Rules of Inference

- **Proof**: valid arguments that establish the truth of a mathematical statement
- **Argument**: a sequence of statements that end with a conclusion
- **Valid**: the conclusion or final statement of the argument must follow the truth of proceeding statements or **premise** of the argument

Argument and inference

- An **argument** is valid *if and only if* it is *impossible* for all the premises to be *true* and the conclusion to be *false*
- Rules of **inference**: use them to deduce (construct) new statements from statements that we already have
- Basic tools for establishing the truth of statements

Valid arguments in propositional logic

- Consider the following arguments involving propositions

"If you have a correct password, then you can log onto the network"

"You have a correct password"

premises

therefore,

"You can log onto the network"

conclusion

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Valid arguments

- $((p \to q) \wedge p) \to q$ is tautology
- When ((p→q) $\wedge$ p) is true, both p→q and p are ture, and thus q must be also be true
- This form of argument is true because when the premises are true, the conclusion must be true

Example

- p: “You have access to the network”
- q: “You can change your grade”
- $p \rightarrow q$: “If you have access to the network, then you can change your grade”

“If you have access to the network, then you can change your grade” ($p \rightarrow q$)

“You have access to the network” (p)

so “You can change your grade” (q)

Example

"If you have access to the network, then you can change your grade" ($p \rightarrow q$) False

"You have access to the network" (p) True

so "You can change your grade" (q)

- Valid arguments
- But the conclusion is not true
- **Argument form**: a sequence of compound propositions involving propositional variables

Inference Rules - General Form

- *Inference Rule* –
  - Pattern establishing that if we know that a set of *hypotheses* are all true, then a certain related *conclusion* statement is true.

Hypothesis 1
Hypothesis 2 …
$\therefore$ *conclusion*

"$\therefore$" means "therefore"

Inference Rules & Implications

- Each logical inference rule corresponds to an implication that is a tautology.
-
  - *Hypothesis 1*
    <u>*Hypothesis 2 …*</u>
    $\therefore$ *conclusion*

  Inference rule

- Corresponding tautology:

  $((Hypoth.\ 1) \wedge (Hypoth.\ 2) \wedge \ldots) \rightarrow conclusion$

Rules of inference for propositional logic

- Can always use truth table to show an argument form is valid
- For an argument form with 10 propositional variables, the truth table requires 2^{10} rows
- The tautology $((p \to q) \wedge p) \to q$ is the rule of inference called **modus ponens** (*mode that affirms*), or the **law of detachment**

$$\begin{array}{l} \quad p \\ \quad p \to q \\ \hline \therefore q \end{array}$$

Proofs: Modus Ponens

I have a total score over 96.

If I have a total score over 96, then I get an A for the class.

$\therefore$ I get an A for this class

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Example

- If both statements “If it snows today, then we will go skiing” and “It is snowing today” are true.
- By modus ponens, it follows the conclusion “We will go skiing” is true

Example

If $\sqrt{2} > \frac{3}{2}$ then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$

Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$

Is it a valid argument? Is conclusion true?

- The premises of the argument are $p \rightarrow q$ and p, and q is the conclusion
- This argument is valid by using modus ponens
- But one of the premises is false, consequently we cannot conclude the conclusion is true
- Furthermore, the conclusion is not true

Proofs: Modus Tollens

If the power supply fails then the lights go out.

The lights are on.

$\therefore$ The power supply has not failed.

$$
\begin{array}{l}
\neg q \\
p \to q \\
\hline
\therefore \neg p
\end{array}
$$

Tautology:

$$(\neg q \land (p \to q)) \to \neg p$$

Proofs: Addition

I am a student.

∴ I am a student or I am a visitor.

$$\frac{p}{\therefore\ p \vee q}$$

Tautology:

$p \rightarrow (p \vee q)$

Proofs: Simplification

I am a student and I am a soccer player.

∴ I am a student.

$$\frac{p \wedge q}{\therefore\ p}$$

Tautology:

$(p \wedge q) \rightarrow p$

Proofs: Conjunction

I am a student.
I am a soccer player.

