Tutorial-2

Mathematics

1.

(i) $f(x,y) = e^x \cos y$

$$f(0+x-0,\ 0+y-0) = f(0,0) + (x-0) f_x(0,0) + (y-0) f_y(0,0) + \frac{1}{2!}\left[(x-0)^2 f_{xx}(0,0) + (y-0)^2 f_{yy}(0,0) + 2(x-0)(y-0) f_{xy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0)\right]$$

a, h, b, k

$f(x,y) = e^x \cos y$, $f(0,0) = 1$

$f_x = (\cos y) e^x$, $f_x(0,0) = 1$

$f_y = -e^x \sin y$, $f_y(0,0) = 0$

$f_{xx} = (\cos y) e^x$, $f_{xx}(0,0) = 1$

$f_{yy} = -e^x \cos y$, $f_{yy}(0,0) = -1$

$f_{xy} = \frac{\partial}{\partial x}(-e^x \sin y) = -e^x \sin y$, $f_{xy}(0,0) = 0$

$f_{xxx} = (\cos y) e^x$, $f_{xxx}(0,0) = 1$

$f_{yyy} = e^x \sin y$, $f_{yyy}(0,0) = 0$

$f_{xxy} = -e^x \sin y$, $f_{xxy}(0,0) = 0$

$f_{xyy} = \frac{\partial f_{yy}}{\partial x} = -e^x \cos y$, $f_{xyy}(0,0) = -1$

$$f(x,y) = 1 + x + \frac{1}{2!}\left[x^2 - y^2\right] + \frac{1}{3!}\left[x^3 - 3xy^2\right]$$

$$\boxed{f(x,y) = 1 + x + \frac{x^2 - y^2}{2} + \frac{x^3 - 3xy^2}{6}}$$ Ans:

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(ii) $\tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$

$$f(1+x-1,\ 1+y-1) = f(1,1) + (x-1) f_x(1,1) + (y-1) f_y(1,1) + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1) f_{xy}(1,1)\right]$$

a, h, b, k

$$+ \frac{1}{3!}\left[(x-1)^3 f_{xxx}(0,0) + (y-1)^3 f_{yyy}(0,0) + 3(x-1)^2(y-1) f_{xxy}(1,1) + 3(x-1)(y-1)^2 f_{xyy}(1,1)\right]$$

$f(x,y) = \tan^{-1}\frac{y}{x}$, $f(1,1) = \frac{\pi}{4}$

$f_x = \frac{x^2\left(\frac{-y}{x^2}\right)}{x^2+y^2} = \frac{-y}{x^2+y^2}$, $f_x(1,1) = -\frac{1}{2}$

$f_y = \frac{x^2\left(\frac{1}{x}\right)}{x^2+y^2} = \frac{x}{x^2+y^2}$, $f_y(1,1) = \frac{1}{2}$

$f_{xx}(1,1) = \frac{y(2x)}{(x^2+y^2)^2} = \frac{2}{4} = \frac{1}{2}$

$f_{yy}(1,1) = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2}{4} = -\frac{1}{2}$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2}$ at $(1,1) = \frac{2-2}{4} = 0$

$f_{xxx} = \frac{\partial}{\partial x}\frac{2xy}{(x^2+y^2)^2} = \frac{(2y)(x^2+y^2)^2 - 2xy(2(x^2+y^2))2x}{(x^2+y^2)^4} = \frac{2\times 4 - 2\times 2\times 2\times 2}{2^4} = \frac{-8}{16} = -\frac{1}{2}$

$f_{yyy} = \frac{\partial}{\partial y}\frac{-2xy}{(x^2+y^2)^2} = \frac{-2x(x^2+y^2)^2 + 2xy(2(x^2+y^2))2y}{(x^2+y^2)^2}$

