Static Games of Incomplete Information

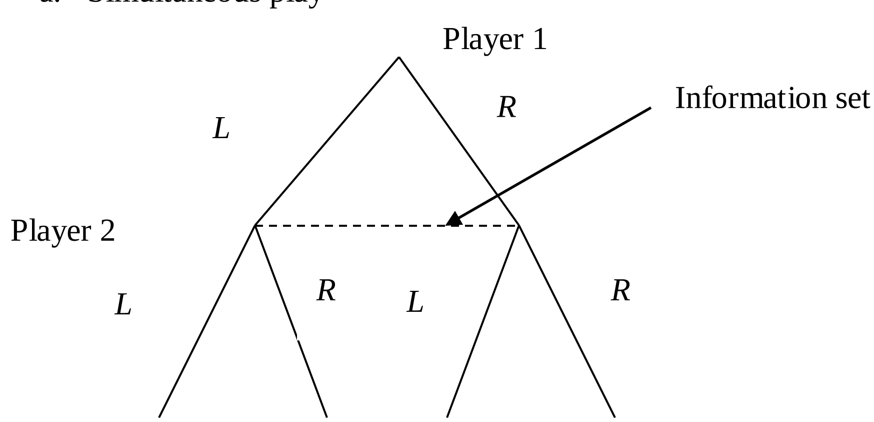
I. Introduction

A. Motivation and definition of incomplete information

1. Information issues have become a major area of investigation in economics: "information economics" raise a profound set of problems especially in combination with considerations of strategic interaction
2. In many realistic situations, one player will know something that other players do not ("**asymmetric information**")
3. Examples:
   a. Oligopoly: firms know own cost (profit) function but do not know rivals' cost (profit function)
   b. Auction: players know value of item to themselves but not to other players (private value auction)
   c. Bargaining: don't know value of an agreement to other player, or the discount factor (impatience) of other player
4. In each case, one player knows information that affects payoffs that rival player does not. When players have private information about payoffs we have a game of **incomplete information**.
5. In the games we have explored to date, every player knew the payoff functions of all players. We refer to these games as games of **complete information**.

B. Incomplete information versus imperfect information

1. Imperfect information games: not all information sets are singletons, i.e., at least one player at some point in the game tree does not know where they are in the tree.
2. Examples:
   a. Simultaneous play



b. Symmetric uncertainty about events. Example: Cournot competition with high and low demand where both players do not know demand

High Demand: $P(Q) = a – Q$

Low Demand: $P(Q) = b – Q$

$a > b$

$Pr(H) = \pi$, $Pr(L) = 1- \pi$

Maximize expected profits:

Max $\pi(a - Q_1 - Q_2)Q_i + (1- \pi) (b - Q_1 - Q_2)Q_i$ for $i = 1, 2$

$= \text{Max } (\pi a + (1- \pi)b- Q_1- Q_2)Q_i$

This game is exactly the same as a perfect information game with the demand intercept equal to the expected value, $\pi a + (1- \pi)b$. To solve the game we first find reaction functions and then solve for a pair of strategies where the reaction functions are best responses to each other (i.e., Nash equilibrium).

3. Incomplete information games with private information about payoffs are much more difficult to analyze than imperfect information games. If one player has private information, they can condition their play on the information, but other players cannot.
4. For example: suppose firm 1 could be a high cost or low cost producer. Cost affects the amount they would find profitable to produce. This would mean they have a different reaction function that depends on cost. How should firm 2 behave given this? How should we proceed with the analysis? Can't just mimic prior game (imperfect information) – because player 1 will condition play on information but player 2 cannot.

C. Harsanyi Transformation: transforming games of incomplete information into games of imperfect information

1. Harsanyi proposed adding a new player to the game: *Nature*
2. Nature gets to start the game by randomly choosing "types" for each player from a probability distribution
3. Each player observes their own type; other players only observe the distribution but not the actual type for other players
4. Game can then be analyzed like a game of imperfect information
5. Example: duopoly asymmetric information game
   a. Firm 1 can be a high cost producer with probability p or low cost producer with probability $(1-p)$
   b. Nature chooses whether firm 1 is the high type or the low type; this information is known by firm 1 but not firm 2
   c. Firms then simultaneously choose quantity
   d. Payoffs are a function of quantities chosen and the cost of production

