ApEc 8001
Applied Microeconomic Analysis: Demand Theory

Lecture 8: Aggregate Demand (MWG, Ch. 4)

I. Introduction

For many fields of economics, especially macroeconomics, the **aggregate demand behavior** of consumers is of more importance than the behavior of individual consumers. While one can always study the behavior of individual consumers and then "add up" the results to obtain results for their aggregate behavior, it may be easier if we could **depict aggregate demand behavior as the behavior of a "representative consumer" who is a rational agent**. More specifically, there are (at least) three questions that economists have about aggregate demand:

1. When can aggregate demand be expressed as a function of prices and aggregate (or average) wealth?

2. When does aggregate demand satisfy the weak axiom of revealed preference?

3. When does aggregate demand provide information about aggregate welfare?

This lecture addresses each of these questions. As you will see, strong assumptions are needed for useful results.

II. Aggregate Demand and Aggregate Wealth

Consider an **economy with I consumers**, each of whom has a distinct preference relation $\succsim_i$, and corresponding demand functions $x_i(p, w_i)$. Assume that everyone faces the same prices $p \in \mathbb{R}^L$ for L goods. **Aggregate demand is the sum of the individual demands** of the I consumers:

$$x(p, w_1, w_2, \ldots w_I) = \sum_{i=1}^{I} x_i(p, w_i)$$

The **main question** addressed in this section is, **under what conditions** can we "simplify" the **aggregate demand** function $x(p, w_1, w_2, \ldots w_I)$ so that instead of being a general function of the entire distribution of wealth it is **a function of only prices and total wealth**:

$$x(p, \sum_{i=1}^{I} w_i)$$

For this ("simplification") property to hold in all possible situations, aggregate demand must be identical for any two distributions of wealth that have the same total wealth. That is, for two distributions of wealth, $(w_1, w_2, \ldots w_I)$ and $(w_1', w_2', \ldots w_I')$, for which $\sum_{i=1}^{I} w_i = \sum_{i=1}^{I} w_i'$, it must be the case that $\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} x_i(p, w_i')$.

To see when this condition is satisfied, let's start with an initial distribution of wealth, $(w_1, w_2, \dots w_I)$, and consider a vector of changes in wealth, $(dw_1, dw_2, \dots dw_I) \in \mathbb{R}^I$, that satisfies $\sum_{i=1}^{I} dw_i = 0$. For this redistribution of wealth to have no effect on aggregate demand the following must hold for all L goods:

$$\sum_{i=1}^{I} \frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} dw_i = 0 \quad \text{for all } \ell = 1, 2, \dots L$$