$= \frac{-2\times 4 + 2\times 2\times 2\times 2}{16} = +\frac{1}{2}$

$$f_{xxy} = \frac{\partial}{\partial x} f_{xy} = \frac{\partial}{\partial x} \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{-2x(x^2+y^2)^2 - (y^2-x^2)(2(x^2+y^2))2x}{(x^2+y^2)^4}$$ $f_{xxy}(1,1)$

$$f_{xyy} = \frac{\partial}{\partial x} f_{yy} = \frac{\partial}{\partial x} \frac{-2xy}{(x^2+y^2)^2} = \frac{-2y(x^2+y^2)^2 + 2xy(2(x^2+y^2))2x}{(x^2+y^2)^4}$$

$$f_{xyy}(1,1) = \frac{-2\times 4 + 2\times 2\times 2\times 2}{16} = \frac{1}{2}$$

Putting all values in expansion

$$f(x,y) = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2!}\left[\frac{(x-1)^2}{2} - \frac{(y-1)^2}{2}\right] + \frac{1}{3!}\left[\frac{-(x-1)^3}{2} + \frac{(y-1)^3}{2}\right.$$
$$\left. + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + \frac{3(x-1)(y-1)^2}{2}\right]$$

(iii) $f(x,y) = x^3 + 3y^3 - xy^2$ about $(1,-1)$

$f(1,-1) = 1-3-1 = -3$

$f_x = 3x^2 - y^2$, $f_x(1,-1) = 3-1 = 2$ $\qquad f_{xxx} = 6$

$f_y = 9y^2 - 2xy$, $f_y(1,-1) = 9+2 = 11$ $\qquad f_{yyy} = 18$

$f_{xx} = 6x$, $f_{xx}(1,-1) = 6$ $\qquad f_{xyy} = -2$

$f_{yy} = 18y - 2x$, $f_{yy}(1,-1) = -18-2 = -20$ $\qquad f_{xxy} = 0$

$f_{xy} = -2y$, $f_{xy}(1,-1) = +2$

$f(x,y) = \cancel{f(1+x-1,}$

$$f(\underset{a}{1}+\underset{h}{x-1}, \underset{b}{-1}+\underset{k}{y+1}) = -3 + (x-1)2 + (y+1)11 + \frac{1}{2!}\left[(x-1)^2 6 - 20(y+1)^2 + 2(x-1)(y+1)(+2)\right]$$
$$+ \frac{1}{3!}\left[(x-1)^3 6 + 18(y+1)^3 + 3(x-1)^2(y+1)(0) + 3(x-1)(y+1)^2(-2)\right]$$

$$= -1 + \left[3(x-1) + 7(y+1)\right] + \left[3(x-1)^2 - 2(x-1)(y+1) - 8(y+1)^2\right]$$
$$+ \left[3(x-1)^3 + (x-1)(y+1)^2 + 3(y+1)^3\right]$$

$f(x,y) = e^x \sin y \quad (0,0)$

$$f(a+x=0,\ b+y=0) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)\right]$$

