ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 11: Money Lotteries and Risk Aversion
(MWG, Ch. 6, pp.183-194)

I. Introduction

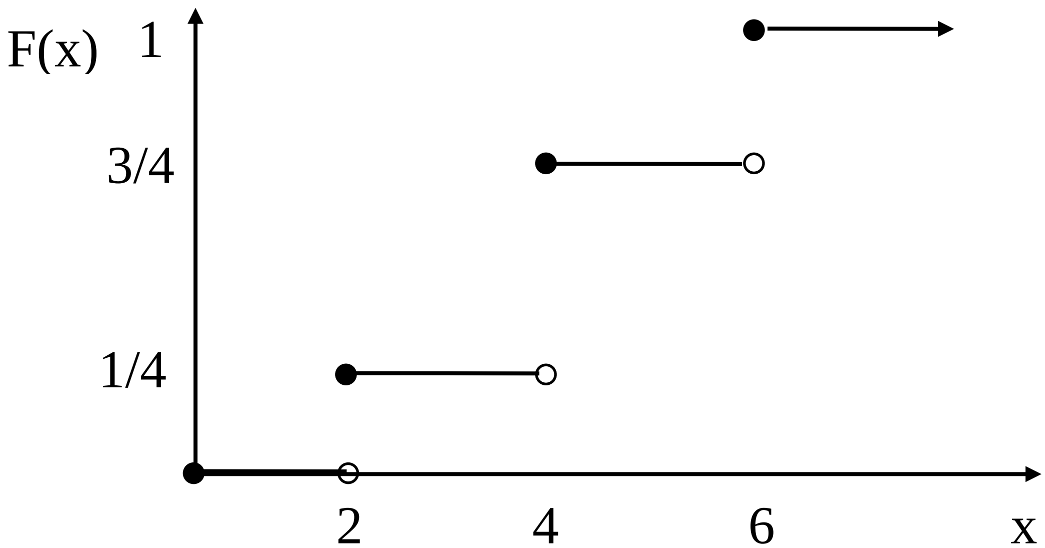
Lecture 10 introduced expected utility theory. The "objects of choice" were deliberately vague. In this lecture we simplify to some extent by assuming that the outcomes are amounts of money. This allows for a relatively intuitive definition of the concept of risk aversion, which is the main subject of this lecture.

II. Lotteries over Money Outcomes and the Expected Utility Framework

Recall that in Lecture 10 that we assumed that there was a finite number of possible outcomes, denoted by N. To be completely general when we switch to monetary outcomes, we want to allow for an infinite number of outcomes.

To start, denote the amounts of **money** by the **continuous variable x**. There is an associated **cumulative distribution function** F: $\mathbb{R} \to [0, 1]$. That is, F(x) must lie between 0 and 1. **For any x, F(x) is the probability that the money amount is less than or equal to x.**

You are probably more used to using a density distribution function, which can be denoted by f(x). Of course, the two are related as follows: $F(x) = \int_{-\infty}^{x} f(t)dt$. The **advantage of using F(x) instead of f(x)** is that F(x) **allows x to take only a finite number of values**. An example of this:



That is, Prob($2) = ¼, Prob($4) = ½, and Prob($6) = ¼.

In this set-up, F(x) is a (modest) generalization of the lotteries we discussed in Lecture 10, where a given lottery is denoted by $\mathscr{L}$. That is, **we can think of each possible function F(x) as a lottery $\mathscr{L}$**. Note that **this approach preserves the "linear structure" of compound lotteries**. That is, if F(x) is a weighted average of K functions the cumulative distribution function of F(x) is:

$$F(x) = \sum_{k=1}^{K} \alpha_k F_k(x), \quad \text{where } \sum_{k=1}^{K} \alpha_k = 1$$

From now on, let the "lottery space" $\mathscr{L}$ be the set of **all distribution functions for nonnegative values of money**.

Preferences over Money Lotteries

As in Lecture 10, the decision maker is assumed to have rational preferences $\succsim$ defined over $\mathscr{L}$. The expected utility theorem can be applied to an infinite number of outcomes of a continuous variable. Thus the expected utility from "owning" a variable x that corresponds to amounts of money in the distribution F(x) is:

$$U(F) = \int_0^\infty u(x)dF(x)$$

[Note: this dF(x) notation is the same as $\int_0^\infty u(x)f(x)dx$.]