$\therefore$ I am a student and I am a soccer player.

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

Tautology:

$((p) \wedge (q)) \rightarrow p \wedge q$

Proofs: Disjunctive Syllogism

I am a student or I am a soccer player.

I am a not soccer player.

$\therefore$ I am a student.

$$p \vee q$$

$$\neg q$$

$$\overline{\qquad\qquad}$$

$$\therefore\ p$$

Tautology:

$$((p \vee q) \wedge \neg q) \rightarrow p$$

Proofs: Hypothetical Syllogism

If I get a total score over 96, I will get an A in the course.

If I get an A in the course, I will have a 4.0 semester average.

$\therefore$ If I get a total score over 96 then
I will have a 4.0 semester average.

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

Tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Proofs: Resolution

I am taking CS1301 or I am taking CS2610.

I am not taking CS1301 or I am taking CS 1302.

$\therefore$ I am taking CS2610 or I am taking CS 1302.

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

Tautology:

$$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$$

It is not sunny this afternoon and it is colder than yesterday.
If we go swimming it is sunny.
If we do not go swimming then we will take a canoe trip.
If we take a canoe trip then we will be home by sunset.
We will be home by sunset

1. It is not sunny this afternoon and it is colder than yesterday.
2. If we go swimming it is sunny.
3. If we do not go swimming then we will take a canoe trip.
4. If we take a canoe trip then we will be home by sunset.
5. We will be home by sunset

p	It is sunny this afternoon
q	It is colder than yesterday
r	We go swimming
s	We will take a canoe trip
t	We will be home by sunset (the conclusion)

1. $\neg p \wedge q$
2. $r \to p$
3. $\neg r \to s$
4. $s \to t$
5. t

propositions

hypotheses

Using the rules of inference to build arguments

An example

p	It is sunny this afternoon
q	It is colder than yesterday
r	We go swimming
s	We will take a canoe trip
t	We will be home by sunset (the conclusion)

1. $\neg p \wedge q$
2. $r \to p$
3. $\neg r \to s$
4. $s \to t$
5. t

	Step	Reason
1.	$\neg p \wedge q$	Hypothesis
2.	$\neg p$	Simplification using (1)
3.	$r \to p$	Hypothesis
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \to s$	Hypothesis
6.	s	Modus ponens using (4) and (5)
7.	$s \to t$	Hypothesis
8.	t	Modus ponens using (6) and (7)

Rule of inference	Tautology	Name
$p \to q$, p, $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$, $p \to q$, $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollen
$p \to q$, $q \to r$, $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$, $\neg p$, $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p, $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$, $\therefore p$	$(p \wedge q) \to p$	Simplification
p, q, $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$, $\neg p \vee r$, $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (p \vee r)$	Resolution

Example

- “If you send me an email message, then I will finish my program” $p \to q$
- “If you do not send me an email message, then I will go to sleep early” $\neg p \to r$
- “If I go to sleep early, then I will wake up feeling refreshed” $r \to s$

$$
\begin{array}{ll}
1)\ p \to q & \text{hypothesis} \\
2)\neg q \to \neg p & \text{contrapositive of (1)} \\
3)\neg p \to r & \text{hypothesis} \\
4)\neg q \to r & \text{hypotheical syllogism using (2) and (3)} \\
5) r \to s & \text{hypothesis} \\
6)\neg q \to s & \text{hypothetical syllogism using (4) and (5)}
\end{array}
$$

- Conclusion
- “If I do not finish writing the program, then I will wake up feeling refreshed” $\neg q \to s$

Inference Rules for Quantified Statements

$$\frac{\forall x\, P(x)}{\therefore P(c)}$$

Universal Instantiation

(for an arbitrary object c from UoD)

$$\frac{P(c)}{\therefore \forall x\, P(x)}$$

Universal Generalization

(for all elements in UoD)

$$\frac{\exists x\, P(x)}{\therefore P(c)}$$

Existential Instantiation

(for some specific object c from UoD)

$$\frac{P(c)}{\therefore \exists x\, P(x)}$$

Existential Generalization

(for some object c from UoD)

