At $x = 1+\sqrt{2}$

$U_n = 1 \longrightarrow$ same divergence

Region of convergence : $(1-\sqrt{2}) < x < (1+\sqrt{2})$

Series Solution

The solution of DE in infinite series form is called a series solution

ordinary and singular point

consider: $Py'' + Qy' + Ry = S \quad \text{—} (1)$

or $\quad y'' + \frac{Q}{P} y' + \frac{R}{P} y = \frac{S}{P}$

The value of x for which P becomes zero is called a singular point.

ex. If $x = a$ makes $P = 0$ in eq (1) then $x = a$ will be a singular point.

and if for $x = a$, $P \neq 0$ then P is called as ordinary point

NOTE: we have $x = 0$ as an ordinary point and as a singular point in our course

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Series solution for $x=0$ is an ordinary point:
Let $y = a_0 + a_1x + a_2x + \dots a_kx^k$ be the series solⁿ of the given DE where $a_0, a_1, a_2 \dots a_n$ and also are the coeff to be obt. These are obt by equating the coeff of x in both sides of the given diff eq. The above eqⁿ can also be written as

$$y = \sum_{k=0}^{\infty} a_k x^k \quad \text{———} \quad (1)$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} k a_k x^{k-1}$$

diff again w.r.t x:

$$\frac{\partial^2 y}{dx^2} = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}$$

Q. Solve in series the eqⁿ:

$$\frac{d^2y}{dx^2} + xy = 0$$

Solⁿ: $P = 1$

$x=0$ will not effect

so $x=0$ is an ordinary solution

then SS $\Rightarrow y = \sum_{k=0}^{\infty} a_k x^k$.

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} k a_k x^{k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}$$

Using all these in the given D.E.

$$\sum_{k=0}^{\infty} k(k-1) a_k x^{k-2} + x \sum_{k=0}^{\infty} k a_x x^{k} = 0 \quad \text{—} \quad (1)$$

Equating to zero the coeff of lowest degree term of x in LHS of (1)

It is obtained by putting $k=0$ in first summat^n

$$\sum_{k=0}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^{k+1} = 0 \quad \text{--- (1)}$$

$k+3$ $\qquad\qquad$ $k+1 \to k-2$

$$2.(2-1) a_2 = 0$$
$$a_2 = 0$$

$k=3$

Now equating to zero coeff of next lowest degree term of x.

It is obtained by putting $k=3$ in first summat^n and k in ~~first~~ summation.

$$3(3-1) a_3 + a_0 = 0$$
$$a_3 = \frac{-a_0}{3.2} \quad -\frac{a_0}{3!}$$

Now equating to zero the coeff of highest degree term of x (in terms of k) i.e. the powers of x^{k+1} here.

for this put $k = k+3$ in 1st summ

$$(k+3)(k+3-1) a_{k+3} + a_k = 0$$

$$\boxed{a_{k+3} = -\frac{a_k}{(k+3)(k+2)}}$$

For $k=0$

$$a_{k_3} = \frac{-a_0}{3.2}$$

for $k=1$ $\quad a_4 = \frac{-a_1}{4.3}$

for $k=2$

$$a_{k_5} = \frac{-a_2}{5 \times 4}$$

since $a_2 = 0$

so $a_5 = 0$

for $k=3$ $\qquad$ for $k=4$

So now the req. series solution is:

$$y = a_0 + a_1 + 0x^2 + \frac{-a_0}{3.2}x^3 + \frac{-a_1}{4.3}x^4 + 0x^5 + \ldots$$

$$y = a_0\left[1 - \frac{x^3}{3.2} + \ldots\right] + a_1\left[x - \frac{x^4}{4.3} + \ldots\right]$$

$$y = a_0\left[1 - \frac{x^3}{\lfloor 3} + \ldots\right] + a_1\left[x - \frac{2x^4}{\lfloor 4} + \ldots\right]$$

Ans;

regular $\leftarrow$ $R_1 \neq \infty$; $R_2 \neq \infty$

irregular $\leftarrow$ $\begin{bmatrix} R_1 \neq \infty & R_2 = \infty \\ R_1 = \infty & R_2 = \infty \\ R_1 = \infty & R_2 \neq \infty \end{bmatrix}$

Types of singular points:

1. Regular
2. Irregular

Let us consider the eqn:

$$y'' + Py' + Qy = 0 \quad \text{— (1)}$$

$x = a$ as singular point.

$R_1 = (x-a)\, P$

$R_2 = (x-a)^2\, Q.$

$\Rightarrow$ Now if $\boxed{R_1 \neq \infty \; ; \; R_2 \neq \infty}$ for $x = a$ then $x = a$ is called regular singular point $\boxed{\text{otherwise}}$ irregular

Q] Determine the singular points of the DE shown below and classify these as regular or irregular

$$(x^2-9)^2 y'' + (x+3) y' + 2y = 0$$

$\rightarrow$

$$y'' + \frac{x+3}{(x^2-9)^2} y' + \frac{2}{(x^2-9)^2} y = 0$$

$x^2 - 9 = 0$

$x^2 = 9$

$\boxed{x = \pm 3}$

Here $x \neq 3$

$P = \dfrac{x+3}{(x^2-9)^2}$

& $Q = \dfrac{2}{(x^2-9)^2}$

for $x = 3$;

$R_1 = (x-3)\, P$

$R_2 = (x-3)^2\, Q.$

$R_1 = (x-3)\, P = \dfrac{x+3}{(x^2-9)^2} \cdot x-3 = \infty$

$R_2 = (x-3)^2 \cdot \dfrac{2}{(x^2-9)^2} = \dfrac{2(x-3)(x-3)}{(x+3)^2 (x-3)^2} \neq \infty.$

Irregular point

* If $x=0$ is irregular singular point, Then solution can not be exposed in form of series.