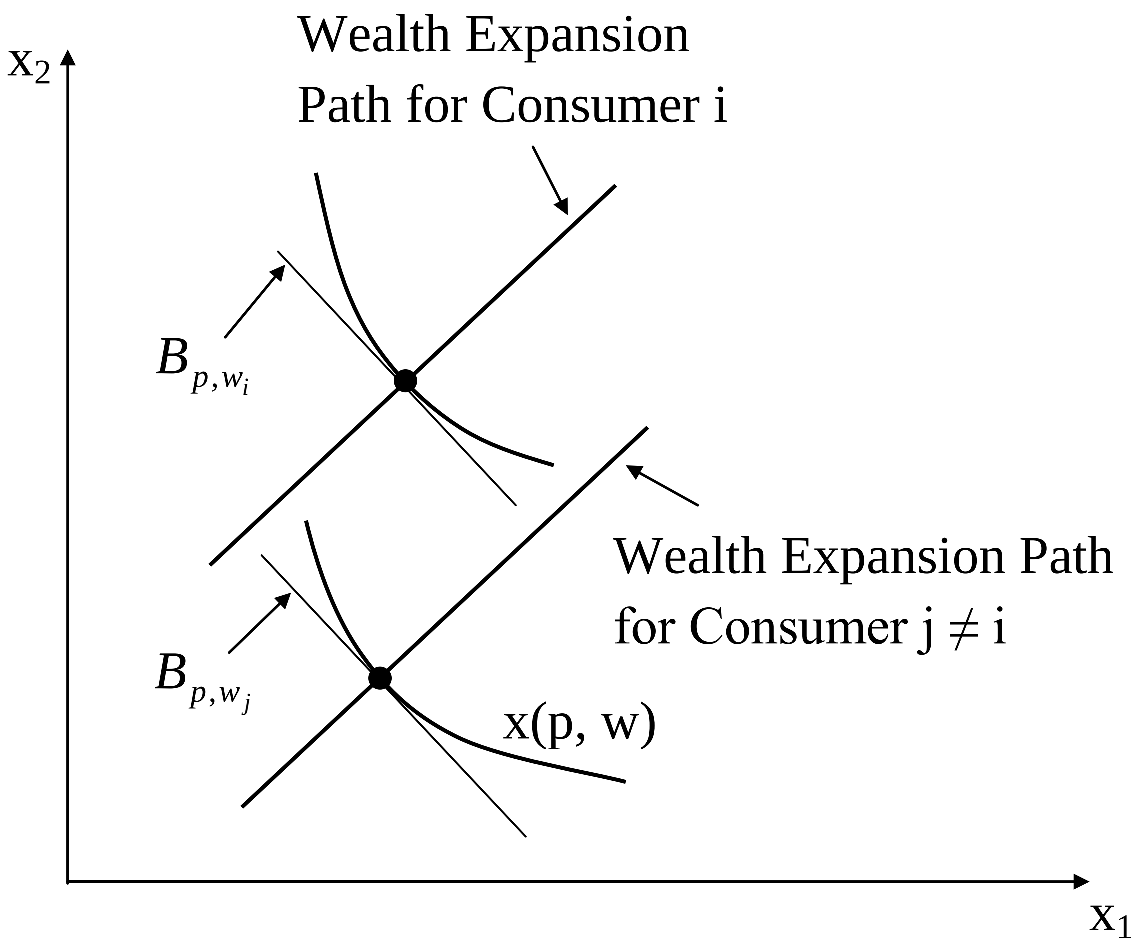
Indeed, this must be true for **all** possible redistribution vectors that satisfy $\sum_{i=1}^{I} dw_i = 0$, and for all possible initial distributions of wealth $(w_1, w_2, \dots w_I)$. **This works if and only if a redistribution of wealth between any two people, i and j, has no effect on aggregate demand**, which implies the following:

$$\frac{\partial x_{\ell i}(p, w_i)}{\partial w_i} = \frac{\partial x_{\ell j}(p, w_j)}{\partial w_j}$$

for all ℓ, all p, and all distributions of wealth $(w_1, w_2, \dots w_I)$.

That is, for any price vector p, and any commodity ℓ, **the wealth effect at p must be the same for all consumers** and for all wealth levels of all consumers. Thus the increase or decrease in demand from a change in wealth must be identical for all consumers at all levels of wealth.

Visually, the wealth expansion paths of all consumers must be parallel lines, as depicted in the diagram below:



This result raises the following question: **What restrictions on preferences lead to demand functions that have this characteristic?** Two examples are: 1. Everyone has the same homothetic preferences; and 2. Everyone has quasi-linear preferences for the same good. These are special cases of a more general assumption about preferences:

Proposition 4.B.1: A necessary and sufficient condition for all consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences lead to **indirect utility functions of the Gorman form**, with the same coefficient on w_i, b(p), for each consumer i:

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

The proof of sufficiency (that this functional form implies parallel, straight wealth expansion paths) is not very hard to show. But the other direction (necessity) is harder; see Deaton and Muellbauer (1980) for that result.

The requirement that changes in the distribution of income will have no effect on aggregate demand is **quite unrealistic**, which means that **the Gorman form** for the indirect utility function **is very restrictive**. This raises the question of whether aggregate demand can be modeled in a less restrictive (more realistic) way by allowing for some other (aggregate) variables in the aggregate demand function. One possibility is the variance of the distribution of wealth. Some economists have examined this; see the references on p.108 of Mas Colell et al.

