Logics

Why study propositional logic?

- A formal mathematical “language” for precise reasoning.
- Start with propositions.
- Add other constructs like negation, conjunction, disjunction, implication etc.
- All of these are based on ideas we use daily to reason about things.

Propositions

- Declarative sentence
- Must be either True or False.

Propositions:

- York University is in Toronto
- York University is in downtown Toronto
- All students at York are Computer Sc. majors.

Not propositions:

- Do you like this class?
- There are x students in this class.

Compound propositions

- new propositions formed from existing propositions using logical operators

- Definition 1: Let p be a proposition. The *negation* of p, denoted by $\neg p$ (or $\bar{p}$), is the statement "It is not the case that p."
  - "not p"

p	$\neg p$
T	F
F	T

Conjunction, Disjunction

- Definition 2: Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q."

- Definition 3: Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q."

Conjunction, Disjunction

- Conjunction: $p \wedge q$ [“and”]
- Disjunction: $p \vee q$ [“or”]

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Exclusive OR (XOR)

- Definition 4: Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

- Definition 5: Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q."
  - p: *hypothesis* (or *antecedent* or *premise*)
  - q: *conclusion* (or *consequence*)
  - Implication
    - "p implies q"
  - $p \rightarrow q$ is false when p is true & q is false . Otherwise true.

Conditional - 2

- $p \rightarrow q$ ["if p then q"]
- Truth table:

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Note the truth table of $\neg p \vee q$

Logical Equivalence

- $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent
- Truth tables are the simplest way to prove such facts.
- We will learn other ways later.

Contrapositive

- Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- Any conditional and its contrapositive are logically equivalent (have the same truth table) – Check by writing down the truth table.
- E.g. The contrapositive of “If you get 100% in this course, you will get an A+” is “If you do not get an A+ in this course, you did not get 100%”.

Converse

- Converse of $p \rightarrow q$ is $q \rightarrow p$
- Not logically equivalent to conditional
- Ex 1: “If you get 100% in this course, you will get an A+” and “If you get an A+ in this course, you scored 100%” are not equivalent.
- Ex 2: If you won the lottery, you are rich.

Other conditionals

Inverse:

- inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$
- The converse is equivalent to the inverse

Biconditionals

- Definition 6: Let *p* and *q* be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "*p* if and only if *q*."
  - "*bi-implications*"
  - "*p* is necessary and sufficient for *q*"
  - "*p* iff *q*"
  - True - when p & q have same truth values , false otherwise.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Compound Propositions

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical operators

- Example: $p \wedge q \vee r$: Could be interpreted as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$
- 1st one is correct.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Example: $p \vee \neg q \wedge r \rightarrow s \vee q$

$$(p \vee ((\neg q) \wedge r)) \rightarrow (s \vee q)$$

Translating English Sentences

- Translation removes ambiguity of sentences.
- Steps to convert an English sentence to a statement in propositional logic
  - Identify propositions and represent using propositional variables.
  - Determine appropriate logical connectives

- "If I go to Harry's or to the country, I will not go shopping."
  - p: I go to Harry's
  - q: I go to the country.
  - r: I will go shopping.

If p or q then not r.

$$(p \vee q) \rightarrow \neg r$$

Another Example

Problem: Translate the following sentence into propositional logic:

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

One Solution: Let a, c, and f represent respectively "You can access the internet from campus," "You are a computer science major," and "You are a freshman."

$$a \rightarrow (c \vee \neg f)$$

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let *p* denote "The automated reply can be sent" and *q* denote "The file system is full."

$$q \rightarrow \neg p$$

Consistent System Specifications

Definition: A list of propositions is *consistent* if it is possible to assign truth values to the proposition variables so that each proposition is true.

Exercise: Are these specifications consistent?

- “The diagnostic message is stored in the buffer or it is retransmitted.”
- “The diagnostic message is not stored in the buffer.”
- “If the diagnostic message is stored in the buffer, then it is retransmitted.”

Solution: Let p denote “The diagnostic message is stored in the buffer.” Let q denote “The diagnostic message is retransmitted” The specification can be written as: $p \lor q, \neg p, p \rightarrow q$,. When p is false and q is true all three statements are true. So the specification is consistent.

- What if “The diagnostic message is not retransmitted is added.”

  Solution: Now we are adding $\neg q$ and there is no satisfying assignment. So the specification is not consistent.

Example

- An island has two kinds of inhabitants, ***knights*, who always tell the truth, and *knaves*, who always lie**.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

Example: **What are the types of A and B?**

Solution: Let p and q be the statements that **A is a knight** and **B is a knight**, respectively. Then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.

