Theoretical Foundations of Computer Science

COUNTING

Chapter Summary

The Basics of Counting

The Pigeonhole Principle

~~Permutations and Combinations~~

Binomial Coefficients and Identities

~~Generalized Permutations and Combinations~~

Basic Counting: The Product Rule

Recall: For a set A, $|A|$ is the cardinality of A (# of elements of A).

For a pair of sets A and B, $A \times B$ denotes their cartesian product:

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Product Rule

If A and B are finite sets, then: $|A \times B| = |A| \cdot |B|$.

Proof: Obvious, but prove it yourself by induction on $|A|$. $\square$

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general Product Rule

If $A_1, A_2, \ldots, A_m$ are finite sets, then

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$$

Proof: By induction on m, using the (basic) product rule. □

Product rule applies when a procedure is made up of separate tasks.

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Product Rule: examples

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Example 2: How many different car license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: 26 choices are available for first 3 letters and 10 choices for each digit. □

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000.$$

Counting Subsets

Number of Subsets of a Finite Set

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Proof: Suppose $S = \{s_1, s_2, \ldots, s_m\}$.
There is a one-to-one correspondence (bijection), between subsets of S and bit strings of length $m = |S|$.
The bit string of length $|S|$ we associate with a subset $A \subseteq S$ has a 1 in position i if $s_i \in A$, and 0 in position i if $s_i \notin A$, for all $i \in \{1, \ldots, m\}$.

$$\{s_2, s_4, s_5, \ldots, s_m\} \quad \cong \quad \underbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 1 & 0 & 1 & 1 & \ldots & 1 \\ \hline \end{array}}_{m}$$

By the product rule, there are $2^{|S|}$ such bit strings. $\square$

Counting Functions

Number of Functions

For all finite sets A and B, the number of distinct functions, $f : A \to B$, mapping A to B is:

$$|B|^{|A|}$$

Proof: Suppose $A = \{a_1, \ldots, a_m\}$. m elements

A function corresponds to a choice of one of the n elements in the co-domain for each of the m elements in the domain.

By the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ *functions* from a set with *m elements to one with n elements.*

$\square$

Sum rule

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Sum Rule

Sum Rule

If A and B are finite sets that are disjoint (meaning $A \cap B = \emptyset$), then

$$|A \cup B| = |A| + |B|$$

Proof. Obvious. (If you must, prove it yourself by induction on $|A|$.) □

general Sum Rule

If $A_1, \ldots, A_m$ are finite sets that are pairwise disjoint, meaning $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, m\}$, then

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

Sum Rule: Examples

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Example 2: Each user on a computer system has a password which must be six to eight characters long.
Each character is an uppercase letter or digit.
Each password must contain at least one digit.
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Each character is an uppercase letter or digit.
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How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, P_8 be the number of passwords of lengths 6, 7, and 8, respectively.

By the sum rule $P = P_6 + P_7 + P_8$.

$P_6 = 36^6 - 26^6$; $P_7 = 36^7 - 26^7$; $P_8 = 36^8 - 26^8$.

So, $P = P_6 + P_7 + P_8 = \sum_{i=6}^{8}(36^i - 26^i)$. □

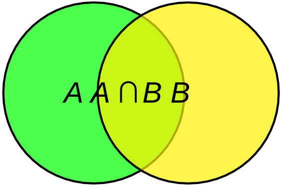
Subtraction Rule (Inclusion-Exclusion for two sets)

Subtraction Rule

For any finite sets A and B (not necessarily disjoint),

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof: Venn Diagram:



|A| + |B| overcounts (twice) exactly those elements in $A \cap B$.

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Subtraction Rule: Example

Example: How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?

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Example: How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?

Solution:

Number of bit strings of length 8 that start with 1: $2^7 = 128$.

Number of bit strings of length 8 that end with 00: $2^6 = 64$.

Number of bit strings of length 8 that start with 1 <u>and</u> end with 00: $2^5 = 32$.

