Network Theorems (AC)

OUTLINES

- Introduction to Network Theorems (AC)
- Thevenin Theorem
- Superposition Theorem
- Maximum Power Transfer Theorem

Network Theorems (AC) - Introduction

This module will deal with network theorems of ac circuit rather than dc circuits previously discussed. Due to the need for developing confidence in the application of the various theorems to networks with controlled (dependent) sources include independent sources and dependent sources. Theorems to be considered in detail include the superposition theorem, Thevinin's theorem, maximum power transform theorem.

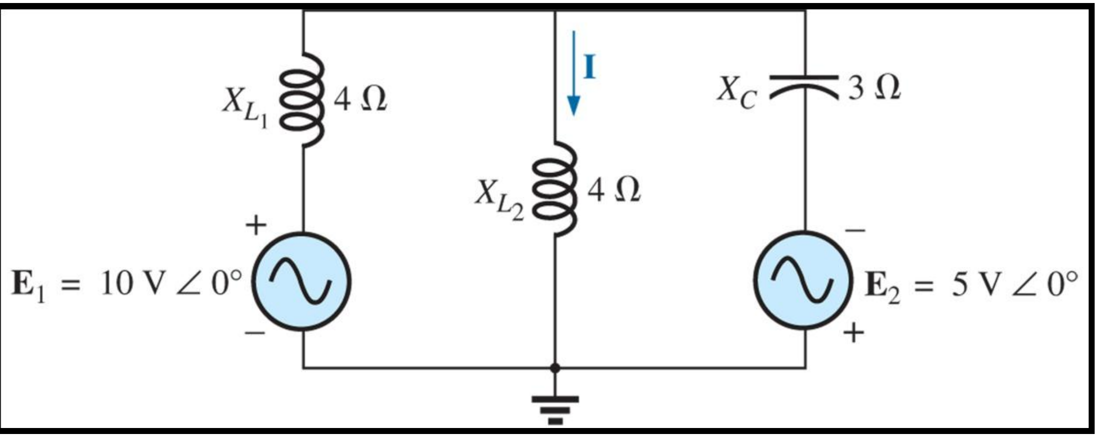
Superposition Theorem

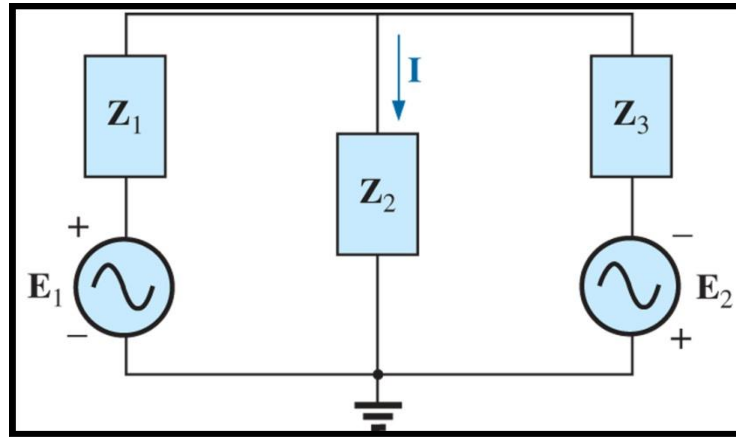
The **superposition theorem** eliminated the need for solving simultaneous linear equations by considering the effects of each source independently in previous module with dc circuits. To consider the effects of each source, we had to remove the remaining sources . This was accomplished by setting ***voltage sources to zero (short-circuit representation)*** and ***current sources to zero (open-circuit representation).*** The current through, or voltage across, a portion of the network produced by each source was then added algebraically to find the total solution for the current or voltage.

The only variation in applying this method to ac networks with independent sources is that we are now working with impedances and phasors instead of just resistors and real numbers.

Independent Sources

Ex. 1 Using the superposition theorem, find the current **I** through the 4Ω resistance (X_{L_2}) in Fig. below.





For the redrawn circuit,

$$Z_1 = +jX_{L_1} = j4\Omega$$

$$Z_2 = +jX_{L_2} = j4\Omega$$

$$Z_3 = -jX_C = -j3\Omega$$

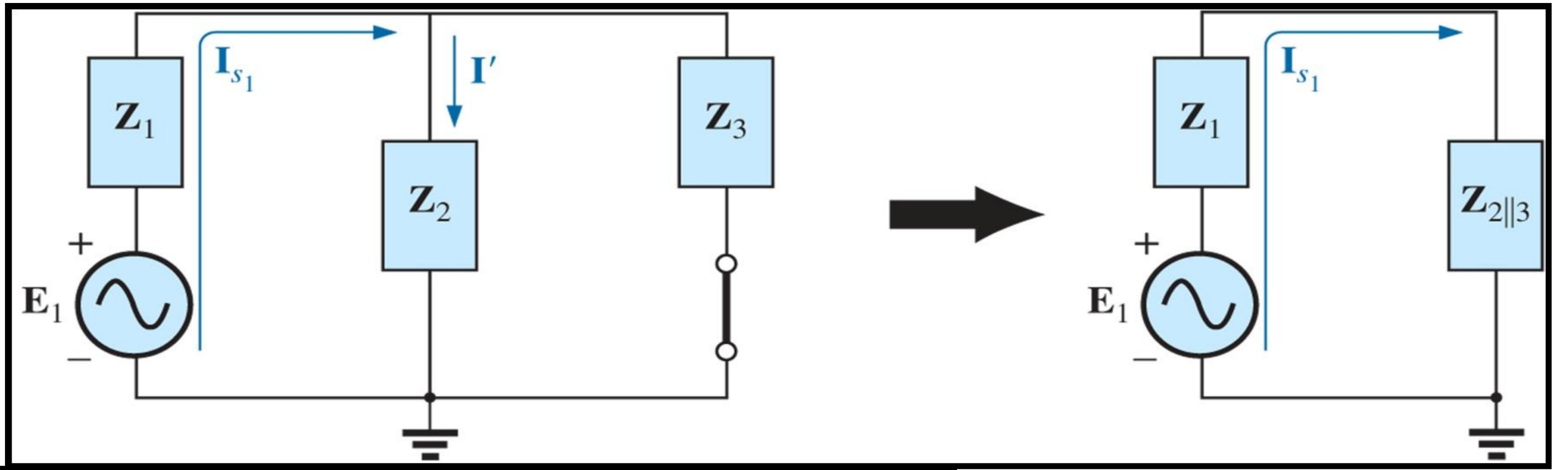
Considering the effects of the voltage source E_1, we have

$$Z_{2//3} = \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{(j4\Omega)(-j3\Omega)}{j4\Omega - j3\Omega}$$

$$= \frac{12\Omega}{j} = -j12\Omega = 12\Omega\angle -90°$$

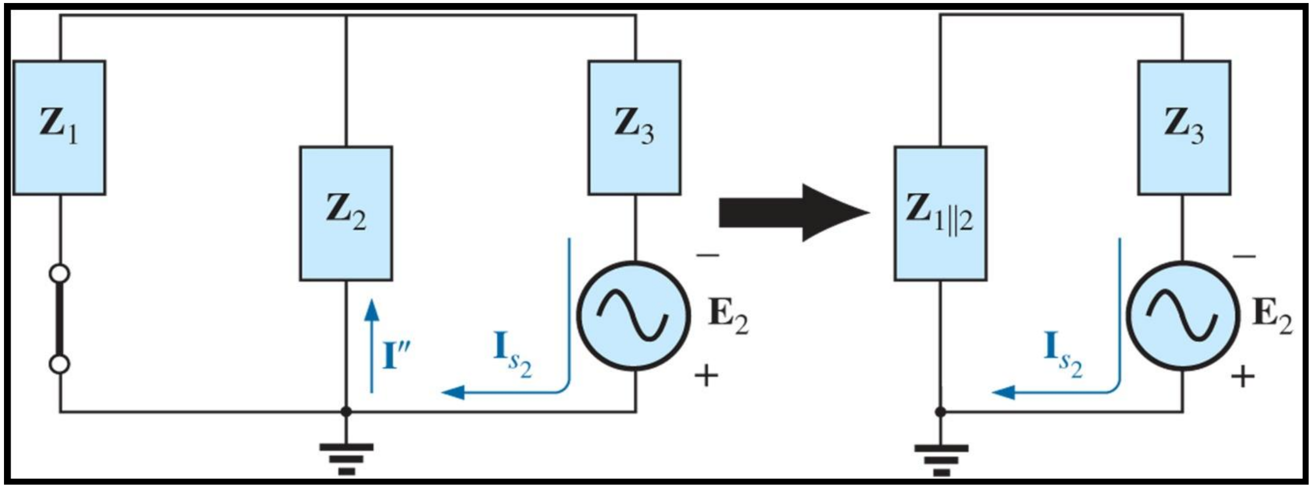
$$I_{s_1} = \frac{E_1}{Z_{2//3} + Z_1} = \frac{10V\angle 0°}{-j12\Omega + j4\Omega}$$

$$= \frac{10V\angle 0°}{-j12\Omega + j4\Omega} = \frac{10V\angle 0°}{8\Omega\angle -90°} = 1.25A\angle 90°$$



$$and \quad I' = \frac{Z_3 I_{s_1}}{Z_2 + Z_3} \quad (current\ divider\ rule)$$

$$= \frac{(-j3\Omega)(j1.25A)}{j4\Omega \text{ - } j3\Omega} = \frac{3.75A}{j1} = 3.75A\angle -90^\circ$$