Example

- "Everyone in this discrete mathematics has taken a course in computer science" and "Marla is a student in this class" imply "Marla has taken a course in computer science"

$1. \forall x(d(x) \rightarrow c(x))$	premise
$2. d(Marla) \rightarrow c(Marla)$	universal instantiation from (1)
$3. d(Marla)$	premise
$4. c(Marla)$	modus ponens from (2) and (3)

Example

- "A student in this class has not read the book", and "Everyone in this class passed the first exam" imply "Someone who passed the first exam has not read the book"

$1. \exists x(c(x) \wedge \neg b(x))$	premise
$2. c(a) \wedge \neg b(a)$	existential instantiation from (1)
$3. c(a)$	simpliciation from (2)
$4. \forall x(c(x) \rightarrow p(x))$	premise
$5. c(a) \rightarrow p(a)$	universal instantiation from (4)
$6. p(a)$	modus ponens from (3) and (5)
$7. \neg b(a)$	simplication from (2)
$8. p(a) \wedge \neg b(a)$	conjunction of (6) and (7)
$9. \exists x(p(x) \wedge \neg b(x))$	existential generalization form (8)

Universal modus ponens

- Use universal instantiation and modus ponens to derive new rule

$$\begin{array}{l} \forall x(p(x) \to q(x)) \\ p(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore q(a) \end{array}$$

- Assume "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. Show $100^2<2^{100}$

Universal modus tollens

- Combine modus tollens and universal instantiation

$$
\begin{array}{l}
\forall x(p(x) \to q(x)) \\
\neg q(a), \text{ where a is a particular element in the domain} \\
\hline
\therefore \neg p(a)
\end{array}
$$

Example

- Is this argument correct or incorrect?
  - "All TAs compose easy quizzes. Ramesh is a TA. Therefore, Ramesh composes easy quizzes."
- First, separate the premises from conclusions:
  - Premise #1: All TAs compose easy quizzes.
  - Premise #2: Ramesh is a TA.
  - Conclusion: Ramesh composes easy quizzes.

Answer

Next, re-render the example in logic notation.

- Premise #1: All TAs compose easy quizzes.
  - Let U.D. = all people
  - Let $T(x)$:≡ "x is a TA"
  - Let $E(x)$:≡ "x composes easy quizzes"
  - Then Premise #1 says: $\forall x, T(x) \rightarrow E(x)$

Answer cont…

- Premise #2: Ramesh is a TA.
  - Let R :≡ Ramesh
  - Then Premise #2 says: T(R)
- Conclusion says: E(R)

The Proof in Detail

- Statement — How obtained

1. $\forall x, T(x) \rightarrow E(x)$ **(Premise #1)**
2. $T(\text{Ramesh}) \rightarrow E(\text{Ramesh})$ **(Universal instantiation)**
3. $T(\text{Ramesh})$ **(Premise #2)**
4. $E(\text{Ramesh})$ **(*Modus Ponens* from statements #2 and #3)**

The argument is correct, because it can be reduced to a sequence of applications of valid inference rules

Another example

- Correct or incorrect? At least one of the 280 students in the class is intelligent. Y is a student of this class. Therefore, Y is intelligent.
- First: Separate premises/conclusion,
  & translate to logic:
  - Premises: (1) $\exists x$ InClass(x) $\wedge$ Intelligent(x)
    (2) InClass(Y)
  - Conclusion: Intelligent(Y)

Answer

- No, the argument is invalid; we can disprove it with a counter-example, as follows:
- Consider a case where there is only one intelligent student X in the class, and X≠Y.
  - Then the premise **∃*x* InClass(*x*) ∧ Intelligent(*x*)** is true, by existential generalization of **InClass(X) ∧ Intelligent(X)**
  - But the conclusion **Intelligent(Y)** is false, since X is the only intelligent student in the class, and Y≠X.
- Therefore, the premises *do not* imply the conclusion.