Technically speaking, u( ) is defined with respect to "sure" amounts of money, while **U( ) is** defined with respect to lotteries and is called **an expected utility function**. Following Mas Colell et al., we call **U( )** the **von-Neumann-Morgenstern (v.N-M) expected utility function**, and we call **u( )** the **Bernoulli utility function.**

Until now there were no restrictions on the **u( ) function**, but from now on assume it is **increasing** and **continuous**.

III. Risk Aversion and Its Measurement

The concept of **risk aversion plays a very important role in economic analysis**. We will start with a very general definition that is not based on the concept of expected utility:

Definition 6.C.1: A decision maker is **risk averse** (that is, displays **risk aversion**), if for any lottery F( ), the "degenerate" lottery that yields the amount $\int_0^\infty x dF(x)$ with certainty is "at least as good as" the lottery F( ) itself. If the decision maker is always (that is, for any possible F( )) indifferent between these two lotteries, we say that he or she is **risk neutral**. Finally, the decision maker is **strictly risk averse** if indifference holds only if the two lotteries are the same (i.e. F( ) is "degenerate").

[Note: $\int_0^\infty x dF(x)$ is E[x], and is also written as $\int_0^\infty x f(x) dx$.]

If we **assume that preferences are consistent with an expected utility representation**, this definition implies that **a decision maker is risk averse if and only if:**

$$\int_0^\infty u(x) dF(x) \leq u\left(\int_0^\infty x dF(x)\right) = u(E[x]) \quad \text{for all } F(\ )$$

Note that this inequality, called Jensen's inequality, is the **defining property of a concave function**. Thus for expected utility theory **risk aversion is equivalent to the concavity of u( )**.

This is quite intuitive, since (strict) concavity implies that the marginal utility of money decreases as wealth increases. In contrast, u( ) is linear for a risk neutral decision maker. **[Draw a picture of each**.]

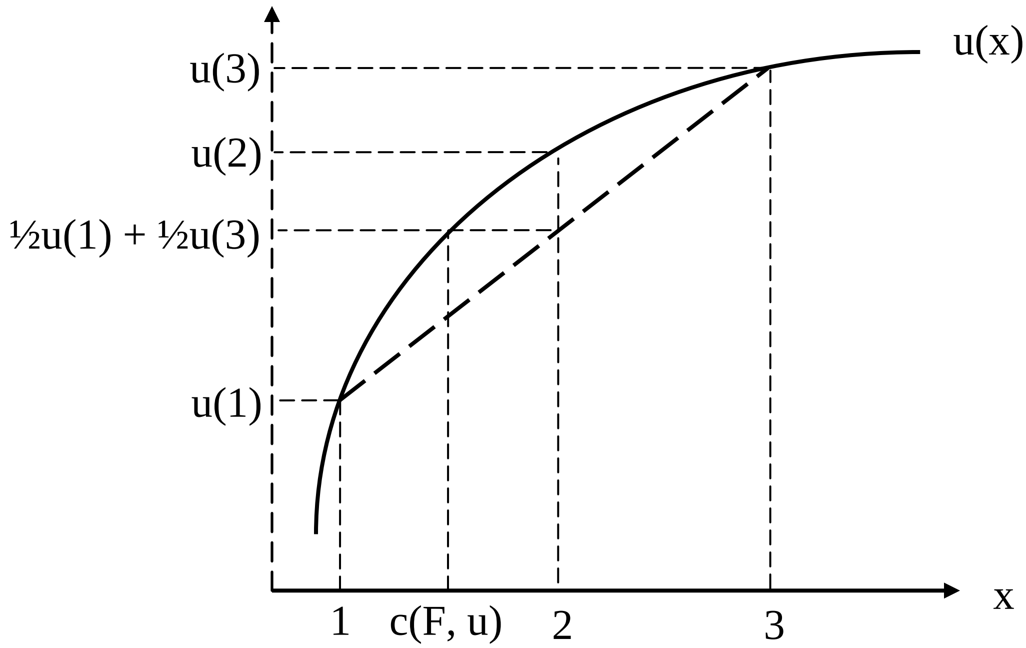
This leads us to **two more useful definitions**:

Definitions 6.C.2: Given a Bernouilli utility function u( ):

The **certainty equivalent of F( )**, denoted by **c(F, u)**, is the amount of money for which the **decision maker is indifferent between** the gamble/lottery **F( ) and** the certain amount **c(F, u)**. That is:

$$u(c(F, u)) = \int_0^\infty u(x)dF(x)$$

This concept is depicted in the diagram at the top of the next page. It depicts an even probability (π = 0.5 for both outcomes) gamble/lottery between 1 and 3 dollars. The result that c(F, u) < 2 shows that the decision maker accepts an outcome with no uncertainty that is less than E[x] in order to avoid the possibility of ending up at u(1).



More generally, if $c(F, u) \leq E[x] = \int_0^\infty x\, dF(x)$ for all F( ) then the decision maker is risk averse:

$$c(F,u) \leq \int_0^\infty x dF(x) \Leftrightarrow u(c(F,u)) \leq u(\int_0^\infty x dF(x)) \Leftrightarrow \int_0^\infty u(x) dF(x) \leq u(\int_0^\infty x dF(x))$$

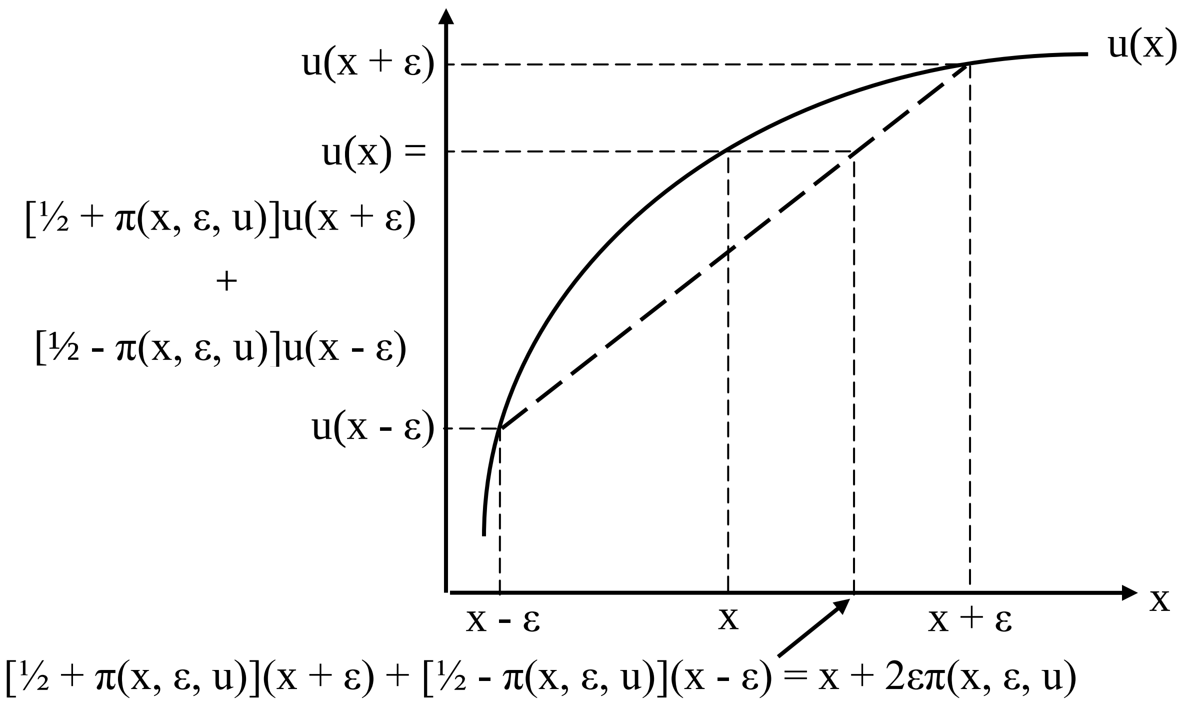
where the last $\Leftrightarrow$ follows from the definition of c(F, u).

