Department of Mathematics

15B11MA211 **Mathematics-II**

Tutorial Sheet 3 **B.Tech. Core**

Series Solutions, Bessel and Legendre functions

1. Find the singular points of the following differential equations and classify them.
   (a) $x^2 y''-5y'+3x^2 y = 0$ (b) $x^2 y''+(\sin x)y'+(\cos x)y = 0$
   (c) $(x^2 + x-2)^2 y''+3(x+2)y'+(x-1)y = 0$ (d) $x^4 y''+4x^3 y'+y = 0$.
2. Solve the following differential equations in series:
   (a) $(1-x^2)y''+2xy'+y = 0$ (b) $xy''+y'+xy = 0$ (c) $x^2 y''+xy'+(x^2 - n^2)y = 0$
   (d) $(x-x^2)y''+(1-5x)y'-4y = 0$ (e) $8x^2 y''+10xy'-(1+x)y = 0$
   (f) $x(1+x)y''+(x+5)y'-4y = 0$.
3. Express the following polynomials in terms of Legendre polynomials
   (a) $f(x) = x^2$ (b) $f(x) = 4x^3 - 2x^2 - 3x + 8$ (c) $f(x) = x^4 + 2x^3 - 6x^2 + 5x - 3$.
4. State and prove Rodrigue's formula and hence obtain $P_0(x), P_1(x), P_2(x)$ & $P_3(x)$.
5. Prove that $P_n(0) = 0$, for n odd and $P_n(0) = \dfrac{(-1)^{n/2}}{2^n}\dfrac{n!}{\{(n/2)!\}^2}$, for n even.
6. Prove the following recurrence relations:
   (a) $nP_n = xP_n' - P_{n-1}'$ (b) $(1+2n)P_n = P_{n+1}' - P_{n-1}'$ (c) $(1-x^2)P_n' = n(P_{n-1} - xP_n)$.
7. Show that $J_2'(x) = (1-\dfrac{4}{x^2})J_1(x) + \dfrac{2}{x}J_0(x)$.
8. Prove that $J_n J_{-n}' - J_{-n} J_n' = -\dfrac{2Sin(n\pi)}{\pi x.}$. Hence deduce that $\dfrac{d}{dx}\left(\dfrac{J_{-n}}{Jn}\right) = -\dfrac{2Sin(n\pi)}{\pi x J_n^2}$.
9. Express $J_2(x), J_3(x)$ & $J_4(x)$ in terms of $J_0(x)$ & $J_1(x)$.
10. Prove that $J_1^{//}(x) = \dfrac{J_2(x)}{x} - J_1(x)$.
11. Prove that $\int J_0(x)\cos x dx = xJ_0(x)\cos x + xJ_1(x)\sin x + c$.

Answers:

1(a) $x = 0$, irregular singular point (b) $x = 0$, regular singular point

(c) $x = 1$, irregular singular point, $x = -2$ regular singular point

(d) $x = 0$, irregular singular point.

2(a) $y(x) = c_0(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + ...) + c_1(x - \frac{1}{2}x^3 + \frac{1}{80}x^5 + ...)$

(b) $y = a(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + ...) + b(u\log x + (\frac{x^2}{2^2} - \frac{x^2}{2^2.4^2}(1 + \frac{1}{2}) + ...))$

(c) $y = ax^n(1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4.8.(n+1).(n+2)} - ...) + bx^{-n}(1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4.8.(1-n).(2-n)} - ...)$

(d) $y = a(1 + 2^2.x + 3^2.x^2 + 4^2.x^3 + ..) + b(u\log x - 2(1.2x + 2.3x^2 + ...))$

(e) $y = ax^{\frac{1}{4}}(1 + \frac{x}{14} + \frac{x^2}{14.44} + ...) + bx^{\frac{-1}{2}}(1 + \frac{x}{2} + \frac{x^2}{220} + ...)$

(f) $y = a(1 + \frac{4}{5}x + \frac{1}{5}x^2 + ...) + bx^{-4}(1 + 4x + \frac{5}{4}x^2 + ...)$

3(a) $f(x) = \frac{2}{3}p_2(x) + \frac{p_0(x)}{3}$ (b) $\frac{8}{5}p_3(x) - \frac{4}{3}p_2(x) - \frac{3}{5}p_1 + \frac{22}{3}$

(c) $f(x) = \frac{8}{35}p_4 + \frac{4}{5}p_3 - \frac{24}{7}p_2 + \frac{31}{5}p_1 - \frac{44}{7}$

9.

$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$, $J_3(x) = (\frac{8}{x^2} - 1)J_1(x) - \frac{4}{x}J_0(x)$

$J_3(x) = (\frac{6}{x^2} - 1)\frac{8}{x}J_1(x) - (\frac{24}{x^2} - 1)J_0(x)$