Department of Mathematics

10B11MA111 **Mathematics-I**
Tutorial Sheet 5 **B.Tech. Core**

Vector Integral Calculus

1. Evaluate the work done by a force $\vec{F} = x^2\hat{i} - 2y\hat{j} + z^2\hat{k}$ over the straight line path from (-1, 2, 3) to (2, 3, 5).

2. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = 3x^2y^2\vec{i} + 2x^3y\vec{j}$ and

   (a) C is the line segment joining the points (-1,1) to (1,1),
   (b) C consists of the line segment from (-1, 1) to (0, 0) followed by the line segment from (0, 0) to (1, 1).
   (c) C consists of the semi circular path joining the points (-1,1) to (1,1) in the positive direction.
   What do you observe. Integral is same for all. Moreover, $\vec{F}$ is conservative so integral doesnot depend upon path

3. Show that $\int_A^B [2xyz^2\,dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz]$ is independent of path of integration joining the points A and B. Also, evaluate the integral for A(0,0,1) and B(1, $\pi/4$, 2).

4. Apply Green's theorem to evaluate the following integral: $\oint_C (e^x \sin y\, dx + e^x \cos y\, dy)$, where C is the ellipse $4(x+1)^2 + 9(y-3)^2 = 36$.

5. Verify Green's theorem for the integral $\oint_C (y^2dx + x^2dy)$, where C is the triangle bounded by lines x = 0, x + y = 1, y = 0.

6. Evaluate $\iint_S \vec{F}.d\vec{S}$ where $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and surface S is the part of plane $2x + 3y + 6z = 12$ included in the first octant.

7. Using Stoke's theorem evaluate $\iint_S curl\ \vec{F}.\hat{n}\,dS$, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and S is the part of surface of paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$.

8. Verify Stoke's theorem, when $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and surface S is part of sphere $x^2 + y^2 + z^2 = 1$ above xy-plane.

9. Use Divergence theorem to evaluate $\iint_S \vec{F}.d\vec{S}$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the open cylinder $x^2 + y^2 = 4$, $0 < z \leq 3$.

10. Verify Divergence theorem, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the planes, $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

C : line segment from $(-1,2,3)$ to $(2,3,5)$

$\vec{F} = x^2\hat{i} - 2y\hat{j} + z^2\hat{k}$

C : $x(t) = -1+3t$ $\quad \frac{z+1}{2+1}$

$y(t) = 2+t$

$z(t) = 3+2t$

$0 \le t \le 1$

$\cancel{ds} = \sqrt{3^2+1^2+2^2} = \sqrt{9+1+4} = \sqrt{14}$

$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 [3(-1+3t)^2 - 2(2+t) + (3+2t)^2 2]\,dt$

$= \int_0^1 x^2\hat{i} - 2 \cdot \quad = \int_0^1 \left[3(9t^2+1-6t) - 4 - 2t + 2(9+4t^2+12t) \right] dt$

$= \int_0^1 [35t^2 + 4t + 17]\,dt \quad -20t$

$= \frac{35}{3}t^3 + 2t^2 + 17t \Big|_0^1$

$= \frac{35}{3} + 2 + 17 = 19 + \frac{35}{3}$

$= \frac{57+35}{3} = \frac{92}{3}$

2) $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = 3x^2y^2\hat{i} + 2x^3y\hat{j}$

a) C is line segment joining $(-1,1)$ & $(1,1)$

$x(t) = -1+2t$

$y(t) = 1$ $\quad , 0 \le t \le 1$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 3(2t-1)^2 \times 2 + 2(2t-1)^3 \times 0 \; dt$

$= 6\int_0^1 (2t-1)^2\,dt = \frac{6}{3\times 2}(2t-1)^3 \Big|_0^1$

$= 1-(-1) = 2$

b) C consists of line segment from $(-1,1)$ to $(0,0)$ followed by line segment from $(0,0)$ to $(1,1)$

