Definition: A real valued function $g(x,y)$ of two real variables is said to be harmonic in a domain D if it has first and second order partial derivatives in a domain D and satisfies the Laplace Equation

$$g_{xx} + g_{yy} = 0.$$

i.e.
$$\boxed{\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0}$$

Ex: Show that the function $h(x,y) = 3x^2y - y^3 + 4$ is harmonic in $\mathbb{C}$.

Sol^n.

$$h(x,y) = 3x^2y - y^3 + 4$$

$$h_x = 6xy \quad ; \quad h_y = 3x^2 - 3y^2$$

$$h_{xx} = 6y \quad ; \quad h_{yy} = -6y$$

It is obvious that h_x, h_y, h_{xx}, h_{yy} are continuous functions (as they are Polynomials).

Now

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 6y - 6y = 0$$

$\Rightarrow$ $h(x,y)$ satisfies Laplace equation

$\Rightarrow$ $h(x,y)$ is an harmonic function.

Harmonic Conjugate: If $u(x,y)$ is a given harmonic function in the domain D and if we can find another harmonic function $v(x,y)$, where the first order partial derivatives of $u(x,y)$ & $v(x,y)$ satisfy the Cauchy-Riemann Equations throughout D, then we say that '$v(x,y)$' is the harmonic conjugate of '$u(x,y)$'.

Result 1: If a complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, Then the functions $u(x,y)$ and $v(x,y)$ are harmonic in D.

Proof: $f(z) = u(x,y) + iv(x,y)$ is analytic $\Rightarrow$ $u(x,y)$ & $v(x,y)$ satisfy C-R equs. at every $z \in D$.

Now C-R equation.

$$u_x = v_y \quad \& \quad u_y = -v_x$$

i.e.

$$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}_{\text{diff w.r.t 'x' on both sides}} \quad \& \quad \underbrace{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{\text{diff w.r.t 'y' on both sides}}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \cdot \partial x} \quad \text{--- (1)} \quad \& \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \cdot \partial y} \quad \text{--- (2)}$$

Now $f(z)$ is anytic in D $\Rightarrow$ the derivatives of all orders are analytic in D

$\Rightarrow$ $u(x,y)$ & $v(x,y)$ have continuous partial derivatives of all orders in D.

$$\Rightarrow \frac{\partial^2 v}{\partial y \cdot \partial x} = \frac{\partial^2 v}{\partial x \cdot \partial y} \quad \text{also} \quad \frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial^2 u}{\partial y \cdot \partial x}$$

Now adding equation (1) & (2) we get.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly we can show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\Rightarrow$ $u(x,y)$ & $v(x,y)$ both satisfies Laplace equation

$\Rightarrow$ $\boxed{u(x,y) \text{ \& } v(x,y) \text{ are harmonic functions}}$.

Result 2 :- A function $f(z) = u(x,y) + iv(x,y)$ is analytic in D $\Leftrightarrow$ $v(x,y)$ is a harmonic Conjugate of $u(x,y)$.

Ex. 1 Given a function $u(x,y) = x^3 - 3xy^2 - 5y$

(a) verify that the function $u(x,y)$ is harmonic in the entire Complex plane.

(b) find the harmonic Conjugate of $u(x,y)$.

(a) $u(x,y) = x^3 - 3xy^2 - 5y \Rightarrow \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2, \; \dfrac{\partial u}{\partial y} = -6xy - 5$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

Now $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$\Rightarrow$ $u(x,y)$ is a harmoic function.

(b) Since Conjugate harmonic function $v(x,y)$ must satisfy C-R equation $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \text{—(1)} \quad \& \quad \frac{\partial v}{\partial x} = 6xy + 5 \quad \text{—(2)}$$

('x' will be treated as a Constant) here

Integrating partially w.r.t. 'y' on both sides, we get

~~Integrating Partially w.r.t 'x' on both sides, we get.~~

$$\therefore v(x,y) = 3x^2y - y^3 + \boxed{h(x)}$$

Unknown Constant of Integration

Now differentiating Partially w.r.t 'x' we get

$$\frac{\partial v}{\partial x} = 6xy + h'(x) \quad \text{—(3)}$$

Now comparing equation (3) & (2), we get $h'(x) = 5$

$$h'(x) = 5$$

Integrating w.r.t 'x' on both sides.

