Department of Mathematics

15B11MA211 | Mathematics-II

Tutorial Sheet 4 | B.Tech. Core

Fourier Series

1. Expand the function $f(x) = x\sin x$ in a Fourier series in the interval $-\pi \le x < \pi$. Use the series obtained to show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} = ... = \frac{\pi - 2}{4}$.

2. Given $f(x) = \begin{cases} -x+1 \; for -\pi < x \le 0 \\ x+1, \; for\, 0 \le x \le \pi \end{cases}$, find a Fourier series for $f(x)$ and hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

3. Find the Fourier series expansion of the function $f(x) = \begin{cases} \pi x \; for\; 0 \le x < 1 \\ 0 \quad x = 1 \\ \pi(x-2) \; for\; 1 < x < 2. \end{cases}$ in the interval [0,2].

4. Find the Fourier series expansion of the function $f(x)=e^{-4x}$ in the interval [-2,2].

5. Find the Fourier series expansion of the function $f(x) = x - x^2$ in the interval $-1 < x \le 1$.

6. Find the half range sine series for the function $f(x) = x^2$ for $0 < x < \pi$.

7. Find the half range cosine series for the function $f(x) = 2x - 1$ for $0 < x < 1$.

8. Find the Fourier series expansion of the function $f(x) = \begin{cases} 0 \text{ for } 0 \le x < l \\ x \text{ for } l \le x < 2l \end{cases}$ in the interval $[0, 2l]$.

Answers. (3) $f(x) = 2(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - ...)$,

(4) $a_0=(e^8-e^{-8})/8$, $a_n=(e^8-e^{-8})\, 8\, (-1)^n/(64+pi^2\, n^2)$, $b_n=(e^8-e^{-8})\, n^*\, pi\, (-1)^n/(64+pi^2\, n^2)$

(5) $f(x) = -\frac{1}{3} + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x}{(2n+1)}$,

(6) $f(x) = \frac{2}{\pi}\left\{(\pi^2 - 4)\sin x - \frac{\pi^2 \sin 2x}{2} + \frac{1}{3}(\pi^2 - \frac{4}{3^2})\sin 3x - ......\right\}$

(7) $f(x) = -\frac{8}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + ...\right)$

(8) $f(x) = \frac{3l}{4} + \frac{l}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x / l}{(2n+1)^2}$

$$-\frac{l}{\pi}\left\{\frac{3\sin \pi x / l}{1} + \frac{\sin 2\pi x / l}{2} + \frac{3\sin 3\pi x / l}{3} + \frac{\sin 4\pi x / l}{4} + ...\right\}$$

Q.1:- $f(x) = x\sin x \quad , -\pi \le x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\, dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\, dx$$

$$= \frac{2}{\pi}\left[(-x\cos x)_0^{\pi} + \int_0^{\pi} 1\cdot\cos x\, dx\right]$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi + (\sin x)_0^{\pi}\right]$$

$$= \frac{2}{\pi}[\pi] = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{x}{2}[\sin(n+1)x + \sin(1-n)x]dx$$

$\because \sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$= \frac{1}{\pi}\left[-x\frac{\cos(n+1)x}{n+1} + \frac{1\cdot\sin(n+1)x}{(n+1)^2}\right]_0^{\pi}$$

$$+ \frac{1}{\pi}\left[-x\frac{\cos(1-n)x}{1-n} + \frac{1\cdot\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n+1)\pi}{n+1}\right] + \frac{1}{\pi}\left[\frac{\pi}{n-1}\cos(1-n)\pi\right]$$

$$= -\frac{\cos(n\pi+\pi)}{n+1} + \frac{1}{n-1}\cos(n\pi-\pi)$$

$$= \frac{-1}{n+1}[\cos n\pi\cos\pi - \sin n\pi\sin\pi] + \frac{1}{n-1}[\cos n\pi\cos\pi + \sin\pi\sin n\pi]$$

$$= \frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1}$$

If n is any integer $\Rightarrow \cos n\pi = (-1)^n$, $\sin n\pi = 0$

$$= (-1)^n\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{-2(-1)^n}{n^2-1}, \quad n \ne 1$$

bcz it is not define for $n=1$ so find $a_1 = ?$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x\sin 2x}{2}dx = \frac{2}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right] = -\frac{1}{2}$$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \sin nx \, dx$ $\qquad \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{2} (\cos(1-n)x - \cos(1+n)x) \, dx$

$= 0 \qquad$ (Odd function)