Considering the effects of the voltage source

E_2 *, we have*

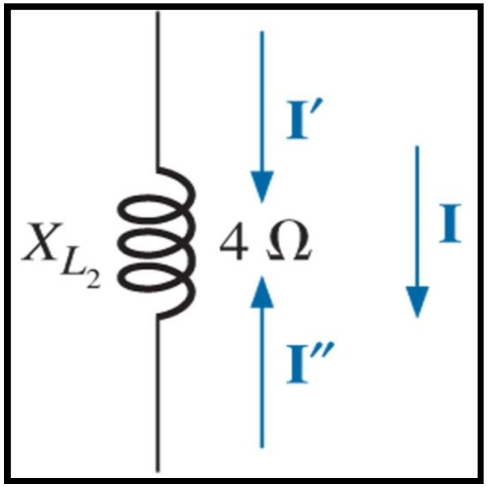
$$Z_{1//2} = \frac{Z_1}{N} = \frac{j4\Omega}{2} = j2\Omega$$

$$I_{s_2} = \frac{E_2}{Z_{1//2} + Z_3} = \frac{5V\angle 0^\circ}{j2\Omega - j3\Omega} = \frac{5V\angle 0^\circ}{1\Omega\angle -90^\circ} = 5A\angle 90^\circ$$

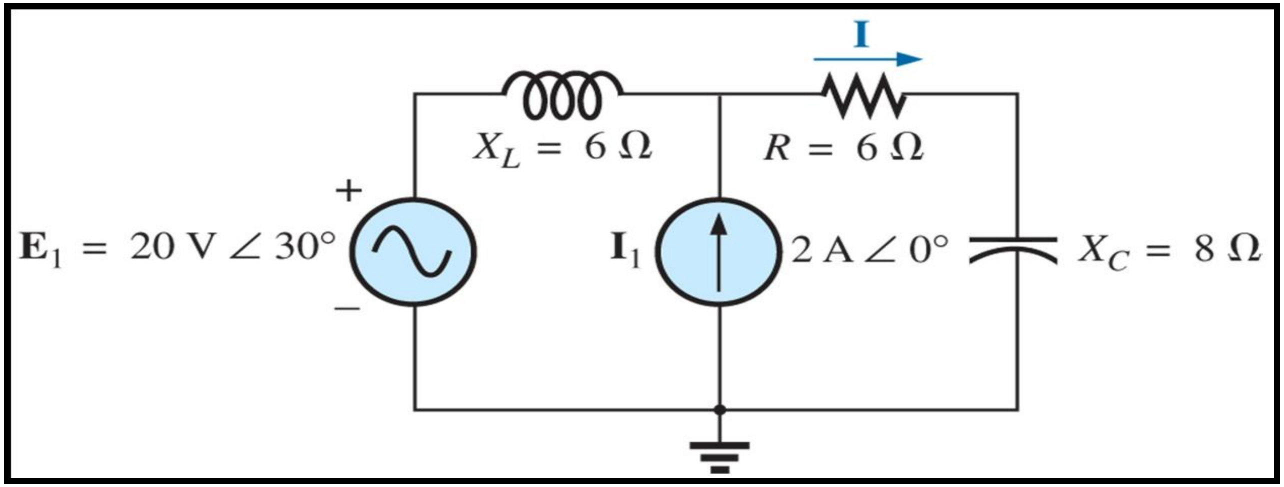
and $\quad I'' = \dfrac{I_{s_2}}{2} = 2.5A\angle 90^\circ$

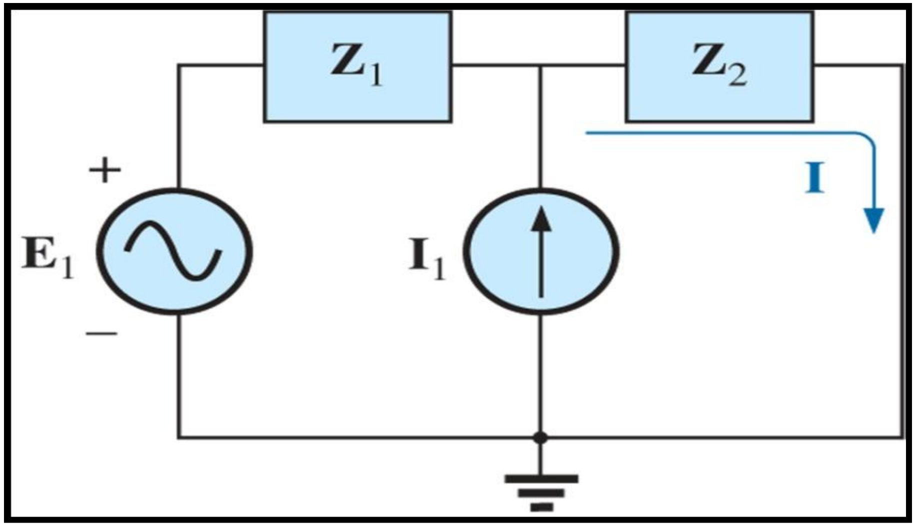
The resultant current through the 4Ω reactance X_{L_2} *is*

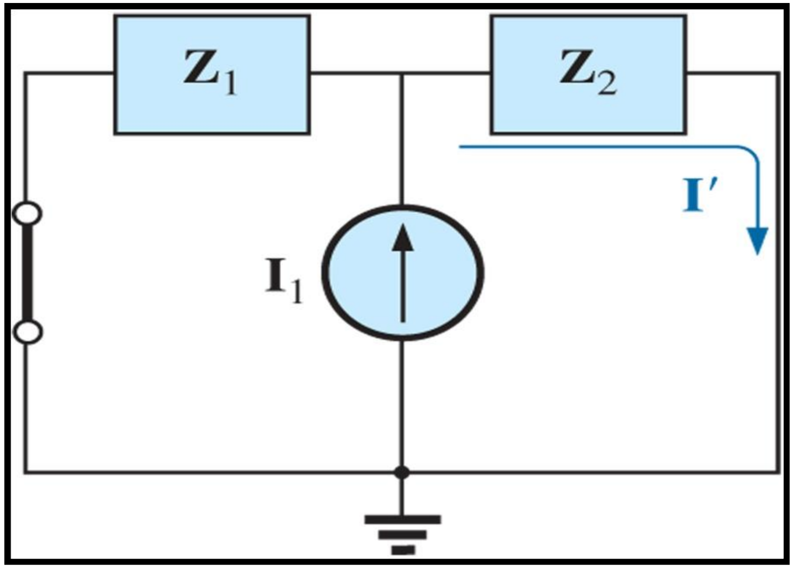
$$I = I' - I'' = 3.75A\angle -90^\circ$$
$$= -j3.75A - j2.50A$$
$$= -j6.25A = 6.25A\angle -90^\circ$$

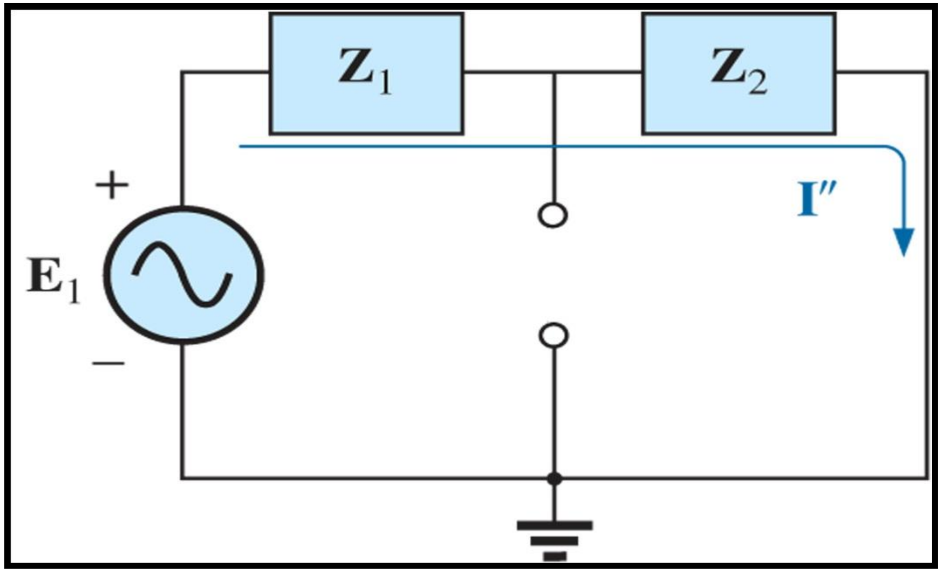


Ex. 2. Using the superposition, find the current **I** through the 6Ω resistor in Fig. shown below.









For the redrawn circuit,

$$Z_1 = j6\Omega \qquad Z_2 = 6\Omega - j8\Omega$$

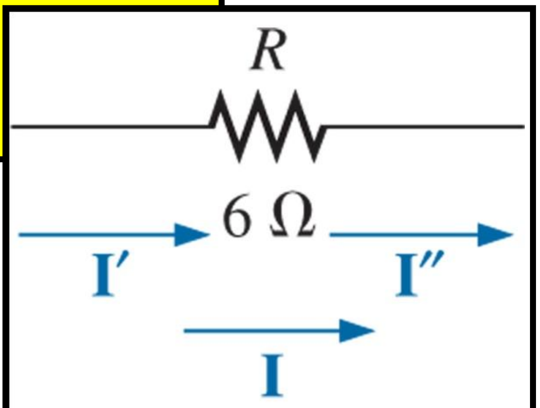
Consider the effects of the voltage source.

