Vibrations of a Stretched String- Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Fourier Half range sine series (Recall)

$f(x) is\ required\ to\ expand\ as\ a\ sine\ series\ in\ the\ range\ 0 < x < L$

$$f(x) = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{L}$$

Where, $$b_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L}\,\mathrm{dx}$$

Method of separation of variables (recap)

- Working Rule:

(1) Assume the solution is going to be of the form $X(x).T(t)$ or $X(x).Y(y)$ etc. This is called separable form.

(2) Substitute that form back into the PDE.

(3) Devide by $X(x)T(t)$ or $X(x)Y(y)$.

(4) Now each term of the equation depends on a different variable so they must both be constants.

(5) For each possible value of the constant (positive, negative, zero), solve the two resulting ODEs and multiply the solutions together to give one specific solution to the PDE.

(6) Form the general solution of the PDE by adding linear combinations of all the specific sols.

Solution of wave equation $\left(\frac{\partial^2 u}{\partial t^2} = c^2 \ \frac{\partial^2 u}{\partial x^2}\right)$

Solution:

Wave eq-

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ------(1)

Let $u = X(x)T(t)$ be the solution of the eq. 1

Then $\frac{\partial^2 u}{\partial t^2} = \mathrm{X}T''$ and $\frac{\partial^2 u}{\partial x^2} = TX''$

Substituting these values in eq. 1

$XT'' = c^2 TX'' \Rightarrow \mathrm{X}''/X = T''/c^2 T$ (Separating variables)

$\Rightarrow \mathrm{X}''/X = T''/c^2 T$= Constant

Three cases may arise for constant –

Solution Case 1

Case 1- If the constant is negative - k^2 (say)-

$\mathrm{X}''/X =$ - $k^2 \Rightarrow \mathrm{X}''+ k^2\mathrm{X} = 0 \ \Rightarrow X = C_1 \cos(kx) + \ C_2 \sin(kx)$

T''/c^2T= - $k^2 \Rightarrow \mathrm{T}''$ + $k^2c^2\mathrm{T} = 0 \ \Rightarrow T = C_3 \cos(kct) + \ C_4 \sin(kct)$

Solution is $u = X(x)T(t)$

$u = (C_1 \cos(kx) + \ C_2 \sin(kx))(C_3 \cos(kct) + \ C_4 \sin(kct))$-------(2)

Solution :Case 2

Case 2- If the constant is positive k^2 (say)-

$\mathrm{X}''/X = k^2 \Rightarrow \mathrm{X}'' - k^2\mathrm{X} = 0 \Rightarrow X = C_5e^{kx} + C_6\, e^{-kx}$

$T''/c^2T = k^2 \Rightarrow \mathrm{T}'' - k^2c^2\mathrm{T} = 0 \Rightarrow T = C_7e^{kct} + C_8\, e^{-kct}$

Solution is $u = X(x)T(t)$

$u = (C_5e^{kx} + C_6\, e^{-kx})(C_7e^{kct} + C_8\, e^{-kct})$-------(3)

Solution : Case 3

Case 3- If the constant =0

$X''/X = 0 \Rightarrow X'' = 0 \Rightarrow X = C_9 + C_{10}x$

$T''/c^2T = 0 \Rightarrow X'' = 0 \Rightarrow X = C_{11} + C_{12}t$

Solution is $u = X(x)T(t)$

$u = (C_9 + C_{10}x)(C_{11} + C_{12}t)$-------(3)

Acceptable solution (Solution given by eq. 2)

Sol 1. contd...

Out of these three solⁿ we have to choose that solⁿ which is consistent with the physical nature of the problem.

As we are dealing with problems on vibrations, u must be a periodic function of X & T.

So the acceptable solⁿ for vibratⁿ of a string is $u = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos kct + C_4 \sin kct)$

~~Obs~~ Wave eq^n TYPE 1 :

when initial velocity is given.

