AGGREGATE SUPPLY

APEC 8002, Fall 2016 **Instructor: Terry Hurley**

INTRODUCTION

Our study of individual producers under cost minimization, revenue maximization, and profit maximization has provided us with some interesting results and testable hypotheses. The question we want to ask now is to what extent these results provide us with useful insights and testable hypotheses for aggregate supply and input demand. Understanding the extent to which aggregate supply and input demand behave like individual supply and input demand is of importance because often times the data we have available is at an aggregate rather than individual level, though this is becoming less of an issue with researchers having greater access to detailed micro level data on individual firms.

AGGREGATE SUPPLY

Suppose we have $j = 1,\ldots, J$ producers each with a possibly different Production Possibility Set $\mathbf{PPS}_j$. To keep our discussion manageable, we will assume the Production Possibility Sets are nonempty, closed, convex and exhibit free disposal. We will also assume that each producer's profit maximization problem has a solution yielding the supply $\mathbf{Y}^j(\mathbf{p})$ and profit function $\pi_j(\mathbf{p})$. Getting aggregate supply and aggregate profit is easy enough:

AS1 $\mathbf{Y}^A(p) = \left\{\sum_{j=1}^{J} \mathbf{y}^j : \mathbf{y} \in \times_{j=1}^{J} \mathbf{Y}^j(\mathbf{p})\right\}$ and

AS2 $\pi^A(\mathbf{p}) = \sum_{j=1}^{J} \pi_j(\mathbf{p})$.

Before moving forward, let us think about equation AS1 and what $\mathbf{y}$ and $\times_{j=1}^{J} \mathbf{Y}^j(\mathbf{p})$ really are. First, $\times_{j=1}^{J} \mathbf{Y}^j(\mathbf{p})$ is the set of all possible combinations of supply with one supply for each producer. Since the supply for each firm is a vector with L elements, $\mathbf{y}$ can be thought of as a matrix with J columns and L rows. The lth and jth element of $\mathbf{y}$ refers to the jth firm's supply of the lth commodity. $\sum_{j=1}^{J} \mathbf{y}^j$ is then just a vector of the sum of firms' supplies for each of the L commodities.

With our assumptions, we know that individual supply and profit functions will satisfy:

(i) $\mathbf{Y}^j(\mathbf{p})$ is homogeneous of degree zero and $\pi_j(\mathbf{p})$ is homogeneous of degree 1 in $\mathbf{p}$;

(ii) $(\mathbf{p}^1 - \mathbf{p}^0) \cdot (\mathbf{y}^j(\mathbf{p}^1) - \mathbf{y}^j(\mathbf{p}^0)) \geq 0$ for all $\mathbf{p}^1$ and $\mathbf{p}^0$;

(iii) $\pi_j(\mathbf{p})$ is convex and continuous in $\mathbf{p}$;

(iv) $\mathbf{Y}^j(\mathbf{p})$ is a convex/singleton set for all $\mathbf{p}$ if $\mathbf{PPS}_j$ is convex/strictly convex;

(v) $\pi_j(\mathbf{p})$ is differentiable with respect to $\mathbf{p}$ at $\mathbf{p}^0$ and $\frac{\partial \pi_j(\mathbf{p}^0)}{\partial p_l} = y_l^j(\mathbf{p}^0)$ for $l = 1,\ldots, L$ if $\mathbf{Y}^j(\mathbf{p}^0)$ is a singleton set (Hotelling's Lemma);

(vi) $\mathbf{D_p y}^j(\mathbf{p}^0) = \mathbf{D_p}^2\pi_j(\mathbf{p}^0)$ is a symmetric and positive semi-definite matrix with $\mathbf{D_p y}^j(\mathbf{p}^0)\,\mathbf{p}^0 = \mathbf{D_p}^2\pi_j(\mathbf{p}^0)\,\mathbf{p}^0 = \mathbf{0}^L$ if $\mathbf{y}^j(\mathbf{p}^0)$ is differentiable at $\mathbf{p}^0$.

