Natural and Step Responses of RLC Circuits

EE3301

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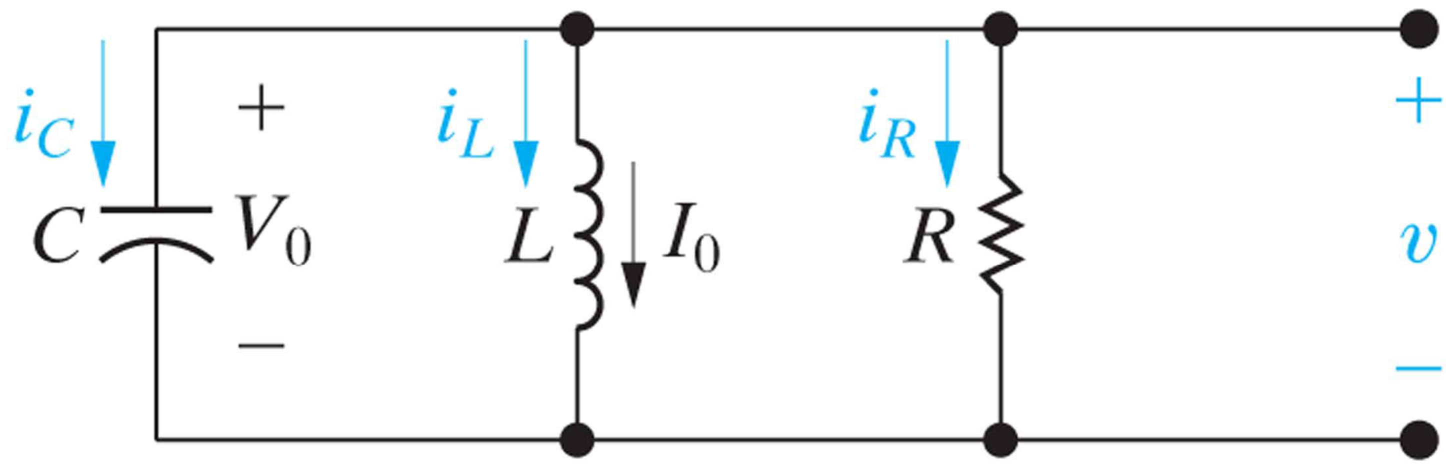
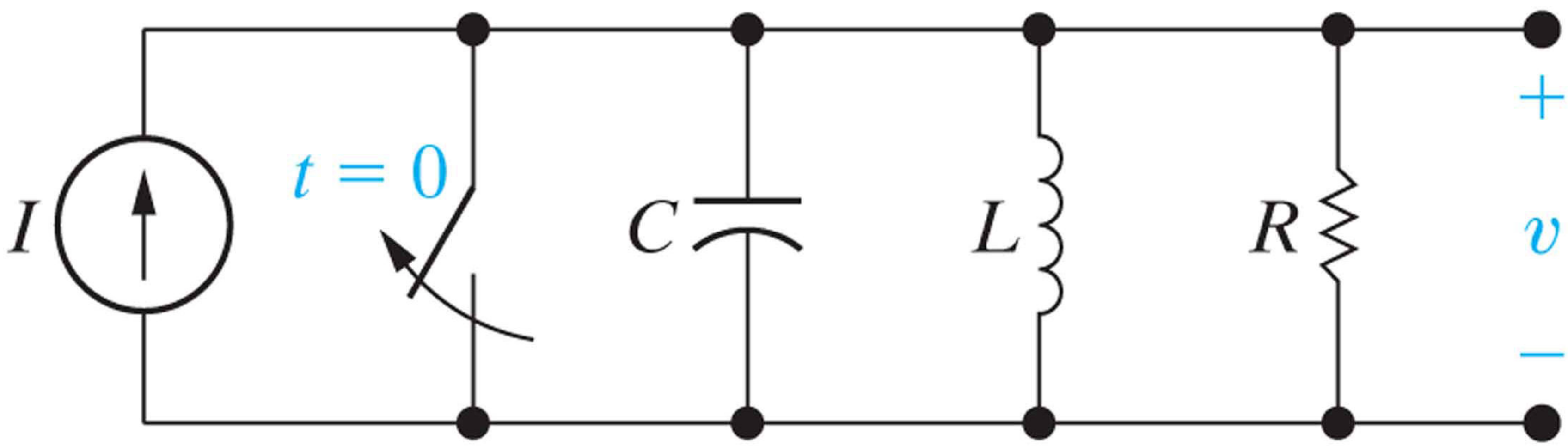
Learning Objectives

1. Be able to determine the natural responses of parallel and series RLC circuits
2. Be able to determine the step responses of parallel and series RLC circuits
3. Be able to determine the responses (both natural and transient) of second order circuits with op amps