Applying the current divider rule, we have

$$I' = \frac{Z_1 I_1}{Z_1 + Z_2} = \frac{(6\Omega)(2A)}{j6\Omega + 6\Omega - j8\Omega}$$

$$= \frac{j12A}{6 - j2} = \frac{12A\angle 90^\circ}{6.32\angle -18.43^\circ}$$

$$= 1.9A\angle 108.43^\circ$$



Conseder the effects of the voltage source

Applying Ohm's law gives us

$$I'' = \frac{E_1}{Z_T} = \frac{E_1}{Z_1 + Z_2} = \frac{20V\angle 30^\circ}{6.32\Omega\angle -18.43^\circ}$$

$$= 3.16A\angle 48.43^\circ$$

The total current through the 6Ω *resistor is*

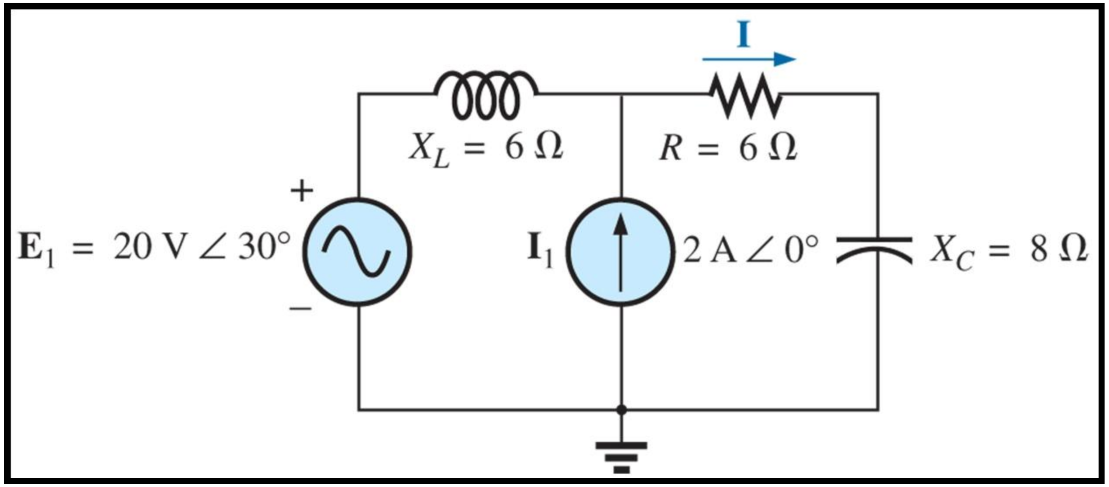
$$I = I' + I''$$

$$= 1.9A\angle 108.43^\circ + 3.16A\angle 48.43^\circ$$

$$= (-0.60A + j1.80A) + (2.10A + j2.36A)$$

$$= 1.50A + j4.16A = 4.42A\angle 70.2^\circ$$

Ex. 3 Using the superposition Theorem, find the voltage across the 6Ω resistor in Fig. shown below. Check the results against $V_{6\Omega} = I(6\Omega)$, where I is the current found through the 6Ω resistor in Example 2.



For the current source,

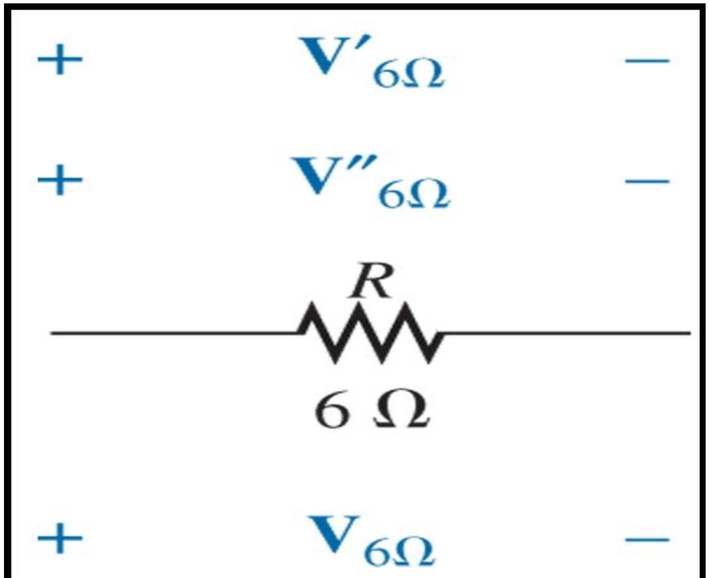
$$V'_{6\Omega} = I'(6\Omega)$$
$$= (1.9A\angle 108.43°)(6\Omega)$$
$$= 11.4V\angle 108.43°$$

For the voltage source,

$$V''_{6\Omega} = I''(6\Omega)$$
$$= (3.16A\angle 48.43°)(6\Omega)$$
$$= 18.96V\angle 48.43°$$

For the total voltage the 6Ω *resistor is*

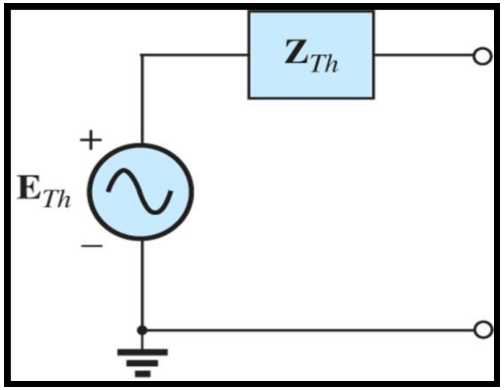
$$V_{6\Omega} = V'(6\Omega) + V''(6\Omega)$$
$$= 11.4V\angle 108.43° + 18.96V\angle 48.43°$$
$$= (-3.60V + j10.82V) + (12.58V + j14.18V)$$
$$= 8.98V + j25.0V = 26.5V\angle 70.2°$$



Check the result, we have

$$V_{6\Omega} = I(6\Omega) = (4.42A\angle 70.2°)(6\Omega)$$
$$= 26.5V\angle 70.2° \quad (checks)$$

Thevenin's Theorem



Since the reactances of a circuit are frequency dependent, the Thevinin circuit found for a particular network is applicable only at one frequency. The steps required to apply this method to dc circuits are repeated here with changes for sinusoidal ac circuits. As before, the only change is the replacement of the term resistance with impedance. Again, dependent and independent sources are treated separately.

Independent Sources

1. ***Remove that portion of the network across which the Thevenin equivalent circuit is to be found.***

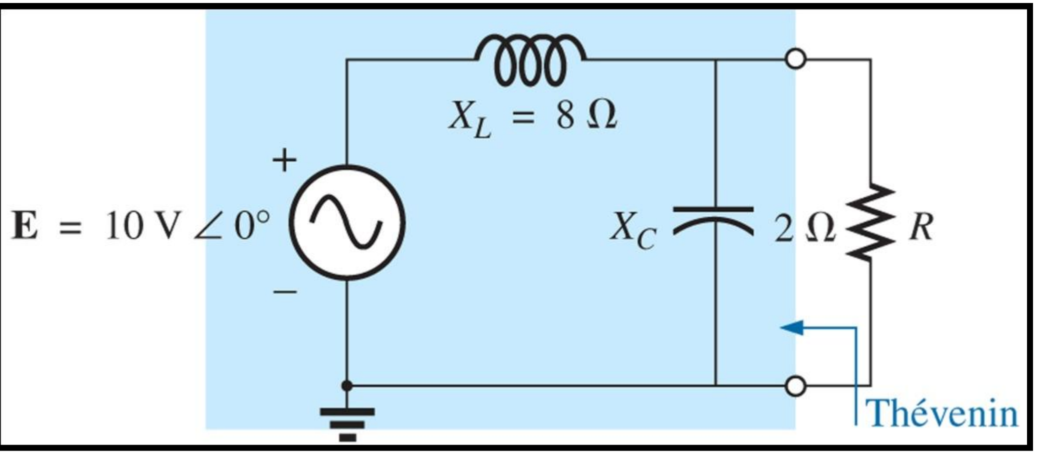
2. ***Mark (o, •, and so on) the terminal of the remaining two-terminal network.***

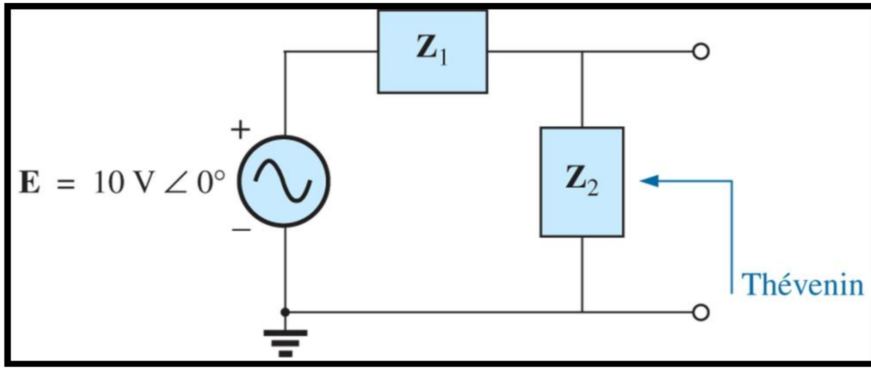
3. ***Calculate Z_{TH} by first setting all voltage and current sources to zero (short circuit and open circuit, respectively) and then finding the resulting impedance between the marked terminals.***

4. ***Calculate E_{TH} by first replacing the voltage and current sources and then finding the open-circuit voltage between the marked terminals.***

5. ***Draw the Thevenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the Thevinin equivalent circuit.***

Ex. 4. Find the Thevenin equivalent circuit for the network external to resistor R in Fig. shown below.