$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$= 1 + \sum_{n=2}^{\infty} \frac{-2(-1)^n}{n^2-1} \cos nx + \left(\frac{-1}{2}\right) \cos x$

Now

$\Rightarrow \quad x \sin x = 1 + \sum_{n=2}^{\infty} \frac{-2(-1)^n}{n^2-1} \cos nx - \frac{1}{2} \cos x$

$= 1 - \frac{1}{2} \cos x - \frac{2}{3} \cos 2x + \frac{2}{8} \cos 3x - - -$

$= 1 - \frac{1}{2} \cos x + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos nx$

Put $x = \frac{\pi}{2}$

$\frac{\pi}{2} = 1 + 2 \left[\frac{-1}{1 \cdot 3} \cos \pi - \frac{\cos 2\pi}{3 \cdot 5} - \frac{\cos 3\pi}{5 \cdot 7} - - - \right]$

$\frac{\pi}{2} - 1 = 2 \left[\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - - - \right]$

$\frac{\pi - 2}{4} = \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - - -$

(2) Given $f(x) = \begin{cases} -x+1 & \text{for } -\pi < x \le 0 \\ x+1 & \text{for } 0 \le x \le \pi \end{cases}$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$

$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-x+1) \, dx + \int_{0}^{\pi} (x+1) \, dx \right]$

$= \frac{1}{\pi} \left[\left(\frac{-x^2}{2} + x\right)_{-\pi}^{0} + \left(\frac{x^2}{2} + x\right)_{0}^{\pi} \right]$

$= \frac{1}{\pi} \left[0 - \left(-\frac{\pi^2}{2} - \pi\right) + \frac{\pi^2}{2} + \pi \right] = \frac{1}{\pi} [\pi^2 + 2\pi] = \pi + 2$

3)
B)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(1-x)\cos nx\,dx + \int_{0}^{\pi}(x+1)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} - \int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx + \frac{\sin nx}{n}\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\sin n\pi}{n}} - \left(\cancel{\frac{x}{n}\sin nx} + \int \frac{1}{n}\sin nx\,dx\right)_{-\pi}^{0}\right] + \frac{1}{\pi}\left[\cancel{\frac{x}{n}\sin nx}\right.$$

$$\left.-\int \frac{\sin nx}{n}\,dx\right]_{0}^{\pi} + \frac{1}{\pi}\cancel{\frac{\sin n\pi}{n}}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2}\cos nx\Big|_{-\pi}^{0} + \frac{\cos nx}{n^2}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2}(1-(-1)^n) + \frac{1}{n^2}((-1)^n-1)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}((-1)^n-1) + \frac{1}{n^2}((-1)^n-1)\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

$$= \begin{cases} \dfrac{-4}{\pi n^2}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(1-x)\sin nx\,dx + \int_{0}^{\pi}(x+1)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} - \left(\frac{\cos nx}{n}\right)_{0}^{\pi} - \int_{-\pi}^{0} x\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(1-(-1)^n) - \frac{1}{n}((-1)^n-1) - \left\{-\frac{x}{n}\cos nx + \int \frac{1\cdot\cos nx}{n}\,dx\right\}_{-\pi}^{0}\right.$$

$$\left. + \left\{-\frac{x}{n}\cos nx + \frac{1\cdot\sin nx}{n^2}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\cos nx - \frac{1}{n^2}\sin nx\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \left(-\frac{\pi}{n}(-1)^n - \frac{1}{n^2}\times 0\right)\right] + \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right] = 0$$

$\therefore \quad f(x) = \frac{\pi+2}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$= \frac{\pi+2}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x - - -\right]$$

Put $x=0$

$$1 = \frac{\pi}{2} + 1 - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

(3) $f(x) = \begin{cases} \pi x & \text{for } 0 \le x \le 1 \\ 0 & x = 1 \\ \pi(x-2) & \text{for } 1 < x < 2 \end{cases}$

$0 \le x \le 2$

$-l \le x \le l$

Here $l = 1$ $\quad l = +ve$ no.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \quad , \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