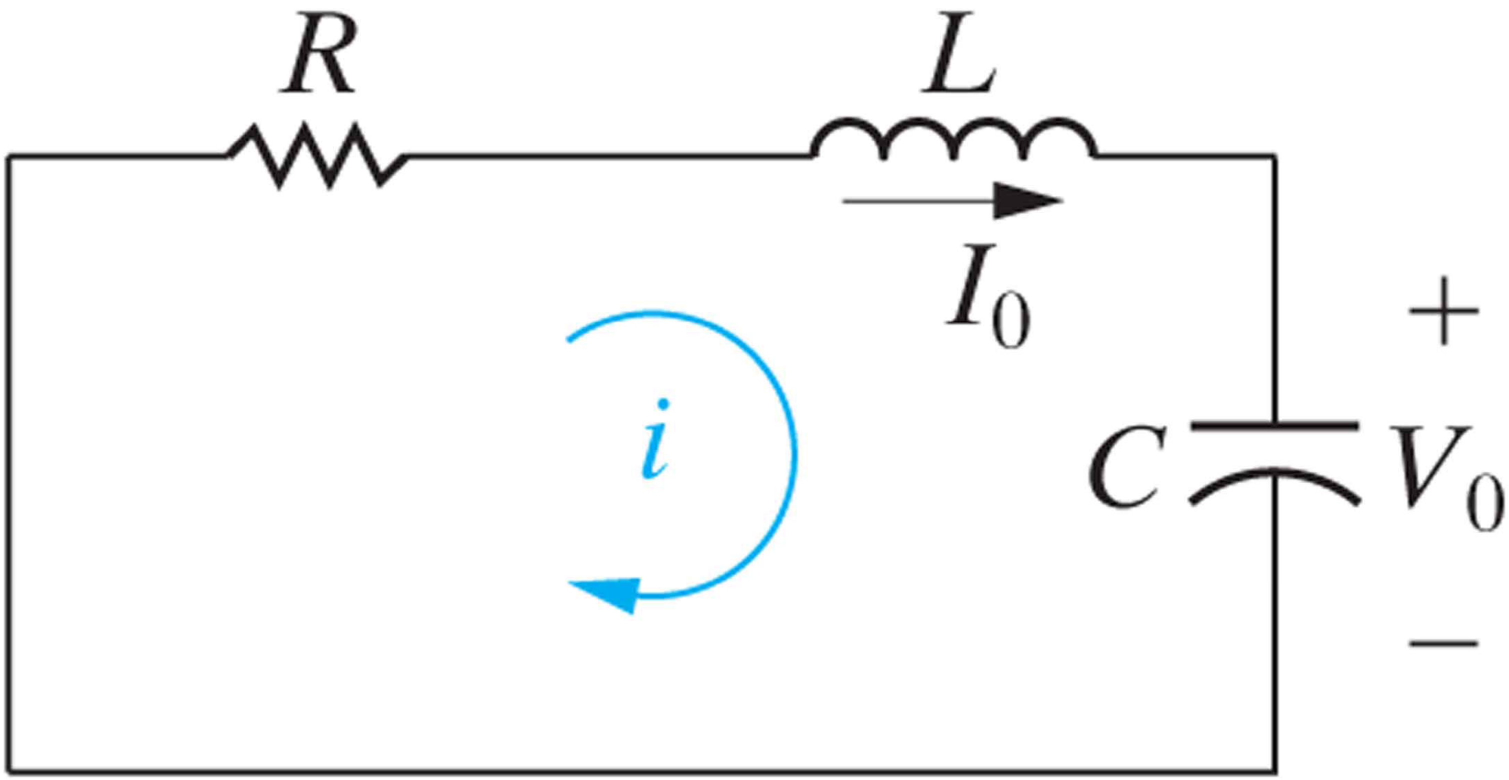
Parallel *RLC circuit*



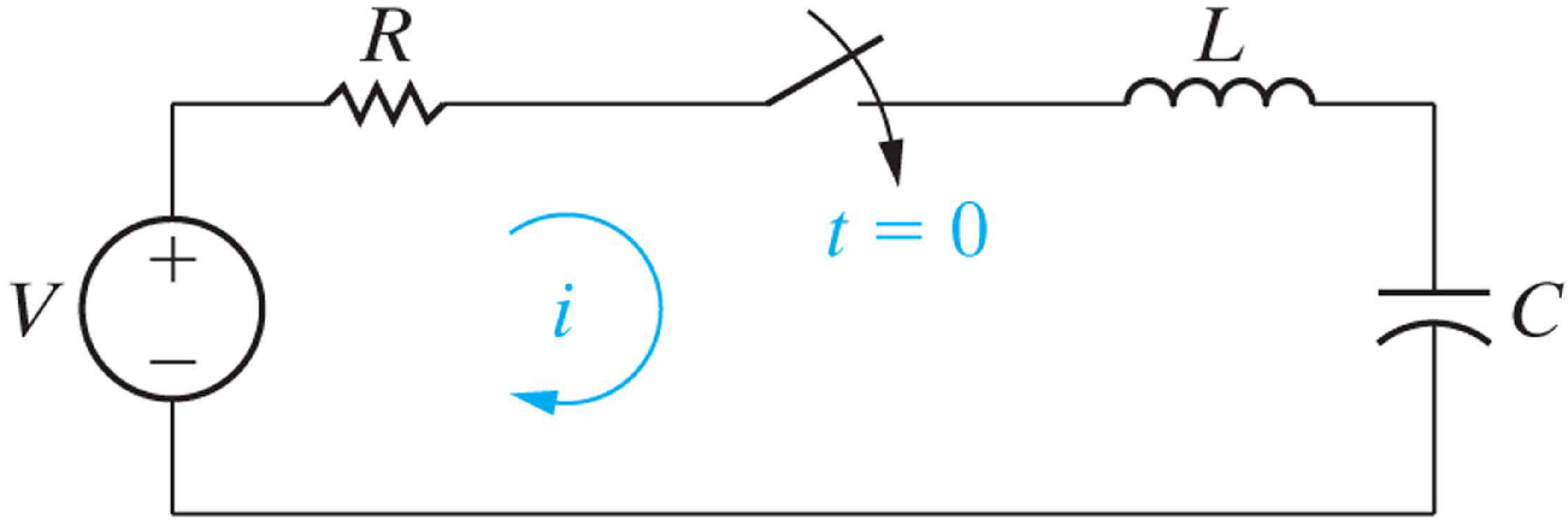
The step response of a parallel *RLC circuit.*



A series *RLC circuit.*



The step response of a series *RLC circuit.*



The Natural Response of a Parallel RLC

1. Using KCL,

This will disappear

$$v(t)/R + \frac{1}{L}\int_0^t v(s)ds + C\frac{dv(t)}{dt} + I_0 = 0$$

$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{v(t)}{LC} = 0$$

The Natural Response of a Series RLC

1. Using KVL,

This will disappear

$$Ri(t) + \frac{1}{C}\int_0^t i(s)\,ds + L\frac{di(t)}{dt} + V_0 = 0$$

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} = 0$$

The Natural Response of a Parallel/Series RLC

$$v(t) = Ae^{st} \Leftarrow parallel$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \Rightarrow$$

$$s = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$i(t) = Ae^{st} \Leftarrow series$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \Rightarrow$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

The Natural Response of a Parallel/Series RLC

There are 3 distinct cases.

Let

$$\alpha = \frac{1}{2RC} \; or \; \alpha = \frac{R}{2L}$$

Neper frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Resonant radiant frequency

Then,

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$1)\ \alpha = \omega_0$ Critically-damped

$2)\ \alpha < \omega_0$ Under-damped

$3)\ \alpha > \omega_0$ Over-damped

The Natural Response of a Parallel RLC

There are 3 distinct cases.

$$1)\ \alpha = \omega_0 \Rightarrow s_1 = s_2 = -\alpha$$

Critically-damped

$$2)\ \alpha < \omega_0 \Rightarrow \begin{cases} s_1 = -\alpha + j\sqrt{\omega_0{}^2 - \alpha^2} \\ s_1 = -\alpha - j\sqrt{\omega_0{}^2 - \alpha^2} \end{cases}$$

Under-damped

$$3)\ \alpha > \omega_0 \Rightarrow \begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0{}^2} \\ s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0{}^2} \end{cases}$$

Over-damped

The Natural Response of a Parallel/Series RLC

How does the response look like?