$$h(x) = 5x + C$$

Hence the harmonic conjugate of $u(x,y)$ is

$$v(x,y) = 3x^2y - y^3 + h(x)$$

$$\boxed{v(x,y) = 3x^2y - y^3 + 5x + C}$$

Ex-2. Find an analytic function whose real part is $u(x,y) = 2xy + 2x$

Sol^n: $f(z) = u + iv$ is analytic $\Rightarrow$ u & v satisfy C-R equ

$$\Rightarrow u_x = v_y \quad \& \quad u_y = -v_x$$

Now

$$u_x = v_y \Rightarrow v_y = 2y + 2 \quad \text{--- (1)}$$

$$u_y = -v_x \Rightarrow v_x = -2x \quad \text{--- (2)}$$

Now integrating equ (1) w.r.t 'y', treating x as Constant, we get

$$v = y^2 + 2y + \phi(x) \quad \text{--- (3)}$$

Now diff (3) w.r.t 'x'

$$v_x = \phi'(x) \quad \text{--- (4)}$$

Comparing equ (4) & (2), we have.

$$\phi'(x) = -2x$$

Now integrating on both sides w.r.t 'x' we get

$$\phi(x) = -x^2 + C$$

Hence

$$v(x,y) = y^2 + 2y + \phi(x) = y^2 + 2y - x^2 + C$$

Ex-2 Contd.. So desired Analytic function is

$$f(z) = 2xy + 2x + i(-x^2 + y^2 + 2y + c) \quad \text{Ans.}$$

Ex-3 (If V is given): find Analytic function $f(z)$ whose imaginary Part is given as: $v(x,y) = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, $[f(z) = u + iv]$

Sol^n.

$$v(x,y) = \cancel{x^2+y} \; x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2} \;\cancel{=2y} \quad \text{—(1)}$$

$$\frac{\partial v}{\partial y} = -2y + \frac{(-2xy)}{(x^2+y^2)^2} \quad \text{—(2)}$$

$f(z)$ is analytic $\Rightarrow$ u & v satisfy the C-R equations

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\left[2x + \frac{y^2 - x^2}{(x^2+y^2)^2}\right] \quad \text{—(3)}$$

$$\& \quad \frac{\partial u}{\partial x} = -2y - \frac{2xy}{(x^2+y^2)^2} \quad \text{—(4)}$$

Now integrating (4) partially w.r.t 'x', treating y as a constant. we get.

$$u = -2y \int dx - y \int \frac{2x}{(x^2+y^2)^2} dx$$

$$u = -2xy + \frac{y}{(x^2+y^2)} + f(y) \quad \text{—(5)}$$

diff (5) w.r.t 'y' Partially, we get.

$$\frac{\partial u}{\partial y} = -2x - \frac{y^2 - x^2}{(x^2+y^2)^2} + f'(y) \quad \text{—(6)}$$

Ex. 1

Now Comparing equations (5) & (6) we get

$$f'(y) = 0 \Rightarrow f(y) = K \rightarrow \text{constant.}$$

Hence

$$u(x,y) = -2xy + \frac{y}{x^2+y^2} + f(y)$$

$$u(x,y) = -2xy + \frac{y}{x^2+y^2} + K$$

Hence desired Analytic function $f(z) = u + iv$ is

Answer $$\boxed{f(z) = -2xy + \frac{y}{(x^2+y^2)} + K + i\left[x^2 - y^2 + \frac{x}{x^2+y^2}\right]}$$

derivative of $f(z)$, $f'(z) = u_x + iv_x$

Constructing $f(z)$ in terms of z :→

1. MILNE-THOMSON'S METHOD : To find $f(z) = u + iv$ in terms of 'z'.

Case I: when Real Part $u(x,y)$ is given.

Step 1: Find u_x and u_y

Step 2: As $f(z) = u + iv \Rightarrow f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = u_x - iu_y \quad \left[\begin{array}{l}\because \text{C-R eqn} \\ u_x = v_y \\ u_y = -v_x\end{array}\right.$$

So we have $f'(z)$ in terms of u_x & u_y

i.e $$\boxed{f'(z) = u_x - iu_y} \quad \text{——— (1)}$$

Step 3: Put $x = z$ and $y = 0$ in $f'(z)$.

