Topics to be Discussed

- DC and AC Currents.
- A Sinusoid.
  - Some Definitions.
  - Phase Difference.
  - Physical Model for a Sinusoid.
- Average of a sine wave.
- RMS or Effective Value.
- Concept of Phasors.
  - Operations on Phasors.
- Additions of Phasors Using Complex Numbers.
- Power and Power Factor.
- Purely Resistive Circuit.
- Purely Inductive Circuit.
  - Inductive Reactance.
- Purely Capacitive Circuit.
  - Capacitive Reactance

Difference between

DC and AC Currents

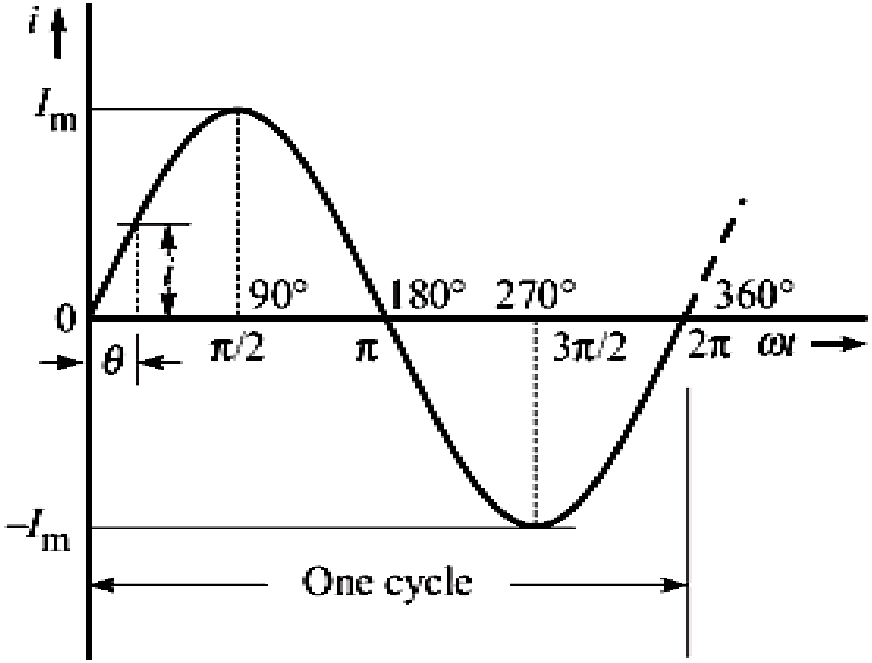
- In dc circuits, when a current of 3 A adds to another current of 4 A, the net current is always 7 A.
- However, an ac current of 3 A when added to another ac current of 4 A can result anything between 1 A and 7 A, depending upon their relative ***phase***.
- It is similar to adding a force of 3 N to another force of 4 N. The result is not necessarily a force of 7 N. It be anything between 1 N to 7 N, depending on their relative ***directions.*** These are added vectorially.

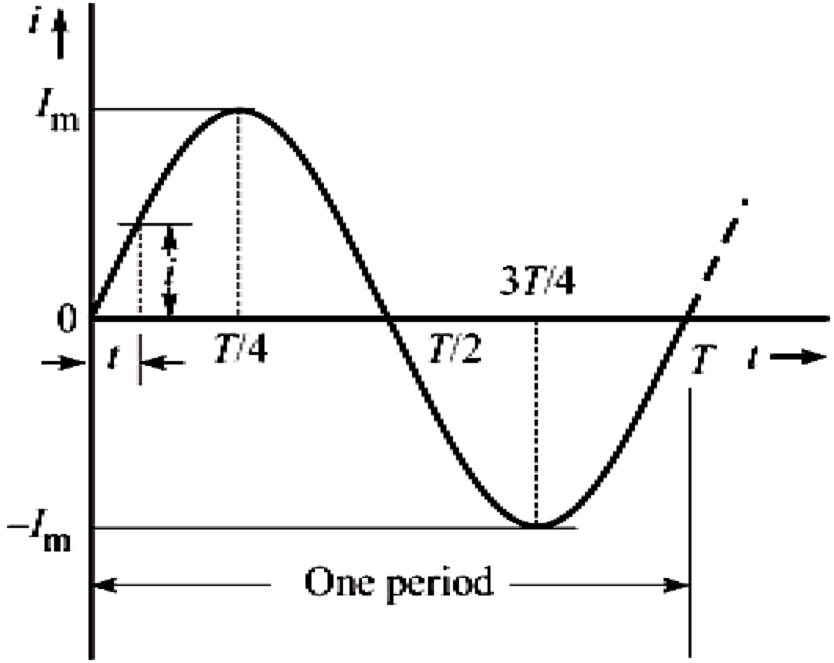
- Does it mean that KCL does not apply to AC Circuits ?

- Ans. :Yes, it does apply. However, the ac currents are added in a Phasor Diagram.

$$\bar{I} = \bar{I}_1 + \bar{I}_2$$

A Sinusoid





(*a*) Current *i* versus angle ωt. (*b*) Current *i* versus time *t*.

$$i = I_m \sin \omega t$$

Some Definitions

- **Amplitude (or peak value) :** It is the maximum value, positive or negative, of an alternating quantity.
- **Instantaneous Value :** It is the value of the quantity at any instant.
- **Time period (*T*) :** It is the duration of one complete cycle.
- **Frequency (*f*) :** It is the number of cycles that occurs in one second.

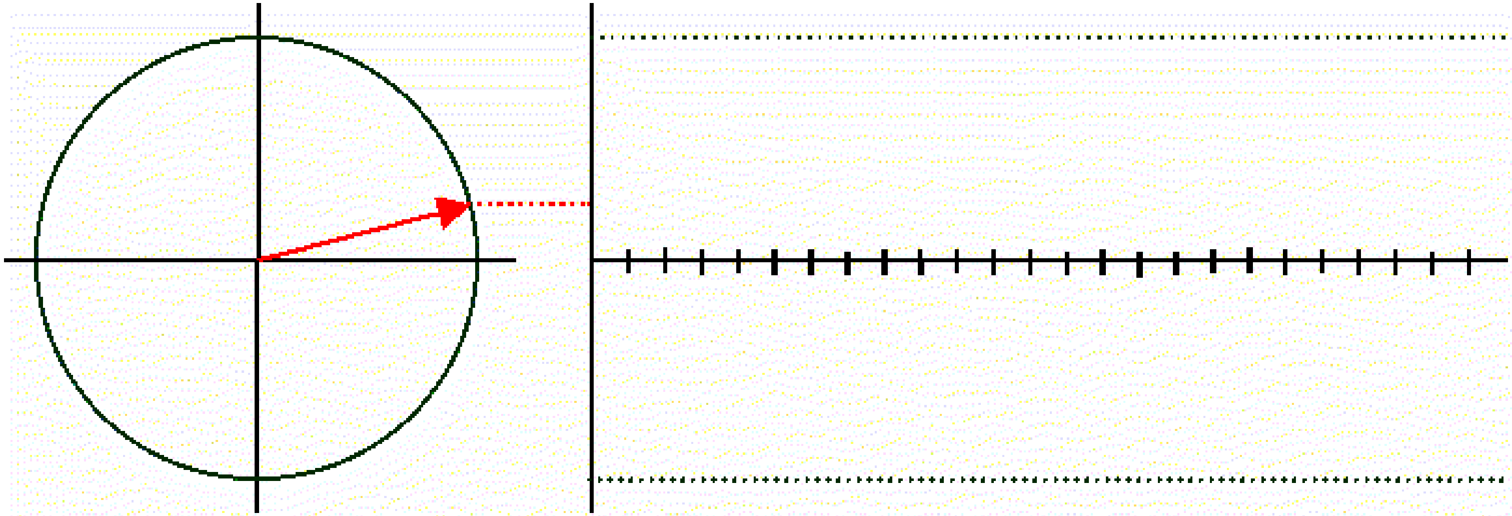
$$f = \frac{1}{T}$$

- **Angular Frequency :** Angular frequency, denoted as ω, is equal to the number of radians covered in one second. Its unit is rad/s. Since one cycle covers 2π radians and there are f cycles in one second, the angular frequency is given as

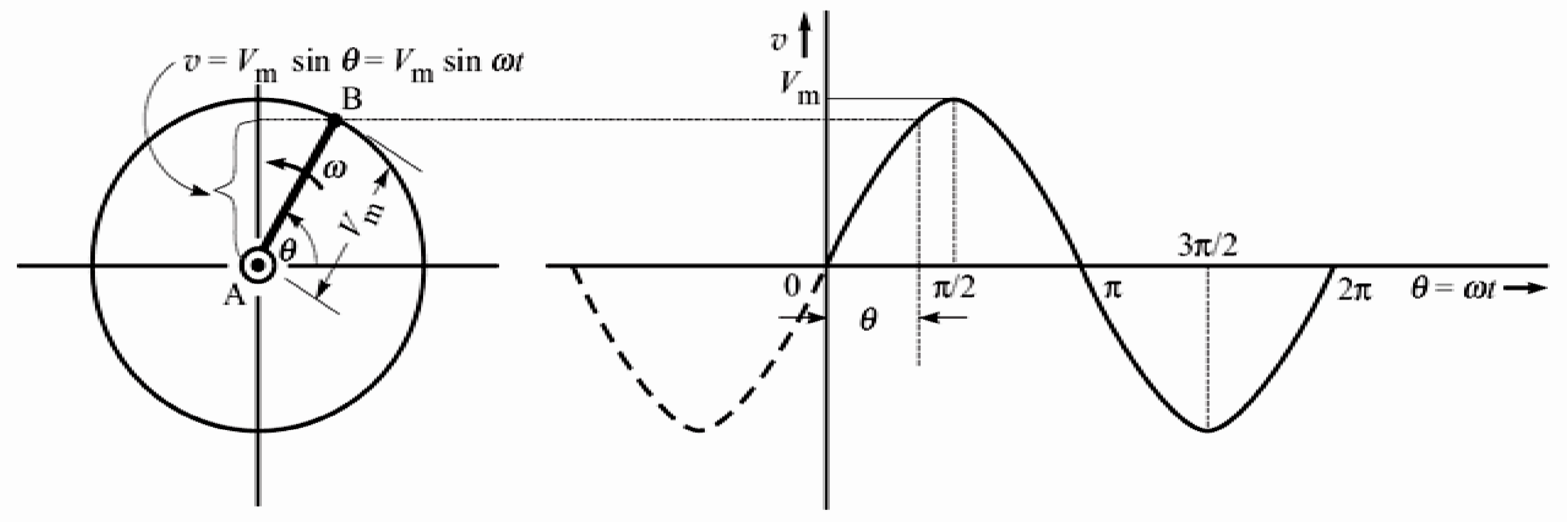
$$\omega = 2\pi f \qquad \text{or} \qquad \omega = \frac{2\pi}{T}$$

- **Phase :** It is the fraction of the time-period or cycle that has elapsed since it last passed from the chosen zero position or origin. The phase at time t from the chosen origin is given by t/T.
- **Phase angle :** It is the equivalent of phase expressed in radians or degrees. It is denoted as θ. Thus, phase angle, $\theta = 2\pi t/T$.
- **Phase difference :** It is the angular displacement between two alternating quantities.

