Counting

Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set of n elements is called an r-permutation
- **Theorem**: The number of r permutations of a set of n distinct elements is

$$P(n,r) = \Pi_{i=0}^{r-1}(n-i) = n(n-1)(n-2)\cdots(n-r+1)$$

- It follows that $$P(n,r) = \frac{n!}{(n-r)!}$$
- In particular $$P(n,n) = n!$$
- Note here that the order is important. It is necessary to distinguish when the order matters and it does not

Application of PIE and Permutations: Derangements (I) (Section 7.6)

- Consider the hat-check problem
  - Given
    - An employee checks hats from n customers
    - However, s/he forgets to tag them
    - When customers check out their hats, they are given one at random
  - Question
    - What is the probability that no one will get their hat back?

Application of PIE and Permutations: Derangements (II)

- The hat-check problem can be modeled using <u>derangements</u>: permutations of objects such that <u>no</u> element is in its original position

  - Example: 21453 is a derangement of 12345 but 21543 is not
- The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!}\right]$$

- Thus, the answer to the hatcheck problem is $\frac{D_n}{n!}$
- Note that

$$e^{-1} = \left[1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!}\right]$$

- Thus, the probability of the hatcheck problem converges

$$\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1} \approx 0.368$$

See textbook, Section 7.6 page 510

Permutations: Example A

- How many pairs of dance partners can be selected from a group of 12 women and 20 men?
  - The first woman can partner with any of the 20 men, the second with any of the remaining 19, etc.
  - To partner all 12 women, we have

$$P(20,12) = 20!/8! = 9.10.11\ldots20$$

Permutations: Example B

- In how many ways can the English letters be arranged so that there are exactly 10 letters between a and z?
  - The number of ways is P(24,10)
  - Since we can choose either a or z to come first, then there are 2P(24,10) arrangements of the 12-letter block
  - For the remaining 14 letters, there are P(15,15)=15! possible arrangements
  - In all there are 2P(24,10).15! arrangements

Permutations: Example C (1)

- How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern *bge* nor *eaf*?
  - The total number of permutations is P(7,7)=7!
  - If we fix the pattern *bge*, then we consider it as a single block. Thus, the number of permutations with this pattern is P(5,5)=5!
  - Fixing the patter eaf, we have the same number: 5!
  - Thus, we have (7! – 2.5!). Is this correct?
  - No! we have subtracted too many permutations: ones containing both *eaf* and *bfe*.

Permutations: Example C (2)

- There are two cases: (1) *eaf* comes first, (2) *bge* comes first
- Are there any cases where *eaf* comes before *bge*?
- No! The letter e cannot be used twice
- If *bge* comes first, then the pattern must be *bgeaf*, so we have 3 blocks or 3! arrangements
- Altogether, we have

$$7! – 2.(5!) + 3! = 4806$$

Outline

- Introduction
- Counting:
  - Product rule, sum rule, Principal of Inclusion Exclusion (PIE)
  - Application of PIE: Number of onto functions
- Pigeonhole principle
  - Generalized, probabilistic forms
- Permutations
- **Combinations**
- Binomial Coefficients
- Generalizations
  - Combinations with repetitions, permutations with indistinguishable objects

Combinations (1)

- Whereas permutations consider order, <u>combinations</u> are used when order does not matter
- **Definition**: A k-combination of elements of a set is an <u>unordered</u> selection of k elements from the set.

  (A combination is imply a subset of cardinality k)

Combinations (2)

- **Theorem**: The number of k-combinations of a set of cardinality n with $0 \leq k \leq n$ is

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

is read ‘n choose k’.

