BILINEAR TRASFORMATION

Definition: The map

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0 \qquad -①$$

is called bilinear transformation or linear fractional transformation or sometimes Möbius transformation. This can also be written as

$$cwz + wd - az - b = 0, \quad ad-bc \neq 0 \qquad -②$$

As equation ② is linear in both w and z, that is why, it is called bilinear transformation.

If $ad-bc=0$ then w is constant.

Proof: Let w_1 & w_2 be ~~~~

$$w = \frac{a(z+b/a)}{c(z+d/c)}$$

$\because$ $ad-bc=0 \Rightarrow b/a = d/c$

hence

$$w = \frac{a(z+b/a)}{c(z+b/a)} = \frac{a}{c} = \text{constant}.$$

A constant function is not linear and hence $ad-bc \neq 0$ is the necessary condition for $w = \dfrac{az+b}{cz+d}$ to be a bilinear transformation.

Ex:-1. Which of the following is a bilinear transformation.

ⓐ $w = \dfrac{2z+1}{4z+2}$ ⓑ $\dfrac{(2+3i)z+i}{-13iz+(2-3i)}$ ⓒ $w=z$.

Sol^n for ⓐ $w = \dfrac{2z+1}{4z+2}$, $a=2,\ b=1,\ c=4,\ d=2 \Rightarrow ad-bc = 4-4=0$

This is not a bilinear transformation

ⓑ $w = \dfrac{(2+3i)z+i}{-13iz+(2-3i)}$, $a = 2+3i,\ b=i,\ c=-13i,\ d=2-3i$

Now $ad-bc = (2-3i)(2+3i) - 13 = 13-13=0$

This is not a bilinear transformation

ⓒ $w=z$, $a=1,\ b=0,\ c=0,\ d=1 \Rightarrow ad-bc = 1 \neq 0$

This is a bilinear transformation.

Fixed Points or invariant Points A bilinear transformation $w = f(z)$ has a fixed Point z_0 if

$$z_0 = f(z_0).$$

fixed Points of $w = f(z)$ are obtained by the equation

$$z = f(z) = \frac{az+b}{cz+d}$$

If $c = 0$ & $a-d=0$ $\Rightarrow$ only one fixed Point which is '∞'
in this case $w = z + b/d$

If $c=0$ & $a-d \neq 0$ $\Rightarrow$ one finite fixed Point & other fixed Point is '∞'.

Ex- find the fixed Points of the following transformations:

(i) $w = \dfrac{z}{z-2}$

Solⁿ fixed Points are given by $w=z$ $\Rightarrow$ $z = \dfrac{z}{z-2}$

$\Rightarrow$ $z^2 - 2z - z = 0$ $\Rightarrow$ $z(z-3) = 0$ $\Rightarrow$ $z = 0, 3$

Hence fixed Points are $0, 3$.

(ii) $w = \dfrac{3iz+1}{z+i}$

Solⁿ fixed Points are given by $z = \dfrac{3iz+1}{z+i}$ $\Rightarrow$ $z^2 - 2iz - 1 = 0$

$\Rightarrow$ $(z-i)^2 = 0$ $\Rightarrow$ $z = i, i$.

Hence there is only one distinct fixed point namely 'i'.

(iii) $w = \dfrac{(2+i)z - 2}{z+i}$

Solⁿ fixed Point $\Rightarrow$ $z = \dfrac{(2+i)z-2}{z+i}$

$\Rightarrow$ $z^2 - 2z + 2 = 0$ $\Rightarrow$ $z = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\Rightarrow$ fixed Points are $1+i$, $1-i$.

(iv) $w = z + \dfrac{2i}{(2-3i)}$

Solⁿ Here $a = (2-3i)$, $b = 2i$, $c = 0$, $d = 2-3i$

$a - d = 0$ & $c = 0$ $\Rightarrow$ only fixed Point is '∞'

(v) $w = \dfrac{2iz + 3}{3i}$

Fixed Points. $a = 2i,\ b = 3,\ c = 0,\ d = 3i.$

here $a - d = 2i - 3i = -i \neq 0$ & $c = 0$ $\Rightarrow$ one finite fixed Point & other fixed Point is '∞'.

The finite fixed Point is given by

$$z = \frac{2iz+3}{3i}$$

$$3iz = 2iz + 3$$

$$iz = 3$$

$$z = 3/i = -3i$$

So the fixed Points are $-3i, \infty$.

CROSS RATIO :- If z_1, z_2, z_3, z_4 are distinct points, then the ratio

$$(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$$

is called the cross ratio of z_1, z_2, z_3, z_4.

This ratio is invariant under the bilinear transformation.

i.e.

