Answers of Tute-7

(1) (i) (a) $\frac{1}{2}\left[\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)-\frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\cdots\right)\right]$.

(b) $\frac{1}{2}\left[\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)-\frac{1}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\cdots\right)\right]$.

(c) $\frac{1}{2(z+1)}-\frac{1}{4}+\frac{z+1}{8}-\frac{(z+1)^2}{16}+\cdots$

(ii) $1+\frac{5}{z-2}-\frac{1}{6(z-2)^2}-\frac{5}{6(z-2)^3}+\frac{1}{120(z-2)^4}+\cdots$

(iii) Case I If $|z+1|<2$

$$f(z)=\frac{1}{2(z+1)^2}-\frac{1}{4(z+1)}+\frac{1}{8}-\frac{1}{16}(z+1)+\frac{1}{32}(z+1)^2+\cdots$$

Case II If $|z+1|>2$

$$\cancel{f(z)=\frac{1}{2(z+1)^2}-\frac{1}{4(z+1)}}$$

$$f(z)=\frac{1}{(z+1)^3}-\frac{2}{(z+1)^4}+\frac{4}{(z+1)^5}+\cdots$$

(2) (i) $z=0$ is simple pole and
$z=1$ is pole of order 2.

Residue at $z=0$ = 1.

Residue at $z=1$ = -1.

(ii) $z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4},$
$e^{7i\pi/4}$ are simple poles.

Residue at $z=e^{i\pi/4}$ $= \frac{1}{4e^{3i\pi/4}}$

Residue at $z=e^{3i\pi/4}$ $= \frac{1}{4e^{9i\pi/4}}$

Similarly others.

(iii) $z = n\pi$ are singular points for
$n = 0, 1, 2 - \cdots$
where $z=0$ is removable singularity
and others are simple poles for
$n = 1, 2, 3 -$

Residue at $z = n\pi$ $= (-1)^n n\pi,$
$n = 1, 2, 3 -$

(d) $z = \frac{1}{n}$ simple poles for $n = 1, 2, 3 - -$

Residue at $z = \frac{1}{n} = \frac{(-1)^n}{n\pi}$

as $n \to \infty$ $z \to 0$, which is non-isolated singularity. Since all neighbourhood points of $z=0$ are singular points.

(e) $z = \pm i$ are simple poles.

Residue at $z = i = \frac{1}{2e}$

Residue at $z = -i = \frac{e}{2}$

(3) (a) $-\frac{1}{2}$ (b) 1.

(6) (a) $\pi\sqrt{2}$ (b) $\frac{\pi}{3}$. (c) $\frac{\pi}{6}$.

(7) (a) $-1-i$ (b) $-i$.

(8) $w = \frac{(1+2i)z-1}{(3-2i)z+1}$

Tutorial Sheet 7

$$\frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$$ is not analytic at $z=-1$ & $z=-3$.

a) $1<|z|<3$.

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$\frac{1}{z+1} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} \qquad \left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$$

$$\Rightarrow \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)$$

$$\frac{1}{z+3} = \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1} \qquad \left|\frac{z}{3}\right|<1 \Rightarrow |z|<3.$$

$$= \frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)$$

$$\therefore \frac{1}{z^2+4z+3} = \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+\cdots\right)\right]$$

b) $|z|>3$.

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$\frac{1}{z+1} = \frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots \qquad |z|>1 \Rightarrow |z|>3.$$

$$\frac{1}{z+3} = \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1} \qquad \left|\frac{3}{z}\right|<1 \Rightarrow |z|>3$$

$$= \frac{1}{z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}-\frac{3^3}{z^3}\cdots\right)$$

$$= \frac{1}{z}-\frac{3}{z^2}+\frac{3^2}{z^3}-\frac{3^3}{z^4}+\cdots$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right)-\left(\frac{1}{z}-\frac{3}{z^2}+\frac{3^2}{z^3}\cdots\right)\right]$$

c) $0<|z+1|<2$.