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \Leftarrow parallel$$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \Leftarrow series$$

Initial conditions must be used to evaluate

How do we solve for the unknowns for over-damped (parallel)?

$$v(0^+) = A_1 + A_2$$

Two equations with two unknowns

$$\frac{dv(0^+)}{dt} = A_1 s_1 + A_2 s_2 \Rightarrow$$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C}\left[-\frac{v(0^+)}{R} - I_0\right]$$

Initial voltage across the capacitor

Initial current through the inductor

How do we solve for the unknowns for the over-damped case(series)?

$$i(0^+) = A_1 + A_2$$

Two equations with two unknowns

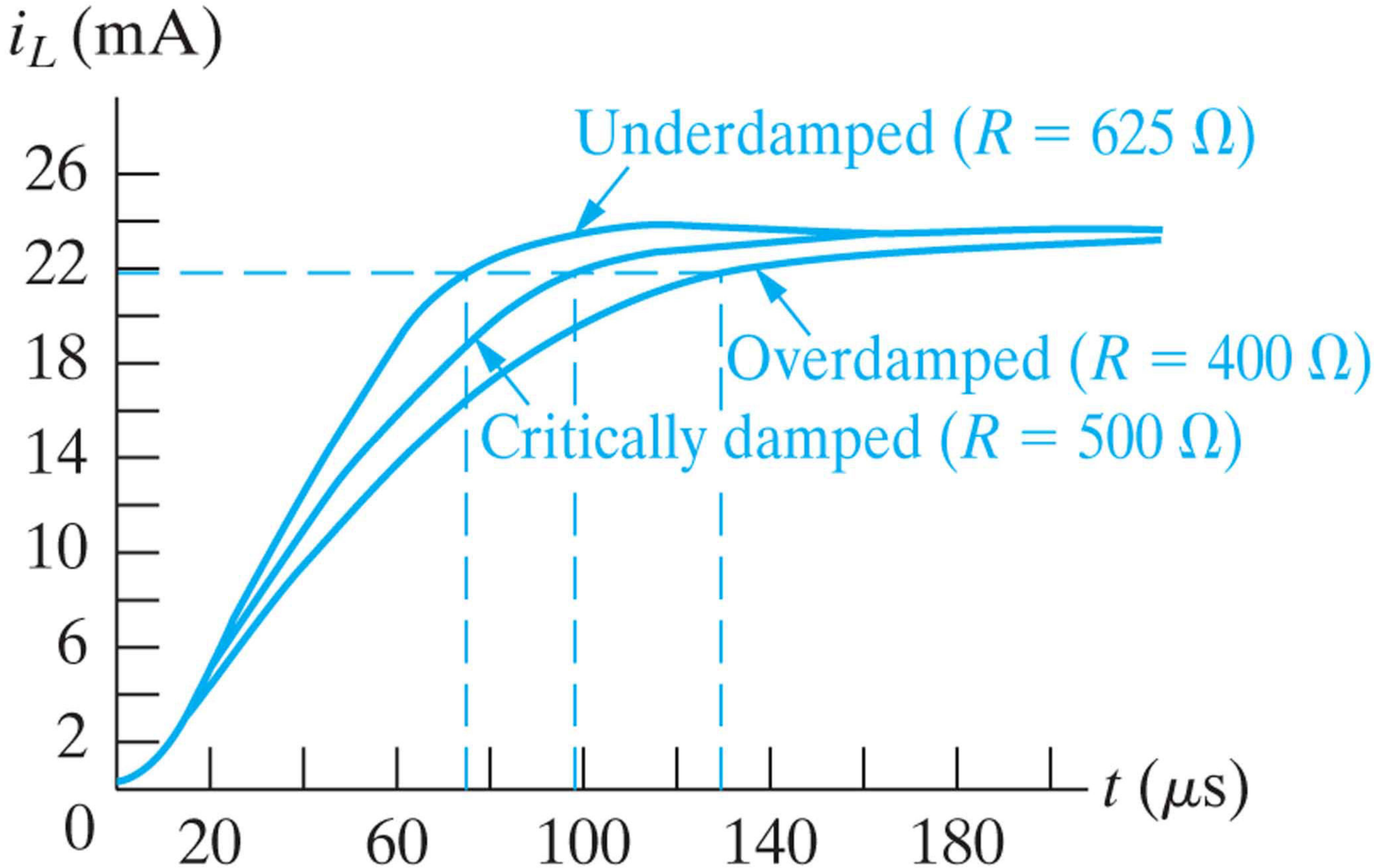
$$\frac{di(0^+)}{dt} = A_1 s_1 + A_2 s_2 \Rightarrow$$

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{1}{L}\left[-Ri(0^+) - V_0\right]$$

Initial current through the inductor

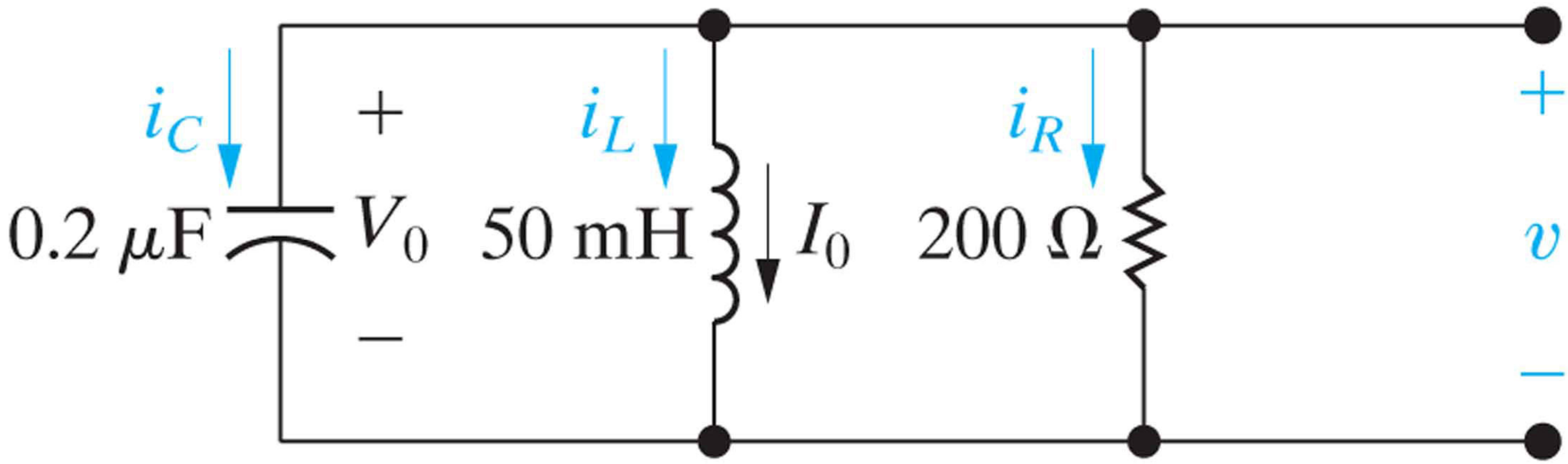
Initial voltage across the capacitor

Example 8.9



i_L (mA)
26
22
18
14
10
6
2
0
20
60
100
140
180
t (μs)
Underdamped (R = 625 Ω)
Overdamped (R = 400 Ω)
Critically damped (R = 500 Ω)