Now turn to the second definition. For any fixed amount of money x, and a positive number ε, the **probability premium**, denoted by **π(x, ε, u)**, is the excess in winning probability over fair (50%) odds that makes the decision maker indifferent between the certain outcome x and a gamble/lottery between the two outcomes x + ε and x – ε. That is:

$$u(x) = (\tfrac{1}{2} + \pi(x, \varepsilon, u))u(x + \varepsilon) + (\tfrac{1}{2} - \pi(x, \varepsilon, u))u(x - \varepsilon)$$

Intuitively, the probability premium measures how much additional (beyond 50%) probability is needed on the "high" outcome so that the expected utility is equal to the utility from obtaining x "for sure".

The following figure depicts the probability premium:



These points are formally summarized as follows:

Proposition 6.C.1: Suppose a decision maker is an expected utility maximizer with a Bernoulli utility function u( ) over amounts of money. Then **the following four properties are equivalent**:

1. The decision maker is risk averse.
2. $u(\ )$ is concave (if it is twice differentiable: $u''(\ ) \leq 0$).
3. $c(F, u) \leq E[x] = \int_0^\infty x dF(x)$ for all $F(\ )$.
4. $\pi(x, \varepsilon, u) \geq 0$ for all x, ε.

Examples

Now let's look at some (hopefully) interesting examples to demonstrate the concept of risk aversion.

Example 6.C.1: Insurance. Consider a **strictly** risk averse decision maker with an initial wealth w who faces a risk of losing D dollars with probability π.

This person can buy as many "insurance claims" as he or she wants, which cost q and pay 1 dollar if the loss occurs. Thus if the person buys an amount of insurance, denoted by α, then his or her wealth will be $w - \alpha q$ if the loss does not occur and $w - \alpha q - D + \alpha$ if the loss occurs. Thus the person's **expected wealth** is:

$$(1-\pi)(w - \alpha q) + \pi[w - D + \alpha(1 - q)]$$

$$= w - \pi D + \alpha(\pi - q)$$

Assume that this person wants to maximize his or her expected utility. The maximization problem is:

$$\max_{\alpha \geq 0} (1-\pi)u(w - \alpha q) + \pi u(w - D + \alpha(1-q))$$

The first order condition is:

$$-q(1-\pi)u'(w - \alpha^* q) + (1-q)\pi u'(w - D + \alpha^*(1-q)) \leq 0$$

with equality holding if $\alpha^* > 0$ (asterisk denotes optimal amount).

Suppose that the insurance is **actuarially fair**, which means that an insurer's net profit is zero. The insurer's profit is $q - \pi\times 1$, which equals zero when $q = \pi$. If we insert this into the first order condition we get:

$$-\pi(1-\pi)u'(w - \alpha^*\pi) + (1-\pi)\pi u'(w - D + \alpha^*(1-\pi)) \leq 0$$

$$(1-\pi)\pi[u'(w - D + \alpha^*(1-\pi)) - u'(w - \alpha^*\pi)] \leq 0$$

$$u'(w - D + \alpha^*(1-\pi)) - u'(w - \alpha^*\pi) \leq 0$$

If $\alpha^* = 0$, then we have $u'(w - D) - u'(w) \leq 0$. But since $u(\ )$ is strictly concave, and $w > w - D$, $u'(w) > u'(w - D)$ and so this contradicts the first order condition. Thus is must be the case that $\alpha^* > 0$.

If $\alpha^* > 0$ then the first order condition is an equality, so it must be the case that $w - D + \alpha^*(1-\pi) = w - \alpha^*\pi$, which implies that $\alpha^* = D$. This means that the strictly risk averse decision maker "insures completely" if the price is actuarially fair, so that his or her income will be $w - D\pi$ in either "outcome".

Example 6.C.2: Demand for a Risky Financial Asset: Suppose that there are two assets. One is "safe" (no risk) in the sense that if you invest \$1 today you are guaranteed \$1 in the next time period. The other is risky, with a random return of \$z per dollar invested, where z is a random variable with a (cumulative) distribution F(z).

