§: Complex Integration

- Integration of functions of a complex variable plays a very important role in many areas of Science.

- For $z = x + iy$, where z is a complex variable made up of two real variables x and y respectively.

- $f(z)$ is denoted as a complex function, i.e., a function whose domain is complex plane $\mathbb{C}$.

- If a function $f(z)$ is analytic function in a domain D, then it possesses derivative of all orders in D, i.e., $f'(z), f''(z), \cdots$ all are analytic in D.

  This result does not exist in the real variable theory.

- The concept of definite integrals for functions of a real variable does not directly extend to the case of complex variable.

- The definite integral

$$\underbrace{\int_a^b f(x)\,dx}_{\text{"Real Variable function"}}$$

represents that the Path of integration is along x-axis from $x = a$ to $x = b$.

- The definite integral $\int_a^b f(z)\,dz$ represents that the path of integration may be along a straight line, or any curve along $z = a$ to $z = b$.

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§: Integral of $f(z)$:

Let $f(z) = u(x) + i\,v(x)$ OR $f(t) = u(t) + i\,v(t)$

Where u and v are real-valued functions of the real variable x or t for $a \le t \le b$. Then

$$\boxed{\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt} \quad -(1)$$

We generally evaluate integrals of this type by finding the antiderivatives of u and v and evaluating the definite integrals on the right hand side of the box equation. That is, if $U'(t) = u(t)$ and $V'(t) = v(t)$, for $a \le t \le b$, we have

$$\int_a^b f(t)\,dt = \Big[U(t) + i\,V(t)\Big]_{t=a}^{t=b}$$

$$= \big[U(b) - U(a)\big] + i\big[V(b) - V(a)\big] \quad \#$$

- Examples: (1) Show that $\int^1 (t-i)^3\,dt = -5/4$.

Solⁿ: We write the integrand in terms of its real and imaginary parts, i.e.,

$$f(t) = (t-i)^3 = (t^3 - 3t) + i(-3t^2 + 1).$$

$$= u(t) + i\,v(t), \text{ where}$$

$$u(t) = t^3 - 3t$$

$$v(t) = -3t^2 + 1.$$

Now, $\int_0^1 u(t)\,dt = \int_0^1 (t^3 - 3t)\,dt = \left(\frac{t^4}{4} - \frac{3t^2}{2}\right)\Big|_0^1$

$$= \left(\frac{1}{4} - \frac{3}{2}\right) - 0 = \frac{-\cancel{10}\,5}{\cancel{8}\,4}$$

$$= -5/4.$$

and
$$\int_0^1 v(t)\,dt = \int_0^1 (-3t^2+1)\,dt$$
$$= \left(-t^3+t\right)\Big|_0^1$$
$$= (-1+1)-0 = 0.$$

$\therefore$ From Eqⁿ (1), we have
$$\int_0^1 (t-i)^3\,dt = \int_0^1 u(t)\,dt + i\int_0^1 v(t)\,dt$$
$$= -\frac{5}{4} + 0 = -\frac{5}{4} \quad \#$$

- **Example (2):** Evaluate the integral
$$\int_0^{\pi/2} \exp(t+it)\,dt.$$

- **Sol^n:** We have $\int_0^{\pi/2} \exp(t+it)\,dt$
$$\Rightarrow \int_0^{\pi/2} \exp(t+it)\,dt = \int_0^{\pi/2} e^t\,dt * e^{it}$$
$$= \int_0^{\pi/2} e^t(\cos t + i\sin t)\,dt$$
$$= \int_0^{\pi/2} e^t \cos t\,dt + i\int_0^{\pi/2} e^t \sin t\,dt \quad \text{--- (1)}$$

- We can evaluate each of the integrals via integration by parts.