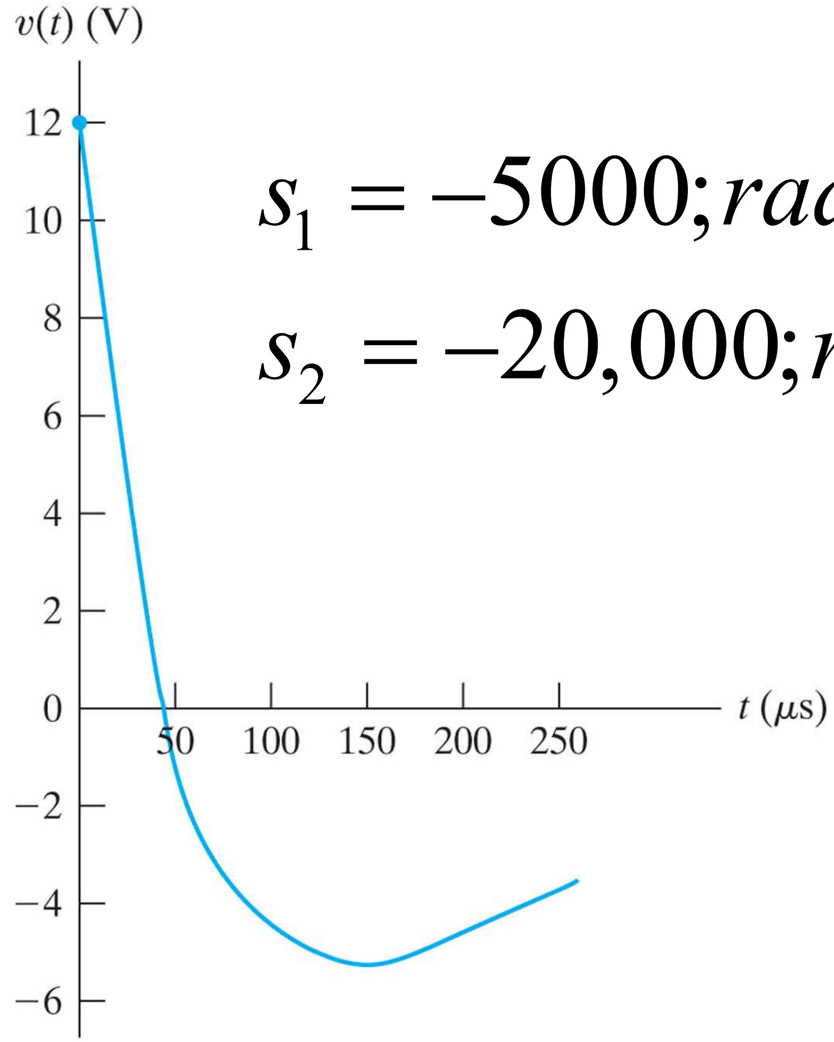
The circuit for Example 8.2.



$$I_0 = 30mA;$$

$$v\left(0^+\right) = 12$$

Figure 8.7 Example 8.2.



$$s_1 = -5000; rad/\sec$$

$$s_2 = -20{,}000; rad/\sec$$

The Natural Response of an under-damped Parallel/Series RLC

How does the response look like?

$$v(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t) \Leftarrow parallel$$

$$i(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t) \Leftarrow series$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Initial conditions must be used to evaluate

How do we solve for the unknowns for under-damped (parallel)?

$$v(0^+) = B_1$$

$$\frac{dv(0^+)}{dt} = \omega_d B_2 - \alpha B_1 \Rightarrow$$

Two equations with two unknowns

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C}\left[-\frac{v(0^+)}{R} - I_0\right]$$

Initial voltage across the capacitor

Initial current through the inductor

How do we solve for the unknowns for under-damped (series)?

$$i(0^+) = B_1$$

Two equations with two unknowns

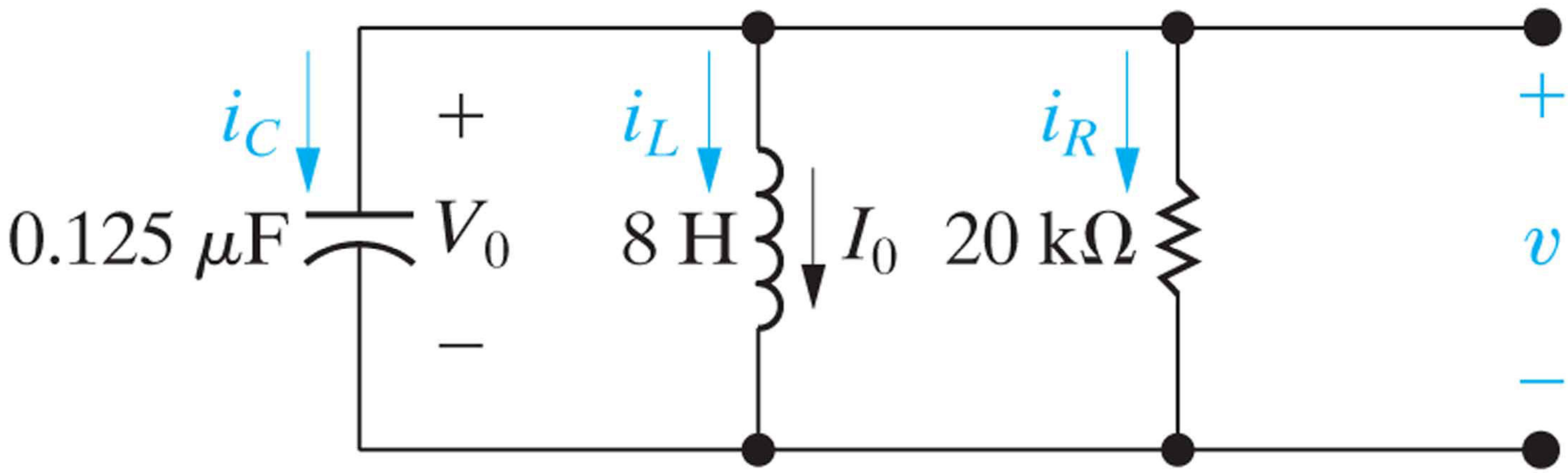
$$\frac{di(0^+)}{dt} = \omega_d B_2 - \alpha B_1 \Rightarrow$$

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{1}{L}\left[-Ri(0^+) - V_0\right]$$