Proof Methods

- Proving $p \rightarrow q$
  - *Direct* proof: Assume p is true, and prove q.
  - *Indirect* proof (Proof by Contraposition): Assume $\neg q$, and prove $\neg p$.
  - *Trivial* proof: Prove q true.
  - *Vacuous* proof: Prove $\neg p$ is true.
- Proving p
  - Proof by *contradiction:* Prove $\neg p \rightarrow (r \wedge \neg r)$

  ($r \wedge \neg r$ is a contradiction); therefore $\neg p$ must be false.

Direct Proof Example

- **Definition:** An integer n is called *odd* iff $n=2k+1$ for some integer k; n is *even* iff $n=2k$ for some k.
- **Axiom:** Every integer is either odd or even.
- **Theorem:** (For all numbers n) If n is an odd integer, then n^2 is an odd integer.
- **Proof**: If n is odd, then $n = 2k+1$ for some integer k. Thus, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Therefore n^2 is of the form $2j + 1$ (with j the integer $2k^2 + 2k$), thus n^2 is odd. □

Another Example

- **Definition:** A real number r is *rational* if there exist integers p and $q \neq 0$, with no common factors other than 1 (i.e., gcd(p,q)=1), such that $r=p/q$. A real number that is not rational is called *irrational.*

- **Theorem:** Prove that the sum of two rational numbers is rational.

Proof

- The sum of two rational numbers is rational:
- Proof: Let r and w be two rational numbers. By definition of rational *r =a/b and w= c/d*

  where a, b,c,d are integers and b≠0 and d≠0

Consider $r + w = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

where bd ≠ 0 is an integer and ad +bc is an integer.

Therefore the sum is rational by definition

Indirect Proof

- Proving $p \rightarrow q$
  - *Indirect* proof: Assume $\neg q$, and prove $\neg p$.
  - Using contraposition

Indirect Proof Example

- **Theorem:** (For all integers n)
  If $3n+2$ is odd, then n is odd.
- **Proof:** Suppose that the conclusion is false, *i.e.*, that n is even. Then $n=2k$ for some integer k. Then $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$. Thus $3n+2$ is even, because it equals $2j$ for integer $j = 3k+1$. So $3n+2$ is not odd. We have shown that $\neg(n \text{ is odd}) \rightarrow \neg(3n+2 \text{ is odd})$, thus its contra-positive $(3n+2 \text{ is odd}) \rightarrow (n \text{ is odd})$ is also true. □

Another Example

- **Theorem:** Prove that if n is an integer and n^2 is odd, then n is odd.

- Proof: Suppose not. [We take the negation of the given statement and suppose it to be true.] Assume, to the contrary, that ∃ an integer n such that n^2 is odd and n is even. [We must deduce the contradiction.] By definition of even, we have

$$n = 2k \text{ for some integer } k.$$

So, by substitution we have

$$n \cdot n = (2k) \cdot (2k)$$

$= 2 (2.k.k)$

Now (2.k.k) is an integer because products of integers are integer; and 2 and k are integers. Hence,

$$n \cdot n = 2 \cdot (\text{some integer})$$

or

$$n^2 = 2. (\text{some integer})$$

and so by definition of n^2 even, is even. So the conclusion is since n is even, n^2, which is the product of n with itself, is also even. This contradicts the supposition that n^2 is odd.

Trivial Proof

- Proving $p \to q$
  - *Trivial* proof: Prove q true.

Trivial Proof Example

- **Theorem:** (For integers n) If n is the sum of two prime numbers, then either n is odd or n is even.
- **Proof:** *Any* integer n is either odd or even. So the conclusion of the implication is true regardless of the truth of the hypothesis. Thus the implication is true trivially. □

Vacuous Proof

- Proving $p \rightarrow q$
  - *Vacuous* proof: Prove $\neg p$ is true.

Vacuous Proof Example

- **Theorem:** (For all *n*) If *n* is both odd and even, then $n^2 = n + n$.
- **Proof:** The statement "*n* is both odd and even" is necessarily false, since no number can be both odd and even. So, the theorem is vacuously true. □

Proof by Contradiction

- Proving p
  - Assume $\neg p$, and prove that $\neg p \rightarrow$ q where q is false
  - Thus $\neg p \rightarrow \mathbf{F}$ is true only if $\neg p = \mathbf{F}$

Note: $r \wedge \neg r$ is a trivial contradiction, equal to **F**

Contradiction Proof Example

- **Theorem:** Prove that $\sqrt{2}$ is irrational.

