# AFRE 802
# Statistical Methods for Agricultural, Food, & Resource Economists

**Introduction to Probability – Part 2 of 2 (WMS Ch. 2.7-2.13)**

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## GAME PLAN

- Collect Ch. 1 HW
- Review and questions from last class
- Graded in-class exercise
- Probability (cont'd)
    a. Conditional probabilities & independence
    b. Some other useful laws of probability
    c. The event-composition method for calculating the probability of an event
    d. The law of total probability & Bayes' Rule

We might not get through all of a-d today, which is fine. We'll pick up where we left off next time.

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## Any questions on additional practice problem?

- Answers:
- 1. Sample median = 4.5
- 2. Sample mean = 4.456
  Sample variance = 0.678
  Sample standard deviation = 0.823

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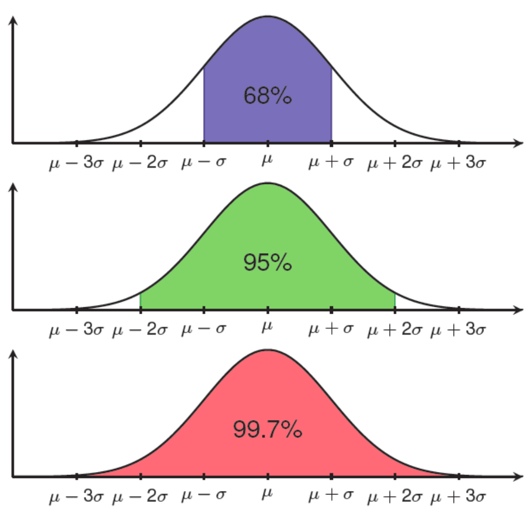
## REVIEW: The "empirical rule"

For a distribution that is <u>approximately</u> normal:

- of obs. are w/in $\sigma$ of $\mu$
- of obs. are w/in $2\sigma$ of $\mu$
- are w/in $3\sigma$ of $\mu$

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## REVIEW

- **Venn diagrams** & **set notation**: any questions?
- **Calculating probabilities:** If a sample space contains $N$ sample points that can occur with equal probability, and compound event $A$ contains $n_A$ of those sample points, then

$$P(A) = \frac{n_A}{N} = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}}$$

- *What are 4 tools that we can use to count sample points?*

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## REVIEW: 4 tools for counting sample points

1. **mn rule**: if there are $p$ groups where the 1st has $n_1$ elements, the 2nd has $n_2$ elements,…, and the $p^{th}$ has $n_p$ elements, the # of distinct sets containing one element from each group is: $\boxed{n_1 \times n_2 \times \ldots \times n_p}$
2. **Permutation**: # of ways of ordering $n$ distinct objects taken $r$ at a time (or # of ways of filling $r$ distinct positions drawing from $n$ distinct objects w/o replacement): $\boxed{P_r^n = \frac{n!}{(n-r)!}}$
3. **Combinations:** # of unordered subsets of size $r$ chosen from $n$ available objects (w/o replacement): $\boxed{\binom{n}{r} = C_r^n = \frac{n!}{r!(n-r)!}}$
4. **Partitioning:** number of ways of partitioning $n$ distinct objects into $k$ distinct groups where $\sum_{i=1}^{k} n_i = n$

$$\boxed{\binom{n}{n_1\ n_2\ \cdots\ n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}}$$

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## Two methods for calculating the probability of an event

1. The **sample point method** (last class)
2. The **event-composition method** (today)

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## REVIEW: The sample-point method for calculating the probability of an event

1. Define the experiment
2. Define the sample space ($S$) by identifying all of the possible <u>simple</u> events / outcomes (call these $E_i$)
3. Assign a probability to each simple event. Be sure these probabilities satisfy:
   $$0 \le P(E_i) \le 1 \quad \text{and} \quad \sum_i P(E_i) = 1$$
4. Define the event of interest (call it $A$) and decompose it into its component simple events
5. Find $P(A)$ by summing the probabilities of these simple events
   - Can do this b/c:

If $A_i \cap A_j = \varnothing$ for all $i \neq j$, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots A_k) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_k) = \sum_{i=1}^{k} P(A_i)$$

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## Sample point method example – any questions?