A rotating bar generates a sinusoidal wave.



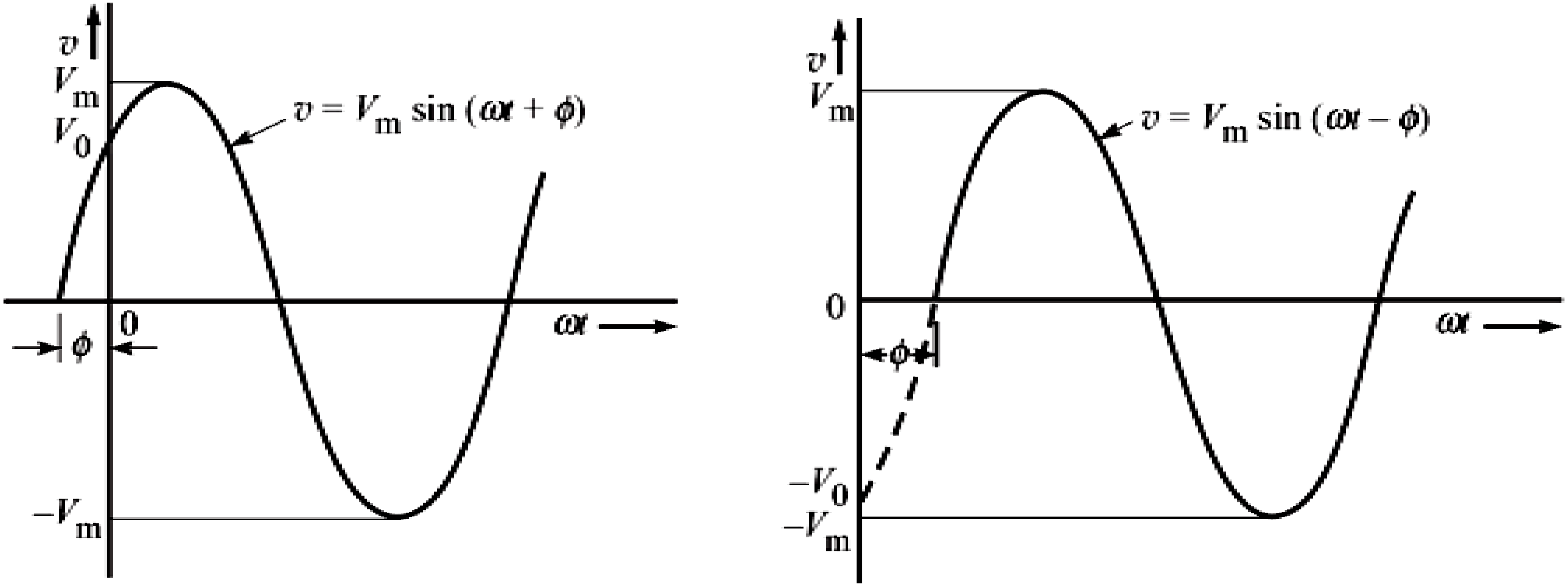
Physical Model for a Sinusoid



(*a*) A rotating crank. **(*b*) Sinusoidal waveform.**

The vertical projection of the rotating crank is its length times the sine of the angle θ.

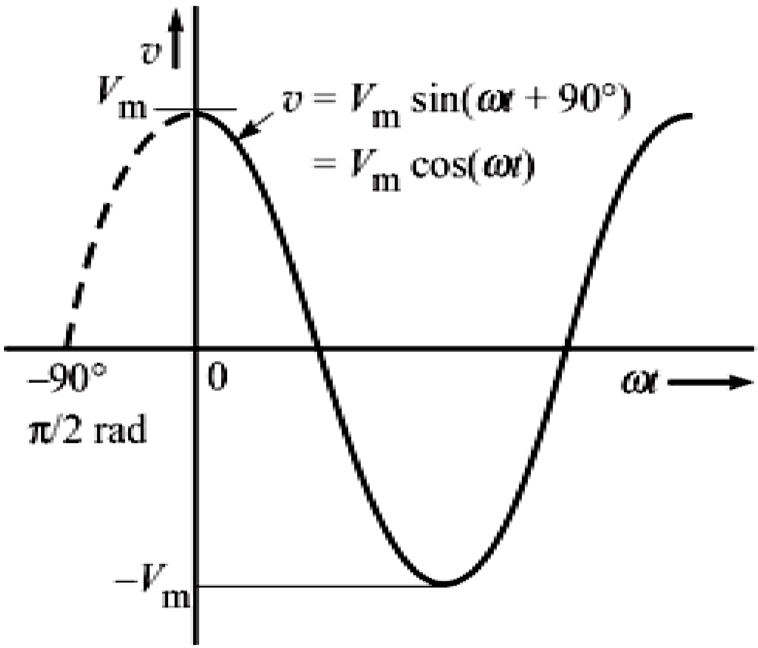
Phase Difference



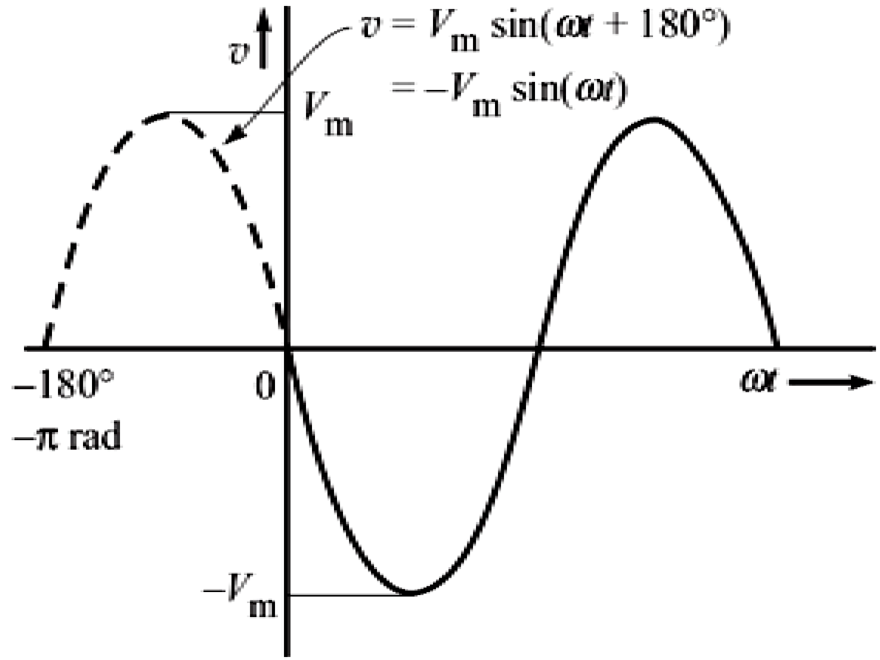
(*a*) Displacement by ϕ to the left

(*b*) Displacement by ϕ to the right

It has become normal practice in electrical engineering to express ωt in radians and the angle in degrees.



(*c*) Displacement by 90° to the left.



(*d*) Displacement by 180°.

Note that the resulting position of the waveform is the same whether the phase shift is 180° (π rad) to the left, or 180° to the right.

Example 1

- A sinusoidal voltage is 20 V peak-to-peak, has a time of 5 ms between consecutive peak and trough, and at *t* = 0 is -3.6 V and decreasing. Find the equation for the instantaneous value of the voltage, and the value of the voltage at *t* = 12 ms.

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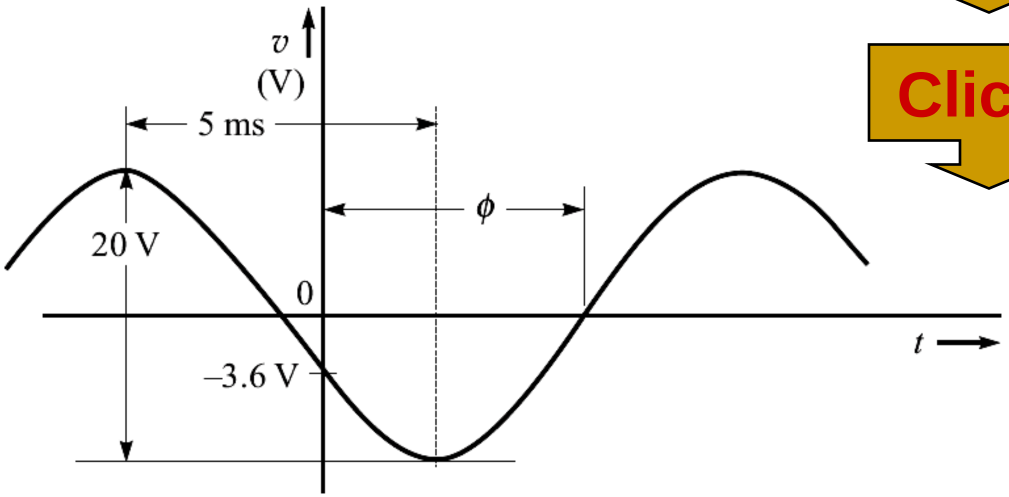
NOTE : First draw the waveshape, then start the solution.



Solution :

$$V_m = \frac{V_{p-p}}{2} = \frac{20}{2}10\,\text{V}; \qquad T = 2\times 5 \text{ ms} = 10 \text{ ms};$$

$$\therefore \quad f = \frac{1}{T} = \frac{1}{10\,\text{ms}} = \frac{1}{10\times 10^{-3}} = 100\,\text{Hz}$$

$$\therefore \ \omega = 2\pi f = 2\pi\times 100 = 628.3 \text{ rad/s.}$$

Therefore, the sinusoid voltage is of the form

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$$v = 10\sin(628.3t + \phi) \text{ V}$$

Now, we are given that at t = 0, v = -3.6 V. Putting these values, we get

$$-3.6 = 10\sin(\phi) \quad \text{or} \quad \sin\phi = -0.36$$

$$\Rightarrow \quad \phi = -158.9^\circ \quad \text{or} \quad 338.9^\circ$$

- As shown in figure, the given sinusoid is shifted to right by angle ϕ, which is less than 180°.