- Let's suppose √2 is a rational number. Then we can write

  √2 = a/b where a, b are whole numbers, b not zero.
- We additionally assume that this a/b is simplified to lowest terms, since that can obviously be done with any fraction.
- From the equality √2 = a/b it follows that $2 = a^2/b^2$, or $a^2 = 2 \cdot b^2$. So the square of a is an even number since it is two times something.
- From this we know that a itself is also an even number. Why? Because it can't be odd; if a itself was odd, then $a \cdot a$ would be odd too. Odd number times odd number is always odd.

- If a itself is an even number, then a is 2 times some other whole number. In symbols, $a = 2k$ where k is this other number.
- If we substitute $a = 2k$ into the original equation $2 = a^2/b^2$, this is what we get:

  $2 = (2k)^2/b^2$ $2 = 4k^2/b^2$ $2 * b^2 = 4k^2$ $b^2 = 2k^2$
- This means that b^2 is even, from which follows again that b itself is even. And that is a contradiction!!!
- WHY is that a contradiction? Because we started the whole process assuming that a/b was simplified to lowest terms, and now it turns out that a and b both would be even. We ended at a contradiction; thus our original assumption (that √2 is rational) is not correct. Therefore √2 cannot be rational.

Disjunctive Normal Form (DNF)

A propositional formula is in disjunctive normal form (DNF) if it consists of a disjunction of (1, … ,*n*) disjuncts where each disjunct consists of a conjunction of (1, …, *m*) atomic formulas or the negation of an atomic formula.

In other words, a DNF is an OR of ANDs.

Example: $(p \land \neg q) \lor (\neg p \land q) \lor (p \land q)$ **DNF**

$(p \land q \land r) \lor (\neg \mathrm{p} \land q \land \neg r)$

$(p \land (q \lor r)) \lor (\neg \mathrm{p} \land q \lor \neg r)$ **Not DNF**

$\neg(\mathrm{p} \lor q)$

Conjunctive Normal Form

A compound proposition is in Conjunctive Normal Form (CNF) if it is a conjunction of disjunctions.

In other words, a CNF is an AND of ORs.

Example:

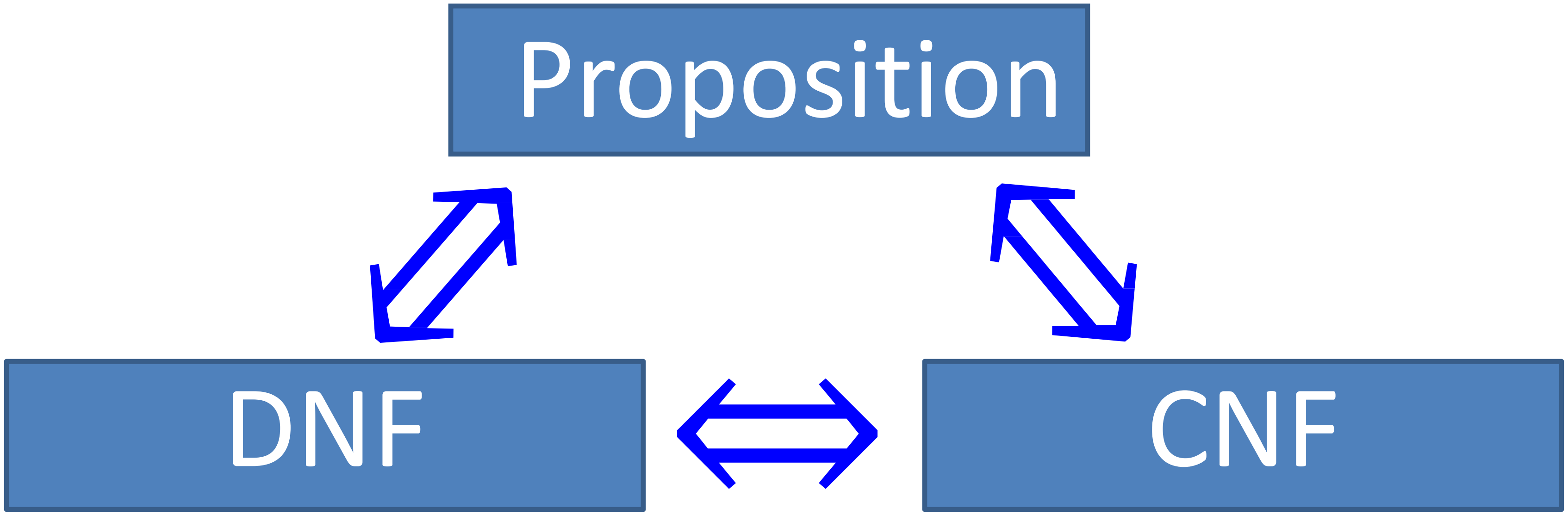
$(p \lor \neg q) \land (\neg p \lor q) \land (p \lor q)$
$(p \lor q \lor r) \land (\neg p \lor q \lor \neg r)$