$|x-0| < 0.1$
$|y-0| \leq 0.1$

$f(0,0) = 0$

$f_x = e^x \sin y,\ f_x(0,0) = 0$

$f_y = e^x \cos y,\ f_y(0,0) = 1$

$f_{xx} = e^x \sin y,\ f_{xx}(0,0) = 0$

$f_{yy} = -e^x \sin y,\ f_{yy}(0,0) = 0$

$f_{xy} = e^x \cos y,\ f_{xy}(0,0) = 1$

Taylor's series

$$f(x,y) = y + \frac{1}{2!}[2xy]$$
$$= y + xy$$

$$|R_2| < \frac{B}{6}[0.1+0.1]^3$$

$\cancel{B = \max[|f_{xx}|, |f_{yy}|, |f_{xy}|]}$

$\cancel{B = \max[e^x \sin y, -e^x \sin y, e^x \cos y]}$

$f_{xxx} = e^x \sin y$

$f_{xxy} = e^x \cos y$

$f_{yyy} = -e^x \cos y$

$f_{xyy} = -e^x \sin y$

$$B = \max[|f_{xxx}|, |f_{yyy}|, |f_{xxy}|, |f_{xyy}|]$$

$|f_{xxx}| = |e^x \sin y| < |e^{0.1} \sin 0.1| = 0.00192$

$|f_{yyy}|_{max} = |-e^{0.1} \times \cos 0|$,
$= 1.105$

$|f_{yyx}|_{max} = |-e^x \sin y|$
$= |-e^{0.1} \sin 0.1| = 0.00192$

$|f_{xxy}| = |e^x \cos y| = |e^{0.1} \cos 0| = 1.105$

$B = 1.105$

$$\text{Error max} = \frac{1.105}{6}[0.1+0.1]^3$$

Ans

3. $f(x,y) = \sqrt{x+y}$

$f(1,3) = 2$

$f_x = \dfrac{1}{2\sqrt{x+y}} \quad f_x(1,3) = \dfrac{1}{4}$

$f_y = \dfrac{1}{2\sqrt{x+y}} \quad f_y(1,3) = \dfrac{1}{4}$

$f_{xx} = \dfrac{-\dfrac{2}{2\sqrt{x+y}}}{4(x+y)} = \dfrac{-\frac{1}{2}}{4\times 2} = \cancel{\tfrac{1}{16}}\ -\dfrac{1}{32}$ at $(1,3)$

$f_{yy} = \dfrac{-\dfrac{2}{2\sqrt{x+y}}}{4(x+y)} = \dfrac{-\frac{1}{2}}{4\times 2} = -\dfrac{1}{32}$ at $(1,3)$

$f_{xy} = \dfrac{d}{dx}\dfrac{1}{2\sqrt{x+y}} = \dfrac{-\dfrac{2}{2\sqrt{x+y}}}{4(x+y)} = \dfrac{-1}{4(x+y)\sqrt{x+y}}$ at $(1,3)$ $= \dfrac{-1}{8\times 4} = -\dfrac{1}{32}$

$f(x,y) = f(\underbrace{1}_{a}+\underbrace{x-1}_{h},\ \underbrace{3}_{b}+\underbrace{y-3}_{k}) = f(1,3)+(x-1)f_x(1,3)+(y-3)f_y(1,3)+\dfrac{1}{2!}\left[(x-1)^2 f_{xx}(1,3)+(y-3)^2 f_{yy}(1,3)+2(x-1)(y-3)f_{xy}(1,3)\right]$

1a.

$= 2 + \dfrac{x-1}{4} + \dfrac{y-3}{4} + \dfrac{1}{2}\left[(x-1)^2\left(-\dfrac{1}{16}\right) + (y-3)^2\left(-\dfrac{1}{16}\right) + 2(x-1)(y-3)\left(-\dfrac{1}{8}\right)\right]$

Now, $f(1.1, 2.9) = f(\underbrace{1}_{a}+\underbrace{0.1}_{h},\ \underbrace{3}_{b}-\underbrace{0.1}_{k})$

$= f(1,3) + 0.1\, f_x(1,3) + (-0.1)\, f_y(1,3) + \dfrac{1}{2!}\left[(0.1)^2 f_{xx}(1,3) + (-0.1)^2 f_{yy}(1,3) + 2(0.1)(-0.1) f_{xy}(1,3)\right]$

$= 2$

$f_{xxy} = f_{xxy} = f_{xyy} = \frac{3}{8(x+y)^{5/2}}$

$|f_{xxx}|_{max} = \frac{3}{8(x-1+y-3+4)^{5/2}} = \frac{3}{8(4-0.2-0.1)^{5/2}}$

$= \frac{3}{8(3.7)^{5/2}}$

$|R_2| \leq \frac{3}{8(3.7)^{5/2}} \frac{1}{6} (0.3)^3$

$|R_2| \leq 0.64 \times 10^{-4}$ Ans:

as $|x-1| \leq 0.2$

Range of $x$ is $[0.8, 1.2]$

Range of $x-1$ is $[-0.2, 0.2]$

min value of $x-1 = -0.2$

Similarly

min. value of $y-3$ is $[-0.1, 0.1]$

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4. (i) $x^4+y^4-y^2-x^2+1$

$f_x = 4x^3-2x = 0 \Rightarrow 4x^2-2 = 0 \quad x = 0, \pm\frac{1}{\sqrt{2}}$

$f_y = 4y^3-2y = 0 \Rightarrow 4y^2-2=0 \quad y = 0, \pm\frac{1}{\sqrt{2}}$

$\cancel{(x,y) = (0,0)\,(0,\tfrac{1}{\sqrt2})\,(0,-\tfrac{1}{\sqrt2})\,(\tfrac{1}{\sqrt2},0)\,(\tfrac{1}{\sqrt2},\tfrac{1}{\sqrt2})\,(\tfrac{1}{\sqrt2},-\tfrac{1}{\sqrt2})}$
$\cancel{(-\tfrac{1}{\sqrt2},0,\ \tfrac{1}{\sqrt2},\tfrac{1}{\sqrt2},\ \tfrac{1}{\sqrt2},\tfrac{1}{\sqrt2})}$

$(x,y) = (0,0), (0,\pm\frac{1}{\sqrt{2}}), (\pm\frac{1}{\sqrt{2}},0)\ (\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})$

$r = 12x^2-2$

$s = 0$

$t = 12y^2-2$

$(0,0) \Rightarrow rt-s^2$
$= rt = 4 \quad r<0 \therefore (0,0)$ is maxima

$(0,\pm\frac{1}{\sqrt2}) \Rightarrow rt = (-2)(4) = -8 < 0 \quad (0,\pm\frac{1}{\sqrt2})$ is neither max nor min.

$(\pm\frac{1}{\sqrt2},0) \Rightarrow rt = (4)(-2) = -8 < 0 \quad (\pm\frac{1}{\sqrt2},0)$ " "

$(\pm\frac{1}{\sqrt2},\pm\frac{1}{\sqrt2}) = (4)(4) = 16 \quad r>0 \therefore (\pm\frac{1}{\sqrt2},\pm\frac{1}{\sqrt2})$ is minima

(ii) $(x^2+y^2)e^{6x+2x^2}$

$\cancel{f_x = (x^2+y^2)e^{6x+2x^2}(6+4x) + e^{6x+2x^2}(2x+0) = 0}$

$(x^2+y^2)(6+4x) + 2x = 0$ $\qquad f_y = 2e^{6x+2x^2} y$

$\cancel{4x^3+6x^2+6y^2+2x+4xy^2=}$ $\qquad f_y = 0 \Rightarrow y = 0$

$$f_x = x^2 \cdot e^{6x+2x^2} \cdot (6+4x) + e^{6x+2x^2} \cdot 2x + y^2 \cdot e^{6x+2x^2} \cdot (6+4x)$$

$$= e^{6x+2x^2}\left((x^2+y^2)(6+4x) + 2x\right)$$

$$= e^{6x+2x^2}(4x^3+4xy^2+6x^2+6y^2+2x)$$

$$f_x = 0 \Rightarrow 4x^3+6x^2+2x = 0$$

$$x(2x^2+3x+1) = 0$$

$$x = 0,\ x = -1,\ x = -\frac{1}{2}$$

$$r = f_{xx} = e^{6x+2x^2}(12x^2+4y^2+12x+2) + \{(x^2+y^2)(6+4x)+2x\}e^{6x+2x^2}(6+4x)$$

$$s = 2y\,e^{6x+2x^2}(6+4x)$$

$$t = 2e^{6x+2x^2}$$

at $(0,0)$ $r = 2$

$s = 0$

$t = 2$

$rt - s^2 = 4 > 0$ $\quad r > 0$

$\therefore (0,0)$ is a point of local minima.