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- Also, we know that shifting rightward means there is a lag of angle ϕ. Therefore, the equation of the given sinusoidal voltage is

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$$v = 10\sin(628.3t - 158.9^\circ)\ \text{V}$$

The value of voltage at 12 ms,

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$$v(t=0.012\,\mathrm{s}) = 10\sin\left(628.3\times 0.012\times\frac{180°}{\pi} - 158.9°\right)\mathrm{V}$$

$$= 10\sin\left(432° - 158.9°\right)\mathrm{V} = \mathbf{-9.985\,V}$$

Different Values of AC

- **Average (**or **Mean) Value :** It is the arithmetic sum of all the values divided by the total number of values.
- **Average Value of AC :** It is equivalent direct current, which transfers the same charge as transferred by that ac current in the same time.

$$\boxed{V_{av} = \frac{1}{2\pi}\int_0^{2\pi} v\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} v\, d(\omega t)}$$

Average of a sine wave

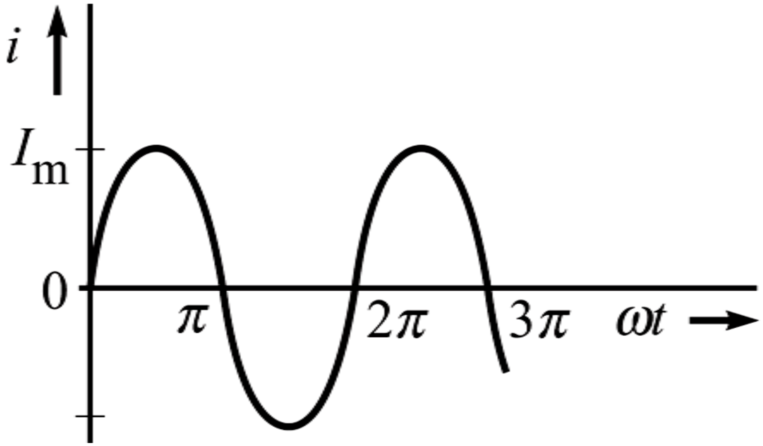
- The average of a sine wave over a full cycle is zero.
- But average value of half-cycle only.

$$I_{av} = I_{av\text{-half-cycle}} = \frac{1}{\pi}\int_0^{\pi} i \, d(\omega t) = \frac{1}{\pi}\int_0^{\pi} I_m \sin \omega t \, d(\omega t)$$

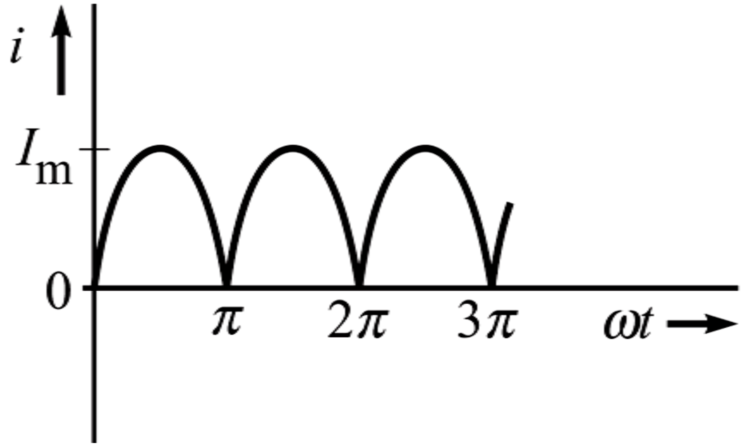
$$= \frac{I_m}{\pi}\Big[-\cos \omega t\Big]_0^{\pi} = \frac{I_m}{\pi}[-\cos \pi + \cos 0] = \frac{2I_m}{\pi}$$

or $$\boxed{I_{av} = \frac{2I_m}{\pi} = 0.637\, I_m}$$

Following two waves have the same average value.

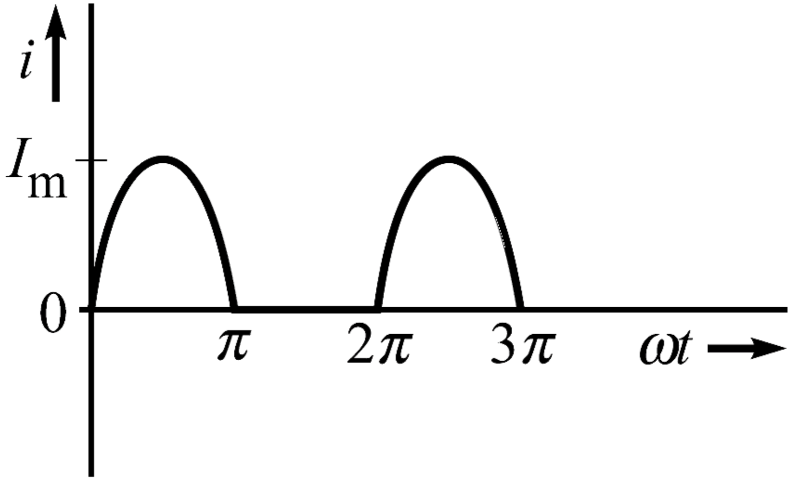


(*a*) Sinusoidal ac current.



(*b*) Full-wave rectified current.

Output of half-wave Rectifier



- The average value is given as

$$= \frac{I_m}{2\pi}\left[-\cos\omega t\right]_0^\pi = \frac{I_m}{2\pi}[-\cos\pi + \cos 0] = \frac{I_m}{\pi}$$

or $$\boxed{I_{av} = \frac{I_m}{\pi} = 0.318\, I_m}$$

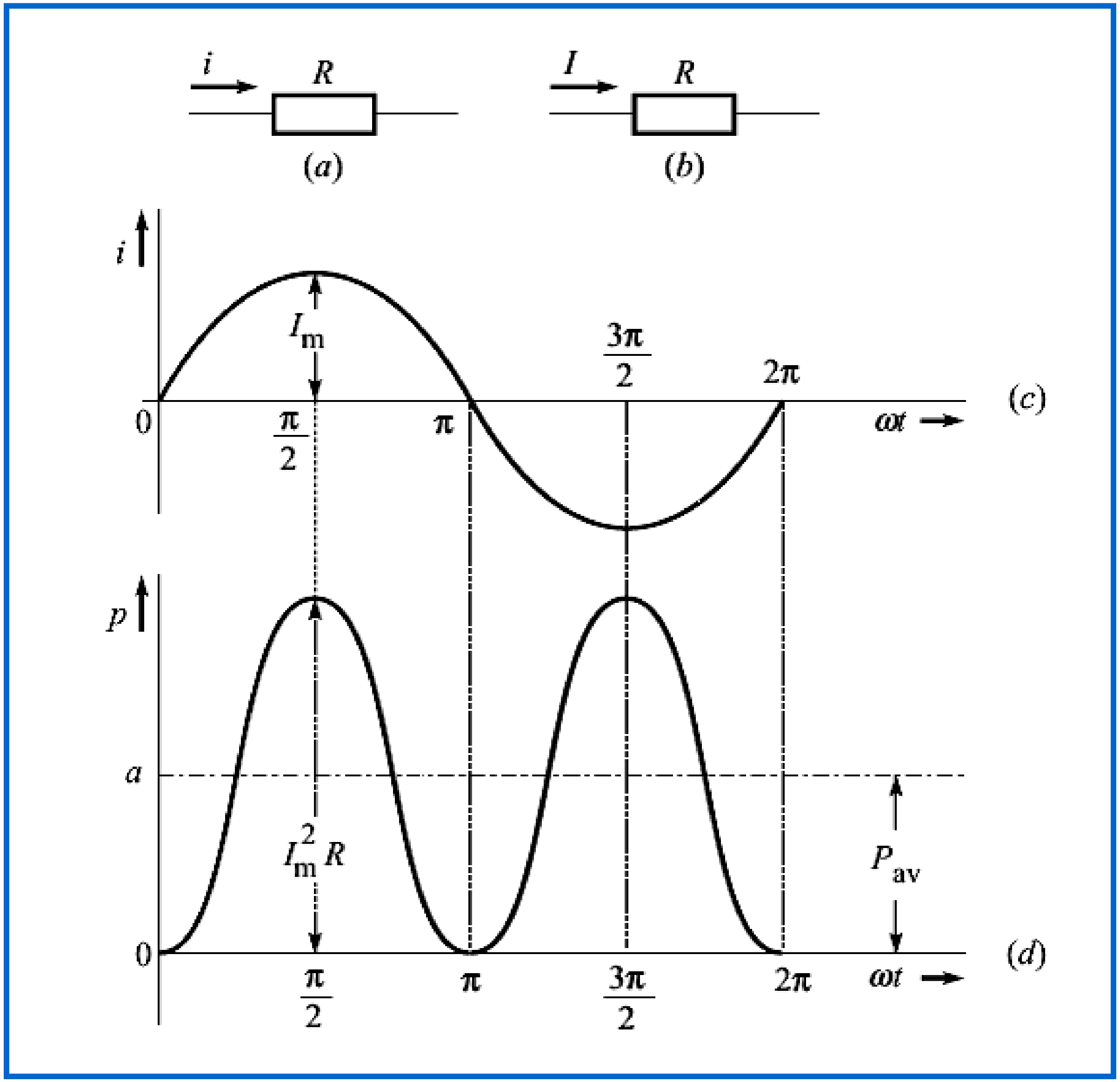
- It is obvious that the average value of half-wave rectified current is just half of the average of full-wave rectified current.

RMS or Effective Value

- It is equivalent dc current, which when flows through the given circuit produces same amount of heating.

Let us try to find the waveform of instantaneous power in a resistor due to an ac current.