Question: Do these properties hold for $\mathbf{y}^A(\mathbf{p})$ and $\pi^A(\mathbf{p})$?

Answer: Yes they do.

Let us start with the homogeneity of aggregate profit. If $\pi^A(\mathbf{p})$ is homogeneous of degree 1 in $\mathbf{p}$ then $\pi^A(\alpha\mathbf{p}) = \alpha\pi^A(\mathbf{p})$ for any $\alpha > 0$. By definition, $\pi^A(\alpha\mathbf{p}) = \sum_{j=1}^J \pi_j(\alpha\mathbf{p})$. Since $\pi_j(\mathbf{p})$ is homogeneous of degree 1 in $\mathbf{p}$, $\pi_j(\alpha\mathbf{p}) = \alpha\pi_j(\mathbf{p})$. Substituting then yields

$$\pi^A(\alpha\mathbf{p}) = \sum_{j=1}^J \alpha\pi_j(\mathbf{p}) = \alpha\sum_{j=1}^J \pi_j(\mathbf{p}) = \alpha\pi^A(\mathbf{p})$$

as desired. Similar arguments can be used to establish that $\mathbf{Y}^A(\mathbf{p})$ is homogeneous of degree zero in $\mathbf{p}$.

For property (ii), $(\mathbf{p}^1 - \mathbf{p}^0) \cdot (\mathbf{y}^j(\mathbf{p}^1) - \mathbf{y}^j(\mathbf{p}^0)) \geq 0$ for all j. Summing then implies

$$\sum_{j=1}^J (\mathbf{p}^1 - \mathbf{p}^0) \cdot \left(\mathbf{y}^j(\mathbf{p}^1) - \mathbf{y}^j(\mathbf{p}^0)\right) =$$

$$(\mathbf{p}^1 - \mathbf{p}^0) \cdot \left(\sum_{j=1}^J \mathbf{y}^j(\mathbf{p}^1) - \sum_{j=1}^J \mathbf{y}^j(\mathbf{p}^0)\right) =$$

$$(\mathbf{p}^1 - \mathbf{p}^0) \cdot \left(\mathbf{y}^A(\mathbf{p}^1) - \mathbf{y}^A(\mathbf{p}^0)\right) \geq 0.$$

For the convexity of $\pi^A(\mathbf{p})$ in $\mathbf{p}$, the key is to recognize that in general the sum of convex functions is convex. For $\pi^A(\mathbf{p})$ to be convex in price

$$\alpha\pi^A(\mathbf{p}^0) + (1 - \alpha)\pi^A(\mathbf{p}^1) \geq \pi^A(\alpha\mathbf{p}^0 + (1 - \alpha)\mathbf{p}^1) \text{ for all } \mathbf{p}^0, \mathbf{p}^1, \text{and } \alpha \in [0, 1].$$

We know that $\pi_j(\mathbf{p})$ is convex in $\mathbf{p}$ for all j. Therefore,

$$\alpha\pi_j(\mathbf{p}^0) + (1 - \alpha)\pi_j(\mathbf{p}^1) \geq \pi_j(\alpha\mathbf{p}^0 + (1 - \alpha)\mathbf{p}^1).$$

If we sum over all j we then get

AS3 $$\sum_{j=1}^{J}\left(\alpha\pi_j(\mathbf{p}^0) + (1-\alpha)\pi_j(\mathbf{p}^1)\right) \geq \sum_{j=1}^{J}\pi_j(\alpha\mathbf{p}^0 + (1-\alpha)\mathbf{p}^1).$$

The left-hand side of equation AS3 can be rewritten as

$$\begin{aligned}\sum_{j=1}^{J}\alpha\pi_j(\mathbf{p}^0) + \sum_{j=1}^{J}(1-\alpha)\pi_j(\mathbf{p}^1) &= \alpha\sum_{j=1}^{J}\pi_j(\mathbf{p}^0) + (1-\alpha)\sum_{j=1}^{J}\pi_j(\mathbf{p}^1)\\ &= \alpha\pi^A(\mathbf{p}^0) + (1-\alpha)\pi^A(\mathbf{p}^1),\end{aligned}$$

while the right-hand side is just $\pi^A(\alpha\mathbf{p}^0 + (1-\alpha)\mathbf{p}^1)$, which is the desired result. For continuity, the sum of continuous function is also continuous.

