

Solving Linear Recurrence Relations

Review

A recursive definition of a sequence specifies

- Initial conditions
- Recurrence relation

Example:

$$a_0=0 \text{ and } a_1=3$$

Initial conditions

$$a_n = 2a_{n-1} - a_{n-2}$$

Recurrence relation

$$a_n = 3n$$

Solution

Linear recurrences

Linear recurrence:

Each term of a sequence is a linear function of earlier terms in the sequence.

For example:

$$a_0 = 1 \qquad a_1 = 6 \qquad a_2 = 10$$

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$$

$$\begin{aligned} a_3 &= a_0 + 2a_1 + 3a_2 \\ &= 1 + 2(6) + 3(10) = 43 \end{aligned}$$

Linear recurrences

Linear recurrences

1. Linear homogeneous recurrences
2. Linear non-homogeneous recurrences

Linear homogeneous recurrences

A **linear homogenous recurrence relation of degree k** with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

a_n is expressed in terms of the previous k terms of the sequence, so its degree is k .

This recurrence includes k initial conditions.

$$a_0 = C_0 \qquad a_1 = C_1 \qquad \dots \qquad a_k = C_k$$

Example

Determine if the following recurrence relations are linear homogeneous recurrence relations with constant coefficients.

- ☐ $P_n = (1.11)P_{n-1}$
a linear homogeneous recurrence relation of degree one
- ☐ $a_n = a_{n-1} + a_{n-2}^2$
not linear
- ☐ $f_n = f_{n-1} + f_{n-2}$
a linear homogeneous recurrence relation of degree two
- ☐ $H_n = 2H_{n-1} + 1$
not homogeneous
- ☐ $a_n = a_{n-6}$
a linear homogeneous recurrence relation of degree six
- ☐ $B_n = nB_{n-1}$
does not have constant coefficient

Solving linear homogeneous recurrences

Proposition 1:

- Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence a_n satisfies the recurrence.
- Assume the sequence a'_n also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence.
(α is any constant)

Proof:

$$\begin{aligned} b_n &= a_n + a'_n \\ &= (c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}) + (c_1 a'_{n-1} + c_2 a'_{n-2} + \dots + c_k a'_{n-k}) \\ &= c_1 (a_{n-1} + a'_{n-1}) + c_2 (a_{n-2} + a'_{n-2}) + \dots + c_k (a_{n-k} + a'_{n-k}) \\ &= c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} \end{aligned}$$

So, b_n is a solution of the recurrence.

Solving linear homogeneous recurrences

Proposition 1:

- Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence a_n satisfies the recurrence.
- Assume the sequence a'_n also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence.
(α is any constant)

Proof:

$$\begin{aligned} d_n &= \alpha a_n \\ &= \alpha (c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}) \\ &= c_1 (\alpha a_{n-1}) + c_2 (\alpha a_{n-2}) + \dots + c_k (\alpha a_{n-k}) \\ &= c_1 d_{n-1} + c_2 d_{n-2} + \dots + c_k d_{n-k} \end{aligned}$$

So, d_n is a solution of the recurrence.

Solving linear homogeneous recurrences

It follows from the previous proposition, if we find some solutions to a linear homogeneous recurrence, then **any linear combination** of them will also be a solution to the linear homogeneous recurrence.

Solving linear homogeneous recurrences

Geometric sequences come up a lot when solving linear homogeneous recurrences.

So, try to find any solution of the form $a^n = r^n$ that satisfies the recurrence relation.

Solving linear homogeneous recurrences

- Recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

- Try to find a solution of form r^n

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \dots - c_k r^{n-k} = 0$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0 \quad (\text{dividing both sides by } r^{n-k})$$

This equation is called the **characteristic equation**.

Example

Example:

The Fibonacci recurrence is

$$F_n = F_{n-1} + F_{n-2}$$

Its characteristic equation is

$$r^2 - r - 1 = 0$$

Solving linear homogeneous recurrences

Proposition 2:

r is a solution of $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ if and only if r^n is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$.

Example:

consider the characteristic equation $r^2 - 4r + 4 = 0$.

$$r^2 - 4r + 4 = (r - 2)^2 = 0$$

So, $r=2$.

So, 2^n satisfies the recurrence $F_n = 4F_{n-1} - 4F_{n-2}$.

$$2^n = 4 \cdot 2^{n-1} - 4 \cdot 2^{n-2}$$

$$2^{n-2} (4 - 8 + 4) = 0$$

Solving linear homogeneous recurrences

Theorem 1:

- Consider the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ and the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$.
- Assume r_1, r_2, \dots and r_m all satisfy the equation.
- Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be any constants.
- So, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_m r_m^n$ satisfies the recurrence.

