Sequences and Summations

Sequences

A sequence is a discrete structure used to represent an ordered list.

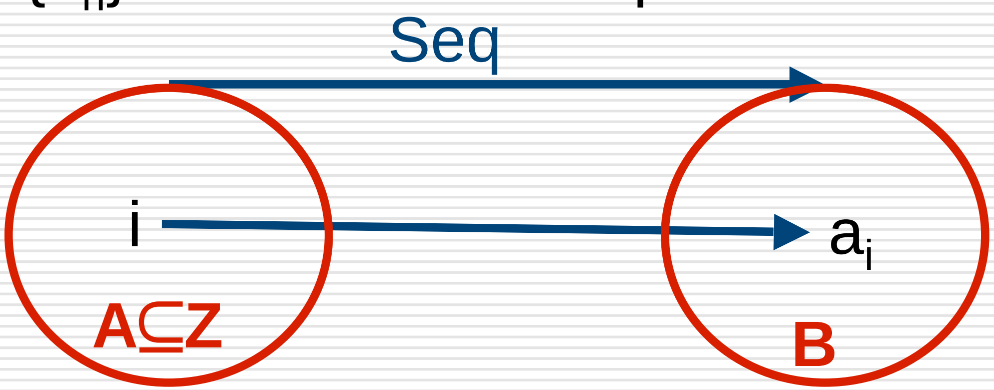
Sequences

A **sequence** is a function from a subset of the set integers (usually {0,1,2,…} or {1,2,3,…}) to a set S.

a_n is denoted the image of integer n.

a_n is called a **term** of the sequence.

The notation $\{a_n\}$ describes the sequence.



Sequences (example)

- Consider is the sequence $\{n^2\}$.

The list of the terms of this sequence is :

1,4,9,16,…

- Consider is the sequence $\{1/n\}$.

The list of the terms of this sequence is :

1,1/2,1/3,1/4,…

Geometric progression

A **geometric progression** is a sequence of the form

$a, ar, ar^2, \ldots, ar^n, \ldots$

Where the initial term a and the **common ratio** r are real numbers.

$\{ar^n\}$ describes a geometric progression.

Geometric progression (example)

- Is $\{(-1)^n\}$ geometric progression?

  1,-1,1,-1,…

  Yes, a=1 and r=-1

- Is $\{2(5)^n\}$ geometric progression?

  2,10,50,250,…

  Yes, a=2 and r=5

- Is $\{6(1/3)^n\}$ geometric progression?

  6,2,2/3,2/9,…

  Yes, a=6 and r=1/3

Arithmetic progression

An **arithmetic progression** is a sequence of the form

a, a+d, a+2d, …., a+nd, …

Where the initial term a and the **common difference** d are real numbers.

{a+dn} describes a geometric progression.

Arithmetic progression (example)

- Is {-1+4n} Arithmetic progression?

  -1,3,7,11,…

  Yes, a=-1 and d=4

- Is {7-3n} Arithmetic progression?

  7,4,1,-2,…

  Yes, a=7 and d=-3

String

Finite sequences of form $a_1, a_2, \ldots, a_n$ are called strings.

The length of a string is the number of terms in the string.

The empty string has no terms and its length is zero.

Example:

'abcd' is a string of length four.

Sequences (example)

Find a formula for the following sequence.

1, 1/2, 1/4, 1/8, 1/16, …

Solution:

$\{1/2^n\}$

It is a geometric progression.

a=1 and r=1/2

Sequences (example)

Find formula for the following sequence.

1, 3, 5, 7, 9, …

Solution:

{1+2n}

It is an arithmetic progression.

a=1 and d=2

Sequences (example)

Find formula for the following sequence.

1, -1, 1, -1, 1, …

Solution:

$\{(-1)^n\}$

It is a geometric progression.

a=1 and r=-1

Sequences (example)

How can you produce the terms of the following sequence?

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …

Solution:

A rule for generating this sequence is that integer n appears exactly n times.

Sequences (example)

How can you produce the terms of the following sequence?

5, 11, 17, 23, 29, 35, 41, …

Solution:

A rule for generating this sequence is 5+6(n).