C_1: $x(t) = -1+t$

$y(t) = 1-t$

$0 \le t \le 1$

C_2: $x(t) = t$

$y(t) = t$

$0 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 3(t-1)^2(1-t)^2 + 2(t-1)^3(1-t)(-1) \quad dt$$

$$+ \int_0^1 3(t\cancel{-1})^2(\cancel{1-}t)^2 + 2(t\cancel{-1})^3(\cancel{1-}t)\, dt$$

$$= \int_0^1 5(t-1)^4\, dt + \int_0^1 5t^4\, dt$$

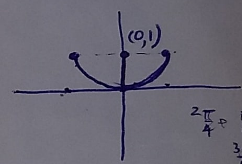
$$= (t-1)^5\Big|_0^1 + t^5\Big|_0^1$$

$$= 0-(-1) + 1 = 2$$

$$= 6\cancel{\int_0^1 (t-1)^4\, dt}$$

$$= \frac{6}{5}\cancel{(t-1)^5\Big|_0^1} = \cancel{\frac{6}{5}(0-(-1)) = \frac{6}{5}}$$

c) C consists of the semi-circular path joining $(-1,1)$ to $(1,1)$ in positive direction



$C: \; x(t) = \cos t$

$y(t) = 1 + \sin t$

$\pi \leq t \leq 0 \qquad \cancel{\frac{3\pi}{4} \leq t \leq \frac{\pi}{4}}$

$\frac{2\pi}{4},\ \frac{\pi}{4},\ \frac{3\pi}{4}$

$x^2 + (y-1)^2 = 1$

$x = \cos\theta$

$y = 1 + \sin\theta$

$0 \leq \theta \leq 2\pi$

$(y-1)^2 = 1 - x^2$

$y = 1 \pm \sqrt{1-x^2}$

$= 1 - \sqrt{1-x^2}$

$$\int_\pi^0 \Big[3\cos^2 t\,(1+\sin t)^2 \sin t$$
$$+ 2\cos^3 t\,(1+\sin t)\cos t\Big]\, dt$$

$$= \int_\pi^0 -3\cos^2 t\,\sin t\,(1+\sin^2 t + 2\sin t)$$
$$+ 2\cos^4 t\,(1+\sin t)\, dt$$

$$= \int_\pi^0 \Big[-3\sin t\cos^2 t - 3\cos^2 t\sin^3 t - 6\cos^2 t\sin^2 t$$
$$+ 2\cos^4 t + 2\sin t\cos^4 t\Big]\, dt$$

$\cos t = \varphi$

$-\sin t\, dt = d\varphi$

$+3\int_{-1}^{+1} \varphi^2\, d\varphi$

$+\varphi^3\Big|_{-1}^{1}$

$+1-(-1) = +2$

$$= +2 + 3\int_{-1}^{1} \varphi^2(1-\varphi^2)\, d\varphi + \frac{6}{4}\int_0^\pi \underbrace{\sin^2 2t}_{\frac{1-\cos 4t}{2}}\, dt +$$

$$+ 2\int_\pi^0 \cos^4 t\, dt - 2\int_{-1}^{1} \varphi^4\, d\varphi$$

$$= +2 + 6\left[\frac{\varphi^3}{3} - \frac{\varphi^5}{5}\right]_0^1 + \frac{3}{2}\left[\frac{t}{2} - \frac{\sin 4t}{8}\right]_0^\pi - \frac{4}{5}\varphi^5\Big|_0^1$$

$$+ 2\int_\pi^0 \cancel{\frac{1}{4}}\,\frac{3}{8} + \frac{\cos 2t}{2} + \frac{\cos 4t}{8}\, dt$$

$\cos 2t = \cos^2 t - \sin^2 t$

$= 1 - 2\sin^2 t$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$\cos^4 t = \frac{1 + \cos^2 2t + 2\cos 2t}{4}$

$= \frac{1 + 2\cos 2t}{4} + \frac{1+\cos 4t}{8}$

$$= +2 + \cancel{6}\left[\cancel{\frac{2}{15}}\right] + \frac{3}{2}\left[\frac{\pi}{2}\right] - \cancel{\frac{4}{5}} + 2\left[\frac{3}{8}(-\pi) + \frac{\sin 2t}{4}\Big|_\pi^0 + \frac{\sin 4t}{32}\Big|_\pi^0\right]$$

$$= \cancel{+2} + \frac{3\pi}{4} - \frac{3\pi}{4} + 2 = +2$$

$$\vec{F} = 2xyz^2\,\hat{i} + (x^2z^2 + z\cos yz)\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}$$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \nabla\phi$$

$$2xyz^2 = \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \phi(x,y,z) = x^2yz^2 + g(y,z)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = x^2z^2 + g_y(y,z)$$