Step 4: Integrate $f'(z) = u_x(z,0) - i\,u_y(z,0)$ w.r.t 'z' to obtain $f(z)$. in terms of 'z' only.

Ex.1 Find the analytic function $f(z) = u + iv$, whose real part is $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2$ by Milne thomson method.

Sol:

$$u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad ; \quad \frac{\partial u}{\partial y} = -6xy - 6y$$

Now

$$f(z) = u + iv$$

C-R equ. $[u_x = -u_y]$

$$\Rightarrow \quad f'(z) = u_x + iv_x = u_x - iu_y$$

$$f'(z) = (3x^2 - 3y^2 + 6x) - i(-6xy - 6y)$$

Now Replace x by z and y by 0, we get

$$f'(z) = 3z^2 + 6z - i(0)$$

$$f'(z) = 3z^2 + 6z$$

Integrating w.r.t. 'z' on both sides, we get

$$f(z) = 3 \cdot \frac{z^3}{3} + 6\frac{z^2}{2} + ic$$

$$\boxed{f(z) = z^3 + 3z^2 + ic}$$ Answer

Case II: Finding f(z) in terms of 'z' when the imaginary part $v(x,y)$ is given:

Step 1: Find v_x & v_y

Step 2: As $f(z) = u + iv \Rightarrow f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = v_y + iv_x \qquad \left[\begin{array}{l}\text{C-R eq}^n \\ u_x = v_y \\ u_y = -v_x\end{array}\right.$$

So, we have $f'(z)$ in terms of v only

$$\boxed{f'(z) = v_y + iv_x} \quad \text{—} (1)$$

Step 3: Replace x by z & y by 0 in (1) & we get

$$f'(z) = v_y(z,0) + iv_x(z,0) \quad \text{—} (2)$$

Step 4: Integrating (2) w.r.t 'z' only, we get

f(z) in terms of 'z' only.

Ex-1: Determine the analytic function $f(z) = u + iv$ whose imaginary part is $v(x,y) = 3x^2y - y^3$

Sol^n

$$v(x,y) = 3x^2y - y^3 \Rightarrow \frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

Now

$$f(z) = u + iv$$

$$f'(z) = u_x + iv_x = v_y + iv_x = 3x^2 - 3y^2 + i\,6xy$$

$$\Rightarrow f'(z) = 3x^2 - 3y^2 + i\,6xy$$

~~Integrating~~ Replace $x = z$ & $y = 0$, we get

$$f'(z) = 3z^2.$$

Integrating w.r.t. 'z', we get

$$\boxed{f(z) = z^3 + c} \quad \text{Ans.}$$

Case III :- finding $f(z) = u+iv$, if $u+v$ is given instead of u or v.

Ex. 1 If $u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$, and $f(z) = u+iv$ is an analytic function of z, then find $f(z)$ in terms of z.

Sol^n

$$u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$$

As we know $f(z) = u+iv \quad -①$

$$if(z) = iu+i^2v = iu-v \quad -②$$

Adding ① & ②, we get

$$f(z)+if(z) = (u-v)+i(u+v)$$

$$f(z)(1+i) = u-v+i(u+v) \quad -③$$

Now let $U = u-v$, $u+v = V$, $F(z) = f(z)(1+i)$

equ ③ becomes,

$$F(z) = U+iV$$

as $V = u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$ is given in the Problem which is the imaginary part of Analytic function $F(z)$. Now we can use Case-II of Milne thomson method to find $F(z)$. and thus $f(z)$.

Now $V = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x} = \dfrac{2\sin 2x}{2\cosh 2y-2\cos 2x} = \dfrac{\sin 2x}{\cosh 2y-\cos 2x}$

$$\frac{\partial V}{\partial x} = \frac{2\cos 2x(\cosh 2y-\cos 2x)-2\sin^2 2x}{(\cosh 2y-\cos 2x)^2}$$

$$\frac{\partial V}{\partial y} = \frac{-\sin 2x(2\sinh 2y)}{(\cosh 2y-\cos 2x)^2}$$

Now As

$$f(z) = u + iv$$

$$f'(z) = u_x + iv_x = v_y + iv_x$$

$$f'(z) = \frac{-\sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2} + i\left[\frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}\right]$$

