- Classification of Second-order partial differential Equations

- **Definition:** An Equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential Equation.

- **Examples:**

1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$; where $z = f(x,y)$.

2) $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$

3) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$; $u = f(x,y,z)$.

- Classification:

Notion: $p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$; $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x \partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$ ...

- Consider a linear, Second-order Equation of the form

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = 0 \quad \text{— (1)}$$

The Equation (1) will be classify in to three Categories, if the discriminant

- $b^2 - 4ac > 0$; Say the Equation (1) is hyperbolic.
- $b^2 - 4ac = 0$; Say the Equation (1) is parabolic.
- $b^2 - 4ac < 0$; Say the Equation (1) is Elliptic.

Examples: Classify the following differential Equations:

1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$

2) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$

3) $x\, u_{xx} + u_{yy} = x^2$

Solution: (1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$

- Comparing this Equation to the general form of 2^{nd}-order pde, we have.

$a = 1, \quad b = 1, \quad c = -2$

The discriminant $b^2 - 4ac = (1)^2 - 4(1)(-2)$

$= 1 + 4 \times 2$

$= 1 + 8 = 9 > 0$

$\therefore$ The given PDE is hyperbolic.

#

(2) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$

- Comparing above Eq^n with the general form of 2^{nd} order pde, we get

~~A~~ $a = x, \quad b = 0, \quad c = y$

The discriminant $b^2 - 4ac = (0)^2 - 4xy$

$= -4xy.$

The classification here depends on the sign of x and y. There are following Cases arise.

- Case(1): If $x > 0;\ y > 0$

$\Rightarrow$ Discriminant $= -4xy < 0$ ($\because x \cdot y > 0$).

$\Rightarrow$ PDE is Elliptic. #

- Case (2): If $x > 0;\ y < 0 \mid x < 0;\ y > 0$

$$\Downarrow$$

$$xy < 0$$

$\Rightarrow$ The discriminant $b^2 - 4ac = -4xy > 0$ because $xy < 0$.

$\Rightarrow$ PDE is hyperbolic #

- Case (3): Either $x = 0$ or $y = 0$.

$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow$ PDE is Parabolic #

③ $x u_{xx} + u_{yy} = x^2$

- Comparing above Eqn with general form of 2nd order differential Equation, we have

$$a = x,\quad b = 0,\quad c = 1$$

- The discriminant $b^2 - 4ac = (0)^2 - 4(x)(1)$

$$= -4x$$

- Here classification depends on the sign of x.

- Case(1): If $x > 0$, then $b^2 - 4ac = -4x < 0$

$\Rightarrow$ PDE is Elliptic. #

- Case(2): If $x < 0$; then $b^2 - 4ac = -4x > 0$

$\Rightarrow$ PDE is Hyperbolic. #

Case(3): If $x=0$ then $b^2-4ac=0$

$\Rightarrow$ PDE is parabolic. #

Q: Classify the following PDEs:

(i) $2u_{xx}+4u_{xy}+3u_{yy}=2$.

(ii) $u_{xx}+4u_{xy}+4u_{yy}=0$

(iii) $xyr-(x^2-y^2)s-xyt+py-qx=2(x^2-y^2)$.

(iv) $2y^2r-2xys+x^2t=(y^2p)/x+(x^2q)/y$.

§: Method of Separation of Variables:

This is a very powerful technique for solving linear PDEs that have no mixed derivatives, i.e., nothing of the form $\partial^2 f/\partial x\partial y$.

- Working Rule:

(1) Assume the solution is going to be of the form $X(x).T(t)$ or $X(x)\cdot Y(y)$ etc. This is called separable form.

(2) Substitute that form back into the PDE.

(3) Devide by $X(x)T(t)$ or $X(x)Y(y)$.

(4) Now each term of the equation depends on a different variable so they must both be constants.

(5) For each possible value of the constant (positive, negative, zero), solve the two resulting ODEs and multiply the solutions together to give one specific solution to the PDE.

