ndre Polynomials

$$P_n(x) = \frac{1}{n!\,2^n} \sum_{r=0}^{n} \frac{(-1)^r\, n!}{r!\,(n-r)!} \frac{d^n}{dx^n}\left(x^{2n-2r}\right)$$

$$= \frac{1}{2^n} \sum_{r=0}^{N} \frac{(-1)^r\,(2n-2r)!\; x^{n-2r}}{r!\,(n-r)!\,(n-2r)!}$$

where $N = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever in an integer

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Q3 a) $f(x) = x^2$

$$= \frac{2}{3}\left(\frac{3}{2}x^2\right)$$

$$= \frac{2}{3}\left(\frac{3}{2}x^2 - \frac{1}{2} + \frac{1}{2}\right)$$

$$= \frac{2}{3}\left(P_2(x)\right) + \frac{1}{3}$$

$$= \frac{1}{3}\left(2P_2(x) + P_0(x)\right)$$

b) $f(x) = 4x^3 - 2x^2 - 3x + 8$

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$x^3 = \frac{2}{5}\left(\frac{5}{2}x^3 - \frac{3}{2}x + \frac{3}{2}x\right)$$

$$= \frac{2}{5}\left(P_3(x) + \frac{3}{2}P_1(x)\right) = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$\therefore f(x) = \frac{4}{5}(2P_3 + 3P_1) - \frac{2}{3}(2P_2 + P_0) - 3P_1 + 8P_0$$

$$= \frac{8}{5}P_3 + \frac{12}{5}P_1 - \frac{4}{3}P_2 - \frac{2}{3}P_0 - 3P_1 + 8P_0$$

$$= \frac{8}{5}P_3 - \frac{4}{3}P_2 - \frac{3}{5}P_1 + \frac{22}{3}P_0$$

c) $f(x) = x^4 + 2x^3 - 6x^2 + 5x - 3$

$1 = P_0$

$x = P_1$

$x^2 = \frac{1}{3}(2P_2 + 1) = \frac{1}{3}(2P_2 + P_0)$

$x^3 = \frac{1}{5}(2P_3 + 3P_1)$

$$x^4 = \frac{8}{35}\left(\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8} + \frac{30}{8}x^2 - \frac{3}{8}\right)$$

$$= \frac{8}{35}(P_4) + \frac{6}{7}x^2 - \frac{3}{35}$$

$$= \frac{8}{35}P_4 + \frac{\cancel{6}^{2}}{7} \times \frac{1}{\cancel{3}}(2P_2 + P_0) - \frac{3}{35}P_0$$

$$= \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0$$

$\frac{2}{7} - \frac{3}{35}$

$\frac{10-3}{35}$

$$f(x) = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0 + \frac{4}{5}P_3 + \frac{6}{5}P_1 - 4P_2 - 2P_0$$
$$+ 5P_1 - 3P_0$$

$$= \frac{8}{35}P_4 + \frac{4}{5}P_3 - \frac{24}{7}P_2 + \frac{31}{5}P_1 - \frac{24}{5}P_0$$

For $x = 0$ $\qquad , t \neq 1$

$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)\, t^n$

$\sum_{n=0}^{\infty} P_n(0)\, t^n = 1 - \frac{1}{2}t^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}t^4 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}t^6$

$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} t^{2k}$

Comparing the coef. on both sides.

$P_1(0) = P_3(0) = P_5(0) = \ldots = 0$ (for all odd n)

$P_n(0) = \frac{\left(\frac{-1}{2}\right)^{n/2}}{(n/2)!}$ $\qquad \frac{(-1)^{n/2}(+\frac{1}{2})(+\frac{1}{2}+1)\ldots}{(n/2)!} = \frac{(-1)^{n/2}(+1)(+3)\ldots}{2^n (n/2)!}$

$\frac{(-1)^{n/2}\; n!}{((n/2)!)^2\, 2^{n/2}\, 2^{n/2}} = \frac{(-1)^{n/2}\; n!}{2^n\, [(n/2)!]^2}$ $\quad n$ even

6) a) $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

Diff. partially w.r.t t.

