Review of set theory

- Variable objects x, y, z; sets S, T, U.
- Literal set {a, b, c} and set-builder $\{x \mid P(x)\}$.
- $\in$ relational operator ("is an element of")
- The empty set $\varnothing$.
- Set relations =, $\subseteq$, $\subset$, $\not\subset$, etc.
- Venn diagrams.
- Cardinality $|S|$ and infinite sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$.
- Power sets P(S).
- Infinite and finite sets

Sets of sets

- More formally:
  $S \notin S$, but $S \in \{S\}$

- The empty set, $\varnothing = \{\}$
- $\varnothing \notin \{\}$
- But $\varnothing \in \{\varnothing\}$
- $\{\varnothing\} = \{\{\}\}$
- $\{\varnothing\} \in \{\{\varnothing\}\}$

Proving Set Identities

- To prove statements about sets, of the form $E_1 = E_2$ (where the *E*s are set expressions), here are three different and useful methods:

1. Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
2. Use set builder notation & logical equivalences.
3. Use a *membership table*.

Method 1: Mutual subsets

- Example:
- Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

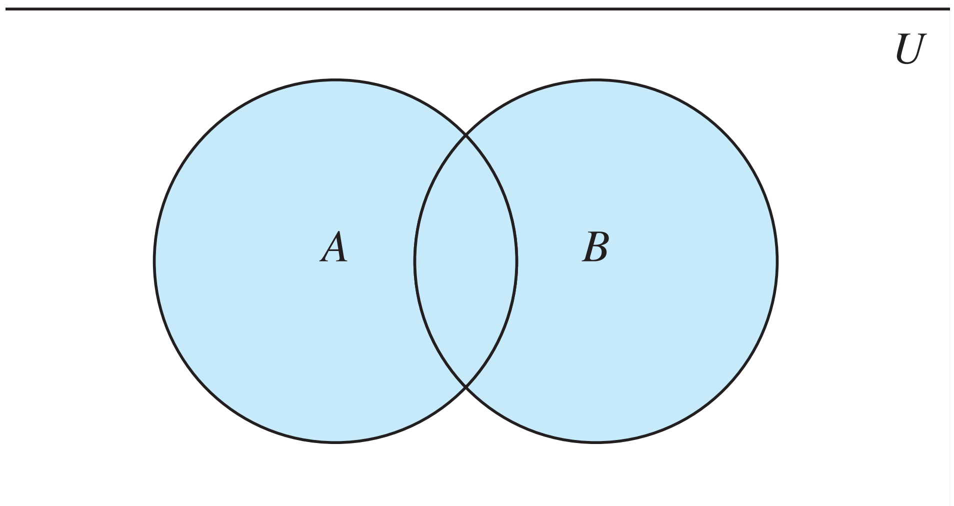
- Part 1: Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
  - Assume $x \in A \cap (B \cup C)$, & show $x \in (A \cap B) \cup (A \cap C)$.
  - We know that $x \in A$, and either $x \in B$ or $x \in C$.
    - Case 1: $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.
    - Case 2: $x \in C$. Then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

- Part 2: Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. …

Method 2: Direct proof

- Consider DeMorgan's rule:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$



$A \cup B$ is shaded.