It is an arithmetic progression.

a=5 and d=6

Useful sequences

$\{n^2\}$

1, 4, 9, 16, 25, 36, 49, …

$\{n^3\}$

1,8,27,64,125,216,343,512,…

$\{2^n\}$

2,4,8,16,32,64,128,256,…

$\{3^n\}$

3,9,27,81,243,729,2187,…

Sequences (example)

Find formula for the following sequence.

1, 7, 25, 79, 241, 727, 2185, …

Solution:

Compare it to $\{3^n\}$.

$\{3^n-2\}$

Summations

The sum of the terms a_m, a_{m+1}, …, a_n from the sequence $\{a_n\}$ is:

$a_m + a_{m+1} + \ldots + a_n$

$$\sum_{j=m}^{n} a_j$$

$\sum_{m \leq j \leq n} a_j$

$\sum$ donates **summation** and j is the **index of summation**.

m is **lower limit** and n is **upper limit.**

Summations (example)

Express the sum of the first 100 terms of the sequence {1/n} for n=1,2,3,.... .

Solution:

$$\sum_{k=1}^{100} 1/k$$

Summations (example)

What is the value of $\sum_{i=1}^{3} i^2$

Solution:

$$\sum_{i=1}^{3} i^2 = 1 + 4 + 9 = 14$$

Shifting the index of summation

Sometimes it is useful to shift the index of summation.

For example:

Consider $\sum_{i=1}^{3} i^2$.

We want the index of summation to run between 0 and 2 rather than from 1 to 3.

i=1,2,3 k=0,1,2

So, k=i-1.

$$\sum_{k=0}^{2} (k+1)^2 = \sum_{i=1}^{3} i^2 = 1+4+9 = 14$$

Shifting the index of summation (example)

Consider the following summation. Change the index of summation to run between 1 and 4.

$$\sum_{j=0}^{3} 2j$$

Solution:

$$\sum_{j=0}^{3} 2j$$

j=0,1,2,3 k=1,2,3,4 k = j+1

$$\sum_{j=0}^{3} 2j = \sum_{k=1}^{4} 2(k-1)$$

Arithmetic laws

The usual laws for arithmetic apply to summations.

Let a and b be real numbers.

$$\sum_{k=1}^{n} (ax_k + by_k) = ax_1 + by_1 + ax_2 + by_2 + \ldots + ax_n + by_n$$

$$= ax_1 + ax_2 + \ldots + ax_n + by_1 + by_2 + \ldots + by_n$$

$$= \sum_{k=1}^{n} ax_k + \sum_{k=1}^{n} by_k \quad = a\sum_{k=1}^{n} x_k + b\sum_{k=1}^{n} y_k$$

Arithmetic laws (example)

$$\sum_{k=1}^{3} (5k + k^2) =$$

$$\sum_{k=1}^{3} 5k + \sum_{k=1}^{3} k^2 =$$

$$(5 + 10 + 15) + (1 + 4 + 9) =$$

$$30 + 14 =$$

$$44$$

Sums of terms of geometric progression

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} (ar^{n+1} - a) / (r-1) & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Sums of terms of geometric progression

Proof:

Let $S = \sum_{j=0}^{n} ar^j$.

$rS = r \sum_{j=0}^{n} ar^j$ (multiply both sides by r)

$= \sum_{j=0}^{n} ar^{j+1}$ (by the distributive property)

$= \sum_{k=1}^{n+1} ar^k$ (shifting the index of summation)

Sums of terms of geometric progression

Proof:

$$rS = \left(\sum_{k=0}^{n} ar^k \right) + (ar^{n+1} - a)$$

(removing k=n+1 term and adding k=0 term)

$$= S + (ar^{n+1} - a)$$

$$rS = S + (ar^{n+1} - a)$$

$$S = (ar^{n+1} - a) / (r - 1) \qquad (\text{if } r \neq 1)$$

If $r = 1$, then $S = \sum_{j=0}^{n} ar^j = (n+1)a$

Double summation

Double summations are used in the analysis of nested loops in computer programs.