- If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight, and therefore, $\neg p$ must be true.
- If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

Logic and Bit Operations

- Bit: binary digit
- Boolean variable: either true or false
  - Can be represented by a bit
- Definition 7: A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

TABLE 9 **Table for the Bit Operators *OR*, *AND*, and *XOR*.**

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Tautology, Contradiction, Contingency

- A tautology is a proposition which is always TRUE.
  - Example: $p \vee \neg p$
- A contradiction is a proposition which is always FALSE.
  - Example: $p \wedge \neg p$
- A contingency is a proposition which is neither a tautology nor a contradiction, such as most previous propositions p we have seen

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

For any contingency p

- $p \vee \neg p$ is a tautology
- $p \wedge \neg p$ is a contradiction

Logically Equivalent

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's Laws

Augustus De Morgan

1806-1871

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

This truth table shows that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

Identity Laws $p \wedge T \equiv p$ $p \vee F \equiv p$

Domination Laws $p \vee T \equiv T$ $p \wedge F \equiv F$

Idempotent laws $p \vee p \equiv p$ $p \wedge p \equiv p$

Double Negation Law $\neg(\neg p) \equiv p$

Negation Laws $p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$

Commutative Laws $p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$

Associative Laws

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

Distributive Laws

$$(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$$
$$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

Absorption Laws $p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$

Exercise: Prove these laws using Truth Table.

More Logical Equivalences

The following logical equivalences are often useful for solving problems. They can be proved using Truth Tables. They can use used to prove more logical equivalences!

Logical Equivalences Involving Conditional Statements

$$p \to q \equiv \neg p \lor q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \lor q \equiv \neg p \to q$$
$$p \land q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \land \neg q$$
$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$
$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$
$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$
$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

Logical Equivalences Involving Biconditional Statements

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Constructing New Logical Equivalence

- How to show logical equivalence
  - Use a truth table
  - Use logical identities that we already know

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law} \\
& && \text{for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

p -> q is logically equivalent to $\neg p \vee q$

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (q \vee \neg q) && \text{by associative and} \\
& && \text{commutative laws} \\
& && \text{laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

Propositional Logic Not Enough

- If we have:

  "All men are mortal."

  "Socrates is a man."

  - Does it follow that "Socrates is mortal?"
- How do you make a statement about all even integers?

  If $x > 2$ then $x^2 > 4$

- Hard to identify "individuals" (e.g., Mary, 3)
- Can't directly talk about properties of individuals or relations between individuals (e.g., "Bill is tall")
- Generalizations, patterns, regularities can't easily be represented (e.g., "all triangles have 3 sides")
- Can't be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.
- So, Predicate Logic

Predicate logic

- *Predicate*: a property that the subject of the statement can have
  - Ex: x>3
    - x: variable
    - >3: predicate
    - P(x): x>3
      - The value of the propositional function P at x
      - Once the value is assigned to variable x, statement P(x) becomes a proposition and has a truth value.
      - So P(1) is false, P(4) is true,….
  - $P(x_1, x_2, \ldots, x_n)$: *n*-place predicate or *n*-ary predicate

Propositional Functions

- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

  $P(-3)$ is false.

  $P(0)$ is false.

  $P(3)$ is true.

- Often the domain is denoted by U. So in this example U is the integers.
- Intuitively, the **universe of discourse (U)** is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.

Examples of Propositional Functions

- Let "*x* + *y* = *z*" be denoted by *R*(*x, y, z*) and *U* (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  Solution: F

  R(3,4,7)

  Solution: T

  R(*x*, 3, *z*)

  Solution: Not a Proposition

- Now let "*x* - *y* = *z*" be denoted by *Q*(*x*, *y*, *z*), with U as the integers. Find these truth values:

  Q(2,-1,3)

  Solution: T

  Q(3,4,7)

  Solution: F

  Q(*x*, 3, *z*)

  Solution: Not a Proposition

Quantifiers

Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier*, "For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

– $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Universal Quantifier: Definition

- **Definition**: The universal quantification of a predicate $P(x)$ is the proposition '<u>$P(x)$ is true for all values of x in the universe of discourse.</u>' We use the notation: $\forall\, x\, P(x)$, which is read 'for all x'.
- If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of the propositions over all the elements

$$\forall\, x\, P(x) \Leftrightarrow P(n_1) \land P(n_2) \land \ldots \land P(n_k)$$

Universal Quantifier

- Examples:
  - $z(z+1)(z+2)$ is divisible by 6 for all integer z
  - q^2 is rational for all rational number q
  - $r^3 > 0$ for all positive real number r

- Important Note: Domain needs to be specified!

What is the truth value of $\forall x\,(x \leq 10)$ when the domain consists of all positive integers not exceeding 3?