II. Bayesian Nash Equilibrium

A. Preliminaries: elements of a static game of incomplete information

1. Previously we described a static game of complete information: $G = \{N, S, U\}$

a. Players: $i = 1, 2, \ldots, n;\ i \in N$

b. Strategies: s_i is a pure strategy (action) for player i, $s_i \in S_i$, where S_i is the set of possible pure strategies for player i. A strategy profile is defined as $s = (s_1, s_2, \ldots, s_n)$ with $s \in S$ where $S = \text{X}_{i \in N} S_i$ is the set of all possible strategy profiles

c. Utility (preferences): $U_i(s)$, describes player *i's* ranking of possible outcomes. $U = \{U_1(s), U_2(s), \ldots, U_n(s)\}$

2. To extend this description for incomplete information games, we need a few additional elements: types and probabilities
   a. Types: t_i is a type for player i, $t_i \in T_i$ is the set of all possible types for player i . Define the vector of types $t = (t_1, t_2, \ldots, t_n)$ with $t \in T$ where $T = \text{X}_{i \in N} T_i$ is the set of all possible combinations of types
   b. We will also find it useful to define types for all other players exclusive of player i's type, $t_{\sim i} = (t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ with $t_{\sim i} \in T_{\sim i} = \text{X}_{i \neq j \in N} T_j$
   c. Probabilities: Define the probability that player i is of type t_i as $p(t_i)$. The probability of any given combination of types, t, is $p(t)$, where $0 \le p(t) \le 1$ for all $t \in T$ and $\sum_{t \in T} p(t) = 1$. The probability distribution $p(t)$ is common knowledge.
   d. Conditional probabilities: $p(t_{\sim i} \mid t_i)$ is the conditional probability of rival players $j \neq i$ being of type $t_{\sim i}$ given that player i is of type t_i.
   e. Finally, we need to modify our definition of strategies and payoffs to note that they are conditional on types:
      - Strategies: $s_i(t_i)$ is a pure strategy for player i - it describes the action chosen by player i for any given type t_i. Important note: strategies are contingent mappings that describe the action chosen by each possible type. Define S_i to be the set of possible pure strategies for player i. A strategy profile is defined as $s(t) = (s_1(t_1), s_2(t_2), \ldots, s_n(t_n))$. The set of all strategy profiles can then be defined just as before.
      - Utility (preferences): $U_i(s_i(t_i), s_{\sim i}(t_{\sim i}), t_i, t_{\sim i})$ is the expected utility for player i. It is important to note that for this expected utility the strategy profile is conditional on types. With this modified definition of an individual's payoff function, the payoff space can be defined just as before.
   f. Note: if you don't like expected utility theory you could modify the definition of a player's payoff function to accommodate a different theory.
   g. Also note that as with mixed strategies, the intensity of preferences matters, not just the ordinal ranking.
   h. With these additions, we can now define a static game of incomplete information as $G = \{N, S, U, T, P\}$.

B. Definition: A strategy profile $s(t)$ is a **Bayesian Nash equilibrium** if for each player i:

$$\sum_{t_i \in T_i} \sum_{t_{\sim i} \in T_{\sim i}} \{ p(t_i, t_{\sim i}) U_i(s_i(t_i), s_{\sim i}(t_{\sim i}), t_i, t_{\sim i}) \} \geq \sum_{t_i \in T_i} \sum_{t_{\sim i} \in T_{\sim i}} \{ p(t_i, t_{\sim i}) U_i(s_i^{'}(t_i), s_{\sim i}(t_{\sim i}), t_i, t_{\sim i}) \}$$

for all $s_i^{'}(t_i) \in S_i$.

In words, in a Bayesian Nash equilibrium each player is maximizing their expected utility (i.e., playing a best response) given the other player's equilibrium strategies. A Bayesian Nash equilibrium is a Nash equilibrium extended to include probabilities and expected utility.

C. Alternative definition (conditional on type t_i): A strategy profile $s(t)$ is a **Bayesian Nash equilibrium** if for each player i:

$$\sum_{t_{\sim i} \in T_{\sim i}} \{ p(t_{\sim i} \mid t_i) U_i(s_i(t_i), s_{\sim i}(t_{\sim i}), t_i, t_{\sim i}) \} \geq \sum_{t_{\sim i} \in T_{\sim i}} \{ p(t_{\sim i} \mid t_i) U_i(s_i^{'}(t_i), s_{\sim i}(t_{\sim i}), t_i, t_{\sim i}) \}$$

for all $s_i^{'}(t_i) \in S_i$, for all $t_i \in T_i$.