$\Rightarrow \quad a_0 = \int_0^1 \pi x\,dx + \int_1^2 (\pi x - 2\pi)\,dx$

$$= \left(\frac{\pi}{2}x^2\right)_0^1 + \left(\frac{\pi}{2}x^2 - 2\pi x\right)_1^2$$

$$= \frac{\pi}{2} + \frac{3\pi}{2} - 2\pi = 0$$

$a_n = \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(x-2)\cos n\pi x\,dx$

$$= \pi\left[\frac{x}{n\pi}\sin n\pi x + 1\cdot\frac{\cos n\pi x}{(n\pi)^2}\right]_0^1 + \pi\left[\frac{(x-2)\sin n\pi x}{n\pi} + \frac{1\cdot\cos n\pi x}{(n\pi)^2}\right]_1^2$$

$$= \pi\left[\frac{(-1)^n}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right] + \pi\left[\frac{\cos 2n\pi}{(n\pi)^2} - \frac{(-1)^n}{(n\pi)^2}\right]$$

$$= \pi\left[\frac{-1}{(n\pi)^2} + \frac{1}{(n\pi)^2}\right] = 0$$

$$b_n = \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi (x-2) \sin n\pi x \, dx$$

$$= \pi \left[\frac{-x \cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2} \right]_0^1 + \pi \left[- \frac{(x-2)\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2} \right]_1^2$$

$$= \pi \left[-\frac{\cos n\pi}{n\pi} - \frac{\cos n\pi}{n\pi} \right]$$

$$= \frac{-2(-1)^n}{n}$$

$$\therefore \; f(x) = -2\left[-\sin \pi x + \frac{1}{2} \sin 2\pi x - \frac{1}{3} \sin 3\pi x \cdots \right]$$

$$= 2\left[\sin \pi x - \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x \cdots \right]$$

(4) $f(x) = e^{-4x}$ in $[-2, 2]$

$$a_0 = \frac{1}{2} \int_{-2}^{2} e^{-4x} \, dx$$

$$= \frac{1}{2 \times -4} \left(e^{-4x} \right)_{-2}^{2} = -\frac{1}{8}\left[e^{-8} - e^{8} \right] = \frac{e^{8} - e^{-8}}{8}$$

$$a_n = \frac{1}{2} \int_{-2}^{2} e^{-4x} \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\left[\left(\cos\left(\frac{n\pi x}{2}\right) \frac{e^{-4x}}{-4} \right)_{-2}^{2} + \frac{1}{-4} \int_{-2}^{2} \sin\left(\frac{n\pi x}{2}\right)\left(\frac{n\pi}{2}\right) e^{-4x} \, dx \right]$$

$$= \frac{1}{2 \times -4}\left[e^{-4x} \cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi}{2} \int e^{-4x} \sin\left(\frac{n\pi x}{2}\right) dx \right]$$

$$= -\frac{1}{8}\left[e^{-4x} \cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi}{2} \left\{ \sin\left(\frac{n\pi x}{2}\right) \frac{e^{-4x}}{-4} + \int \cos\left(\frac{n\pi x}{2}\right)\left(\frac{n\pi}{2}\right) \frac{e^{-4x}}{4} \, dx \right] \right.$$

$$= -\frac{1}{8}\left[e^{-4x} \cos\left(\frac{n\pi x}{2}\right) - \frac{n\pi}{8} \sin\left(\frac{n\pi x}{2}\right) e^{-4x} + \frac{(n\pi)^2}{16} \underline{\int \cos\left(\frac{n\pi x}{2}\right) e^{-4x} \, dx} \right.$$

$$= -\frac{e^{-4x}}{8} \cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi}{64} \sin\left(\frac{n\pi x}{2}\right) e^{-4x} - \frac{(n\pi)^2}{128} \times 2 a_n$$

$$a_n \left(1 + \left(\frac{n\pi}{8}\right)^2\right) = e^{-4x}\left[\frac{n\pi}{64} \sin\left(\frac{n\pi x}{2}\right) - \frac{1}{8} \cos\left(\frac{n\pi x}{2}\right) \right]_{-2}^{2}$$

$$= e^{-8}\left[-\frac{1}{8} (-1)^n \right] - e^{8}\left[-\frac{1}{8} (-1)^n \right] = \frac{(-1)^n}{8} (e^{8} - e^{-8})$$

$$a_n = \frac{64}{64+(n\pi)^2} \cdot \frac{(-1)^n}{8}(e^8 - e^{-8})$$

$$= \frac{(-1)^n\, 8\,(e^8 - e^{-8})}{64+(\pi n)^2}$$

or

$$\int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2} e^{ax}(a\cos bx + b\sin bx)$$

$$= \frac{1}{2}\left[\frac{1}{16+\left(\frac{n\pi}{2}\right)^2} e^{-4x}\left(-4\cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi x}{2}\right)\right)\right]_{-2}^{2}$$

$$= \frac{1}{2}\left[\frac{4}{64+(n\pi)^2} e^{-4\times 2}\left(-4(-1)^n\right)\right] - \frac{2}{64+(n\pi)^2} e^{8}\left(-4(-1)^n\right)$$