What is the truth value of $P(1) \wedge P(2) \wedge P(3)$?

Universal Quantifier: Example

- Let $P(x)$: ‘x must take a discrete mathematics course’ and $Q(x)$: ‘x is a CS student.’
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statements:
  - “Every CS student must take a discrete mathematics course.”

    $$\forall x\ Q(x) \rightarrow P(x)$$

  - “Everybody must take a discrete mathematics course or be a CS student.” $\forall x\ (P(x) \vee \mathrm{Q(x)})$
  - “Everybody must take a discrete mathematics course and be a CS student.” $\forall x\ (P(x) \wedge \mathrm{Q(x)})$

Universal Quantifier: Example

1) Let $P(x)$: $x + 1 > x$

What is the truth value for $\forall x\, (P(x)$

where the domain consists of all real numbers ? TRUE

2) Let Q(x) be the statement "x<2". What is the truth value for

$\forall x\, Q(x)$

where the domain consists of all real numbers ? FALSE

Universal Quantifier: Example

- Express the statement: 'for every x and every y, $x+y>10$'
- Answer:
  - Let $P(x,y)$ be the statement $x+y>10$
  - Where the universe of discourse for x, y is the set of integers
  - The statement is: $\forall x \, \forall y \, P(x,y)$
- Shorthand: $\forall x,y \, P(x,y)$

Existential Quantifier: Definition

- **Definition**: The existential quantification of a predicate $P(x)$ is the proposition 'There exists a value x in the universe of discourse such that $P(x)$ is true.' We use the notation: $\exists x\, P(x)$, which is read 'there exists x'.
- If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of the propositions over all the elements

$$\exists x\, P(x) \Leftrightarrow P(n_1) \vee P(n_2) \vee \ldots \vee P(n_k)$$

Existential Quantifier: Example

1) Let $P(x)$: $x > 10$

What is the truth value for $\exists x P(x)$

where the domain consists of all real numbers ? TRUE

2) Let Q(x) be the statement "x=x+1". What is the truth value for

$\exists x P(x)$ where the domain consists of all real numbers ? FALSE

Existential Quantifier: Example

- Let $P(x,y)$ denote the statement 'x+y=5'
- What does the expression $\exists x \, \exists y \, P(x,y)$ mean?
- Which universe(s) of discourse make it true?

Existential Quantifier: Example

- Express the statement: ‘there exists a real solution to $ax^2+bx-c=0$’
- Answer:
  - Let $P(x)$ be the statement $x= (-b\pm\sqrt{(b^2-4ac)})/2a$
  - Where the universe of discourse for x is the set of real numbers. Note here that a, b, c are fixed constants.
  - The statement can be expressed as $\exists x\, P(x)$
- What is the truth value of $\exists x\, P(x)$?
  - It is false. When $b^2<4ac$, there are no real number x that can satisfy the predicate
- What can we do so that $\exists x\, P(x)$is true?
  - Change the universe of discourse to the complex numbers, $\mathbb{C}$

Quantifiers: Truth values

- In general, when are quantified statements true or false?

Statement	True when…	False when…
$\forall x\, P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

Quantifiers with Restricted Domain

- Sometimes, we want to simplify the writing by using short-hand notation
- Assuming the domain consists of all integers, guess what does each of the following mean?
  - $\forall x < 0 \, (x^2 > 0)$
  - $\forall y \neq 0 \, (y^3 \neq 0)$
  - $\exists z > 0 \, (z^2 = 10)$

Quantifiers with Restricted Domain

- $\forall x < 0\ (x^2 > 0)$ means

  "For every x in the domain with $x < 0$, $x^2 > 0$."

  The proposition is the same as:

  $$\forall x\ (x < 0 \rightarrow x^2 > 0)$$

- $\exists z > 0\ (z^2 = 10)$ means

  "There is some z in the domain with $z > 0$, $z^2 = 10$."

  The proposition is the same as:

  $$\exists z\ (z > 0 \wedge z^2 = 10)$$

Quantifiers with restricted domains

- Restriction of a universal quantification ➔ universal quantification of a conditional statement
- Restriction of a existential quantification ➔ existential quantification of a conjunction

Negation

- We can use negation with quantified expressions as we used them with propositions
- **Lemma**: Let $P(x)$ be a predicate. Then the followings hold:

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg\, (\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

- Rules for negations for quantifiers are called De Morgan's Laws for quantifiers.

Negating Quantified Expressions

"Every student in your class has taken a course in calculus."