Steps 1 *and* 2:

$$Z_1 = jX_L = j8\Omega \qquad Z_2 = -jX_L = -j2\Omega$$

Step 3:

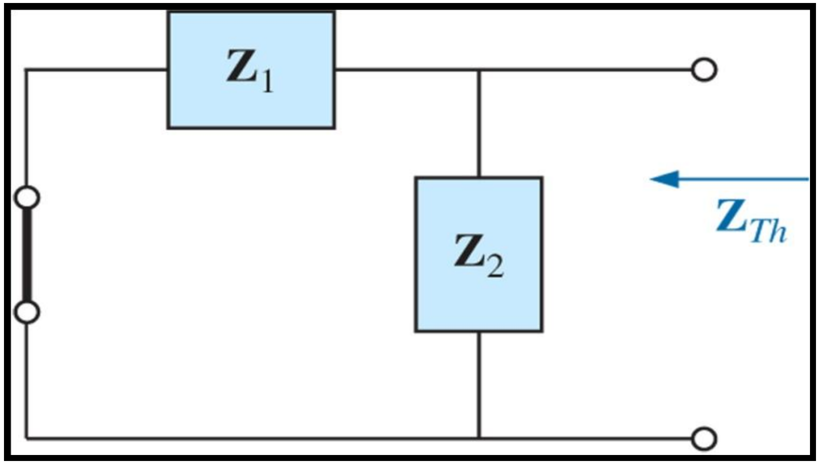
$$Z_{Th} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(j8\Omega)(-j2\Omega)}{j8\Omega - j2\Omega}$$

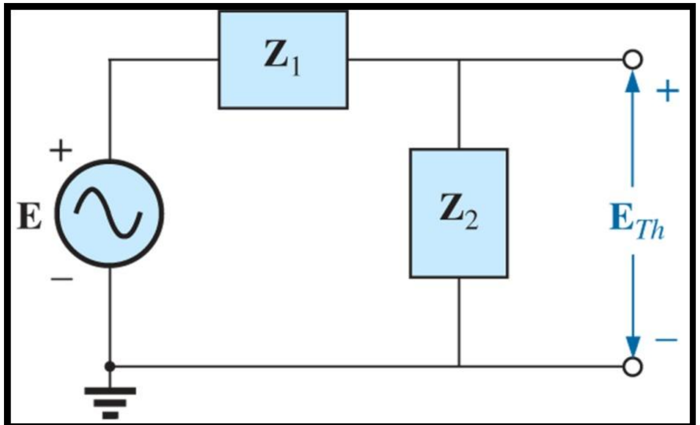
$$= \frac{-j^2 16\Omega}{j6\Omega} = \frac{16\Omega}{6\angle 90°} = 2.67\Omega\angle -90°$$

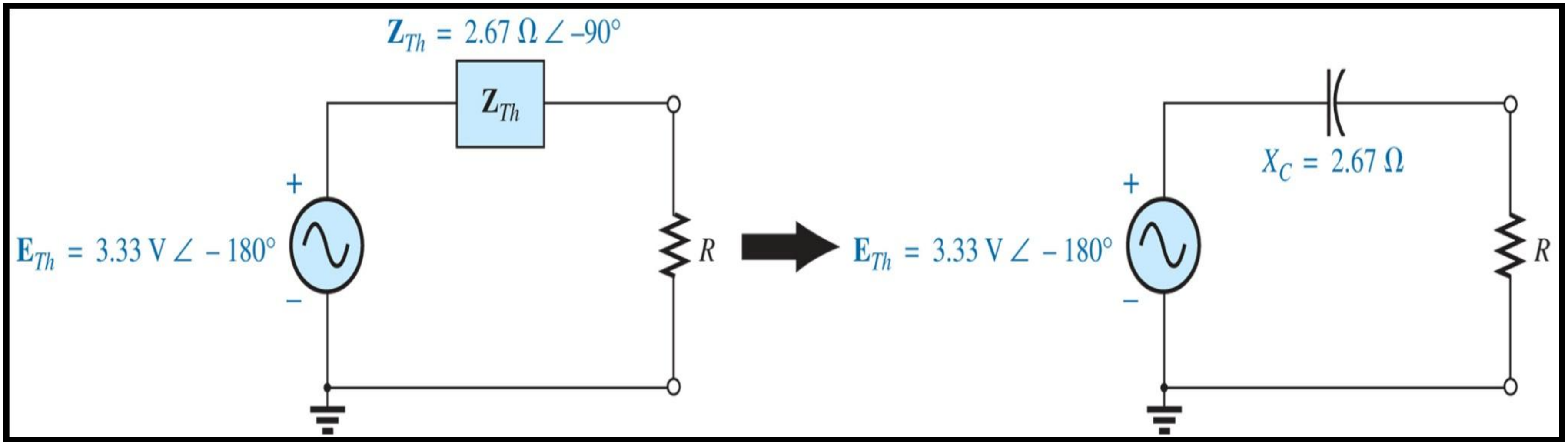
Step 4:

$$E_{Th} = \frac{Z_2 E}{Z_1 + Z_2} \qquad (\textit{voltage divider rule})$$

$$= \frac{(-j2\Omega)(10V)}{j8\Omega - j2\Omega} = \frac{-j20V}{j6} = 3.33V\angle -180°$$

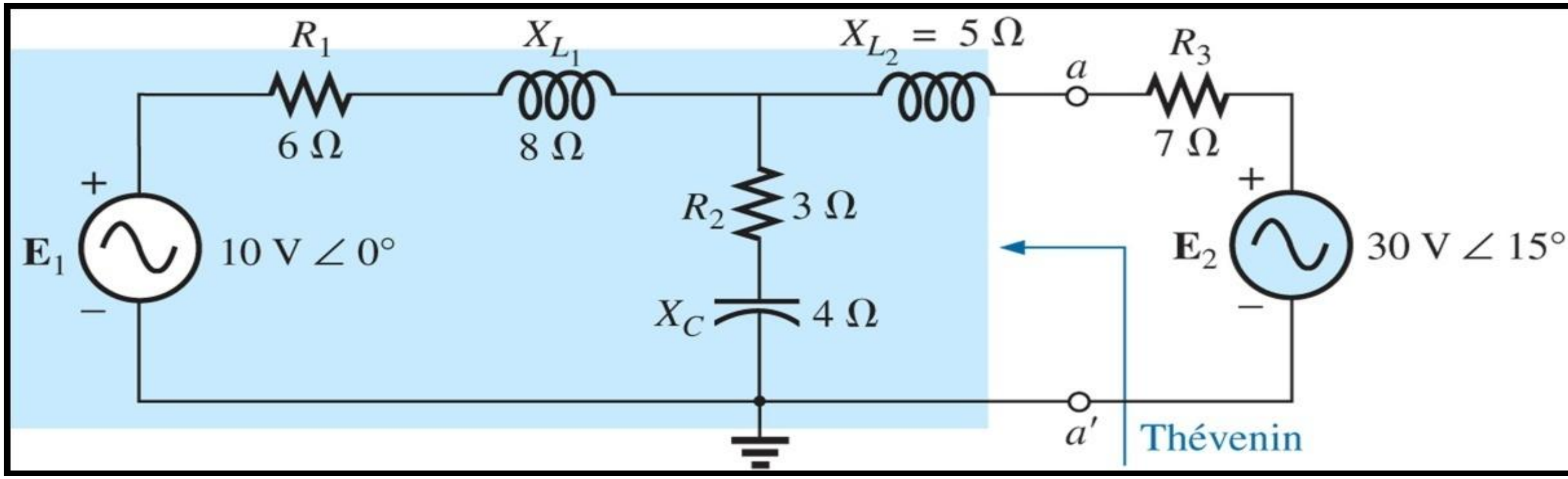


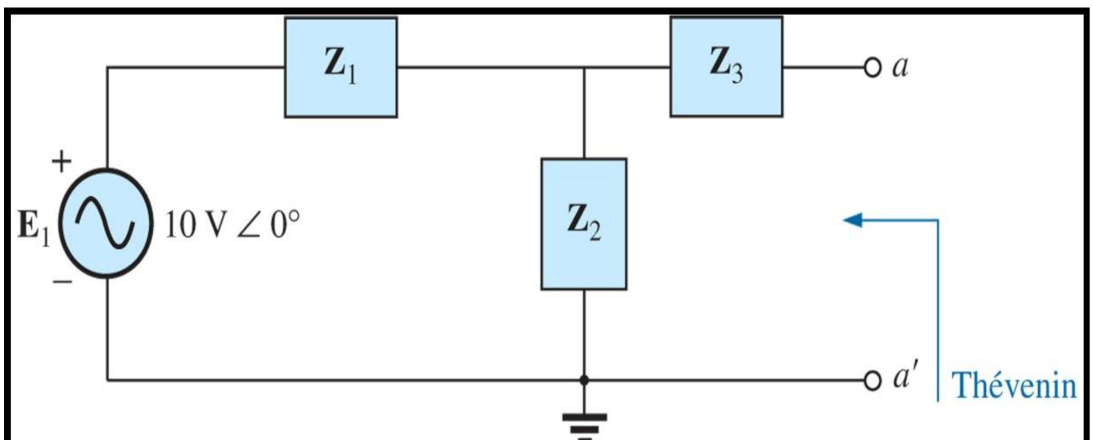




$\mathbf{Z}_{Th} = 2.67\ \Omega \angle -90°$
$\mathbf{Z}_{Th}$
+
$\mathbf{E}_{Th} = 3.33\ \text{V} \angle -180°$
–
R
$X_C = 2.67\ \Omega$
+
$\mathbf{E}_{Th} = 3.33\ \text{V} \angle -180°$
–
R

Ex. 5. Find the Thevenin equivalent circuit for the network external to resistor to branch a-a´ in Fig. shown below.





