Department of Mathematics
15B11MA111
Tutorial Sheet 4

Mathematics-I
B.Tech. Core

Beta and Gamma functions

1. Using Gamma and Beta functions, evaluate the following integrals(use $\Gamma(1/3) = 2.6791$) wherever needed)

   i) $\int_0^1 (-\ln(y))^{2.5} dy$ (ii) $\int_0^\infty 5^{-2x^3} dx$ (iii) $\int_0^\infty \frac{x^a}{a^x} dx$ $(a>1)$ (iv) $\int_0^b x\sqrt{b^3-x^3}\,dx$ (v) $\int_0^\infty \frac{1}{1+x^4} dx$

2. Using Beta and Gamma functions, show that (assume $m>0$, $n>0$, $a>0$)

   i) $\int_0^\infty \frac{x^{m-1}}{1+x} dx = \Gamma(m)\Gamma(1-m)$ (ii) $\int_0^\infty x^m \exp(-\underbrace{2x^n}_{y})dx = \frac{\Gamma((m+1)/n)}{n\, 2^{(m+1)/n}}$ (iii) $\int_0^{\pi/2} \sin^{2m-1}(\theta)d\theta = \frac{(2m-2)(2m-4)...2}{(2m-1)(2m-3)...3}$

3. Evaluate $\iiint x^3 dxdydz$ over the volume bounded by the three coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

4. Find the C.G. of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ if the density at any point (x,y,z) is xyz.

   (Use Dirichlet's integral)

Vector Differential Calculus

5. Using vectors, find the following equations in vector and Cartesian forms:
   (i) A line through the points A (1, 2, -3) and B (-3, 1, -2). What is the parametric form of the line?
   (ii) A line through (1, 2, -3) and normal to the plane $5x-3y+2z=3$. Also parametrize it.
   (iii) A plane passing through the points P(1, 1, -2), Q(3, 0, 2) and R(-2, 1, 0).
6. Find the equation of the tangent plane and normal line to the surface $2x^2+3y^2-z+2x-5=0$ at the point (-1, 1, 2).
7. Determine the angle between the normals to the surface $xy+y^2=ze^{-2x+y}$ at the points (1, 2, 6) and (-3, 3, 0).
8. Find the unit vector normal to the surface $z=x^2+y^2$ at the point (-1, -2, 5).
9. Find the values of constants a, b so that the surface $ax^2-byz-ax-2x=0$ will be orthogonal to the surface $4x^2y+z^3-8=0$ at the point (2, 1, -2).
10. In what direction the directional derivative of $f=x^2yz^3+4$ at the point (2, 1, -1) is maximum? What is the magnitude of this maximum?
11. Find the directional derivative of $\phi=4e^{4x-2y+2z}$ at the point (-1, -1, 1) in the direction towards the point (3, 3, 8).
12. If $div(r\vec{\nabla}(1/r^3))=(a+1)r^b$, where $r=|\vec{r}|$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find the values of a and b.
13. Find the most general function $f(r)$ such that $f(r)\vec{r}$ is solenoidal.
14. Define an irrotational field. Is the field $\vec{F}=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ irrotational? Justify your answer.
15. What is a conservative field? Prove that $\vec{F}=(2xy+z^3)\hat{i}+x^2\hat{j}+3xz^2\hat{k}$ is a conservative force field. Also find the scalar potential.
16. If v and w are two vector point functions, then prove that
    (i) $div(v\times w)=w.curl(v)-v.curl(w)$
    (ii) $curl(v\times w)=(w.\nabla)v-(v.\nabla)w+v\,div(w)-w\,div(v)$

$12a+5b$
$5a+5b=10$
$7a=-10$
$a=-\frac{10}{7}$
$-120+\frac{120}{7}$

$(2ax-a)\hat{i}-bz\hat{j}-by\hat{k}$

$3a\hat{i}+2b\hat{j}-b\hat{k}$

$8xy\hat{i}+4x^2\hat{j}+3z^2\hat{k}$

$16\hat{i}+16\hat{j}+12\hat{k}$

$a=-2/7$
$b=16/7$

$b=\frac{24}{7}$

$48a+32b-12b=32$
$48a+20b=32$
$24a+10b=16$
$12a+5b=8$

$4a+2b-2a-4=0$
$2a+2b=4$
$a+b=2$

$12a+5(2-a)=0$
$7a=-10$
$a=-10/7$

...mma function

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \;,\; n>0$

$\Gamma(1) = 1$

- $\Gamma(n+1) = n\,\Gamma(n)$
  $= n!$
- $\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\, dx$
- $\Gamma(n) = \dfrac{\Gamma(n+1)}{n} = \dfrac{\Gamma(n+2)}{n(n+1)} \;,\; n \neq 0,-1$
  $= \dfrac{\Gamma(n+k+1)}{n(n+1)\ldots(n+k)} \;,\; n \neq 0,-1,-2,\ldots -k$

This defines $\Gamma(n)$ for $n<0$, k being the least positive integer such that $n+k+1>0$