Instantaneous Power



$$p = i^2 R = R I_m^2 \sin^2 \omega t$$

$$= \frac{R I_m^2}{2}\left(1 - \cos 2\omega t\right) = \frac{R I_m^2}{2} - \frac{R I_m^2}{2}\cos 2\omega t$$

$$\therefore \quad P_{av} = \frac{I_m^2}{2} R$$

Comparing the dc and ac average power,

$$I^2 R = \frac{1}{2} I_m^2 R \qquad \text{or} \qquad I^2 = \frac{1}{2} I_m^2$$

$$\therefore \qquad I_{eff} = \frac{I_m}{\sqrt{2}}$$

Let us again examine the procedure of finding the I_{eff}

$$P_{av} = \frac{1}{T}\int_0^T i^2 R\,dt = \left[\frac{1}{T}\int_0^T i^2\,dt\right]R$$

This power, we equated to $I_{eff}^2 R$. Thus, we find that

$$\boxed{I_{eff}^2 = \frac{1}{T}\int_0^T i^2\,dt \quad \text{or} \quad I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt}}$$

- **Note** that the rms value is always greater than the average value, except for a rectangular wave.
- For a rectangular wave, both the rms and average values are same.
- For ***full-wave rectifier,*** I_{rms} is same as that for AC.

$$I_{\mathrm{rms}} = \frac{I_{\mathrm{m}}}{\sqrt{2}}$$

- For ***half-wave rectifier***, I_{rms} can be determined as

$$I_{\mathrm{rms}} = \frac{I_{\mathrm{m}}}{2}$$

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Important Factors

- **Form Factor :** It is the ratio of rms value to the average value of an alternating quantity.

$$K_f = \frac{I_{rms}}{I_{av}} = 1.11 \text{ for sinusoidal.}$$

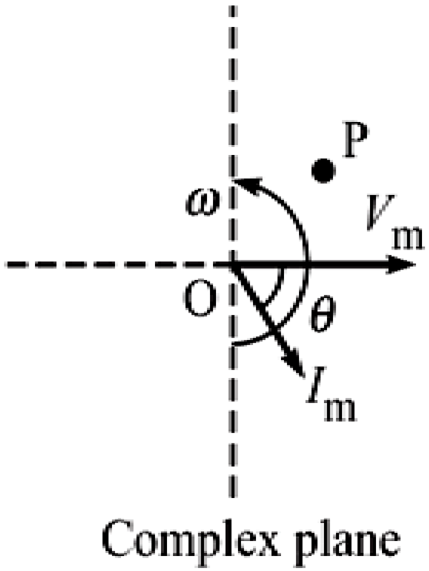
- Form factor has less value for less peaky wave. For square wave it is 1.0, for triangular wave it is 1.15.

- **Peak (**or **Amplitude) Factor :** It is the ratio of maximum value to the rms value.

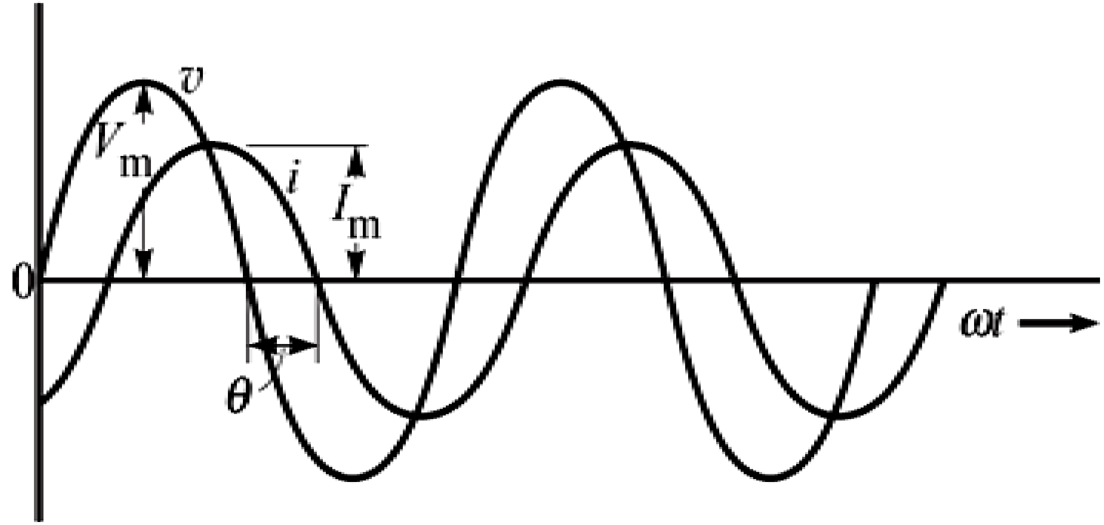
$$K_p = \frac{I_m}{I_{rms}} = 1.414 \text{ for sinusoidal.}$$

Concept of Phasors

- The rotating bar that generates a sinusoid can be taken as a phasor.



(a) Rotating voltage and current phasors.



(b) Waveforms produced.

The phasor V_m is in reference direction, but phasor I_m lags behind V_m by θ.

Thus, we can write,

$$v(t) = V_m \sin \omega t \qquad \text{and} \qquad i(t) = I_m \sin(\omega t - \theta)$$

- **Phasor Diagram :** It is a diagram containing the phasors of inter-related sinusoidal voltages and currents, with their phase differences indicated,

- It can be drawn either in terms of peak values or rms values.

- **Note** that the phase of two sine waves can be compared only if

  1. Both have the same frequency.

  2. Both are written with positive amplitude.

  3. Both are written as sine functions, or as cosine functions.

Algebraic Operations on Phasors

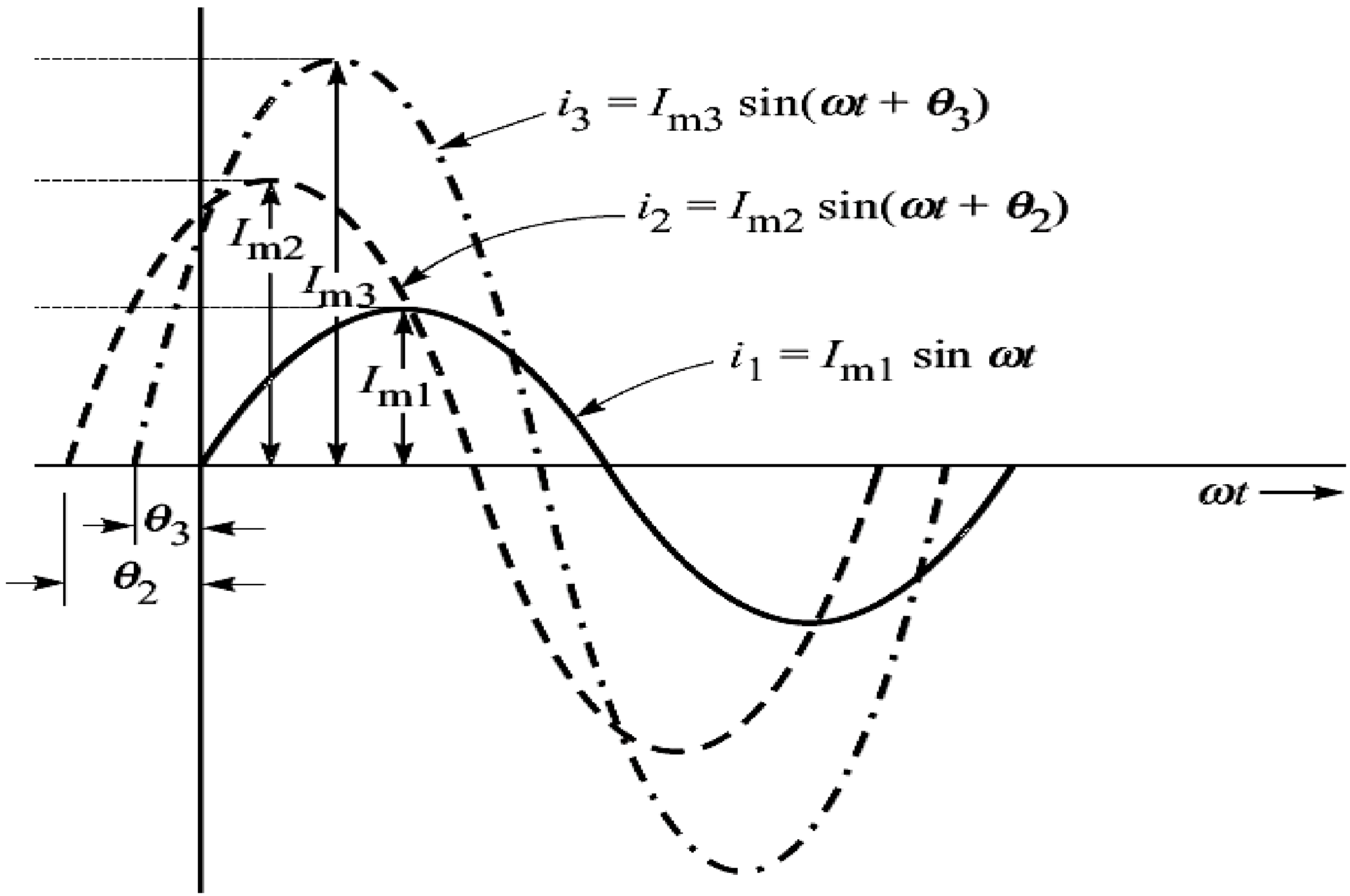
Suppose, we wish to add two phasors,

$$i_1 = I_{m1} \sin \omega t \qquad \text{and} \qquad i_2 = I_{m2} \sin(\omega t + \theta_2)$$

We can find the sum of i_1 and i_2 in following three ways :

1. By using the plots of waveforms.
2. By using trigonometrical identities.
3. By using the concept of phasors.

(1) By Using The Plots of Waveforms.



(2) By Using Trigonometrical Identities

$$
\begin{aligned}
i_1 = i_1 + i_2 &= I_{m1}\sin\omega t + I_{m2}\sin\ (\omega t + \theta_2) \\
&= I_{m1}\sin\omega t + I_{m2}[\sin\omega t\cos\theta_2 + \cos\omega t\sin\theta_2] \\
&= I_{m1}\sin\omega t + (I_{m2}\cos\theta_2)\sin\omega t + (I_{m2}\sin\theta_2)\cos\omega t \\
&= [I_{m1} + (I_{m2}\cos\theta_2)]\sin\omega t + (I_{m2}\sin\theta_2)\cos\omega t
\end{aligned}
$$

