MICHIGAN STATE UNIVERSITY

# AFRE 802
# Statistical Methods for Agricultural, Food, & Resource Economists

**Discrete random variables & their probability distributions**

**(Part 1 of 3)**

**(WMS Ch. 3.1-3.3)**

September 12, 2017

Nicole Mason
Michigan State University
Fall 2017

MICHIGAN STATE UNIVERSITY

## GAME PLAN

- Hand back remaining HWs and in-class exercises
- Collect Ch. 2 HWs
- **Request on future HWs**: please staple your pages together and order the questions in the order assigned. (It's fine to answer them out of order, but if you do, please use separate sheets of paper and order then when compiling your HW before handing it in.) I'd greatly appreciate it!
- Review and questions from last class
- **Graded in-class exercise**
- **Discrete Random Variables (Ch. 3) – Part 1 of 3**
  - Random variables (RV) defined
  - General notion of prob. distribution for a discrete RV
  - Expected values and variances of RVs

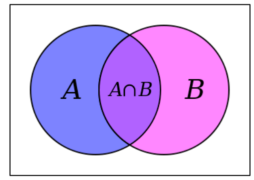
1

## REVIEW: Key formulas from last class (minus variants)

MICHIGAN STATE UNIVERSITY

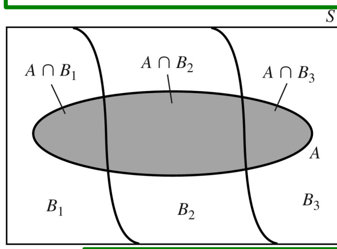
$$P(A) = 1 - P(\bar{A})$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$




$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A)$$




**Law of total probability**

If $S = B_1 \cup B_2 \cup \ldots \cup B_k$,

then $P(A) = \sum_{i=1}^{k} P(B_i)P(A \mid B_i)$

and $P(B_j \mid A) = \dfrac{P(B_j)P(A \mid B_j)}{P(A)} = \dfrac{P(B_j)P(A \mid B_j)}{\sum_{i=1}^{k} P(B_i)P(A \mid B_i)}$

**Bayes' Rule**

2

## REVIEW

MICHIGAN STATE UNIVERSITY

**Independence of events**: Two events are said to be independent if…

$$P(A \mid B) = P(A)$$
$$P(B \mid A) = P(B)$$
$$P(A \cap B) = P(A)P(B)$$

*Any other questions from last class?*

3

MICHIGAN STATE UNIVERSITY

## Graded in-class exercise

4

MICHIGAN STATE UNIVERSITY

## Where are we going from here?

- We're now equipped with tools from probability theory for identifying sample spaces and calculating the probabilities of events
- Now we're going to use these tools to study the probability distributions of various discrete and continuous random variables (variables whose values are the outcomes of random experiments)
- Later we'll use the probability distributions and data from a sample of the population to make statistical inferences about that population

5

## Random variable (RV) defined

- A variable that takes on numerical values and has an outcome that is determined by a random experiment (Wooldridge)
- A real-valued function defined over a sample space (WMS)
- Example #1
  - Experiment: Flip a coin 10 times & count the # of Ts
  - RV = # of Ts
  - Sample space = {0, 1, …, 10}
- Example #2
  - Experiment: Interview 100 shoppers as they leave Meijer and record how much $ they just spent on groceries. (Ignore people that were there to get refunds.)
  - RV = $ spent on groceries
  - Sample space: $\geq 0$

6

## Discrete vs. continuous RVs

- **Discrete**: the RV takes on a finite or countably infinite # of distinct values
  - *Examples?*
    - Binary/dichotomous variables (only 2 values, e.g., 0/1, male/female)
    - Categorical/nominal variables (e.g., race)
    - Rankings / ordinal variables (small, medium, large)
    - Count variables (integers: 0, 1, 2, …)
    - Etc.

***We would assign a # to each qualitative outcome***

- **Continuous**: an RV that takes on an infinite number of values (in its range). It takes on each value with zero probability (infinite # of possible values so can't count or assign a positive probability to them).
  - *Examples?*

7

MICHIGAN STATE UNIVERSITY

## Probability distributions

- To draw inferences about a population, we need to know the probability of observed events
- We can often describe an event as an RV → so need to know the RV's probability distribution
- **Probability distributions link each value of an RV to its probability of occurrence**
- Some types of RVs are very common, so we'll study their specific probability distributions the next several classes

8

MICHIGAN STATE UNIVERSITY

## Notation for probability distributions of discrete RVs

- RVs: uppercase letters (e.g., $Y$ )
- Particular values of the RV: lowercase letters ($y$)
- Example:
  - You interview 100 MSU undergrads & ask if they watched the MSU home opener football game or not
  - ***Y*** is the # of the 100 MSU undergrads that could have possibly watched the game (can be any value between 0 and 100) → **random**
  - ***y*** is the # of the 100 undergrads that actually watched the game, i.e., the observed value → **not random**
- ***P(Y=y)*** or ***p(y)*** is the probability that the RV takes on a specific value
  - EX) $P(Y=20)$ and $p(20)$ denote the probability that exactly 20 of the 100 undergrads watched the game