For (iv), note that $\mathbf{Y}^A(\mathbf{p})$ is a convex set if for all $\mathbf{y}^0, \mathbf{y}^1 \in \mathbf{Y}^A(\mathbf{p})$ and $\alpha \in [0, 1]$, then $\alpha\,\mathbf{y}^0 + (1 - \alpha)\,\mathbf{y}^1 \in \mathbf{Y}^A(\mathbf{p})$. Suppose this is not the case such that there is some $\alpha^0 \in [0, 1]$ where $\alpha^0\,\mathbf{y}^0 + (1 - \alpha^0)\,\mathbf{y}^1 \notin \mathbf{Y}^A(\mathbf{p})$. By equation AS1, $\mathbf{y}^0 = \sum_{j=1}^{J}\mathbf{y}^{j^0}$ where $\mathbf{y}^{j^0} \in \mathbf{Y}^j(\mathbf{p})$ for all j and $\mathbf{y}^1 = \sum_{j=1}^{J}\mathbf{y}^{j^1}$ where $\mathbf{y}^{j^1} \in \mathbf{Y}^j(\mathbf{p})$ for all j. By the convexity of $\mathbf{Y}^j(\mathbf{p})$, $\alpha^0\mathbf{y}^{j^0} + (1-\alpha^0)\mathbf{y}^{j^1} \in \mathbf{Y}^j(\mathbf{p})$. Summing and equation AS1 then imply $\sum_{j=1}^{J}\left(\alpha^0\mathbf{y}^{j^0} + (1-\alpha^0)\mathbf{y}^{j^1}\right) \in \mathbf{Y}^A(\mathbf{p})$. But

$$\sum_{j=1}^{J}\left(\alpha^0\mathbf{y}^{j^0}+(1-\alpha^0)\mathbf{y}^{j^1}\right)=\alpha^0\mathbf{y}^0+(1-\alpha^0)\mathbf{y}^1$$

yielding contradiction that $\alpha^0\mathbf{y}^0+(1-\alpha^0)\mathbf{y}^1\in\mathbf{Y}^A(\mathbf{p})$. Similar arguments can be used to establish that $\mathbf{Y}^A(\mathbf{p})$ is a singleton set with strict convexity.

Property (v) is easy enough to establish by differentiating equation AS2:

AS4 $\frac{\partial\pi^A(\mathbf{p})}{\partial p_l}=\sum_{j=1}^{J}\frac{\partial\pi_j(\mathbf{p})}{\partial p_l}=\sum_{j=1}^{J}y_l^j(\mathbf{p})=y_l^A(\mathbf{p})$ for all l.

Similarly for property (vi), note that

AS5
$$\mathbf{D_p}\mathbf{y}^A(\mathbf{p})=\mathbf{D}_{\mathbf{p}}^2\pi^A(\mathbf{p})=\begin{bmatrix}\frac{\partial^2\pi^A(\mathbf{p})}{\partial {p_1}^2} & \cdots & \frac{\partial^2\pi^A(\mathbf{p})}{\partial p_1\partial p_L}\\ \vdots & \ddots & \vdots\\ \frac{\partial^2\pi^A(\mathbf{p})}{\partial p_L\partial p_1} & \cdots & \frac{\partial^2\pi^A(\mathbf{p})}{\partial {p_L}^2}\end{bmatrix}$$

$$=\begin{bmatrix}\sum_{j=1}^{J}\frac{\partial^2\pi_j(\mathbf{p})}{\partial {p_1}^2} & \cdots & \sum_{j=1}^{J}\frac{\partial^2\pi_j(\mathbf{p})}{\partial p_1\partial p_L}\\ \vdots & \ddots & \vdots\\ \sum_{j=1}^{J}\frac{\partial^2\pi_j(\mathbf{p})}{\partial p_L\partial p_1} & \cdots & \sum_{j=1}^{J}\frac{\partial^2\pi_j(\mathbf{p})}{\partial {p_L}^2}\end{bmatrix}$$