$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n P_n(x)\, t^{n-1}$ — ①

Diff w.r.t x

$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} P_n'(x)\, t^n$ — ②

① / ②

$\frac{-2x+2t}{-2t} \Rightarrow \frac{\sum_{n=0}^{\infty} n P_n(x)\, t^{n-1}}{\sum_{n=0}^{\infty} P_n'(x)\, t^n}$

$(x-t)\sum_{n=0}^{\infty} P_n'(x)\, t^n = \sum_{n=0}^{\infty} n P_n(x)\, t^n$

$\sum_{n=0}^{\infty} x P_n'(x)\, t^n - \sum_{n=0}^{\infty} P_n'(x)\, t^{n+1} = \sum_{n=0}^{\infty} n P_n(x)\, t^n$

comparing coef. of t^n

$$x P_n'(x) - P_{n-1}'(x) = n P_n(x)$$

b) $(1+2n) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Diff w.r.t t,

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$

$$(x-t)\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$
$$-2x\sum_{n=0}^{\infty} n P_n(x) t^n$$
$$+\sum_{n=0}^{\infty} n P_n(x) t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} + \sum_{n=0}^{\infty} n P_n(x) t^{n+1}$$
$$-2x \sum_{n=0}^{\infty} n P_n(x) t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (1+2n) x P_n(x) t^n = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} + \sum_{n=0}^{\infty} (n+1) P_n(x) t^{n+1}$$

$$\Rightarrow (1+2n) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

Diff w.r.t x.

$$x(1+2n) P_n'(x) + (2n+1) P_n(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x) \quad -(1)$$

from (a) $n P_n(x) = x P_n'(x) - P_{n-1}'(x)$

$$\Rightarrow x P_n'(x) = n P_n(x) + P_{n-1}'(x) \quad -(2)$$

Put (2) in (1)

$$(2n+1)[n P_n(x) + P_{n-1}'(x)] + (2n+1) P_n(x) = (n+1) P_{n+1}'(x)$$
$$+ n P_{n-1}'(x)$$

$$\Rightarrow (2n+1)(n+1) P_n(x) = (n+1) P_{n+1}'(x) + (n-2n-1) P_{n-1}'(x)$$

$$\Rightarrow (2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$$ [Dividing each term by $(n+1)$]

$(1-x^2) P_n'(x) = n(P_{n-1} - x P_n)$

From (a) $n P_n = x P_n' - P_{n-1}'$

$\Rightarrow x P_n' = n P_n + P_{n-1}'$ — ①

Also, $P_n' - x P_{n-1}' = n P_{n-1}$ — ②

① $\times x$ $\Rightarrow x^2 P_n' = x^2 P_{n-1}' + nx P_n$ — ③

From ② & ③.

$n \cancel{P_n} + \cancel{P_{n-1}'} = x^2 \cancel{P_{n-1}'} + nx \cancel{P_{n-1}}$

$(1-x^2) P_{n-1}'(x)$

$P_n' - n P_{n-1} = x^2 P_n' - nx P_n$.

$(1-x^2) P_n'(x) = n(P_{n-1} - x P_n)$.

Hence proved.

Proof of ② $P_n' - x P_{n-1}' = n P_{n-1}$

5) For $x = 0$

$$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)\, t^n$$

$$\sum_{n=0}^{\infty} P_n(0)\, t^n = 1 + (-1/2)t^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}t^4 + \dots + \frac{(-\frac{1}{2})\dots(-\frac{1}{2}}{n!}$$

$$= 1 + \frac{(-\frac{1}{2})t^2}{1!} + \frac{(\frac{1}{2})(\frac{3}{2})t^4}{2!} - \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})t^6}{3!}$$

$$+ \dots \frac{(-1)^n(\frac{1}{2})(\frac{3}{2})\dots(\frac{2n-1}{2})\, t^{2n}}{n!}$$

$$\Rightarrow 1 + \sum_{n=1}^{\infty} P_n(0)\, t^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1.3.5\dots(2n-1)}{2^n\, n!} t^{2n}$$

$\because P_0(0) = 1$

If n is odd, $n = 2m+1$

$P_n(0) = 0$ ($\because$ we have even terms)

If n is even $n = 2m$

$m = n/2$.

then
$$P_n(0) = \sum_{n=0}^{\infty} \frac{(-1)^{n/2}\, 1.3.5\dots(n-1)}{2^{n/2}\,(\frac{n}{2})!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n/2}\, 1.2.3.4\dots(n-1)(n)}{2^{n/2}\, 2.4.6\dots n \cdot (\frac{n}{2})!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n/2}\, n!}{2^{n/2}\, 2^{n/2}(1.2\dots\frac{n}{2})\,(\frac{n}{2})!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n/2}\, n!}{2^n\,[(\frac{n}{2})!]^2}$$

OR
$$P_n(x) = \frac{1}{2^n}\sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!\, x^{n-2r}}{r!\,(n-r)!\,(n-2r)!}, \quad N = \frac{n}{2} \text{ or } \frac{n-1}{2}$$ whichever is integ

If n is even, $N = n/2$. (all terms except for $N = n/2$ contain x & so they will be zero)

$$P_n(0) = \frac{1}{2^n}\frac{(-1)^{n/2}\, n!}{[(n/2)!]^2}$$

If n is odd, $N = n-1/2$. so $x^{n-2r} = x^{n-n+1} = x$.