- $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$\Gamma(-\tfrac{1}{2}) = \dfrac{\Gamma(\tfrac{1}{2})}{-\tfrac{1}{2}} = -2\sqrt{\pi}$

- $\Gamma(n) = k^n \int_0^\infty e^{-kx} x^{n-1}\, dx$
- $\Gamma(n) = \int_0^1 \left[\log\left(\tfrac{1}{x}\right)\right]^{n-1} dx \;;\; n>0$
- $\Gamma(n) = \dfrac{1}{n} \int_0^\infty e^{-x^{1/n}}\, dx$

Beta function

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \;,\; m>0, n>0$

- $\beta(m,n) = \beta(n,m)$
- $\beta(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}\, dx$
- $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Relation b/w Beta & Gamma functions

$\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

* $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$, where $0 < n < 1$

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$$

$$\Gamma(1/3)\,\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}$$

$$5 = \log e^5$$

$\frac{1}{2}$ $\Gamma\left(\frac{3+1}{2}\right)$

$\Gamma\left(\frac{m+n+2}{2}\right)$

$\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

where $0<n<1$

i) $\int_0^1 (-\ln(y))^{2.5}\, dy$

$= \int_0^1 \left(\ln\left(\frac{1}{y}\right)\right)^{2.5} dy$

$= \int_0^1 \left(\ln\left(\frac{1}{y}\right)\right)^{\frac{7}{2}-1} dy$

$= \Gamma(7/2) = 5/2\, \Gamma(5/2) \cdot 5/2 \cdot 3/2\, \Gamma(3/2)$

$= 5/2 \cdot 3/2\, \cancel{\Gamma(3/2)} \cdot 1/2\, \Gamma(1/2)$

$= \frac{15}{8}\sqrt{\pi}$

$\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx$

$n>0$

Put

$t = -\ln y$

$y = e^{-t}$

$dy = -e^{-t} dt$

$y=0$

$t=\infty$

$y=1$

$t=0$

$-\int_\infty^0 t^{5/2} e^{-t}\, dt$

$= \int_0^\infty t^{5/2} e^{-t}\, dt$

$= \Gamma(7/2)$

$a^x = e^{x \ln a}$

ii) $\int_0^\infty 5^{-2x^3}\, dx$

$= \int_0^\infty e^{-2x^3 \ln 5}\, dx$

$2x^3 \ln 5 = y$

$x = \left(\frac{y}{2\ln 5}\right)^{1/3}$

$dx = \frac{1}{(2\ln 5)^{1/3}} \cdot \frac{1}{3} y^{-\frac{2}{3}} dy$

$\frac{1}{5^{2x^3}}$

$\frac{1}{(5^2)^{x^3}}$

25^{x^3}

$e^{\log 25^{x^3}}$

$e^{x^3 \log 25}$

$e^{2x^3 \log 5}$

$\log e^{5^{-2x^3}}$

$= \int_0^\infty e^{-y} \frac{1}{(2\ln 5)^{1/3}} \cdot \frac{1}{3} y^{-2/3}\, dy$

$= \frac{1}{3(2\ln 5)^{1/3}} \int_0^\infty e^{-y} y^{\frac{1}{3}-1}\, dy$

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$= \frac{1}{3(2\ln 5)^{1/3}} \Gamma(1/3)$

$= \frac{1}{3(2\ln 5)^{1/3}} (2.6791) = \frac{2.6791}{4.429513}$

$= 0.6049$

(iii) $\int_0^\infty \frac{x^a}{a^x}\,dx \quad (a>1)$

$= \int_0^\infty e^{-x\log a}\, x^a\, dx$

Put $x \log a = y$ $\qquad a^x = e^{\log a^x}$

$x = \frac{y}{\log a}$

$dx = \frac{dy}{\log a}$

$= \int_0^\infty e^{-y} \left(\frac{y}{\log a}\right)^a \frac{dy}{\log a}$

$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-y}\, y^a\, dy$

$= \frac{1}{(\log a)^{a+1}}\, \Gamma(a+1) \;, \; a>1$

iv) $\int_0^{b} x\sqrt{b^3-x^3}\,dx$

Put $x^3 = b^3 y$

$x = b\, y^{1/3}$

$dx = \frac{b}{3} y^{-2/3}\, dy$

$\int_0^1 b\, y^{1/3} \sqrt{b^3(1-y)} \left(\frac{b}{3}\right)\left(y^{-2/3}\right) dy$

$= \frac{b^2}{3} \cdot b^{3/2} \int_0^1 \sqrt{1-y}\; y^{-1/3}\, dy$ $\qquad \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$= \frac{b^{7/2}}{3} \int_0^1 (y)^{\frac{2}{3}-1} (1-y)^{\frac{3}{2}-1}\, dy$

$= \frac{b^{7/2}}{3}\, \beta\left(\frac{2}{3}, \frac{3}{2}\right) = \frac{b^{7/2}}{3} \frac{\Gamma(2/3)\,\Gamma(3/2)}{\Gamma(13/6)}$

