Vibrations of a Stretched String- Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Fourier Half range sine series (Recall)

f(x) is required to expand as a sine series in the range 0 < x < L

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Where,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Method of separation of variables (recap)

- Working Rule:
 - (1) Assume the solution is going to be of the form X(z).T(t) or X(z).Y(y) etc. This is called Separable form.
 - (2) Substitute that form back into the PDE.
 - (3) Devide by X(2) T(+) or X(2) Y(y).
 - (4) Now each term of the equation depends on a different variable so they must both be constants.
 - (5) For each possible value of the constant (positive, megative, zero), solve the two resulting ODEs and multiply the folitions together to give one specific solution to the PDE.
 - (6) Form the general volution of the PDE by adding linear combinations of all the specific sold,

Solution of wave equation $\left(\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\right)$

Solution:

Wave eq-

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad -----(1)$$

Let u = X(x)T(t) be the solution of the eq. 1

Then
$$\frac{\partial^2 u}{\partial t^2} = XT''$$
 and $\frac{\partial^2 u}{\partial x^2} = TX''$

Substituting these values in eq. 1

$$XT'' = c^2 TX'' \Rightarrow X''/X = T''/c^2 T$$
 (Separating variables)

$$\Rightarrow$$
 X''/X = T''/c²T = Constant

Three cases may arise for constant –

Solution Case 1

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Case 1- If the constant is negative - k^2 (say)-
X''/X = -k^2 \Rightarrow X'' + k^2X = 0 \Rightarrow X = C_1 \cos(kx) + C_2 \sin(kx)
T''/c^2T = -k^2 \Rightarrow T'' + k^2c^2T = 0 \Rightarrow T = C_3 \cos(kct) + C_4 \sin(kct)
Solution is u = X(x)T(t)
u = (C_1 \cos(kx) + C_2 \sin(kx))(C_3 \cos(kct) + C_4 \sin(kct))-----(2)
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Solution: Case 2

Case 2- If the constant is positive k^2 (say)- $X''/X = k^2 \Rightarrow X'' - k^2X = 0 \Rightarrow X = C_5 e^{kx} + C_6 e^{-kx}$ $T''/c^2T = k^2 \Rightarrow T'' - k^2c^2T = 0 \Rightarrow T = C_7 e^{kct} + C_8 e^{-kct}$ Solution is u = X(x)T(t)

 $u = (C_5 e^{kx} + C_6 e^{-kx})(C_7 e^{kct} + C_8 e^{-kct}) - - - - (3)$

Solution: Case 3

Case 3- If the constant =0 $X''/X = 0 \Rightarrow X'' = 0 \Rightarrow X = C_9 + C_{10}x$ $T''/c^2T = 0 \Rightarrow X'' = 0 \Rightarrow X = C_{11} + C_{12}t$ Solution is u = X(x)T(t) $u = (C_9 + C_{10}x)(C_{11} + C_{12}t) -----(3)$

Acceptable solution (Solution given by eq. 2)

841.	contd
=100001==	out of these three sol ne have to choose that son
	The state of the s
	or we are dealing with peroblems on vibrations.
	The state of the s
	so the acceptable son for viberating a string
	is u= (c, cos kn + c, sin kn) (Czcos kct + G sin Kot)
-	us u= (G, Cos Kra TC, SITC Kra

#	when Enitial velo	itu is	given.			
	(3u) = v60.		0	h.h.	1000	1
	u(z,0) = 0.		- E	1 1-14	F "= "	
	pos time	5 5	Se -	11/4-	- Py	

Question 1 (We will learn step by step)-

A tightly stretched string of length 1 meter with fixed end points is initially in equilibrium position. It is set vibrating by giving each point a

velocity
$$g(x) = \begin{cases} x, & 0 < x < 1/2 \\ 1 - x, & 1/2 < x < 1 \end{cases}$$
. Find the displacement.