$$\int_0^{\pi/2} \underbrace{e^t}_{u} \underbrace{\cos t}_{v}\, dt = e^t.\sin t\Big|_0^{\pi/2} - \int_0^{\pi/2} e^t.\sin t\, dt$$

$$= \left(e^{\pi/2}\sin\left(\frac{\pi}{2}\right) - e^0.\sin 0\right) - \int_0^{\pi/2} e^t \sin t\, dt$$

$$= e^{\pi/2} - \int_0^{\pi/2} \underbrace{e^t}_{u} \underbrace{\sin t}_{v}\, dt$$

$$= e^{\pi/2} - \Big[e^t.(-\cos t)\Big]_0^{\pi/2} + \int_0^{\pi/2} e^t(-\cos t)\, dt$$

$$= e^{\pi/2} - \left[e^{\pi/2}\left(-\cos\frac{\pi}{2}\right) - e^0(-\cos 0)\right] + \int_0^{\pi/2} e^t(-\cos t)\, dt$$

$$= e^{\pi/2} - 1 - \int_0^{\pi/2} e^t \cos t\, dt.$$

$$\Rightarrow 2\int_0^{\pi/2} e^t \cos t\, dt = e^{\pi/2} - 1$$

$$\Rightarrow \int_0^{\pi/2} e^t \cos t\, dt = \frac{1}{2}(e^{\pi/2} - 1).$$

Similarly, we have $\int_0^{\pi/2} e^t \sin t\, dt = \frac{1}{2}(1 + e^{\pi/2})$

$$\therefore \int_0^{\pi/2} \exp(t + it)\, dt = \int_0^{\pi/2} e^t \cos t\, dt + i\int_0^{\pi/2} e^t \sin t\, dt$$

$$= \frac{1}{2}(e^{\pi/2} - 1) + i\frac{1}{2}(1 + e^{\pi/2})$$

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§: Properties of Complex Integration:

Let $f(t) = u(t) + iv(t)$ and $g(t) = p(t) + iq(t)$ be continuous on $a \le t \le b$.

① The integral of their sum is the sum of their integrals, i.e.,

$$\int_a^b (f(t) + g(t))\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$$

② If we devide the interval $a \le t \le b$ into $a \le t \le c$ and $c \le t \le b$ and integrate $f(t)$ over subintervals, then we get

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

③ If $z = c + id$, denotes a complex constant, then

$$\int_a^b (c+id) f(t)\,dt = (c+id) \int_a^b f(t)\,dt$$

④ If the limits of integration reversed, then

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

⑤ The integral of the product $f * g$ becomes

$$\int_a^b fg\,dt = \int_a^b f(t) g(t)\,dt = \int_a^b [u(t)p(t) - v(t)q(t)]\,dt$$

$$+$$

$$i \int_a^b [u(t)q(t) + v(t)p(t)]\,dt$$

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- Questions: Evaluate the following complex integrals

① $\int_0^1 (t+it^2)\,dt$ ② $\int_0^{\pi} t e^{-it}\,dt$

③ $\int_0^1 (te^{-t^2} + 2i/\sqrt{t})\,dt$ ④ $\int_0^1 (t-i)^{-1}\,dt$; $t \neq i$

→ $\boxed{\text{Do it by yourself.}}$

§: Contours and Contour Integrals:

Earlier we evaluate integral of the form $\int_a^b f(t)\,dt$ where f was complex-valued and $[a, b]$ was interval on real axis (so that 't' was real with $t \in [a, b]$).

- Here we define and evaluate integrals of the form

$\int_C f(z)\,dz$, where f is a complex-valued and C is a contour in plane (so that z is complex, with $z \in \mathbb{C}$).

- Complex definite integrals are called the "line integrals" and are written as $\int_C f(z)\,dz$. The integrand $f(z)$ is integrated over a given curve C in the complex plane called the "path of integration" represented by a parametric representation

$$z(t) = x(t) + i y(t) \ ; \ a \leq t \leq b.$$

- The sense of increasing t is called the positive sense on C.

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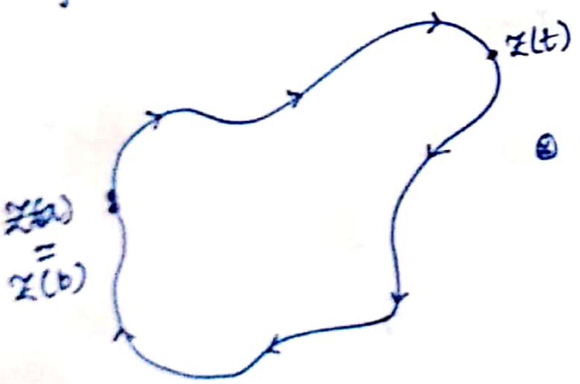
- The following discussion lead to the concept of a contour, which is a type of curve that is adequate for the study of integration.