Another approach to continue modeling aggregate demand as a function of only prices and average (or total) wealth but not to require that the aggregate demand function be valid for all possible prices and distributions of wealth.

To see how this could be done, consider an admittedly **extreme case**. Assume that individual i's wealth, w_i, is generated by a process that is a function of only prices (p) and aggregate wealth (denoted by w). That is:

$$w_i = w_i(p, w)$$

Note that the $w_i(\ )$ function can be different for each person, as indicated by the i subscript. While this may seem hard to believe, it may be possible. If a person's wealth is determined only by his or her labor and capital, both of which have a price, and by government taxes and/or transfers, which may depend only on wages and aggregate wealth), then this may be reasonable.

If this assumption is correct, then aggregate demand is a function of only p and aggregate wealth (w):

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} x_i(p, w_i(p, w)) = \sum_{i=1}^{I} x_i(p, w) = x(p, w)$$

If it is really true that $w_i = w_i(p, w)$, then $\sum_{i=1}^{I} w_i(p, w) = w$, where w is aggregate wealth, and we call the "family" of functions $w_1 = w_1(p, w)$, $w_2 = w_2(p, w)$, … $w_I = w_I(p, w)$ the **wealth distribution rule**. Thus if such a wealth distribution rule exists, aggregate demand can always be written as a function of only prices and aggregate wealth.

Question: How does this restrict p and the distrib. of w?

III. Aggregate Demand and the Weak Axiom

Turn now to the second question. In the most general terms, this question is: What properties of individual demand carry over to the aggregate demand function

$x(p, w_1, w_2, \ldots w_I) = \sum_{i=1}^{I} x_i(p, w_i)$?

In fact, three properties hold for aggregate demand if they hold for individual demand:

1. Continuity

2. Homogeneity of degree zero

3. Walras' law [$p\cdot x(p, w_1, w_2, \ldots w_I) = \sum_{i=1}^{I} w_i$]

But what about the weak axiom of revealed preference?

To have any possibility that aggregate demand satisfies the weak axiom, **we will need some more assumptions**. We need aggregate demand in the form $x(p, w)$. Assume that there exists a wealth distribution rule of the type given on the previous page (i.e. for all i, $w_i = w_i(p, w)$). Thus aggregate demand can be expressed as:

$$x(p, w) = \sum_{i=1}^{I} x_i(p, w_i(p, w))$$

To make further progress we **assume a more specific (and quite simple) distribution rule**: The wealth of each consumer is a fixed proportion of total wealth (and so does not depend on prices):

$$w_i(p, w) = \alpha_i w \text{ for all } w \in \mathbb{R}$$

This implies that we can write aggregate demand as:

$$x(p, w) = \sum_{i=1}^{I} x_i(p, \alpha_i w)$$

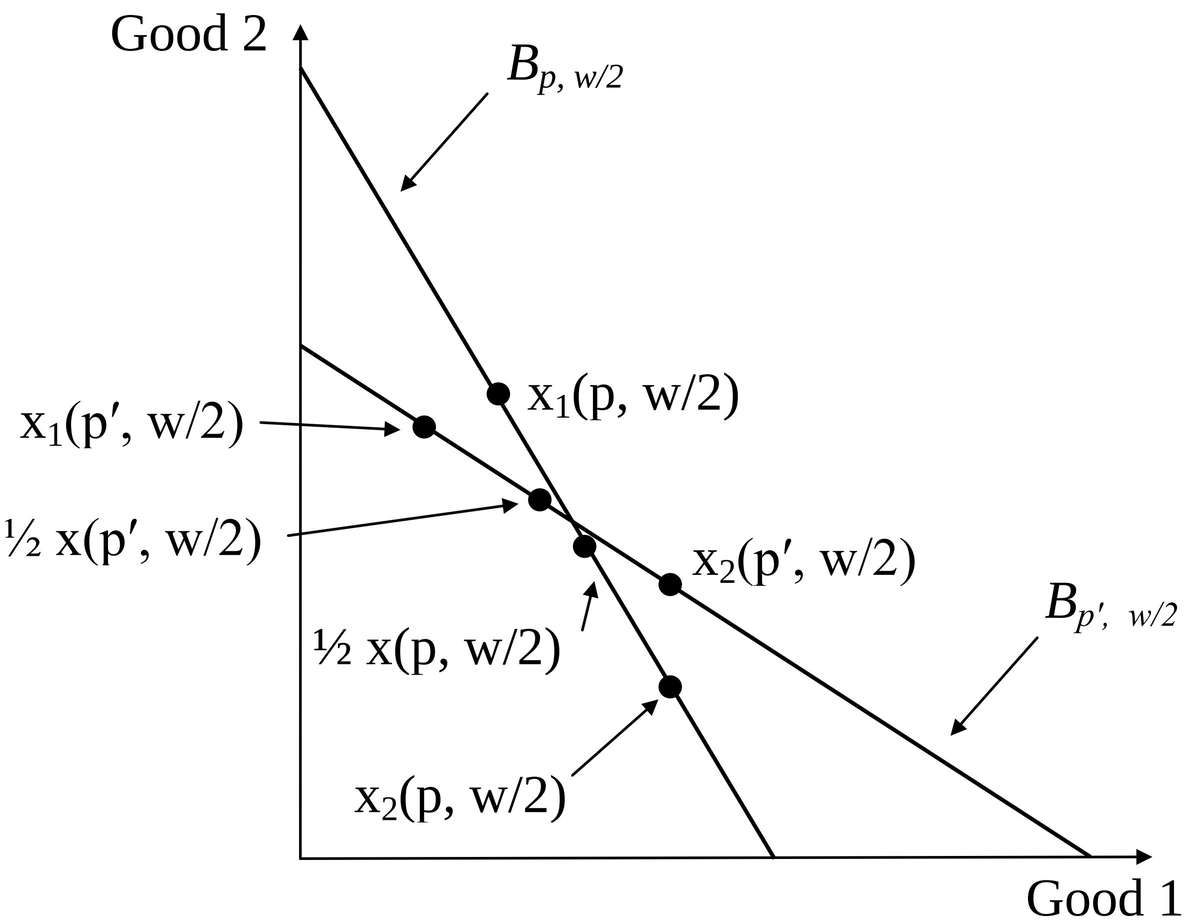
Here is the weak axiom (see p.2 of Lecture 3), applied to aggregate demand:

Definition: The ***aggregate*** (Walrasian) demand function $x(p, w)$ satisfies the **weak axiom of revealed preference** (WARP) if, for any two price-wealth pairs (p, w) and (p', w'):

$$\text{If } p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w),$$
$$\text{then } p' \cdot x(p, w) > w'$$

Even with this distribution rule assumption, it is possible that the aggregate demand function does not satisfy the weak axiom. This can be seen by examining the following example.

Example: Failure of Aggregate Demand to Satisfy the Weak Axiom. Suppose that there are 2 commodities and 2 consumers. Wealth is equally distributed: $w_1 = w_2 = w/2$. Consider the following diagram, with two price vectors, p and p′, and corresponding individual demands $x_1(p, w/2)$ & $x_2(p, w/2)$ under p, and $x_1(p', w/2)$ & $x_2(p', w/2)$ under p′:



The behavior each consumer satisfies the weak axiom. For example, the bundle chosen by consumer 1 for the price vector p´ is affordable under the vector p, but the bundle chosen for the vector p is not affordable under the price vector p′. But this is **not** the case for aggregate demand.

So what happened that makes individual consumers behave in a way that satisfies the weak axiom while aggregate behavior does not? This can be explained in terms of the **wealth effects of price changes**.

Recall from Lecture 3 (Proposition 2.F.1) that a demand function $x(p, w)$ **satisfies the weak axiom if and only if** it satisfies the law of demand for **compensated** price changes. That is, if (and only if) the price change to p' is compensated, so that $w' = p' \cdot x(p, w)$, then the following holds:

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

with the strict inequality holding if $x(p', w') \neq x(p, w)$.