Let us put

$$a = I_{m1} + (I_{m2}\cos\theta_2) \quad \text{and} \quad b = I_{m2}\sin\theta_2$$

$$\therefore \qquad i_3 = a\sin\omega t + b\cos\omega t$$

This can be written as

$$i_3 = \sqrt{a^2 + b^2}\,\sin(\omega t + \theta_3)$$

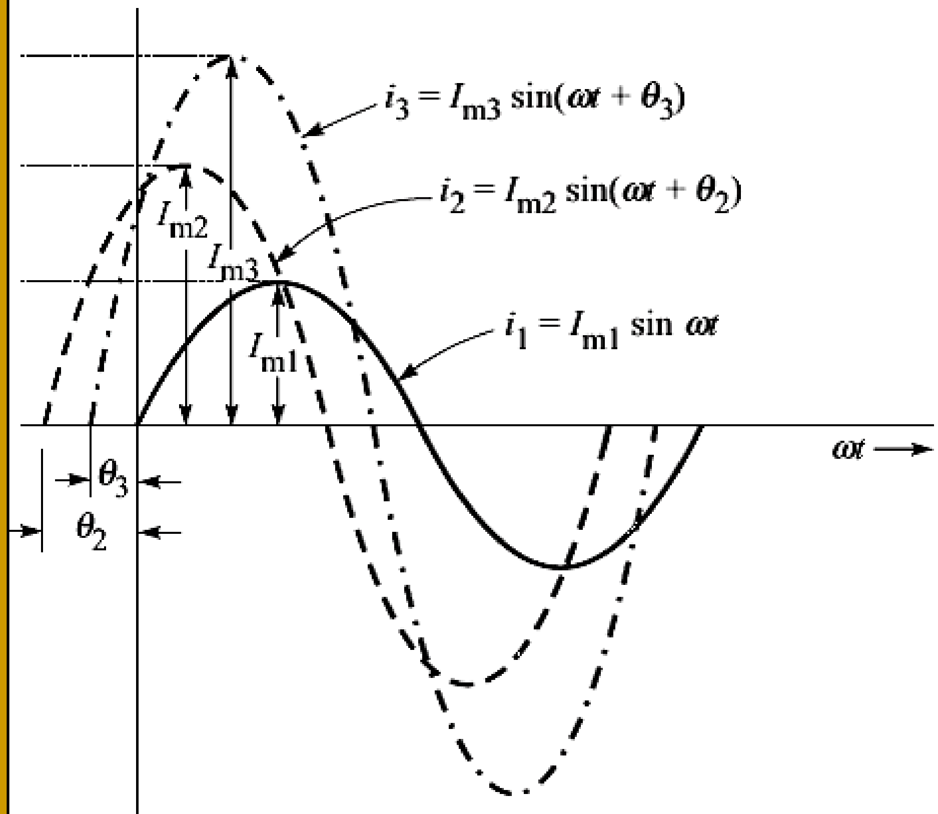
Where,

$$I_{m3} = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \theta_3 = \tan^{-1}\frac{b}{a}$$

- Though this procedure does not require us to plot the sinusoid curves, it is

too cumbersome.

(3) By Using The Concept of Phasors



(*b*) Addition using plots of waveforms.

- Thus, we find that two sinusoids can easily be added by a phasor diagram.
- Since we usually refer to their effective values and not their peak values, the phasor diagram can be drawn in rms values directly.
- Even more important is that we need not even draw the phasor diagram.

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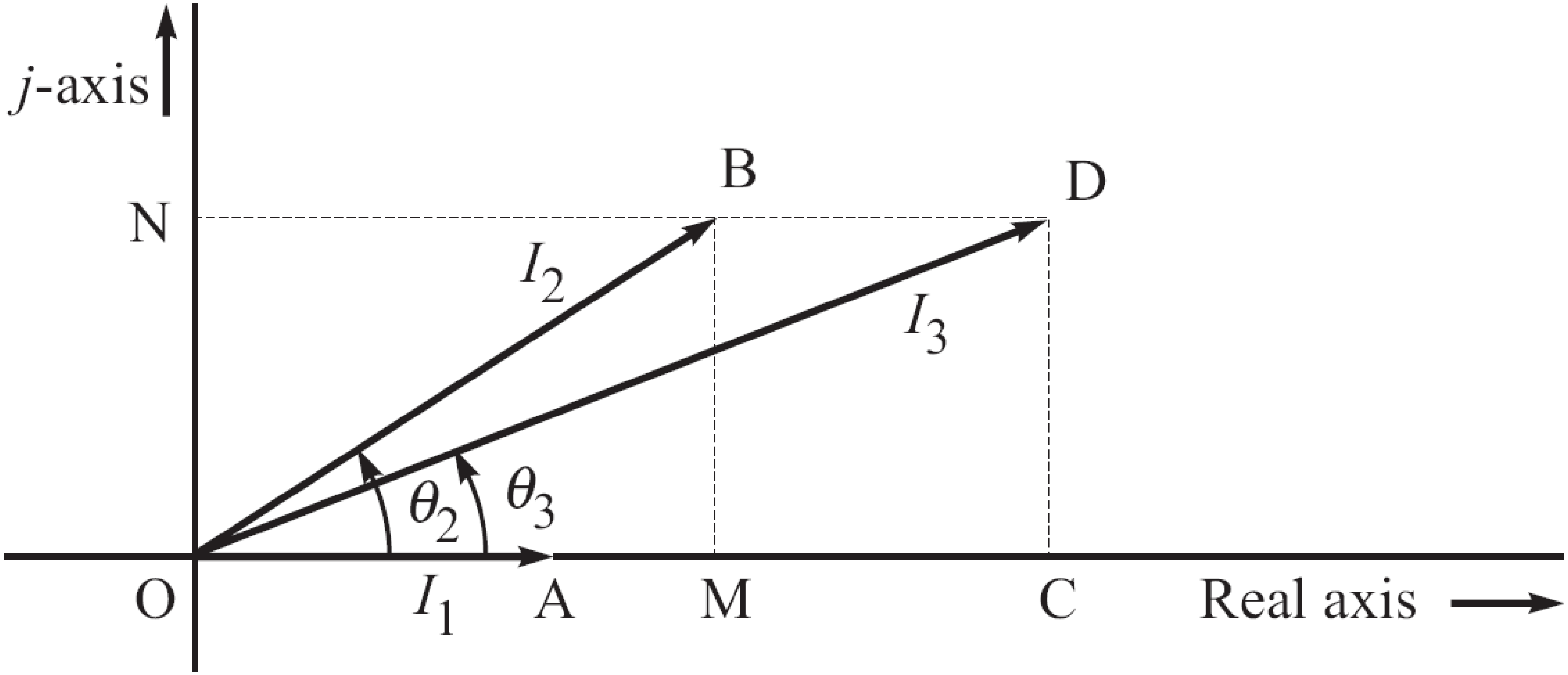
- Phasors can be added simply by using ***complex numbers***.

Additions of Phasors Using Complex Numbers

Let us add two phasors $\mathbf{I}_1$ and $\mathbf{I}_2$,

$$\mathbf{I}_1 = I_1\angle 0^\circ \qquad \text{and} \qquad \mathbf{I}_2 = I_2\angle \theta^\circ$$



Finding the components along the real and imaginary axes,

$$\mathbf{I_1} = I_1 + j\,0; \qquad \mathbf{I}_2 = I_2\cos\theta_2 + j\,I_2\sin\theta_2$$

$$\therefore \quad \mathbf{I}_3 = \mathbf{I}_1 + \mathbf{I}_2 = (I_1 + j0) + (I_2\cos\theta_2 + j\,I_2\sin\theta_2)$$

$$= (I_1 + I_2\cos\theta_2) + j\,I_2\sin\theta_2$$

We can write the result as

$$\mathbf{I_3} = I_3\sin(\omega t + \theta_3)$$

where, $$I_3 = \sqrt{(I_1 + I_2\cos\theta_2)^2 + (I_2\sin\theta_2)^2}$$

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$$\theta_3 = \tan^{-1}\frac{I_2\sin\theta_2}{I_1 + I_2\cos\theta_2}$$

Example 2

Find the expression for the sum of the currents

$$i_1 = 10\sqrt{2}\sin\omega t \text{ A} \quad \text{and} \quad i_2 = 20\sqrt{2}\sin(\omega t + 60°) \text{ A}$$

Also, determine the rms value of the sum of these two currents.

Solution : Since we need to find the expression, there is no need to first find the rms values of the two phasors. We can directly write the peak values as complex numbers,

$$\mathbf{I}_{\text{m1}} = 10\sqrt{2}\angle 0° \text{ A} \qquad \text{and} \qquad \mathbf{I}_{\text{m2}} = 20\sqrt{2}\angle 60° \text{ A}$$

The summation of these two currents is given as

$$\mathbf{I}_{\mathrm{m}} = \mathbf{I}_{\mathrm{m1}} + \mathbf{I}_{\mathrm{m2}} = (10\sqrt{2}\angle 0^\circ + 20\sqrt{2}\angle 60^\circ)\ \mathrm{A}$$

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$$= 37.42\angle 40.9^\circ\ \mathrm{A}$$

The resultant current *i* can be written in time domain as

$$i = \mathbf{37.42\ sin}(\omega t + \mathbf{40.9^\circ})\ \mathbf{A}$$

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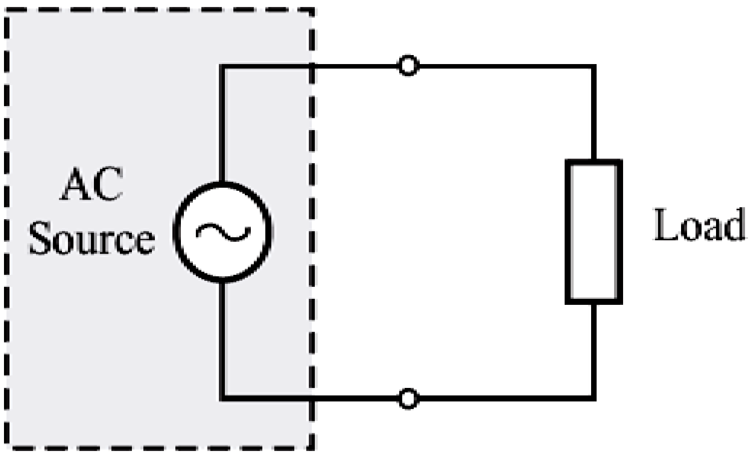
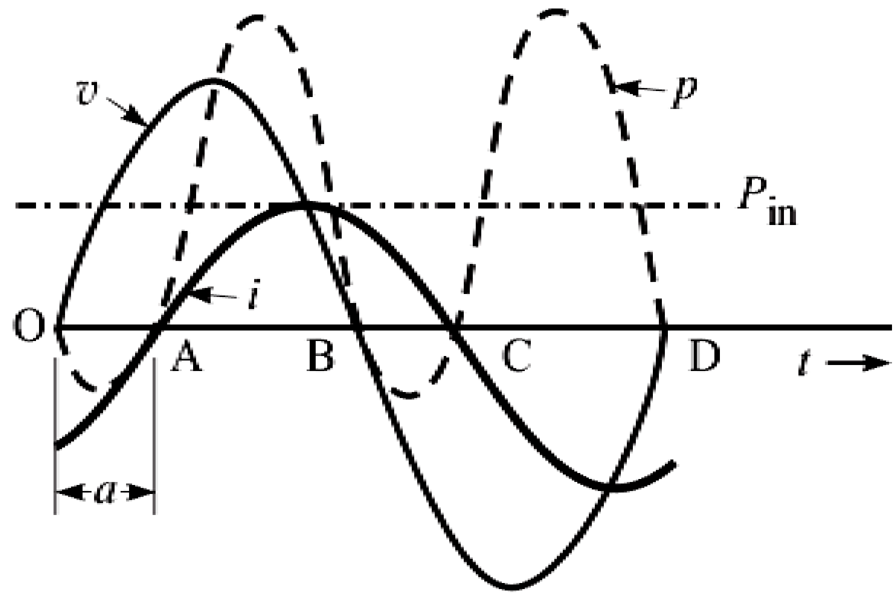
The rms value,

$$I = \frac{I_{\mathrm{m}}}{\sqrt{2}} = \frac{37.42}{\sqrt{2}} = \mathbf{26.46\ A}$$

Power and Power Factor



(a) AC source delivering power to a load.



