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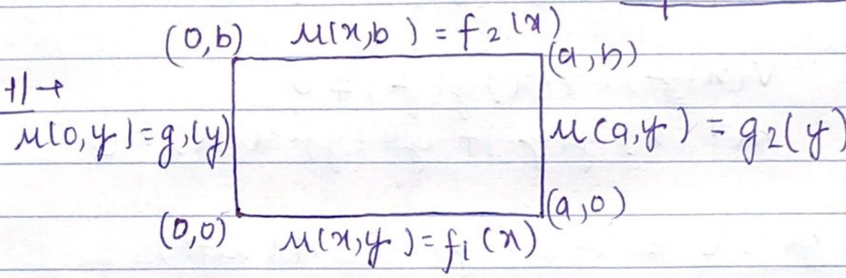
MATHS ASSIGNMENT - 2 [Partial differential equations]

Ques 1 →



$$u_{xx} + u_{yy} = 0$$

Boundary conditions:

$u(x,0) = f_1(x)$

$u(a,y) = g_2(y)$

$u(x,b) = f_2(x)$

$u(0,y) = g_1(y)$

$u = u_1 + u_2 + u_3 + u_4$

For u_1 :-

$u(0,y) = 0$

$u(x,0) = 0$

$u(x,b) = 0$

$u(a,y = g_2 y$

For u_2 :-

$u(0,y) = g_1(y)$

$u(x,0) = 0$

$u(a,y) = 0$

$u(x,b) = 0$

For u_3 :-

$u(x,0) = f_1(x)$

$u(0,y) = 0$

$u(x,b) = 0$

$u(a,y) = 0$

For u_4 :-

$u(x,b) = f_2(x)$

$u(0,y) = 0$

$u(x,y) = 0$

$u(a,y) = 0$

Subject: solution for u_1 :-

Date

$$v_{xx} + v_{yy} = 0 \quad , \quad (x,y) \in D$$

$$\text{BCS} \begin{cases} v(0,y) = 0 \\ v(x,0) = 0 \\ v(x,b) = 0 \\ v(a,y) = g_2(y) \end{cases}$$

$$v(x,y) = x(x)\,y(y) \neq 0$$

$$x''(x)\,y(y) + y''(y) \cdot x(x) = 0$$

$$\frac{x''(x)}{x(x)} = \frac{-y''(y)}{y(y)} = \lambda \text{ (left)}$$

$$x''(x) - \lambda(x) = 0$$

$$y''(y) + \lambda y(y) = 0$$

→ using boundary conditions :

① $v(x,0) = x(x)\,y(0) = 0 \Rightarrow y(0) = 0$

② $v(x,b) = x(x)\,y(b) = 0 \Rightarrow y(b) = 0$

③ $v(a,y) = x(a)\,y(y) = 0 \Rightarrow x(a) = 0$

④ $v(0,y) = x(0)\,y(y) = 0 \Rightarrow x(0) = 0$

$$(1 \cdot y')' + 0 \cdot y = \lambda \cdot 1 \cdot y \quad , \quad 0 < y < b$$

$$y(0) = 0 = y(b)$$

λ is real

$\Rightarrow \lambda = \mu^2$ and $\lambda = 0$

or

$\lambda = -\mu^2$ with $\mu > 0$

Case-I :- $\lambda = \mu^2$

$$y'' - \mu^2 y = 0$$

$\cancel{y(0)} = 0 \Rightarrow$

$$y(y) = c_1 e^{\mu y} + c_2 e^{-\mu y}$$

Subject: $y(0)=0 \Rightarrow c_1=-c_2$ Date

$y(y) = 2c_1 \sinh \mu y$

$y(b) = 0 \Rightarrow 2c_1 \sinh \mu y = 0$

$\Rightarrow c_1 = 0$

$\Rightarrow y(y) = 0, \quad \forall \; y \in (0,b)$

$\lambda = \mu^2$ is not eigen value

Case 2: $\lambda = 0$

$y'' = 0$

$\boxed{y(y) = c_1 y + c_2 = 0}$

$y(0) = 0 \Rightarrow c_2 = 0$

$y(b) = 0 \Rightarrow c_1 = 0$

$\Rightarrow \lambda = 0$ not an eigen value

Case 3:- $\lambda = -\mu^2$

$y'' + \mu^2 y = 0$

$y(y) = c_1 \cos \mu y + c_2 \sin \mu y$

$y(y) = 0 \Rightarrow c_1 = 0$

$y(b) = 0 \Rightarrow c_2 \sin \mu b = 0$

$\mu = \frac{n\pi}{b}, \quad n = 1, 2, 3 ---$

eigen value:- $\lambda_n = \frac{-n^2\pi^2}{b^2}, \quad n = 1, 2, 3 --$

$y_0(y) = \sin \frac{n\pi x}{b}, \quad n = 1, 2, 3$

$x'' - \frac{n^2\pi^2}{b^2} x(n)(n) = 0$

$\Rightarrow x_n(x) = A_n e^{\frac{n\pi x}{b}} + B_n e^{-\frac{n\pi}{b}}$

$v(x,y) = (A_n e^{\frac{n\pi x}{b}} + B_n e^{-\frac{n\pi}{b}}) \cdot \sin \frac{n\pi x}{b}$