Direct Proof ---- on blackboard

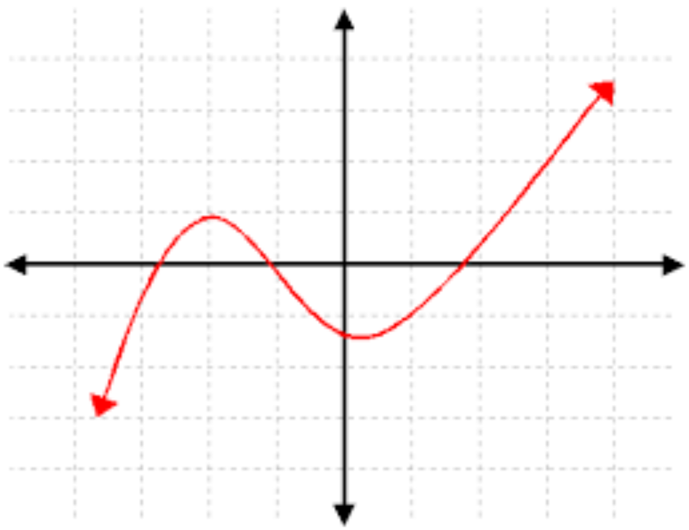
Method 3: Membership Tables

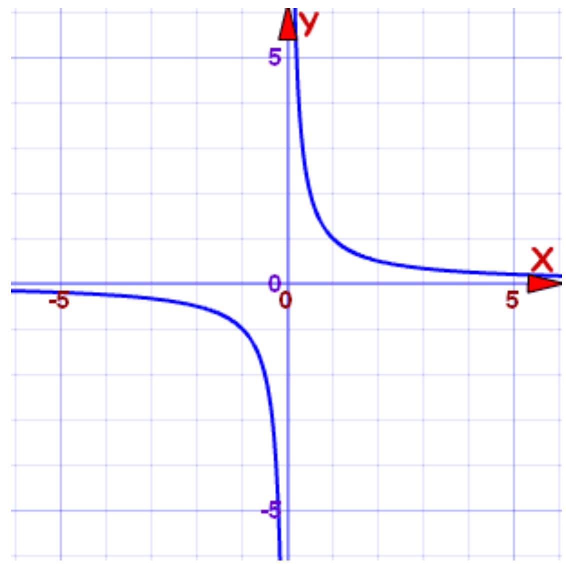
- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use "1" to indicate element is a member of the specified set, and "0" for non-membership. This is a function.
- Prove equivalence with identical columns.

More on this latter --- after we discuss functions

Functions

- From calculus, you are familiar with the concept of a real-valued function *f*, which assigns to each number $x \in \mathbb{R}$ a particular value $y = f(x)$, where $y \in \mathbb{R}$.





- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *map*.)

Function: Formal Definition

- Def. For any sets A, B, we say that a *function f* *(or "mapping") from A to B* is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$. We can write

$$f{:}A \rightarrow B$$

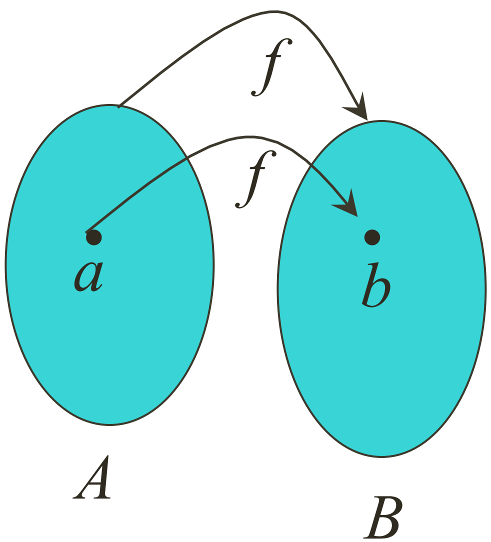
  as short-hand notation to denote function f maps elements from set A to set B.

- To implement the function for a particular $x \in A$ we write $f(x)$.

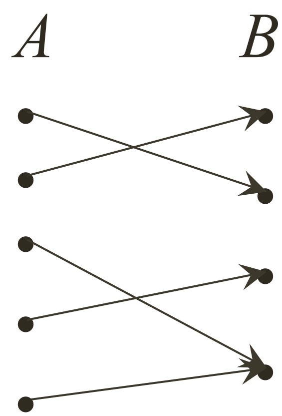
- For $a \in A$ and $b \in B$, we can also write $f(a)=b$.

Graphical Representations

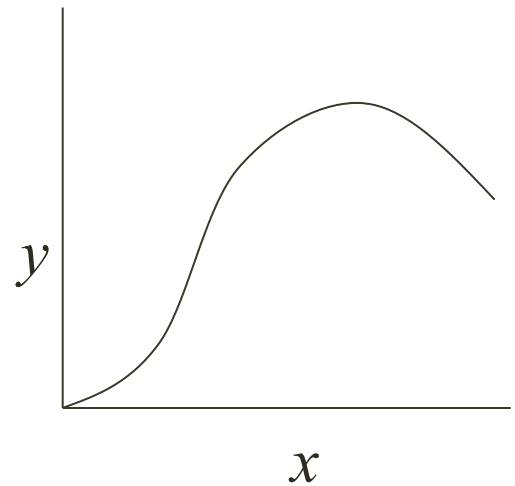
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Note: EVERY element of set A has to be mapped to ONE (and only one) element in B.

Functions and sets

- A *set S* over universe *U* can be viewed as a function from the elements of *U* to {**T**, **F**}, saying for each element of *U* whether it is in *S*.
- Ex.