At $x=-3$;

$$R_1 = (x+3)\,p = (x+3)\cdot\frac{(x+3)}{(x^2-9)^2}$$

$$= \frac{(x+3)(x+3)}{(x-3)(x+3)\,(x-3)(x+3)}$$

$$= \frac{1}{(x-3)^2} \neq \infty$$

$$R_2 = (x+3)^2\cdot\frac{2}{(x-3)(x+3)\,(x-3)(x+3)}$$

$$= \frac{2}{(x-3)^2\cdot\cancel{(x+3)}} \neq \infty$$

$\therefore$ Regular singular points.

Series solution of a D.E having $x=0$ as a singular point. (Frobenious method). (Frobenius)

For $x=0$ to be singular point, consider the series solution as

To make $x=0$ a singular point.

$$y = \boxed{x^m}\,(a_0 + a_1x + a_2x^2 + \dots a_kx^k + \dots$$

$$y = \sum_{k=0}^{\infty} a_k x^{m+k} \quad \text{——— (1)}$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} (m+k)\,a_k\,x^{m+k-1} \quad \text{——— (2)}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (m+k)(m+k-1)\,a_k\,x^{m+k-2} \quad \text{——— (3)}$$

x^{-2} ~~to~~
k+2

Puting these summations ~~to~~ ie eq^n ① ② & ③ in the given D.E we find the DE in summation form.

On equating the coeff of lowest degree term in x: in that eq^n, we get a quadratic equation in m called indicial equation.

This indicial eq^n in m will give two values of m: maybe real, distinct or equal, so we have following cases i.e —

★ CASE I: ~~(fro~~ (when roots m_1 & m_2 are distinct and not differ by integer.

$m = 1, 2$ $\qquad m_1 \sim m_2 \neq 0, 1 \ldots$ (Integral form)

In this case: the complete solution will be:

$$\boxed{y = C_1 (y_{m_1}) + C_2 (y_{m_2})}$$

Q] Solve $3xy'' + 2y' + y = 0$

→ Let the required series solution is $y = \sum_{n=0}^{\infty} a_k x^{m+k}$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Now; $\sum_{k=0}^{\infty} 3a_k(m+k)(m+k-1)x^{m+k-1} +$

$$\sum_{k=0}^{\infty} 2a_k(m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k(m+k)\left[3(m+k-1)+2\right]x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k}$$

eq. to zero the coeff of lowest degree term in x, i.e the coeff of $x^{m+k-1} \Rightarrow x^{m-1}$ $\{k=0\}$

$$a_k(m+k)[3(m+k-1)+2]\,x^{m+k-1} = 0$$

$$a_0 m[3m-3+2]$$
$$a_0 m[3m-1] = 0$$
$m = 0, \frac{1}{3}$ doesnot differ by integer.

Now equating to zero the next lowest degree term in x i.e x^m.

First summation mein $k=1$ & second summation mein $k=0$ rakhne par x^m milega ; hence

$$a_1(m+1)[3m+2] + a_0 = 0$$

$$a_1 = \frac{-a_0}{(m+1)(3m+2)}$$

Now equating to zero, the coeff of x^{m+k}

$$a_{k+1}[m+k+1][3(m+k)+2] + a_k = 0.$$

$$\boxed{a_{k+1} = \frac{-a_k}{(m+k+1)(3(m+k)+2)}}$$ recurrence relation

For $m=0$

$$a_1 = \frac{-a_0}{2}$$

$$a_{k+1} = \frac{-a_k}{(k+1)(3k+2)}$$

$$a_2 = \frac{-a_1}{2(5)} = \frac{-a_1}{10}$$

$$a_2 = \frac{a_0}{20}$$

$$a_3 = \frac{-a_2}{3 \cdot 6} = \frac{-a_2}{24} = \frac{-a_0}{480}.$$

$$(y)_{m=0} = a_0\left[1 - \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{480}x^3 + \quad\right]$$

$$(y)_{m=1/3} = a_0 x^{1/3}\left[1 - \frac{1}{4}x + \frac{1}{56}x^2 - \frac{1}{1680}x^3 + \dots\right]$$

Complete solution is: $\boxed{y = c_1(y)_{m=0} + c_2(y)_{m=1/3}}$

For $m = 1/3$

★ Case - II : when the roots are equal: $m = m_1 = m_2$.
complete solution will be $\boxed{y = c_1(y)_{m_1} + c_2\left(\frac{dy}{\partial m}\right)_{m_2}}$