$= \frac{b^{7/2}}{\cancel{3}} \frac{\Gamma(2/3)\,\frac{1}{\cancel{2}}\Gamma(1/2)}{\frac{7}{6}\cdot\frac{1}{\cancel{6}}\Gamma(1/6)}$ $\qquad \frac{13-6}{6}$

$= \frac{6}{7}\, b^{7/2}\, \frac{\Gamma(2/3)\sqrt{\pi}}{\Gamma(1/6)}$

$\int_0^\infty \frac{1}{1+x^4}\,dx$

$1+x^4 = e^t$ $\quad x^4 = e^t - 1$

$4x^3\,dx = e^t\,dt$ $\quad x = (e^t-1)^{1/4}$

$\frac{1}{e^t}\quad e^t\,dt\quad \frac{1}{4x^3}$

Put $y = x^4$

$x = y^{1/4}$

$dx = \frac{1}{4} y^{-3/4}\,dy$

$= \int_0^\infty \frac{1}{1+y} \cdot \frac{1}{4} y^{-3/4}\,dy$

$-\frac{3}{4} = n-1$

$= \frac{1}{4} \int_0^\infty \frac{y^{-3/4}}{1+y}\,dy$

$n = \frac{1}{4}$

$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$

$= \frac{1}{4} \cdot \frac{\pi}{\sin \pi/4} = \boxed{\frac{\sqrt{2}\,\pi}{4}}$

$\log(\tan\theta + \sec\theta)$

$\frac{1}{\tan + \sec}\quad \sec^2 + \sec\tan\theta$

$\frac{\log \sec\theta}{\sec\theta}$

2) (i) Put $\dfrac{x}{1+x} = y$

$$x = y + xy$$
$$x(1-y) = y$$
$$x = \frac{y}{1-y} \quad \Rightarrow dx = \frac{1-y+y}{(1-y)^2}\,dy$$
$$= \frac{dy}{(1-y)^2}$$

$x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = 1$

$$\int_0^\infty \frac{x^{m-1}}{1+x}\,dx$$

$$\frac{x^{m-1}}{(1+x)^{m-1}} \cdot \frac{1}{(1+x)^{2-m}}$$

$$= \cancel{\int_0^\infty \left(\frac{x}{1+x}\right)^{m-1} \cdot \frac{1}{(1+x)}}$$

$$1+x = 1 + \frac{y}{1-y}$$
$$= \frac{1-y+y}{1-y}$$
$$= \frac{1}{1-y}.$$

$$= \int_0^1 y^{m-1} \frac{1}{(1-y)^{-2+m}} \cdot \frac{1}{(1-y)^2}\,dy$$

$$= \int_0^1 y^{m-1}(1-y)^{-m}\,dy$$

$n-1 = -m$

$n = 1-m$

$$= \beta(m, 1-m)$$

$$= \frac{\Gamma(m)\,\Gamma(1-m)}{\Gamma(m+1-m)}$$

$$= \Gamma(m)\,\Gamma(1-m)$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)}\,dx = \Gamma(m)\,\Gamma(1-m)$$

We have $\displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$m+n=1$$

$$n = 1-m$$

$$\therefore \int_0^\infty \frac{x^{m-1}}{1+x}\,dx = \beta(m, 1-m) = \Gamma(m)\,\Gamma(1-m)$$

(ii) $\displaystyle\int_0^\infty x^m \exp(-2x^n)\,dx = \frac{\Gamma((m+1)/n)}{n\,2^{(m+1)/n}}$

Put $y = 2x^n$

$$x = \left(\frac{y}{2}\right)^{1/n}$$

$$dx = \frac{1}{2^{1/n}}\cdot\frac{1}{n}\,y^{\frac{1}{n}-1}\,dy$$

$$\int_0^\infty \frac{y^{m/n}}{2^{m/n}}\,e^{-y}\,\frac{1}{2^{1/n}}\cdot\frac{1}{n}\cdot y^{\frac{1}{n}-1}\,dy$$

$$= \frac{1}{n\cdot 2^{(m+1)/n}}\int_0^\infty e^{-y}\,y^{\left(\frac{m+1}{n}\right)-1}\,dy$$

$$= \frac{1}{n\cdot 2^{(m+1)/n}}\,\Gamma\left(\frac{m+1}{n}\right)$$

(iii) We have $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\beta(m,n)}{2}$