  Let S={3}

  Then *S*(0)=**F**, *S*(3)=**T**, etc.

- A *set operator* such as ∩,∪ can be viewed as a function from pairs of sets to a new set.
- Ex.

  ∩(({1,3},{3,4})) = {3}

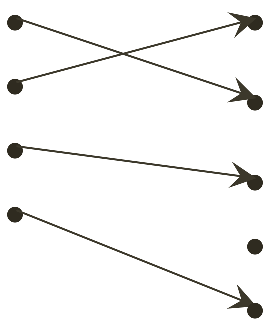
Important Function Terminology

- Def. Let f:$A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$). Then
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R \subseteq B$ of f is $R=\{b \mid \exists a\, f(a)=b\}$.

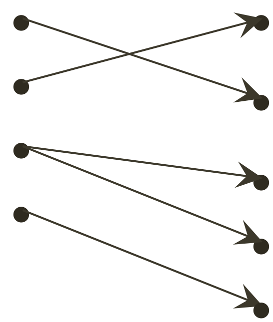
We also say the *signature* of f is $A \rightarrow B$.

Basics of functions

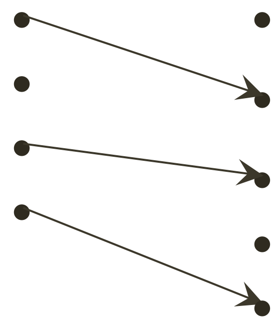
- $f{:}A \rightarrow B$ means that EVERY element of set A has to be mapped to ONE (and only one) element in B.
- Set A is the domain
- Set B is the codomain



A valid function!



Not a function!



Not a function!

- e.g., Let the *domain* A be the set of students in this class
- Let the *codomain* B be the letter grades {A,B,C,D,F}
- Calculation of the final grade is a function that maps all the scores of each student in A to a letter grade in B.

B^A: Set of all functions

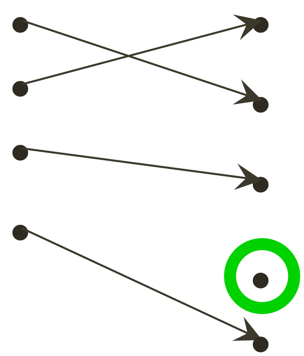
- Def. The set of *all* possible functions $f{:}A \rightarrow B$ is B^A.
- Theorem. For finite A, B, $|B^A| = |B|^{|A|}$.
- Remark. For finite set S, the cardinality of the power set $|P(S)| = \mathbf{2}^S$.

Are these functions?

	Function?	Domain	Codomain
• $f(a) = a^2$		$\mathbb{Z}$	
• $f(a) = \sin(a)$		$\mathbb{R}$	
• $f(a) = \pm a$		$\mathbb{Z}$	
• $f(a) = 1/a$		$\mathbb{Z}$	
• $f(a) = 1/a$		$\mathbb{Z}^+$	

Range

- The actually values in codomain B that are mapped onto.
- The range of function f is the set { f(a) | $a \in A$}



Although this element is in the *codomain*, it is not in the *range* of *f*.

e.g. Even though the codomain of grades for this class is the set {A,B,C,D,F}, we hope that the range is smaller.

Range versus Codomain

- Remarks.
- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Ex. Suppose I declare to you that:

"*f* is a function mapping students in this class to their weight measured to the closest pound."

- At this point, you know *f*'s codomain is: ___________.
- Suppose each student's weight is between 80-300 pounds.
- Then the range of *f* is __________, but its codomain is ____________________.

Images of Sets under Functions

- Def. Let $f{:}A \to B$, and $S \subseteq A$.
-

  The *image* of S under f is simply the set of all images (under f) of the elements of S.

  $$f(S) :\equiv \{f(s) \mid s \in S\}$$
  $$:\equiv \{b \mid \exists\, s \in S{:}\ f(s)=b\}.$$

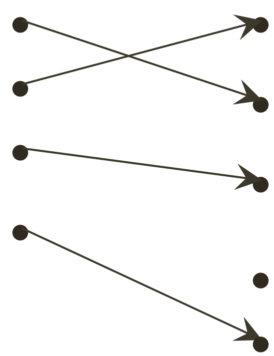
- Note the range of f can be defined as simply the image (under f) of f's domain!