$$= \frac{2}{64+(n\pi)^2} e^{-8}\left(-4(-1)^n\right) - \frac{2}{64+(n\pi)^2} e^{8}\left(-4(-1)^n\right)$$

$$= \frac{(-1)^n\, 8\,(e^8 - e^{-8})}{64+(n\pi)^2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} e^{-4x}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{1}{16+\frac{n^2\pi^2}{4}} e^{-4x}\left(-4\sin\left(\frac{n\pi x}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi x}{2}\right)\right)\right]_{-2}^{2}$$

$$\int e^{ax}\sin bx = \frac{1}{a^2+b^2} e^{ax}(a\sin bx - b\cos bx)$$

$$= \frac{1}{2}\left[\frac{4}{64+n^2\pi^2} e^{-4x}\left(-4\sin\left(\frac{n\pi x}{2}\right) - \left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)\right)\right]_{-2}^{2}$$

$$= \frac{2}{64+n^2\pi^2} e^{-8}\left(-\frac{n\pi}{2}(-1)^n\right) - \frac{2}{64+n^2\pi^2} e^{8}\left(-\frac{n\pi}{2}(-1)^n\right)$$

$\because \cos n\pi = (-1)^n, \sin n\pi = 0$

$$= \frac{2}{64+n^2\pi^2} \cdot \frac{n\pi(-1)^n}{2}\left[-e^{-8} + e^{8}\right]$$

$$= \frac{(-1)^n n\pi}{64+n^2\pi^2}(e^8 - e^{-8}) \quad \text{Ans}$$

$$f(x) = \frac{e^8 - e^{-8}}{8\times 2} + \sum_{n=1}^{\infty} \frac{(-1)^n\, 8\,(e^8 - e^{-8})}{64+(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n n\pi}{64+n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\cdot(e^8 - e^{-8})$$

5) $f(x) = x - x^2$ in interval $-1 < x \leq 1$.

$$a_0 = \int_{-1}^{1} (x - x^2)\,dx = + \int_{-1}^{1} \frac{x^2}{2}dx - \int_{-1}^{1} \frac{x^3}{3}\,dx$$

$$= \left[\frac{1}{2} - \frac{1}{2}\right] - 2\int_0^1 \frac{x^3}{3}\,dx = -\frac{2}{3}\left[x^3\right]_0^1 = -\frac{2}{3}$$

$$a_n = \int_{-1}^{1} (x - x^2) \cos(n\pi x)\,dx$$

$$= \int_{-1}^{1} \underbrace{x \cos n\pi x}_{\text{odd fun}^n}\,dx - \int_{-1}^{1} x^2 \cos(n\pi x)\,dx \quad \text{even}$$

$$= -2\int_0^1 x^2 \cos(n\pi x)\,dx$$

$$= -2\left[\frac{x^2}{n\pi} \sin(n\pi x)\Big|_0^1 - \int_0^1 \frac{2x}{n\pi} \sin(n\pi x)\,dx\right]$$

$$= \frac{4\cancel{x}}{n\pi} \int_0^1 x \sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi}\left[\frac{-x}{n\pi} \cos n\pi x + \frac{1}{(n\pi)^2} \sin n\pi x\right]_0^1$$

$$= \frac{4}{(n\pi)^2}\left[-x \cos n\pi x + \frac{\sin n\pi x}{n\pi}\right]_0^1$$

$$= \frac{4}{(n\pi)^2}\left[-(-1)^n\right]$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin(n\pi x)\,dx$$

$$= \int_{-1}^{1} x \sin(n\pi x)\,dx - \int_{-1}^{1} \underbrace{x^2 \sin n\pi x}_{\text{odd}}\,dx$$

$$= 2\int_0^1 x \sin(n\pi x)\,dx$$

$$= 2\left[\frac{-x}{n\pi} \cos(n\pi x) + \frac{1 \cdot \sin n\pi x}{(n\pi)^2}\right]_0^1$$

$$= 2\left[\frac{-(-1)^n}{n\pi}\right]$$

$$f(x) = -\frac{2}{3}\left[\frac{1}{2}\right] + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}(-(-1)^n) \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} 2\left(\frac{-(-1)^n}{n\pi}\right) \sin\left(\frac{n\pi x}{2}\right)$$