Note that a risk averse investor will put some of his or her money into z only if $\int_0^\infty z\, dF(z) = E[z] > 1$.

Question: Could z to take some negative values?

The investor has a wealth w, of which he or she needs to decide how much to invest in the safe asset, β, and how much to invest in the risky asset, α. Thus $w = \alpha + \beta$.

The investor's maximization problem is to choose α and β to maximize expected utility:

$$\underset{\alpha,\beta\geq 0}{\text{Max}} \int_0^\infty u(\alpha z + \beta)dF(z), \quad \text{subject to } \alpha + \beta = w$$

Since $\beta = w - \alpha$, this can be expressed in terms of a single parameter α:

$$\underset{0\leq\alpha\leq w}{\text{Max}} \int_0^\infty u(w + \alpha(z - 1))dF(z)$$

The optimal value of α, α^*, must satisfy the Kuhn-Tucker conditions:

$$\int_0^\infty [u'(w + \alpha(z - 1))(z - 1)]dF(z) \begin{cases} \leq 0 \text{ if } \alpha^* < w \\ \geq 0 \text{ if } \alpha^* > 0 \end{cases}$$

Suppose $\alpha^* = 0$. Then the u′( ) term in this expression becomes a constant, that is $u'(w)$. This can be pulled out of the integral sign, so the only thing left to integrate is $\int_0^\infty (z - 1)]dF(z) = \int_0^\infty zdF(z) - \int_0^\infty dF(z) = E[z] - \int_0^\infty f(z)dz = E[z] - 1$. Since we assumed that $E[z] > 1$, $\alpha^* = 0$ violates the "top" first order condition, so it must be that $\alpha^* > 0$.

This implies that a **risk averse investor will accept at least a small amount of risk** as long as the expected value of the risky asset is greater than the value of the "safe" asset.

Example 6.C.3: General Asset Problem. In the previous example, we could have defined expected utility as a function of α and β:

$$U(\alpha, \beta) = \int_0^\infty u(\alpha z + \beta)dF(z)$$

The Bernouli utility function u( ) is increasing, continuous and concave in $\alpha z + \beta$, and these three properties carry over to $U(\alpha, \beta)$.

Next, assume that there are N assets (one of which may be a riskless asset), denoted by n = 1, 2, … N. Each asset n has a return z_n per dollar invested, where z_n is a random variable. In the most general case, the returns to these N assets follow a joint distribution function $F(z_1, z_2, \dots z_N)$.

The utility of holding a **portfolio** of N assets is:

$$U(\alpha_1, \alpha_2, \dots \alpha_N) = \int_0^\infty u(\alpha_1 z_1 + \alpha_2 z_2 + \dots + \alpha_N z_N)dF(z_1, z_2, \dots z_N)$$

This utility function for portfolios, $U(\alpha_1, \alpha_2, \dots \alpha_N)$, is also increasing, continuous and concave in all of the α's. Thus the demand for assets can be treated exactly like the demand for goods that has been studied in Lectures 2-8. Thus risk aversion leads to indifference curves in "asset space" that are convex. (See p.189 of Mas Colell et al. for more discussion.)

IV. The Measurement of Risk Aversion

For many types of economic analysis, it is useful to quantify the extent of a decision maker's risk aversion. We start with absolute risk aversion, and then turn to relative risk aversion.

Absolute Risk Aversion

Definition 6.C.3: Given twice-differentiable Bernoulli utility function u( ), defined over sums of money, the **Arrow-Pratt coefficient of <u>absolute</u> risk aversion** at x is defined as:

$$r_A(x) = -u''(x)/u'(x)$$

The reasoning behind this definition is **fairly intuitive**. First, it is very intuitive that a utility function that is linear in x shows no risk aversion, and has $u''(x) = 0$. This suggests that a measure of risk aversion should be related to the **curvature** of u( ), which suggests defining it in terms of u″( ). Yet using u″( ) alone is not sufficient because choices under uncertainty are unaffected by linear transformations of u( ). **To "control for" linear transformations** the easiest thing to do is **divide u″( ) by u′( )**, which is what $r_A(x)$ does. The last thing to do is to make $r_A(x)$ increasing as u″( ) increases in absolute value, so we put a negative sign in front of $u''(x)/u'(x)$.