Solution: Step 1

JUOI	i:step i
	Ulbrat" of string is governed by foll. ogn:
	Vibrat of string as
SA"	Using wave Equation 3-
	du - K2 Dec.
	Using wave equation of dt2 ore
	let u= xT
	Let $u = x_1$ $x = c^2 x'' = x'' = x'' = x'' = -k^2, 0, +k^2$
_	X C' I
	Case I: U= [C, cos (kx) + 5 sin (kx)][C3 cos ket + C4 sin kct]
	Case 1: U= C1 cos (11)
	Conse III $u = (c_0 + c_0 x)(c_1 + c_0 t)$ Conse III $u = (c_0 + c_1 x)(c_1 + c_0 t)$ Conse III $u = (c_0 + c_1 x)(c_1 + c_0 t)$ Therefore Therefore $c_0 + c_1 x = c_1 t = c_1 $
,	1 = C+C10 (cq.exn+C,e)(C110+G16
0	Case W. M. Co.
(1)	ment the objected not
	Only case I is acceptable with the physical not
	u = [c, cos(kx) + c, sin(kx)][c, cos ckt + Cq sin ckt]
\sim	m - [c] cos (

Solution Step 2: Find boundary conditions(B.C.)

Given that ---

B.C.

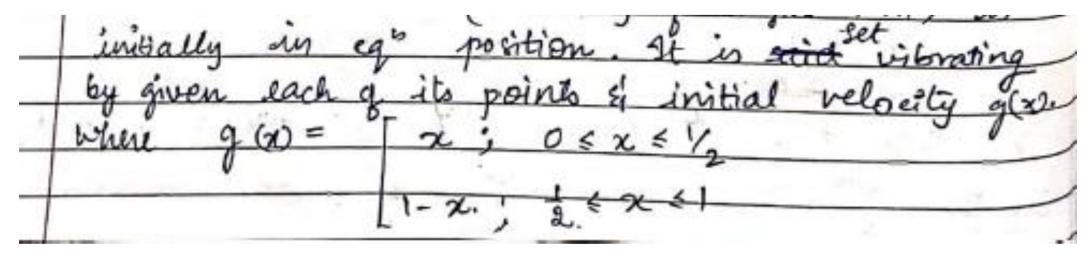
$$u(0,t) = 0$$
 -----(Cond. 1)

and

$$u(1, t) = 0$$
-----(Cond. 2)

Solution Step 3: Find initial Condition (I.C.)

Given that ---

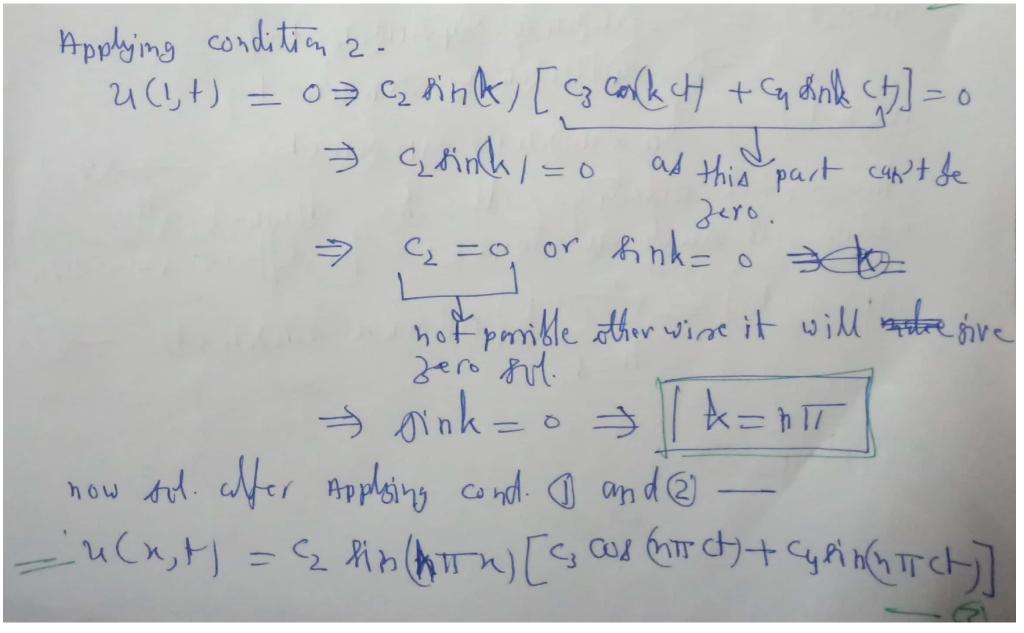


I.C.

$$u(x,0)=0$$
 (initially its in equilibrium position)----cond. (3) $\left(\frac{\partial u}{\partial x}\right)at(x,0)=g(x)$ -------Cond. (4)

Step 4: Apply all above 4 conditions on acceptable solution (eq. 1)

Condition 2



Condition 3-

Condition 4

Here L=1(hTTC) = 2 (g(n) finnin dn U(x,t) = 2 4 (AhnTru Ahnrt + JAHNT) Type 2: U(x,0) is given.