$$\Rightarrow x^2z^2 + z\cos yz = x^2z^2 + g_y(y,z)$$

$$\Rightarrow g_y(y,z) = z\cos yz$$

$$\Rightarrow g(y,z) = \frac{z}{z}\sin yz + \psi(z)$$

$$= \sin yz + \psi(z)$$

$$\therefore \phi(x,y,z) = x^2yz^2 + \sin yz + \psi(z)$$

$$2x^2yz + y\cos yz = \frac{\partial \phi}{\partial z} = 2x^2yz + y\cos yz + \psi'(z)$$

$$\psi'(z) = 0$$

$$\psi(z) = c$$

$$\phi(x,y,z) = x^2yz^2 + \sin yz + c$$

$\vec{F}$ is conservative with scalar potential $\phi(x,y,z)$. so integral is independent of path.

$$\text{Integral} = \phi(x,y,z)\Big|_{A(0,0,1)}^{B(1,\frac{\pi}{4},2)}$$

$$= 1^2\left(\frac{\pi}{4}\right)(2^2) + \sin\left(\frac{\pi}{4_2}\times 2\right) + \cancel{c} -$$

$$[(0) + \sin(0) + \cancel{c}]$$

$$= \pi + 1$$

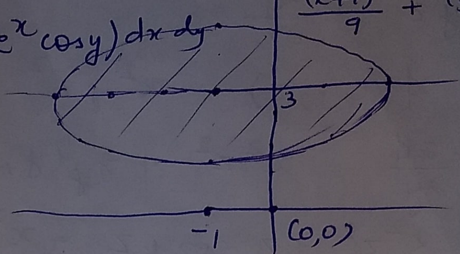
4) $\oint_C e^x \sin y \, dx + e^x \cos y \, dy$

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$= \int_{-4}^{2}\int_{3-\frac{2}{3}\sqrt{9-(x+1)^2}}^{3+\frac{2}{3}\sqrt{9-(x+1)^2}} (e^x \cos y - e^x \cos y)\,dx\,dy$$

$$= 0$$

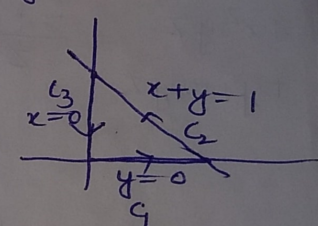
$$\frac{(x+1)^2}{9} + \frac{(y-3)^2}{4} =$$



$$(y-3)^2 = \pm\frac{\sqrt{36-4(x+1)^2}}{3}$$

$$= \pm\frac{2}{3}\sqrt{9-(x+1)^2}$$

5) $\oint_C y^2 dx + x^2 dy$, C is triangle bdd by lines $x=0$, $x+y=1$, $y=0$



C_1	C_2	C_3
$x = t$	$x = t$	$x = 0$
$y = 0$	$y = 1-t$	$y = t$
$0 \le t \le 1$	$1 \le t \le 0$	$1 \le t \le 0$

$$\oint_C y^2 dx + x^2 dy = \oint_{C_1} y^2 dx + x^2 dy + \oint_{C_2} y^2 dx + x^2 dy + \int_{C_3} y^2 dx + x^2 dy$$

$$= \int_0^1 t^2 \times 0 \, dt + \int_1^0 [(1-t)^2 + t^2 \times -1]\,dt + \int_1^0 t^2 \times 0 + 0 \, dt$$

$$= \cancel{\int_0^1 t^2 (1-t)^2 dt} = \cancel{\int_0^1 t^2 (1+t^2-2t)\,dt}$$

$$\cancel{= \frac{t^3}{3} + \frac{t^5}{5} - \frac{2t^4}{4}\Big|_0^1 = \frac{1}{3} + \frac{1}{5} - \frac{1}{2}}$$