The alternative definition defines best responses for player i given that they are of type t_i.

D. Summary: Bayesian Nash equilibrium is an extension of Nash equilibrium to the case with probability distributions over types. It requires that each player does at least as well in expected utility terms when playing the equilibrium strategy as playing any other strategy (i.e., no profitable deviations in expected utility terms).

III. Examples: applying Bayesian Nash equilibrium to solve static games of incomplete information

A. Two player game where one player can be of two types

1. Consider the game where we have two players *A* and *B*. Suppose player *B* has an equal probability of being one of two types: type 1 (Cooperative) and type 2 (Uncooperative).
2. Both types of player *B* have two strategies either *L* or *R*. Player *A* can choose between *U* and *D*.
3. The payoffs for each type are summarized in the tables below

Player *B* Type 1 (Cooperative)

	L	*R*
U	30, 25	0, 0
D	0, 0	20, 25

Player *B* Type 2 (Uncooperative)

	L	*R*
U	30, 0	0, 25
D	0, 25	20, 0

4. Question: What are the pure strategy Bayesian Nash equilibria for this game?
5. To answer this question, we start by defining the strategy space for each player: $S_A = \{U, D\}$ and $S_B = \{(L, L), (L, R), (R, L), (R, R)\}$. The space of strategy profiles is then $S = S_A \times S_B$.
6. Next we need to define the payoffs over strategy profiles shown in the following table:

	(*L*,*L*)	(*L*,*R*)	(*R*,*L*)	(*R*,*R*)
U	30/2+30/2=30, (25,0)	30/2+0/2=15, (25,25)	30/2+0/2=15, (0,0)	0/2+0/2=0, (0,25)
D	0/2+0/2=0, (0, 25)	0/2+20/2=10, (0, 0)	20/2+0/2=10, (25, 25)	20/2+20/2=20, (25, 0)

Note: payoffs read as follows – the first row calculates the expected payoff for *A*, the second pair of numbers is the payoff for player *B* of type 1 and type 2.

7. We can now look for which strategies are best responses for each player, given the strategy played by all other players. For player *A*, *U* is a best response when $s_B = (L, L), (L, R)$, or (R, L) and *D* is a best response when $s_B = (R, R)$. Player *B*'s best response depends on its type. For type 1, *L* is a best response when player *A* chooses *U* and *R* is a best response when player *A* chooses *D*. For type 2, *L* is a best response when player *A* chooses *D* and *R* is a best response when player *A* chooses *U*. Denote each player's best responses by putting asterisks next to the corresponding payoffs and then look for a strategy profile where there is an asterisk next to the payoff for player *A* and the payoff to both types of player *B*

	(*L*,*L*)	(*L*,*R*)	(*R*,*L*)	(*R*,*R*)
U	30*, (25*,0)	15*, (25*,25*)	15*, (0,0)	0, (0,25*)
D	0, (0, 25*)	10, (0, 0)	10, (25*, 25*)	20*, (25*, 0)

8. The only strategy profile that is a mutual best response for both players (and both types for player *B*) is $s = \{U, (L, R)\}$, which is the unique Bayesian Nash equilibrium for this game.

B. Cournot Duopoly with Asymmetric Information about Cost

1. Model setup:
   a. Each firm chooses a quantity: q_1 for firm 1 and q_2 for firm 2
   b. Inverse demand: $P(Q) = a - Q$, with $Q = q_1 + q_2$
   c. Cost:
      - Firm 1: c_1
      - Firm 2: $c_2 = \begin{cases} c_2^H \\ c_2^L \end{cases}$

   d. $\Pr(c_2 = c_2^H) = \Pr(c_2 = c_2^L) = 0.5$
   e. Payoffs:
      - Define $t_i = a - c_i$
      - Assume that: $\begin{cases} t_1 = 1 \\ t_2^H = 3/4 \\ t_2^L = 5/4 \end{cases}$
      - $\pi_i = (t_i - q_1 - q_2)q_i$
2. Solving for Bayesian Nash equilibrium
   a. Firm 1:

$$Max_{q_1}\ p(t = t_2^H)(1 - q_1 - q_2^H)q_1 + p(t = t_2^L)(1 - q_1 - q_2^L)q_1$$

$$= Max_{q_1}\ (1 - q_1 - \frac{q_2^H}{2} - \frac{q_2^L}{2})q_1$$

Best response function:

$$q_1 = \frac{1 - \dfrac{q_2^H + q_2^L}{2}}{2} = \frac{1 - E(q_2)}{2}$$

   b. Firm 2: find a strategy for each type

$$Max_{q_2^\omega}\ (t_2^\omega - q_1 - q_2^\omega)q_2^\omega \quad \text{for } \omega = H, L$$