9

## Probability distributions for discrete RVs

- The **probability distribution** for a **discrete RV** *Y* **describes *p(y)=P(Y=y)* for all *y***
  - i.e., it describes the probability that the RV takes on a particular value for all possible values of the RV
  - If a given value is not possible, implicitly *p(y)=0*
- *p(y)* can be described by a formula, table, or graph
  - Don't need to explicitly list values for which *p(y)=0*

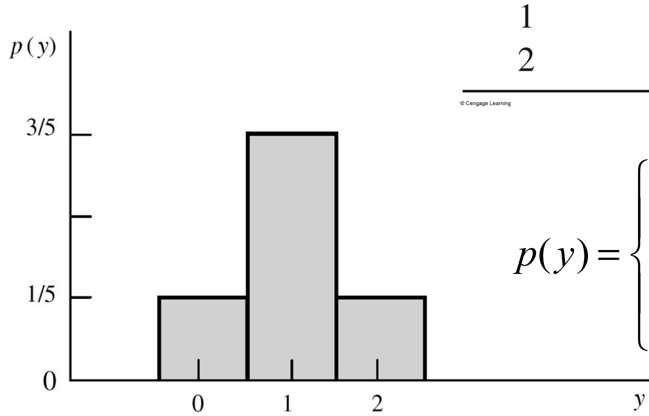




## Probability distribution – tables, graphs, formulas

| $y$ | $p(y)$ |
|---|---|
| 0 | 1/5 |
| 1 | 3/5 |
| 2 | 1/5 |








$$p(y) = \begin{cases} 1/5 \text{ for } y = 0 \\ 3/5 \text{ for } y = 1 \\ 1/5 \text{ for } y = 2 \end{cases}$$

MICHIGAN STATE UNIVERSITY

*What characteristics must these values of p(y) have (range, sum, etc.)?*

- For any discrete probability distribution, the following must be true:

  1. $0 \le p(y) \le 1$ for all $y$
  2. $\sum_i p(y_i) = 1$, for all $y_i$ with $p(y_i) > 0$

12

MICHIGAN STATE UNIVERSITY

## Example

- Experiment: Flip a coin 3 times & observe the # of Hs
- Let $Y$ = the # of Hs observed
- *What is the probability distribution of Y?*
- S={HHH, HTH, HHT, THH, THT, TTH, HTT, TTT}
- 8 sample points by the **mn rule** = 2*2*2

| *y* | *p(y)* |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

13

## Recall from class #1

- Talked about means, variances, and standard deviations of samples and populations
- Similar concepts apply to RVs
  - Mean = "**expected value**" – measure of central tendency of an RV
  - **Variance & standard deviation** – measures of dispersion/spread/variability of an RV

14

## The expected value of a discrete RV – intuition

- Consider our previous example. *What is the expected value (mean) of Y?*

| *y* | *p(y)* |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

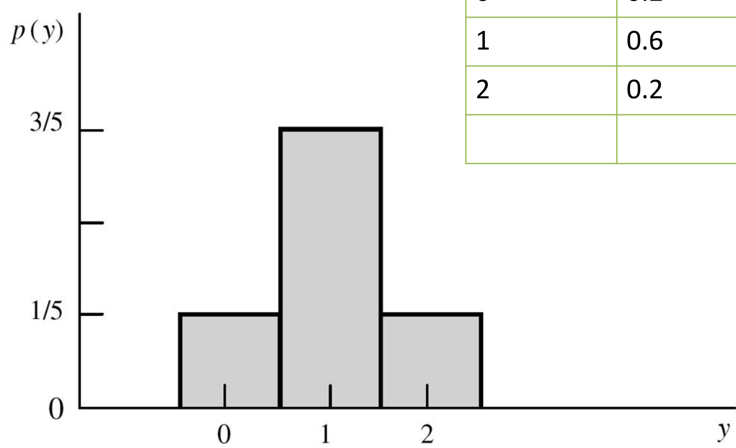
**Probability distribution for *Y***
(# of Hs observed when toss a coin 3 times)

- The expected value is the weighted average, where each possible value of *Y* is weighted by its probability
- 0*(1/8)+1*(3/8)+2*(3/8)+3*(1/8)=1.5

15

## Expected value – graphically – what is the mean (average) of $Y$?

| y | p(y) | yp(y) |
|---|---|---|
| 0 | 0.2 | 0 |
| 1 | 0.6 | 0.6 |
| 2 | 0.2 | 0.4 |
| | | **E(Y)=1** |