Applying the subtraction rule, the number is $128 + 64 - 32 = 160$. $\square$

The Pigeonhole Principle

Pigeonhole Principle

For any positive integer k , if $k + 1$ objects (pigeons) are placed in k boxes (pigeonholes), then at least one box contains two or more objects.

Proof: Suppose no box has more than 1 object. Sum up the number of objects in the k boxes. There can't be more than k .
Contradiction. $\square$

Pigeonhole Principle (rephrased more formally)

If a function $f : A \to B$ maps a finite set A with $|A| = k + 1$ to a finite set B, with $|B| = k$, then f isnotone-to-one.

Pigeonhole Principle: Examples

Example 1: At least two students registered for this course will receive exactly the same final exam mark. Why?

Pigeonhole Principle: Examples

Example 1: At least two students registered for this course will receiveexactly the samefinal exam mark. Why?

Reason: There are at least 102 students registered for TFCS , so, at least 102 objects. Final exam marks are integers in the range 0-100 (so, exactly 101 boxes).

$\square$

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle (GPP)

If $N \geq 0$ objects are placed in $k \geq 1$ boxes, then at least one box contains at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

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Generalized Pigeonhole Principle (GPP)

If $N \geq 0$ objects are placed in $k \geq 1$ boxes, then at least one box contains at least $\lceil \frac{N}{k} \rceil$ objects.

Proof: Suppose no box has more than $\lceil \frac{N}{k} \rceil - 1$ objects. Sum up the number of objects in the k boxes. It is at most

$$k \cdot \left(\left\lceil \frac{N}{k} \right\rceil - 1 \right) < k \cdot \left(\left(\frac{N}{k} + 1 \right) - 1 \right) = N$$

Thus, there must be fewer than N. Contradiction.
(We are using the fact that $\lceil \frac{N}{k} \rceil < \frac{N}{k} + 1$.) $\square$

Exercise: Rephrase GPP as a statement about functions $f : A \rightarrow B$ that map a finite set A with $|A| = N$ to a finite set B, with $|B| = k$.

Generalized Pigeonhole Principle: Examples

Example 1: Consider the following statement:

"At least d students in this course were born in the same month." (1)

Suppose the actual number of students registered for TFCS is 250. What is the maximum number d for which it is certain that statement (1) is true?

Generalized Pigeonhole Principle: Examples

Example 1: Consider the following statement:

"At least d students in this course were born in the same month." (1)

Suppose the actual number of students registered for TFCS is 250. What is the maximum number d for which it is certain that statement (1) is true?

Solution: Since we are assuming there are 250 registered students in TFCS.

$\left\lceil \frac{250}{12} \right\rceil = 21$, so by GPP we know statement (1) is true for $d = 21$.

Statement (1) need not be true for $d = 22$, because if 250 students are distributed *as evenly as possible* into 12 months, the maximum number of students in any month is 21, with other months having only 20. $\square$

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GPP: more Examples

Example 2: How many cards must be selected from a standard deck of 52 cards to guarantee that at least thee cards of the same suit are chosen?

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Solution: There are 4 suits. (In a standard deck of 52 cards, every card has exactly one suit. There are no jokers.) So, we need to choose N cards, such that $\left\lceil \frac{N}{4} \right\rceil \geq 3$. The smallest integer N such that $\left\lceil \frac{N}{4} \right\rceil \geq 3$ is $2 \cdot 4 + 1 = 9$. □

Binomial Coefficients

Consider the polynomial in two variables, x and y, given by:

$$(x + y)^n = \underbrace{(x + y) \cdot (x + y) \ldots (x + y)}_{n}$$

By multiplying out the n terms, we can expand this polynomial and write it in a standard sum-of-monomials form:

$$(x + y)^n = \sum_{j=0}^{n} c_j x^{n-j} y^j$$

Question: What are the coefficients c_j? (These are called binomial coefficients.)

Examples:

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

The Binomial Theorem

Binomial Theorem

For all $n \geq 0$:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{N-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n} y^n$$

Pascal's Identity

Theorem (Pascal's Identity)

For all integers $n \geq 0$, and all integers r, $0 \leq r \leq n + 1$:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$