Steps 1 and 2: Note the reduced complexity with subscripted impedances:

$$Z_1 = R_1 + jX_{L_1} = 6\Omega + j8\Omega$$

$$Z_2 = R_2 - jX_C = 3\Omega - j4\Omega$$

$$Z_3 = +jX_{L_2} = j5\Omega$$

$$\text{Step 3:}$$

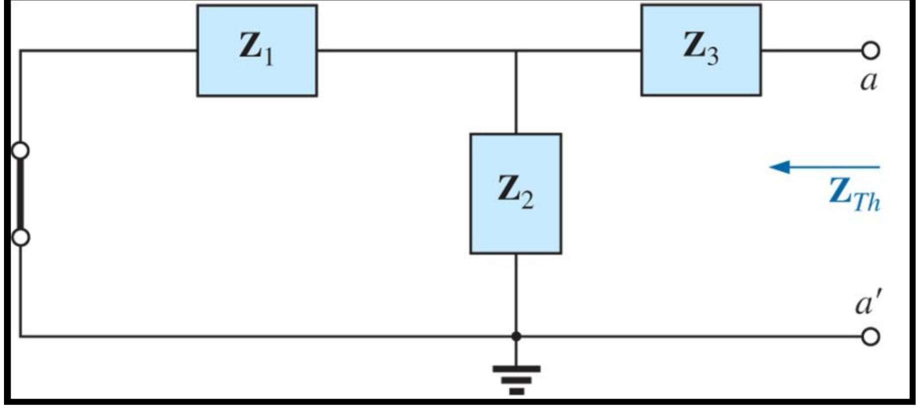
$$Z_{Th} = Z_3 + \frac{Z_1 Z_2}{Z_1 + Z_2}$$

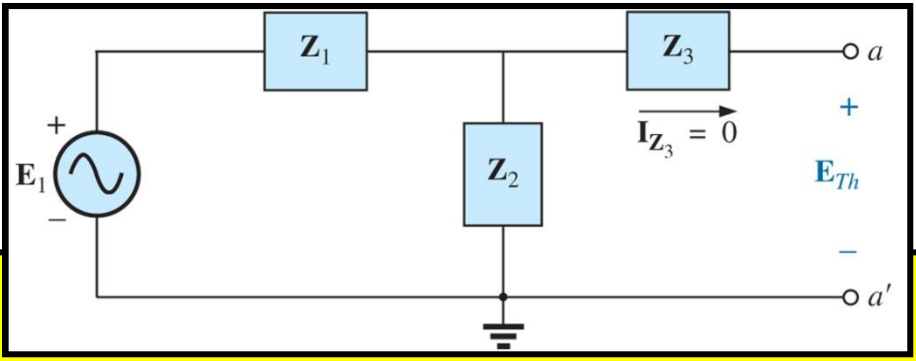
$$= j5\Omega + \frac{(10\Omega\angle 53.13^\circ)(5\Omega\angle -53.13^\circ)}{(6\Omega + j8\Omega) + (3\Omega - j4\Omega)}$$

$$= j5 + \frac{50\angle 0^\circ}{9 + j4} = j5 + \frac{50\angle 0^\circ}{9.85\angle 23.96^\circ}$$

$$= j5 + 5.08\angle -23.96^\circ = j5 + 4.64 - j2.06$$

$$= 4.64\Omega + j2.94\Omega = 5.49\Omega\angle 32.36^\circ$$



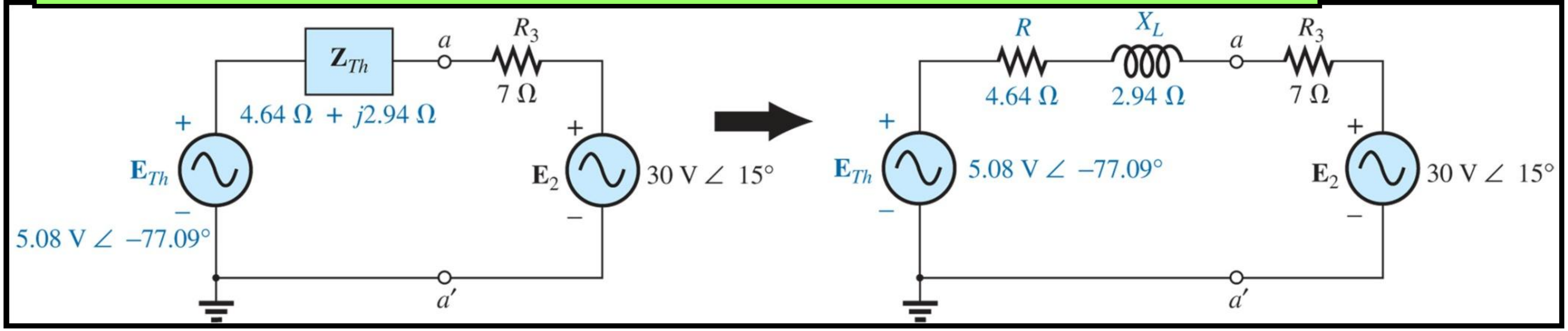


Step 4: Since $a - a'$ is an open circuit,

$I_{Z_3} = 0$. Then E_{Th} is the voltage drop across Z_2:

$$E_{Th} = \frac{Z_2 E}{Z_2 + Z_1} \quad (\textit{voltage divider rule}) = \frac{(5\Omega\angle -53.13^\circ)(10V\angle 0^\circ)}{9.85\Omega\angle 23.96^\circ} = \frac{50V\angle -53.13^\circ}{9.85\angle 23.96^\circ} = 5.08V\angle -77.09^\circ$$

Step 5: The Thevenin equivalent circuit is shown in Fig. shown below.

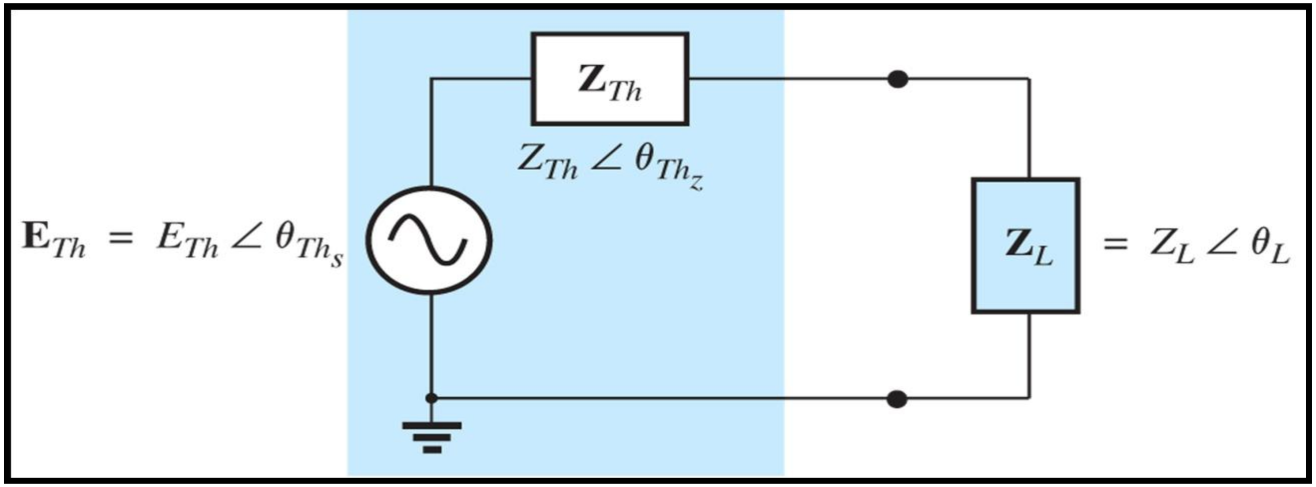


Maximum Power Transfer Theorem

When applied to ac circuits, the maximum power transfer theorem states that

maximum power will be delivered to a load when the load impedance is the conjugate of the Thevenin impedance across its terminals.