(b) Waveforms of voltage, current and power.

$$v = V_m \sin \omega t \qquad \text{and} \qquad i = I_m \sin(\omega t - \theta)$$

Let the effective values be

$$V = V_{\mathrm{m}} / \sqrt{2} \quad \text{and} \quad I = I_{\mathrm{m}} / \sqrt{2}$$

Apparently, it seems that the power going to the load should be equal to *VI*.

But the real power is the average of the instantaneous power.

Let us write the expression of instantaneous power,

$$p = vi = [V_{\mathrm{m}} \sin \omega t][I_{\mathrm{m}} \sin(\omega t - \theta)] = V_{\mathrm{m}} I_{\mathrm{m}} \sin \omega t \sin(\omega t - \theta)$$

$$= \frac{V_{\mathrm{m}} I_{\mathrm{m}}}{2} [\cos\{\omega t - (\omega t - \theta)\} - \cos\{\omega t + (\omega t - \theta)\}]$$

$$= \frac{V_{\mathrm{m}}}{\sqrt{2}} \cdot \frac{I_{\mathrm{m}}}{\sqrt{2}} [\cos\theta - \cos(2\omega t - \theta)] = VI[\cos\theta - \cos(2\omega t - \theta)]$$

$$= VI \cos\theta - VI \cos(2\omega t - \theta)$$

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- The average value of the second term is zero.
- Thus, the average power or actual power or real power consumed by the load is given as

$$\boxed{P = VI \cos\theta}$$

Power Factor

It is defined as the factor by which the apparent power is to be multiplied so as to get the real power. Thus,

$$\text{power factor}\,(pf) = \cos\theta$$

In case the phase angle is zero, the circuit is said to have **unity pf.** *The real power is then same as the apparent power.*

Example 3

In an ac circuit, the instantaneous voltage and current are given as

$$v = 55 \sin \omega t \text{ V} \quad \text{and} \quad i = 6.1 \sin(\omega t - \pi/5) \text{ A}$$

Determine the average power, the apparent power, the instantaneous power when ωt (in radians) equals 0.3, and the power factor in percentage

Solution : The phase angle, $\theta = \pi/5$

The rms values are

$$V = \frac{V_m}{\sqrt{2}} = \frac{55}{\sqrt{2}} = 38.89\,\text{V}$$

$$I = \frac{I_m}{\sqrt{2}} = \frac{6.1}{\sqrt{2}} = 4.31\,\text{A}$$

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Therefore, the average power,

$$P_{av} = VI\cos\theta = 38.89 \times 4.31 \times \cos\pi/5$$

$$= 167.62 \times 0.809 = \mathbf{135.6\,W}$$

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The apparent power,

$$P_a = VI = 38.89 \times 4.31 = \mathbf{167.62\,VA}$$

Click

The instantaneous power at ωt = 0.3,

Click

$$p = VI\cos\theta - VI\cos(2\omega t - \theta)$$
$$= 135.6 - 167.62 \times \cos(2 \times 0.3 - \pi/5) = \mathbf{-31.95\ W}$$

The power factor,

Click

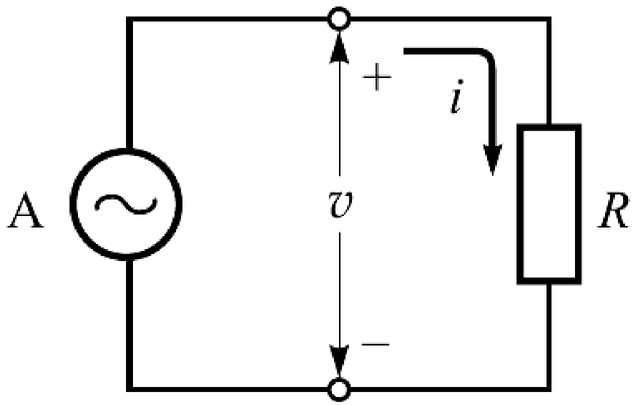
$$pf = \cos\theta = \cos\pi/5 = 0.809 = \mathbf{80.9\,\%}$$

Next

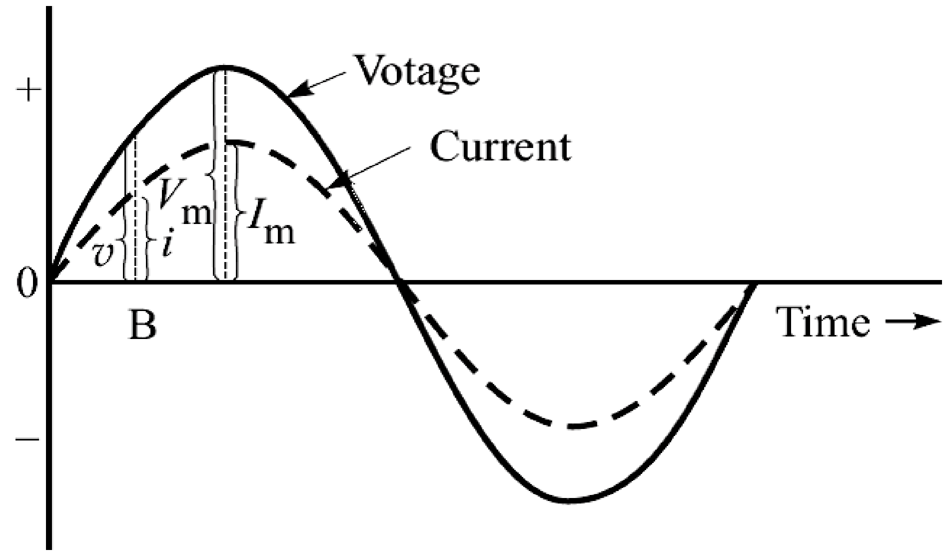
Behaviour of *R*, *L* and *C* in AC Circuits

- We shall study the steady-state response of *R*, *L* and *C* to a sustained sinusoidal function.
- We shall establish the phase angle relationships between *V* and *I*.
- These relationships remain fixed irrespective of how the components are connected in a circuit.

Purely Resistive Circuit



(a) Resistive circuit.

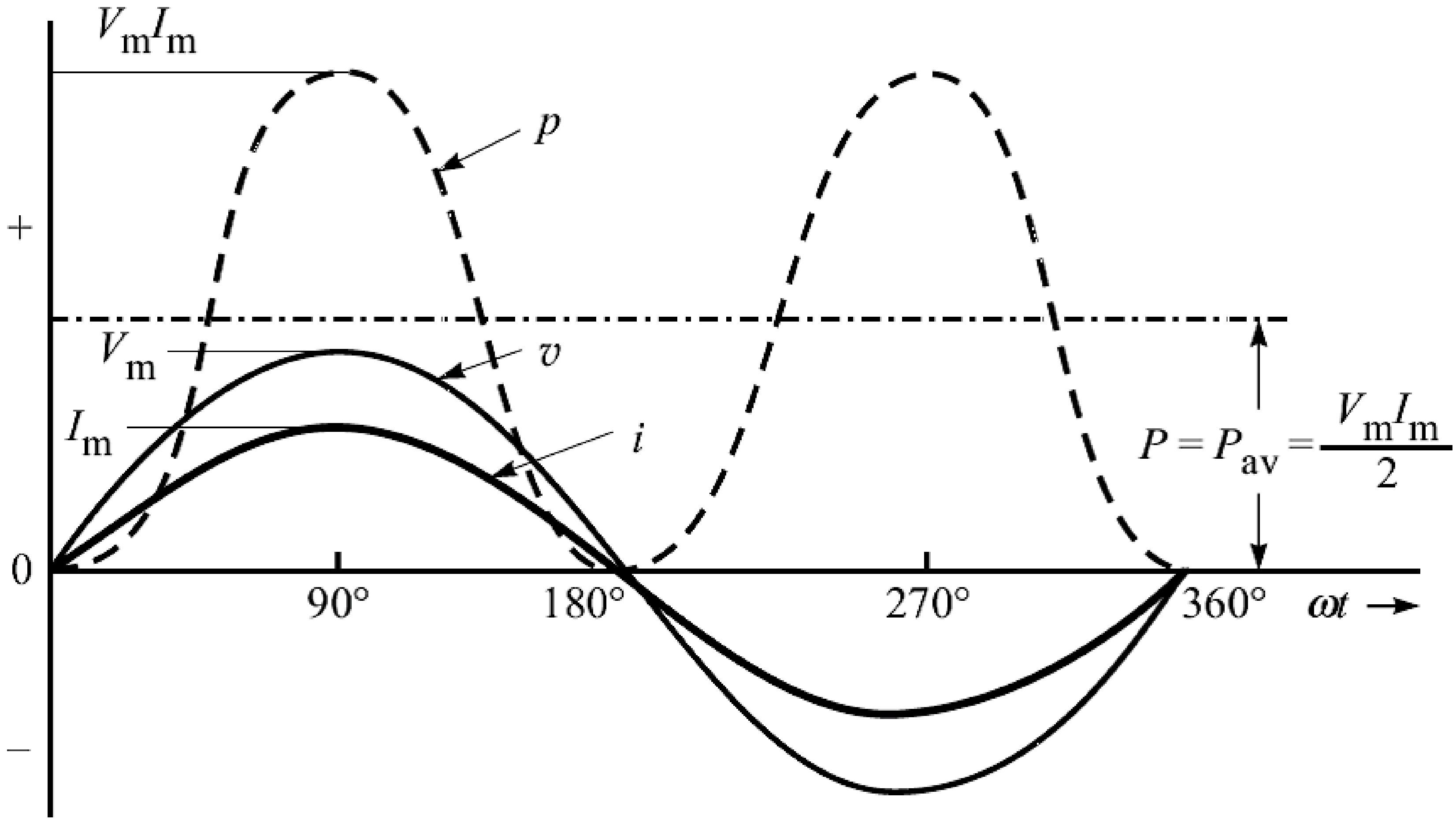


(b) Voltage and current waveforms.