This statement is a universal quantification, namely,

$$\forall x\, P(x),$$

where $P(x)$ is the statement "x has taken a course in calculus"

Negation of this statement :
It is not the case that every student in your class has taken a course in calculus.
Equivalent to :
There is a student in your class who has not taken a course in calculus.

$$\exists x\, \neg P(x).$$

This example illustrates the following logical equivalence:

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x).$$

Another Example

Suppose we wish to negate an existential quantification. For instance, consider the proposition "There is a student in this class who has taken a course in calculus." This is the existential quantification

$$\exists x\, Q(x),$$

where *Q(x) is the statement "x has taken a course in calculus."*

Negation of this statement :
"It is not the case that there is a student in this class who has taken a course in calculus."
This is equivalent to
"Every student in this class has not taken calculus,"

$$\forall x\, \neg Q(x).$$

This example illustrates the equivalence

$$\neg\exists x\, Q(x) \equiv \forall x\, \neg Q(x).$$

Negation: Truth

Truth Values of Negated Quantifiers

Statement	**True when...**	**False when...**
$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$	$P(x)$ is false for every x	There is an x for which $P(x)$ is true
$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

The Order of Quantifiers

- Order in which quantifiers appear is important
- Example:

  Suppose that the domain for both x and y are integers. What are the truth values of the following?

  1. $\forall y \, \exists x \, (x + y = 1)$
  2. $\exists x \, \forall y \, (x + y = 1)$

1. For all y there exists an x such that x+y=1 holds.
we can find at AT LEAST ONE x based on y   TRUE
2. There exists an x for all y such that x+y=1 holds
AT LEAST ONE x can be found BEFORE any other variable is set.   FALSE

The Order of Quantifiers

- Two special cases where the order of quantifiers is not important are:
  1. All quantifiers are universal quantifiers
  2. All quantifiers are existential quantifiers

- Example:

$$\exists x \exists y (x + y = 1)$$

  means the same as

$$\exists y \exists x (x + y = 1)$$

English into Logic

- Logic is more precise than English
- Transcribing English into Logic and vice versa can be tricky
- When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:

Use $\forall$ with $\Rightarrow$

$\forall x\ \mathit{Lion}(x) \Rightarrow \mathit{Fierce}(x)$: Every lion is fierce

$\forall x\ \mathit{Lion}(x) \wedge \mathit{Fierce}(x)$: Everyone is a lion and everyone is fierce

Use $\exists$ with $\wedge$

$\exists\, x\ \mathit{Lion}(x) \wedge \mathit{Vegan}(x)$: Holds when you have at least one vegan lion

Applications: English Translation

- How to translate the following sentence

  "Every student in this class has studied Calculus."

  into a logical expression, if

  Q(x) denotes "x has studied Calculus", and
  the domain of x is all students in this class?

$$\forall x Q(x).$$

- If we change domain to all people, our statement:

  "For every person x, if person x is a student in this class then x has studied calculus"
- *S(x) represents the statement that person x is in this class.*
- $\forall x(S(x) \rightarrow Q(x))$.
- *the statement cannot be expressed as*

  $\forall x(S(x) \wedge Q(x))$
- *because this statement says that all people are students in this class and have studied* calculus

Applications: English Translation

- How to translate the following sentences
  1. "All lions are fierce."
  2. "Some lion does not drink coffee."
  3. "Some fierce creatures do not drink coffee."

  into logical expressions, if

  P(x) := "x is a lion", Q(x) := "x is fierce",
  R(x) := "x drinks coffee",
  and the domain of x consists of all creatures?

- We can express these statements as:
- $\forall x(P(x) \rightarrow Q(x))$.
- $\exists x(P(x) \land \neg R(x))$.
- $\exists x(Q(x) \land \neg R(x))$.
- 2nd statement cannot be written as $\exists x(P(x) \rightarrow \neg R(x))$.
- *Reason :* $P(x) \rightarrow \neg R(x)$ *is true whenever x is not a lion, so that* $\exists x(P(x) \rightarrow \neg R(x))$ *is true as long as* there is at least one creature that is not a lion, even if every lion drinks coffee.
- Similarly, the third statement cannot be written as
- $\exists x(Q(x) \rightarrow \neg R(x))$.

Applications: English Translation

- How to translate the following sentence

  "If a person is a female and is a parent, then this person is someone's mother"

  into a logical expression, if

  $F(x)$:= "x is a female", $P(x)$:= "x is a parent",

  $M(x, y)$:= "x is a mother of y",

  and the domain consists of all people?

- It can be expressed as "For every person *x,* if *x is female and x is a parent, then there* exists a person *y such that x is the mother of y".*

$$\forall \mathbf{x}\ ((F(x) \wedge P(x)) \rightarrow \exists y\ M(x, y))$$

$$\forall \mathbf{x} \exists y\ ((F(x) \wedge P(x)) \rightarrow M(x, y))$$