Now $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta$ means $2n-1=0$, $n = 1/2$

$$= \frac{\beta(m, 1/2)}{2}$$

$$= \frac{\Gamma(m)\,\Gamma(1/2)}{2\,\Gamma(m+1/2)}$$

$$= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(m)}{\Gamma\left(\frac{2m+1}{2}\right)} = \frac{\sqrt{\pi}}{2}\,\frac{(m-1)!}{\left(\frac{2m-1}{2}\right)\left(\frac{2m-3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}$$

$$= \sqrt{\pi}\,\frac{(2m-2)(2m-4)\cdots 2}{(2m-1)(2m-3)\cdots 3}$$

$$= \frac{\sqrt{\pi}\,(m-1)!}{\cancel{2}\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots\left(\cancel{\frac{1}{2}}\right)\Gamma\left(\cancel{\frac{1}{2}}\right)}$$

$$= \frac{(m-1)!}{\frac{1}{2^{m-1}}(2m-1)(2m-3)\cdots 3}$$

$$= \frac{(2m-2)(2m-4)\cdots 2}{(2m-1)(2m-3)\cdots 3}$$

$\frac{3}{2}\cdot\frac{5}{2}\cdots\left(m-\frac{1}{2}\right)$

$m-\frac{1}{2} = \frac{3}{2}+(n-1)(1)$

$m = 2+n-1$

$= n+1$

3) $I_1 = \iiint x^3\, dx\, dy\, dz$ over volume bdd by three coordinate planes & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$

If surface is $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$

$$I = \iiint_T x^{\alpha-1} y^{\beta-1} z^{\gamma-1}\, dx\, dy\, dz$$

$\alpha, \beta, \gamma, a, b, c, p, q, r$ are positive

then $$I = \frac{a^\alpha b^\beta c^\gamma}{pqr}\,\frac{\Gamma(\alpha/p)\,\Gamma(\beta/q)\,\Gamma(\gamma/r)}{\Gamma\left(\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}+1\right)}$$

$$\boxed{\text{If } p=q=r=1 \quad a^\alpha b^\beta c^\gamma\,\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma+1)}}$$

$\therefore\ p = q = r = 1$

$\alpha = 4,\ \beta = \gamma = 1$

$$I_1 = a^4 bc\,\frac{\Gamma(4)\Gamma(1)\Gamma(1)}{\Gamma(4+1+1+1)}$$

$$= a^4 bc\,\frac{3!}{6!} = \frac{a^4 bc}{120}$$

4) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$, $\rho(x,y,z) = xyz$

Mass $= \iiint_T xyz\, dx\, dy\, dz$

$\alpha = \beta = \gamma = 2,\ p = q = r = 2$

$$= \frac{(abc)^2}{2.2.2}\,\frac{\Gamma(1)^3}{\Gamma(4)} = \frac{(abc)^2}{48}$$

$$\bar{x} = \frac{1}{M}\iiint_T x\,\rho(x,y,z)\,dx\,dy\,dz$$

$$= \frac{48}{(abc)^2}\iiint_T x^2 yz\,dx\,dy\,dz$$

$\beta = \gamma = p = q = r = 2, \quad \alpha = 3$

$$\bar{x} = \frac{48}{a^2b^2c^2}\left[\frac{a^3b^2c^2}{8}\,\frac{\Gamma(3/2)(\Gamma(1))^2}{\Gamma(\frac{3}{2}+1+1+1)}\right]$$

$3+\frac{3}{2}$

$$= 6a\,\frac{\cancel{\frac{1}{2}\Gamma(\frac{1}{2})}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\cancel{\frac{1}{2}\Gamma(\frac{1}{2})}}$$

$$= \frac{\overset{2}{\cancel{6}}a\cdot 8}{35\cdot\cancel{3}} = \frac{16a}{35}$$

$$\bar{y} = \frac{48}{a^2b^2c^2}\iiint_T xy^2z\,dx\,dy\,dz$$

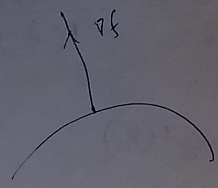
$$= \frac{\overset{6}{\cancel{48}}}{a^2b^2c^2}\left[\frac{a^2b^3c^2}{8}\,\frac{\Gamma(1)\Gamma(3/2)\Gamma(1)}{\Gamma(1+\frac{3}{2}+1+1)}\right]$$

$$= 6b\,\frac{\cancel{\frac{1}{2}\Gamma(\frac{1}{2})}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\cancel{\frac{1}{2}\Gamma(\frac{1}{2})}}$$

$$= \frac{16b}{35}$$

$$\bar{z} = \frac{16c}{35}$$

Centre of gravity $= \left(\frac{16a}{35}, \frac{16b}{35}, \frac{16c}{35}\right)$



$$\frac{x-x_1}{\left(\frac{\partial f}{\partial x}\right)_P} = \frac{y-y_1}{\left(\frac{\partial f}{\partial y}\right)_P} = \frac{z-z_1}{\left(\frac{\partial f}{\partial z}\right)_P}$$