Replace $x = z$, & $y = 0$, we get

$$f'(z) = \frac{\sin 2z(2\sinh 0)}{(\cosh 0 - \cos 2z)^2} + i\left[\frac{2\cos 2z(\cosh 0 - \cos 2z) - 2\sin^2 2z}{(\cosh 0 - \cos 2z)^2}\right]$$

$\sinh 0 = 0$
$\cosh 0 = 1$

$$f'(z) = \cancel{\sin 2z}\ 0 + i\left[\frac{2\cos 2z(1 - \cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2}\right]$$

$$(1+i)f'(z) = i\frac{(2\cos 2z - 2)}{(1 - \cos 2z)^2} = \frac{-2i}{1 - \cos 2z} = -i\operatorname{cosec}^2 z$$

~~Integration~~

$$(1+i)f'(z) = -i\operatorname{cosec}^2 z$$

Integrating w.r.t. 'z', we get

$$(1+i)f(z) = -\int i\operatorname{cosec}^2 z + c$$

$$(1+i)f(z) = i\cot z + c$$

$$f(z) = \frac{i}{(1+i)}\cot z + \frac{c}{(1+i)}$$

$$= \frac{i(1-i)}{2}\cot z + \frac{c}{(1+i)}$$

Answer $\quad f(z) = \frac{1}{2}\cot z + \frac{i}{2}\cot z + A$

Case IV: finding $f(z) = u+iv$, if $u-v$ is given instead of u or v.

Ex: Find the analytic function $f(z) = u+iv$, if
$$u-v = (x-y)(x^2+4xy+y^2)$$

Sol:
$$f(z) = u+iv \quad -①$$
$$i f(z) = iu + i^2 v = iu - v \quad -②$$

Adding ① & ②, we get $\quad f(z)(1+i) = (u-v) + i(u+v) \quad -③$

let $\quad (1+i) f(z) = F(z), \quad u-v = U, \quad u+v = V$

Now equ ③ becomes, $\quad F(z) = U + iV.$, where $F(z)$ is an analytic fun of U & V and here in the Problem $U = u-v$ is given and we have to find $F(z)$ {using Milne-thomson's Case I}

$$U = u-v = (x-y)(x^2+4xy+y^2)$$
$$U_x = x^2+4xy+y^2 + (x-y)(2x+4y) = 3x^2-3y^2-6xy$$
$$U_y = -(x^2+4xy+y^2) + (x-y)(4x+2y) = 3x^2-3y^2-6xy$$

Now
$$F(z) = U+iV \Rightarrow F'(z) = U_x + iV_x$$

So
$$\Rightarrow F'(z) = U_x - iU_y$$

$$F'(z) = 3x^2-3y^2-6xy - i[3x^2-3y^2-6xy]$$
$$\Rightarrow (1+i) f'(z) = 3x^2-3y^2-6xy - i[3x^2-3y^2-6xy]$$

Now, Replace $x = z$ & $y = 0$, we get
$$(1+i) f'(z) = 3z^2 - i3z^2 = 3(1-i)z^2$$

integrating on both sides w.r.t 'z', we get

$$(1+i)f(z) = 3(1-i)\frac{z^3}{3} + c$$

$$f(z) = \frac{(1-i)}{(1+i)} z^3 + c.$$

$$\boxed{f(z) = -iz^3 + c}$$ Ans.

Exercise: ① ⓐ verify that the given function

$u(x,y) = e^{-x}(x\sin y - y\cos y)$ is harmonic

ⓑ find v such that $f(z) = u + iv$ is analytic

ⓒ find $f(z)$ in terms of 'z' only.

Ans (b): $v = e^{-x}(y\sin y + x\cos y) + c$

(c) $f(z) = ize^{-z} + c$

② If ~~u=xy~~ $u = x^3 - 3xy^2$, then find v such that $f(z) = u + iv$ is analytic

Ans: $v(x,y) = 3x^2y - y^3 + c$.

③ Given that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate.

Ans

$v = \tan^{-1}(y/x) + c$

④ If $f(z) = u + iv$ is an analytic function of z, and

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$$

find $f(z)$ subject to condition $f(\pi/2) = 0$

Ans:

$$f(z) = \frac{1}{2} - \frac{1}{2}\cot\frac{z}{2}$$