All terms will contain x

$P_n(0) = 0$.

Show that $J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$

using recurrence relation

$J_{n+1} = \frac{2n}{x} J_n - J_{n-1}$

$J_n' = J_{n-1} - 2 J_{n+1}$

$J_n' = \frac{n}{x} J_n - J_{n+1}$

$x J_n' = -n J_n + x J_{n-1}$ — (1)

Put $n = 2$.

$x J_2' = -2 J_2 + x J_1$

$\Rightarrow J_2'(x) = -\frac{2}{x} J_2(x) + J_1(x)$ — (2)

using recurrence relation

$x J_n' = n J_n - x J_{n+1}$ — (3)

from (1) & (3)

$-n J_n + x J_{n-1} = n J_n - x J_{n+1}$

$x J_{n-1} = 2n J_n - x J_{n+1}$

Put $n = 1$

$x J_0(x) = 2 J_1(x) - x J_2(x)$

$\Rightarrow J_2(x) = -J_0(x) + \frac{2}{x} J_1(x)$ — (4)

using (4) in (2)

$J_2'(x) = -\frac{2}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) + J_1(x)$

$= \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$

8) $J_n J_{-n}' - J_{-n} J_n' = -\frac{2 \sin(n\pi)}{\pi x}$

Bessel's equation is

$x^2 y'' + x y' + (x^2 - n^2) y = 0$

$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right) y = 0$

$\Rightarrow J_n'' + \frac{J_n'}{x} + \left(1 - \frac{n^2}{x^2}\right) J_n = 0$ — (1)

& $J_{-n}'' + \frac{J_{-n}'}{x} + \left(1 - \frac{n^2}{x^2}\right) J_{-n} = 0$ — (2)

① $\times J_{-n}$ − ② $\times J_n$

$$J_{-n} J_n'' + \frac{J_{-n} J_n'}{x} + \left(1 - \frac{n^2}{x^2}\right) J_n J_{-n} = 0$$

$$J_n J_{-n}'' + \frac{J_n J_{-n}'}{x} + \left(1 - \frac{n^2}{x^2}\right) J_n J_{-n} = 0$$

$$J_{-n} J_n'' - J_n J_{-n}'' + \frac{1}{x}\left(J_{-n} J_n' - J_n J_{-n}'\right) = 0$$

$$\Rightarrow \frac{J_{-n} J_n'' - J_n J_{-n}''}{J_{-n} J_n' - J_n J_{-n}'} = -\frac{1}{x}$$

Derivative of deno = $J_{-n} J_n'' + \cancel{J_n' J_{-n}'} - J_n J_{-n}'' - \cancel{J_{-n}' J_n'}$ = Numerator

Integrating w.r.t x

$$\log\left(J_{-n} J_n' - J_n J_{-n}'\right) = -\log x + \log C$$

$$J_{-n} J_n' - J_n J_{-n}' = \frac{C}{x}.$$

Now $J_n(x) = \dfrac{1}{2^n \Gamma(n+1)} \left[x^n - \dfrac{x^{n+2}}{2(2n+2)} + \dfrac{x^{n+4}}{2\cdot4(2n+2)(2n+4)} \cdots\right]$

$$J_{-n} = \frac{2^n}{\Gamma(1-n)}\left[x^{-n} - \frac{x^{2-n}}{2(2-2n)} + \frac{x^{4-n}}{2\cdot4(2-2n)(4-2n)}\right]$$

$$J_n'(x) = \frac{1}{2^n \Gamma(n+1)}\left[n x^{n-1} - \frac{(n+2)x^{n+1}}{2(2n+2)} + \frac{(n+4)\, x^{n+3}}{2\cdot4(2n+2)(2n+4)} \cdots\right]$$

$$J_{-n}'(x) = \frac{2^n}{\Gamma(1-n)}\left[-n x^{-n-1} - \frac{(2-n)\, x^{1-n}}{2(2-2n)} + \frac{(4-n)\, x^{3-n}}{2\cdot4(2-2n)(4-2n)}\right]$$

