subject to $D_O(\mathbf{q}, \mathbf{z}) \leq 1$.

The *Revenue Function* is defined as

RM6 $R(\mathbf{p}, \mathbf{z}) = \mathbf{p} \cdot \mathbf{q}^* = \mathbf{p} \cdot \mathbf{q}(\mathbf{p}, \mathbf{z})$.

It represents the maximum revenue possible given output prices and inputs.

Like the conditional factor demand and cost function, the conditional supply and revenue function satisfy a variety of properties. Showing these properties hold requires arguments that are similar to those made above for the conditional input demands and cost function.

If $\mathbf{Q}(\mathbf{p}, \mathbf{z})$ is the conditional supply and $R(\mathbf{p}, \mathbf{z})$ is the revenue function derived from a feasible output set that is nonempty, closed, and satisfies weak free disposal,

(i) $\mathbf{Q}(\mathbf{p}, \mathbf{z})$ is homogeneous of degree zero and $R(\mathbf{p}, \mathbf{z})$ is homogeneous of degree one in $\mathbf{p}$;

(ii) $(\mathbf{p}^1 - \mathbf{p}^0)\cdot(\mathbf{q}(\mathbf{p}^1, \mathbf{z}) - \mathbf{q}(\mathbf{p}^0, \mathbf{z})) \geq 0$ for all $\mathbf{p}^1$ and $\mathbf{p}^0$;

(iii) $R(\mathbf{p}, \mathbf{z})$ is a convex and continuous function of $\mathbf{p}$;

(iv) $R(\mathbf{p}, \mathbf{z}^1) \geq R(\mathbf{p}, \mathbf{z}^0)$ for $\mathbf{z}^1 \geq \mathbf{z}^0$ if $\mathbf{IRS}(\mathbf{q})$ satisfies strong free disposal and $R(\mathbf{p}, \alpha\mathbf{z}) \geq R(\mathbf{p}, \mathbf{z})$ for $\alpha \geq 1$ if $\mathbf{IRS}(\mathbf{q})$ satisfies weak free disposal;

(v) $\mathbf{Q}(\mathbf{p}, \mathbf{z})$ is a convex/singleton set for all $\mathbf{p}$ if $\mathbf{FOS}(\mathbf{z})$ is convex/strictly convex;

(vi) $R(\mathbf{p}, \mathbf{z})$ is differentiable with respect to $\mathbf{p}$ at $\mathbf{p}^0$ and $\frac{\partial R(\mathbf{p}^0, \mathbf{z})}{\partial p_m} = q_m(\mathbf{p}^0, \mathbf{z})$ for all m if $\mathbf{Q}(\mathbf{p}^0, \mathbf{z})$ is a singleton set;

(vii) $\mathbf{D_p}^2\mathbf{R}(\mathbf{p}^0, \mathbf{z}) = \mathbf{D_p}\mathbf{q}(\mathbf{p}^0, \mathbf{z})$ are symmetric and positive semi-definite matrices with $\mathbf{D_p}\mathbf{q}(\mathbf{p}^0, \mathbf{z}) \cdot \mathbf{p}^0 = \mathbf{0}_M$ if $\mathbf{q}(\mathbf{p}^0, \mathbf{z})$ is differentiable at p^0; and

(viii) $\mathbf{Q}(\mathbf{p}, \mathbf{z})$ and $R(\mathbf{p}, \mathbf{z})$ are homogeneous of degree one in $\mathbf{z}$ if **PPS** exhibits constant returns to scale.