$$=\sum_{j=1}^{J}\mathbf{D_p}\mathbf{y}^j(\mathbf{p}).$$

We know that $\mathbf{D_p}\mathbf{y}^j(\mathbf{p})$ is symmetric and positive semi-definite for all j. Adding a symmetric and positive semi-definite matrix to another symmetric and positive semi-definite matrix yields a symmetric and positive semi-definite matrix — the desired result. Also,

AS6 $\mathbf{D_p}\mathbf{y}^A(\mathbf{p})\mathbf{p}=\sum_{j=1}^{J}\mathbf{D_p}\mathbf{y}^j(\mathbf{p})\mathbf{p}=0$

because $\mathbf{D_p y}^j(\mathbf{p})\mathbf{p} = 0$ for all j.

To summarize, if $\mathbf{Y}^A(\mathbf{p})$ and $\pi^A(\mathbf{p})$ are the aggregate supply and profit function derived from individual producers with Production Possibility Sets that are nonempty, closed, and satisfy free disposal, then:

(i) $\mathbf{Y}^A(\mathbf{p})$ is homogeneous of degree zero and $\pi^A(\mathbf{p})$ is homogeneous of degree 1 in $\mathbf{p}$;

(ii) $(\mathbf{p}^1 - \mathbf{p}^0) \cdot (\mathbf{y}^A(\mathbf{p}^1) - \mathbf{y}^A(\mathbf{p}^0)) \geq 0$ for all $\mathbf{p}^1$ and $\mathbf{p}^0$;

(iii) $\pi^A(\mathbf{p})$ is convex and continuous in $\mathbf{p}$;

(iv) $\mathbf{Y}^A(\mathbf{p})$ is a convex/singleton set for all $\mathbf{p}$ if $\mathbf{PPS}_j$ is convex/strictly convex for all j;

(v) $\pi^A(\mathbf{p})$ is differentiable with respect to $\mathbf{p}$ at $\mathbf{p}^0$ and $\frac{\partial \pi^A(\mathbf{p}^0)}{\partial p_l} = y_l{}^A(\mathbf{p}^0)$ for $l = 1,\ldots, L$ if $\mathbf{Y}^A(\mathbf{p}^0)$ is a singleton set; and

(vi) $\mathbf{D_p y}^A(\mathbf{p}^0) = \mathbf{D_p}^2\pi^A(\mathbf{p}^0)$ is a symmetric and positive semi-definite matrix with $\mathbf{D_p y}^A(\mathbf{p}^0)\,\mathbf{p}^0 = \mathbf{D_p}^2\pi^A(\mathbf{p}^0)\,\mathbf{p}^0 = \mathbf{0}^L$ if $\mathbf{y}^A(\mathbf{p}^0)$ is differentiable at $\mathbf{p}^0$.

Another implication of all this is that if we define the aggregate production possibilities set as

AS7 $\mathbf{PPS}^A = \left\{\sum_{j=1}^{J} \mathbf{y}_j : \mathbf{y} \in \times_{j=1}^{J} \mathbf{PPS}_j\right\}$,

then

AS1' $\mathbf{Y}^A(\mathbf{p}) = \{\mathbf{y} \in \mathbf{PPS}^A : \mathbf{p} \cdot \mathbf{y} \geq \mathbf{p} \cdot \mathbf{y}'$ for all $\mathbf{y}' \in \mathbf{PPS}^A\}$ and

AS2' $\pi^A(\mathbf{p}) = \mathbf{p} \cdot \mathbf{y}^A(\mathbf{p})$.

