Simply Connected Domain – A connected domain is simply connected if every simple closed curve inside D encloses only points of D.

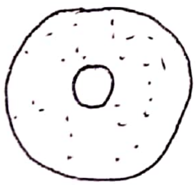
or

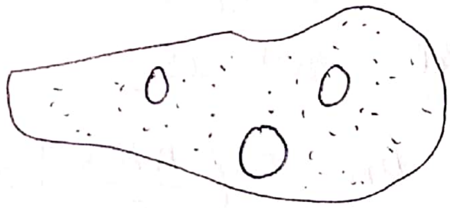
If any simple closed curve in the domain can be contracted (Shrunk) to a point within the domain.

Ex – Circle, Ellipse, Rectangle.

Multiply Connected Domains – A domain which is not simply connected is called a Multiply Connected Domain.

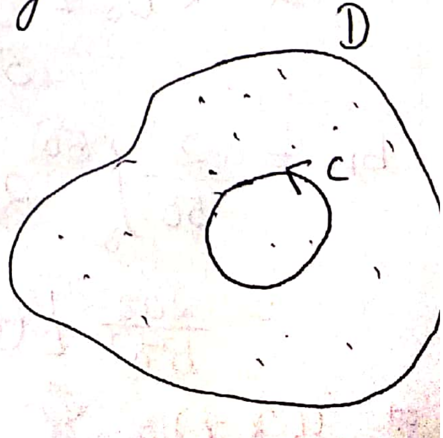
Ex – Annulus, Any domain with holes





Cauchy's Integral Theorem – If $f(z)$ is an analytic function in a simply connected domain D and $f'(z)$ is continuous at each point inside and on a closed curve C where C is any closed curve in D then

$$\boxed{\int_C f(z)\,dz = 0}$$



Proof- Let $f(z) = u+iv$

as $z = x+iy \Rightarrow dz = dx + i\,dy$

Now, $\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$

$= \int_C [u\,dx - v\,dy] + i\int_C [v\,dx + u\,dy]$

By Green's theorem,

$$\oint_C [M\,dx + N\,dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

$$\Rightarrow \int_C f(z)\,dz = \iint_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy + i\iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\,dy$$

As, $f(z)$ is an analytic function, so u and v will satisfy C-R equations

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \int_C f(z)\,dz = \iint_R \left[\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right] dx\,dy + i\iint_R \left[\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right] dx\,dy$$

$= 0$

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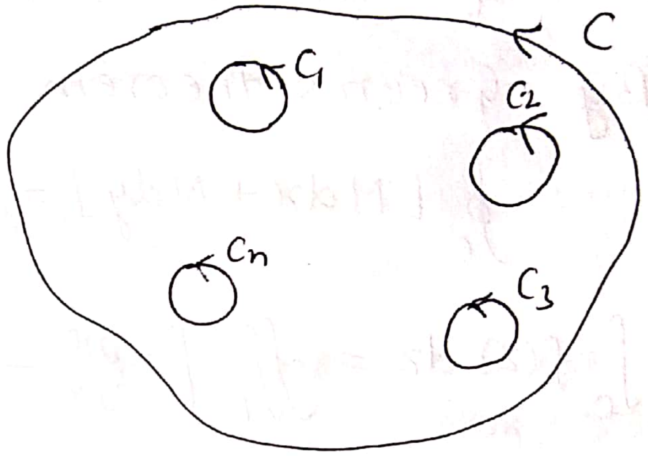
Cauchy's integral thm for multiply connected domain

Let $f(z)$ be an analytic function in a domain D bounded by non-intersecting simple closed curves $C, C_1, C_2, --- C_n$ where $C_1, C_2, --- C_n$ lies inside C then

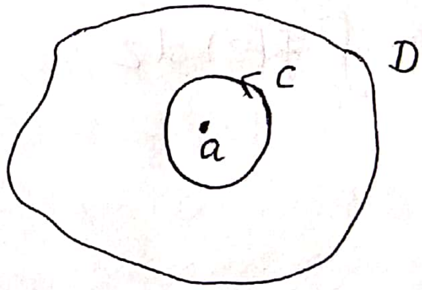
$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ---- + \int_{C_n} f(z)dz$$



Cauchy's Integral Formula

Let $f(z)$ be an analytic function in a simply connected domain D and let a be any point in D and C be any simple closed curve in D enclosing the point $z=a$ then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz}$$



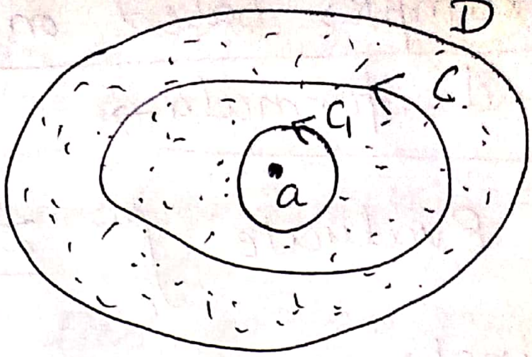
where C is traversed in anticlockwise direction.

Proof- Let $F(z) = \dfrac{f(z)}{z-a}$

which is analytic at all points within C except at $z=a$.

Now, let us draw a small circle 'c_1' with centre at a and radius 'ρ' such that C_1

lies entirely in C.