*If you randomly selected 2 days (without replacement) from a given 7-day week this semester, what is the probability that you would have AFRE 801 or 802 lecture both days? (Ignore weeks w/ holidays and the first week of class.)*

1. List the sample space and assign probabilities to each sample point.
   - {MTu}, {MW}, {MTh}, {MF}, {MSa}, {MSu}
     {TuW}, {TuTh}, {TuF}, {TuSa}, {TuSu}, {WTh}, {WF}, {WSa}, {WSu},
     {ThF}, {ThSa}, {ThSu}, {FSa}, {FSu}, {SaSu} = 21
     so prob of each is 1/21
   - Can check with combination: ${}_7C_2$=7!/(2!5!)=(7*6)/2=21
2. Event A is that 2 of the days picked are Mon-Thurs. Count the number of sample points that correspond to event A.
   - 6 sample points or use combination: ${}_4C_2$=4!/(2!2!)=(4*3)/2=6
3. Calculate the probability of event A.
   - 6/21=2/7

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## Graded in-class exercise

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## Summary

- An essential step in the **sample point method of calculating the probability of an event** is to identify the total number of sample points in the sample space and the number of sample points that correspond to the event of interest
- When there is a **small number of sample points**, we can **count** the sample points **by hand**
- But when there are **many sample points**, tools from **combinatorial analysis** (the *mn* rule, permutations, combinations, etc.) can help us do this **more efficiently and accurately**
- Alternative approach to calculating the probability of an event: the **event-composition method**. But first we need to go over some definitions/concepts.

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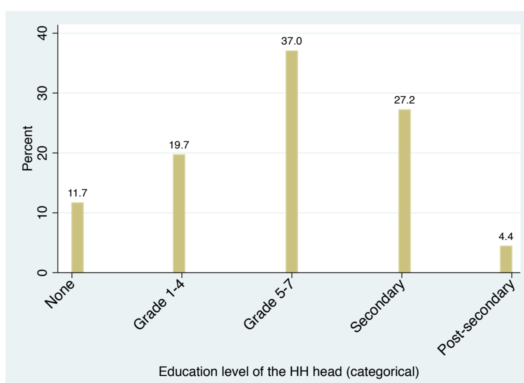
## Unconditional vs. conditional probability

- **Unconditional probability** –does not take into account information on other events that have already occurred
- **Conditional probability** – does take into account such info

Unconditional probability: e.g., probability of secondary education = 27.2%

Conditional probability: e.g., *what is the probability of secondary education* ***given that*** *the HH head has at least upper primary (grades 5-7) education* ?



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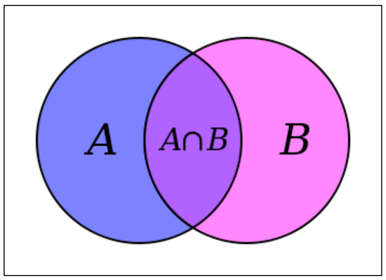
## Conditional probability - formally

*Using the Venn diagram as a guide, what is the probability of event A, given that event B has already occurred (i.e, what is P(A|B))?*

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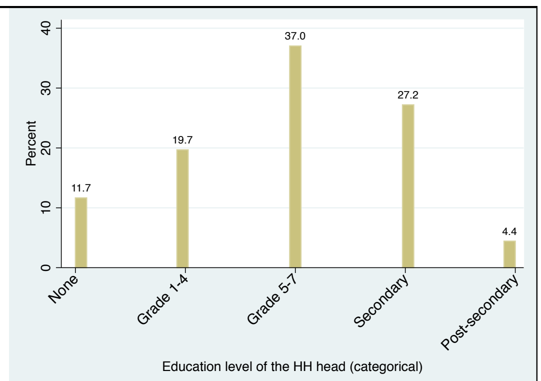
- P(A|B): “Given B” so focus on B (all pink)
- P(A|B): What’s the probability of getting A, given that you’re limited to B (all pink)?

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
assuming $P(B) \neq 0$

## Conditional probability - example

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$




- In our previous example, define events:
  - A: HH head has secondary education
  - B: HH head has at least upper primary education (grades 5-7)
  - *What is* $A \cap B$ *?*

b/c A is a subset of B [Venn diagram]