$$\frac{1}{(z+1)(z+3)} = \frac{1}{(z+1)(z+1+2)} \qquad |z+1|<2$$

$$= \cancel{\frac{1}{(z+1)(z+1)\left[1+\frac{2}{z+1}\right]}} = \frac{1}{2(z+1)\left(1+\frac{z+1}{2}\right)}$$

$$= \frac{1}{2(z+1)}\left[1-\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2-\left(\frac{z+1}{2}\right)^3+\cdots\right]$$

$$= \frac{1}{2(z+1)} - 1 + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \cdots$$

(ii) $(z+3)\sin\left(\frac{1}{z-2}\right)$ about $z_0 = 2$.

$$= (z-2+5)\sin\left(\frac{1}{z-2}\right)$$

$$= (z-2)\sin\frac{1}{z-2} + 5\sin\frac{1}{z-2}.$$

$$= (z-2)\left[\frac{1}{z-2} - \frac{1}{3!(z-2)^3} + \frac{1}{5!(z-2)^5} - \cdots\right]$$

$$+5\left[\frac{1}{z-2} - \frac{1}{3!(z-2)^3} + \frac{1}{5!(z-2)^5} \cdots\right]$$

$$= 1 - \frac{1}{3!(z-2)^2} + \frac{1}{5!(z-2)^4} \cdots$$

$$+5\left[\frac{1}{z-2} - \frac{1}{3!(z-2)^3} + \cdots\right]$$

(iii) $\frac{1}{(z+1)^2(z+3)}$ about $z_0 = -1$

is not analytic at $z = -1, -3$.

so we have two case

(a) $|z+1| < 2$ and (b) $|z+\cancel{2}1| > 2$.

Case (a) $|z+1| < 2$.

$$\frac{1}{(z+1)^2(z+3)} = \frac{1}{(z+1)^2(z+1+2)} = \frac{1}{2(z+1)^2}\left(1+\frac{z+1}{2}\right)^{-1}$$

$$= \frac{1}{2(z+1)^2}\left[1 - \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 - \cdots\right]$$

$$= \frac{1}{2(z+1)^2} - \frac{1}{4(z+1)} + \frac{1}{8} - \frac{z+1}{16} + \frac{(z+1)^2}{32} \cdots$$

Case b $|z+1| > 2$.

$$\frac{1}{(z+1)^2(z+3)} = \frac{1}{(z+1)^3}\left(1+\frac{2}{z+1}\right)^{-1}$$

$$= \frac{1}{(z+1)^3}\left[1 - \frac{2}{z+1} + \left(\frac{2}{z+1}\right)^2 - \frac{2^3}{(z+1)^3} + \cdots\right]$$

$$= \frac{1}{(z+1)^3} - \frac{2}{(z+1)^4} + \frac{4}{(z+1)^5} - \frac{8}{(z+1)^6} + \cdots$$

2) a) $\dfrac{z+1}{z(z-1)^2} = \dfrac{p(z)}{Q(z)}$

$Q(z) = z(z-1)^2 = 0$

when $z = 0, 1$ so they are isolated sing.

Laurant series expansion about $z=0$

$$\frac{1}{(z-1)^2} + \frac{1}{z(z-1)^2}$$

$$=(1+2z+3z^2+4z^3+\cdots) \qquad \frac{1}{(z-1)^2} = \frac{1}{1+z^2-2z}$$

$$+\frac{1}{z}(1+2z+3z^2+4z^3+\cdots) \quad \text{for } |z|<1$$

$(1-z)^{-2}$

$1+2z+3z^2$

$$=(1+2z+3z^2+\cdots) + \frac{1}{z} + 2 + 3z + 4z^2 + \cdots$$

$z=0$ is a simple pole.