at $(-1,0)$

$rt - s^2 = 4$ $4$ $r > 0$

$\therefore (-1,0)$ point of local minima

at $(-\frac{1}{2}, 0)$

$r = -e^{-5/2}$

$s = 0$

$t = 2e^{-5/2}$

$rt - s^2 = -2e^{-5} < 0$

Saddle point

$z = 6x^2 - 2x^3 + 3y^2 + 6xy$

$f_x = 12x - 6x^2 + 6y$

$f_y = 6y + 6x \qquad f_y = 0 \Rightarrow y = -x$

$\therefore f_x = 0$

$2x - x^2 + y = 0$

$2x - x^2 - x = 0$

$x - x^2 = 0$

$x = 0 \quad x = 1$

$y = 0 \quad y = -1$

$(0,0)\ (1,-1)$

$r = 12 - 12x$

$s = 6$

$t = 6$

at $(0,0)$

$r = 12$

$rt - s^2 > 0$

$12 \times 6 - 6 \times 6 > 0$ & $r > 0$

So $(0,0)$ is point of loc. minima.

at $(1,-1)$

$r = 12 - 12 \times 1$

$= 0$

$rt - s^2$

$0 \times 6 - 36 < 0$ so $(1,-1)$ is saddle point.

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5. $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$

$f_x = 4x - 4 \quad = 0 \Rightarrow x = 1$

$f_y = 2y - 4 \quad = 0 \Rightarrow y = 2$

$f_{xx} = 4 = r$

$f_{yy} = 2 = t$

$f_{xy} = 0 = s$

$rt - s^2$

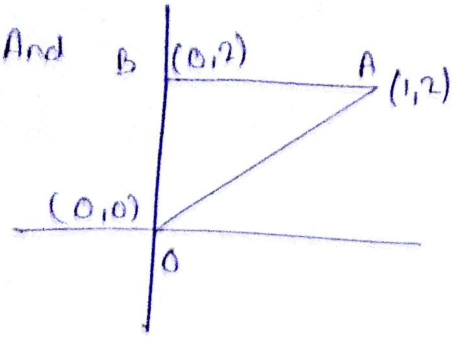
$4 \times 2 - 0 = 8 > 0$ & $r = 4 > 0$

so $(1,2)$ is point of local minima.

$f(1,2) = 2 - 4 + 4 - 8 + 1$

$= -5$ minima.

And




$f(0,0) = 1$

$f(1,2) = -5$

$f(0,2) = 4 - 8 + 1$

$= -3$

Along OB, $x = 0$

$f(x,y) = y^2 - 4y + 1 = \phi(x)$

$\phi'(x) = 2y - 4 = 0$ $\quad$ $\phi''(x) = 2$ $\quad$ ~~minima (0,2)~~

$y = 2$,

Along AB $\quad$ (0,2) is point of ~~absolute~~ minima.

$y = 2$

$f(x,y) = 2x^2 - 4x - 3 = \phi_1(x)$

$\phi_1'(x) = 4x - 4$

$x = 1$ $\quad$ (1,2) is point of ~~absolute~~ minima.

$\phi_1''(x) = 4$

Along OA:

$y = 2x$

$f(x,y) = 6x^2 - 12x + 1 = \phi_2(x)$

$\phi_2'(x) = 12x - 12$ $\quad$ So, (1,2) is point of ~~absolute~~ minima.

$x = 1$

$\phi_2''(x) = 12$

So,

$f(0,0) = 1$ maxima. $\quad$ $f(0,2) = -3$

$f(1,2) = -5$ $\quad$ $f(1,2) = -5$ minima

$f(0,2) = -3$

(0,0) is point of absolute maxima.