0

(6) Half range sine series for the function $f(x) = x^2$ for $0 < x < \pi$

Soln:-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx)\, dx$$

$$= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx\Big|_0^{\pi} + \int_0^{\pi} \frac{2x}{n}\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^2}{n}(-1)^n + \frac{2}{n}\left\{\frac{x}{n}\sin nx + \frac{1}{(n)^2}\cos nx\right\}_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1}\pi^2}{n} + \frac{2}{n}\left\{\frac{1}{n^2}((-1)^n - 1)\right\}\right]$$

$$= \frac{2(-1)^{n+1}\pi}{n} + \frac{4}{\pi n^3}((-1)^n - 1)$$

$$b_1 = 2\pi + \frac{4}{\pi}(-2) = 2\pi - \frac{8}{\pi}$$

$$b_2 = -\frac{2\pi}{2} = -\pi$$

$$b_3 = \frac{2\pi}{3} + \frac{4}{27\pi} \times -2 = \frac{2\pi}{3} - \frac{8}{27\pi}$$

$$\therefore \quad f(x) = \left(2\pi - \frac{8}{\pi}\right)\sin x - \pi \sin 2x + \left(\frac{2\pi}{3} - \frac{8}{27\pi}\right)\sin 3x$$

(7) Half range cosine series for $f(x) = 2x - 1$, $0 < x < 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx, \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$a_0 = 2\int_0^1 (2x-1)\, dx$$

$$= 2[x^2 - x]_0^1$$

$$= 2[1-1]$$

$$= 0$$

$$a_n = 2\int_0^1 (2x-1)\cos(n\pi x)\,dx$$

$$= 4\left[\int_0^1 x\cos(n\pi x)\,dx\right] - \frac{2}{n\pi}\sin(n\pi x)\Big|_0^1$$

$$= 4\left[\frac{x}{n\pi}\sin(n\pi x)\Big|_0^1 + \frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_0^1\right]$$

$$= 4\left[\frac{1}{(n\pi)^2}\left((-1)^n - 1\right)\right]$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \dfrac{-8}{(n\pi)^2} & \text{when } n \text{ is odd} \end{cases}$$

$$f(x) = -\frac{8}{\pi^2}\left(\cos\pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

(8) $$f(x) = \begin{cases} 0 & \text{for } 0 \le x < l \\ x & \text{for } l \le x < 2l \end{cases} \quad \text{in } [0, 2l] \qquad L = \frac{2l}{2} = l$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\left[\int_l^{2l} x\,dx\right] = \frac{1}{2l}x^2\Big|_l^{2l} = \frac{1}{2l}\left[4l^2 - l^2\right] = \frac{3l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_l^{2l} x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{x\sin\left(\frac{n\pi x}{l}\right)}{n\pi/l}\Bigg|_l^{2l} - \int 1.\frac{\sin\left(\frac{n\pi x}{l}\right)}{n\pi/l}\,dx\right]$$

$$= \frac{1}{l}\left[\left[0 - \frac{2l^2}{n\pi}\sin(n\pi)\right] + \frac{l}{l\cdot n\pi}\frac{\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\Bigg|_l^{2l}\right]$$

$$= \frac{-2l}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{l}{(n\pi)^2}\left(\cos(2n\pi) - \cos(n\pi)\right)$$

$$= \frac{l}{n^2\pi^2}\left(1 - (-1)^n\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{2l}{(2n+1)^2 \pi^2} & n \text{ odd} \end{cases}$$

$$a_n = \frac{2l}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi x}{l}\right)$$

$$b_n = \frac{1}{l} \int_{l}^{2l} x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l}\left[\frac{-x \cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} + \frac{1 \cdot \sin(n\pi x/l)}{\left(\frac{n\pi}{l}\right)^2} \right]_l^{2l}$$

$$= \frac{1}{l}\left[\frac{-l}{n\pi}\right]\left[x \cos\left(\frac{n\pi x}{l}\right)\right]_l^{2l}$$

$$= \frac{-1}{n\pi}\left[2l - l(-1)^n\right]$$

$$= -\frac{l}{n\pi}\left[2 - (-1)^n\right]$$

$$= \begin{cases} \dfrac{-l}{n\pi} & \text{when } n \text{ is even} \\ \dfrac{-3l}{n\pi} & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{3l}{4} + \frac{2l}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi x}{l}\right)$$

$$- \frac{l}{\pi}\left[3 \sin\left(\frac{\pi x}{l}\right) + \frac{1}{2} \sin\left(\frac{2\pi x}{l}\right) + \frac{3 \sin(3\pi x/l)}{3} + \frac{\sin(4\pi x/l)}{4} + \cdots\right]$$

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