$2m-1=p$

$p=p$

$q=0$

$$\int_0^{\pi/2} \sin^{p}\theta \, d\theta$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 1 & -2 \\ 2 & -1 & 4 \end{vmatrix}$$

$$= \hat{i}(2) - \hat{j}(-20+4) + \hat{k}(3)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 0 & 2 \\ -5 & 1 & -2 \end{vmatrix}$$

$$\hat{i}(-2) - \hat{j}(6+10) + \hat{k}(-3)$$

$A(1,2,-3)$, $B(-3,1,-2)$

$$\frac{x-1}{-3-1} = \frac{y-2}{1-2} = \frac{z+3}{-2+3}$$

$$\frac{x-1}{-4} = \frac{y-2}{-1} = \frac{z+3}{1} = t$$

$x = 1-4t$

$y = 2-t$

$z = -3+t$

V. F: $(1-4t)\hat{i} + (2-t)\hat{j} + (t-3)\hat{k}$

ii) $f: 5x - 3y + 2z - 3 = 0$

$\nabla f = 5\hat{i} - 3\hat{j} + 2\hat{k}$

$$\frac{x-1}{5} = \frac{y-2}{-3} = \frac{z+3}{2} = t$$

$x = 1+5t$, $y = 2-3t$, $z = -3+2t$

VF: $(1+5t)\hat{i} + (2-3t)\hat{j} + (-3+2t)\hat{k}$

(iii) $P(1,1,-2)$, $Q(3,0,2)$, $R(-2,1,0)$

eq of plane is given by

$a(x-1) + b(y-1) + c(z+2) = 0$

where $a\hat{i} + b\hat{j} + c\hat{k}$ is a vector perpendicular to plane

$\vec{PQ} = 2i - \hat{j} + 4\hat{k}$ $\qquad \vec{QR} = -5\hat{i} + \hat{j} - 2\hat{k}$

$\vec{PR} = -3\hat{i} + 2\hat{k}$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 4 \\ -3 & 0 & 2 \end{vmatrix} = \hat{i}(-2) - j(4+12) + \hat{k}(-3)$$

$= -2\hat{i} - 16\hat{j} - 3\hat{k}$

$= a\hat{i} + b\hat{j} + c\hat{k}$

$\therefore$ eq: $-2(x-1) - 16(y-1) - 3(z+2) = 0$

$2(x-1) + 16(y-1) + 3(z+2) = 0$

$2x + 16y + 3z - 2 - 16 + 6 = 0$

$2x + 16y + 3z = 12$

6) $f(x,y,z): 2x^2 + 3y^2 - z + 2x - 5 = 0$

Normal $\nabla f = (4x+2)\hat{i} + 6y\,\hat{j} - \hat{k}$

$$\nabla f\big|_{(-1,1,2)} = -2\hat{i} + 6\hat{j} - \hat{k}$$

Eq of tangent plane

$$(x+1)(-2) + (y-1)6 + (z-2)(-1) = 0$$

$$-2x - 2 + 6y - 6 + 2 - z = 0$$

$$-2x + 6y - z = 6$$

Eq of normal line

$$\frac{x+1}{-2} = \frac{y-1}{6} = \frac{z-2}{-1} \cancel{= t}$$

7) $xy + y^2 = z e^{-2x+y}$

$f: xy + y^2 - z e^{-2x+y} = 0$

$$\nabla f = (y - z(-2) e^{-2x+y})\hat{i} + (x + 2y - z e^{-2x+y})\hat{j} - e^{-2x+y}\hat{k}$$

$$= \cancel{y + 2z e^{-2x+y}}$$

$$\vec{n}_1 = \nabla f\big|_{(1,2,6)} = (2 + 12e^{-2+2})\hat{i} + (1 + 4 - 6e^0)\hat{j} - \hat{k}$$

$$= 14\hat{i} - \hat{j} - \hat{k}$$

$$\vec{n}_2 = \nabla f\big|_{(-3,3,0)} = 3\hat{i} + (-3+6)\hat{j} - e^{6+3}\hat{k}$$

$$= 3\hat{i} + 3\hat{j} - e^9\hat{k}$$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} = \left|\frac{42 - 3 + e^9}{\sqrt{196+1+1}\,\sqrt{9+9+e^{18}}}\right|$$