$$\cancel{= \frac{8}{15} - \frac{1}{2} = \frac{1}{30}}$$

$$\int_1^0 (1 + \cancel{t^2} - 2t - \cancel{t^2})\,dt$$

$$= t - t^2 \Big|_1^0$$

$$= 0 - (1-1) = 0$$

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (2x - 2y)\, dx\, dy$$

$$= 2\int_0^1 xy - \frac{y^2}{2}\Big|_0^{1-x} dx$$

$$= 2\int_0^1 x(1-x) - \frac{(1-x)^2}{2}\, dx$$

$$= 2\int_0^1 x - x^2 - \frac{1}{2}(1+x^2-2x)\, dx$$

$$= 2\int_0^1 x - x^2 - \frac{1}{2} - \frac{x^2}{2} + x\, dx$$

$$= 2\int_0^1 \left(-\frac{3x^2}{2} + 2x - \frac{1}{2}\right) dx$$

$$= 2\left[-\frac{x^3}{2} + x^2 - \frac{x}{2}\right]_0^1$$

$$= 2\left[-\frac{1}{2} + 1 - \frac{1}{2}\right] = 0$$

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

Hence, verified.

6) $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

$\hat{n} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{49}} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$

R: $2x+3y=12$

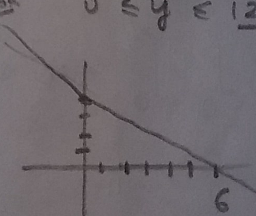
$0 \le x \le 6$

$0 \le y \le \frac{12-2x}{3}$

$f: 2x + 3y + 6z - 12 = 0$

$\vec{n} = \nabla f = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{7\,dx\,dy}{6}$



$$\iint_S \vec{F}\cdot d\vec{s} = \iint_R \vec{F}\cdot\hat{n}\, dS$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (18z\hat{i} - 12\hat{j} + 3y\hat{k})\cdot\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{\cancel{7}}\,\frac{\cancel{7}}{6}\,dx\,dy$$

$$= \frac{1}{6}\int_0^6 \int_0^{\frac{12-2x}{3}} \left(\frac{36}{\cancel{6}}(12-2x-3y) - 36 + 18y\right) dx\,dy$$

$$= \frac{1}{6}\int_0^6 \int_0^{\frac{12-2x}{3}} 72 - 12x - \cancel{18y} - 36 + \cancel{18y}\, dy\,dx$$

$$= \frac{1}{6}\int_0^6 36y - 12xy \Big|_0^{12-2x/3} dx$$

$= \frac{1}{6}\int_0^6 \cancel{36}^{12}\left(\frac{12-2x}{\cancel{3}}\right) - \cancel{12}^{4} x \left(\frac{12-2x}{\cancel{3}}\right) dx$

$= \frac{1}{6}\int_0^6 144 - 24x - 48x + 8x^2 \, dx$

$= \frac{1}{6}\left[\frac{8}{3}x^3 - \frac{72}{2}x^2 + 144x\right]\Big|_0^6$

$= \left[\frac{8}{\cancel{3}} \times \cancel{6}^{2} \times 36 - 36(36) + 144(6)\right]\frac{1}{6}$

$= [576 - 1296 + 864]\frac{1}{6}$

$= \frac{1}{6}[1440 - 1296] = 144 \times \frac{1}{6}$

$= 24$

7) $\iint \text{curl } \vec{F} \cdot \hat{n} \, ds$

$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

S is part of surface of paraboloid

$z = 1 - x^2 - y^2, \quad z \geq 0$

$\iint \text{curl } \vec{F} \cdot \hat{n} \, ds = \oint_c \cancel{\nabla} \vec{F} \cdot d\vec{r}$

$= \oint_c y\,dx + z\,dy + x\,dz$

$= \cancel{\int_0^{2\pi}\int_0^1 \left[r^2 \sin^2 t + (1-r^2) r \cos t + (r^2 \cos t)(-2)\right] dt}$

$x = \cancel{r} \cos t$

$y = \cancel{r} \sin t$

$z = \cancel{1-r^2}\ 0$

~~$0 \leq r \leq 1$~~

$0 \leq t \leq 2\pi$

$= \cancel{\int_0^{2\pi}\int_0^1 -r^2 \sin^2 t + r \cos t - r^3 \cos t - 2r^2 \cos t \, dt}$

$= \int_0^{2\pi} -\sin^2 t \, dt$

$= \int_0^{2\pi} \frac{\cos 2t - 1}{2} dt = \frac{\sin 2t}{4} - \frac{t}{2}\Big|_0^{2\pi}$

$= 0 - \frac{1}{2}(2\pi) = -\pi$

Check :

Check: $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$= \hat{i}(0-1) - \hat{j}(1) + \hat{k}(-1)$