AGGREGATE COST

With the profit function, aggregation was a relatively straightforward exercise, which makes it tempting to presume it will be equally straightforward for cost functions. Unfortunately, this is not the case. With the profit function, we assume producers choose the optimal level of outputs and inputs based on the competitive prices for outputs and inputs. In our previous discussions, we have seen that this implies producers will equate the MRTS to the ratio of input prices and the MRT to the ratio of output prices. If all firms face the same prices, this will result in all of them equating their MRTSs and MRTs, which can be shown to result in efficient production across producers. With the cost function, we assume producers choose the optimal level of inputs given competitive input prices and some allocation of output. As with profit maximization, this will imply that producers equate their MRTSs, but it will not necessarily mean that they equate their MRTs, which can be shown to result in inefficient production across firms.

To develop a better understanding of this issue and its implications, let $D_I^j(\mathbf{q}^j, \mathbf{z}^j)$ be the jth firm's differentiable input distance function and $\mathbf{r}$ be a vector of input prices faced by all firms. As before, we will assume these input distance functions are derived from input requirement sets that are nonempty, closed, satisfy weak free disposal, and satisfy strict convexity. We know from our cost minimization problem that

AS8 $\frac{r_{n'}}{r_n} \geq \frac{\frac{\partial D_I^j(\mathbf{q}^j, \mathbf{z}^{j^*})}{\partial z_{n'}^j}}{\frac{\partial D_I^j(\mathbf{q}^j, \mathbf{z}^{j^*})}{\partial z_n^j}}$ for all $z_n^{j^*} > 0$ and $z_{n'}^{j^*} \geq 0$, and

AS9 $D_I(\mathbf{q}^j, \mathbf{z}^{j^*}) = 1$ for all j.

The solution to equations AS8 and AS9 is a set of conditional input demands for each producer that depend on input prices and its output: $\mathbf{z}^j(\mathbf{r}, \mathbf{q}^j)$ for all j. We know that this solution minimizes each producer's cost given $\mathbf{r}$ and $\mathbf{q}^j$.

Now let us consider the alternative problem of minimizing an industry's cost of producing $\mathbf{q}^A = \sum_{j=1}^{J} \mathbf{q}^j$:

AS10 $\min_{\mathbf{z}^1 \geq 0,\ldots,\mathbf{z}^J \geq 0, \mathbf{q}^1 \geq 0,\ldots,\mathbf{q}^J \geq 0}\ \mathbf{r} \cdot \sum_{j=1}^{J} \mathbf{z}^j$ subject to $\mathbf{q}^A = \sum_{j=1}^{J} \mathbf{q}^j$ and

$$D_I^j\left(\mathbf{q}^j, \mathbf{z}^j\right) \geq 1 \text{ for } j = 1,\ldots, J.$$

The Lagrangian and first order conditions for this problem are

AS11 $L = r \cdot \sum_{j=1}^{J} z^j + \sum_{j=1}^{J} \lambda^j \left(1 - D_I^j\left(\mathbf{q}^{j^*}, \mathbf{z}^{j^*}\right)\right) + \sum_{m=1}^{M} \mu^m \left(q_m^A - \sum_{j=1}^{J} q_m^j\right),$

AS12 $\frac{\partial L}{\partial z_n^j} = r_n - \lambda^{j^*} \frac{\partial D_I^j\left(\mathbf{q}^{j^*}, \mathbf{z}^{j^*}\right)}{\partial z_n^j} \geq 0,\ \frac{\partial L}{\partial z_n^j} z_n^{j^*} = 0$, and $z_n^{j^*} \geq 0$ for all n and j,

AS13 $\frac{\partial L}{\partial q_m^j} = \mu^{m^*} - \lambda^{j^*} \frac{\partial D_I^j\left(\mathbf{q}^{j^*}, \mathbf{z}^{j^*}\right)}{\partial q_m^j} \geq 0,\ \frac{\partial L}{\partial q_m^j} q_m^{j^*} = 0$, and $q_m^{j^*} \geq 0$ for all m and j,

AS14 $D_I\left(\mathbf{q}^{j^*}, \mathbf{z}^{j^*}\right) = 1$ and $\lambda^{j^*} \geq 0$for all j, and

AS15 $q_m^A = \sum_{j=1}^{J} q_m^j$ and $\mu^{m^*} \geq 0$ for all m.