$$= \frac{39 + e^9}{3\sqrt{22}\,\sqrt{18 + e^{18}}}$$

$$\theta = \cos^{-1}\left(\frac{e^9 + 39}{3\sqrt{22(18 + e^{18})}}\right)$$

$$\begin{array}{r|l} 2 & 198 \\ 3 & 99 \\ 3 & 33 \\ & 11 \end{array}$$

$z = x^2 + y^2$ at $(-1,-2,5)$

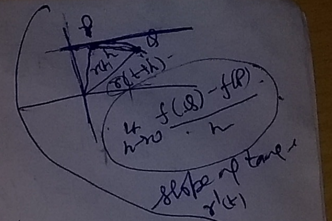
$f: z - x^2 - y^2 = 0$

$\nabla f = -2x\hat{i} - 2y\hat{j} + \hat{k}$

$= 2\hat{i} + 4\hat{j} + \hat{k}$

$|\nabla f| = \sqrt{4+16+1} = \sqrt{21}$

unit Normal vector $= \dfrac{2\hat{i} + 4\hat{j} + \hat{k}}{\sqrt{21}}$



9) $ax^2 - byz - ax - 2x = 0$

$\nabla f_1 = (2ax - a - 2)\hat{i} + (-bz)\hat{j} - by\hat{k}$

$= (4a - a - 2)\hat{i} + 2b\hat{j} - b\hat{k}$

$4x^2y + z^3 - 8 = 0$

$\nabla f_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

$= 16\hat{i} + 16\hat{j} + 12\hat{k}$

Orthogonal if $\nabla f_1 \cdot \nabla f_2 = 0$

$(3a - 2)16 + 2b(16) - b(12) = 0$

$48a - 32 + 20b = 0$

$24a + 10b = 16$

$12a + 5b = 8$

$-5a + 5b = -10$

$7a = -2$

$a = -\frac{2}{7}$

$b = 2 - a \Rightarrow b = 2 + \frac{2}{7}$

$b = \cancel{\ldots} = \frac{16}{7}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$\hat{i}(0) - \hat{j}(0) + \hat{k}(1)$

$\hat{k}$

$\Rightarrow$ $ax^2 - byz - ax - 2x = 0$

At $(2,1,-2)$

$4a + 2b - 2a = 4$

$2a + 2b = 4$

~~$10a + 5b = 20$~~

$a + b = 2$

$5a + 5b = 10$

10) $f = x^2yz^3 + 4$ at $(2,1,-1)$

$\nabla f = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$

$\nabla f|_{(2,1,-1)} = 2(2)(-1)\hat{i} + 4(-1)\hat{j} + 3(4)(1)\hat{k}$

$= -4\hat{i} - 4\hat{j} + 12\hat{k}$

Max in direction of ∇f

Magnitude of Max $= \sqrt{16+16+144} = \sqrt{176}$

$= 4\sqrt{11}$

2	176
2	88
4	44
	11

11) $\phi = 4e^{4x - 2y + 2z}$

$\nabla \cdot \phi = 4e^{4x-2y+2z}(4\hat{i} - 2\hat{j} + 2\hat{k})$

$\nabla \phi|_{(-1,-1,1)} = 4e^{-4+2+2}(4\hat{i} - 2\hat{j} + 2\hat{k})$

$= 16\hat{i} - 8\hat{j} + 8\hat{k}$

$\vec{b} = (3-(-1))\hat{i} + (3-(-1))\hat{j} + (8-1)\hat{k}$

$= 4\hat{i} + 4\hat{j} + 7\hat{k}$

$$\hat{b} = \frac{4\hat{i} + 4\hat{j} + 7\hat{k}}{\sqrt{16+16+49}} = \frac{4\hat{i}+4\hat{j}+7\hat{k}}{9}$$

32

81

$$D_b \phi = \nabla\phi \cdot \hat{b}$$

$$= (16\hat{i} - 8\hat{j} + 8\hat{k}) \cdot \frac{(4\hat{i} + 4\hat{j} + 7\hat{k})}{9}$$

$$= \frac{1}{9}\left[64 - 32 + \underset{32}{56}\right]$$

$$= \frac{1}{9}[88] = \frac{88}{9}$$

12) $\operatorname{div}\left(r \nabla\left(\frac{1}{r^3}\right)\right) = (a+1) r^b$

where $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla\left(\frac{1}{r^3}\right) = \nabla\left[(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2} \times \cancel{2}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$r\nabla\left(\frac{1}{r^3}\right) = -3(x^2+y^2+z^2)^{-\frac{5}{2}+\frac{1}{2}}[\vec{r}]$$

$$= -3\frac{1}{(x^2+y^2+z^2)^2}\vec{r}$$

$$= -\frac{3}{r^4}\vec{r}$$

$$= \frac{-3}{r^4}(x\hat{i} + y\hat{j} + z\hat{k}) \qquad r^{-4}$$

$$\operatorname{div}\left(r\nabla\left(\frac{1}{r^3}\right)\right) = \left(\frac{-3x-4}{r^5}\frac{\partial r}{\partial x}x - \frac{3}{r^4}\right) + \left(\frac{12}{r^5}\frac{\partial r}{\partial y}y - \frac{3}{r^4}\right)$$

$$+\left(\frac{12}{r^5}\frac{\partial r}{\partial z}z - \frac{3}{r^4}\right)$$

$$= \frac{12}{r^5}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right) - \frac{9}{r^4}$$

$$= \frac{3}{r^4} = 3r^{-4}$$

$a + 1 = 3$ $\qquad b = -4$

$a = 2$

$$\nabla\left(\frac{1}{r^3}\right) = \frac{-3}{r^4}\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right)$$

$$= \frac{-3}{r^4}\left(\frac{x\hat{i}}{r} + \frac{y\hat{j}}{r} + \frac{z\hat{k}}{r}\right)$$

$$\frac{\partial r}{\partial x} = \frac{1\times 2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$= \frac{-3}{r^5}(\vec{r})$$

$$r\,\nabla\left(\frac{1}{r^3}\right) = \frac{-3}{r^4}\vec{r}$$

$$\operatorname{div}\left(\frac{-3}{r^4}\vec{r}\right) = \frac{-3\times -4}{r^5}\frac{\partial r}{\partial x}\cdot x - \frac{3}{r^4} + \frac{12}{r^5}\frac{\partial r}{\partial y}y - \frac{3}{r^4} + \frac{12}{r^5}\frac{\partial r}{\partial z}z - \frac{3}{r^4}$$