$= -\hat{i} - \hat{j} - \hat{k}$ ✓

$z + x^2 + y^2 - 1 = 0$

$\nabla f = \vec{n} = 2x\hat{i} + 2y\hat{j} + \hat{k}$

$\hat{n} = \dfrac{2x\hat{i} + 2y\hat{j} + \hat{k}}{\sqrt{4x^2+4y^2+1}}$

$ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$

$= \dfrac{dx\,dy}{\frac{1}{\sqrt{4x^2+4y^2+1}}}$

$\displaystyle\iint_R \frac{1}{\sqrt{4x^2+4y^2+1}}(-2x-2y-1)\sqrt{4x^2+4y^2+1}\,dx\,dy$

$x^2+y^2=1$

$\displaystyle = \int_0^1\int_0^{2\pi}(-2r\cos\theta - 2r\sin\theta - 1)\,r\,d\theta\,dr$

$\displaystyle = \int_0^1 -2r^2\sin\theta + 2r^2\cos\theta - r\theta \Big|_0^{2\pi} dr$

$\displaystyle = \int_0^1 2r^2(1-1) - r(2\pi)\,dr$

$\displaystyle = -\frac{2\pi}{2} r^2 \Big|_0^1 = -\pi$

8) $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

S part of sphere $x^2+y^2+z^2=1$ above xy plane

$z = \sqrt{1-x^2-y^2}, \quad z \geq 0.$

$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \hat{i}(-1) - \hat{j}(1) + \hat{k}(-1)$

$= -\hat{i} - \hat{j} - \hat{k}$

$\nabla\vec{r} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\hat{n} = \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}}$

$= x\hat{i} + y\hat{j} + z\hat{k}$

$ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$

$= \dfrac{dx\,dy}{z} = \dfrac{dx\,dy}{\sqrt{1-x^2-y^2}}$

$\iint \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_R \dfrac{-x-y-\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}\,dx\,dy$

$= \int_0^1\int_0^{2\pi} \left(\dfrac{-r(\cos\theta+\sin\theta)-\sqrt{1-r^2}}{\sqrt{1-r^2}}\right) r\,d\theta\,dr$

$= \int_0^1 \dfrac{-r^2}{\sqrt{1-r^2}}(\sin\theta-\cos\theta) - r\theta\Big|_0^{2\pi} dr$

$= \int_0^1 \dfrac{-r^2}{\sqrt{1-r^2}}(0-0-(1-1)) - 2\pi r\,dr$

$= -2\pi\int_0^1 r\,dr = -\pi r^2\Big|_0^1 = -\pi$

$\oint_c \vec{F}\cdot d\vec{r} = \oint_c y\,dx + z\,dy + x\,dz$

$z = \sqrt{1-x^2-y^2}$

c $x = \cos t$

$y = \sin t$

$z = 0$

$= -\int_0^{2\pi} \sin^2 t\,dt$

$\int_0^{2\pi} \dfrac{\cos 2t - 1}{2}\,dt$

$= \dfrac{1}{2}\left[\dfrac{\sin 2t}{2} - t\right]_0^{2\pi}$

$= \dfrac{1}{2}\left[-2\pi\right] = -\pi$

$= \iint \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$$\iint_S \vec{F}\cdot d\vec{s} \quad , \quad \vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$$

surface $x^2+y^2 = 4$, $0 < z \leq 3$.

$$\iint_R \vec{F}\cdot\hat{n}\, dA \qquad f: x^2+y^2-4=0$$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

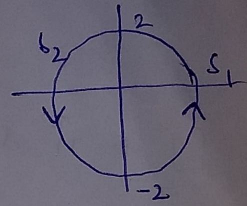
$$= \iint_R 4x\cdot\frac{x}{2} - 2y^2\cdot\frac{y}{2}\, dA \qquad \hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$$= \iint_R (2x^2 - y^3)\, dA \qquad = \frac{x\hat{i}+y\hat{j}}{2}$$