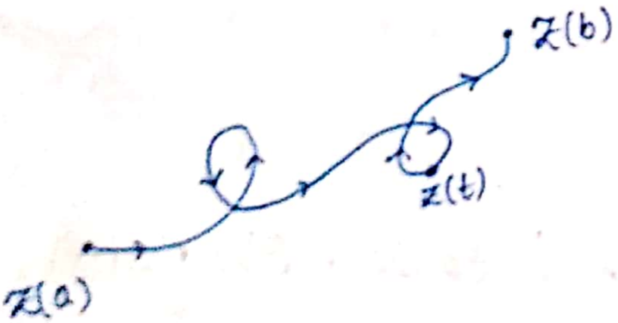
- C is simple if it does not cross itself, which means $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$ except possibly when $t_1 = a$ and $t_2 = b$.

• C (Simple Curve).

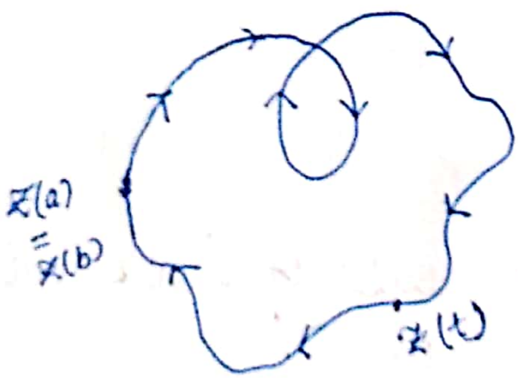
- A curve C with the property $z(b) = z(a)$ is a closed curve. If $z(b) = z(a)$ is the only point of intersection, then we say that C is a "simple closed curve".



• C (a simple closed curve)



* a curve that is not simple and not closed.



* a closed curve but not simple.

- A synonym for contour is Path.
- As parameter t increases from the value a to the value b, the point $z(t)$ starts at the <u>initial point</u> $z(a)$, moves along a curve C and ends up at the <u>terminal point</u> $z(b)$.
- If C is simple, then $z(t)$ moves <u>continuously</u> from $z(a)$ to $z(b)$ as <u>t increases</u> and the curve is given an <u>orientation</u>, which we indicate by drawing arrows along curve.

- <u>Remark</u>: ⊛ $\int_a^b f(t)\,dt$; $f : [a,b] \longrightarrow \mathbb{C}$ (Complex-valued)

$$f(t) = u(t) + i\,v(t), \text{ where}$$

$$u, v : [a,b] \longrightarrow \mathbb{C} \quad \text{(Complex-valued)}$$

⊛ $\int_C f(z)\,dz$; $f : \mathbb{C} \to \mathbb{C}$ be continuous.

Path:

$$C = \{\gamma(t) : t \in [a,b]\}$$

$$\gamma : [a,b] \longrightarrow \mathbb{C}$$

$$\gamma(t) = x(t) + i\,y(t);$$

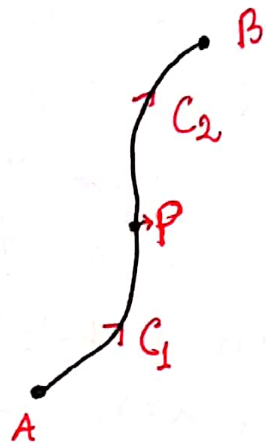
<u>where</u> $x, y : [a,b] \longrightarrow \mathbb{R}$. continuous.

$$\boxed{\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$$

- **Properties:**

① **Sense Reversal:** $\int_{z_0}^{z} f(z)\,dz = -\int_{z}^{z_0} f(z)\,dz$

② **Partitioning of Path:**



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

Where the curve C consists of two smooth curves C_1 and C_2 joined end to end.

③ **ML-Inequality:**

$$\left| \int_C f(z)\,dz \right| \leq ML,$$

Where M is a constant such that $|f(z)| \leq M$ everywhere on C and L is the length of the curve.

→ $L(C) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b |z'(t)|\,dt.$