CNF

$(p \lor (q \land r)) \land (\neg p \lor q \lor$
$\neg r)$
$\neg(p \land q)$

Not CNF

Proposition to CNF and DNF



Every compound proposition
can be rewritten in CNF or DNF.

Method to construct DNF

- Construct the Truth Table for the proposition
- Pick each row that evaluates to T

–If a variable r in this row is T then write it as it; otherwise, write the negation of it, i.e., $\neg r$

–OR these written literals (literal = variable or its complement)

How to find the DNF of $(p \vee q) \to \neg r$

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \to \neg r$
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

There are five sets of input that make the statement true. Therefore there are five minterms.

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \rightarrow \neg r$
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

From the truth table we can set up the DNF

$$(p \vee q) \rightarrow \neg r \iff (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

Converting Expressions to DNF or CNF

The following procedure converts an expression to DNF or CNF:

1. Remove all $\Rightarrow$ and $\Leftrightarrow$.
2. Move $\neg$ inside. (Use De Morgan's law.)
3. Use distributive laws to get proper form.

Simplify as you go. (e.g. double-neg., idemp., comm., assoc.)

Example: Transformation into CNF

Transform the following formula into CNF.

$$\neg(p \to q) \vee (r \to p)$$

1. Express implication by disjunction and negation.

$$\neg(\neg p \vee q) \vee (\neg r \vee p)$$

2. Push negation inwards by De Morgan's laws and double negation.

$$(p \wedge \neg q) \vee (\neg r \vee p)$$

3. Convert to CNF by associative and distributive laws.

$$(p \vee \neg r \vee p) \wedge (\neg q \vee \neg r \vee p)$$

4. Optionally simplify by commutative and idempotent laws.

$$(p \vee \neg r) \wedge (\neg q \vee \neg r \vee p)$$

and by commutative and absorbtion laws

$$(p \vee \neg r)$$

Example Transformation to DNF

I $(p \rightarrow q) \wedge \neg q$

- $(\neg p \vee q) \wedge \neg q$
- $(\neg p \wedge \neg q) \vee (q \wedge \neg q)$

II $\neg (p \wedge q) \leftrightarrow (p \vee q)$

Another method to construct CNF

We may construct a CNF formula A' which is equivalent to A by means of the following truth table method. For each truth assignment τ (for exactly the atoms in A) with $\tau(A) = false$, include in A' the clause which is the set of all literals L with $\tau(L) = false$. An equivalent DNF can be constructed dually: by including a conjunction for each assignment which satisfies A.

Example 1. *Let* $A = ((P \wedge Q) \vee (\neg P \wedge \neg Q))$*. The truth table for* A*, adorned with the clauses to include in the CNF, is:*

P	Q	A	*clause to include in* A'
true	*true*	*true*	*none*
true	*false*	*false*	$(\neg P \vee Q)$
false	*true*	*false*	$(P \vee \neg Q)$
false	*false*	*true*	*none*