Initial current through the inductor

Initial voltage across the capacitor

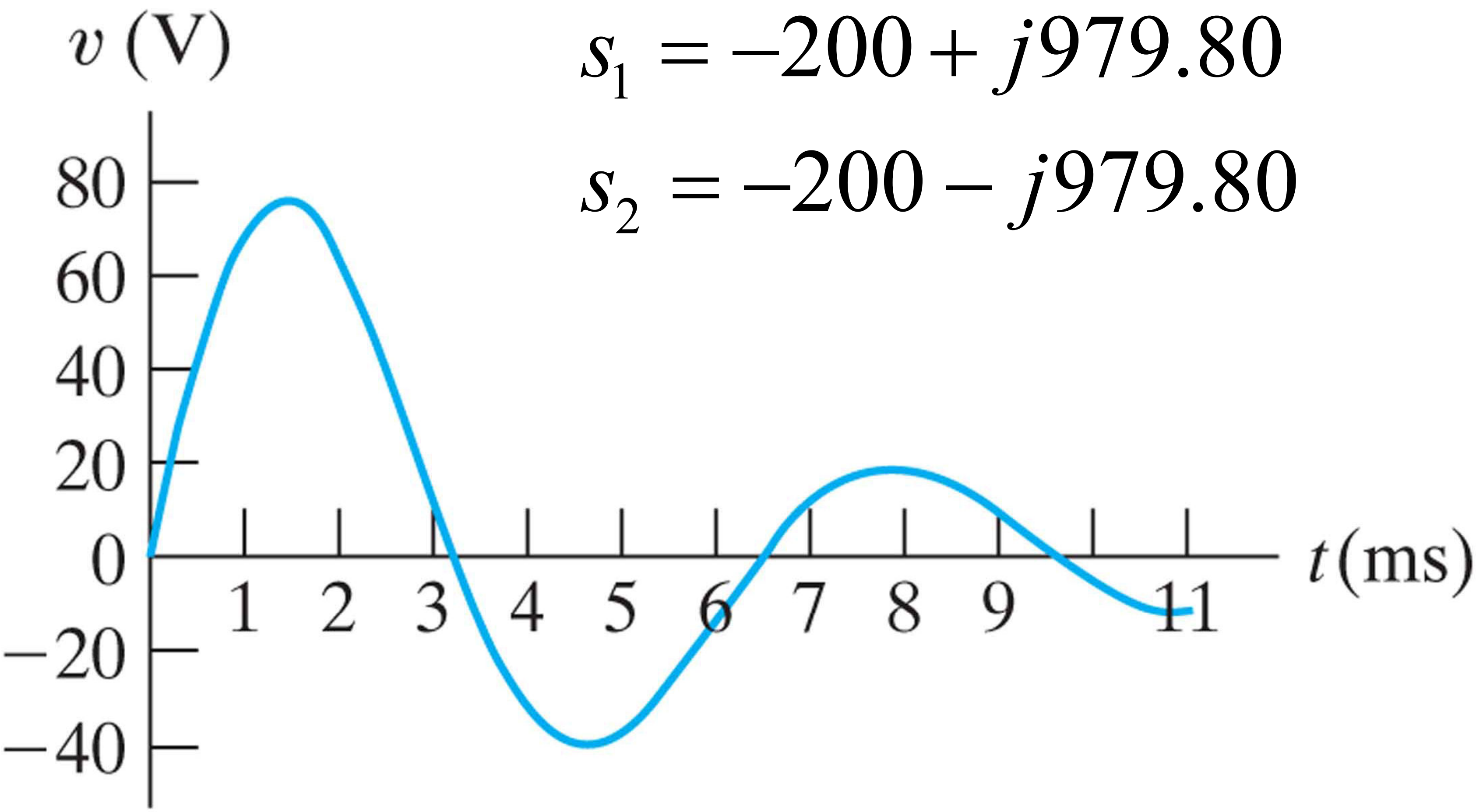
The circuit for Example 8.4



$$I_0 = -12.25mA;$$

$$v(0^+) = 0$$

The voltage response for Example 8.4



$$s_1 = -200 + j979.80$$

$$s_2 = -200 - j979.80$$

The Natural Response of a Critically Damped Parallel/Series RLC

How does the response look like? $s_1 = s_2 = s$

$$v(t) = D_1 t e^{st} + D_2 e^{st}$$

Initial conditions must be used to evaluate

How do we solve for the unknowns for critically-damped (parallel)?

$$v(0^+) = D_2$$

Two equations with two unknowns

$$\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 \Rightarrow$$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{1}{C}\left[-\frac{v(0^+)}{R} - I_0\right]$$

Initial voltage across the capacitor

Initial current through the inductor

How do we solve for the unknowns for the critically-damped (series)?

$$i(0^+) = D_2$$

Two equations with two unknowns

$$\frac{di(0^+)}{dt} = D_1 - \alpha D_2 \Rightarrow$$

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{1}{L}\left[-Ri(0^+) - V_0\right]$$

Initial current through the inductor

Initial voltage across the capacitor

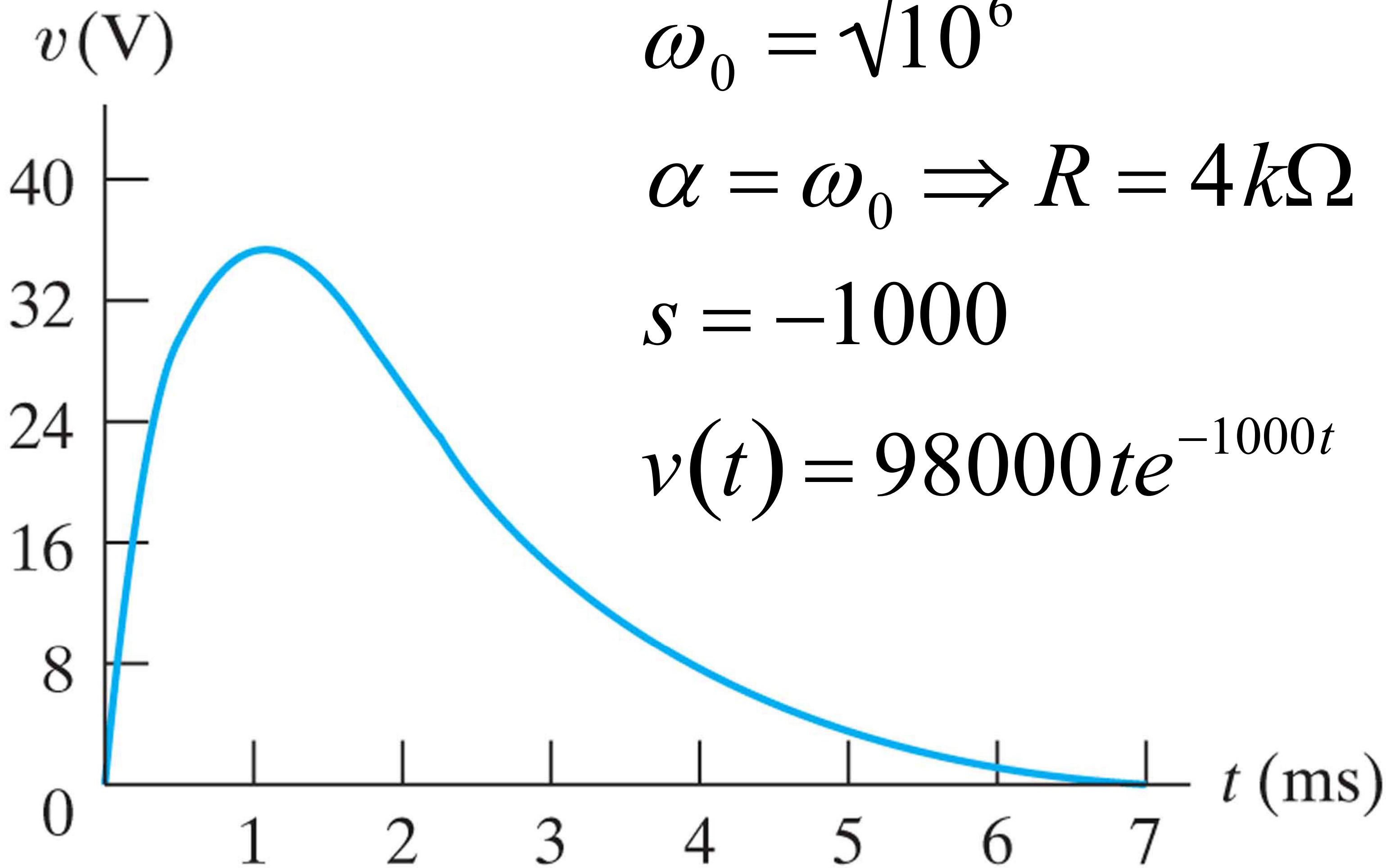
Example 8.5 (critically-damped)-R has been changed to make this happen