Q] solve by power series method.

$$x(x-1)y'' + (3x-1)y' + y = 0$$

→ $$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$y' = \sum_{k=0}^{\infty} (m+k)\, a_k x^{m+k-1}$$

$$y'' = \sum_{k=0}^{\infty} (m+k)(m+k-1)\, a_k x^{m+k-2}$$

$$\Rightarrow (x^2-x)\left[\sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k-2}\right] + (3x-1)\sum_{k=0}^{\infty}(m+k)a_k x$$

$$+ \sum_{k=0}^{\infty} a_k x^{m+k}.$$

$$\Rightarrow \sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m+k} - \sum_{k=0}^{\infty}(m+k)(m+k-1)a_k x^{m}$$

$$+ \sum_{k=0}^{\infty} 3(m+k)a_k x^{m+k} - \sum_{k=0}^{\infty}(m+k)a_k x^{m+k-1}$$

$$+ \sum_{k=0}^{\infty} a_k x^{m+k}.$$

$$\Rightarrow \sum_{k=0}^{\infty} x^{m+k} a_k (m+k)[m+k-1+3-1] - \sum_{k=0}^{\infty} x^{m+k-1} a_k (m+k)$$

$$[m+k-1+1]$$

$$\Rightarrow \sum_{k=0}^{\infty} x^{m+k} a_k (m+k)[m+k+3] - \sum_{k=0}^{\infty} x^{m+k-1} a_k (m+k)$$

Eq to zero the coeff of lowest degree term in x

$$-a_k(m+k)^2 \;\cancel{+}\; -a_0(m^2) = 0$$

$$\boxed{m = 0, 0}$$

$$a_k(m+k)(m+k+3) - a_{k+1}(m+k+1)^2 = 0$$

$$a_{k+1} = \frac{a_k(m+k)(m+k+3)}{(m+k+1)^2}$$

$$a_1 = \frac{a_0(m)(m+3)}{(m+1)^2} = a_0\left(\frac{m^2+3m}{(m+1)^2}\right)$$

$$a_1 = a_0$$

$$\boxed{a_{k+1} = a_k}$$

$$y = x^m [a_0 + a_1 x + a_2 x^2 + \dots]$$

$$y = a_0 x^m [1 + x + x^2 + \dots]$$

$$\frac{dy}{dm} = a_0 x^m \log x [1 + x + x^2 + \dots]$$

$$(y)_{m=0} = a_0 [1 + x + x^2 + \dots]$$

$$\left(\frac{dy}{dm}\right)_{m=0} = a_0 \log x [1 + x + x^2 + \dots]$$

★ CASE III : when roots are distinct and differ by an integer ; solution will be: $\boxed{y = c_1 (y)_{m_1} + c_2 \left(\frac{dy}{dm}\right)_{m_2}}$

NOTE: { if $m = 0, -2$ then if $a_2 = \frac{a_0}{(m+2)(m+4)}$

we take $a_0 = b_0 (m+2)$ }

In this case some coeff of power of x in the series solution becomes ∞ for a value of m, if that root is m_2, then we put $a_0 = b_0 (m - m_2)$ to remove this ∞ condition.

we consider $\frac{dy}{dx}$ for m_2 in this case.

Q] solve by power series method $x^2 y'' + 5xy' + x^2 y = 0$

→ $x^2 y'' + 5xy' + x^2 y = 0$

$x = 0$ is singular point then;

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$y' = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$y'' = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Now, $\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k} + \sum_{k=0}^{\infty} 5a_k(m+k)x^{m+k}$

$$+ \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} x^{m+k} a_k(m+k)[m+k-1+5] + \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

Now eq to lowest degree term;

$a_0(m)(m+4)=0$; $m=0,-4$ $a_0 \neq 0$

Now eq to second highest degree term.

$a_0(m+1)(m+5)=0$

$a_1=0$

Now, eq to x^{m+k+2}

$a_{k+2}(m+k+2)(m+k+6)+a_k=0$

$$\boxed{a_k = -a_{k+2}(m+k+2)(m+k+6)}$$

$$a_{k+2} = \frac{-a_k}{(m+k+2)(m+k+6)} \quad \text{— (2)}$$

$k=0$; $a_2 = \dfrac{-a_0}{(m+2)(m+6)}$

$m=0,-4$ $a_2 = \dfrac{-a_0}{12}$

$m=0$

$k=-4$; $a_{-2} = \dfrac{-a_0}{-2 \cdot 2} = \dfrac{-a_0}{4} \Rightarrow \boxed{\dfrac{a_0}{4} - a_2}$

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$$(y)_{m=0} = a_0\left[1-\frac{x^2}{12}+\frac{x^2}{3\cdot 3\cdot 4}-\frac{x^6}{2\cdot 3\cdot 4}+\dots\right]$$