Instial disp. is given & initial net. i-e. 20/=0

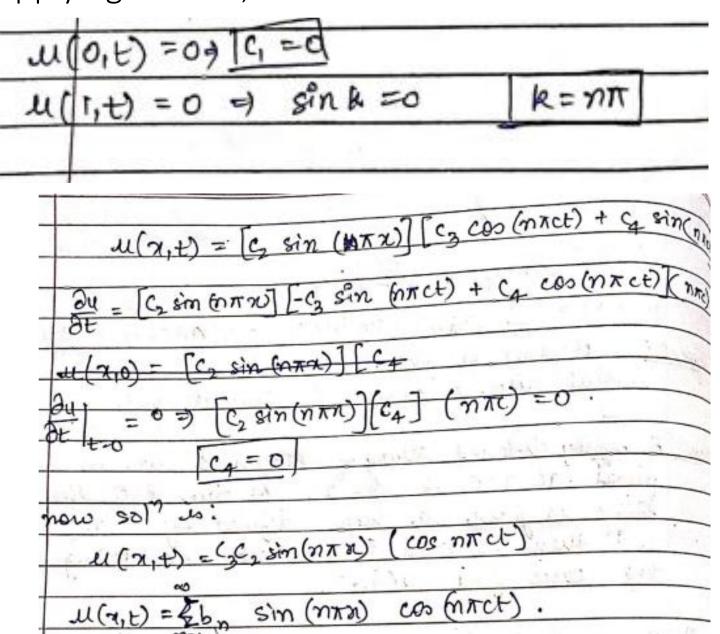
otle=0

Duest: A tightly stretched string of length 1 has its ends
fixed at z=0 & n=1. At time, f=0, the
string is given a shape defined by $u(z_0) = z(1-z)$ and then released from rest. Considering all
the cases find $u(z_0,t)$.

Solution Q2- Step 1,2 and 3

gan: The partial diff. used.	
Case I: (c, cos Kx + c2 sin Kn) (ca cos ckt + cy sinch
Care II:	Control to the second
Cense III:	*x*- uni 1 %
caseI is considerable onl	y as it is periodic.
- Court Court	(0,t) =0
. St	1 = 0 t=0
u(x	$0) = \chi - \chi^2$.

Applying cond. 1, 2and 3



Condition 4 (with use of Fourier half range sine series

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,	<i>o</i> •			117
=2 ((x - x2) sin	(DKK)	da	
J.	65	1000	4.31	
b= 4	[1-(-1)]		30	
N ₃ M ₃		1.2+		-
		ATTENDED TO		10
	D- 101 0			
	145 7	- 17 17 -		

Q.3. A tightly stretched string of length l with fixed end points is initially in equilibrium position. It is set vibrating by giving each point a velocity $v = \sin^3\left(\frac{px}{l}\right)$. Find the displacement.

Solution:

After Applying step 1, 2 & 3 and conditions 1,2 & 3-

The solution is –

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l}$$

Applying condition 4

$$\left(\frac{\partial u}{\partial x}\right)at(x,0) = \sin^3\left(\frac{px}{l}\right)$$

$$\sin^3\left(\frac{\pi x}{l}\right) = \left[\sum_{n=1}^{\infty} \frac{b_n n\pi c}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}\right] at \ t = 0$$

$$\Rightarrow \sin^3\left(\frac{px}{l}\right) = \sum_{n=1}^{\infty} \frac{b_n n\pi c}{l} \sin\frac{n\pi x}{l}$$

$$\Rightarrow \frac{1}{4} \left(3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right) \right) = \frac{b_1 \pi c}{l} \sin\frac{\pi x}{l} + \frac{2b_2 \pi c}{l} \sin\frac{2\pi x}{l} + \frac{3b_3 \pi c}{l} \sin\frac{3\pi x}{l}$$

+...

$$\Rightarrow b_1 = \frac{3l}{4c\pi}, b_3 = -\frac{l}{12c\pi}$$

$$u(x,t) = \frac{3l}{4c\pi} \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \frac{l}{12c\pi} \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l}$$