$$\omega_0 = \sqrt{10^6}$$

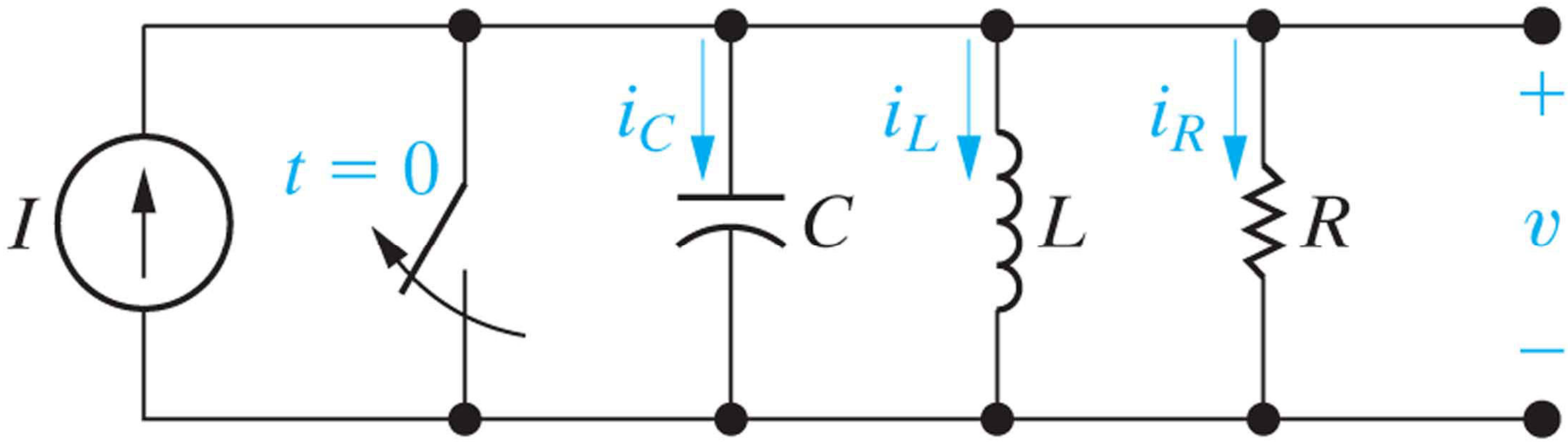
$$\alpha = \omega_0 \Rightarrow R = 4k\Omega$$

$$s = -1000$$

$$v(t) = 98000te^{-1000t}$$



A circuit used to describe the step response of a parallel *RLC circuit*



The Step Response of a Parallel RLC

1. Using KCL,

$$\frac{v_L(t)}{R} + i_L(t) + C\frac{dv_L(t)}{dt} = I$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) + LC\frac{d^2 i_L(t)}{dt^2} = I$$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{I}{LC}$$

The Step Response of a Parallel RLC (direct method)

1. First find the natural response
2. Add to the natural response the final value
3. Use the initial conditions to solve for coefficients

$$\frac{d^2 i_{L,n}(t)}{dt^2} + \frac{1}{RC}\frac{di_{L,n}(t)}{dt} + \frac{1}{LC} i_{L,n}(t) = 0$$

$$i_L(t) = I_f + i_{L,n}(t)$$

The Step Response of a Parallel RLC (direct method)

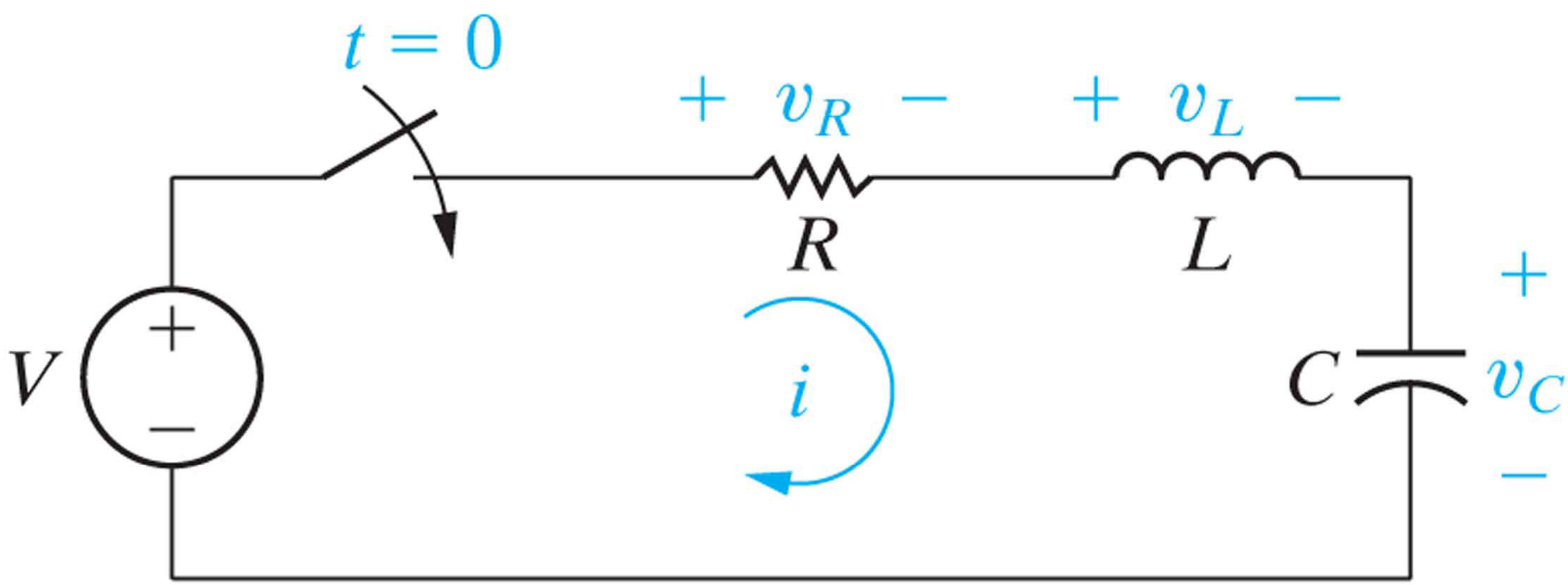
$$i_L(0^+), \frac{di_L(0^+)}{dt} \Leftarrow known$$