A more rigorous motivation for $r_A(x)$ as a measure of risk aversion comes from examining the behavior of the probability premium $\pi(x, \varepsilon, u)$ as $\varepsilon \rightarrow 0$. Recall that $\pi(x, \varepsilon, u)$ is defined as:

$$u(x) = (\tfrac{1}{2} + \pi(x, \varepsilon, u))u(x + \varepsilon) + (\tfrac{1}{2} - \pi(x, \varepsilon, u))u(x - \varepsilon)$$

Differentiating this twice with respect to ε (try this at home) and evaluating this expression at $\varepsilon = 0$ gives:

$$4\pi'_{\varepsilon}(x, 0, u)u'(x) + u''(x) = 0$$

Rearranging this expression gives:

$$-u''(x)/u'(x) = r_A(x) = 4\pi'_{\varepsilon}(x, 0, u)$$

Thus $r_A(x)$ measures the rate at which the probability premium increases at certainty ($\varepsilon = 0$) as ε increases.

A final interesting property of $r_A(x)$ is that the **utility function u( ) can be recovered** by integrating $r_A(\ )$ twice. The two constants of integration are irrelevant since they yield only linear transformations of u( ). **Thus $r_A(x)$ contains all the information about u( )** and thus contains all information about behavior under uncertainty.

Example 6.C.4: Start with the utility function $u(x) = -e^{-ax}$ for $a > 0$. Clearly, $u'(x) = ae^{-ax}$ and $u''(x) = -a^2e^{-ax}$. Thus $r_A(x) = a$ for all values of x. Since $r_A(x)$ fully characterizes the underlying utility function, it follows that **any** utility function that has a **constant absolute risk aversion** (CARA) property **must take the form**:

$$u(x) = -\alpha e^{-ax} + \beta \text{ for some } \alpha > 0 \text{ and some } \beta$$

(In fact, β has no effect on behavior, so it can be set to 0.)

Comparisons across People and across Wealth Levels

The Arrow-Pratt coefficient of absolute risk aversion is useful for comparing risk across different people and comparing risk aversion for a given person at different levels of wealth. Let's start with comparisons across people.

Comparisons across People. Suppose we have two Bernoulli utility functions (for two different people): $u_1(\ )$ and $u_2(\ )$. When can we state that, for example, $u_2(\ )$ is unambiguously **more risk averse** than $u_1(\ )$? Here are five plausible definitions of "more risk averse":

1. $r_A(x, u_2) \geq r_A(x, u_1)$ for all values of x.

2. There exists an increasing concave function $\psi(\ )$ such that $u_2(x) = \psi(u_1(x))$ for all x; that is, $u_2(\ )$ is a concave transformation of u_1 (is "more concave" than u_1).

3. $c(F, u_2) \leq c(F, u_1)$ for any $F(\ )$.

4. $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

5. Whenever $u_2(\ )$ finds a lottery $F(\ )$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\ )$ also finds $F(\ )$ at least as good as a riskless outcome $\bar{x}$. That is, $\int_0^\infty u_2(x)dF(x) \geq u_2(\bar{x})$ implies $\int_0^\infty u_1(x)dF(x) \geq u_1(\bar{x})$ for any $F(\ )$ and $\bar{x}$. (Note: $\bar{x} \neq E[x]$.)

In fact, all five of these are mathematically equivalent:

Proposition 6.C.2: Definitions 1-5 of the **more-risk-averse-than** relation are equivalent.

Mas Colell et al. give a proof for the equivalence of 1. and 2. on pp.191-192. The others are left as exercises.

Note: The "more-risk-averse-than" relation between two Bernoulli utility functions is only a **partial ordering**. That is, it is quite possible that one utility function is "more risk averse" at some levels of income while the other is “more risk averse” at other levels. In this case, it is not possible, in general, to rank the two utility functions in terms of their risk aversion.