16



## The expected value of a discrete RV – formally

For a discrete RV, $Y$, with probability function $p(y)$, the expected value of $Y$, $E(Y)$ or $\mu$, is defined as:

$$\boxed{\mu = E(Y) = \sum_i y_i p(y_i)}$$

17

MICHIGAN STATE UNIVERSITY

## Example – expected value of a discrete RV

- Find E(Y) for a discrete RV that has the following probability distribution:

| $y$ | $p(y)$ |
|---|---|
| 0 | 1/4 |
| 1 | 1/2 |
| 2 | 1/4 |



18

MICHIGAN STATE UNIVERSITY

## The variance of a discrete RV

$$\boxed{s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y})^2}$$

- Recall the formula for the sample variance from 1st class
- *What is the interpretation of variance?*
- *E(.)* is our population notion of an average
- $\mu$ is the population mean of random variable $Y$

→ *What would be a sensible formula for the variance of Y, $\sigma^2$?*

$$\boxed{\sigma^2 = Var(Y) = V(Y) = E\left[(Y-\mu)^2\right]}$$

- *What is the standard deviation of Y?*

$$\boxed{\sigma = \sqrt{\sigma^2}}$$

19

MICHIGAN STATE UNIVERSITY

## Expected value of a function of a RV

- In order to find the variance, we need to take the expected value of a function of $Y$:

$$\boxed{Var(Y) = V(Y) = E\left[(Y-\mu)^2\right]}$$

- Recall that $\boxed{E(Y) = \sum_i y_i p(y_i)}$
- Let $g(Y)$ be a function of $Y$. If $Y$ is a discrete RV, then:

$$\boxed{E\left[g(Y)\right] = \sum_i g(y_i)p(y_i)}$$

- *Given this formula, what is V(Y) for a discrete RV?*

20

MICHIGAN STATE UNIVERSITY

## The variance of a discrete RV (cont'd)

$$\boxed{\sigma^2 = V(Y) = E\left[(Y-\mu)^2\right] = \sum_i \underbrace{(y_i-\mu)^2}_{g(y_i)} p(y_i)}$$

21

Example: Now find the variance and standard deviation of $Y$. (Same probability distribution as the expected value example.)

| $y$ | $p(y)$ |
|---|---|
| 0 | 1/4 |
| 1 | 1/2 |
| 2 | 1/4 |



22



Other useful rules for expected values of discrete RVs *(see textbook for proofs if you're interested):*

1. $E(c) = c$ for any constant $c$

A constant is known with certainty (it's not an RV)

2. $E(cY) = cE(Y)$, or more generally, $E\left[cg(Y)\right] = cE\left[g(Y)\right]$

Can pull constants outside of *E(.)*, again b/c they're not RVs

3. $E(bY + c) = bE(Y) + c$ for any constants $b$ and $c$

*E(.)* is a linear operator

4. $E\left[g_1(Y) + g_2(Y)\right] = E\left[g_1(Y)\right] + E\left[g_2(Y)\right]$

MICHIGAN STATE UNIVERSITY

Example: use these results to derive these other very useful formulas for *V(Y)*:

$$V(Y) = E\left[(Y-\mu)^2\right] = E(Y^2) - \mu^2$$

$$= E(Y^2) - \left[E(Y)\right]^2$$

$$= \left[\sum_i y_i^2 p(y_i)\right] - \left[E(Y)\right]^2$$

24

MICHIGAN STATE UNIVERSITY

Other useful rules for variances

1. $V(c) = 0$, where $c$ is a constant.

*Why?*

2. $V(cY) = c^2 V(Y)$

3. $V(bY + c) = b^2 V(Y)$ for any constants $b$ and $c$

*Time-permitting, let's prove this on the next slide*

25

## Example

Use the formula for *V(Y)* and the rules for expected values to show that:

$$V(bY+c)=b^2V(Y) \qquad \text{for any constants } b \text{ and } c$$

26

## Homework:

- WMS Ch. 3 (part 1 of 3)
  - Probability distributions of discrete RVs (general): 3.1, 3.5, 3.11
  - Expected values & variances of discrete RVs: 3.12, 3.19, 3.21, 3.22, 3.25
- **Ch. 3 HW will be due the class after we finish Ch. 3

## Next class:

- Discrete random variables (Part 2 of 3) – specific distributions (Bernoulli, binomial, geometric)

## Reading for next class:

- WMS Ch. 3: 3.4 & 3.5

## Application to look into for next class:

- Find an example of how the Bernoulli, binomial, or geometric distribution is applied in your field

27

## Additional in-class exercise #1

**3.2** You and a friend play a game where you each toss a balanced coin. If the upper faces on the coins are both tails, you win \$1; if the faces are both heads, you win \$2; if the coins do not match (one shows a head, the other a tail), you lose \$1 (win (−\$1)). Give the probability distribution for your winnings, $Y$, on a single play of this game.

Then calculate the mean, variance, and standard deviation of $Y$.





## Additional in-class exercise #2

Find the mean, variance, and standard deviation of a random variable, $Y$, with the following probability distribution.

| $y$ | $p(y)$ |
|---|---|
| 0 | 1/8 |
| 1 | 1/4 |
| 2 | 3/8 |
| 3 | 1/4 |

Learning