Some Definitions

Suppose that $a_1, a_2, \ldots a_n$ is a sequence of real numbers.

- A **subsequence** of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$, where $1 \le i_1 < i_2 < \ldots < i_m \le n$
- A sequence is called **strictly increasing** if each term is larger than the term that precedes it.
- A sequence is called **strictly decreasing** if each term is smaller than the one that precedes it.
  – Example: {1, 5, 6, 2, 3, 9} is a sequence.
  – {5,6,9} is a subsequence that is strictly increasing

Theorem: Every sequence of n^2+1 distinct real numbers contains a subsequence of (at least) length n+1 that is strictly increasing or strictly decreasing.

Example: 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

10 = 3^2+1 terms so must be a subsequence of length 4 that is either strictly increasing or strictly decreasing.

1,4,6,12

1,4,6,7

11,9,6,5

……

Theorem: Every sequence of n^2+1 <u>distinct</u> real numbers contains a subsequence of at least length n+1 that is strictly increasing or strictly decreasing.

Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of n^2+1 distinct numbers. Associate an ordered pair (i_k,d_k) with each term of the sequence where i_k is the length of the longest increasing subsequence starting at a_k and d_k is length of the longest decreasing subsequence starting at a_k.

Example: 8, 11, 9, 1, 4, 6, 12, 10, 5, 7

$a_2 = 11$, (2,4)

$a_4 = 1$, (4,1)

Proof by contradiction: Now suppose that there are no increasing or decreasing subsequences of length n+1 or greater. Then i_k and d_k are both positive integers $\leq n$, for $k=1$ to n^2+1.

By the <u>product rule</u>, there are n^2 <u>possible</u> ordered pairs for (i_k,d_k).
Why? Because each has the range from 1 to n.

By the pigeonhole principle, since we have n^2+1 ordered pairs (one for each element in the sequence) two of them must be identical. Formally $\exists$ terms a_s and a_t in the sequence, with $s<t$ such that $i_s = i_t$ and $d_s = d_t$.

We will show that this is impossible.

Because the terms in the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$. If $a_s < a_t$, an increasing subsequence of length i_t+1(or greater) can be constructed starting at a_s, by taking a_s followed by an increasing subsequence of length i_t, beginning at a_t. But we have said that $i_s = i_t$. Thus this is a contradiction.

Similarly, if $a_s > a_t$, it can be shown that d_s must be greater than d_t, which is also a contradiction.