Complex Variables

- The solution to the equation $x^2 = -1$; $x^2 + 4 = 0$ led to the introduction of complex numbers.
- A complex no z is an ordered pair (x, y) of real numbers, written as $z = x + iy$

real part — $\text{Re } z$; Imaginary part — $\text{Im } z$

where $i^2 = -1$.

- Complex Variable — If x and y are two real variables then the variable of the form $z = x + iy$ is called complex variable.

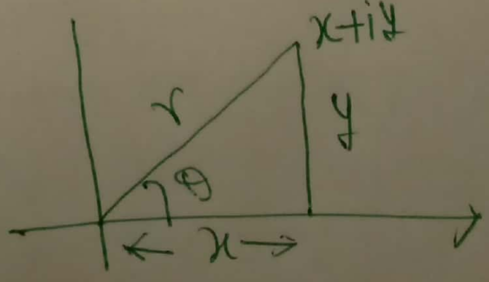
* If $z = x + iy$; then $\bar{z} = x - iy$

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

* $e^{i\theta} = \cos\theta + i\sin\theta$

* $z = x + iy = re^{i\theta} = r\cos\theta + ir\sin\theta$

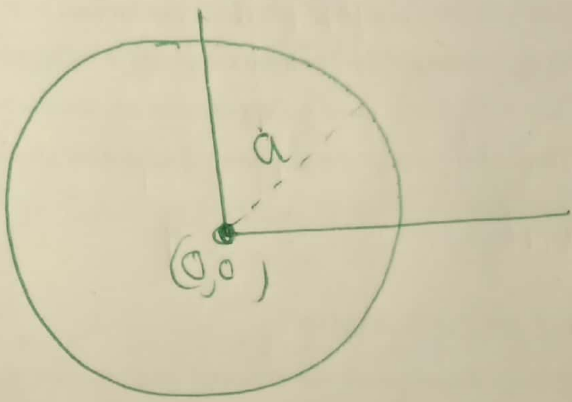
$$r = \sqrt{x^2 + y^2} \; ; \; \theta = \tan^{-1}(y/x)$$



$$|z| = a \Rightarrow \sqrt{x^2+y^2} = a$$

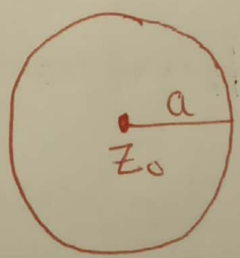
$$\Rightarrow x^2+y^2 = a^2$$

i.e. $|z| = a$ ~~represents~~ is an eq. of circle with center ~~of~~ at $(0,0)$ and radius a



$|z-z_0| = a \Rightarrow (x-x_0)^2+(y-y_0)^2 = a^2$

is an eq. of circle with center at (x_0, y_0) and radius a



Complex Function — A function f defined on $\mathbb{C}$ is a rule that assigns every z in $\mathbb{C}$ a complex no w, ~~called the~~ written as—

$$w = f(\underbrace{z}_{\text{Complex variable}})$$

If to each value of z, there corresponds ~~only~~ one and only one value of w, then w is said to be single valued function of z otherwise multivalued.

$\odot$ $w = f(z) = u(x,y) + i\,v(x,y)$

$\Downarrow$

$u(x,y)$ and $v(x,y)$ are real functions

examples –

1. Let $w = f(z) = z^2 + 3z$ find u and v and calculate the value of f at $z = 1+3i$.

Sol →

$$w = z^2 + 3z = (x+iy)^2 + 3(x+iy)$$

$$= x^2 - y^2 + 2ixy + 3x + i3y$$

$$= (x^2 - y^2 + 3x) + i(2xy + 3y)$$

Comparing with

$$w = u + iv$$

$$u = x^2 - y^2 + 3x$$

$$v = 2xy + 3y$$

w at $1+3i$ means w at $x=1, y=3$

$$w = (1^2 - 9 + 3) + i(2\times1\times3 + 3\times3)$$

$$= \underline{-5 + 15i}$$

② Try your own →

Let $w = 2iz + 6\bar{z}$. Find u and v and value of f at $z = \frac{1}{2} + 4i$.