Proof:

By Proposition 2, $\forall i$ r_i^n satisfies the recurrence.

So, by Proposition 1, $\forall i$ $\alpha_i r_i^n$ satisfies the recurrence.

Applying Proposition 1 again, the sequence $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_m r_m^n$ satisfies the recurrence.

Example

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0=2$ and $a_1=7$?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2) = 0 \quad r_1 = 2 \text{ and } r_2 = -1$$

- So, by theorem $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ is a solution.
- Now we should find α_1 and α_2 using initial conditions.

$$a_0 = \alpha_1 + \alpha_2 = 2$$

$$a_1 = \alpha_1 2 + \alpha_2 (-1) = 7$$

- So, $\alpha_1 = 3$ and $\alpha_2 = -1$.
- $a_n = 3 \cdot 2^n - (-1)^n$ is a solution.

Example

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0=0$ and $f_1=1$?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

$$r^2 - r - 1 = 0$$

$$r_1 = (1+\sqrt{5})/2 \text{ and } r_2 = (1-\sqrt{5})/2$$

- So, by theorem $f_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$ is a solution.
- Now we should find α_1 and α_2 using initial conditions.

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1(1+\sqrt{5})/2 + \alpha_2(1-\sqrt{5})/2 = 1$$

- So, $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$.
- $a_n = 1/\sqrt{5} \cdot ((1+\sqrt{5})/2)^n - 1/\sqrt{5}((1-\sqrt{5})/2)^n$ is a solution.

Example

What is the solution of the recurrence relation

$$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$$

with $a_0=8$, $a_1=6$ and $a_2=26$?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

$$r^3 + r^2 - 4r - 4 = 0$$

$$(r+1)(r+2)(r-2) = 0$$

$$r_1 = -1, r_2 = -2 \text{ and } r_3 = 2$$

- So, by theorem $a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3 2^n$ is a solution.
- Now we should find α_1 , α_2 and α_3 using initial conditions.

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$$

$$a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$$

$$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$$

- So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.
- $a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n$ is a solution.

Solving linear homogeneous recurrences

If the characteristic equation has k distinct solutions r_1, r_2, \dots, r_k , it can be written as

$$(r - r_1)(r - r_2) \dots (r - r_k) = 0.$$

If, after factoring, the equation has $m+1$ factors of $(r - r_1)$, for example, r_1 is called a solution of the characteristic equation with multiplicity $m+1$.

When this happens, not only r_1^n is a solution, but also $nr_1^n, n^2r_1^n, \dots$ and $n^m r_1^n$ are solutions of the recurrence.

Solving linear homogeneous recurrences

Proposition 3:

- Assume r_0 is a solution of the characteristic equation with multiplicity at least $m+1$.
- So, $n^m r_0^n$ is a solution to the recurrence.

Solving linear homogeneous recurrences

When a characteristic equation has fewer than k distinct solutions:

- We obtain sequences of the form described in Proposition 3.
- By Proposition 1, we know any combination of these solutions is also a solution to the recurrence.
- We can find those that satisfies the initial conditions.

Solving linear homogeneous recurrences

Theorem 2:

- Consider the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ and the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$.
- Assume the characteristic equation has $t \leq k$ distinct solutions.
- Let $\forall i (1 \leq i \leq t)$ r_i with multiplicity m_i be a solution of the equation.
- Let $\forall i, j (1 \leq i \leq t \text{ and } 0 \leq j \leq m_i - 1)$ α_{ij} be a constant.
- So, $a_n = (\alpha_{10} + \alpha_{11} n + \dots + \alpha_{1, m_1 - 1} n^{m_1 - 1}) r_1^n$
+ $(\alpha_{20} + \alpha_{21} n + \dots + \alpha_{2, m_2 - 1} n^{m_2 - 1}) r_2^n$
+ ...
+ $(\alpha_{t0} + \alpha_{t1} n + \dots + \alpha_{t, m_t - 1} n^{m_t - 1}) r_t^n$
satisfies the recurrence.

Example

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0=1$ and $a_1=6$?

Solution:

- First find its characteristic equation

$$r^2 - 6r + 9 = 0$$

$$(r - 3)^2 = 0 \quad r_1 = 3 \quad (\text{Its multiplicity is 2.})$$

- So, by theorem $a_n = (\alpha_{10} + \alpha_{11}n)(3)^n$ is a solution.

- Now we should find constants using initial conditions.

$$a_0 = \alpha_{10} = 1$$

$$a_1 = 3\alpha_{10} + 3\alpha_{11} = 6$$

- So, $\alpha_{11} = 1$ and $\alpha_{10} = 1$.

- $a_n = 3^n + n3^n$ is a solution.

Example

What is the solution of the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with $a_0=1$, $a_1=-2$ and $a_2=-1$?