If the price-wealth change from (p, w) to (p', w') had been a compensated price change for **each** consumer, that is if $\alpha_i w' = p' \cdot x_i(p, \alpha_i w)$ for all i, then because individual demands satisfy the weak axiom we would see that:

$$(p' - p) \cdot [x_i(p', \alpha_i w') - x_i(p, \alpha_i w)] \leq 0 \quad \text{for all } i = 1, 2, \ldots I$$

with the strict inequality holding if $x_i(p', \alpha_i w') \neq x_i(p, \alpha_i w)$. Adding this expression over all i shows that, **when each person is compensated, aggregate demand satisfies the weak axiom**: $(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$.

The problem is that **a price-wealth change that is compensated in the aggregate**, so that $w' = p' \cdot x(p, w)$, **need not be compensated for each individual**. That is, we may have $\alpha_i w' \neq p' \cdot x_i(p, \alpha_i w)$ for some or all i.

The Uncompensated Law of Demand

So are there any restrictions on individuals' preferences that, if imposed, imply that aggregate demand satisfies the weak axiom of revealed preference? Given the above explanation, a "natural" restriction could be that **uncompensated** changes in prices satisfy the inequality given above. In fact, this is exactly what we need to assume to get aggregate preferences to satisfy the weak axiom. To start, we need to define this restriction:

Definition: The individual demand function $x_i(p, w_i)$ satisfies the **uncompensated law of demand (ULD)** property if:

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0$$

for any p, p′ and w, with the strict inequality holding if $x_i(p', w_i) \neq x_i(p, w_i)$. This definition can also apply to the aggregate demand function (just delete the i subscripts).

Note that wealth has not changed; that is what we mean by **uncompensated**.

Similar to the discussion of the weak axiom in Lecture 3, if $x_i(p, w_i)$ is differentiable we can state the ULD property as:

> If $x_i(p, w_i)$ is differentiable and satisfies the ULD property, then $D_p x_i(p, w_i)$ is negative semidefinite; that is $dp \cdot D_p x_i(p, w_i) dp \leq 0$ for all dp.

As in Lecture 3, there is also a converse relationship:

> If $D_p x_i(p, w_i)$ is negative semidefinite for all p, then $x_i(p, w_i)$ satisfies the ULD property.

Both of these statements also apply to the aggregate demand function.

All of this leads to the following proposition:

Proposition 4.C.1: If every consumer's Walrasian demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property, so does the aggregate demand $x(p, w) = \sum_{i=1}^{I} x_i(p, \alpha_i w)$. Consequently, the aggregate demand function $x(p, w)$ satisfies the weak axiom.

Proof: Consider two price-wealth pairs for aggregate demand, (p, w) and (p', w), with $x(p, w) \neq x(p', w)$. It must be the case that, for at least one person i:

$$x_i(p, \alpha_i w) \neq x_i(p', \alpha_i w)$$

Since ULD holds for all i, for each such person it must be that $(p' - p)\cdot[x_i(p', \alpha_i w) - x_i(p, \alpha_i w)] < 0$. Summing this over all i implies ULD holds for aggregate demand:

$$(p' - p)\cdot[x(p', w) - x(p, w)] < 0$$

To prove that the weak axiom holds, use any (p, w) and (p′, w′), with $\mathbf{x(p, w) \neq x(p', w')}$ and $\mathbf{p\cdot x(p', w') \leq w}$. Define $p'' = (w/w')p'$. By homogeneity of degree zero, we have $x(p'', w) = x(p', w')$, which implies $p\cdot x(p'', w) \leq w$. ULD implies $(p'' - p)\cdot[x(p'', w) - x(p, w)] < 0$, or alternatively $p''\cdot x(p'', w) - p\cdot x(p'', w) - p''\cdot x(p, w) + p\cdot x(p, w) < 0$. Combining this with $p\cdot x(p'', w) \leq w$ and Walras' law implies that $p''\cdot x(p, w) > w$. Finally, the definition of p″ implies that $\mathbf{p'\cdot x(p, w) > w'}$. **Q.E.D.**

While this result shows that the ULD property implies that aggregate demand satisfies the weak axiom, we need to think about the restrictiveness (reasonableness) of ULD. The following two propositions give two situations under which ULD holds for individual preferences:

Proposition 4.C.2: If $\succsim_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property.