For $m=-4$, the coeff. of x^4 onwards will become ∞, so consider $a_0 = b_0(m+4)$ in eq^n ③

$$y = x^m b_0\left[(m+4) - \frac{(m+4)x^2}{(m+2)(m+6)} + \frac{1}{(m+2)(m+6)(m+8)}x^4 - \frac{1}{(m+2)(m+6)^2(m+8)(m+16)} + \dots\right] \quad \text{④}$$

$$\frac{dy}{dm} = b_0 x^m \log x\left[\dots\right] + b_0 x^m\left[1 - \frac{x^2}{(m+2)^2(m+6)^2} \times (m+2)(m+3) - (2m+8) + \dots\right]$$

Now, put $m=-4$

$$\left(\frac{dy}{dm}\right)_{m=-4} = b_0 x^{-4}\log x\left[\frac{x^4}{(-2)(2)(4)} - \frac{x^6}{(-2)(4)(+4)(6)}\right]$$

$$+ b_0 x^{-4}\left[1 - x^2\left[\frac{(-2)(2)-(4)+\dots}{4(4)}\right]\right]$$

$$\left(\frac{dy}{dx}\right)_{m=-4} = b_0 x^{-4}\log x\left[\frac{-x^4}{16}+\frac{x^6}{192}\right] + b_0 x^{-4}\left[1-x^2\left[\frac{-1}{2}\right]\right]$$

Bessel's function →

standard formula for gamma function : Γn

(i) $\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$; $n > 0$ (mostly)

(ii) $\Gamma 1 = 1$

(iii) $\Gamma n+1 = n\Gamma n = n \int_0^\infty e^{-x} x^{n-1} \, dx$

(iv) $\Gamma n+1 = n!$; when n is an integer
(+ve integer)

(v) $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$

(vi) $\Gamma \frac{1}{2} = \sqrt{\pi}$

Bessel's equation →

$$x^2y'' + xy' + (x^2 - n^2)y = 0 ;$$

is called Bessel equation where n is a non-negative constant.

Solution of Bessel's equation →

- CASE I: If $n \neq 0$ and n is not an integer then the general solution of bessel's equation will be

$$y = AJ_n(x) + BJ_{-n}(x)$$

where;

$$\boxed{\begin{aligned} J_n(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma n+r+1} \left(\frac{x}{2}\right)^{n+2r} \\ J_{-n}(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma -n+r+1} \left(\frac{x}{2}\right)^{-n+2r} \end{aligned}}$$

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Q] If the bessel's equation is $x^2y'' + xy' + \left(x^2 - \frac{1}{16}\right)y = 0$

Then find the solution.

→ $n = 1/4$

$y = AJ_n(x) + BJ_{-n}(x)$

$= AJ_{1/4}(x) + BJ_{-1/4}(x)$

- CASE II : If $n \neq 0$; n is an integer

$$y = AJ_n(x) + BY_n(x)$$

$$\boxed{Y_n(x) = \frac{(\cos n\pi)J_n(x) - J_{-n}(x)}{\sin n\pi}} \quad ; \text{ for all } n$$

Here $J_n(x)$ is called bessel's function of first kind
$Y_n(x)$ is called bessel's function of second kind

- CASE III : If $n = 0$, then the general solution is

integer or not

$$y = AJ_0(x) + BY_0(x)$$

In this case $\boxed{Y_0(x) = \lim_{n\to 0} \frac{(\cos n\pi)J_n(x) - J_{-n}(x)}{\sin n\pi}}$

Relation b/w $J_n(x)$ & $J_{-n}(x)$ →

$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$; n is an integer.

NOTE: If n is an integer ; we may write

$$\boxed{\Gamma(n+r+1) = (n+r)!}$$

so in that case $J_n(x)$ can also be written as →

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r}.$$

series expansion of

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \left(\text{for all } x\right)$$

Q. P.T $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$\cancel{J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^{1/2}}{\frac{1}{2}!}}$$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left(\frac{3}{2}+r\right)} \left(\frac{x}{2}\right)^{1/2+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\left(\frac{3}{2}+r\right)} \left(\frac{x}{2}\right)^{2r+1/2}$$

11.10 P.T $J_{-1/2}(x) = \sqrt{2/\pi x} \cdot \cos x$

$J_n(x) = J_n$

Recurrence relation ⟶

(I) $x J_n'$ = $n J_n - x J_{n+1}$ — kam wale mein se zyada kosum.

(II) $x J_n'$ = $-n J_n + x J_{n-1}$

(III) $2 J_n' = J_{n-1} - J_{n+1}$

Imp (iv) $2n J_n = x [J_{n-1} + J_{n+1}]$ ⟶ without any derivative

(v) $[x^{-n} J_n]' = - x^{-n} J_{n+1}$

(vi) $[x^n J_n]' = x^n J_{n-1}$

Q] P.T $J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$

$2n J_n = x [J_{n-1} + J_{n+1}]$ —— (1)

for $n = 2$

$4 J_2 = x [J_1 + J_3]$

$4 J_2 = x J_1 + x J_3$

$x J_3 = 4 J_2 - x J_1$

$J_3 = \left(\frac{4}{x}\right) J_2 - J_1$ —— (2)

for $n = 1$ in eq^n (1)

$2 J_1 = x [J_0 + J_2]$

$2 J_1 - x J_0 = x J_2$

$J_2 = \left(\frac{2}{x}\right) J_1 - J_0$ —— (3)

using (3) in eq^n (2)

$J_3 = \frac{4}{x} \left[\left(\frac{2}{x}\right) J_1 - J_0\right] - J_1$

$= \left(\frac{8}{x^2}\right) J_1 - \frac{4}{x} J_0 - J_1$

$J_3 = (8/x^2 - 1) J_1 - 4/x \, J_0$ ⟶ Hence proved

Q2] Find the following in terms of $\sin x$ & $\cos x$.