Laurant series exp. about $z=1$

$$\frac{1}{(z-1)^2}\frac{z+1}{z} = \frac{1}{(z-1)^2}\left(1+\frac{1}{z}\right) = \frac{1}{(z-1)^2} + \frac{1}{(z-1)^2(1+(z-1))}$$

$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^2}\left[1-(z-1)+(z-1)^2 \cdots\right]$$

$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 + (z-1) + (z-1)^2 + \cdots$$

$$= \frac{2}{(z-1)^2} - \frac{1}{z-1} + 1 + (z-1) + (z-1)^2 + \cdots$$

$\therefore$ $z=1$ is a pole of order 2.

$$\text{Res (at } z=0) = \lim_{z\to 0} z\, f(z)$$

$$= \lim_{z\to 0} z\,\frac{z+1}{z(z-1)^2} = 1.$$

$$\text{Res (at } z=1) = \cancel{\lim_{z\to 1} (z-1)\frac{z+1}{z(z-1)^2}} \quad \frac{d}{dz}\left[(z-1)^2 f(z)\right] \text{ at } z=1$$

$$= \frac{d}{dz}\left(\frac{z+1}{z}\right)\Big|_{at\ z=1}$$

$$= \frac{-1}{z^2} \text{ at } z = 1$$

$$= -1 \ . \ = \text{coef of } \frac{1}{z-1}$$

b) $$\frac{1}{z^4+1} = \frac{\phi(z)}{\psi(z)}$$

$$z^4 = -1 = e^{(2n+1)\pi i}$$

$z = e^{(2n+1)\pi i/4}$. are simple poles.

$n = 0, 1, 2, 3$.

$$\psi'(z) = 4z^3 .$$

$$\text{Res}\left(\text{at } z = e^{\frac{\pi i}{4}}\right) = \frac{1}{4z^3}\Big|_{z = e^{\frac{\pi i}{4}}}$$

$$= \frac{1}{4e^{3\pi i/4}}$$

$$\text{Res}\left(\text{at } z = e^{3\pi i/4}\right) = \frac{1}{4e^{9\pi i/4}}$$

$$\text{Res}\left(\text{at } z = e^{5\pi i/4}\right) = \frac{1}{4e^{15\pi i/4}}$$

$$\text{Res}\left(\text{at } z = e^{7\pi i/4}\right) = \frac{1}{4e^{21\pi i/4}} .$$

$$\frac{z}{\sin z}$$

$\sin z = 0$ where $z = n\pi$, $n = 0, 1, 2, \ldots$

$z = 0$ at $z = 0$.

$\frac{z}{\sin z}$ has removable sing at $z = 0$.

$z = n\pi$, $n = 1, 2, \ldots$ are isolated sing.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$$

$$\frac{z}{\sin z} = \frac{z}{z}\left(1 - \left(\frac{z^2}{3!} \mp \frac{z^4}{5!} + \ldots\right)\right)^{-1}$$

$$= 1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} - \ldots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right)^2 + \ldots$$

so removable sing at $z = 0$.

$z = n\pi$ are simple poles

$$\text{Res (at } z = n\pi) = \frac{\phi(n\pi)}{\psi'(n\pi)}$$

$\phi = z$

$\psi = \sin z$

$\psi' = \cos z$

$$= \frac{n\pi}{\cos(n\pi)}.$$

$$= \frac{n\pi}{(-1)^n} = (-1)^n n\pi$$

$n = 1, 2, 3, \ldots$

OR $\lim_{z \to n\pi} (z - n\pi) \cdot \frac{z}{\sin z}$ $\left(\frac{0}{0}\right)$

$$= \lim_{z \to n\pi} \frac{2z - n\pi}{\cos z}$$

$$= \frac{2n\pi - n\pi}{\cos n\pi} = (-1)^n n\pi.$$

d) $\frac{1}{\sin(\pi/z)}$.