${n \choose k}$

Combinations (3)

- A useful fact about combinations is that they are symmetric

$$\binom{n}{1} = \binom{n}{n-1} \qquad \binom{n}{2} = \binom{n}{n-2} \qquad \binom{n}{3} = \binom{n}{n-3}$$

- **Corollary**: Let n, k be nonnegative integers with k $\leq$ n, then

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinations: Example A

- In the Powerball lottery, you pick
  - five numbers between 1 and 55 and
  - A single 'powerball' number between 1 and 42

  How many possible plays are there?
- Here order does not matter
  - The number of ways of choosing 5 numbers is $\binom{55}{5}$
  - There are 42 possible ways to choose the powerball
  - The two events are not mutually exclusive: $42\binom{55}{5}$
  - The odds of winning are $\frac{1}{42\binom{55}{5}} < 0.000000006845$

Combinations: Example B

- In a sequence of 10 coin tosses, how many ways can 3 heads and 7 tails come up?
  - The number of ways of choosing 3 heads out of 10 coin tosses is $\binom{10}{3}$
  - It is the same as choosing 7 tails out of 10 coin tosses $\binom{10}{7} = \binom{10}{3} = 120$
  - … which illustrates the corollary $\binom{n}{k} = \binom{n}{n-k}$

Combinations: Example C

- How many committees of 5 people can be chosen from 20 men and 12 women
  - If exactly 3men must be on each committee?
  - If at least 4 women must be on each committee?
- *If exactly three men must be on each committee?*
  - We must choose 3 men and 2 women. The choices are <u>not</u> mutually exclusive, we use the product rule

$$\binom{20}{3} \cdot \binom{12}{2}$$

- *If at least 4 women must be on each committee?*
  - We consider 2 cases: 4 women are chosen and 5 women are chosen. Theses choices are mutually exclusive, we use the addition rule:

$$\binom{20}{1} \cdot \binom{12}{4} + \binom{20}{0} \cdot \binom{12}{5} = 10,692$$

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Generalized Combinations & Permutations (1)

- Sometimes, we are interested in permutations and combinations in which repetitions are allowed
- **Theorem**: The number of r-permutations of a set of n objects with repetition allowed is n^r

…which is easily obtained by the product rule

- **Theorem**: There are

$$\binom{n+r-1}{r}$$

r-combinations from a set with n elements when repetition of elements is allowed

Generalized Combinations & Permutations: Example

- There are 30 varieties of donuts from which we wish to buy a dozen. How many possible ways to place your order are there?
- Here, n=30 and we wish to choose r=12.
- Order does not matter and repetitions are possible
- We apply the previous theorem
- The number of possible orders is

$$\binom{n+r-1}{r} = \binom{30+12-1}{12} = \binom{17}{12}$$

Generalized Combinations & Permutations (2)

- **Theorem:** The number of different permutations of n objects where there are n_1 indistinguishable objects of type 1, n_2 of type 2, and n_k of type k is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

An equivalent ways of interpreting this theorem is the number of ways to

  - distribute n distinguishable objects
  - into k distinguishable boxes
  - so that n_i objects are place into box i for i=1,2,3,…,k

Example

- How many permutations of the word Mississipi are there?
- 'Mississipi' has
  - 4 distinct letters: m,i,s,p
  - with 1,4,4,2 occurrences respectively
  - Therefore, the number of permutations is

$$\frac{11!}{1!4!4!2!}$$

Distinguishable objects and distinguishable boxes

- How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards? C(52,5), C(47,5), C(42,5), C(37,5), C(32,5).
- The answer is ______52!______
- $(5! \times 5! \times 5! \times 5! \times 32!)$
-

Indistinguishable objects and distinguishable boxes

- How many ways are there to place 10 distinguishable balls into 8 distinguishable bins.
- There are C(n+r-1,n-1) ways to place r indistinguishable objects into n distinguishable boxes.
- C(8+10-1,7) = C(17,7) = C(17,10)

Distinguishable objects and indistinguishable boxes.

- How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees.

- How many ways are there to 4 employees into 3 indistinguishable offices.
- 14

Indistinguishable Objects and Indistinguishable Boxes

- How many ways are there to pack six copies of the same book into 4 identical boxes, where a box can contain as many as six books?
- 6
- 5,1
- 4,2
- 4,1,1
- 3,3

3,2,1

3,1,1,1

2,2,2

2,2,1,1