Solution:

- Find its characteristic equation

$$r^3 + 3r^2 + 3r + 1 = 0$$

$$(r + 1)^3 = 0 \quad r_1 = -1 \quad (\text{Its multiplicity is 3.})$$

- So, by theorem $a_n = (\alpha_{10} + \alpha_{11}n + \alpha_{12}n^2)(-1)^n$ is a solution.
- Now we should find constants using initial conditions.

$$a_0 = \alpha_{10} = 1$$

$$a_1 = -\alpha_{10} - \alpha_{11} - \alpha_{12} = -2$$

$$a_2 = \alpha_{10} + 2\alpha_{11} + 4\alpha_{12} = -1$$

- So, $\alpha_{10}=1$, $\alpha_{11}=3$ and $\alpha_{12}=-2$.
- $a_n = (1 + 3n - 2n^2)(-1)^n$ is a solution.

Example

What is the solution of the recurrence relation

$a_n = 8a_{n-2} - 16a_{n-4}$, for $n \geq 4$,
with $a_0=1$, $a_1=4$, $a_2=28$ and $a_3=32$?

Solution:

- Find its characteristic equation

$$r^4 - 8r^2 + 16 = 0$$

$$(r^2 - 4)^2 = (r-2)^2 (r+2)^2 = 0$$

$$r_1 = 2 \quad r_2 = -2 \quad (\text{Their multiplicities are 2.})$$

- So, by theorem $a_n = (\alpha_{10} + \alpha_{11}n)(2)^n + (\alpha_{20} + \alpha_{21}n)(-2)^n$ is a solution.
- Now we should find constants using initial conditions.

$$a_0 = \alpha_{10} + \alpha_{20} = 1$$

$$a_1 = 2\alpha_{10} + 2\alpha_{11} - 2\alpha_{20} - 2\alpha_{21} = 4$$

$$a_2 = 4\alpha_{10} + 8\alpha_{11} + 4\alpha_{20} + 8\alpha_{21} = 28$$

$$a_3 = 8\alpha_{10} + 24\alpha_{11} - 8\alpha_{20} - 24\alpha_{21} = 32$$

- So, $\alpha_{10} = 1$, $\alpha_{11} = 2$, $\alpha_{20} = 0$ and $\alpha_{21} = 1$.

- $a_n = (1 + 2n) 2^n + n (-2)^n$ is a solution.

Linear non-homogeneous recurrences

A **linear non-homogenous recurrence relation** with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n),$$

where c_1, c_2, \dots, c_k are real numbers, and $f(n)$ is a function depending only on n .

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

is called the **associated homogeneous recurrence relation**.

This recurrence includes k initial conditions.

$$a_0 = C_0 \quad a_1 = C_1 \quad \dots \quad a_k = C_k$$

Example

The following recurrence relations are linear non-homogeneous recurrence relations.

□ $a_n = a_{n-1} + 2^n$

□ $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

□ $a_n = a_{n-1} + a_{n-4} + n!$

□ $a_n = a_{n-6} + n2^n$

Linear non-homogeneous recurrences

Proposition 4:

- Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$ be a linear non-homogeneous recurrence.
- Assume the sequence b_n satisfies the recurrence.
- Another sequence a_n satisfies the non-homogeneous recurrence if and only if $h_n = a_n - b_n$ is also a sequence that satisfies the associated homogeneous recurrence.

Linear non-homogeneous recurrences

Proof:

Part1: if h_n satisfies the associated homogeneous recurrence then a_n is satisfies the non-homogeneous recurrence.

$$\blacksquare \quad b_n = c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} + f(n)$$

$$\blacksquare \quad h_n = c_1 h_{n-1} + c_2 h_{n-2} + \dots + c_k h_{n-k}$$

$$\begin{aligned} b_n + h_n \\ = c_1 (b_{n-1} + h_{n-1}) + c_2 (b_{n-2} + h_{n-2}) + \dots + c_k (b_{n-k} + h_{n-k}) + f(n) \end{aligned}$$

Since $a_n = b_n + h_n$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$.

So, a_n is a solution of the non-homogeneous recurrence.

Linear non-homogeneous recurrences

Proof:

Part2: if a_n satisfies the non-homogeneous recurrence then h_n is satisfies the associated homogeneous recurrence.

$$\blacksquare \quad b_n = c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} + f(n)$$

$$\blacksquare \quad a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$$

$$\begin{aligned} a_n - b_n \\ = c_1 (a_{n-1} - b_{n-1}) + c_2 (a_{n-2} - b_{n-2}) + \dots + c_k (a_{n-k} - b_{n-k}) \end{aligned}$$

Since $h_n = a_n - b_n$, $h_n = c_1 h_{n-1} + c_2 h_{n-2} + \dots + c_k h_{n-k}$

So, h_n is a solution of the associated homogeneous recurrence.