One-to-One Functions

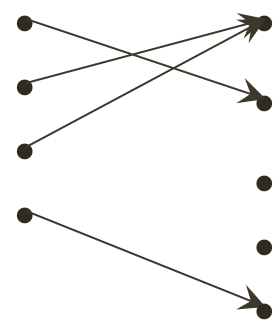
- Def. A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
  - Formally: given f:$A \rightarrow B$, "f is injective" :≡ ($\neg\exists x,y$: $x \neq y \wedge f(x)=f(y)$).
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have same cardinality. What about codomain?
- Memory jogger: Each element of the domain is <u>injected</u> into a different element of the range.
  - Compare "each dose of vaccine is injected into a different patient."

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:

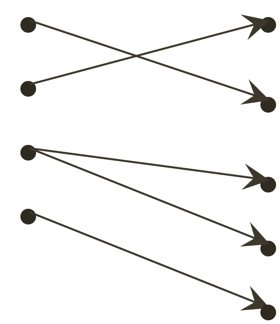


One-to-one



Not one-to-one

Consider $f(x) = x^2$ where $x \in \mathbb{Z}$



Not even a function!

Sufficient Conditions for 1-1ness

- For functions f over numbers, we say:
  - f is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all x,y in domain;
  - f is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all x,y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one.
- Ex.
- $f(x) = x^3$
- $f(x) = 1/x$ (Converse is not necessarily true)

Onto (Surjective) Functions

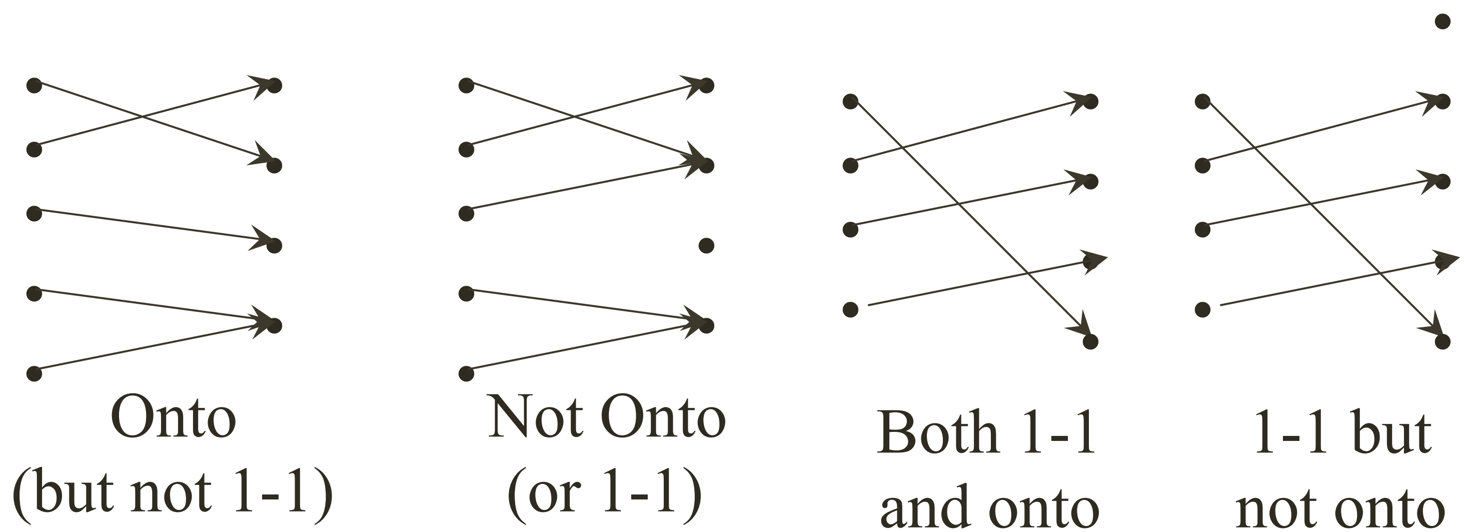
- Def. A function $f{:}A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b \in B, \exists a \in A: f(a)=b$).

- Remark. An *onto* function maps the set *A* onto (over, covering) the *entirety* of the set *B*, not just over a piece of it.

- Ex. Let $f{:}\mathbb{R} \rightarrow \mathbb{R}$.
- $f(x) = x^3$ is onto,
- $f(x) = x^2$ is not onto. (Why?)