For example:

loop 1: for i=1 to 4

loop 2: for j=1 to 3

x = x + 1

$$\sum_{i=1}^{4} \sum_{j=1}^{3} 1$$

Double summation (example)

To evaluate the double summation, first expand the inner summation and then continue by computing the outer summation

$$\sum_{i=1}^{4} \sum_{j=1}^{3} i(j) =$$

$$\sum i(1 + 2 + 3) =$$

$$\sum (i + 2i + 3i) =$$

$$(1+2+3) + (2+4+6) + (3+6+9) + (4+8+12) =$$

$$9 + 12 + 18 + 24 = 60$$

Summations and functions

The summation notation can be used to add some values of a function.

Example:

Assume f(x) = x+2. $\sum_{x \in \{2,3,5\}} f(x) = ?$

$$\sum_{x \in \{2,3,5\}} f(x) = f(2) + f(3) + f(5)$$
$$= 4 + 5 + 7 = 16$$

Useful summation formula

$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$(ar^{n+1} - a) / (r-1)\ ,\ (r \neq 1)$
$\sum_{k=1}^{n} k$	$n(n+1) / 2$
$\sum_{k=1}^{n} k^2$	$n(n+1)(2n+1) / 6$

Useful summation formula

$\sum_{k=1}^{n} k^3$	$n^2(n+1)^2 / 4$

Useful summation formula (example)

Find $\sum_{k=50}^{100} k^2$.

Solution:

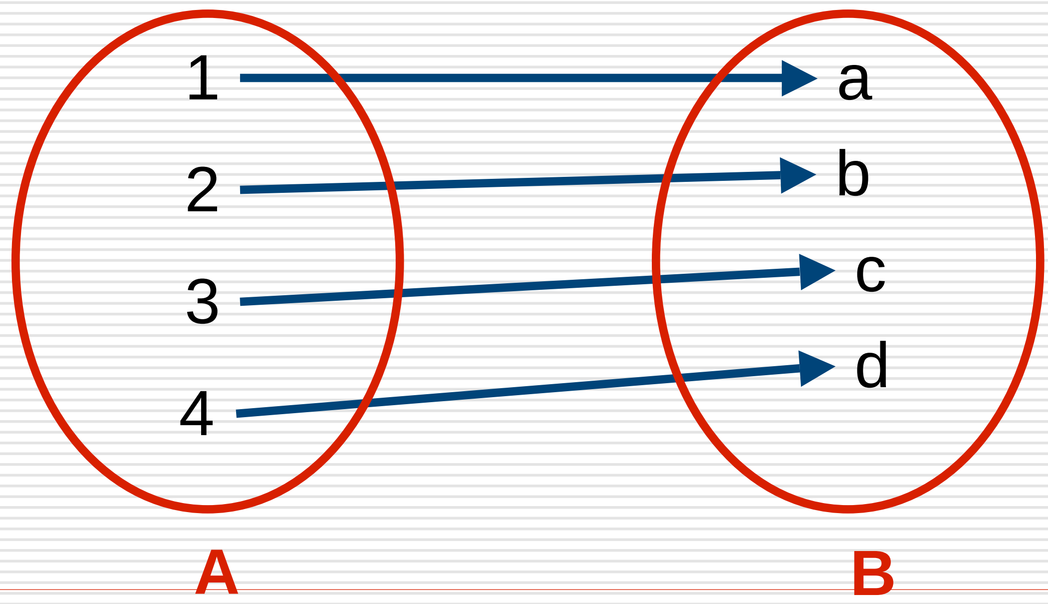
$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2 =$$