Now, F(z) is a analytic in a multiply connected region bounded by C and C_1. So, by Cauchy's integral th^m for multiply connected region

$$\int_C \frac{f(z)}{z-a}\,dz = \int_C \frac{f(z)}{z-a}\,dz$$

$\because$ C_1 is a circle with radius ρ and centre at a so,

$$z-a = \rho e^{i\theta}$$

$$\Rightarrow \quad z = a+\rho e^{i\theta} \quad \Rightarrow \quad dz = i\rho e^{i\theta}\,d\theta$$

$$\Rightarrow \int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\,d\theta$$

$$= i\int_{C_1} f(a+\rho e^{i\theta})\,d\theta$$

Now as $\rho \to 0$, the circle C_1 shrinks to the point a. Hence,

$$\int_C \frac{f(z)\,dz}{z-a} = i\int_{C_1} f(a)\,d\theta = i\,f(a)\int_0^{2\pi} d\theta$$

$$= 2\pi i\, f(a)$$

$$\Rightarrow \quad f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

<u>Note</u> -

$$\boxed{f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz}$$

Also

$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Examples based on Cauchy's integral theorem and formula →

① Evaluate $\int_C \frac{z^2-z+1}{z-1}\,dz$ where C is a circle (i) $|z|=1$ (ii) $|z|=\frac{1}{2}$

Solⁿ – (i) Here $f(z) = z^2-z+1$ and $a=1$

Since $f(z)$ is analytic within and on the circle $|z|=1$ and $z=1$ lies on C.

So, by Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$\Rightarrow \quad f(1) = \frac{1}{2\pi i}\int_{|z|=1} \frac{z^2-z+1}{z-1}\,dz$$

as $f(z) = z^2-z+1 \Rightarrow f(1)=1$

So

$$1 = \frac{1}{2\pi i}\int_C \frac{z^2-z+1}{z-1}\,dz$$

$$\Rightarrow \int_C \frac{z^2-z+1}{z-1}\,dz = 2\pi i$$

(ii) When $|z|=\frac{1}{2}$ the C is a circle with radius $\frac{1}{2}$ and $z=1$ lies outside C.

and $f(z) = \frac{z^2-z+1}{z-1}$ is analytic inside and on C

So, by Cauchy's integral theorem,

$$\int_C f(z)\,dz = 0 \quad \Rightarrow \quad \int_C \frac{z^2-z+1}{z-1}\,dz = 0$$

$C: |z| = \frac{1}{2}$

② Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$ where c is the circle $|z|=3$.

Sol- $f(z) = \frac{e^{2z}}{(z-1)(z-2)}$ is not analytic at $z=1, z=2$ and both of these points lies within the Circle $|z|=3$.

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C e^{2z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz$$

$$I = \int_C \frac{e^{2z}}{z-2} dz - \int_C \frac{e^{2z}}{z-1} dz$$

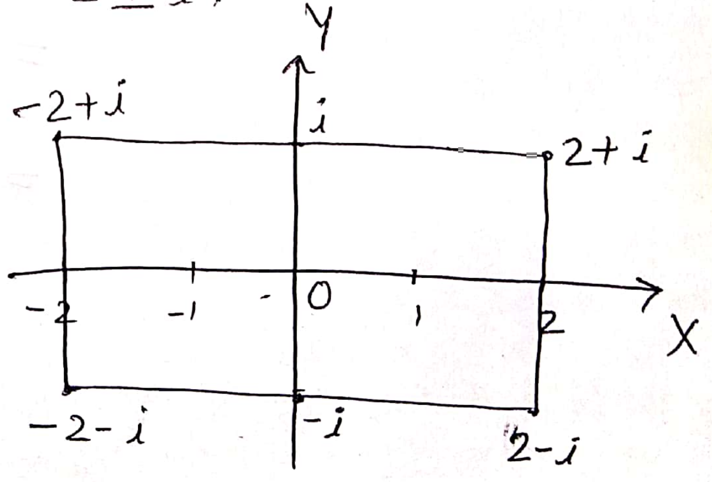
By Cauchy's integral formula,

$$I = 2\pi i\, e^{2\times 2} - 2\pi i\, e^{2\times 1} = 2\pi i\, e^2(e^2-1)$$

③ Evaluate $\int_C \frac{\cos \pi z}{(z^2-1)} dz$ around a rectangle with vertices $2 \pm i$, $-2 \pm i$.

Soln- $\int_C \frac{\cos \pi z}{(z-1)(z+1)} dz$

$$= \frac{1}{2}\int_C \left[\frac{\cos \pi z}{z-1}\right] dz - \frac{1}{2}\int_C \left[\frac{\cos \pi z}{z+1}\right] dz$$



$f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and $z=-1$, $z=1$ lie inside the given region.

So, by Cauchy's integral formula,

$$I = \pi i \cos \pi(1) - \pi i \cos \pi(-1) = 0$$

Ex ④ Evaluate $\int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is the circle $|z|=1$.

Sol - $f(z) = \sin^2 z$ is analytic inside the circle $|z|=1$ and $z = \frac{\pi}{6} = 0.5235$ (approx) lies inside $|z|=1$.

So, by Cauchy's integral formula,

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\Rightarrow f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$$

$$\Rightarrow \int_C \frac{f(z)}{(z-a)^3} dz = \pi i f''(a)$$

$$\Rightarrow \int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \pi i \left[\frac{d^2(\sin^2 z)}{dz^2}\right]_{z=\frac{\pi}{6}}$$

$$= \pi i \left[2\cos 2z\right]_{z=\frac{\pi}{6}}$$

$$= 2\pi i \cos\frac{\pi}{3} = \underline{\underline{\pi i}}$$

Exercise

① Evaluate $\int_C \frac{e^{3z}}{z-i\pi}\, dz$ where $C: |z|=4$

② Evaluate $\int_{|z|=4} \frac{e^{3z}}{(z+i\pi)^7}\, dz$

③ Evaluate $\int_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$

Ans ① $-2\pi i$

② $-\frac{81\pi i}{40}$

③ $4\pi i$