Take projection on yz plane

thus $R: -2 \leq y \leq 2$

$0 < z \leq 3$



$$dA = \frac{dy\,dz}{\hat{n}\cdot\hat{i}} = \frac{dy\,dz}{x/2}$$

$$= \frac{2}{x}\,dy\,dz$$

Along S_1 : $x = \sqrt{4-y^2}$

$$= \int_0^3 \int_{-2}^{2} \left(2(4-y^2) - y^3\right)\frac{2}{\sqrt{4-y^2}}\, dy\,dz$$

$$= \int_0^3 \int_{-2}^{2} \left[4\sqrt{4-y^2} - \frac{2y^3}{\sqrt{4-y^2}}\right] dy\,dz$$

even f^n odd f^n

$$= 8\int_0^3 \int_0^2 \sqrt{4-y^2}\, dy\,dz$$

$$= 8\times 3\left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\left(\frac{y}{2}\right)\right]_0^2$$

$$= 24\left[2\times\frac{\pi}{2}\right] = 24\pi$$

Along S_2 :

$x = -\sqrt{4-y^2}$

$$\int_0^3 \int_2^{-2} \left(2(4-y^2) - y^3\right)\frac{2\,dy\,dz}{-\sqrt{4-y^2}}$$

$$= 24\pi$$

Along S_3 :

$x^2+y^2 = 4$ $z = 3$.

$\hat{n} = \hat{k}$

$dA = dx\,dy$

$$= \iint_{\theta=0\, r=0}^{2\pi\;\; 2} z^2\, dx\,dy \quad (z^2 = 9)$$

$$= 9\iint dx\,dy$$

$$= 9(\pi 4)$$

$$= 36\pi$$

So $24\pi + 24\pi + 36\pi = 84\pi$

$$\operatorname{div}\vec{F} = 4 - 4y + 2z$$

$$\int_0^3 \iiint 4 - 4y + 2z \; dx\,dy\,dz .$$

$$\int_0^3 \int_0^{2\pi} \int_0^2 (4r - 4r^2\sin\theta + 2zr)\, dr\,d\theta\,dz$$

$$= \int_0^3 \int_0^{2\pi} 2r^2 - \frac{4}{3}r^3\sin\theta + zr^2\Big|_0^2 \, d\theta\,dz$$

$$= \int_0^3 \int_0^{2\pi} 8 - \frac{32}{3}\sin\theta + 4z \; d\theta\,dz$$

$$\int_0^3 8(2\pi) + 8\pi z \; dz$$

$$16\pi(3) + 4\pi z^2\Big|_0^3$$

$$48\pi + 36\pi$$

$$84\pi$$

10) $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

S is surface of cube bdd by the planes

$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

$$\iiint_V \nabla\cdot\vec{F}\,dV = \int_0^1\int_0^1\int_0^1 4z - 2y + y \; dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (4z-y)\,dy\,dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= 2z^2 - \frac{z}{2}\Big|_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

$\iint_S \vec{F}\cdot\hat{n}\,dA$

When S is $x=0$

$\hat{n} = -\hat{i}$

$\vec{F}\cdot\hat{n} = -4xz = 0$.

$\iint_S \vec{F}\cdot\hat{n}\,dA = 0$

When S is $x=1$

$\hat{n} = \hat{i}$ $\qquad dy\,dz = \frac{dA}{\hat{n}\cdot\hat{i}} = dA$

$\vec{F}\cdot\hat{n} = 4xz = 4z$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \int_0^1\int_0^1 4z\,dy\,dz = \int_0^1 4z\,dz$$
$$= 2z^2\Big|_0^1 = 2.$$

When S is $y=0$

$\hat{n} = -\hat{j}$, $\vec{F}\cdot\hat{n} = y^2 = 0$

When S is $y=1$

$\hat{n} = \hat{j}$ $\quad \vec{F}\cdot\hat{n} = -y^2 = -1$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \int_0^1\int_0^1 -\,dx\,dz \qquad \frac{dx\,dz}{\hat{n}\cdot\hat{j}} = dA$$

$= -1$

When S is $z=0$

$\hat{n} = -\hat{k}$, $\vec{F}\cdot\hat{n} = -yz = 0$

When S is $z = 1$

$$\hat{n} = \hat{k} \qquad \vec{F}\cdot\hat{n} = yz = y$$

$$\iint_S \vec{F}\cdot\hat{n}\, dA = \int_0^1\int_0^1 y \, dx\, dy \qquad dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$= \int_0^1 y\, dy = \frac{1}{2}.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, dA = 0+2+0-1+0+\frac{1}{2}$$

where S is cube

$$= 3/2.$$

$$= \iiint_V (\nabla\cdot\vec{F})\, dV$$

$\therefore$ Divergence theorem is verified.