(i) $J_{3/2}(x)$ (ii) $J_{-3/2}(x)$

$\{J_{-n}(x) = (-1)^n J_n(x)\}$

Considering:

$2n J_n = x[J_{n-1} + J_{n+1}]$

for $n = 1/2$

$$J_{1/2} = x J_{-1/2} + x J_{3/2}$$

$$J_{3/2} = \frac{1}{x} \cdot J_{1/2} - J_{-1/2}$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \left[\sqrt{\frac{2}{\pi x}} \cdot \cos x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

$$\boxed{J_{3/2} = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]}$$

Similarly:

$$\boxed{J_{-3/2} = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]}$$

Q] Show that : $\int x J_0^2(x)\,dx = \frac{1}{2}x^2[J_0^2(x) + J_1^2(x)]$

LHS

$$\int x J_0^2(x)\,dx = \underbrace{J_0^2(x)\cdot\frac{x^2}{2}}_{\text{firstpart}} - \underbrace{\int 2 J_0 J_0' \frac{x^2}{2}\,dx}_{\text{we are req to find this}}$$

$$J_0^2\cdot\frac{x^2}{2} - \int 2 J_0 J_0' \frac{x^2}{2}\,dx \quad \text{——} \ (1)$$

$$J_0\cdot(\cancel{2}J_0')\cdot\frac{x^2}{\cancel{2}}\cdot dx$$

$$J_0\cdot x^2 J_0'\,dx$$

using $2J_n' = J_{n-1} + J_{n+1}$ —— (2)

$$J_0\cdot(2J_0')\cdot\frac{x^2}{2}$$

for $n = 0$

$$2J_0' = J_{-1} + J_1 \quad \text{——} \ (3)$$

$$= (-1)^0 J_1 + J_1$$

$$2J_0' = 2J_1 \qquad \boxed{J_{-1} = -J_1}$$

$$\int x J_0^2(x)\,dx = J_0^2\cdot\frac{x^2}{2} - \int J_0\cdot 2J_1\cdot\frac{x^2}{2}\,dx$$

$$= J_0^2\frac{x^2}{2} - \int J_0 J_1\cdot x^2\,dx$$

$$= J_0^2\frac{x^2}{2} - \int J_0 J_{-1} x^2\,dx$$

$$(x J_0)(x J_{-1})$$

$$= x J_1$$

$$= J_0^2\frac{x^2}{2} - \int (x J_0)(x J_1)\,dx \quad \text{——} \ (4)$$

$$[x^n J_n]' = x^n J_{n-1} \quad \text{—— (5)}$$

for $n=1$

$$(xJ_1)' = x J_0$$

$\therefore$ (4) becomes:

$$\int x \cdot J_0^2(x)\, dx = J_0^2 \cdot \frac{x^2}{2} + \int [xJ_1][xJ_1]' dx$$

$$= J_0 \frac{x^2}{2} + \int (xJ_1) \cdot \frac{d}{dx}(xJ_1)\, dx$$

$$= J_0 \frac{x^2}{2} + \frac{(xJ_1)^2}{2}$$

Orthogonality of Bessel's function $\longrightarrow$

α, β are roots of the function $J_n(x)=0$

$J_n(x) = 0$ that means $\longrightarrow$

(i) $J_n(\alpha)=0$

(ii) $J_n(\beta)=0$

$$\int_0^1 x J_n(\alpha x)\, J_n(\beta x) \cdot dx = 0$$

NOTE: For a DE of the type $x^2y'' + xy' + (p^2x^2 - n^2)y = 0$

The complete solution will be;

(3 marks)

$$\boxed{y = C_1 J_n(px) + C_2 J_{-n}(px)}$$

Generating function for $J_n(x)$ $\longrightarrow$

The bessel f^n $J_n(x)$ is the coeff of z^n in the expansion of $e^{x/2(z - 1/z)}$

$$e^{x/2(z - 1/z)} = \sum_{n=-\infty}^{\infty} J_n(x)\, z^n$$

$$P_n(x) = \frac{1.3.5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right]$$

Legendre Polynomial →

- Legendre's Differential equation →

The D.E:

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is called Legendre's D.E.

The particular solution of this eqn are called Legendre's functions of first kind and second kind.

∴ the solution will be:

$$\boxed{y = C_1 P_n(x) + C_2 Q_n(x)}$$

$P_n(x)$ is Legendre's solution of first kind called as Legendre's polynomial where n is a non-negative integer.