$\sin\left(\frac{\pi}{z}\right) = 0 = \sin n\pi$.

when $z = \frac{1}{n}$, $n = 1, 2, \ldots$

are singular (isolated pts)

Now $n \to \infty \Rightarrow z = 0$ .is non-isolated singular point.

$$\text{Res}\left(\text{at } z = \frac{1}{n}\right) = \frac{1}{\cos(\pi/z) \times -\frac{1}{z^2}}$$

$$= \lim_{z \to \frac{1}{n}} \frac{\left(z - \frac{1}{n}\right)}{\sin(\pi/z)} \left[\frac{0}{0}\right] = \left.\frac{-1}{\frac{\cos(\pi/z)}{z^2}}\right|_{z = 1/n}.$$

$$= \frac{-1}{\frac{\cos n\pi}{1/n^2}}$$

$$= \frac{1}{n^2 \cos n\pi}$$

$$= \frac{(-1)^n}{n^2}$$

e) $\frac{z e^{iz}}{1+z^2}$

$1 + z^2 = 0 \Rightarrow z = \pm i$ are isolated singularities

About $z = i$

$$\frac{z e^{iz}}{z^2+1} = \frac{(z-i+i)\, e^{i(z-i+i)}}{(z-i)(z-i+2i)} = \frac{(z-i+i)\, e^{i(z-i)}}{e(z-i)(z-i+2i)}$$

$$= \frac{e^{i(z-i)}}{2ie}\left(1 + \frac{z-i}{2i}\right)^{-1} + \frac{i}{2ie}\frac{e^{i(z-i)}}{(z-i)}\left(1 + \frac{z-i}{2i}\right)^{-1}$$

$$\frac{e^{i(z-i)}}{2ie}\left(1-\frac{z-i}{2i}+\left(\frac{z-i}{2i}\right)^2\cdots\right).$$

$$+\frac{1}{2e}\frac{e^{i(z-i)}}{z-i}\left(1-\frac{z-i}{2i}+\frac{(z-i)^2}{(2i)^2}-\cdots\right).$$

$\times$

$$\text{Res (at } z=i) = \frac{z\,e^{iz} \text{ at } z=i}{2z \text{ at } z=i}$$

$=.$

$$= \frac{i\,e^{i^2}}{2i} = \frac{e^{-1}}{2} = \frac{1}{2e}$$

$$\text{Res (at } z=-i) = \frac{z\,e^{iz} \text{ at } z=-i}{2z \text{ at } z=-i}$$

$$= \frac{\cancel{+i}\,e^{-i^2}}{\cancel{+2i}} = \frac{e}{2}.$$

~~Res (at z = -i)~~

OR $\displaystyle\lim_{z\to -i}(z+i)\frac{z\,e^{iz}}{1+z^2}$

$$= \lim_{z\to -i}\frac{z\,e^{iz}}{z-i} = \frac{+\cancel{i}\;e^{-i^2}}{-2\cancel{i}}$$

$$= \frac{e}{2}.$$

③ a) $z\cos\left(\frac{1}{z}\right)$

$$= z\left(1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \cdots\right)$$

$$= z - \frac{1}{2!\,z} + \frac{1}{4!\,z^3} - \frac{1}{6!\,z^5} + \cdots$$

Residue $= -\frac{1}{2}$.

b) $\dfrac{1+e^z}{\sin z + z\cos z}$.

OR $\dfrac{1+e^0}{\cos z + \cos z - z\sin z}$

$\dfrac{1+1}{2\cos 0} = \dfrac{2}{2} = 1$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sin z + z\cos z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$
$$+ z - \frac{z^3}{2!} + \frac{z^5}{4!} - \cdots$$

$$= 2z - z^3\left(\frac{1}{3!} + \frac{1}{2!}\right) + z^5\left(\frac{1}{5!} + \frac{1}{4!}\right) - \cdots$$

$$= 2z\left(1 - \frac{z^2}{2}\left(\frac{1}{3!} + \frac{1}{2!}\right) + \frac{z^4}{2}\left(\frac{1}{5!} + \frac{1}{4!}\right) \cdots\right).$$

$$\frac{1+e^z}{\sin z + z\cos z} = \frac{1+e^z}{2z\left(1 - \left\{\frac{z^2}{2}\left(\frac{1}{3!} + \frac{1}{2!}\right) - \frac{z^4}{2}\left(\frac{1}{5!} + \frac{1}{4!}\right) \cdots\right\}\right)}.$$