coef of $\frac{1}{x}$ in $J_{-n} J_n'$ $= \dfrac{\cancel{2^n}}{\Gamma(1-n)} \times \dfrac{n}{\cancel{2^n}\Gamma(n+1)}$

coef of $\frac{1}{x}$ in $J_n J_{-n}'$ $= \dfrac{1}{\cancel{2^n}\Gamma(n+1)} \times \dfrac{-n\, \cancel{2^n}}{\Gamma(1-n)}$

coef of $\frac{1}{x}$ in $J_{-n} J_n' - J_n J_{-n}'$ $= \dfrac{n}{\Gamma(n+1)\Gamma(1-n)} + \dfrac{n}{\Gamma(n+1)\Gamma(1-n)}$

$$\therefore \quad C = \frac{2n}{\Gamma(n+1)\Gamma(1-n)}$$

$$c = \frac{2}{\sqrt{n}\sqrt{1-n}}$$

$$= \frac{2 \sin n\pi}{\pi}$$

$$\left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$\therefore J_{-n} J_n' - J_n J_{-n}' = \frac{2\sin(n\pi)}{\pi x} \quad \text{— (i)}$$

$$\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right) = -\frac{2\sin(n\pi)}{\pi x J_n^2}$$

$$\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right) = \frac{J_n J_{-n}' - J_{-n} J_n'}{J_n^2}$$

$$= -\frac{2\sin(n\pi)}{\pi x J_n^2} \quad [\text{using (i)}]$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + c$$

$$\int \cancel{x^{-n} J_n(x)\,dx} = \int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)$$

$$\int x J_0(x)\,dx = x J_1(x) + c$$

9) $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$ using $n=1$ in

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Put $n = 2$.

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$= \frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x)$$

$$= \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

Put $n = 3$

$$J_4(x) = \frac{2 \times 3}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

10) Prove $J_1''(x) = \frac{J_2(x)}{x} - J_1(x)$

We have

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad - (1)$$

& $$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \quad - (2)$$

Diff (1) w.r.t x.

$$x J_n''(x) + J_n'(x) = n J_n'(x) - x J_{n+1}'(x) - J_{n+1}(x)$$

Put $n = 1$

$$x J_1''(x) + J_1'(x) = J_1'(x) - x J_2'(x) - J_2(x)$$

$$\Rightarrow x J_1''(x) = -x J_2'(x) - J_2(x)$$

$$\Rightarrow J_1''(x) = -J_2'(x) - \frac{J_2(x)}{x} \quad - (3)$$

Put $n = 2$ in (2)

$$x J_2'(x) = -2 J_2(x) + x J_1(x)$$

$$J_2'(x) = -\frac{2}{x} J_2(x) + J_1(x) \quad - (4)$$

...ing (4) in (3)

$$J_1''(x) = +\frac{2}{x} J_2(x) - J_1(x) - \frac{J_2(x)}{x}$$

$$= \frac{1}{x} J_2(x) - J_1(x).$$

Hence, proved.

11) $\int J_0(x) \cos x \, dx = x J_0(x) \cos x + x J_1(x) \sin x + c$

$$\int \underbrace{J_0(x) \cos x}_{\mathrm{I}} . \underbrace{1}_{\mathrm{II}} \, dx = J_0(x) \cos x \, . \, x - \int \frac{d}{dx} (J_0(x) \cos x) \, x \, dx + c$$

$$= x J_0(x) \cos x - \int \left[x (J_0'(x) \cos x - J_0(x) \sin x) \right] dx + c$$

$$= x J_0(x) \cos x - \int x J_0'(x) \cos x \, dx + \int \underbrace{x J_0(x)}_{\mathrm{II}} \underbrace{\sin x}_{\mathrm{I}} \, dx + c$$

$J_n' = \frac{n}{x} J_n - J_{n+1}$

Put $n = 0$

$J_0' = - J_1$

$$= x J_0(x) \cos x + \int x \; J_1(x) \cos x \, dx + \sin x \int x J_0(x) \, dx - \int \left[\cos x \int x J_0(x) \, dx \right] dx + c$$

$$\boxed{\because \int x J_0(x) \, dx = x J_1(x)}$$

$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$

$$= x J_0(x) \cos x + \int x J_1(x) \cancel{\cos} x \, dx + x \sin x \, J_1(x) - \int x \, J_1(x) \cancel{\cos} x \, dx + c$$

$$= x J_0(x) \cos x + x J_1(x) \sin x + c$$

$x J_n' = n J_n - x J_{n+1}$

Put $n = 0$

$x J_0' = - x J_1$