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

The bilinear transformation which maps three distinct Points z_1, z_2, z_3 in z-plane onto three distinct Points w_1, w_2, w_3 is given by

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

If one of the numbers z_1, z_2, z_3 is infinite (say $z_3 = \infty$) then in the cross ratio, the factors involving z_3 will be replaced by '1', i.e. $(z_2 - z_3) = -1$, $(z_3 - z) = -1$

Similarly, if any of the numbers w_1, w_2, w_3 is infinite, then we will replace the factor involving that number by '-1'.

Ex-1. find the bilinear transformation which maps points $-1, 0, 1$ onto $0, i, 3i$.

Sol^n.

$z_1 = -1,\ z_2 = 0,\ z_3 = 1,\ \&\ w_1 = 0,\ w_2 = i,\ 3i.$

the bilinear transformation is given by

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\Rightarrow \frac{w(\cancel{-}2i)}{\cancel{-}i(3i-w)} = \frac{\cancel{-}(z+1)}{\cancel{-1}(1-z)}$$

$$\Rightarrow \frac{2w}{3i-w} = \frac{1+z}{1-z}$$

$$\Rightarrow 2w - 2wz = 3i - w + 3iz - zw$$

$$2w(1-z) = 3i + 3iz - w(1+z)$$

$$2w(1-z) + w(1+z) = 3i(1+z)$$

$$w[2(1-z) + 1+z] = 3i(1+z)$$

$$w(3-z) = 3i(1+z)$$

$$w = \frac{3i(1+z)}{3-z}$$

we have to find w ~~ecel~~ in terms of z

$$\boxed{w = \frac{3i(1+z)}{3-z}}$$

→ This is required bilinear transformation

Ex-2. find the bilinear transformation which maps point $i, 1, -1$ onto $1, 0, \infty$ respectively

Sol^n

$z_1 = i,\ z_2 = 1,\ z_3 = -1\ ;\ w_1 = 1,\ w_2 = 0,\ w_3 = \infty$

the desired bilinear trans is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)},$$

Since $w_3 = \infty$ so we will Replace factors $(w_2 - w_3) = -1$ & $(w_3 - w) = -1$

So

$$\frac{(w-1)(-1)}{(1-0)(-1)} = \frac{(z-i)\{1-(-1)\}}{(i-1)(-1-z)}$$

$$\frac{(w-1)}{1} = \frac{(z-i)(2)}{(1+z)(1-i)}$$

$$w = 1+\frac{2(z-i)}{(1+z)(1-i)} = \frac{(1+z)(1-i)+2(z-i)}{(1+z)(1-i)}$$

$$w = \frac{1-i+z-iz+2z-2i}{(1+z)(1-i)} = \frac{1-3i+3z-iz}{(1+z)(1-i)}$$

$$\boxed{w = \frac{i(1-z)}{1+z}}$$

which is required bilinear transformation.

Ex:-3 Find the bilinear transformation that maps the points $z_1=\infty$, $z_2=i$, $z_3=0$ into the points $w_1=0$, $w_2=i$ & $w_3=\infty$.

Soln. The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Now

$$\frac{(w-0)(-1)}{(0-i)(-1)} = \frac{(-1)\cdot(i-0)}{(-1)\cdot(0-z)}$$

$$\Rightarrow \frac{w}{-i} = \frac{i}{-z}$$

$$\Rightarrow \boxed{w = -\frac{1}{z}}$$

Ans

Exercise Q.1 ① find the fixed Points of the following bilinear transformations

(1) $w = \frac{z-1}{z+1}$ (2) $w = \frac{z}{2-z}$. (3) $w = z + \frac{1}{i}$

(4) $w = \frac{8z+3i}{7i}$

Q.2 find the bilinear transformation which maps).

(a) $\{\infty, i, 0\}$ onto $\{0, 1, \infty\}$

(b) $\{-1, 0, 1\}$ onto $\{-i, 1, i\}$

(c) $\{0, 1, \infty\}$ onto $\{w_1, w_2, w_3\}$.

(d) $\{1, i, -1\}$ onto $\{i, 0, -i\}$.

Q.3 find all the bilinear transformations which have fixed Points as $-i$ and i.

More examples on fixed Points.

Ex: find ~~the~~ all the bilinear transformations which have fixed Points as -1 and 1.

Sol: by definition of fixed Points. $w = z$. $\Rightarrow w = \frac{az+b}{cz+d}$

for fixed Point $w=z=1$ $\Rightarrow 1 = \frac{a+b}{c+d}$ $(\because w=z=1)$

$\Rightarrow a+b = c+d$ —①.

Now for fixed Point $w=z=-1$ $\Rightarrow -1 = \frac{-a+b}{-c+d}$

$\Rightarrow c-d = -a+b$ —②

Now solving equation ① & ② we get

$a = d$ & $b = c$.

Hence required transformation is given by

$$w = \frac{az+b}{cz+d} = \frac{az+b}{bz+a}, \quad ad-bc \neq 0 \text{ i.e. } a^2-b^2 \neq 0$$

$$\boxed{w = \frac{az+b}{bz+a}, \quad a^2-b^2 \neq 0}$$ Ans