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{0.272}{0.686} = 0.397$$

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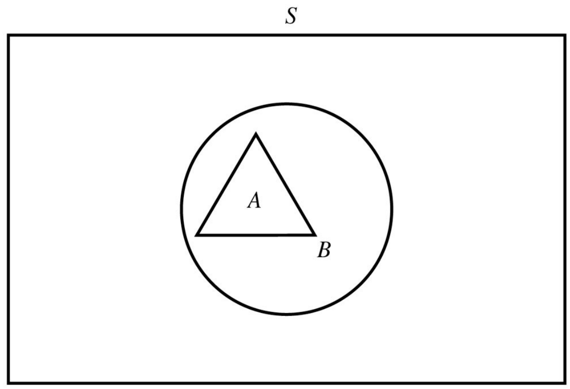
$$P(A \cap B) = P(A) \text{ if } A \subset B$$



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## Independence

- Intuitively – taking into account info on the other event doesn't affect the probability of the event in question
- Formally:

Two events $A$ and $B$ are **independent** if any one of the following holds:

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = P(B)$$

$$P(A \cap B) = P(A)P(B)$$

Otherwise, the events are **dependent**.

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## Independence - example

- A card is selected at random from a deck (52 cards). Define the following events:
  - *A*: The card is an ace
  - *D*: The card is a diamond
- *Are events A and D independent? (Divide class & check i vs. ii. vs. iii on previous slide)*

$$P(A)=\frac{4}{52}=\frac{1}{13},\qquad P(D)=\frac{13}{52}=\frac{1}{4}$$

$$(i)\ P(A\,|\,D)=\frac{P(A\cap D)}{P(D)}=\frac{1/52}{13/52}=\frac{1}{13}=P(A)$$

$$(ii)\ P(D\,|\,A)=\frac{P(D\cap A)}{P(A)}=\frac{1/52}{4/52}=\frac{1}{4}=P(D)$$

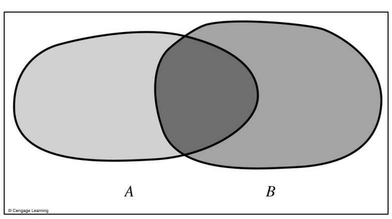
$$(iii)\ P(A\cap D)=\frac{1}{52},\qquad P(A)P(D)=\frac{4}{52}\cdot\frac{13}{52}=\frac{52}{52\cdot 52}=\frac{1}{52}$$





## Some additional useful laws of probability

$\boxed{1.\ P(A)=1-P(\bar{A})}$

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Using the Venn diagram as a guide, what is $P(A\cup B)$?

$\boxed{2a.\ P(A\cup B)=P(A)+P(B)-P(A\cap B)}$

What is $P(A\cap B)$ if $A$ and $B$ are mutually exclusive events?

$\rightarrow$ If $A$ and $B$ are mutually exclusive, $P(A\cup B)=P(A)+P(B)$

If rearrange terms in 2a, what is $P(A\cap B)$?

$\boxed{2b.\ P(A\cap B)=P(A)+P(B)-P(A\cup B)}$

## Some additional useful laws of probability

$$\boxed{\text{3a. } P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A)}$$

Recall that $P(A \cap B) = P(A)P(B)$ for independent events

3a comes from rearranging terms in our earlier results that:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ and likewise } P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

We can also rearrange the terms of the RHS of 3a to obtain:

$$\boxed{\text{3b. } P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)} \text{ and } P(B \mid A) = \frac{P(B)P(A \mid B)}{P(A)}}$$

We'll see a version of this result again when we get to Bayes' Rule.





## Examples

- Two events A and B are such that *P(A)*=0.2, *P(B)*=0.3, and *P(A* U *B)*=0.4. Find the following:

a. $P(A \cap B)$

b. $P(\overline{A} \cup \overline{B})$. Hint: use DeMorgan's Laws or Venn diagram.

c. $P(\overline{A} \cap \overline{B})$. Hint: use DeMorgan's Laws or Venn diagram.

d. $P(\overline{A} \mid B)$. Hint: use a Venn diagram to find $P(\overline{A} \cap B)$

## Review: The sample-point method for calculating the probability of an event

1. Define the experiment
2. Define the sample space ($S$) by identifying all of the possible simple events / outcomes (call these $E_i$)
3. Assign a probability to each simple event. Be sure these probabilities satisfy:

$$0 \le P(E_i) \le 1 \quad \text{and} \quad \sum_i P(E_i) = 1$$

4. Define the event of interest (call it $A$) and decompose it into its component simple events
5. Find $P(A)$ by summing the probabilities of these simple events

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## Another approach: The event-composition method for calculating the probability of an event

1. Define the experiment (same as sample-point method)

2. Visualize the sample points (e.g., w/ Venn diagram) if it helps clarify the problem and/or the info you've been provided.

