Recurrence Relations

1. Find the first five terms of the sequence, defined by the recurrence relation:
$$u_{n+2} = 3u_{n+1} - 2u_n \text{ with } u_1 = 1, u_2 = 2$$

2. A sequence is defined by the recurrence relation $u_{n+2} = 3u_{n+1} - 2u_n$ with $u_1 = 1, u_2 = 2$.
   Prove by induction that $u_n = 2^{n-1}$.

3. Find the first five terms of the sequences defined by the following recurrence relations:
   (a) $u_{n+1} = 4u_n + 1, u_1 = 1$
   (b) $u_n = 4 - u_{n-1}, u_1 = 3$
   (c) $u_{n+2} = u_{n+1}u_n, u_1 = 1, u_2 = 2$
   (d) $u_n = nu_{n-1}, u_1 = 2$

4. The terms of a sequence satisfy the recurrence relation $u_{n+1} = u_n + 2n - 1$ with $u_1 = 1$. Show that $u_n = n^2$ for all $n \geq 1$.

5. Find the formula for u_n in terms of n for the sequences below:
   (a) (i) $u_{n+1} = 2u_n - 1, u_1 = 3$ (ii) $u_{n+1} = 2u_n + 3, u_1 = 1$
   (b) (i) $u_{n+1} = -3u_n + 2, u_1 = 1$ (ii) $u_{n+1} = -3u_n + 5, u_1 = -2$
   (c) (i) $u_{n+1} = \frac{1}{3}u_n + 1, u_1 = 3$ (ii) $u_{n+1} = \frac{1}{3}u_n - 2, u_1 = 8$

6. Solve the recurrence relation $u_{n+2} = u_{n+1} + u_n$ with $u_1 = u_2 = 1$.

7. Solve the recurrence relation $u_{n+2} + 4u_n = 0$ with $u_1 = 0, u_2 = 1$.

8. Two sequences, a_n and b_n, satisfy the recurrence relations
$$a_{n+1} = 3a_n + b_n \qquad b_{n+1} = 5a_n - b_n$$
   with $a_1 = 6, b_1 = -6$.
   (a) Find the value of a_2.
   (b) Show that $a_{n+2} = 2a_{n+1} + 8a_n$.
   (c) Solve the second order recurrence relation for a_n.
   (d) Hence find an expression b_n in terms of n.

9. A magic crystal produces several new crystals every day. The crystals that were produced the previous day produce only one new crystal, but the older ones produce 9 new crystals each.
   (a) If c_n is the number of crystals on day n:
      (i) Write down the numbers of crystals on days $n - 1$ and $n - 2$.
      (ii) Write down the number of crystals which are exactly one day old on day n.
      (iii) Find an expression for the total number of new crystals created on day n.
      (iv) Hence explain why $u_n = 2u_{n-1} + 8u_{n-2}$.
   (b) Harry was given two newly-formed magic crystals on the first day of term.
      (i) How many crystals does he have on the second day?
      (ii) Find an expression for the number of crystals Harry has on the nth day of term.