$$= \frac{12}{r^5}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right) - \frac{9}{r^4}$$

$$= \frac{12r^2}{r^6} - \frac{9}{r^4} = \frac{3}{r^4} = 3r^{-4}$$

$$= (a+1)r^b$$

$$a+1 = 3 \qquad \boxed{b=-4}$$

$$\boxed{a=2}$$

13) $\nabla\cdot(f(r)\vec{r}) = 0$

$$\Rightarrow f(r)(\nabla\cdot\vec{r}) + (\nabla f(r))\cdot\vec{r} = 0$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow 3f(r) + f'(r)\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}\right)\cdot\vec{r} = 0$$

$$\Rightarrow 3f(r) + f'(r)\frac{r^2}{r} = 0$$

$$\Rightarrow 3f(r) + rf'(r) = 0$$

$$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$$

$$\Rightarrow \ln f(r) = -3\ln r + \ln c$$

$$\Rightarrow f(r) = \frac{c}{r^3}.$$

$$\boxed{\text{Ans: } \frac{c}{r^3}}$$

$$\nabla\left(\frac{1}{r^3}\right) = \frac{-3}{r^4}\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right)$$

$$= \frac{-3}{r^4}\left(\frac{x\hat{i}}{r} + \frac{y\hat{j}}{r} + \frac{z\hat{k}}{r}\right)$$

$$= \frac{-3}{r^5}(\vec{r})$$

$$\frac{\partial r}{\partial x} = \frac{1 \times 2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$r\,\nabla\left(\frac{1}{r^3}\right) = \frac{-3}{r^4}\vec{r}$$

$$\operatorname{div}\left(\frac{-3}{r^4}\vec{r}\right) = \frac{-3\times -4}{r^5}\frac{\partial r}{\partial x}\cdot x - \frac{3}{r^4} + \frac{12}{r^5}\frac{\partial r}{\partial y}y - \frac{3}{r^4} + \frac{12}{r^5}\frac{\partial r}{\partial z}z - \frac{3}{r^4}$$

$$= \frac{12}{r^5}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right) - \frac{9}{r^4}$$

$$= \frac{12 r^2}{r^6} - \frac{9}{r^4} = \frac{3}{r^4} = 3r^{-4}$$

$$= (a+1)r^b$$

$$a+1 = 3 \qquad \boxed{b=-4}$$

$$\boxed{a=2}$$

13) $\nabla\cdot(f(r)\vec{r}) = 0$

$\Rightarrow f(r)(\nabla\cdot\vec{r}) + (\nabla f(r))\cdot\vec{r} = 0$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow 3f(r) + f'(r)\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}\right)\cdot\vec{r} = 0$$

$$\Rightarrow 3f(r) + f'(r)\frac{r^2}{r} = 0$$

$$\Rightarrow 3f(r) + r f'(r) = 0$$

$$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$$

$$\Rightarrow \ln f(r) = -3\ln r + \ln c$$

$$\Rightarrow f(r) = \frac{c}{r^3}.$$

$$\boxed{\underline{\text{Ans}}:\ \frac{c}{r^3}}$$

14) $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$

curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$

$= \hat{i}(-1 + 1) - \hat{j}(3z^2 - 3z^2) + \hat{k}(6x - 6x)$

$= 0$

$\therefore \vec{F}$ is irrotational.

15) $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ ~~$f = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$~~

$\vec{F} = \text{grad } f = \nabla f$

$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\frac{\partial f}{\partial x} = 2xy + z^3$

$f = x^2y + xz^3 + g(y, z)$

$x^2 = \frac{\partial f}{\partial y} = x^2 + g_y(y, z)$

$\Rightarrow g_y(y, z) = 0$

~~$g(y,z)$~~ $= g(z)$.

$\therefore f(x, y, z) = x^2y + xz^3 + g(z)$

$3xz^2 = \frac{\partial f}{\partial z} = 3xz^2 + g'(z)$

$g'(z) = 0$

$g(z) = c$, constant.

$\therefore f(x, y, z) = x^2y + xz^3 + c$ (Scalar potential).

curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = \hat{i}(0 - 0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$.

$\therefore$ conservative vector field.