The solution to this problem is a set of conditional input demands and supplies for each producer that depend on input prices and aggregate output: $\mathbf{z}^j(\mathbf{r}, \mathbf{q}^A)$ and $\mathbf{q}^j(\mathbf{r}, \mathbf{q}^A)$. Note that the implications of equation AS12 are identical to equation AS8. Similarly, the implications of equation AS14 are identical to equation AS9. What this tells us is that minimizing industry cost requires minimizing each producer's cost. However, equations AS12 and AS13 also tell us that

AS16 $$\frac{\frac{\partial D_I^{j'}\left(\mathbf{q}^{j'^*}, \mathbf{z}^{j'^*}\right)}{\partial z_n^{j'}}}{\frac{\partial D_I^{j}\left(\mathbf{q}^{j^*}, \mathbf{z}^{j^*}\right)}{\partial z_n^{j}}} = \frac{\frac{\partial D_I^{j'}\left(\mathbf{q}^{j'^*}, \mathbf{z}^{j'^*}\right)}{\partial q_m^{j'}}}{\frac{\partial D_I^{j}\left(\mathbf{q}^{j^*}, \mathbf{z}^{j^*}\right)}{\partial q_m^{j}}} \text{ for all } z_n^{j^*} > 0,\ z_n^{j'^*} > 0,\ q_m^{j^*} > 0 \text{ and } q_m^{j'^*} > 0$$

at the industry cost minimizing solution. Intuitively, minimizing each producer's cost is necessary, but not sufficient to minimize industry cost because how much each producer produces will matter. Producers that have a relatively low marginal cost should produce more than producers with higher marginal costs. Given our solution, we can write the industry cost function as

AS17 $$C^A(\mathbf{r},\mathbf{q}^A) = \mathbf{r}\cdot\sum_{j=1}^{J}\mathbf{z}^j(\mathbf{r},\mathbf{q}^A) = \sum_{j=1}^{J}C_j\left(\mathbf{r},\mathbf{q}^j(\mathbf{r},\mathbf{q}^A)\right)$$

because we know industry cost minimization requires individual producers to minimize cost.

Before looking at the properties of this industry cost function, it is useful to frame the problem from a little different perspective now that we know it is optimal for individual producers to minimize costs:

AS18 $$\min_{\mathbf{q}^1\geq 0,\ldots,\mathbf{q}^J\geq 0}\sum_{j=1}^{J}C_j\left(\mathbf{r},\mathbf{q}^j\right) \text{ subject to } \mathbf{q}^A = \sum_{j=1}^{J}\mathbf{q}^j.$$

The Lagrangian for this problem is

AS19 $$L = \sum_{j=1}^{J}C_j\left(\mathbf{r},\mathbf{q}^j\right) + \sum_{m=1}^{M}\gamma^m\left(q_m^A - \sum_{j=1}^{J}q_m^j\right)$$

which has the first order conditions

AS20 $$\frac{\partial L}{\partial q_m^j} = \frac{\partial C_j\left(\mathbf{r},\mathbf{q}^{j^*}\right)}{\partial q_m^j} - \gamma^{m*} \geq 0,\ \frac{\partial L}{\partial q_m^j}q_m^{j\,*} = 0, \text{ and } q_m^{j\,*} \geq 0 \text{ for all } m \text{ and } j,$$

AS21 $$q_m^A = \sum_{j=1}^{J}q_m^{j\,*} \text{ and } \gamma^{m*} \geq 0 \text{ for all } m.$$

Equation AS20 says that $\frac{\partial C_j\left(\mathbf{r},\mathbf{q}^{j^*}\right)}{\partial q_m^j} = \frac{\partial C_j\left(\mathbf{r},\mathbf{q}^{j'^*}\right)}{\partial q_m^{j'}}$ for all *j* and *j*’ that produce commodity *m*. That is, for industry cost minimization to occur, producers must equate their marginal cost of production for each output.