Proof: For simplicity, assume that $\succsim_i$ is represented by a differentiable utility function $x_i(p, w_i)$. (Differentiability is not necessary, but the proof without this assumption is more complicated.) Consider the matrix $D_p x_i(p, w_i)$:

$$D_p x_i(p, w_i) = S_i(p, w_i) - (1/w_i)\, x_i(p, w_i)\, x_i(p, w_i)^T$$

where $S_i(p, w_i)$ is consumer i's Slutsky matrix. Because $[dp \cdot x_i(p, w_i)]^2 > 0$ (except when $dp \cdot x_i(p, w_i) = 0$) and $dp \cdot S_i(p, w_i)dp < 0$ (except when dp is proportional to p), we can conclude that $D_p x_i(p, w_i)$ is negative semidefinite, and so the ULD condition holds. **Q.E.D.**

Question: Where was the homotheticity assumption used in this proof?

Unfortunately, the assumption of homotheticity is very doubtful, and tests using real consumer demand data virtually always reject it. Mas Colell et al. do not mention this, but they do point out that homotheticity "works" because it has very simple wealth effects (all goods are normal goods), and this proof does not need to make use of any substitution effects. The following proposition may be more credible:

Proposition 4.C.2: Suppose that $\succsim_i$ is defined on the consumption set $X = \mathbb{R}^L_+$, and can be represented by a twice differentiable concave function $u_i(\ )$. If

$$-\frac{x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4 \quad \text{for all } x_i$$

then $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property.

The proof of this is left as an exercise for "the courageous reader".

This condition may not be very restrictive, but I am not aware that anyone has derived its theoretical implications or tested it with real data.

On page 113, Mas-Colell et al. point out that under certain (quite restrictive) assumptions, it is possible that aggregate demand satisfies ULD even though individual consumers' demands may not satisfy it (it is not required to hold for the proof once other assumptions are made). It is not clear to me how useful this result is. Thus the discussion on pp.113-115 of Mas-Colell is optional.

IV. Aggregate Demand and the Existence of a Representative Consumer

When can we measure aggregate welfare using the aggregate demand function and the welfare measurement methods discussed in Lecture 7 that were used for individual consumers? That is, when can we treat the aggregate demand function as a relationship that is generated by a (fictional) **representative consumer** whose preferences can then be used to measure aggregate (social) welfare? This section examines this question.

To start, we need to make some assumption about how wealth is distributed to ensure that an aggregate demand function exists. To do this, assume there is a distribution rule, denoted by $(w_1(p, w), w_2(p, w), \ldots w_I(p, w))$, that assigned "wealths" to each person in society based only on prices and aggregate wealth. Assume that this rule distributes all the wealth, so that $\sum_{i=1}^{I} w_i(p, w) = w$ for all values of p and w, and that every $w_i(p, w)$ function is continuous and homogenous of degree one.

As seen above, these assumptions imply that aggregate demand $x(p, w) = \sum_{i=1}^{I} x_i(p, w)$ has the properties of a standard demand function, so that x(p, w) is continuous, homogenous of degree one, and satisfies Walras' law. Of course, the particular characteristics of x(p, w) will depend on the distribution of wealth, as determined by the distribution rule (w_1(p, w), w_2(p, w), … w_I(p, w)).

There are two senses in which a "representative consumer" exists, one is "positive" (just to describe behavior, not to make and welfare judgments) and the other is "normative" (used to make welfare judgments). We start by defining the positive representative consumer:

Definition: A **positive representative consumer** exists if there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the *aggregate* demand function x(p, w) is exactly the Walrasian demand function generated by this preference relation. That is $x(p, w) \succ x'$ whenever $x' \neq x(p, w)$ and $p \cdot x' \leq w$.