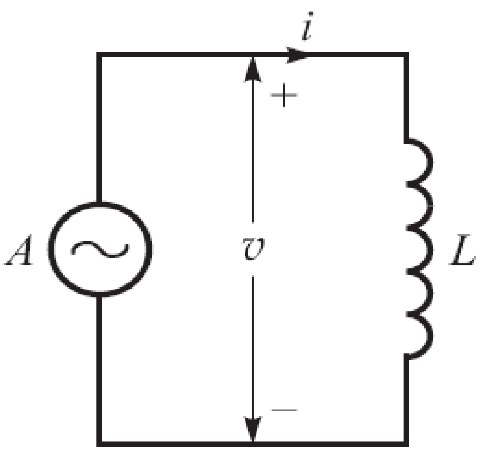
(c) Phasor diagram

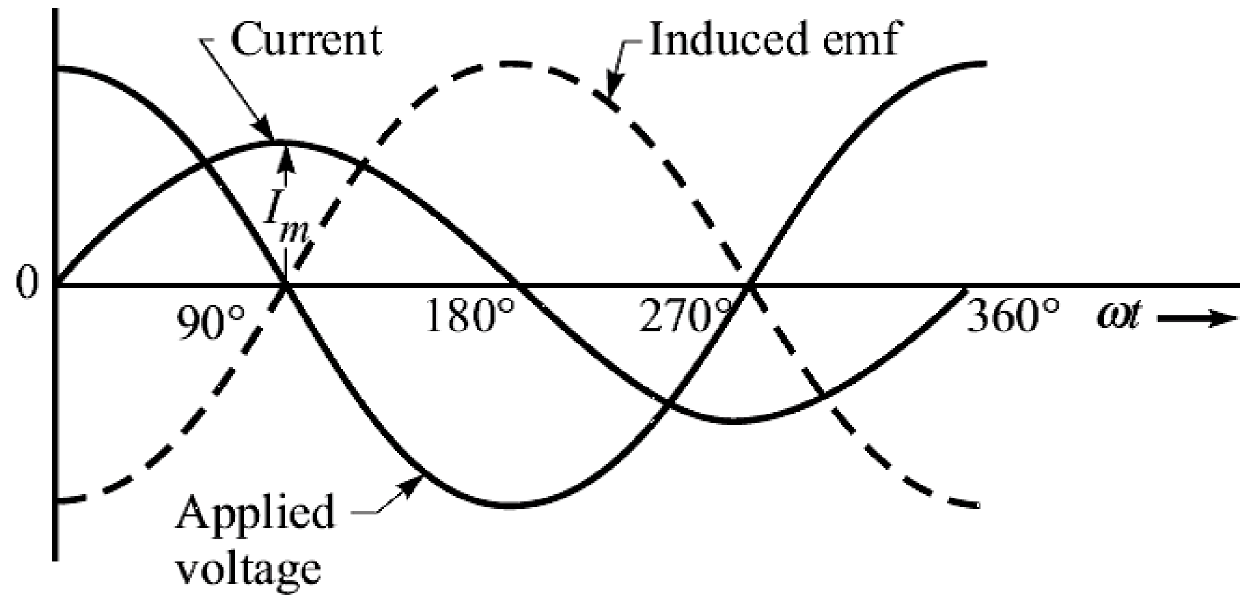
The two waveforms are ***in phase*** with each other; the circuit has **unity** *pf.*

Power waveform for a resistive circuit.

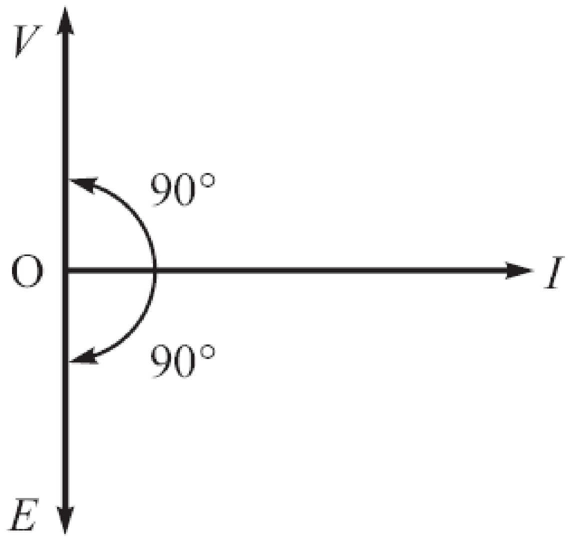


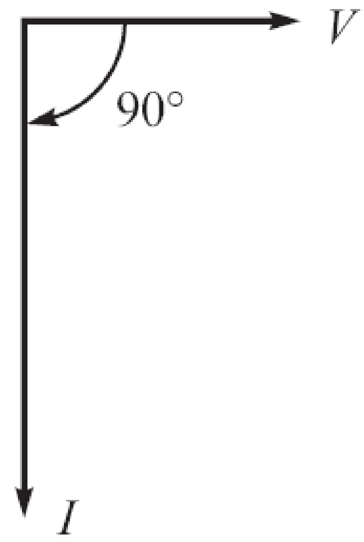
Purely Inductive Circuit





(a) Inductive circuit. (b) Voltage and current waveforms.





(*c*) Phasor diagram (with **I** reference).

(*d*) Phasor diagram (with **V** reference).

- **Note : W**e are at full liberty to take any phasor as reference

- The current lags the applied voltage by $\pi/2$.

Analysis :

We start with the assumption that the current is given as

$$i = I_{\mathrm{m}} \sin \omega t$$

$$e = -L\frac{di}{dt} = -L\frac{d}{dt}(I_{\mathrm{m}} \sin \omega t)$$

Click

$$= -\omega L I_{\mathrm{m}} \cos \omega t = \omega L I_{\mathrm{m}} \sin(\omega t - \pi/2)$$

- The induced emf *e* lags the current *i* by π/2.

- The induced emf e opposes the applied voltage v. Hence,

$$v = -e = \omega L I_{\mathrm{m}} \cos \omega t = \omega L I_{\mathrm{m}} \sin(\omega t + \pi/2)$$

- It means that the applied voltage leads the current by π/2.

$$\text{If} \quad \mathbf{I} = I\angle 0^\circ = I + j0;$$

$$\mathbf{V} = V\angle 90^\circ = 0 + jV$$

Inductive Reactance

From $v = \omega L I_{\mathrm{m}} \sin(\omega t + \pi/2)$, we can write

$$V_{\mathrm{m}} = \omega L I_{\mathrm{m}} \quad \Rightarrow \quad \frac{V_{\mathrm{m}}}{I_{\mathrm{m}}} = \omega L$$

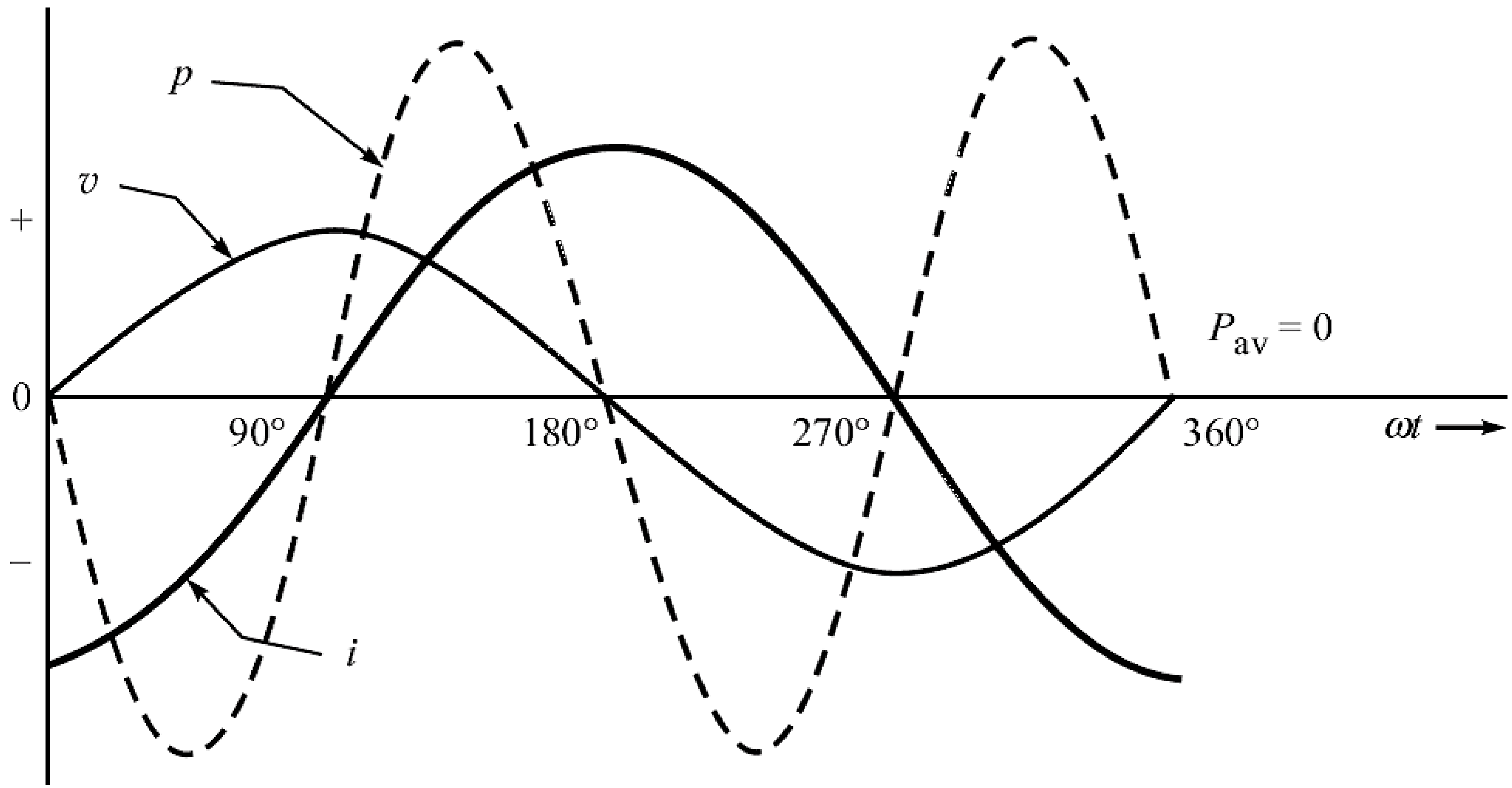
That is, the ratio of rms voltage to rms current,

$$\frac{V}{I} = \frac{V_{\mathrm{m}}/\sqrt{2}}{I_{\mathrm{m}}/\sqrt{2}} = \frac{V_{\mathrm{m}}}{I_{\mathrm{m}}} = \omega L$$

This is called ***inductive reactance,***

$$X_L = \omega L = 2\pi f L$$

Power waveform for a purely inductive circuit.