Linear non-homogeneous recurrences

Proposition 4:

- Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$ be a linear non-homogeneous recurrence.
- Assume the sequence b_n satisfies the recurrence.
- Another sequence a_n satisfies the non-homogeneous recurrence if and only if $h_n = a_n - b_n$ is also a sequence that satisfies the associated homogeneous recurrence.

- We already know how to find h_n .
- For many common $f(n)$, a solution b_n to the non-homogeneous recurrence is similar to $f(n)$.
- Then you should find solution $a_n = b_n + h_n$ to the non-homogeneous recurrence that satisfies both recurrence and initial conditions.

Example

What is the solution of the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + 3n + 1 \text{ for } n \geq 2,$$

with $a_0=2$ and $a_1=3$?

Solution:

- Since it is linear non-homogeneous recurrence, b_n is similar to $f(n)$

Guess: $b_n = cn + d$

$$b_n = b_{n-1} + b_{n-2} + 3n + 1$$

$$cn + d = c(n-1) + d + c(n-2) + d + 3n + 1$$

$$cn + d = cn - c + d + cn - 2c + d + 3n + 1$$

$$0 = (3+c)n + (d-3c+1)$$

$$c = -3 \quad d = -10$$

- So, $b_n = -3n - 10$.

(b_n only satisfies the recurrence, it does not satisfy the initial conditions.)

Example

What is the solution of the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + 3n + 1 \text{ for } n \geq 2,$$

with $a_0=2$ and $a_1=3$?

Solution:

□ We are looking for a_n that satisfies both recurrence and initial conditions.

□ $a_n = b_n + h_n$ where h_n is a solution for the associated homogeneous recurrence: $h_n = h_{n-1} + h_{n-2}$

□ By previous example, we know $h_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$.

$$a_n = b_n + h_n$$

$$= -3n - 10 + \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$$

□ Now we should find constants using initial conditions.

$$a_0 = -10 + \alpha_1 + \alpha_2 = 2$$

$$a_1 = -13 + \alpha_1(1+\sqrt{5})/2 + \alpha_2(1-\sqrt{5})/2 = 3$$

$$\alpha_1 = 6 + 2\sqrt{5} \qquad \alpha_2 = 6 - 2\sqrt{5}$$

So, $a_n = -3n - 10 + (6 + 2\sqrt{5})((1+\sqrt{5})/2)^n + (6 - 2\sqrt{5})((1-\sqrt{5})/2)^n$.

Example

What is the solution of the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} + 2^n \text{ for } n \geq 2,$$

with $a_0=1$ and $a_1=2$?

Solution:

- Since it is linear non-homogeneous recurrence, b_n is similar to $f(n)$

Guess: $b_n = c2^n + d$

$$b_n = 2b_{n-1} - b_{n-2} + 2^n$$

$$c2^n + d = 2(c2^{n-1} + d) - (c2^{n-2} + d) + 2^n$$

$$c2^n + d = c2^n + 2d - c2^{n-2} - d + 2^n$$

$$0 = (-4c + 4c - c + 4)2^{n-2} + (-d + 2d - d)$$

$$c = 4 \quad d = 0$$

- So, $b_n = 4 \cdot 2^n$.

(b_n only satisfies the recurrence, it does not satisfy the initial conditions.)

Example

What is the solution of the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} + 2^n \text{ for } n \geq 2,$$

with $a_0=1$ and $a_1=2$?

Solution:

- We are looking for a_n that satisfies both recurrence and initial conditions.
- $a_n = b_n + h_n$ where h_n is a solution for the associated homogeneous recurrence: $h_n = 2h_{n-1} - h_{n-2}$.
 - Find its characteristic equation
$$r^2 - 2r + 1 = 0$$
$$(r - 1)^2 = 0$$
$$r_1 = 1 \quad (\text{Its multiplicity is 2.})$$
- So, by theorem $h_n = (\alpha_1 + \alpha_2 n)(1)^n = \alpha_1 + \alpha_2 n$ is a solution.

Example

What is the solution of the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} + 2^n \text{ for } n \geq 2,$$

with $a_0=1$ and $a_1=2$?

Solution:

- $a_n = b_n + h_n$
- $a_n = 4 \cdot 2^n + \alpha_1 + \alpha_2 n$ is a solution.
- Now we should find constants using initial conditions.

$$a_0 = 4 + \alpha_1 = 1$$

$$a_1 = 8 - \alpha_1 + \alpha_2 = 2$$

$$\alpha_1 = -3 \qquad \alpha_2 = -3$$

So, $a_n = 4 \cdot 2^n - 3n - 3$.