$$= \frac{1+e^z}{2z}\left(1 + \left\{\frac{z^2}{2}\left(\frac{1}{3!} + \frac{1}{2!}\right) - \frac{z^4}{2}\left(\frac{1}{5!} + \frac{1}{4!}\right) + \cdots\right\}\right.$$
$$\left. + \left\{\frac{z^2}{2}\left(\frac{1}{3!} + \frac{1}{2!}\right) - \frac{z^4}{2}\left(\frac{1}{5!} + \frac{1}{4!}\right) + \cdots\right\}^2 + \cdots\right)$$

$$= \frac{2 + z + \frac{z^2}{2!} + \cdots}{2z}\left(1 + \frac{z^2}{2}\left(\frac{1}{3!} + \frac{1}{2!}\right) - \cdots\right) + \left(\frac{z^2}{2}\left(\frac{1}{3!} + \frac{1}{2!}\right) - \cdots\right)^2 + \cdots$$

$$= \left(\frac{1}{z} + \frac{1}{2} + \frac{z}{4} + \cdots\right)\left(1 + \frac{z^2}{2}\left(\frac{1}{3!} + \frac{1}{2!}\right) + \cdots\right).$$

Coef of $\frac{1}{z} \to 1$

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a) $\int_C \tan z \, dz = -4\pi i \qquad C : |z| = 2$

$\tan z = \dfrac{\sin z}{\cos z}$,

$\cos z = 0 = \cos(2n+1)\frac{\pi}{2}$

$z = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

$z = \pi/2, -\pi/2$ at $n = 0, -1$ lies inside C.
is a simple pole

$\text{Res}(\text{at } z = \pi/2) = \lim_{z \to \pi/2} (z - \pi/2) \tan z$

$= \lim_{z \to \pi/2} \dfrac{(z - \pi/2)\sin z}{\cos z} \quad \left(\frac{0}{0}\right)$

$= \lim_{z \to \pi/2} \dfrac{(z - \pi/2)\cos z + \sin z}{-\sin z}$

$= \lim_{z \to \pi/2} (\pi/2 - z)\dfrac{\cos z}{\sin z} - 1$

$= -1$.

$\text{Res}(\text{at } z = -\pi/2) = \lim_{z \to -\pi/2} \dfrac{(z + \pi/2)\sin z}{\cos z} \quad \left(\frac{0}{0}\right)$

$= \lim_{z \to -\pi/2} \dfrac{(z + \pi/2)\cos z + \sin z}{-\sin z}$

$= -1$.

sum of residues $= -1 - 1 = -2$.

$\therefore \int_C \tan z \, dz = 2\pi i$ (sum of residues)

$= 2\pi i(-2) = -4\pi i$

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$C : |z| = 1.$

b) $\int_C z e^{1/z}\, dz$

$z = 0$ is sing. pt which lies in C.

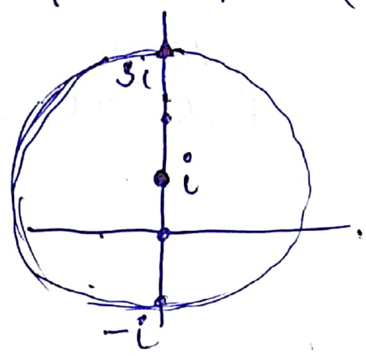
$z e^{1/z} = z\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!\ldots}\right)$

$= z + 1 + \frac{1}{2z} + \frac{1}{3!z^2} + \ldots$

$\int_C z e^{1/z}\, dz = 2\pi i\,(\text{Res } z=0)$

$= 2\pi i\left(\frac{1}{2}\right) = \pi i.$

c) $\int_C \frac{z-1}{(z+1)^2 (z-2)}\, dz \qquad C : |z - i| = 2$



It has simple poles at $z = 2$ (lies outside) and a pole of order 2 at $z = -1$ (lies inside C)

