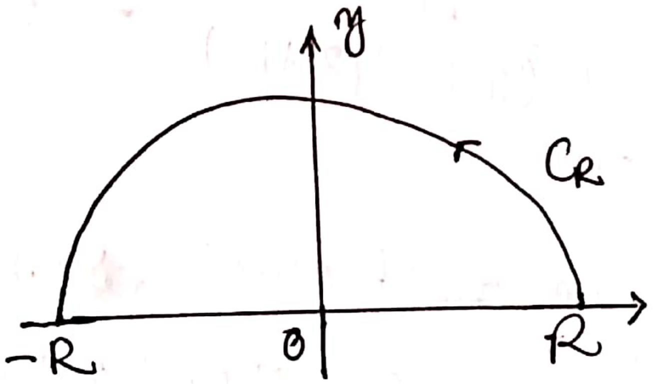
§: Evaluation of the Integral of the type $\int_{-\infty}^{\infty} f(x)\,dx$.

Theorem: If $f(z)$ is a function which is analytic in the upper half of z-plane except at a finite number of poles in it, having no poles on the real-axis and if further $z f(z)$ tends to zero as $|z| \to \infty$, then by contour integration

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^{+},$$

where $\sum R^{+}$ represents the sum of the residues at poles in upper half plane.



Another varient:

$$\int_{C} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum R^{+}.$$

Q1: Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx \quad ; \quad a>0,\ b>0.$$

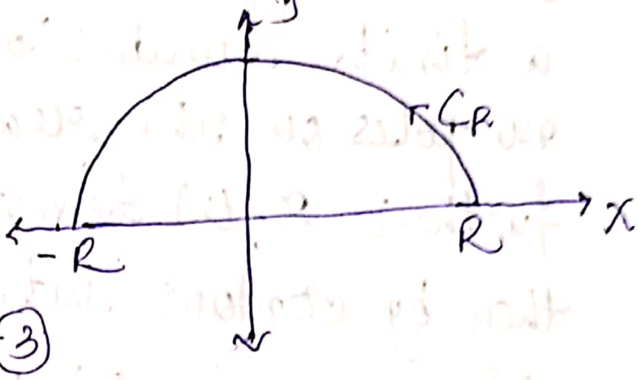
Solⁿ:

$$\int_{C} \frac{dz}{(z^2+a^2)(z^2+b^2)} = \int_{-R}^{R} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx + \int_{C_R} \frac{1}{(z^2+a^2)(z^2+b^2)}\,dz \quad \text{--- (1)}$$

Now,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} \quad \text{--- (2)}$$

As $R\to\infty$, the poles lies outside of the circle, this means



$$\int_{C_R} \frac{dz}{(z^2+a^2)(z^2+b^2)} = 0. \quad \text{--- (3)}$$

Using (2) and (3) in Equation (1), we get

$$\int_{C} \frac{dz}{(z^2+a^2)(z^2+b^2)} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} \quad \text{--- (4)}$$

Since C is an upper half circle, so we need to find the poles and corresponding to poles inside the C we compute the sum of the residues in C rest are taken to be zero.

Now, $(z^2+a^2)(z^2+b^2)=0.$

$\Rightarrow (z+ai)(z-ai)(z+bi)(z-bi)=0$

$\Rightarrow z = +ai, -ai, +bi, -bi$

Since $-ai, -bi$ lies lower half region and thus their residues becomes zero.

Further, we compute the residues corresponding to $+ai, +bi$ as follows:

$$R_1 = \lim_{z \to ai} (z-ai) f(z) = \lim_{z \to ai} (z-ai) \frac{1}{(z^2+a^2)(z^2+b^2)}$$

$$= \lim_{z \to ai} \frac{\cancel{(z-ai)}}{(z+ai)\cancel{(z-ai)}(z^2+b^2)} = \lim_{z \to ai} \frac{1}{(z+ai)(z^2+b^2)}$$

$$\boxed{R_1 = \frac{1}{2ai(-a^2+b^2)}}$$

Now, $R_2 = \lim_{z \to bi} (z-bi) f(z) = \lim_{z \to bi} \frac{1}{(z^2+a^2)(z+bi)}$

$$\boxed{R_2 = \frac{1}{(-b^2+a^2)2bi}}$$

Therefore by theorem:

$$\int_C \frac{dz}{(z^2+a^2)(z^2+b^2)} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i (R_1+R_2)$$

$$= 2\pi i \left[\frac{1}{2ai(b^2-a^2)} + \frac{1}{2bi(a^2-b^2)} \right]$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$$

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Q2: Evaluate the Integral

$$\int_0^{\infty} \frac{1}{(x^2+1)^2}\,dx .$$

Solⁿ: Rewrite the given integrals as:

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$$ (By the Properties of integrals)

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} + \frac{1}{2}\int_{C_R} \frac{dz}{(z^2+1)}$$

As $z \to \infty$.

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int_C \frac{1}{(z^2+1)^2}\,dz$$

$$= \frac{1}{2}\int_C \frac{dz}{(z-i)^2(z+i)^2}$$ ($-i, -i$ lies outside the upper half plane).

$\therefore$ Poles are $z = \pm i, \pm i$

$\therefore$ Residues are corresponds to $+i, +i$.

$$\therefore R = \lim_{z\to i} \frac{1}{1!}\frac{d}{dz}\cancel{(z-i)^2}\frac{1}{(z+i)^2\cancel{(z-i)^2}}$$ (Repeated Roots)

$$= \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{-8i}$$

$$\Rightarrow \boxed{R = \frac{1}{4i}}$$

$$\therefore \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{4} \quad \#$$

Exercise: Solve $\int_0^\infty \frac{1}{(x^4+1)} dx$. (Use Demoivre's to get Roots).

Ans: $\frac{\pi}{2\sqrt{2}}$ #