That is, for Fig. shown below, for maximum power transfer to the load,



$$Z_L = Z_{Th} \quad and \quad \theta_L = -\theta_{Th_Z}$$

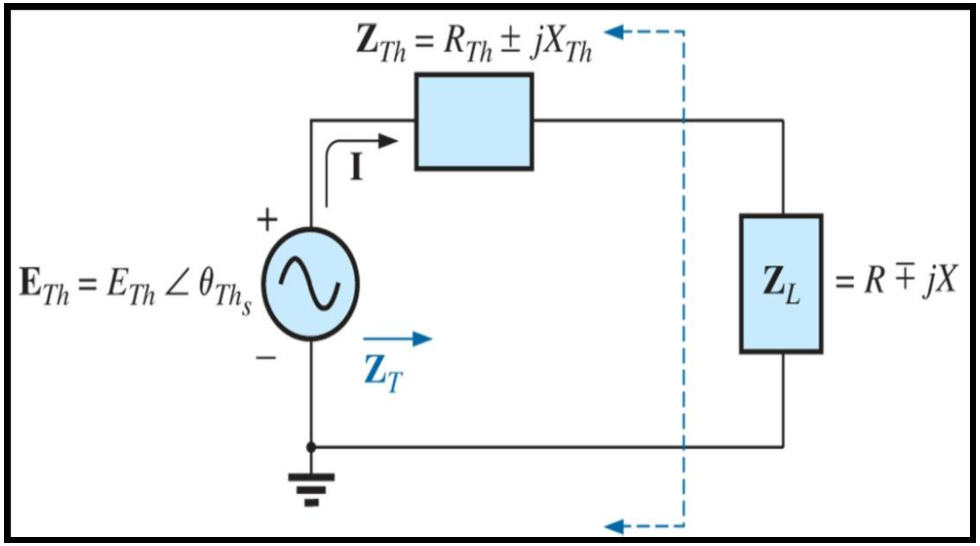
or, in rectangular form,

$$R_L = R_{Th} \quad and \quad \pm jX_{load} = \mp jX_{Th}$$

The conditions just mentioned will make the total impedance of the circuit appear purely resistive, as indicated in Fig.

$$Z_T = (R \pm jX) + (R \mp jX)$$

and $\quad Z_T = 2R$



Since the circuit is purely resistive, the power factor of the circuit under maximum power conditions is 1: that is,

$$\text{PF} = 1 \quad \text{(maximum power transfer)}$$

The magnitude of the current I in *above* Fig. is

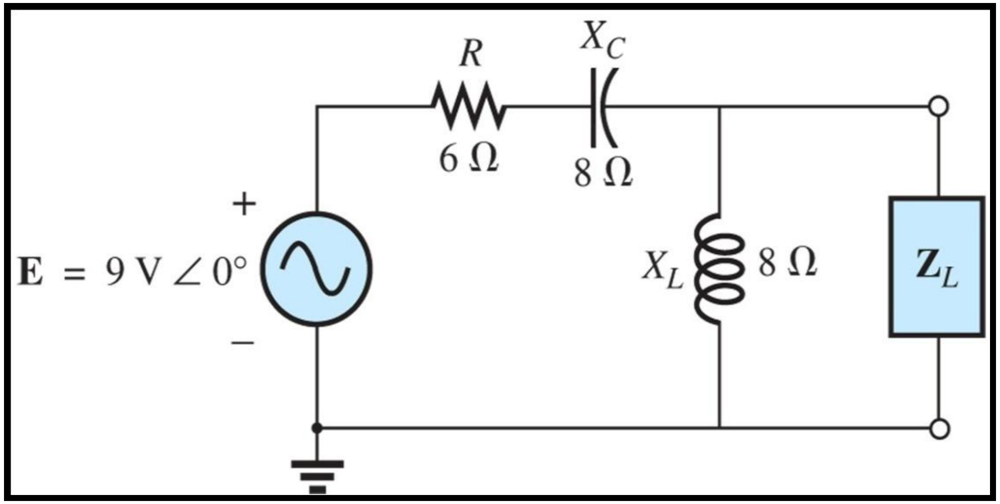
$$I = \frac{E_{Th}}{Z_T} = \frac{E_{Th}}{2R}$$

The maximum power to the load is

$$P_{max} = I^2R = \left(\frac{E_{Th}}{2R}\right)^2 R$$

and $\quad P_{max} = \dfrac{E_{Th}^2}{4R}$

Ex. 6 Find the load impedance in Fig. shown below for maximum power to the load, and find the maximum power.



Determine Z_{Th} :

$$Z_1 = R - jX_C = 6\Omega - j8\Omega = 10\Omega\angle -53.13^\circ$$

$$Z_2 = +jX_L = j8\Omega$$

$$Z_{Th} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(10\Omega\angle -53.13^\circ)(8\Omega\angle 90^\circ)}{6\Omega - j8\Omega + j8\Omega}$$

$$= \frac{80\Omega\angle 36.87^\circ}{6\angle 0^\circ} = 13.33\Omega\angle 36.87^\circ = 10.66\Omega + j8\Omega$$

and $Z_L = 13.33\Omega\angle -36.87^\circ = 10.66\Omega - j8\Omega$

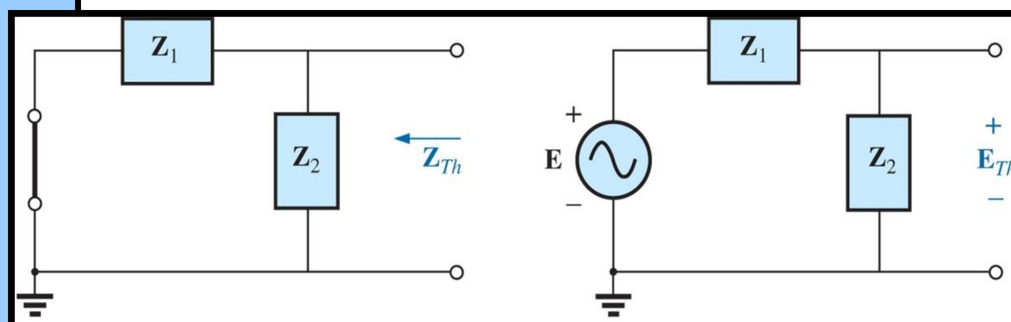
To find the maximum power, we must find

$$E_{Th} = \frac{Z_2 E}{Z_2 + Z_1} \quad (\textit{voltage divider rule})$$

$$= \frac{(8\Omega\angle 90^\circ)(9V\angle 0^\circ)}{j8\Omega + 6\Omega - j8\Omega} = \frac{72V\angle 90^\circ}{6\angle 0^\circ} = 12V\angle 90^\circ$$

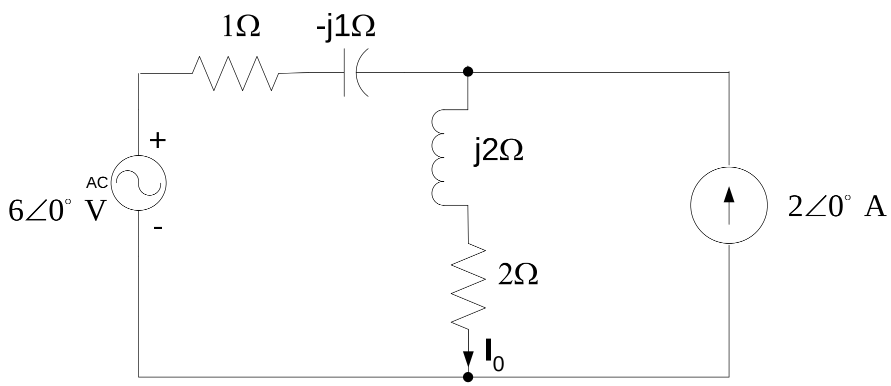
Then $P_{\max} = \frac{E_{Th}^2}{4R} = \frac{(12V)^2}{4(10.66\Omega)}$

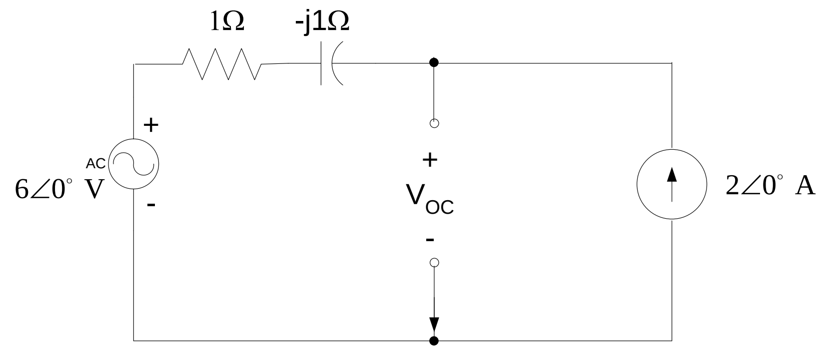
$$= \frac{144}{42.64} = 3.38\,W$$



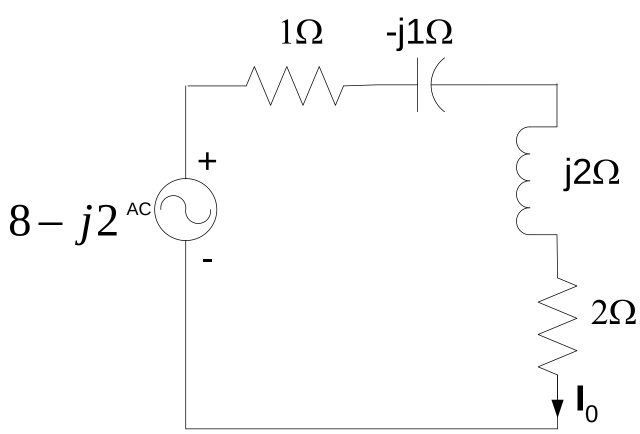
(a) (b)