$\text{Res}(\text{at } z=2) = \lim_{z\to 2} \frac{z-1}{(z+1)^2}$

$= \frac{1}{9}.$

$\text{Res}(\text{at } z=-1) = \frac{1}{1!} \frac{d}{dz}\left(\frac{z-1}{z-2}\right)$ at $z = -1$

$= \frac{(z-2)-(z-1)}{(z-2)^2}$ at $z = -1$

$= \frac{-1}{(z-2)^2}\Big|$ at $z = -1$

$= \frac{-1}{9}.$

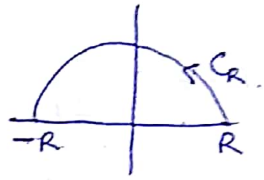
$\therefore \int_C \frac{z-1}{(z+1)^2 (z-2)}\, dz = 2\pi i\left(-\frac{1}{9}\right)$

$= -\frac{2\pi i}{9}.$

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx$$

$$\int_C \frac{z^2+1}{z^4+1}\, dz = \int_C f(z)\, dz$$

$$z^4+1 = 0 \Rightarrow z^4 = -1$$
$$= e^{(2n+1)\pi i}$$
$$\Rightarrow z = e^{(2n+1)i\pi/4}$$



$n = 0, 1, 2, 3$

$z = e^{i\pi/4}, e^{3i\pi/4}$ only lie inside C

Res (at $z = e^{i\pi/4}$)

$$= \frac{e^{i\pi/2}+1}{4e^{i3\pi/4}} = \frac{i+1}{4\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}$$

$$e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$
$$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Res (at $z = e^{3i\pi/4}$)

$$= \frac{e^{3i\pi/2}+1}{4e^{i9\pi/4}} = \frac{1-i}{4\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}$$

~~Res (at $z = e^{5i\pi/4}$)~~

$$e^{i3\pi/2} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$
$$= 0 + i(-1) = -i$$

$$e^{i9\pi/4} = \cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}$$
$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

Sum of residues
$$= \frac{i+1}{4\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} + \frac{1-i}{4\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{1}{\frac{4}{\sqrt{2}}}\left[\frac{i+1}{(-1+i)} + \frac{1-i}{1+i}\right]$$

$$= \frac{\sqrt{2}}{4}\left[\frac{\cancel{1}+\cancel{i^2}+2i + (-1+2i+1)}{i^2-1}\right]$$

$$= \frac{\sqrt{2}}{4}\left[\frac{4i}{-2}\right] = -\frac{i}{\sqrt{2}}.$$

$$\int_C f(z)\,dz = 2\pi i\left(-\frac{i}{\sqrt{2}}\right) = \sqrt{2}\,\pi$$

Now

$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} \frac{x^2+1}{x^4+1}\,dx.$$

when $R \to \infty$

$|z| \to \infty$.

$$|f(z)| = \left|\frac{z^2+1}{z^4+1}\right| = \frac{z^2\,|1+1/z^2|}{|z^4|\,|1+1/z^4|}$$

$$= \frac{1}{|z^2|}\frac{|1+1/z^2|}{|1+1/z^4|} = \frac{1}{R^2}\left(\frac{1+1/R^2}{1+1/R^4}\right)$$

$$\to 0.$$

$$\therefore \int_{C_R} f(z)\,dz = 0.$$

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = \sqrt{2}\,\pi.$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} \frac{|z^2+1|}{|z^4+1|}\,|dz|.$$

$$\le \int_{C_R} \frac{|z|^2+1}{|z^4|-1}\,|dz|.$$

$z = Re^{i\theta}$

$|dz| = |Re^{i\theta}\,i\,d\theta|$

$= R\,d\theta.$

$$\le \int_0^{\pi} \frac{R^2+1}{R^4-1}\,R\,d\theta.$$

$$= \frac{R(R^2+1)}{R^4-1}\,\pi.$$

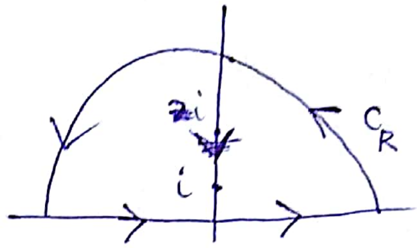
$$= \pi\frac{R^3(1+1/R^2)}{R^4(1-1/R^4)} \to 0 \quad \text{as } R\to\infty$$

So $\int_{C_R} f(z)\,dz = 0.$

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

Consider $\int_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz$

$= \int_C f(z)\,dz$



where C is the contour consisting of ~~quarter~~ semi circle C_R of radius R together with the part of real axis from 0 to R ~~and part of imaginary axis from R to 0~~.