Illustration of Onto

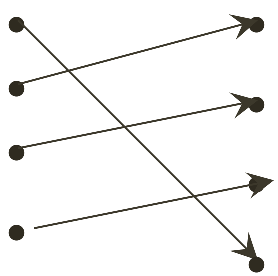
- Some functions that are, or are not, *onto* their codomains:



Bijections

- Def. A function f is said to be *a* *bijection*, (or *a* *one-to-one correspondence*, or *reversible*, or *invertible*,) iff
  it is <u>both</u> one-to-one <u>and</u> onto.

- Def. For bijections $f{:}A \rightarrow B$, there exists an *inverse of f*, written $f^{-1}{:}B \rightarrow A$, which is the unique function such that
  - (where I_A is the identity function on A)



$$f^{-1} \circ f = I_A$$

Both 1-1
and onto

|Domain| = |Codomain| = |Range|

We can invert the function!!

The Identity Function

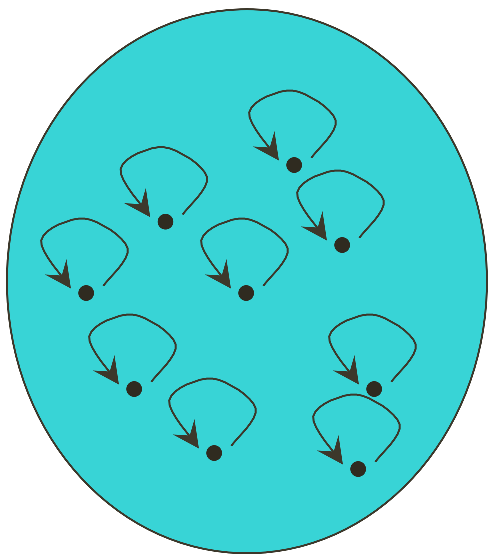
- Def. For any domain A, the *identity function* $I{:}A\rightarrow A$ (variously written, I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a\in A,\ I(a)=a$.

- Some identity functions you've seen:
  - f(x) = x+0, or f(x) = 1 x,
  - $\wedge$ with **T**, $\vee$ with **F**,
  - $\cup$ with $\varnothing$, $\cap$ with U.

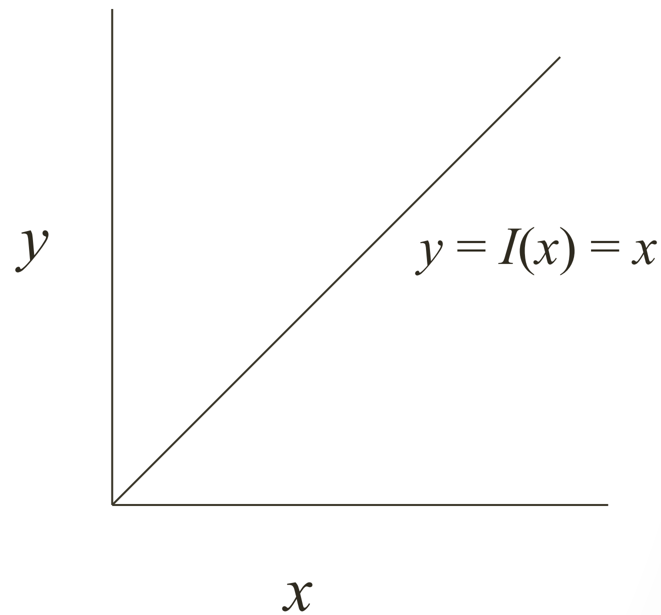
- Remark. The identity function is always both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



Graphs of Functions

- We can represent a function $f:A \rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$.

- Note that $\forall a$, there is only 1 pair (a,b).
  - Later (ch.6): *relations* loosen this restriction.

- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one y for each x.

A Couple of Key Functions

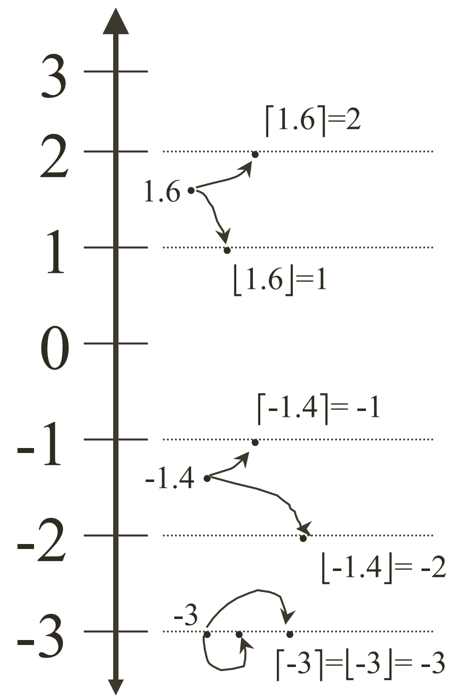
- In discrete math, we will frequently use the following two functions over real numbers:

- Def. The *floor* function $\lfloor\cdot\rfloor:\mathbb{R}\rightarrow\mathbb{Z}$, where $\lfloor x\rfloor$ ("floor of x") means the largest (most positive) integer $\leq x$. *Formally,* $\lfloor x\rfloor :\equiv \max(\{i\in\mathbf{Z} \mid i\leq x\})$.

- Def. The *ceiling* function $\lceil\cdot\rceil:\mathbb{R}\rightarrow\mathbb{Z}$, where $\lceil x\rceil$ ("ceiling of x") means the smallest (most negative) integer $\geq x$. *Formally,* $\lfloor x\rfloor :\equiv \min(\{i\in\mathbf{Z} \mid i\geq x\})$

Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$,
  $\lfloor -x \rfloor \neq - \lfloor x \rfloor$ &
  $\lceil -x \rceil \neq - \lceil x \rceil$
- Note that if $x \in \mathbf{Z}$,
  $\lfloor x \rfloor = \lceil x \rceil = x$.

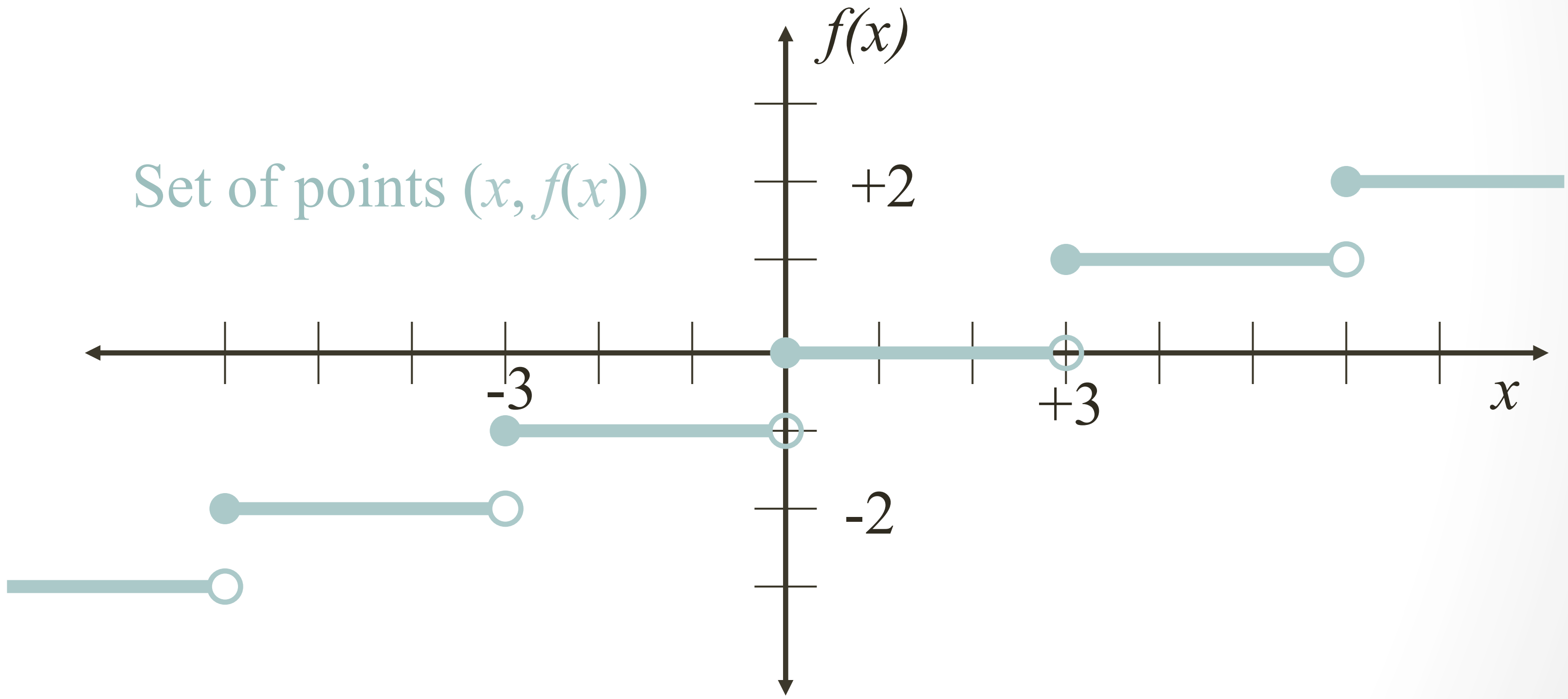


Plots with floor/ceiling

- Note that for $f(x)=\lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a \geq 0$ and $a<1$, but not for the value $a=1$.
- We say that the set of points $(a,0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$.
  - Sets that do not include all of their limit points are generally called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Operators (general definition)

- Def. An *n-ary operator over* (or *on*) the set *S* is any function from the set of ordered *n*-tuples of elements of *S*, to *S* itself.
- Ex. If $S=\{\mathbf{T},\mathbf{F}\}$,
  ¬ can be seen as a unary operator, and
  ∧,∨ are binary operators on *S*.