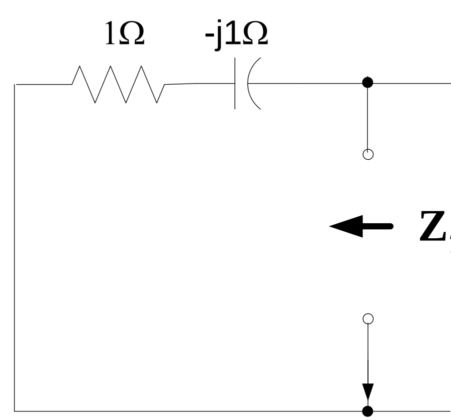
Ex. 7 Using Thevenin's theorem find the current Io for the circuit shown in the fig. below.





$$\mathbf{V}_{OC} = 6 + 2(1 - j) = 8 - j2$$



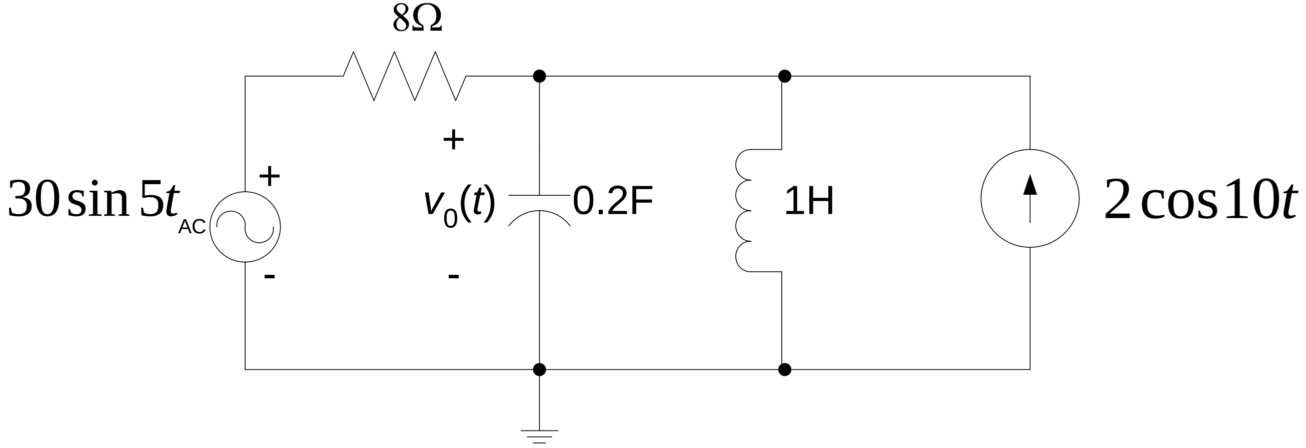


$$\mathbf{Z}_{TH} = 1 - j1$$

$$\mathbf{I}_0 = \frac{8 - j2}{1 - j1 + j2 + 2} = \frac{8 - j2}{3 + j1} = \frac{8.246\angle -14.04^\circ}{3.162\angle 18.43^\circ}$$

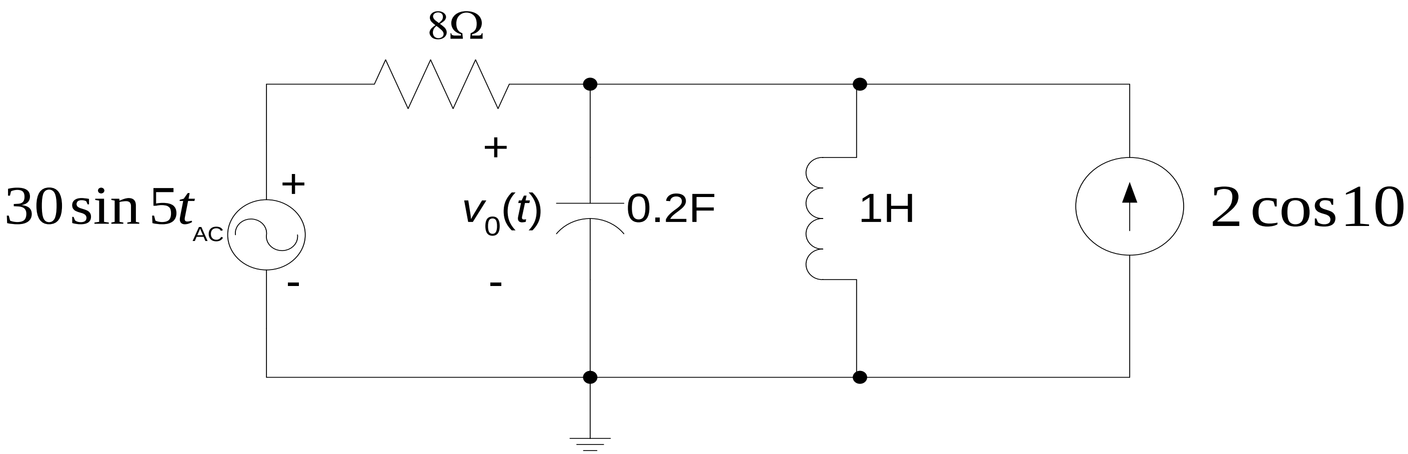
$$\mathbf{I}_0 = 2.608\angle -32.47^\circ$$

Ex. 8 Solve the following problem for Vo(t).

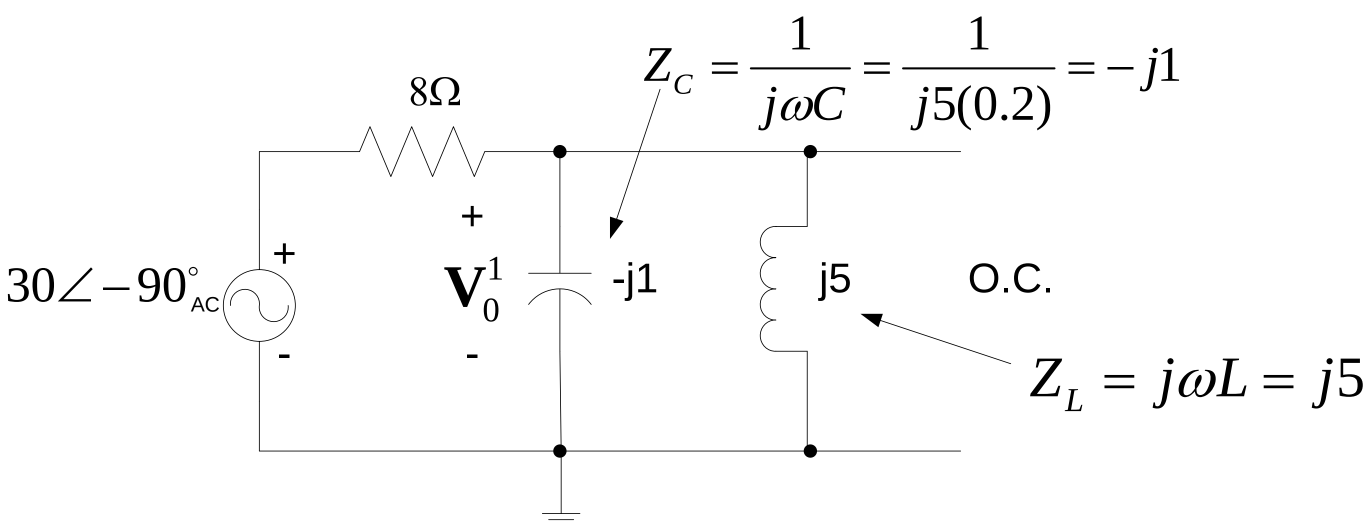


Note that the voltage source and the current source have ***two different frequencies***. Thus, if we want to use phasors, the only way we've solved sinusoidal steady-state problems, we MUST use superposition to solve this problem. We will consider each source acting alone, and then find $v_0(t)$ by superposition.