The integrand has simple poles at $z = i, -i, 2i, -2i$ of which only $z = i, 2i$ lie inside C.

$$\text{Res}(\text{at } z=i) = \lim_{z\to i} (z-i)\,f(z).$$

$$= \lim_{z\to i} \frac{\cancel{(z-i)}\,z^2}{(z+i)\cancel{(z-i)}(z^2+4)}$$

$$= \frac{i^2}{2i(i^2+4)} = \frac{-1}{2i(3)} = -\frac{1}{6i}$$

$$\text{Res}(\text{at } z=2i) = \lim_{z\to 2i} (z-2i)\,f(z)$$

$$= \lim_{z\to 2i} \frac{z^2}{(z^2+1)(z+2i)}$$

$$= \frac{4i^2}{(4i^2+1)(4i)} = \frac{-4}{(-3)(4i)}$$

$$= \frac{1}{3i}.$$

$\therefore$ By Residue thm

$$\int_C f(z)\,dz = 2\pi i\left(\frac{1}{3i} - \frac{1}{6i}\right)$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{6}\right) = 2\pi\left(\frac{1}{6}\right) = \frac{\pi}{3}.$$

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$$\int_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \cancel{+ \int_R^0 f(y)}$$

Now as $R \to \infty$

$$\frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{z^4\left(1+\frac{1}{z^2}\right)\left(1+\frac{4}{z^2}\right)}$$

$$= \frac{1}{z^2(1+z^{-2})(1+4z^{-2})}$$

$\frac{1}{z^2}$ decrease for any point on and tends to zero

C_R as $|z| \to \infty$

$$\int_{C_R} f(z)\,dz = 0 .$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3} .$$

$$\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6} .$$

2e) $\displaystyle\int_0^{\pi} \frac{4\cos 2\theta}{5-4\cos\theta}\,d\theta .$

Put $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta .$

$d\theta = \frac{dz}{iz} .$

$\cos\theta = \frac{z + 1/z}{2}$ $\qquad$ $\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2}$

$= \frac{z^2 + 1/z^2}{2} .$

$$I = \int_C \frac{2(z^2 + 1/z^2)}{5 - 2(z + 1/z)} \frac{dz}{iz} . \qquad C : |z| = 1$$

$$= \frac{2}{i}\int_C \frac{(z^4+1)/z^2}{5 - 2\left(\frac{z^2+1}{z}\right)} \cdot \frac{dz}{z} .$$

$$= \frac{2}{i}\int_C \frac{(z^4+1)/z^{\cancel{2}}}{(5z - 2z^2 - 2)/\cancel{z}} \frac{dz}{z} = -\frac{2}{i}\int_C \frac{z^4+1}{z^2(2z^2-5z+2)} dz$$

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$$= -\frac{2}{i}\int_C \frac{z^4+1}{z^2(2z-1)(z-2)}\,dz$$

Res (at $z=0$) $= \frac{d}{dz}\left(\frac{z^4+1}{2z^2-5z+2}\right)$ at $z=0$

$$= \frac{(2z^2-5z+2)(4z^3)-(z^4+1)(4z-5)}{(2z^2-5z+2)^2}$$

$$= \frac{-(+1)(-5)}{2^2} = \frac{5}{4}$$

Res (at $z=\frac{1}{2}$) $= \lim_{z\to\frac{1}{2}} (z-\frac{1}{2})\frac{z^4+1}{2z^2(z-\frac{1}{2})(z-2)}$

$$= \frac{\frac{1}{16}+1}{2\times\frac{1}{4}\left(\frac{1}{2}-2\right)} = \frac{17/16}{-3/4}$$

$$= -\frac{17}{12}.$$

~~Res~~ and $z=2$ doesnot lie inside $|z|=1$.