- Ex. ∪ and ∩ are binary operators on the set of all sets.

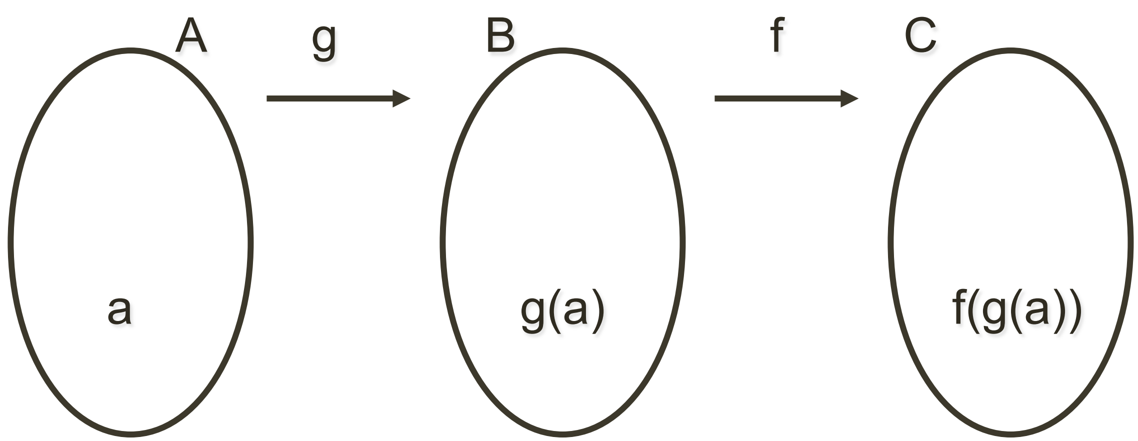
Constructing Function Operators

- If • (“dot”) is any operator over B, then we can extend • to also denote an operator over functions $f:A\rightarrow B$.
- Ex. Given any binary operator $\bullet:B\times B\rightarrow B$, and functions $f,g:A\rightarrow B$, we define $(f \bullet g):A\rightarrow B$ to be the function defined by: $\forall a\in A, (f \bullet g)(a) = f(a)\bullet g(a)$.

Function Operator Example

- +,× ("plus", "times") are binary operators over $\mathbb{R}$. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions*
- Def. Let $f, g: \mathbb{R} \to \mathbb{R}$.
  - $(f + g): \mathbb{R} \to \mathbb{R}$, where $(f + g)(x) = f(x) + g(x)$
  - $(f \times g): \mathbb{R} \to \mathbb{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Function Composition Operator

- Def. Let g:$A \to B$ and f:$B \to C$. *The composition of f and g*, denoted by $f \circ g$, is defined by $(f \circ g)=f(g(a))$.
- 

- Remark. $\circ$ (like Cartesian $\times$, but unlike $+, \wedge, \cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$.)

Review of §2.3 (Functions)

- Function variables f, g, h, …
- Notations: $f{:}A \rightarrow B$, $f(a)$, $f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1},
  binary operators +, -, *etc.*, and $\circ$.
- The $\mathbf{R} \rightarrow \mathbf{Z}$ functions $\lfloor x \rfloor$ and $\lceil x \rceil$.