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = v(x).$$

$$\underset{\text{pos.}}{u(x}, \underset{\text{time}}{0)} = 0.$$

Question 1 (We will learn step by step)-

A tightly stretched string of length *1 meter with fixed end points is initially in equilibrium position. It is set vibrating by giving each point a velocity* $g(x) = \begin{cases} x, & 0 < x < 1/2 \\ 1 - x, & 1/2 < x < 1 \end{cases}$. *Find the displacement.*

Solution : Step 1

Vibratⁿ of string is governed by foll. eqⁿ:

Solⁿ: Using wave Equation :-

$$\frac{d^2u}{dt^2} = K^2 \frac{\partial^2 u}{\partial x^2}$$

Let $u = XT$

$$XT'' = c^2 X'' T \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -K^2, 0, +K^2$$

Case I: $u = [c_1 \cos(kx) + c_2 \sin(kx)] \cdot [c_3 \cos kct + c_4 \sin kct]$

Case II: $u = (c_5 + c_6 x)(c_7 + c_8 t)$

Case III: $u = \cancel{c_9 + c_{10}}\ (c_9 e^{kx} + c_{10} e^{-kx})(c_{11} e^{kct} + c_{12} e^{-kct})$

1 mark

Only case I is acceptable with the physical natu

$$u = [c_1 \cos(kx) + c_2 \sin(kx)][c_3 \cos ckt + c_4 \sin ckt]$$

Solution Step 2: Find boundary conditions(B.C.)

Given that ---

A tightly stretched string with tightly fixed pts at $x=0$ & $x=1$ (or string of length 1 m)

B.C.

$u(0,t) = 0$ ---------(Cond. 1)

and

$u(1,t) = 0$--(Cond. 2)

Solution Step 3: Find initial Condition (I.C.)

Given that ---

initially in eq^n position. It is ~~said~~ set vibrating by given each of its points & initial velocity $g(x)$.

where $g(x) = \begin{cases} x ; & 0 \leq x \leq 1/2 \\ 1-x ; & \frac{1}{2} \leq x \leq 1 \end{cases}$

I.C.

$u(x,0) = 0$ (initially its in equilibrium position)----cond. (3)

$\left(\frac{\partial u}{\partial x}\right) at(x,0) = g(x)$---Cond. (4)

Step 4: Apply all above 4 conditions on acceptable solution (eq. 1)

Applying condition. 1 –

$u(0,t) = 0 \Rightarrow [c_1 \cos(0) + c_2 \sin(0)][c_3 \cos(kct) + c_4 \sin(kct)] = 0$

$\Rightarrow c_1 [c_3 \cos(kct) + c_4 \sin(kct)] = 0$

$\Rightarrow c_1 = 0$ or $\underbrace{c_3 \cos(kct) + c_4 \sin(kct) = 0}$

this can't be zero as it will give zero solution

so $\boxed{c_1 = 0}$

now solution after App. condition 1 —

$u(x,t) = c_2 \sin(kx) [c_3 \cos(kct) + c_4 \sin(kct)]$ — (2)

Condition 2

Applying condition 2 -

$$u(l,t) = 0 \Rightarrow c_2 \sin(kl) \underbrace{[c_3 \cos(kct) + c_4 \sin(kct)]}_{} = 0$$

$\Rightarrow c_2 \sin kl = 0$ as this part can't be zero.

$\Rightarrow \underbrace{c_2 = 0}_{}$ or $\sin k = 0$ $\Rightarrow$ ~~$k=$~~

not possible otherwise it will ~~make~~ give zero sol.

$\Rightarrow \sin k = 0 \Rightarrow \boxed{k = n\pi}$

now sol. after Applying cond. 1 and 2 —

$$u(x,t) = c_2 \sin(n\pi x)[c_3 \cos(n\pi ct) + c_4 \sin(n\pi ct)]$$

— (2)

Condition 3-

Applying condition 3 —

$u(x,0)=0 \Rightarrow c_2 \sin(n\pi x)\cdot c_3 = 0 \Rightarrow \boxed{c_3=0}$

as $c_2 \sin n\pi x \neq 0$

Now

sol. after applying first three cond —

$u(x,t) = c_2 \sin(n\pi x)\cdot c_4 \sin(n\pi ct)$

$= (c_2 c_4) \sin(n\pi x)\cdot \sin(n\pi ct)$

$u(x,t) = b_n \sin(n\pi x) \sin(n\pi ct)$ —— (4)

the general solution will be obtained after adding all such solutions (i.e. for different values of n)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\cdot \sin(n\pi ct) \quad \text{—(5)}$$

Condition 4

Applying condition 4 →

$$\left(\frac{\partial u}{\partial t}\right)_{(x,0)} = g(x)$$

$$\Rightarrow \left[\sum_{n=1}^{\infty} b_n \sin(n\pi x)\cdot(n\pi c)\cdot\cos(n\pi c t)\right]_{(x,0)} = g(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} (b_n n\pi c)\cdot \sin n\pi x = g(x)$$

$$\Rightarrow (b_n n\pi c) = \frac{2}{L}\int_0^L g(x)\cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

(using Fourier half range sine series)