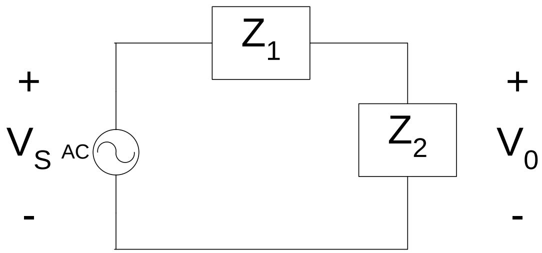
Remember that $\sin \omega t = \cos\left(\omega t - 90^\circ\right)$

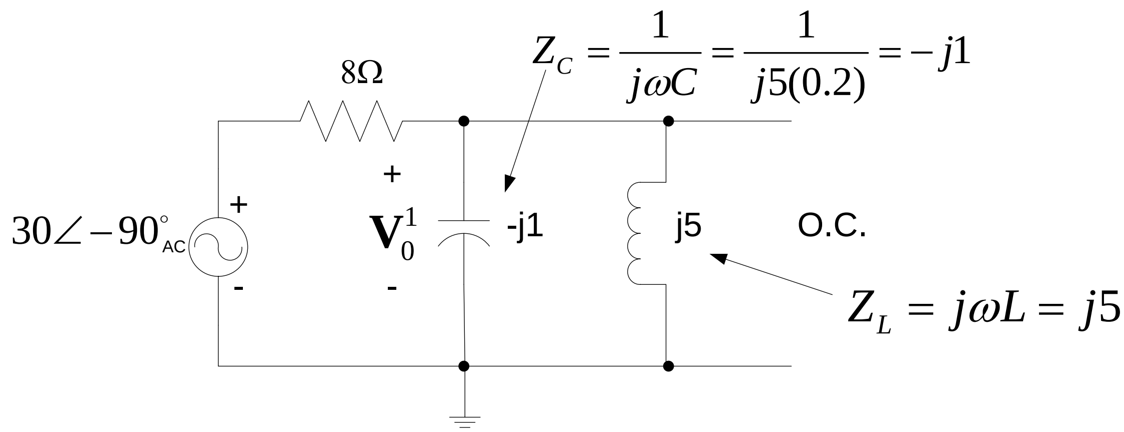


Consider first the $30\sin 5t$ acting alone.
Since, $30\sin 5t = 30\cos\left(5t - 90^\circ\right)$,we have w = 5 and



Use voltage division





$$\mathbf{V}_0^1 = \frac{Z_2}{Z_1 + Z_2}\mathbf{V}_S$$

$$Z_2 = \frac{(-j1)(j5)}{-j1 + j5} = \frac{5}{j4} = -j1.25$$

$$Z_1 = 8$$

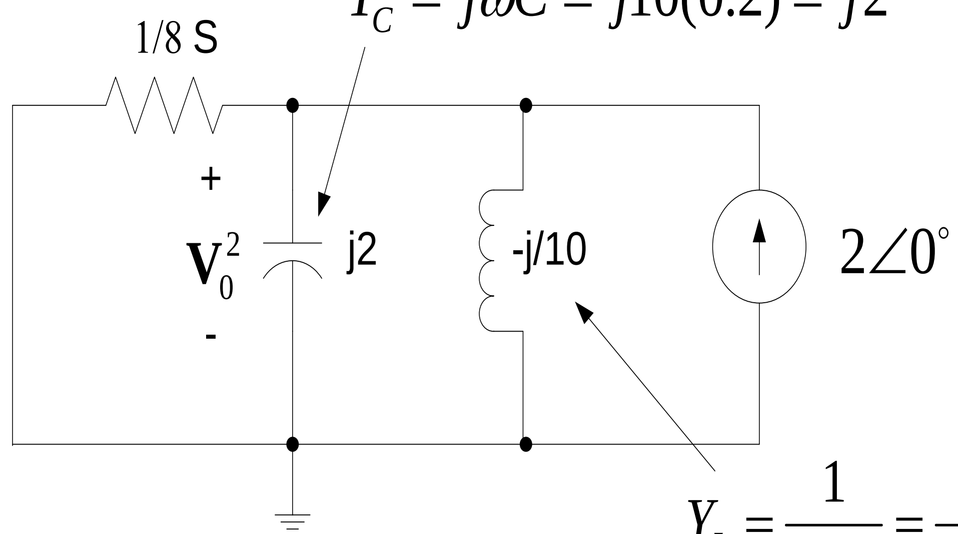
$$\mathbf{V}_0^1 = \frac{-j1.25}{8 - j1.25}\left(30\angle -90^\circ\right) = \frac{1.25\angle -90^\circ}{8.097\angle -8.881^\circ}\left(30\angle -90^\circ\right)$$

$$\mathbf{V}_0^1 = 4.631\angle -171.1^\circ$$

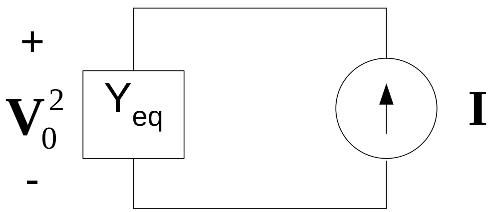
$$v_0^1(t) = 4.631\cos\left(5t - 171.12^\circ\right) = 4.631\sin\left(5t - 81.12^\circ\right)$$

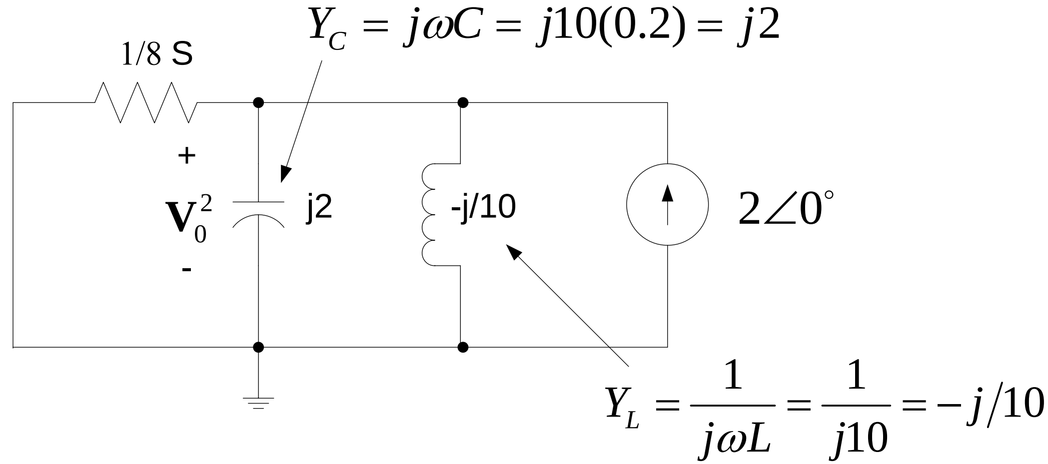
Now consider first the $2\cos 10t$ acting alone.
We have ω = 10 and

$$Y_C = j\omega C = j10(0.2) = j2$$



$$Y_L = \frac{1}{j\omega L} = \frac{1}{j10} = -j/10$$





$$\mathbf{Y}\mathbf{V} = \mathbf{I} \qquad \mathbf{V}_0^2 = \frac{\mathbf{I}}{\mathbf{Y}}$$

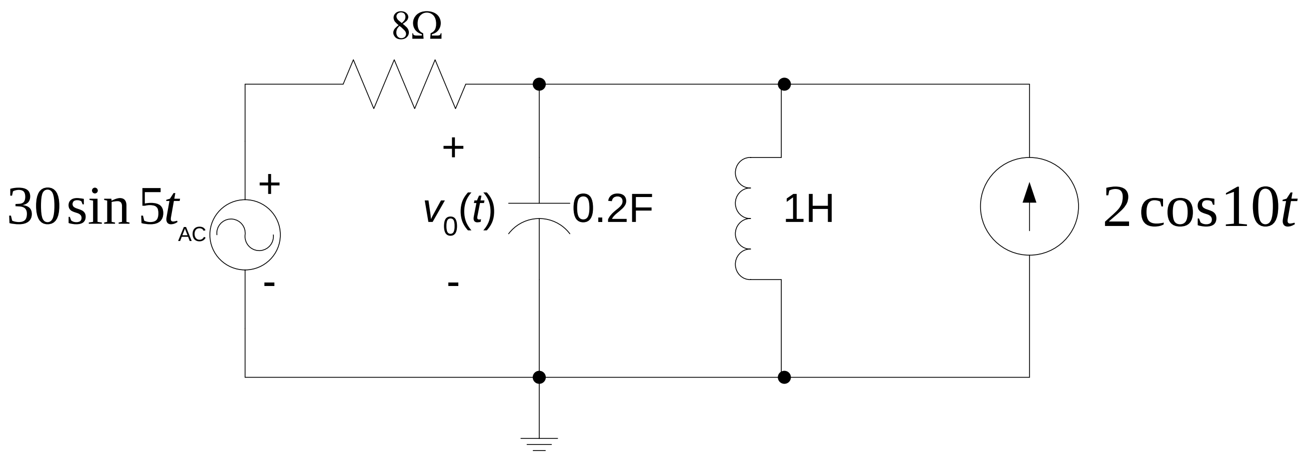
For a parallel combination of Y's we have

$$\mathbf{Y}_{eq} = \sum \mathbf{Y}_i = 1/8 + j2 - j0.1 = 0.125 + j1.90$$

$$\mathbf{Y}_{eq} = 1.904\angle 86.24^\circ$$

$$\mathbf{V}_0^2 = \frac{2\angle 0^\circ}{1.904\angle 86.24^\circ} = 1.05\angle -86.24^\circ$$

$$v_0^2(t) = 1.05\cos\left(10t - 86.24^\circ\right)$$



$$v_0^1(t) = 4.631\sin\left(5t - 81.12^\circ\right)$$

$$v_0^2(t) = 1.05\cos\left(10t - 86.24^\circ\right)$$

By superposition $$v_0(t) = v_0^1(t) + v_0^2(t)$$

$$v_0(t) = 4.631\sin\left(5t - 81.12^\circ\right) + 1.05\cos\left(10t - 86.24^\circ\right)$$