Best response function:

$$q_2^{\omega} = \frac{t_2^{\omega} - q_1}{2}$$

$$q_2^H = \frac{t_2^H - q_1}{2} = \frac{3/4 - q_1}{2}$$

$$q_2^L = \frac{t_2^L - q_1}{2} = \frac{5/4 - q_1}{2}$$

$$E(q_2) = \frac{q_2^H + q_2^L}{2} = \frac{1 - q_1}{2}$$

Note: the last line (E(q_2)) is found by taking expectation over the strategy of player 2 for various types: $E(q_2) = 0.5q_2^L + 0.5q_2^H$

c. Solving for equilibrium: use E(q_2) in the reaction function for firm 1 to solve for q_1, then use this to solve for q_2^L and q_2^H.

$$q_1^* = 1/3$$

$$q_2^{H*} = 5/24$$

$$q_2^{H*} = 11/24$$

d. Note: this is a different outcome than if firm 1 knew what firms 2's costs were when it chose its strategy. Then firm 1 could adjust its strategy based on the actual costs of firm 2.

C. Provision of a Public Good (free-rider problem)

1. Setup
   a. Two players i = 1, 2
   b. Strategies: each player chooses either to contribute or not to contribute
   c. Payoffs:
      - If at least one player contributes then the public good is provided
      - The benefits of the public good are 1 for each player
      - Costs of provision for player i: c_i
      - Cost of provision for each player is drawn from a probability distribution function $p(c)$. Let $P(c)$ be the cumulative distribution. Assume the support on the distribution is $[\underline{c}, \bar{c}]$. Assume that $\underline{c} < 1 < \bar{c}$.
      - Information about the cost of information is private information: player i knows their own costs but not the cost of the player j.

Payoff matrix of the public goods contribution game

	Contribute	Don't contribute
Contribute	1-c_1, 1-c_2	1-c_1, 1
Don't contribute	1, 1-c_2	0, 0

2. Solving for Bayesian Nash equilibrium
   a. Strategy as a function of type: $s_i(c_i)$, where type is the cost of provision (c_i)
   b. For a given type, there are two possible pure strategies – contribute or don't contribute
   c. Payoffs as a function of choosing each strategy:
      - Payoffs from contribution: 1-c_i
      - Payoffs from not contributing: 1*z_j, where z_j,is the probability that the other player contributes
   d. Many possible ways to try to find equilibrium: one reasonable approach is to look for "cutoff strategies" where the player contributes if costs of contribution are sufficiently low and to not contribute otherwise.
   e. Characterizing cutoff strategies
      - At the cutoff cost level, c_i^*, a player should be indifferent between contributing and not contributing: $1-c_i^* = z_j$
      - Note that $z_j = \Pr(c_j < c_j^*) = P(c_j^*)$
      - Therefore, $1-c_i^* = P(c_j^*)$
      - We also have $1-c_j^* = P(c_i^*)$; or $c_j^* = 1-P(c_i^*)$
      - So: $1-c_i^* = P(1-P(c_i^*))$
      - This is an implicit function that characterizes the optimal cutoff rule: c_i^* (but without further assumptions this is all that we can say)
   f. To make further progress, let's restrict attention to symmetric equilibrium:
      $c_i^* = c_j^* = c^*$
      $c^* = 1-P(c^*)$
   g. Assume a uniform distribution on the range (0, 2): $[\underline{c}, \bar{c}] = [0,2]$
      - P(c*) = c*/2
      - c* = 1 – P(c*) = 1 – c*/2
      - c* = 2/3
3. Summary – intuition of the cutoff rule equilibrium. At c* = 2/3, the value to the player is 1/3 by contributing. At c* = 2/3, the expected value to the player is 1/3 from not contributing, because this is the probability that the other player will contribute. There will be some free riding – because c* < 1 – a player will choose not to contribute even when the benefits of having the public good to them exceed the cost of contribution.