(1,2) " " " " minima

$(x,y) = 4x^2 - 4xy + y^2 \quad -(1)$

$\phi_1(x,y) = x^2+y^2-5 \quad -(2)$

$F(x,y) = 4x^2 - 4xy + y^2 + \lambda_1 (x^2+y^2-5)$

$\frac{\partial F}{\partial x} = 8x - 4y + 2\lambda_1 x = 0 \quad -(3)$

$\frac{\partial F}{\partial y} = -4x + 2y + 2\lambda_1 y = 0 \quad -(4)$

equ. (3) + $2\times$ eq (4)

$2\lambda_1 x + 4\lambda_1 y = 0$

$\lambda_1 = 0, \ x = -2y$

putting $\lambda = 0$ & $x = -2y$ in (3)

putting $x = -2y$ in (2)

$(-2y)^2 + y^2 = 25$

$y = \pm\sqrt{5}$

$x = \pm 2\sqrt{5}$

$\frac{\partial f}{\partial x} = 2x - y = 0 \Rightarrow 2x = y$

$\frac{\partial f}{\partial y} = -4x + 2y$

$x^2 + 4x^2 = 25$

$x = \pm\sqrt{5}$

$y = \pm 2\sqrt{5}$

$(\sqrt{5}, 2\sqrt{5}), (\sqrt{5}, -2\sqrt{5}), (-\sqrt{5}, 2\sqrt{5}), (-\sqrt{5}, -2\sqrt{5}), (2\sqrt{5}, \sqrt{5}), (2\sqrt{5}, -\sqrt{5}), (-2\sqrt{5}, \sqrt{5}),$
$(-2\sqrt{5}, -\sqrt{5})$

$f(\sqrt{5}, 2\sqrt{5}) = 20 - 40 + 20 = 0$ minima

$f(\sqrt{5}, -2\sqrt{5}) = 20 + 40 + 20 = 80$

$f(-\sqrt{5}, 2\sqrt{5}) = 20 + 40 + 20$
$= 80$

$f(-\sqrt{5}, -2\sqrt{5}) = 20 - 40 + 20$
$= 0$ minima

$f(2\sqrt{5}, \sqrt{5}) = 45$

$f(2\sqrt{5}, -\sqrt{5}) = 125$ maxima

$f(-2\sqrt{5}, \sqrt{5}) = 125$ maxima

$f(-2\sqrt{5}, -\sqrt{5}) = 45$

7(a) $f(x,y,z) = xyz + \lambda_1 (x+y+z^2-16)$ & $x,y,z > 0$

$$\frac{\partial f}{\partial x} = yz + \lambda_1 = 0$$

$$\frac{\partial f}{\partial y} = xz + \lambda_1 = 0$$

$$\frac{\partial f}{\partial z} = xy + 2\lambda_1 z = 0$$

$$\lambda_1 = -yz$$

$$\lambda_1 = -xz$$

$$\lambda_1 = \frac{-xy}{2z}$$

$x = y$ & $z = 0$
× (not possible)

$$-xz = \frac{-xy}{2z} \qquad -yz = \frac{-xy}{2z}$$

$$x(2z^2 - y) = 0 \qquad y(2z^2 - x) = 0$$

$$\underset{\times}{x=0} \quad \boxed{y = 2z^2} \qquad \underset{\times}{y=0} \quad \boxed{x = 2z^2}$$

$$2z^2 + 2z^2 + z^2 = 16$$

$$z = \frac{4}{\sqrt{5}}$$

$$x = \frac{32}{5}, \; y = \frac{32}{5}$$

$f\;xyz = \dfrac{4096}{25\sqrt{5}}$

2) Let the distance of point from origin be $= \sqrt{x^2+y^2+z^2}$

$$f(x,y,z) = x^2+y^2+z^2$$

$$F(x,y,z) = x^2+y^2+z^2+\lambda_1(y+2z-12)+\lambda_2(x+y-6)$$

$$\frac{\partial F}{\partial x} = 2x+\lambda_2 = 0$$

$$\frac{\partial F}{\partial y} = 2y+\lambda_1+\lambda_2 = 0$$

$$\frac{\partial F}{\partial z} = 2z+2\lambda_1 = 0$$

$$\boxed{\lambda_1 = -z} \quad \& \quad \boxed{\lambda_2 = -2x}$$

then

$$2y = -(\lambda_1+\lambda_2)$$

$$2y = -(-z-2x)$$

$$\boxed{2y = 2x+z}$$

Solving given equation with above equation

$2y = 2x+z$ — (1)