- Generating function for Legendre's polynomial

The function $(1-2xh+h^2)^{-1/2}$ is called the generating function for for $P_n(x)$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} P_n(x)\,h^n = (1-2xh+h^2)^{-1/2}}$$

Some results for Legendre's polynomial:

1. $P_n(1) = 1$
2. $P_n(-x) = (-1)^n P_n(x)$
3. $P_n'(-x) = (-1)^{n+1} P_n(x)$

Proof

1. We know that $\sum_{n=0}^{\infty} P_n(x)\,h^n = (1-2xh+h^2)^{-1/2}$ —— (1)

$$\sum_{n=0}^{\infty} P_n(x) h^n = (1-2xh+h^2)^{-1/2}$$

Putting $x=1$

$$\sum_{n=0}^{\infty} P_n(1) h^n = (1-2h+h^2)^{-1/2}$$

$$\sum_{n=0}^{\infty} P_n(1) h^n = (1-h)^{-1}$$
$$= 1+h+h^2+h^3+\dots+h^n.$$

Now comparing coeff of h^n on both sides; we get

$$\boxed{P_n(1) = 1}$$

Hence proved the first result.

Proof ②

To prove: $P_n(-x) = (-1)^n P_n(x)$

$$\sum_{n=0}^{\infty} P_n(x) h^n = (1-2xh+h^2)^{-1/2}$$

for $x = (-x)$

$$\sum_{n=0}^{\infty} P_n(-x) h^n = (1+2xh+h^2)^{-1/2} \quad \text{—— ②}$$

On putting $h \longrightarrow (-h)$ in equat^n ①

$$\sum_{n=0}^{\infty} P_n(x) (-h)^n = (1+2hx+h^2)^{-1/2} \quad \text{—— ③}$$

Comparing ② & ③ (LHS).

$$\sum_{n=0}^{\infty} P_n(-x) h^n = \sum_{n=0}^{\infty} P_n(x) (-h)^n$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

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Equating coeff of h^n both sides:

$$\boxed{P_n(-x) = (-1)^n P_n(x)}$$

Proof ③:

To prove: $P_n(-x) = (-1)^{n+1} P_n'(x)$

From result ② we have →

$$P_n(-x) = (-1)^n P_n(x)$$

Differentiating both sides:

$$(-1) P_n'(-x) = (-1)^n P_n'(x)$$

$$\Rightarrow P_n'(-x) = (-1)^{n+1} P_n'(x)$$

Comparing the coeff

$$(-1)^2 (-1)^{n-1} = (-1)^{n+1}$$

[Q] Prove that

4 mark

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$$

→ $$\sum_{n=0}^{\infty} P_n(x) h^n = (1-2xh+h^2)^{-1/2} \quad \text{—— ①}$$

put $x=0$:

$$\sum_{n=0}^{\infty} P_n(0) h^n = (1+h^2)^{-1/2}$$

expanding the RHS

$$\sum_{n=0}^{\infty} P_n(0) h^n = 1 + \left(-\frac{1}{2}\right)h^2 + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2!} h^4 + \ldots$$

$$= 1 - \frac{1}{2}h^2 + \frac{5}{2} \cdot \frac{h^2}{2!}$$

$-\frac{1}{2} \cdot -\frac{3}{2}$

$-5/2$

35

$-\frac{5}{2 \times 2}$

$-\frac{1}{2} - \frac{2}{2} - \frac{1}{4}$

Q. To show that $P_n(x)$ is coeff of z^n in expansion of $(1-2xz+z^2)^{-1/2}$ in ascending power of z

(or) $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n h(x)$

$$\sum_{n=0}^{\infty} P_n(0) h^n = 1 + \left(\frac{-1}{2}\right) h^{2.1} + \frac{(-1/2)(-3/2)}{2!} h^{2.2} + \frac{(-1/2)(-3/2)(-5/2)}{3!} h^{2.3} + \ldots + \frac{(-1)^n (1/2)(3/2)(5/2)\ldots\left(\frac{2n-1}{2}\right)}{n!} h^{2n}$$

Comparing coeff of h^{2n} on both sides:

$$P_{2n}(0) = \frac{(-1)^n}{n!}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{2n-1}{2}\right)$$

$$= \frac{(-1)^n}{2^n . n!} 1.3.5\ldots(2n-1) \times \frac{2.4.6\ldots(2n)}{2.4.6\ldots(2n)}$$

$$= \frac{(-1)^n}{2^n. n!} \frac{(2n)!}{(2\times1)(2\times2)(2\times3)\ldots(2\times n)}$$

$$= \frac{(-1)^n (2n)!}{2^n n! . 2^n . n!}$$

$$= \frac{(-1)^n (2n)!}{(2^n . n!)^2}$$

Rodrigue's formula $\longrightarrow$

~~The~~

$$\boxed{P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n}(x^2-1)^n}$$

Some standard results of Rodrigue's formula

Putting $n = 0, 1, 2, 3, 4, 5$

$$P_0(x) = \frac{1}{2^0 0!} \frac{\partial^0}{\partial x^0}(x^2-1)^0 = 1$$

$$P_1(x) = x$$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

Q] Express $4x^3 + 6x^2 + 7x + 2$ in terms of legendre's polynomial.