D. First Price Sealed-Bid Auction with two bidders

1. Setup
   a. Two players (bidders): $i = 1, 2$
   b. Strategy: each player bids: b_i
   c. Payoffs $U_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i \text{ if } b_i > b_j \\ (v_i - b_i)/2 \text{ if } b_i = b_j \\ 0 \text{ if } b_i < b_j \end{cases}$

      where v_i is the value of the item to player i.
      Note: in what follows we will ignore ties (probability 0 event with continuous strategies and continuous types)
   d. Information on valuation is private information (value determines type). Assume that value is uniformly distributed on [0, 1].

2. Solving for Bayesian Nash equilibrium
   a. Restrict attention to symmetric solutions: there may well be additional asymmetric solutions
   b. The problem facing each bidder given that they have value v_i is:

      $$Max_{b_i}\ (v_i - b_i)\ \Pr(b_i > b(v_j))$$

      where $b(v_j)$ is the bid function for player j as a function of player j's type (v_j) (the strategy is a mapping from value (type) to bid)
   c. Two methods of solution – "guess" the answer and show that it is right (easier with the right guess); or derive it using general principles (harder but general)
   d. Finding the solution via guessing: suppose that the function $b(v_j) = v_j/2$
   e. Since v_j is uniformly distributed on [0, 1], $b(v_j)$ is uniformly distributed on [0, ½]. Therefore: $\Pr(b_i > b(v_j)) = \begin{cases} 2b_i \text{ for } 0 \le b_i \le 1/2 \\ 1 \text{ for } b_i > 1/2 \end{cases}$
   f. The payoff maximization problem for player i is (focus on $0 \le b_i \le 1/2$):

      $$Max_{b_i}\ (v_i - b_i)\ \Pr(b_i > b(v_j))$$
      $$= Max_{b_i}\ (v_i - b_i)\ 2b_i$$

   g. The solution to this problem yields $b(v_i) = v_i/2$. So, in fact, the "guess" was correct and a Bayesian Nash equilibrium is to bid $b(v_i) = v_i/2$.

h. Find a solution via the general methods: assume a monotonic bid as a function of value so we can invert $b(v_j)$: $b^{-1}(b_j) = v_j$ (map from bid to value).

i. Simplifying the probability statement:

$$\Pr(b_i > b(v_j)) = \Pr(b^{-1}(b_i) > v_j) = b^{-1}(b_i).$$

Note: given a uniform distribution on [0, 1], the cumulative probability of x is equal to x: $P(x) = x$. In words, what this shows is that the probability that $b_i > b_j$ is equal to the probability that $v_i > v_j$.

j. The problem facing each bidder substituting in for the probability statement is:

$$Max_{b_i}\ (v_i - b_i)\, b^{-1}(b_i)$$

The first order necessary condition for an optimal solution:

$$-b^{-1}(b_i)) + (v_i - b_i)\frac{d}{db_i} b^{-1}(b_i) = 0$$

k. Some useful calculations that allow substitution for expressions in the first order condition

- $b^{-1}(b_i)) = v_i$
- $\dfrac{d}{db_i} b^{-1}(b_i) = \dfrac{dv_i}{db_i}$
- $\dfrac{db_i}{dv_i} = b'(v_i)$
- $\dfrac{dv_i}{db_i} = \dfrac{1}{\dfrac{db_i}{dv_i}} = \dfrac{1}{b'(v_i)}$

l. Using these facts, we can rewrite the first order condition as:

$$-v_i + (v_i - b_i(v_i))\frac{1}{b'(v_i)} = 0$$

$$b'(v_i)v_i + b_i(v_i) = v_i$$

m. This last expression is a first-order differential equation in $b(v_i)$. Solving this differential equation:

$$\int b'(v_i)v_i + b_i(v_i) = \int v_i$$

$$b(v_i)v_i = \frac{{v_i}^2}{2} + k$$

where k is a constant of integration. We know that when $v_i = 0$, that $b_i = 0$, so $k = 0$. Therefore, we have:

$$b(v_i) = \frac{v_i}{2}$$

3. Summary – intuition for equilibrium bid function: there is a tradeoff between the probability of winning and the amount you win ($v_i - b_i$). The optimal solution is to bid half the value. This is virtually the same problem as that for a monopolist who faces a linear demand curve and the solution is the same (price halfway down the demand curve).