In other words, the "positive representative consumer" is an "imaginary" individual who, by maximizing his or her utility given a budget set $\{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$, generates the (aggregate) demand function x(p, w) that is observed for the economy as a whole.

If we want to use aggregate data **to conduct any welfare analysis** of the type discussed in Lecture 7, we need to have a positive representative consumer because **we need a utility function** that is consistent with the aggregate data. Yet while the existence of a positive representative consumer is a necessary condition for welfare analysis, it is not a sufficient condition. We also **need to link the welfare of the positive representative consumer to the welfare of the actual population**. Before we draw that link, we need to define a **social welfare function** that aggregates the welfare of each member of the population into an aggregate welfare function for society as a whole:

Definition: A (Bergson-Samuelson) **social welfare function** is a function W: $\mathbb{R}^I \to \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, u_2, \ldots u_I) \in \mathbb{R}^I$ of the utility levels of the I consumers in the economy.

We can write the social welfare function as $W(u_1, u_2, \ldots u_I)$ and we will assume that it is increasing in all its arguments, concave and (when useful) differentiable.

Suppose that there is a benevolent central authority that, for any given prices p, redistributes aggregate wealth w to all I individuals in society to maximize the social welfare function. In other words, suppose that the **wealth distribution rule** is designed to **maximize social welfare**:

$$\max_{w_1, w_2 \ldots w_I} W(v_1(p, w_1), v_2(p, w_2), \ldots v_I(p, w_I))$$

$$\text{subject to } \sum_{i=1}^{I} w_i = w$$

where $v_i(p, w_i)$ is individual i's indirect utility function.

This optimization of the social welfare produces the **social indirect utility function**, which can be denoted by $v(p, w)$. The following proposition shows that **this social indirect utility function can be used to "produce" a positive representative consumer for the aggregate demand function** $x(p, w) = \sum_{i=1}^{I} x_i(p, w_i(p, w))$:

Proposition 4.D.1: Assume that, for each level of prices p and aggregate wealth w, the wealth distribution function $(w_1(p, w), w_2(p, w), \ldots w_I(p, w))$ solves the above social welfare maximization problem. Then the maximized value, which can be denoted by $v(p, w)$, is an indirect utility function for a positive representative consumer whose behavior yields the aggregate demand function $x(p, w) = \sum_{i=1}^{I} x_i(p, w_i(p, w))$.

The proof is given on pp.117-118 of Mas Colell et al. The basic idea is to apply Roy's identity to this aggregate indirect utility function to derive an aggregate (Walrasian) demand function, and then show that this function indeed satisfies $x(p, w) = \sum_{i=1}^{I} x_i(p, w_i(p, w))$.

With this "aggregate" indirect utility function we can now define a normative representative consumer:

Definition 4.D.1: The positive representative consumer with a preference relation $\succsim$ that yields the aggregate demand function $x(p, w) = \sum_{i=1}^{I} x_i(p, w_i(p, w))$ is a **normative representative consumer** that corresponds to the social welfare function W( ) **if** for every (p, w), the distribution of wealth $(w_1(p, w), w_2(p, w), \ldots w_I(p, w))$ solves the above social welfare maximization problem **and** the solution, v(p, w), is an indirect utility function for $\succsim$.

Example: Assume that all consumers have homothetic preferences that are represented by utility functions that are homogenous of degree one. Let the social welfare function be $W(u_1, u_2, \ldots u_I) = \sum_{i=1}^{I} \alpha_i \ln(u_i)$, with all $\alpha_i > 0$ and $\sum_{i=1}^{I} \alpha_i = 1$. Then one can show that the optimal wealth distribution is the (price independent) rule that $w_i(p, w) = \alpha_i w$. Thus, in the homothetic case, the aggregate demand function $x(p, w) = \sum_{i=1}^{I} x_i(p, w_i(p, \alpha_i w))$ can be viewed as originating from the normative representative consumer that is generated by this social welfare function.

The Bottom Line: If a normative representative consumer exists, then a positive representative consumer exists, but **not** vice versa. To obtain a normative representative consumer you need to assume (impose?) a social welfare function and find a wealth distrib. rule that maximizes it.