Application of Residue Theorem to evaluate real integrals –

Type 1 – Evaluation of real definite integral of rational function of $\cos\theta$ and $\sin\theta$

Put $z = e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \quad dz = i e^{i\theta} d\theta$

$\Rightarrow \quad \boxed{d\theta = \frac{dz}{iz}}$

and $\frac{1}{z} = \cos\theta - i\sin\theta$

So, $\quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \Rightarrow \quad \boxed{\cos\theta = \frac{z^2+1}{2z}}$

$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \quad \Rightarrow \quad \boxed{\sin\theta = \frac{z^2-1}{2iz}}$

$$|z| = |e^{i\theta}| = 1$$

Put these values in $\quad I = \int_C f(z)\,dz$

where c is unit circle $|z| = 1$.

Ex-1 Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$ using contour integration.

Sol – Put $z = e^{i\theta}$, $\quad d\theta = \frac{dz}{iz}$

So, $\quad z^3 = e^{3i\theta} = \cos 3\theta + i\sin 3\theta$

So, $\quad \cos 3\theta = $ Real part of z^3

and $\quad \cos\theta = \frac{z^2+1}{2z}$

So, $I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$

$= \text{Real part of } \int_C \frac{z^3 \cdot \frac{dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)}$ where $C: |z|=1$

$= \text{Real part of } \frac{1}{i}\int_C \frac{z^3\, dz}{5z-2z^2-2}$

$= \text{R.P. of } \frac{-1}{i}\int_C \frac{z^3\, dz}{2z^2-5z+2}$

Let $f(z) = \frac{z^3}{2z^2-5z+2}$

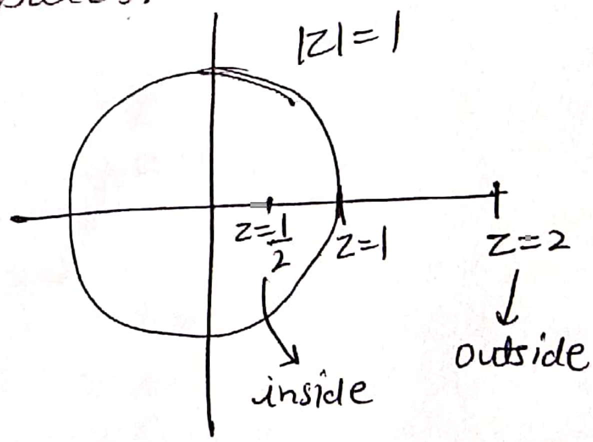
The poles are given by $2z^2-5z+2=0$

$\Rightarrow \quad 2z^2-4z-z+2=0$

$\Rightarrow \quad 2z(z-2)-(z-2)=0$

$\Rightarrow \quad (z-2)(2z-1)=0$

$\Rightarrow \quad z=\frac{1}{2}, 2$ are simple poles.



Now, $z=\frac{1}{2}$ lies inside $C: |z|=1$

Residue $\left[f(z) : z=\frac{1}{2}\right]$

$= \lim_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right) f(z)$

$= \lim_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right)\frac{z^3}{2\left(z-\frac{1}{2}\right)(z-2)} = -\frac{1}{24}$

$\Rightarrow \int_C f(z)\,dz = 2\pi i\left(\frac{-1}{24}\right) = \frac{-\pi i}{12}$

$I = \text{R.P. of } \frac{-1}{i}\left(\frac{-\pi i}{12}\right) = \frac{\pi}{12}$

Ex-2 Apply Cauchy's Residue theorem, to prove that $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2} = \frac{2\pi}{1-p^2}$ $(0<p<1)$

Sol

$$I = \int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}$$

Let $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$$I = \int_C \frac{dz}{iz\left[1 - \frac{p}{i}\left(z-\frac{1}{z}\right) + p^2\right]}$$

$$= \int_C \frac{dz}{[zi - p(z^2-1) + ip^2 z]}$$

$$= \int_C \frac{dz}{iz + i^2 p z^2 + p + ip^2 z}$$

$$= \int_C \frac{dz}{iz(1+ipz) + p(1+ipz)} = \int_C \frac{dz}{(p+iz)(1+ipz)}$$

The poles of $f(z)$ are given by

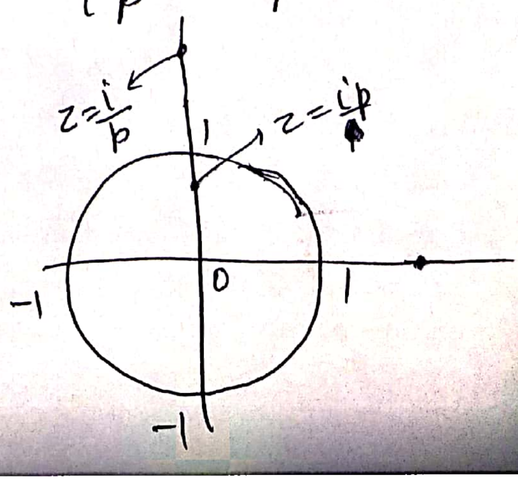
$$(p+iz)(1+ipz) = 0$$

$$\Rightarrow \quad z = -\frac{p}{i} = ip \quad \text{and} \quad z = \frac{-1}{ip} = \frac{i}{p}$$

Now $0<p<1$

So $z = ip$ lies inside $|z|=1$

But $z = \frac{i}{p}$ lies outside $|z|=1$



So, $\operatorname{Res}[f(z) : z = ip] = \lim_{z \to ip} \left\{ (z-ip) \dfrac{1}{(p+iz)(1+ipz)} \right\}$

$= \lim_{z \to ip} \left\{ \cancel{(z-ip)} \dfrac{1}{i\cancel{(z-ip)}(1+ipz)} \right\}$

$= \dfrac{1}{i(1+ip\,ip)} = \dfrac{1}{i(1-p^2)}$

By Residue theorem,

$\int_C f(z)\,dz = 2\pi i \times \dfrac{1}{i(1-p^2)}$

$= \dfrac{2\pi}{1-p^2}, \quad 0 < p < 1$