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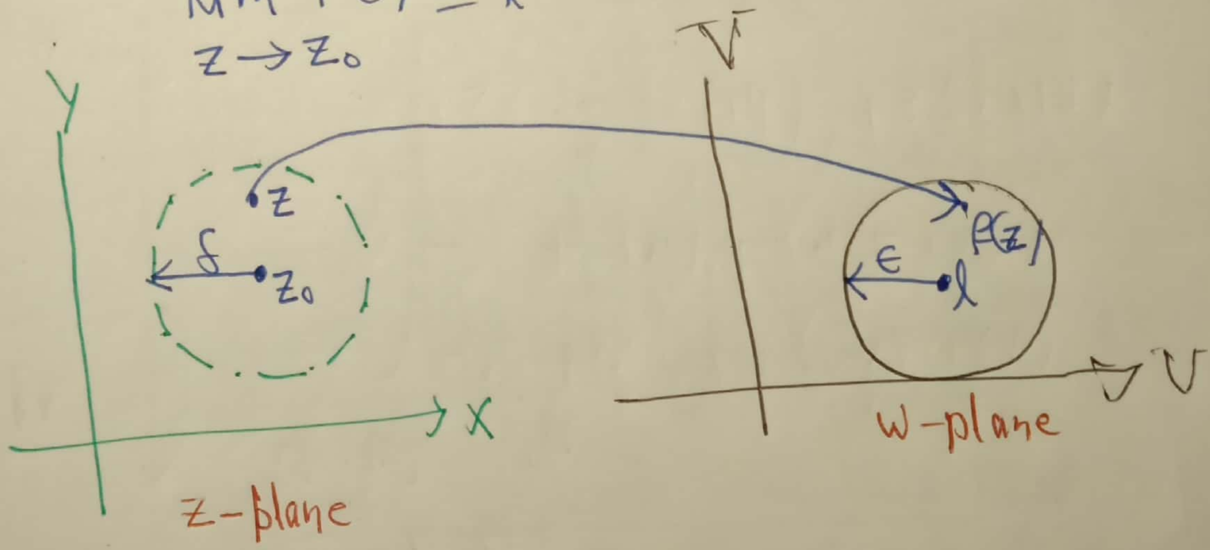
Limit –

A function $w = f(z)$ is said to tend to limit l as z approaches a point z_0, if for every real ϵ, we can find a positive real δ such that $|f(z) - l| < \epsilon$ for $|z - z_0| < \delta$.

written as –

$$\lim_{z \to z_0} f(z) = l$$



Continuity of f(z) – A function $w = f(z)$ is said to be continous at $z = z_0$, if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- f(z) is said to be continous in a domain if it is continous at each point of this domain.

Derivative—

derivative of complex function $w = f(z)$ at z_0 is —

$$f'(z_0) = \left(\frac{dw}{dz}\right)_{\text{at } z = z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Imp. Note → Differentiability of z_0 means that, along whatever path z approaches z_0, ∴ $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ always approaches a certain value and all these values are equal.

(Recall (limit, differentiability) in case of functions of several variables in Maths -1)

Ex. 1. Check whether the function $f(z) = z^2$ is differentiable at any point?

Sol →

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z.\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z)$$

$$= 2z \Rightarrow f(z) \text{ is differentiable for all } z.$$

Ex 2 P.T. $\bar{z}$ is not differentiable at any z.

Sol.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

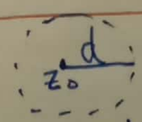
$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

which is path dependent so limit doesn't exist $\Rightarrow$ $f'(z)$ does not exist at any z.

Analytic Function

$\boxed{}$ → $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

Neighbourhood of a point z_0 →



$$N_d(z_0) = N(d, z_0) = \{ z : |z - z_0| < d \}$$

→ $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighbourhood of z_0.

(i) Entire function — If $f(z)$ is ~~differentiable~~ analytic at every point in a complex plane then $f(z)$ is called entire function.

e.g. e^z, z^2, z^3.

Theorem — (Cauchy-Riemann Equations)

Necessary condition for $f(z)$ to be analytic —

If $f(z) = u(x,y) + i v(x,y)$ is analytic function ~~at a in a domain D, then~~ at a point z_0, then u_x, u_y, v_x, v_y exists and satisfy the Cauchy-Riemann equations —

$$\boxed{\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}}$$

; at every point in some neighbourhood of z_0.

Theorem (Sufficient condition) —

If (i) $f(z)$ is defined at every point in some neighbourhood of z_0.

(ii) u and v satisfy the C.R. equations at every point in the neighbourhood of z_0.

(iii) u_x, v_x, u_y and v_y are continuous at every point in the neighbourhood of z_0

Then $f(z)$ is analytic at z_0 and $\boxed{f'(z) = u_x + i v_x}$

Ex./. Test the analyticity of $f(z)$

(a). $f(z) = x + e^x \cos y + iy + ie^x \sin y$

(b). $f(z) = \bar{z}$

Procedure — Step 1 :- Find u, v

Step 2 — Find u_x, u_y, v_x and v_y

Step 3 — Check for C-R equations

i.e. $u_x = v_y$ and $v_x = -u_y$

Step 4 — Check the continuity of u_x, u_y, v_x, v_y.

If step 3 and 4 satisfied then $f(z)$ is analytic.

Sol. (a). $u = x + e^x \cos y$; $v = y + e^x \sin y$

$u_x = 1 + e^x \cos y$; $v_x = e^x \sin y$

$u_y = -e^x \sin y$; $v_y = 1 + e^x \cos y$

$\Rightarrow$ $u_x = v_y$ and $v_x = -u_y$ (C.R. eqns satisfied)

u_x, v_x, u_y and v_y are continuous

$\Rightarrow$ $f(z)$ is analytic function.

(b) Try your own