→ Let $4x^3 + 6x^2 + 7x + 2 = AP_3(x) + BP_2(x) + CP_1(x) + DP_0(x)$

$= A$ — ①

$= A\frac{1}{2}[5x^3 - 3x] + B\frac{1}{2}[3x^2 - 1] + Cx + D$

Comparing coeff bothsides;

$A = 8/5, \quad B = 4, \quad C = -\frac{47}{5}, \quad D = 4,$

Recurrence relations on $P_n(x)$:

1) $nP_n(x) = (2n-1)x\,P_{n-1}(x) - (n-1)P_{n-2}(x)$

2) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

3) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

Imp 4) $(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

Imp 5) $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$

6) $(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$

7) $(1-x^2)P_n'(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$

- **Orthogonal Property of Legendre's Polynomial**

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0 & ; m \neq n \\ \dfrac{2}{2n+1} & ; m = n \end{cases}$$

e.g. $\int_{-1}^{1} P_2(x)\,P_3(x)\,dx = 0$

Eg $\int_{-1}^{1} [P_2(x)]^2 \cdot dx = \dfrac{2}{2(2)+1} = \dfrac{2}{5}$

2016 a] State the orthogonal property of Legendre's Polynomial. Hence using it, evaluate the following integral →

$$\int_{-1}^{1} (3x^2 - 1)\,[P_4(x)]^2 \cdot dx$$

→ We know;

$$\int_{-1}^{1} P_m(x) \cdot P_n(x) \cdot dx = \begin{cases} 0 & ; m \neq n \\ \dfrac{2}{2n+1} & ; m = n \end{cases}$$

Now;

$$\int_{-1}^{1} [2P_2(x)][P_4(x)]^2 \cdot dx = [2P_2(x)] \quad \times$$

(WRONG APPROACH)

Right approach! →

$$\int_{-1}^{1} 3x^2 (P_4(x))^2 \cdot dx - \int [P_4(x)]^2 \cdot dx$$

$= I_1 + I_2$

$$I_2 = \frac{-2}{2(4)+1} = -\frac{2}{9}$$

Now:

$$I_1 = \int_{-1}^{1} 3x^2 [P_4(x)]^2 . dx$$

$$= 3x^2 . \int [P_4(x)]^2 - 2\int_{-1}^{1} x.[P_4(x)]^2 .dx$$

X wrong Approach

NOT 1LATE

$$I_1 = \int_{-1}^{1} 3[(x) P_4(x)]^2 . dx \quad \text{--- (1)}$$

$$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

$$x P_n(x) = \frac{1}{2n+1} [(n+1) P_{n+1}(x) + n P_{n-1}(x)]$$

$$x P_4(x) = \frac{1}{9} [5 P_5(x) + 4 P_3(x)]$$

$$I_1 = \int_{-1}^{1} 3 . \frac{1}{9} [5 P_5(x) + 4 P_3(x)]^2 . dx$$

$$= \frac{1}{27} \int_{-1}^{1} 25 [P_5(x)]^2 + 16 [P_3(x)]^2 + 40 P_5(x) P_3(x). dx$$

$$= \frac{1}{27} \left[25 \times \frac{2}{11}\right] + \frac{1}{27} \left[16 \times \frac{2}{7}\right] + 0$$

$$= \frac{50}{27 \times 11} + \frac{32}{7 \times 27} \Rightarrow \frac{50}{297} + \frac{32}{189} = \frac{26}{77}$$

$$I = I_1 + I_2 = \frac{26}{77} - \frac{2}{9} = \frac{80}{693}.$$

Q] Apply orthogonal property of $P_n(x)$ to determine a, b & c such that:

$x^4 = a P_4(x) + b P_2(x) + c P_0(x)$ } Applying orthogonal property (absolute constant).

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Fourier Series →

A series of sines and cosines of an angle and its multiple of the form

$$\boxed{f_x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin n\frac{\pi x}{c}}$$

is called fourier series, where $a_0, a_1, a_2 \ldots b_1, b_2 \ldots$ are called fourier constants, while c may have any value and it is an arbitrary constant.

└ (The value may change as per the question)

Fourier series expansion →

(Values a_0, a_n, b_n)

CASE I:

If the function $f(x)$ be defined in the interval 0 to $2c$ then:

$$a_0 = \frac{1}{c}\int_0^{2c} f(x).dx$$

$$a_n = \frac{1}{c}\int_0^{2c} f(x).\cos\frac{n\pi x}{c}.dx$$

$$b_n = \frac{1}{c}\int_0^{2c} f(x).\sin\frac{n\pi x}{c}.dx.$$

CASE II:

If the funct^n $f(x)$ defined in the interval $-c$ to c

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x).\frac{\cos n\pi x}{c}.dx$$

$\int_{-a}^{a} \rightarrow$ special case.

even (+) odd (−)

$\oplus \times \ominus = \ominus$
$\oplus \times \oplus = \oplus$
$\ominus \times \ominus = \oplus$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x).\sin\frac{n\pi x}{c}.dx$$

On the basis of nature of funcⁿ f(x). we may also write →

(i) If f(x) is an even function in interval (−c, c)

$$\boxed{a_0 = \frac{2}{c}\int_{0}^{c} f(x).dx \qquad a_n = \frac{2}{c}\int_{0}^{c} f(x).\cos\frac{n\pi x}{c}.dx}$$

$$b_n = \frac{2}{c}\int_{0}^{c} f(x).\sin\frac{n\pi x}{c}.dx \quad \times$$

$\cos\frac{n\pi x}{c} \rightarrow$ even function
$f(x) \rightarrow$ even $\rightarrow$ given
$\} \rightarrow$ multiplicatⁿ will give an even functⁿ

$\sin\frac{n\pi x}{c} \rightarrow$ odd function
$f(x). \rightarrow$ even $\rightarrow$ given
$\} \rightarrow$ multiplicatⁿ will give an odd funct.