The first thing to recognize is that $\mathbf{z}^j(\mathbf{r}, \mathbf{q}^A)$ and $\mathbf{q}^j(\mathbf{r}, \mathbf{q}^A)$ will be homogenous of degree zero in **r**. To find these values, we can use equations AS8, AS16, AS14, and AS15. Of these equations, only AS8 contains prices or more specifically the ratio of prices. Multiplying all prices by the same positive number does not change this ratio. Also, since $\mathbf{z}^j(\mathbf{r}, \mathbf{q}^A)$ and $\mathbf{q}^j(\mathbf{r}, \mathbf{q}^A)$ are homogeneous of degree zero in **r**, it should be clear from AS17 that $C^A(\mathbf{r}, \mathbf{q}^A)$ is homogenous of degree 1 in **r**.

Let $\mathbf{z}^A(\mathbf{r}, \mathbf{q}^A) = \sum_{j=1}^{J} \mathbf{z}^j(\mathbf{r}, \mathbf{q}^A)$. We can then show just as before that

$$(\mathbf{r}^1 - \mathbf{r}^0) \cdot \left(\mathbf{z}^A(\mathbf{r}^1, \mathbf{q}^A) - \mathbf{z}^A(\mathbf{r}^0, \mathbf{q}^A)\right) \le 0$$

for all $\mathbf{r}^1$ and $\mathbf{r}^0$.

Concavity of $C^A(\mathbf{r}, \mathbf{q}^A)$ in **r** implies that for all $\alpha \in [0, 1]$, $\mathbf{r}^0$ and $\mathbf{r}^1$,

$$C^A(\alpha\mathbf{r}^0 + (1-\alpha)\mathbf{r}^1, \mathbf{q}^A) \ge \alpha C^A(\mathbf{r}^0, \mathbf{q}^A) + (1-\alpha) C^A(\mathbf{r}^1, \mathbf{q}^A)$$

which can also be written as

$$\alpha\left(\mathbf{r}^0 \cdot \mathbf{z}^A(\alpha\mathbf{r}^0 + (1-\alpha)\mathbf{r}^1, \mathbf{q}^A) - C^A(\mathbf{r}^0, \mathbf{q}^A)\right) + \\ (1-\alpha)\left(\mathbf{r}^1 \cdot \mathbf{z}^A(\alpha\mathbf{r}^0 + (1-\alpha)\mathbf{r}^1, \mathbf{q}^A) - C^A(\mathbf{r}^1, \mathbf{q}^A)\right) \ge 0.$$

This expression must be true by the definition of the cost function.

Differentiating aggregate cost with respect to q_m^A yields

AS22 $$\frac{\partial C^A(\mathbf{r}, \mathbf{q}^A)}{\partial q_m^A} = \sum_{m'=1}^{M} \sum_{j=1}^{J} \frac{\partial C_j\left(\mathbf{r}, \mathbf{q}^j(\mathbf{r}, \mathbf{q}^A)\right)}{\partial q_{m'}^j} \frac{\partial q_{m'}^j(\mathbf{r}, \mathbf{q}^A)}{\partial q_m^A}.$$

Equation AS20 implies

$$\frac{\partial c_j\left(\mathbf{r},\mathbf{q}^j(\mathbf{r},\mathbf{q}^A)\right)}{\partial q^j_{m'}}\frac{\partial q^j_{m'}(\mathbf{r},\mathbf{q}^A)}{\partial q^A_m} = \gamma^{m'}(\mathbf{r},\mathbf{q}^A)\frac{\partial q^j_{m'}(\mathbf{r},\mathbf{q}^A)}{\partial q^A_m}.$$

Also, totally differentiating $\sum_{j=1}^J q^j_m(\mathbf{r},\mathbf{q}^A) = q^A_m$ with respect to q^A_m implies $\sum_{j=1}^J \frac{\partial q^j_m(\mathbf{r},\mathbf{q}^A)}{\partial q^A_m} = 1$, while totally differentiating $\sum_{j=1}^J q^j_m(\mathbf{r},\mathbf{q}^A) = q^A_m$ with respect to $q^A_{m'}$ implies $\sum_{j=1}^J \frac{\partial q^j_m(\mathbf{r},\mathbf{q}^A)}{\partial q^A_{m'}} = 0$. Therefore, equation AS22 can be written as