3. Express the event of interest (say, $A$) as a composition of 2 or more events (unions, intersections, subsets, complements, etc.)

4. Apply the laws of probability to the composition in step #3 to find $P(A)$

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## The event-composition method - example

- Suppose 60% of Michigan voters are Democrats (*D*) and 40% are Republicans (*R*)
  - *P(D)*=0.6, *P(R)*=0.4
- Suppose 70% of Michigan Republicans and 80% of Democrats are in favor (*F*) of an amendment to the state constitution. *Express this in conditional probabilities.*
  - $P(F\,|\,R)=0.7,\quad P(F\,|\,D)=0.8$
- *What is the probability that a randomly selected Michigan voter will be in favor of the amendment, i.e., what is P(F)?*
  - Let's apply the event decomposition method

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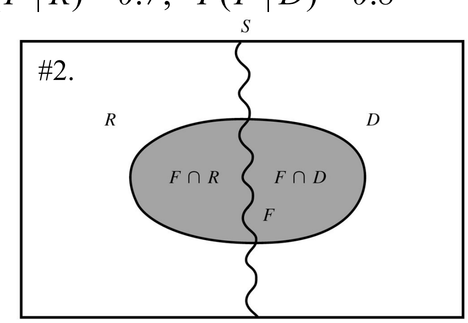
## The event-composition method – example (cont'd)

- *P(D)*=0.6, *P(R)*=0.4, $P(F\,|\,R)=0.7,\quad P(F\,|\,D)=0.8$
- $P(F)=?$

1. Define the experiment
2. Visualize the sample points
3. Express the event (F) as a composition
4. Apply the laws of probability




$$\#3.\ P(F)=P\big[(F\cap R)\cup(F\cap D)\big]=P(F\cap R)+P(F\cap D).\ Why?$$

$$\#4.\ P(F\cap R)=\ P(R)P(F\,|\,R)=(0.4)(0.7)=0.28$$

$$P(F\cap D)=\ P(D)P(F\,|\,D)=(0.6)(0.8)=0.48$$

$$P(F)=P(F\cap R)+P(F\cap D)=0.28+0.48=0.76$$

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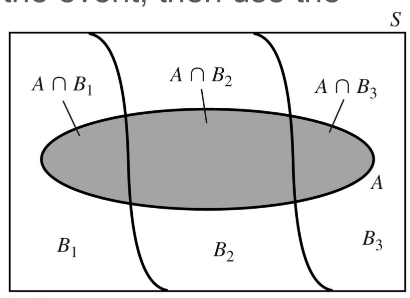
## The event-decomposition method - partitioning

- Sometimes it's easier to apply the event-decomposition method if we first "**decompose**" the event, then use the "**law of total probability**"

**Decomposition**:

1. Partition $S$ into mutually exclusive subsets $B_i$ such that: $S=B_1 \cup B_2 \cup \ldots \cup B_k$
2. Decompose $A$ and find $P(A)$:




$$A=(A\cap B_1)\cup(A\cap B_2)\cup\ldots\cup(A\cap B_k)$$

These are mutually exclusive, so

$$(I)\ P(A)=P(A\cap B_1)+P(A\cap B_2)+\ldots+P(A\cap B_k)$$

Recall that $P(A\cap B_i)=P(B_i)P(A|B_i)$, so we can write $(I)$ as:

$$\boxed{P(A)=\sum_{i=1}^{k}P(B_i)P(A|B_i)}$$

**Law of total probability**

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Recall from earlier

$$\boxed{P(B|A)=\frac{P(B)P(A|B)}{P(A)}}$$

## Bayes' Rule

- From the law of total probability and our earlier results on conditional probabilities, we can get **Bayes' Rule**:

If $\{B_1,B_2,\ldots,B_k\}$ is a partition of $S$, then

$$\boxed{P(B_j|A)=\frac{P(B_j)P(A|B_j)}{P(A)}=\frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{k}P(B_i)P(A|B_i)}}$$

via Law of total probability

- Bayes' Rule is the foundation of **"Bayesian learning"** (e.g., in technology adoption studies)
  - Gist: our beliefs (say, about a technology) can be expressed in probabilities, and as we learn, we update/modify our beliefs (probabilities)
- *What other examples of applications did you find?*

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## Decomposition and Bayes' Rule - example