$$\therefore -\frac{2}{i}\int_C \frac{z^4+1}{z^2(2z-1)(z-2)}\,dz$$

$$= -\frac{2}{i}(2\pi i)\left(\frac{5}{4}-\frac{17}{12}\right)$$

$$= -4\pi\left(\frac{15-17}{12_3}\right)$$

$$= \frac{2\pi}{3}.$$

as $R\to\infty$ $\int_{-\infty}^{\infty}\frac{1}{z^2+1}\,dz \neq 0$ $\quad \int_{-\infty}^{\infty}\frac{1}{z^2+1}\,dz = 0$

$|z|=R$, $z=Re^{i\theta}$, $dz = iRe^{i\theta}$

$\frac{1}{z^2+1}$, $|z|=R$ $\quad \leq \frac{1}{|z|^2+1}$

$\left|\frac{1}{z^2+1}\right| = \frac{1}{|z^2+1|} \leq \frac{1}{|z|^2-1}$

$\frac{1}{|z^2+1|}$

$|z|=R$

$\int_{-R}^{R}\frac{1}{|z^2+1|}\,dz \leq \int_{-R}^{R}\frac{1}{R^2-1}\,dz$

$2z^2-5z+2$

$2z^2-4z-z+2$

$2z(z-2)-(z-2)$

$(2z-1)(z-2)$

7) a) $w = \dfrac{2z+2i}{2i+z}$

Fixed points are :

$$z = \frac{2z+2i}{2i+z}$$

$$\Rightarrow z^2+(2i-2)z-2i = 0 .$$

$$\Rightarrow z = \frac{2-2i \pm \sqrt{(2i-2)^2-4\times-2i}}{2}$$

$$= \frac{2-2i \pm \sqrt{-4+4-8i+8i}}{2}$$

$$= 1-i, \; 1-i \quad \text{(repeated root)}$$

Fixed point is $\underline{1-i}$

b) $w = \dfrac{1-3iz}{z-i}$

$$z = \frac{1-3iz}{z-i}$$

$$z^2-iz+3iz-1 = 0$$

$$z^2+2iz+i^2 = 0$$

$$(z+i)^2 = 0$$

$$z = -i, -i$$

Fixed point is $\underline{-i}$

8) (z_1) $-1 \longrightarrow i$ (w_1)

(z_2) $i \longrightarrow 1$ (w_2)

(z_3) $0 \longrightarrow -1$ (w_3)

Transformation is

$$\frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1} = \frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1}$$

$$\frac{w-i}{w-1} \cdot \frac{-1-1}{-1-i} = \frac{z+1}{z-i} \cdot \frac{0-i}{0+1}$$

$$\frac{w-i}{w-1} \cdot \frac{+2}{+1+i} = -i\,\frac{z+i}{z-i}$$

$$\frac{w-i}{w-1} = \frac{1-i}{2}\left(\frac{z+i}{z-i}\right)$$

$$w-i = \left(\frac{1-i}{2}\right)\left(\frac{z+i}{z-i}\right)(w-1).$$

$$w\left(1-\left(\frac{1-i}{2}\right)\left(\frac{z+i}{z-i}\right)\right) = -\left(\frac{1-i}{2}\right)\left(\frac{z+i}{z-i}\right)+i$$

$$w\left(\frac{2(z-i)-(1-i)(z+i)}{\cancel{2(z-i)}}\right) = \frac{2i(z-i)-(1-i)(z+i)}{\cancel{2(z-i)}}$$

$$w\left(2z-2i-(z+i-iz+1)\right)$$

$$= 2iz+2-(z+i-iz+1)$$

$$w = \frac{(3i-1)z+1-i}{(1+i)z-3i-1}$$

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