$\operatorname{div}(v \times w) = w \cdot \operatorname{curl} v - v \cdot \operatorname{curl}(w)$

$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$

$\vec{w} = w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k}$

$$\vec{v} \times \vec{w} = \hat{i}(v_2 w_3 - v_3 w_2) - \hat{j}(v_1 w_3 - v_3 w_1) + \hat{k}(v_1 w_2 - v_2 w_1)$$

$$\operatorname{div}(\vec{v} \times \vec{w}) = \frac{\partial (v_2 w_3 - v_3 w_2)}{\partial x} + \frac{\partial (v_3 w_1 - v_1 w_3)}{\partial y} + \frac{\partial (v_1 w_2 - v_2 w_1)}{\partial z}$$

$$= v_2 \frac{\partial w_3}{\partial x} + w_3 \frac{\partial v_2}{\partial x} - v_3 \frac{\partial w_2}{\partial x} - w_2 \frac{\partial v_3}{\partial x} +$$

$$+ v_3 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial v_3}{\partial y} - v_1 \frac{\partial w_3}{\partial y} - w_3 \frac{\partial v_1}{\partial y} +$$

$$+ v_1 \frac{\partial w_2}{\partial z} + w_2 \frac{\partial v_1}{\partial z} - v_2 \frac{\partial w_1}{\partial z} - w_1 \frac{\partial v_2}{\partial z}$$

$$= -v_1 \left(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z} \right) - v_2 \left(\frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x} \right) - v_3 \left(\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} \right)$$

$$+ w_1 \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + w_2 \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) + w_3 \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)$$

$$= -\vec{v} \cdot \operatorname{curl} \vec{w} + \vec{w} \cdot \operatorname{curl} \vec{v}.$$

(ii) $\vec{v} \times \vec{w} = \hat{i}(v_2 w_3 - w_2 v_3) + \hat{j}(v_3 w_1 - v_1 w_3) + \hat{k}(v_1 w_2$

$$\operatorname{curl}(\vec{v} \times \vec{w}) = \left(\frac{\partial(v_1 w_2 - v_2 w_1)}{\partial y} - \frac{\partial(v_3 w_1 - v_1 w_3)}{\partial z} \right) \hat{i}$$

$$+ \left(\frac{\partial(v_2 w_3 - w_2 v_3)}{\partial z} - \frac{\partial(v_1 w_2 - v_2 w_1)}{\partial x} \right) \hat{j}$$

$$+ \left(\frac{\partial(v_3 w_1 - v_1 w_3)}{\partial x} - \frac{\partial(v_2 w_3 - w_2 v_3)}{\partial y} \right) \hat{k}$$

$$= \Big[v_1 \frac{\partial w_2}{\partial y} + w_2 \frac{\partial v_1}{\partial y} - v_2 \frac{\partial w_1}{\partial y} - w_1 \frac{\partial v_2}{\partial y}$$

$$- \left(v_3 \frac{\partial w_1}{\partial z} + w_1 \frac{\partial v_3}{\partial z} - v_1 \frac{\partial w_3}{\partial z} - w_3 \frac{\partial v_1}{\partial z} \right) \Big] \hat{i}$$

$$+ \Big[v_2 \frac{\partial w_3}{\partial z} + w_3 \frac{\partial v_2}{\partial z} - w_2 \frac{\partial v_3}{\partial z} - v_3 \frac{\partial w_2}{\partial z}$$

$$- \left(v_1 \frac{\partial w_2}{\partial x} + w_2 \frac{\partial v_1}{\partial x} - v_2 \frac{\partial w_1}{\partial x} - w_1 \frac{\partial v_2}{\partial x} \right) \Big] \hat{j}$$

$$+ \Big[v_3 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial v_3}{\partial x} - v_1 \frac{\partial w_3}{\partial x} - w_3 \frac{\partial v_1}{\partial x}$$

$$- \left(v_2 \frac{\partial w_3}{\partial y} + w_3 \frac{\partial v_2}{\partial y} - w_2 \frac{\partial v_3}{\partial y} - v_3 \frac{\partial w_2}{\partial y} \right) \Big] \hat{k}$$

$$= v \operatorname{div}(w) - w \operatorname{div}(v)$$

$$+ \hat{i} \left(w_2 \frac{\partial v_1}{\partial y} - v_2 \frac{\partial w_1}{\partial y} - v_3 \frac{\partial w_1}{\partial z} + w_3 \frac{\partial v_1}{\partial z} - v_1 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial v_1}{\partial x} \right)$$

$$+ \hat{j} \left(w_3 \frac{\partial v_2}{\partial z} - v_3 \frac{\partial w_2}{\partial z} - v_1 \frac{\partial w_2}{\partial x} + w_1 \frac{\partial v_2}{\partial x} - v_2 \frac{\partial w_2}{\partial y} + w_2 \frac{\partial v_2}{\partial y} \right)$$

$$+ \hat{k} \left(w_1 \frac{\partial v_3}{\partial x} - v_1 \frac{\partial w_3}{\partial x} - v_2 \frac{\partial w_3}{\partial y} + w_2 \frac{\partial v_3}{\partial y} - v_3 \frac{\partial w_3}{\partial z} + w_3 \frac{\partial v_3}{\partial z} \right)$$

$$= v \operatorname{div}(w) - w \operatorname{div}(v) + \left(w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y} + w_3 \frac{\partial}{\partial z} \right) \vec{v}$$

$$- \left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z} \right) \vec{w}$$

$$= v \operatorname{div}(w) - w \operatorname{div}(v) + (\vec{w} \cdot \nabla) \vec{v} - (\vec{v} \cdot \nabla) \vec{w}$$