The power consumed by the circuit is zero. Same amount of energy keeps on flowing alternately from source to the load and then from load to the source.

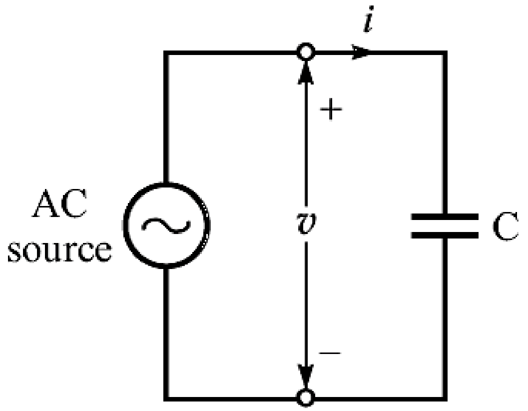
The real power,

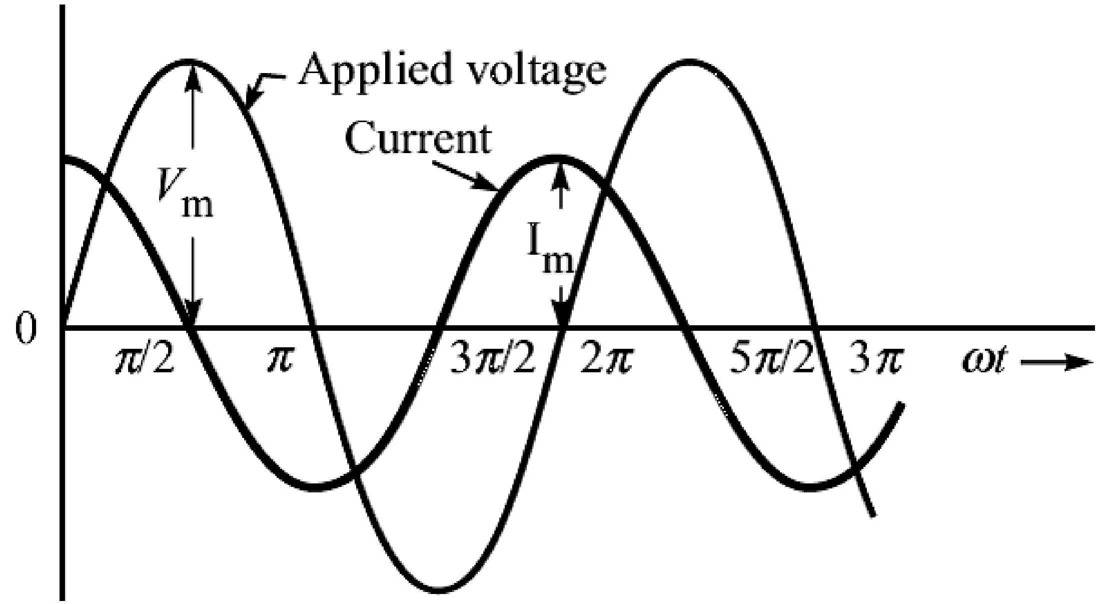
$$P = VI\cos\theta = VI\cos 90^\circ = 0$$

The power factor of the circuit,

$$pf = \cos\theta = \cos 90^\circ = 0\,(\text{lagging})$$

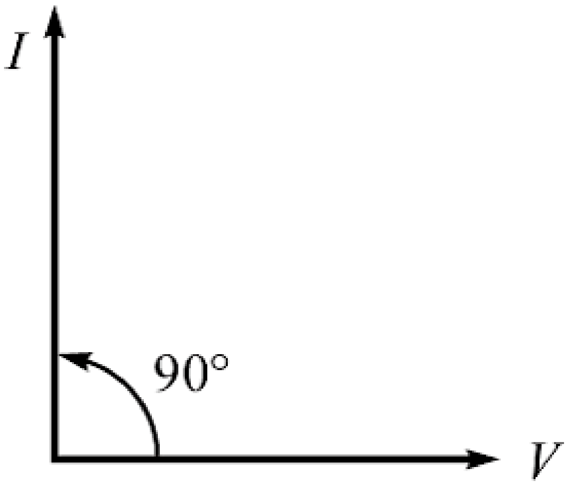
Purely Capacitive Circuit



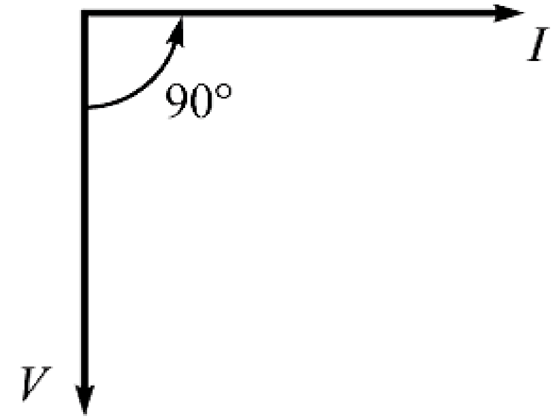


(*a*) Capacitive circuit.

(*b*) Voltage and current waveforms.



(*c*) Phasor diagram (with **V** reference).



(*d*) Phasor diagram (with **I** reference).

- The current leads applied voltage by π/2.

Analysis

We start with the applied voltage,

$$v = V_{\mathrm{m}} \sin \omega t$$

$$\therefore \qquad i = C\frac{dv}{dt} = C\frac{d}{dt}(V_{\mathrm{m}} \sin \omega t)$$

$$= CV_{\mathrm{m}}\omega \cos \omega t = \omega CV_{\mathrm{m}} \cos \omega t$$

$$= \omega CV_{\mathrm{m}} \sin(\omega t + \pi/2)$$

If $\quad \mathbf{V} = V\angle 0° = V + j0; \quad \mathbf{I} = I\angle 90° = 0 + jI$

Capacitive Reactance

From $i = \omega C V_m \sin(\omega t + \pi/2)$, we can write

$$I_m = \omega C V_m$$

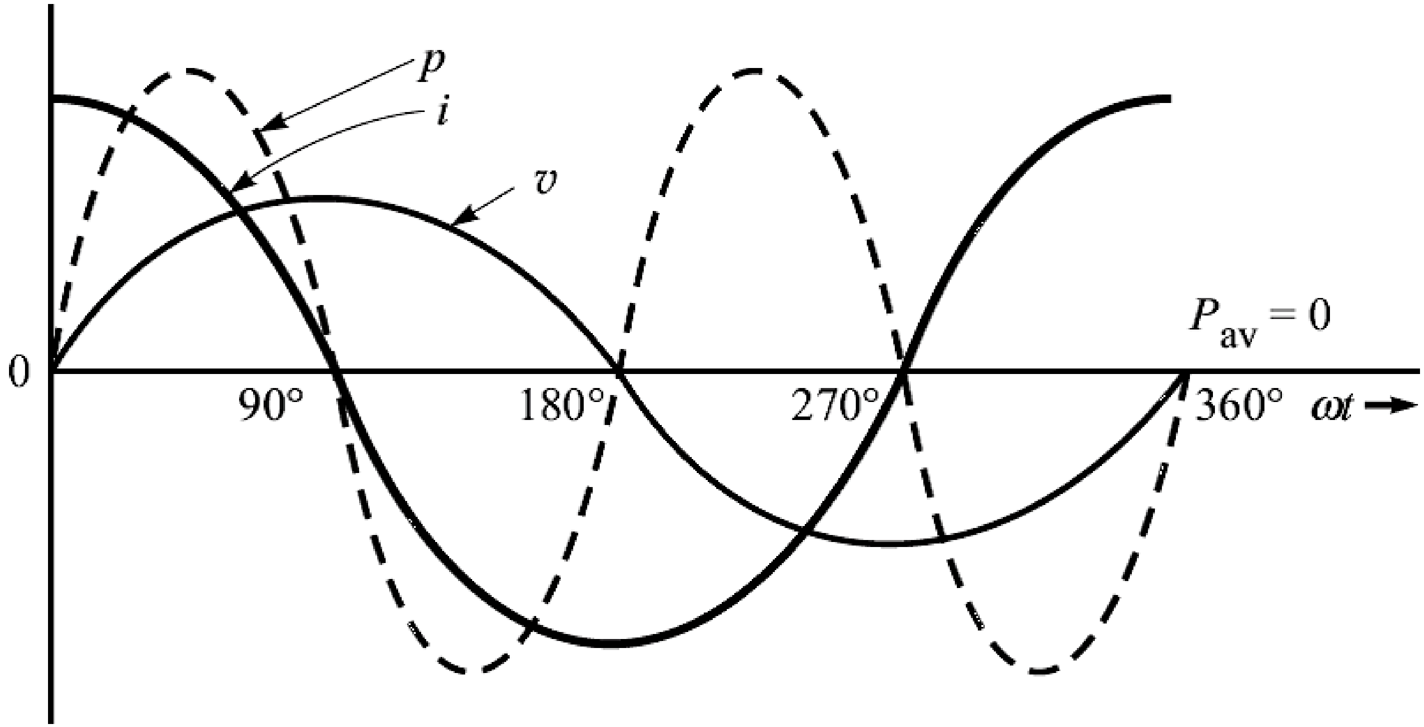
Hence, if V and I are the rms values, the ratio

$$\frac{V}{I} = \frac{V_m/\sqrt{2}}{I_m/\sqrt{2}} = \frac{V_m}{I_m} = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

This is called ***capacitive reactance,***

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

Power waveform for a purely capacitive circuit.



- The average power and the power factor are given as

$$P = VI\cos\theta = VI\cos 90° = 0$$

$$pf = \cos\theta = \cos 90° = 0\,(\text{leading})$$

Review

- DC and AC Currents.
- A Sinusoid.
  - Some Definitions.
  - Phase Difference.
  - Physical Model for a Sinusoid.
- Average of a sine wave.
- RMS or Effective Value.
- Concept of Phasors.
  - Operations on Phasors.
- Additions of Phasors Using Complex Numbers.
- Power and Power Factor.
- Purely Resistive Circuit.
- Purely Inductive Circuit.
  - Inductive Reactance.
- Purely Capacitive Circuit.
  - Capacitive Reactance