$$i_L(t) = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$i_L(t) = I_f + B_1' e^{-\alpha t}\cos(\omega_d t) + B_2' e^{-\alpha t}\sin(\omega_d t)$$

$$i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

A circuit used to illustrate the step response of a series *RLC circuit.*



The Step Response of a Series RLC

1. Using KVL,

$$RC\frac{dv_c(t)}{dt}+v_c(t)+LC\frac{d^2v_c(t)}{dt^2}=V$$

$$L\frac{di_L(t)}{dt}+Ri_L(t)+\frac{1}{C}\int^t i_L(t^{'})dt^{'}=V$$

$$\frac{d^2v_c(t)}{dt^2}+\frac{R}{L}\frac{dv_c(t)}{dt}+\frac{1}{LC}v_c(t)=\frac{V}{LC}$$

The Step Response of a Series RLC (direct method)

$$v_c(0^+), \frac{dv_c(0^+)}{dt} \Leftarrow known$$

$$v_c(t) = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

$$v_c(t) = V_f + B_1' e^{-\alpha t}\cos(\omega_d t) + B_2' e^{-\alpha t}\sin(\omega_d t)$$

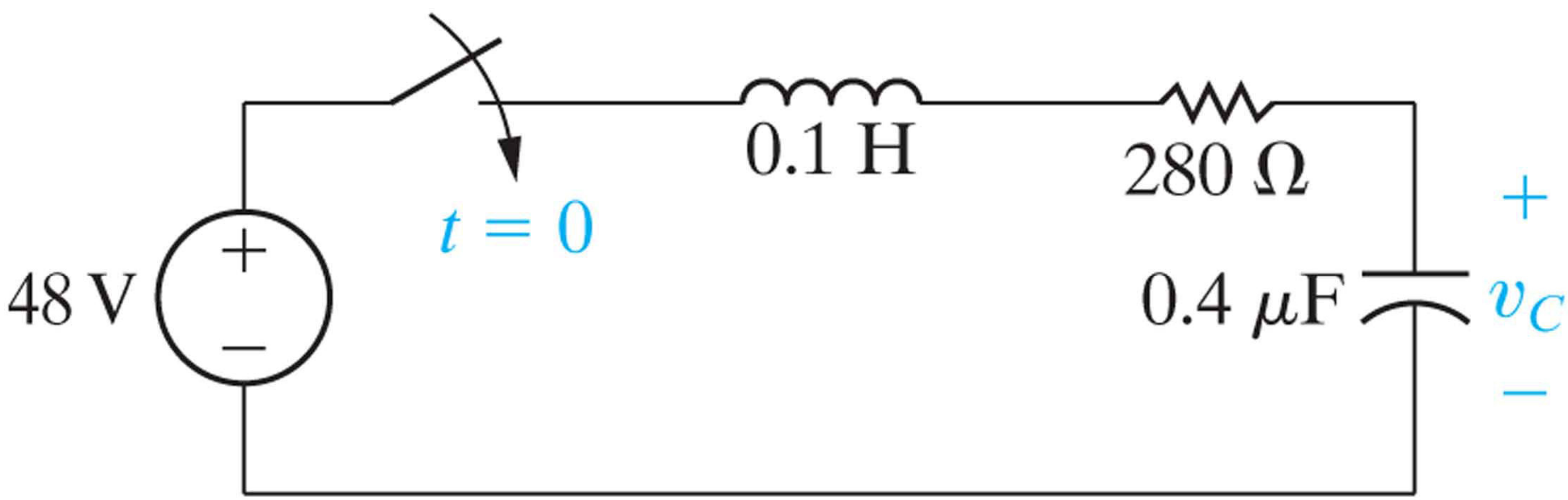
$$v_c(t) = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}$$

The circuit for Example 8.12

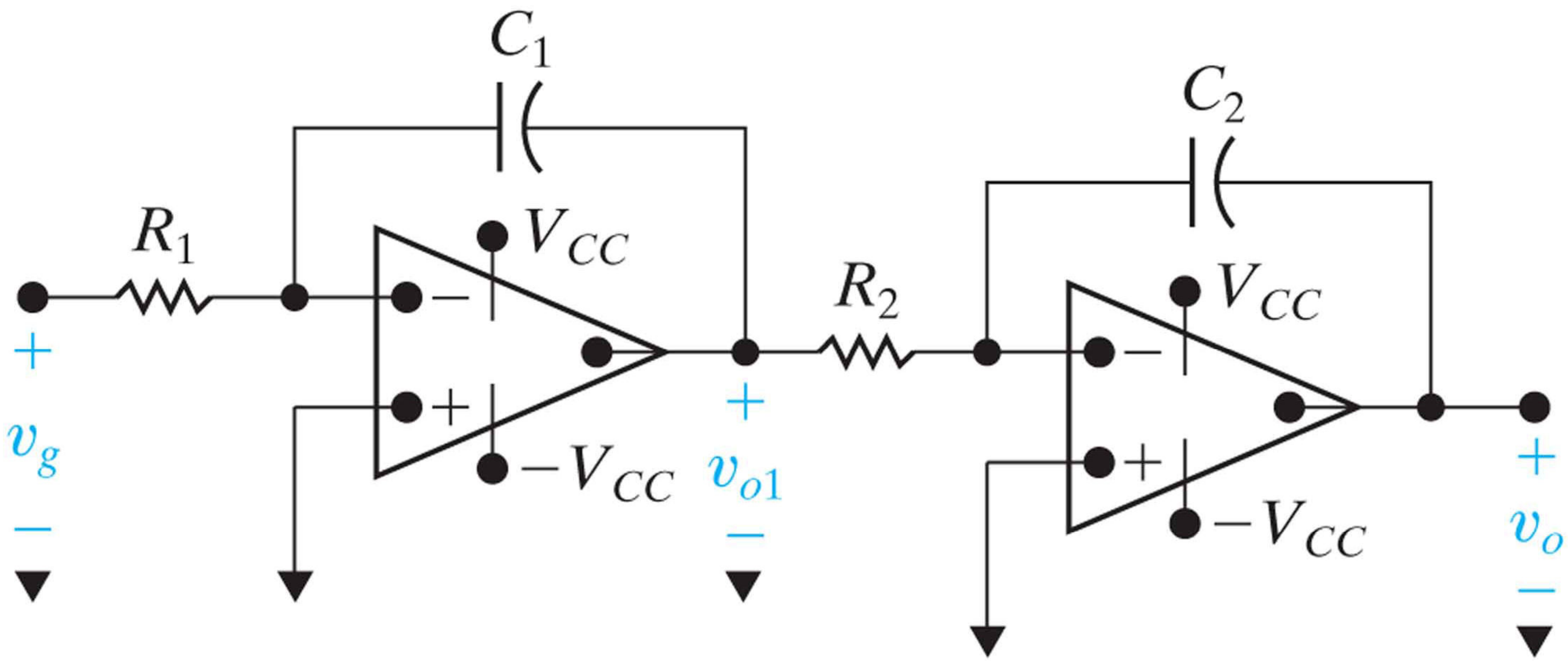
$$v_c(0^+) = 0, \frac{dv_c(0^+)}{dt} = 0$$