On p.192, Mas Colell et al. continue with Example 6.C.2. They show that if a Bernoulli utility function u_2 is more risk averse than another such utility function u_1, then the fraction of wealth invested in the risky asset will be smaller. That is $\alpha_2^* < \alpha_1^*$. This is quite intuitive.

Comparisons across Wealth Levels. It is often assumed that wealthier individuals are more willing to accept risk, because they can “afford to take chances”. This will depend on the functional form of u( ). To investigate this more, we need to define this concept:

Definition 6.C.4: The Bernoulli utility function u( ) exhibits **decreasing absolute risk aversion** if $r_A(x, u)$ is a decreasing function of x. That is, $\partial r_A(x, u)/\partial x < 0$.

This definition can be expressed in many ways. The following proposition explains this, where z denotes an increase or decrease in wealth for a given wealth x:

Proposition 6.C.3: The following properties are equivalent:

1. The Bernoulli utility function u( ) exhibits decreasing absolute risk aversion.

2. Whenever $x_2 < x_1$, $u_2(z) \equiv u(x_2 + z)$ is a concave transformation of $u_1(z) \equiv u(x_1 + z)$. That is, u( ) is "more concave" at lower values of x.

3. For any risk F(z), which is the distribution of z, the **certainty equivalent** of the lottery formed by adding risk z to wealth level x, given by the amount $c_x = \int_0^\infty u(x + z)dF(z)$, is such that $(x - c_x)$ **is decreasing in x**. Thus the higher x is, the less the person is willing to pay to get rid of the risk.

4. The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.

5. For any z with distribution F(z), if $\int_0^\infty u(x_2 + z)dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int_0^\infty u(x_1 + z)dF(z) \geq u(x_1)$

Relative Risk Aversion

For some uses, decreasing absolute risk aversion is too "weak" of an assumption, and it also may not be analytically convenient. Thus it is useful to examine a

stronger assumption: **nonincreasing relative risk aversion**. The intuition is that analysis of absolute risk aversion focuses on additive/absolute changes in wealth. in contrast, relative risk aversion focuses on percentage changes in wealth.

To see how this works, let $t > 0$ be **proportional** increases or decreases in wealth. We can think of t as a random variable and consider how its distribution affects utility given a certain "initial" wealth. It is convenient to define:

$$\tilde{u}(t) \equiv u(tx)$$

Starting with $t = 1$, the risk associated with small changes in t could be measured by $\tilde{u}''(1)/\tilde{u}'(1)$. Using the definition of $\tilde{u}(t)$ implies that:

$$\tilde{u}''(1)/\tilde{u}'(1) = xu''(x)/u'(x)$$

This leads us to our formal definition:

Definition 6.C.5: For a Bernoulli utility function u( ), the **coefficient of relative risk aversion** at x is given by:

$$r_R(x, u) = -xu''(x)/u'(x)$$

This is a stronger (more restrictive) concept than absolute risk aversion because it is less likely to decrease as x increases (less likely to display decreasing risk aversion as x increases). To see this, note that $r_R(x, u) = xr_A(x, u)$. If a small increase in x decreases $r_R(x, u)$ it must certainly decrease $r_A(x, u)$, but if a small increase in x decreases $r_A(x, u)$ it may or may not decrease $r_R(x, u)$.

An example of a functional form that has a **constant relative risk aversion** (CRRA) property is:

$$u(x) = x^{1-\alpha}, \text{ where } \alpha < 1$$

In this case $r_R(x, u) = \alpha$. Another example is $u(x) = \log(x)$.

Finally, it is worth pointing out that there are two other equivalent concepts of decreasing relative risk aversion:

Proposition 6.C.4: The following conditions for a Bernoulli utility function u( ) are equivalent:

1. $r_R(x, u)$ is decreasing in x.

2. Whenever $x_2 < x_1$, $\tilde{u}_2(t) \equiv u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) \equiv u(tx_1)$.

3. Given any risk F(t) on $t > 0$, the certainty equivalent $\bar{c}_x$ defined by $u(\bar{c}_x) = \int_0^\infty u(tx)dF(t)$ has the property that $x/\bar{c}_x$ is decreasing in x.

See p.194 of MWG for a short proof that 1. implies 3.