Here L =1

$$b_n(n\pi c) = 2\int_0^1 g(x)\sin n\pi x\,dx$$

$$= 2\int_0^{1/2} x \sin n\pi x\,dx$$

$$+ 2\int_{1/2}^1 (1-x)\sin n\pi x\,dx$$

$$\boxed{b_n = \frac{4}{n^3\pi^3 c}\sin\left(\frac{n\pi}{2}\right)}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3 c}\left[\sin n\pi x \cdot \sin n\pi ct\right]\sin\left(\frac{n\pi}{2}\right)$$

Type 2: $u(x,0)$ is given.

initial disp. is given & initial vel. i.e. $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

Quest: A tightly stretched string of length 1 has its ends fixed at $x=0$ & $x=1$. At time $t=0$, the string is given a shape defined by $u(x,0) = x(1-x)$ and then released from rest. Considering all the cases find $u(x,t)$.

Solution Q2- Step 1,2 and 3

Solⁿ: The partial diff. used is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Case I: $(c_1 \cos kx + c_2 \sin kx)(c_3 \cos ckt + c_4 \sin ckt)$

Case II:

Case III:

case I is considerable only as it is periodic.

Boundary conditions: $u(0,t) = 0$

$u(1,t) = 0$

Initial condition: $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

$u(x,0) = x - x^2$.

Applying cond. 1, 2and 3

$u(0,t) = 0 \Rightarrow \boxed{c_1 = 0}$

$u(1,t) = 0 \Rightarrow \sin k = 0 \quad \boxed{k = n\pi}$

$u(x,t) = [c_2 \sin(n\pi x)][c_3 \cos(n\pi ct) + c_4 \sin(n\pi ct)]$

$\frac{\partial u}{\partial t} = [c_2 \sin(n\pi x)][-c_3 \sin(n\pi ct) + c_4 \cos(n\pi ct)](n\pi c)$

~~$u(x,0) = [c_2 \sin(n\pi x)][c_4$~~

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \Rightarrow [c_2 \sin(n\pi x)][c_4](n\pi c) = 0$

$\boxed{c_4 = 0}$

now solⁿ is:

$u(x,t) = c_3 c_2 \sin(n\pi x)(\cos n\pi ct)$

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \cos(n\pi ct)$.

Condition 4 (with use of Fourier half range sine series

$$u(x,0) = b_n \sin(n\pi x)$$

$$b_n = 2\int_0^1 g(x) \sin(n\pi x)\, dx.$$

$$= 2\int_0^1 (x - x^2) \sin(n\pi x)\, dx$$

$$b_n = \frac{4}{n^3\pi^3}\left[1 - (-1)^n\right]$$

Q.3. A tightly stretched string of length *l with fixed end points is initially in equilibrium position. It is set vibrating by giving each point a velocity* $v = sin^3\left(\frac{px}{l}\right)$*. Find the displacement.*

Solution:

After Applying step 1, 2 &3 and conditions 1,2 & 3-

The solution is –

$$u(x,t) = \sum_{n=1}^{\infty} b_n \, sin\frac{n\pi x}{l} sin\frac{n\pi ct}{l}$$

Applying condition 4

$$\left(\frac{\partial u}{\partial x}\right) at(x,0) = sin^3\left(\frac{px}{l}\right)$$

$$sin^3\left(\frac{\pi x}{l}\right) = \left[\sum_{n=1}^{\infty} \frac{b_n n\pi c}{l} sin\frac{n\pi x}{l} cos\frac{n\pi ct}{l}\right] at\ t = 0$$

$$\Rightarrow sin^3\left(\frac{px}{l}\right) = \sum_{n=1}^{\infty} \frac{b_n n\pi c}{l} sin\frac{n\pi x}{l}$$

$$\Rightarrow \frac{1}{4}\left(3sin\left(\frac{\pi x}{l}\right) - sin\left(\frac{3\pi x}{l}\right)\right) = \frac{b_1\pi c}{l} sin\frac{\pi x}{l} + \frac{2\ b_2\pi c}{l} sin\frac{2\pi x}{l} + \frac{3\ b_3\pi c}{l} sin\frac{3\pi x}{l}$$

$+\ldots$

$$\Rightarrow b_1 = \frac{3l}{4c\pi}, b_3 = -\frac{l}{12c\pi}$$

$$u(x,t) = \frac{3l}{4c\pi} sin\frac{\pi x}{l} sin\frac{\pi ct}{l} - \frac{l}{12c\pi}\ sin\frac{3\pi x}{l} sin\frac{3\pi ct}{l}$$