AS22' $$\frac{\partial c^A(\mathbf{r},\mathbf{q}^A)}{\partial q^A_m} = \gamma^m(\mathbf{r},\mathbf{q}^A),$$

which equation AS21 implies must be non-decreasing. Now recall that to show a similar property for the individual cost function we need to assume strong free disposal of outputs. This assumption is still floating around. For AS20 to hold with positive output for some producer, the marginal cost must be increasing in that output.

Differentiating aggregate cost with respect to r_n yields

AS23 $$\frac{\partial c^A(\mathbf{r},\mathbf{q}^A)}{\partial r_n} = \sum_{j=1}^J z^j_n\left(\mathbf{r},\mathbf{q}^j(\mathbf{r},\mathbf{q}^A)\right) + \sum_{j=1}^J\sum_{m'=1}^M \frac{\partial c_j\left(\mathbf{r},\mathbf{q}^j(\mathbf{r},\mathbf{q}^A)\right)}{\partial q^j_{m'}}\frac{\partial q^j_{m'}(\mathbf{r},\mathbf{q}^A)}{\partial r_n}.$$

Equation AS20 implies

$$\frac{\partial c_j\left(\mathbf{r},\mathbf{q}^j(\mathbf{r},\mathbf{q}^A)\right)}{\partial q^j_{m'}}\frac{\partial q^j_{m'}(\mathbf{r},\mathbf{q}^A)}{\partial r_n} = \gamma^{m'}(\mathbf{r},\mathbf{q}^A)\frac{\partial q^j_{m'}(\mathbf{r},\mathbf{q}^A)}{\partial r_n}.$$

Since $\sum_{j=1}^J q^j_{m'}(\mathbf{r},\mathbf{q}^A) = q^A_{m'}$, totally differentiating with respect to r_n yields $\sum_{j=1}^J \frac{\partial q^j_{m'}(\mathbf{r},\mathbf{q}^A)}{\partial r_n} = 0$. Therefore, equation AS23 becomes

AS23' $$\frac{\partial c^A(\mathbf{r},\mathbf{q}^A)}{\partial r_n} = \sum_{j=1}^J z^j_n\left(\mathbf{r},\mathbf{q}^j(\mathbf{r},\mathbf{q}^A)\right) = z^A_n(\mathbf{r},\mathbf{q}^A),$$

which is the aggregate cost version of Shephard's lemma.

Given the concavity of $C^A(\mathbf{r}, \mathbf{q}^A)$ in $\mathbf{r}$, if $C^A(\mathbf{r}, \mathbf{q}^A)$ is twice continuously differentiable in $\mathbf{r}$, $\mathbf{D_r}^2 C^A(\mathbf{r}, \mathbf{q}^A) = \mathbf{D_r}\mathbf{z}^A(r, q^A)$ are symmetric and negative semi-definite matrices with $\mathbf{D_r}\mathbf{z}^A(\mathbf{r}, \mathbf{q}^A) \cdot \mathbf{r}^0 = \mathbf{0}^N$.

To summarize, an aggregate cost function will satisfy all the typical properties of an individual firm's cost function provided the output of each producer equates the marginal cost of production across all producers, which will be the case if we are looking at a perfectly competitive industry. Similar results can be derived for the revenue function, provided inputs are distributed across producers to equate their marginal revenue.

Before leaving to look at production under uncertainty, I should warn you that these results work out reasonably well because we have assumed all firms face the same price. Unfortunately, sometimes the data we get includes industry costs, output, and the average price of inputs, or it might include industry profit and an average price of input and outputs. Instances like these provide additional challenges for aggregation that we do not have time to discuss. Fortunately, this is a topic that is usually discussed in some detail in books focused on production (e.g., Chambers 1994).