E. Double Auction (Buyer and Seller with Private Information)

1. Setup
   a. Players: one buyer and one seller
   b. Strategy: buyer offers a price P_b; seller offers a price P_s.
   c. Trade occurs at price $P = \frac{P_b + P_s}{2}$ when $P_b \geq P_s$, and does not occur otherwise
   d. Payoffs:

$$\text{Buyer: } U_b = \begin{cases} v_b - P \text{ with trade} \\ 0 \text{ with no trade} \end{cases}$$

$$\text{Seller: } U_s = \begin{cases} P - v_s \text{ with trade} \\ 0 \text{ with no trade} \end{cases}$$

   where v_b is the value of the item to the buyer and v_s is the value of the item to the seller.
   e. Types: v_b and v_s are independently distributed uniformly on [0, 1]. Each player knows their own value but not the value of the other player.

2. Solving for Bayesian Nash equilibrium
   a. Strategies as a function of types: $P_b(v_b)$ & $P_s(v_s)$
   b. Note: since a player does not know the rival's value, they don't know their price offered and so they don't know if trade will occur or at what price if it does.
   c. The utility maximization problem facing the buyer and the seller is as follows:

$$\text{Buyer: } Max_{p_b}\ (v_b - P) \mid P_b \geq P_s$$

$$\text{Seller: } Max_{p_s}\ (P - v_s) \mid P_b \geq P_s$$

d. We can rewrite this (and make it significantly uglier, though more operational) as:

$$\text{Buyer: } Max_{p_b}\left[v_b - \frac{P_b + E(P_s(v_s) \mid P_b \geq P_s)}{2}\right]\Pr(P_b \geq P_s(v_s))$$

$$\text{Seller : } Max_{p_s}\left[\frac{P_s + E(P_b(v_b) \mid P_b \geq P_s)}{2} - v_s)\right]\Pr(P_b \geq P_s(v_s))$$

e. Note: there are many possible forms the equilibrium can take. We will explore two types of equilibria:
   - Single price equilibrium,
   - Linear strategy equilibrium

f. General warning: incomplete information games can generate lots of equilibria (this is even more of a problem in dynamic versions). One game theorists said: "You can get anything as an equilibrium result with the right kind of imperfect information."

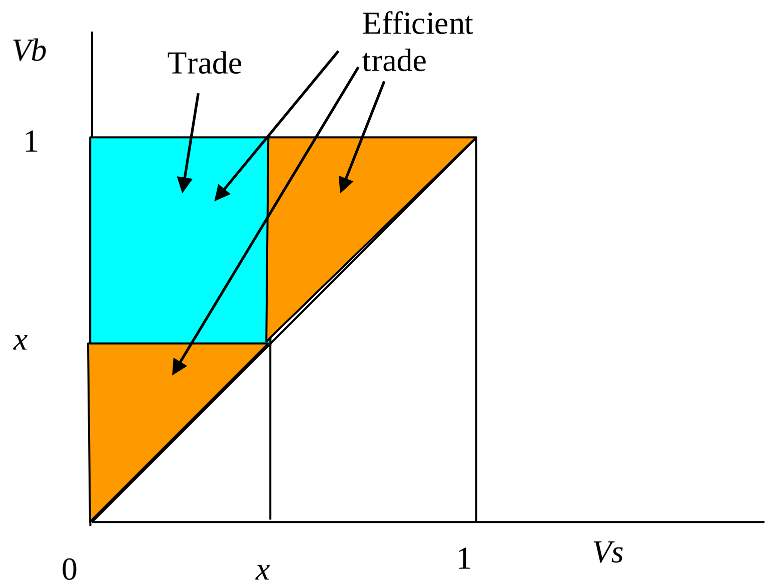
3. Single price equilibrium
   a. Let $x \in [0,1]$ be the single price
   b. Strategies:

$$\text{Buyer: } P_b = \begin{cases} x & \text{if } v_b \geq x \\ 0 & \text{if } v_b < x \end{cases}$$

$$\text{Seller : } P_s = \begin{cases} x & \text{if } v_s \leq x \\ 1 & \text{if } v_s > x \end{cases}$$