- In a very dry area, it only rains 5 days/year but the weatherman has forecasted rain for tomorrow. When it rains, he forecasts rain 90% of the time. When it doesn't rain, he forecasts rain 10% of the time. What is the probability that it will rain tomorrow given that the weatherman has predicted it will rain tomorrow?
- Mutually exclusive events: it rains tomorrow ($B_1$); it does not rain tomorrow ($B_2$). $P(B_1)=$ , $P(B_2)=$
- Let $A$ = weatherman predicts it will rain tomorrow.
  $P(A|B_1)=$ , $P(A|B_2)=$
- We want to find $P(B_1|A)$:

$$P(B_1 \mid A) = \frac{P(B_1)P(A \mid B_1)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2)}$$

$$= \frac{(5/365)*0.90}{(5/365)*0.90 + (360/365)*0.1} = 0.111$$

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## Where are we going from here?

- We're now equipped with tools from probability theory for identifying sample spaces and calculating the probabilities of events
- Next we'll put these tools to use to study the probability distributions of various discrete and continuous random variables (variables whose values are the outcomes of random experiments)
- Later we'll use the probability distributions and data from a sample of the population to make statistical inferences about that population

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## Homework:

- WMS Ch. 2 (part 2 of 2)
  - Conditional probabilities & independence: 2.71, 2.76, 2.83
  - Some additional probability formulas: 2.91, 2.96, 2.102
  - The event-composition method: 2.110, 2.115, 2.121
  - Law of total probability/Bayes' Rule: 2.125, 2.129
- **If we finish Ch. 2 today, then all Ch. 2 HW is due on Tues., Sep. 12**

## Next class:

- Discrete random variables (Part 1 of 3)

## Reading for next class:

- WMS Ch. 3: 3.1 through 3.3

## Application to look into for next class:

- What are some discrete outcome variables relevant to your research interests?

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## Additional in-class exercise #1

Gregor Mendel was a monk who, in 1865, suggested a theory of inheritance based on the science of genetics. He identified heterozygous individuals for flower color that had two alleles (one r = recessive white color allele and one R = dominant red color allele). When these individuals were mated, 3/4 of the offspring were observed to have red flowers, and 1/4 had white flowers. The following table summarizes this mating; each parent gives one of its alleles to form the gene of the offspring.

| | Parent 2 | |
|---|---|---|
| Parent 1 | r | R |
| r | rr | rR |
| R | Rr | RR |

We assume that each parent is equally likely to give either of the two alleles and that, if either one or two of the alleles in a pair is dominant (R), the offspring will have red flowers. What is the probability that an offspring has

**a** at least one dominant allele?

**b** at least one recessive allele?

**c** one recessive allele, given that the offspring has red flowers?

## Additional in-class exercise #2

In the definition of the independence of two events, you were given three equalities to check: $P(A|B) = P(A)$ or $P(B|A) = P(B)$ or $P(A \cap B) = P(A)P(B)$. If any one of these equalities holds, $A$ and $B$ are independent. Show that if any of these equalities hold, the other two also hold.



## Additional in-class exercise #3

A smoke detector system uses two devices, $A$ and $B$. If smoke is present, the probability that it will be detected by device $A$ is .95; by device $B$, .90; and by both devices, .88.

**a** If smoke is present, find the probability that the smoke will be detected by either device $A$ or $B$ or both devices.

**b** Find the probability that the smoke will be undetected.

## Additional in-class exercise #4

A population of voters contains 40% Republicans and 60% Democrats. It is reported that 30% of the Republicans and 70% of the Democrats favor an election issue. A person chosen at random from this population is found to favor the issue in question. Find the conditional probability that this person is a Democrat.

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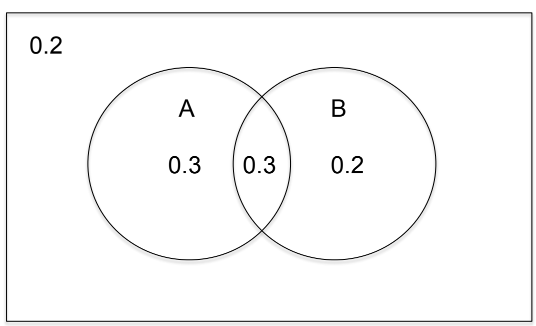
## Venn diagram – independence (http://youtu.be/mX2D1NffRI8)




$$P(A) = 0.6, \quad P(B) = 0.5$$
$$P(A \cap B) = 0.3$$
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = 0.6 = P(A)$$
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = 0.5 = P(B)$$

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