$$(100 \cdot 101 \cdot 201) / 6 - (49 \cdot 50 \cdot 99) / 6 =$$

$$338350 - 40425 = 297925$$

Cardinality

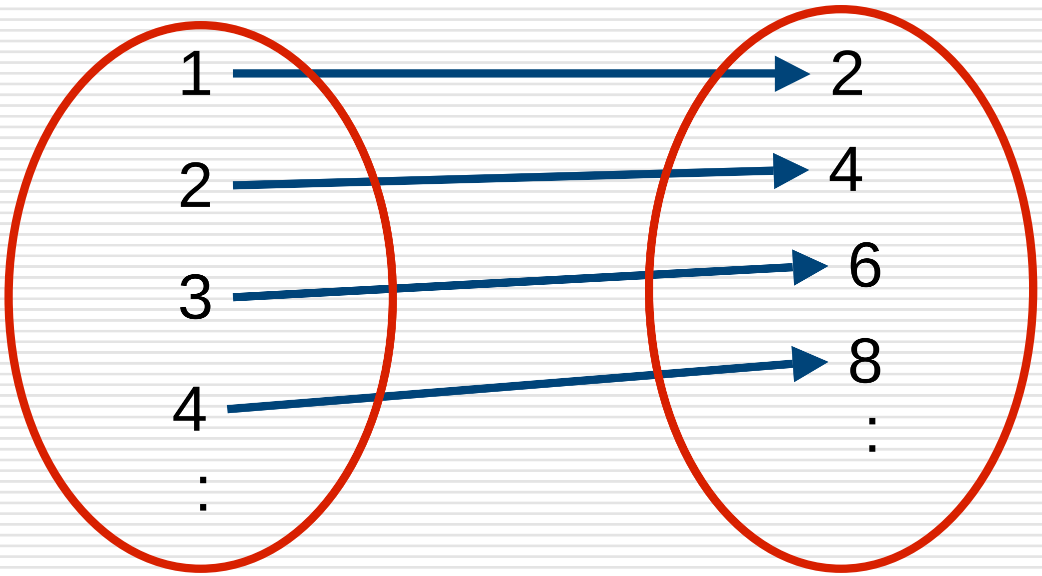
The set A and B have the same cardinality if and only if there is a bijection from A to B.



Cardinality

A set that is either finite or has the same cardinality as the set of positive integers is called **countable**.

A set that is not countable is **uncountable**.



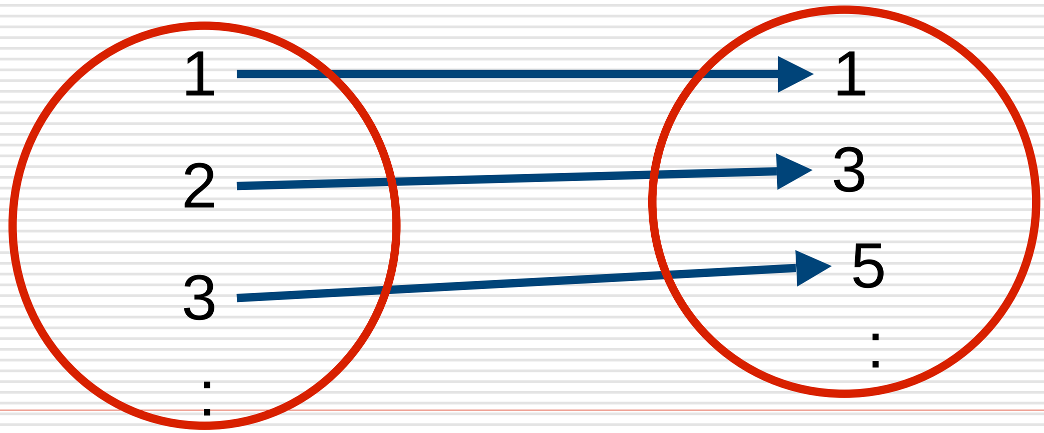
Cardinality (example)

Show that the set of odd integers is a countable set.

Solution:

- Show it has the same cardinality as positive integers. (There is a bijection from positive integers to odd integers.)

$$f(n) = 2n-1$$



Cardinality (example)

Show that the set of positive rational numbers is a countable set.

Solution:

- Show it has the same cardinality as positive integers. (There is a bijection from positive integers to odd integers.)

f(n) = 2n-1

Assume f(n)=f(m).

2n-1 = 2m-1

So, n=m.

So, f is one-to-one.

Assume m is an integer.

f(m) = 2m -1 = n.

∃ n that is odd integer and f(m)=n.

So, f is onto. Thus, f is bijection.