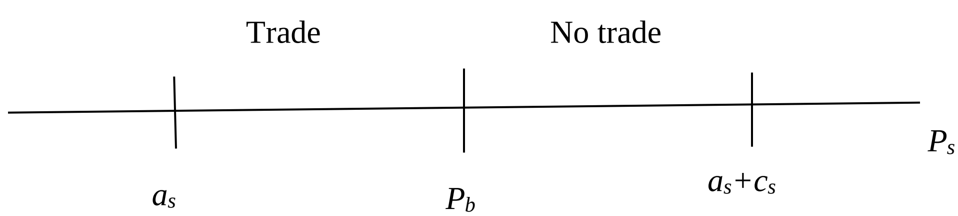
   c. Show that this is a Bayesian Nash equilibrium: to show this we need to check that each player's strategy is a best response to the other player's strategy
   d. Buyer:
      - Assume that $v_s \leq x$ (otherwise payoffs will be 0)
      - Payoff: $Max_{p_b}\left[v_b - \frac{P_b + x}{2}\right]\Pr(P_b \geq x)$
      - $\Pr(P_b \geq x)$ is either equal to 1 or 0: 1 if $P_b \geq x$, otherwise 0
      - If $v_b < x$, then buyer does not want to trade and one way to guarantee this occurring is to set $P_b = 0$.
      - If $v_b \geq x$, then trade occurring is beneficial for the buyer. Note that the payoff function is declining in P_b as long as $P_b \geq x$. The best response is to set $P_b = x$.
      - Therefore, the strategy for the buyer: $P_b = \begin{cases} x & \text{if } v_b \geq x \\ 0 & \text{if } v_b < x \end{cases}$ is a best response to the seller's strategy $P_s = \begin{cases} x & \text{if } v_s \leq x \\ 1 & \text{if } v_s > x \end{cases}$.

e. Similar logic shows that seller's strategy is a best response to the buyer's single price strategy (but will not be shown here).
f. Since both players' strategy is a best response to the rival player's strategy so this is a Bayesian Nash equilibrium.
g. Pattern of trade
   Trade occurs when $v_b \geq x$ and $v_s \leq x$ (blue box in the figure below)
   Efficient trade occurs whenever $v_b \geq v_s$ (blue and orange areas above the 45 degree line)



Note: trade only occurs for a subset of conditions when it is efficient to do so.

4. Linear strategy equilibrium
   a. Strategy: Buyer $P_b(v_b) = a_b + c_b v_b$
      Seller $P_s(v_s) = a_s + c_s v_s$
   b. Derive buyer's best response to the seller's strategy
      - $$Max_{p_b}\left[v_b - \frac{P_b + E(P_s(v_s) \mid P_b \geq P_s)}{2}\right]\Pr(P_b \geq P_s(v_s))$$
      - Note: v_s is distributed uniformly on [0, 1], and $P_s(v_s) = a_s + c_s v_s$, therefore, P_s is distributed uniformly on [a_s, a_s+c_s].



- $\Pr(P_b \geq P_s) = \dfrac{P_b - a_s}{a_s + c_s - a_s} = \dfrac{P_b - a_s}{c_s}$
- $E(P_s(v_s) \mid P_b \geq P_s) = \dfrac{a_s + P_b}{2}$

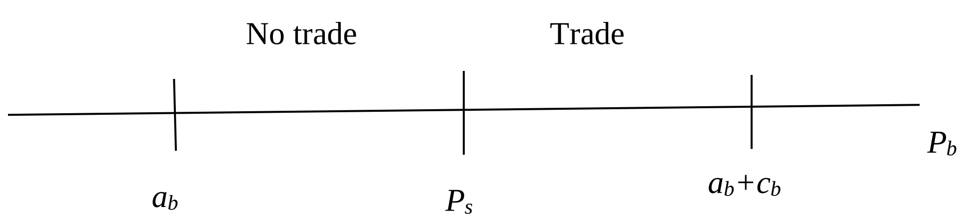
  $Max_{p_b}\left[v_b - \dfrac{P_b + E(P_s(v_s) \mid P_b \geq P_s)}{2}\right]\Pr(P_b \geq P_s(v_s))$
- $= Max_{p_b}\left[v_b - \dfrac{P_b + \dfrac{a_s + P_b}{2}}{2}\right]\dfrac{P_b - a_s}{c_s}$
- The first order necessary condition for an optimal solution is (after re-arranging):

  $P_b = \dfrac{1}{3}a_s + \dfrac{2}{3}v_b$

  Therefore, we have: $a_b = \dfrac{a_s}{3}$; $c_b = \dfrac{2}{3}$

c. Derive seller's best response to the buyer's strategy

- $Max_{p_s}\left[\dfrac{P_s + E(P_b(v_b) \mid P_b \geq P_s)}{2} - v_s)\right]\Pr(P_b \geq P_s(v_s))$
- Note: v_b is distributed uniformly on [0, 1], and $P_b(v_b) = a_b + c_b v_b$, therefore, P_b is distributed uniformly on [a_b, a_b+c_b].