Subject: Next → $V(0,y) = 0 \Rightarrow \sum_{n=1}^{\infty} (A_n + B_n) \sin \frac{\pi n y}{b} = 0$ Date

$$A_n + B_n = 0 \quad , \quad n = 1, 2, 3 \, ----$$

$$A_n = -B_n$$

$$V(a,y) = 0 \Rightarrow \sum_{n=1}^{\infty} 2A_n \sinh \frac{n\pi a}{b} , \sin \frac{n\pi y}{b} = g(y)$$

$$2A_n \sin \frac{by\pi a}{b} = \frac{\int_a^b g(y) \sin \frac{n\pi y}{b} dy}{\int_0^b \sin^3 n\pi y \, dy}$$

$$A_n = \frac{1}{b \sin \frac{n\pi a}{b}} \int_0^b g(y) \sin \frac{n\pi y}{b} \cdot dy$$

General solution :-

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \mu_n x \sinh \mu_n y$$

Also if $A_n = \frac{2}{n\pi} \frac{1}{\sinh \left(\frac{n\pi b}{a} \right)} \int_0^a f(x) \cos \frac{n\pi x}{a} dx$

we have :- $u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{a} \cosh \frac{n\pi y}{a}$

$$A_n = \frac{2}{n\pi} \sinh \left(\frac{n\pi b}{a} \right) \int_0^a x(x-a) \cos \frac{n\pi x}{a}$$

$$\int_0^a x(x-a) \cos \frac{n\pi x}{a} dx =$$

$$\Rightarrow \frac{2a^3 \cos n\pi}{n^2\pi^2} - \frac{a^3 \cos n\pi}{n^2\pi^2}$$

$$\frac{a^3 \cos n\pi}{n^2\pi^2} \Rightarrow \frac{a^3}{n^2\pi^2} (-1)^{2n}$$

Subject: n is even :-

$A_n = \frac{a^3}{n^2\pi^2}$, $A_2 = \frac{a^3}{4\pi^2}$, $A_4 = \frac{a^3}{16\pi^2}$

n is odd :-

$A_n = \frac{-a^3}{n^2\pi^2}$ $A_1 = \frac{-a^3}{\pi^2}$, $A_3 = \frac{-a^3}{9\pi^2}$

$A_0 = \frac{2}{a}\int_0^a x(x-a)\,dx$.

$= \frac{2}{a}\left[\frac{x^3}{3} - \frac{ax^2}{2}\right]_0^a = \frac{2}{a}x - \frac{a^3}{6} = \frac{-a^2}{3}$

$u(x,y) = \frac{-a^2}{3} + \left[\frac{-a^3}{\pi^2}\cos\frac{\pi x}{a}\cosh\frac{\pi y}{a} + \frac{a^3}{4\pi^2}\,2\cos\frac{\pi x}{a}\right.$
$\left.\cosh\frac{2\pi n}{a} - \frac{a^3}{9\pi^2}\cos\frac{3\pi x}{a}\cos\frac{3h\pi y}{a} + ----\right]$

$=$ Ans.

Ques 12 → $T(x,t) = e^{-\alpha\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$

$\therefore \frac{dT}{dx} = e^{-\alpha\lambda^2 t}(-A\lambda\sin\lambda x + B\lambda\cos\lambda x)$

using (11) Boundary condition
i.e $\left.\frac{dT}{dx}\right|_{x=0} = 0$

$B = 0$

using $\left.\frac{dT}{dx}\right|_{x=0} = 0$

$e^{-\alpha\lambda^2 t}(-A\lambda\sin\lambda a + B\lambda\cos\lambda a) = 0$

$\sin\lambda = a = 0$

Subject : $\lambda a = n\pi$ Date

$\lambda = \frac{n\pi}{a}$, $n = 0, 1, 2, 3 - - -$

now using principle of superposition:

$$T(x,t) = \sum A_n e^{-\alpha\lambda^2 t} \cos \lambda x$$
$$= \sum_{n=0}^{\infty} A_n e^{-\alpha\left(\frac{n\pi}{a}\right)^2 t} \cos \frac{n\pi x}{a}$$

using $T(x,0) - x(a-x)$ & $0 < x < a$

$$T(x,0) = x(a-x) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\alpha\left(\frac{n\pi}{a}\right)^2 t}$$

$$A_0 = \frac{2}{a}\int_0^a x(a-x)\,dx$$

$$= \frac{2}{a} = \left[\frac{a^3}{2} - \frac{a^3}{3}\right] = \frac{2}{a} \times \frac{a^3}{6} \qquad A_0 = \frac{a^3}{3}$$

$$A_n = \frac{2}{a}\int_0^a x(a-x)\cos\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\left[\int_0^a x\,a\cos\left(\frac{n\pi x}{a}\right)dx - \int_0^a x^2\cos\left(\frac{n\pi x}{a}\right)\right]$$

$$= \frac{-2a^2}{n^2\pi^2}(1+\cos n\pi) = \frac{-2a^2}{n^2\pi^2}(1+(-1)^n))$$

If n is even :- $A_n = \frac{-4a^2}{n^2\pi^2}$

If n is odd :- $A_n = 0$

$$T(x,t) = \frac{a^2}{3} - \frac{4a^2}{\pi^2}\sum_{n=2,4,6---}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{a}\right) e^{-\alpha\left(\frac{n\pi}{a}\right)^2 t}$$

= **Ans**