(6) Form the general solution of the PDE by adding linear combinations of all the specific Sol^{s},

- Examples: ① Solve (by the method of Separation of Variables):

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

② Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \;; \text{ where } u(x,0) = 6e^{-3x}.$$

③ Solve the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t} \;;\quad 0<x<3,\ t>0$$

for the B.Cs: $u(0,t) = u(3,t) = 0$ and the initial condition: $u(x,0) = 5\sin 4\pi x$.

Sol^n ①: Suppose the solution $z = X(x).Y(y)$. where X is a function of x alone and Y that of y alone.

- Using the value of z in the given PDE:

$\partial z/\partial x = X'(x).Y(y)$; $\partial^2 z/\partial x^2 = X''(x).Y(y)$

$\partial z/\partial y = X(x).Y'(y)$;

We have.

$\Rightarrow X''Y - 2X'Y + XY' = 0$

$\Rightarrow \boxed{\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}}$ (Separating the variables).

$\therefore \dfrac{X'' - 2X'}{X} = a$; i.e., $X'' - 2X' - aX = 0$ — (A)

and $-\dfrac{Y'}{Y} = a$; i.e., $Y' + aY = 0$ — (B)

(A) $\Rightarrow X'' - 2X' - aX = 0$ (2^{nd} order linear DE).

A.E.: $m^2 - 2m - a = 0$

$\Rightarrow m = \dfrac{2 \pm \sqrt{4+4a}}{2} = \dfrac{2 \pm 2\sqrt{1+a}}{2} = 1 \pm \sqrt{1+a}$

$\Rightarrow m_1 = 1+\sqrt{1+a}$

$m_2 = 1-\sqrt{1+a}$

$\therefore$ The solution $X(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow X(x) = c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}$

(B) $\Rightarrow Y(y) = c_3 e^{-ay}$

$\therefore$ The general solution is

$z = X(x).Y(y)$

$= \left[c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}\right]. c_3 e^{-ay}$

$\Rightarrow \boxed{z = \left[a e^{(1+\sqrt{1+a})x} + b e^{(1-\sqrt{1+a})x}\right] e^{-ay}}$; $\left.\begin{array}{l} a = c_1 c_3 \\ b = c_2 c_3 \end{array}\right\}$ constants.

#

Solⁿ (2): $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$; $u(x,0) = 6e^{-3x}$

Suppose the solution is

$u(x,t) = X(x).T(t)$

$\Rightarrow u_x' = X'(x).T(t)$ or $X'T$

$u_t = X(x).T'(t)$ or XT'

Substituting in the given PDE, we have.

$\Rightarrow (X' - X)*T = 2XT'$

OR $\frac{X'-X}{2X} = \frac{T'}{T} = K$ (say)

$\therefore \frac{X'-X}{2X} = K$ and $\frac{T'}{T} = K$ — (B)

$\Rightarrow X' - X - 2KX = 0$

$\Rightarrow X'/X = 1+2K$ — (A)

Ⓐ $\Rightarrow \int \frac{X'}{X} = \int 1+2k$

$\Rightarrow \log X = (1+2k)x + \log c$

$\Rightarrow \boxed{X = c\, e^{(1+2k)x}}$

Ⓑ $\Rightarrow \int \frac{T'}{T} = \int k$

$\Rightarrow \log T = kt + \log c'$

$\Rightarrow \boxed{T = c' e^{kt}}$

$\therefore$ The solution is $u(x,t) = X(x).T(t)$.

$\Rightarrow \boxed{u(x,t) = cc' e^{(1+2k)x} \cdot e^{kt}}$ —(*)

Using initial condition

$u(x,0) = 6e^{-3x} = cc' e^{(1+2k)x}$

$\Rightarrow cc' = 6$ and $1+2k = -3$

$\Rightarrow \boxed{k = -2}$

$\therefore$ Solⁿ is $u(x,t) = 6e^{-3x} \cdot e^{-2t}$

$\Rightarrow \boxed{u(x,t) = 6e^{-(3x+2t)}}$ #