- $\Pr(P_b \geq P_s) = \dfrac{a_b + c_b - P_s}{a_b + c_b - a_b} = \dfrac{a_b + c_b - P_s}{c_b}$
- $E(P_s(v_s) \mid P_b \geq P_s) = \dfrac{a_b + c_b + P_s}{2}$

  $Max_{p_s}\left[\dfrac{P_s + E(P_b(v_b) \mid P_b \geq P_s)}{2} - v_s)\right]\Pr(P_b \geq P_s(v_s))$
- $= Max_{p_s}\left[\dfrac{P_s + \dfrac{a_b + c_b + P_s}{2}}{2} - v_s\right]\dfrac{a_b + c_b - P_s}{c_b}$

- The first order necessary condition for an optimal solution is (after re-arranging):

  $P_s = \frac{1}{3}(a_b + c_b) + \frac{2}{3}v_s$

  Therefore, we have: $a_s = \frac{1}{3}(a_b + c_b) = \frac{1}{3}(\frac{a_s}{3} + \frac{2}{3}) = \frac{1}{4}$; $c_s = \frac{2}{3}$

  Using this we can solve for a_b: $a_b = \frac{a_s}{3} = \frac{1}{12}$

d. Pulling this information together, we have the Bayesian Nash equilibrium:

Buyer $P_b(v_b) = \frac{1}{12} + \frac{2}{3}v_b$

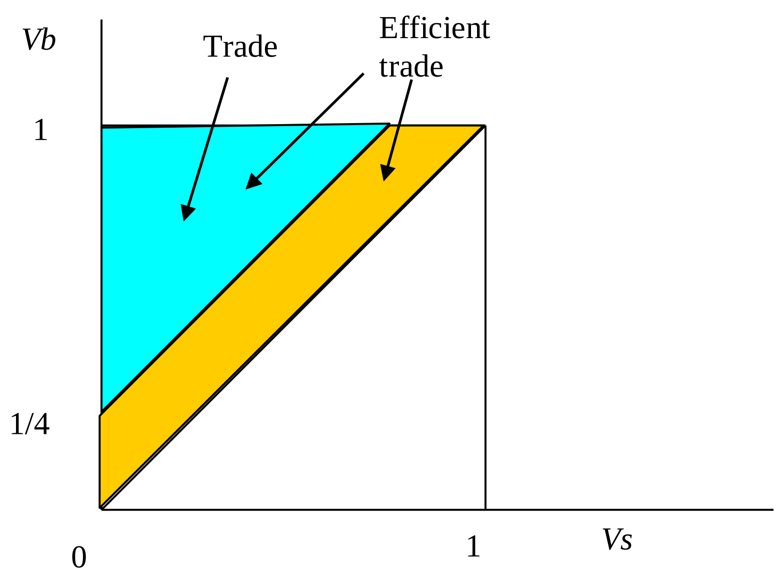
Seller $P_s(v_s) = \frac{1}{4} + \frac{2}{3}v_s$

e. Pattern of trade

Trade occurs if and only if: $P_b \geq P_s$

$P_b(v_b) = \frac{1}{12} + \frac{2}{3}v_b \geq \frac{1}{4} + \frac{2}{3}v_s = P_s(v_s)$

$v_b \geq v_s + 1/4$



f. Summary

- Double auction is a model of a very general trade problem: typically buyer and seller do not know the valuation of the other side of the transaction. With small numbers, as with buying a car or a house, the double auction is an appropriate model
- With asymmetric information, there is an incentive to downplay valuation by the buyer or overplay the value for the seller in order to get more of the value from any trade that occurs
- But, trying to drive a harder bargain may result in trade not occurring at all.
- Must weigh the value of a trade against the probability of a trade occurring
- Results shown for single price and linear strategies each result in inefficient outcomes in the sense that some trades that should occur ($v_b \geq v_s$) do not in fact occur.
- Myerson and Satterthwaite (1983) prove that the double auction game will generate an inefficient outcome in the case where there is some probability that a trade should not occur. They also show that the linear strategy solved for above constitutes the best Bayesian Nash equilibrium in the sense of minimizing the inefficiency (i.e., the linear strategy solves the mechanism design problem in the double auction case).
- Note: Coase theorem – with no transactions costs and perfect information, trade will lead to an efficient outcome. Results here show that incomplete information will lead to inefficient outcomes.