∴ for odd function while integrating from $\int_{-a}^{a}$

The value of integral will be zero

$$\therefore \boxed{b_n = 0}$$

(ii) If $f(x)$ is an odd function

$\longrightarrow a_0 = 0$

$\longrightarrow a_n = 0$

$\longrightarrow b_n = \frac{2}{c} \int_0^c f(x). \sin \frac{n\pi x}{c} . dx.$

CASE III : (Half Range series case) $\longrightarrow$

If functn $f(x)$ is defined in the interval 0 to c then $\longrightarrow$

(i) To get the series of cosines only; we assume $f(x)$ as even function in the interval $-c$ to c

Then we have:

$\longrightarrow a_0 = \frac{2}{c} \int_0^c f(x). dx$

$\longrightarrow a_n = \frac{2}{c} \int_0^c f(x). \cos \frac{n\pi x}{n} . dx$

$\longrightarrow b_n = 0$

(ii) To get the series of sines only; we assume $f(x)$ as odd functions in the interval $-c$ to c

Then we have:

$\longrightarrow a_0 = 0$

$\longrightarrow a_n = 0$

$\longrightarrow b_n = \frac{2}{c} \int_0^c f(x). \sin \frac{n\pi x}{n} . dx.$

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Some Important Results →

1. $\sin n\pi = 0$
2. $\cos n\pi = (-1)^n$
3. $\int_0^{2\pi} \sin nx \, dx = 0$

Q] Expand the f^n $f(x) = x^2$ as fourier series in $[-\pi, \pi]$. Hence deduce that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

→ $f(x) = x^2$ → even function

Let the req. fourier series be →

$$x^2 = f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=0}^{\infty} b_n \frac{\sin n\pi x}{c}$$

$$a_0 = \frac{2}{c}\int_0^{\pi} x^2 \, dx \quad ; \quad b_n = 0$$

$$= \frac{2}{c} \left.\frac{x^3}{3}\right|_0^{\pi} = \frac{2}{c} \cdot \frac{\pi^3}{3} = \frac{2}{3c}\pi^3$$

$$a_n = \frac{2}{c}\int_0^{\pi} x^2 \cdot \cos\frac{n\pi x}{c} \, dx$$

$\because b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{c}$$

$$a_0 = \frac{2}{3c}\pi^3 = \frac{2}{3\pi} \cdot \pi^3 = \frac{2\pi^2}{3}$$

$b_n = 0$

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cdot \cos nx.$$

$\because c = \pi$; $a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx.$

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$f(-x) = f(x) \longrightarrow$ even

$f(-x) = -f(x) \longrightarrow$ odd.

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$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \cdot dx$$

$$= \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx\right]$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - \left[2x \cdot \frac{-\cos nx}{n^2} + \int_0^{\pi} 2 \cdot \frac{\cos nx}{n^2}\right.\right.$$

$$\therefore x^2 = \frac{\pi^2}{3} + \sum_{n=0}^{\infty}$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \left[-2x\left(\frac{\cos nx}{n^2}\right) + \left[2 \cdot \frac{\sin nx}{n^3} - \int_0 0\right]\right.\right.$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + 2x\left(\frac{\cos nx}{n^2}\right) - \frac{2 \sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2(0) + 2\pi\left(\frac{\cos n\pi}{n^2}\right) - 0\right]$$

$$= \frac{2}{\cancel{\pi}}\left[2\cancel{\pi}\left(\frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{4 \cos n\pi}{n^2}$$

$$\boxed{a_n = \frac{4 \cos n\pi}{n^2}} - \frac{4(-1)^n}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$\therefore \boxed{x^2 = \frac{\pi^2}{3} + \sum_{n=0}^{\infty} \frac{4}{n^2}(-1)^n \cos nx}$$

put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=0}^{\infty} \frac{4}{n^2}(-1)^n \cancel{\cos(\pi x)} \rightarrow (-1)^n$$

$$\left(\pi^2 - \frac{\pi^2}{3}\right) = \sum_{n=0}^{\infty} \frac{4}{n^2}(-1)^{2n}$$

$$\frac{2\pi^2}{3} = \sum_{n=0}^{\infty} \frac{4}{n^2} = 4 \sum_{n=0}^{\infty} \frac{1}{n^2}$$

putting values of n

$$\frac{\cancel{2}\pi^2}{3} = \cancel{4}2 \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence deduced //.