$$v_c(t) = 48 + B_1' e^{-\alpha t} \cos(\omega_d t) + B_2' e^{-\alpha t} \sin(\omega_d t)$$

$$\alpha = -1400; \omega_d = 4800$$



Second order circuits with op amps



$$\frac{d^2 v_0(t)}{dt^2} = \frac{1}{R_1 C_1} \frac{1}{R_2 C_2} v_g(t)$$

Second order circuits with op amps

- This is a variation of the second order system
- The output is the double integration of the input
- Depending on the initial charges on the capacitors, the response will vary
- For a constant input, the output will increase indefinitely

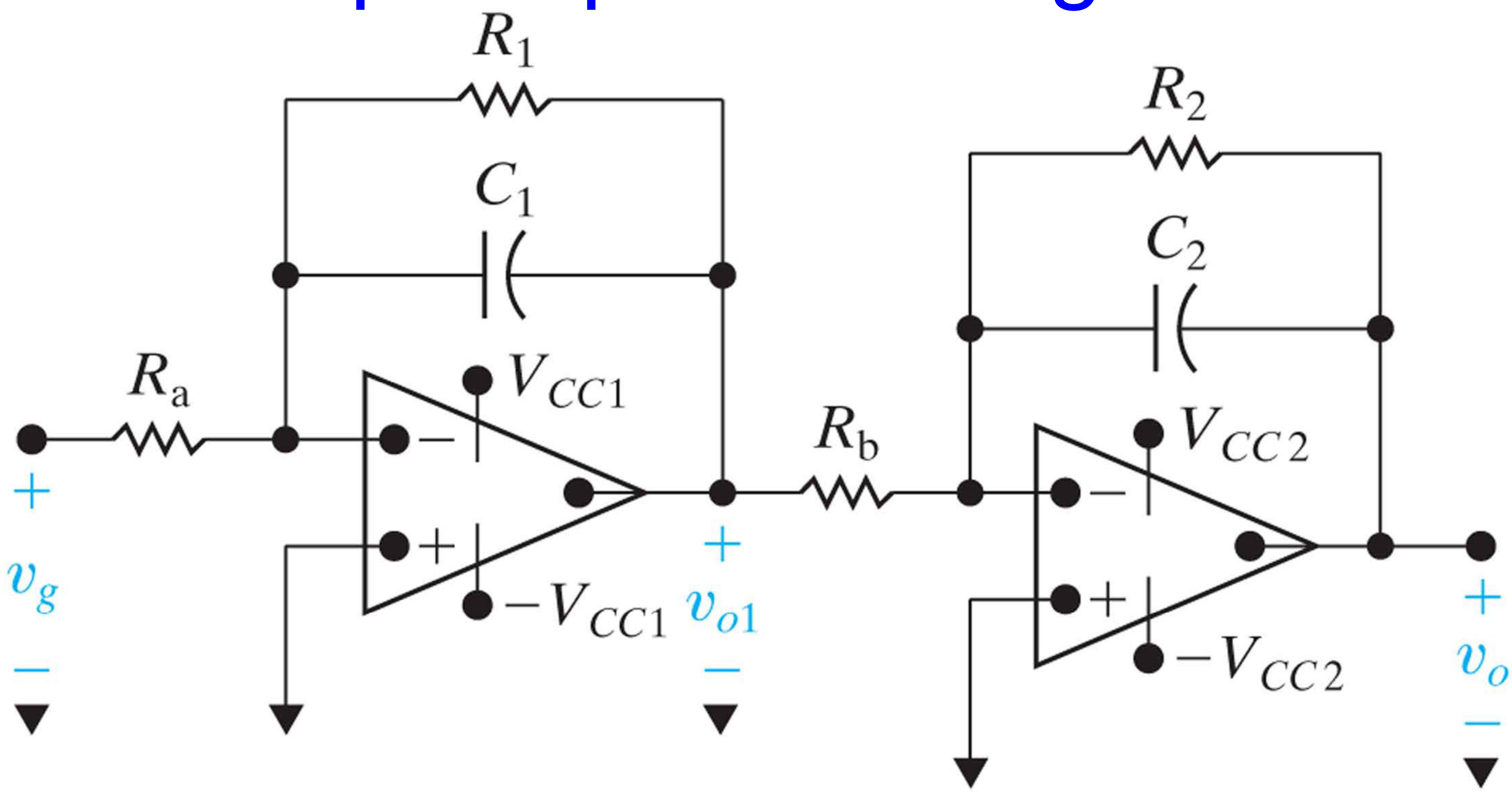
$$\frac{d^2 v_0(t)}{dt^2} = \frac{1}{R_1 C_1} \frac{1}{R_2 C_2} v_g(t)$$

$$v_g(t) = V_0$$

Trouble

$$v_0(t) = \frac{V_0}{2R_1C_1R_2C_2} t^2$$

Second order circuits with op amps-imperfect integrator



$$v_{01}(t) = -R_b C_2 \frac{dv_0(t)}{dt} - \frac{R_b C_2}{\tau_2} v_0(t)$$

$$v_g(t) = -R_a C_1 \frac{dv_{01}(t)}{dt} - \frac{R_a C_1}{\tau_1} v_{01}(t)$$

$$\tau_1 = R_1 C_1 ; \tau_2 = R_2 C_2$$

$$\frac{d^2 v_0(t)}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dv_0(t)}{dt} + \left(\frac{1}{\tau_1 \tau_2}\right) v_0(t) = \frac{v_g(t)}{R_a C_1 R_b C2}$$

$$s_1 = \frac{-1}{\tau_1} ; s_2 = \frac{-1}{\tau_2}$$

$$v_g(t) = V_0 U(t)$$

$$v_0(t) = V_f + A_1 e^{s_1 t} + A_2 e^{s_2 t} \Leftrightarrow v_0(0^+) = \frac{dv_0(0^+)}{dt} = 0$$

$$V_f = \frac{V_0 \tau_1 \tau_2}{R_a C_1 R_b C2} = \frac{R_1 R_2 V_0}{R_a R_b}$$

If it is less than the supply voltage