$y+2z = 12$ — (2)

$x+y = 6$ — (3)

in (1)

$$2y = 12-2y+\frac{12-y}{2}$$

$$4y = 24-4y+12-y$$

$$9y = 36$$

$\underline{y=4}$, $x=2$, $z=4$

$$(x,y,z) = (2,4,4)$$

$$d = \sqrt{x^2+y^2+z^2} = \sqrt{4+16+16}$$

$$= \sqrt{36} = 6$$

(c)

$$f(x,y,z) = x^2+y^2+z^2 + \lambda_1(ax+by+cz-1) + \lambda_2(\alpha x+\beta y+\gamma z-1)$$

$$\frac{\partial f}{\partial x} = 2x + a\lambda_1 + \alpha\lambda_2 = 0 \quad -(1)$$

$$\frac{\partial f}{\partial y} = 2y + b\lambda_1 + \beta\lambda_2 = 0 \quad -(2)$$

$$\frac{\partial f}{\partial z} = 2z + c\lambda_1 + \gamma\lambda_2 = 0 \quad -(3)$$

Multiplying equ (1) by 'b' & equ (2) by 'a'. ~~adding both of them~~ subtracting (2) from (1)

$$2xb - 2ay + \lambda_2(\alpha b - \beta a) = 0$$

$$\lambda_2 = \frac{2(ay - xb)}{(\alpha b - \beta a)} \quad -(4)$$

Multiplying (1) by (β) & (2) by (α) & subtracting (2) from (1)

$$\cancel{2x\alpha - 2y\beta}\ 2x\beta - 2y\alpha + \lambda_1(a\beta - b\alpha)$$

$$\lambda_1 = \frac{2(\alpha y - \beta x)}{(a\beta - b\alpha)} \quad -(5)$$

Putting value of ~~equ~~ $\lambda_1$ & $\lambda_2$ in equ (3)

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

$$(u,v) \to (x,y) \to (u,v)$$

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = u_x x_u + u_y y_u$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = u_x x_v + u_y y_v$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = v_x x_v + v_y y_v$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} = v_x x_u + v_y y_u$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

P.T.O

9. $u = \frac{x+y}{1-xy}$ $\quad v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial u}{\partial x} = \frac{(1-xy) - (x+y)(-y)}{(1-xy)^2} = \frac{1+\cancel{xy}+y^2-\cancel{xy}}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{x^2+xy+1-xy}{(1-xy)^2} = \frac{x^2+1}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{x^2+1}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Jacobian = 0 ⇒ Thelm must exist some functional relationship.

$v = \tan^{-1}x + \tan^{-1}y$

$v = \tan^{-1}\frac{x+y}{1-xy}$

$v = \tan^{-1}u$

$u = \tan v$

10. $u = \frac{x+y}{z}$, $v = \frac{y+z}{x}$, $w = \frac{y(x+y+z)}{xz} = \frac{xy+y^2+zy}{xz} = \frac{y}{z} + \frac{y^2}{xz} + \frac{y}{x}$

$\frac{\partial u}{\partial x} = \frac{1}{z}$, $\frac{\partial u}{\partial y} = \frac{1}{z}$, $\frac{\partial u}{\partial z} = \frac{-(x+y)}{z^2}$ $\qquad \frac{\partial w}{\partial y} = \frac{1}{z} + \frac{2y}{xz} + \frac{1}{x}$

$\frac{\partial v}{\partial x} = \frac{-(y+z)}{x^2}$, $\frac{\partial v}{\partial y} = \frac{1}{x}$, $\frac{\partial v}{\partial z} = \frac{1}{x}$ $\qquad \frac{\partial w}{\partial x} = \frac{-y^2}{x^2 z} - \frac{y}{x^2}$

$$\frac{\partial w}{\partial z} = \frac{-y}{z^2} - \frac{y^2}{xz^2}$$