Applications of Partial Differential Equations

'Laplace's Equation' or 'Potential Equation' or 'Two-Dimensional Steady-State Heat Flow' →

The two-dimensional heat equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

reduces to Laplace's equation given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (1)}$$

when the heat-flow is in the steady-state (i.e. $\frac{\partial u}{\partial t} = 0$).

The solution of Laplace's equation (1), $u(x,y)$, in a rectangular region can be obtained by the separation of variable technique in both type of problems —

(i) Dirichlet Problem — where $u(x,y)$ is prescribed on the boundary

(ii) Neumann Problem — where derivative of $u(x,y)$ in the normal direction to the boundary is prescribed

Solved Examples on Laplace's Equation —

Type I → When $u(x,y)$ is given as function of $f(y)$ on a side of rectangle.

Ex. 1 Solve the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in a rectangle in the x-y plane, $0 < x < a$ and $0 < y < b$ satisfying the following boundary conditions.

$$u(x,0) = 0 \quad , \quad u(x,b) = 0$$
$$u(0,y) = 0 \quad \& \quad u(a,y) = f(y).$$

Sol^n -

Consider

$$u(x,y) = X(x).\,Y(y) \quad \text{——— (1)}$$

substituting (1) in Laplace' eqn, we get

$$X''Y + X\ddot{Y} = 0$$

where $'$ denotes diff. w.r.to x & $\cdot$ diff. w.r. to y. *

So, $\boxed{-\dfrac{\ddot{Y}}{Y} = +\dfrac{X''}{X} = k \;(\text{Let})}$

which gives

$$\ddot{Y} - kY = 0 \quad \text{——— (2)}$$
$$\& \; X'' + kX = 0 \quad \text{——— (3)}$$

The boundary conditions are

$$u(x,0) = 0 \quad \text{——— (4)}$$
$$u(x,b) = 0 \quad \text{——— (5)}$$
$$u(0,y) = 0 \quad \text{——— (6)}$$
$$u(a,y) = f(y) \quad \text{——— (7)}$$

* For type I take '−' sign with Y terms.

Using (1) in (4)

$$0 = u(x,0) = X(x).Y(0) \Rightarrow Y(0) = 0 \quad \text{—— (8)}$$

Using (1) in (5)

$$0 = u(x,b) = X(x).Y(b) \Rightarrow Y(b) = 0 \quad \text{—— (9)}$$

Using (1) in (6)

$$0 = u(0,y) = X(0).Y(y) \Rightarrow X(0) = 0 \quad \text{—— (10)}$$

If $K \geq 0$, so (2) will have only trivial solutions. so let $K = -\lambda^2 < 0^*$, so (2) becomes

$$\ddot{Y} + \lambda^2 Y = 0 \quad \text{—— (11)}$$

The general solution of (11) will be

$$Y(y) = A\cos\lambda y + B\sin\lambda y$$

Using (8) & (9) it gives $A = 0$

and $\sin\lambda b = 0 \Rightarrow \lambda b = n\pi$

Such that

$$\lambda_n = \frac{n\pi}{b}, \text{ for } n = 1,2,3,\ldots$$

Thus we get infinitely many solutions.

$$Y_n(y) = B\sin\frac{n\pi y}{b} \quad \text{—— (12)}$$

Now the solution of (3) is

$$X(x) = Ce^{-\lambda x} + De^{\lambda x}$$

Using (6), we get $C = -D$

So $X(x) = 2D\,\dfrac{e^{\lambda x} - e^{-\lambda x}}{2} = E\sinh\lambda x$

$$\Rightarrow X_n(x) = E_n\sinh\left(\frac{n\pi x}{b}\right) \quad \text{—— (13)}$$

Imp * For Type I & Type II this condn will remain same

From (12) & (13) the solution of (1) will be

$$u_n(x,y) = B.E_n \sinh\left(\frac{n\pi x}{b}\right).\sin\left(\frac{n\pi y}{b}\right)$$

or $$u_n(x,y) = B_n \sinh\left(\frac{n\pi x}{b}\right).\sin\left(\frac{n\pi y}{b}\right)$$

We have infinitely many solutions for $n=1,2,3,\ldots$
So the complete solution will be sum of all these solutions, such that

$$u(x,y) = \sum u_n(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi x}{b}\right).\sin\left(\frac{n\pi y}{b}\right) \quad \text{--- (14)}$$

Now to find B_n, considering boundary condition (7), we get

$$f(y) = \sum B_n \sinh\left(\frac{n\pi a}{b}\right).\sin\left(\frac{n\pi y}{b}\right)$$

Here B_n's are Fourier coefficients of the half range series of $f(y)$ in $(0,b)$, such that

$$B_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y).\sin\left(\frac{n\pi y}{b}\right) dy$$

$$\Rightarrow B_n = \frac{2}{b.\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y).\sin\left(\frac{n\pi y}{b}\right) dy$$

$n=1,2,3,\ldots$

--- (15)

The eqn (14) & (15) represents the required solution.

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Type II ⇒ When $u(x,y)$ is given as function of $f(x)$ on a side of rectangle.

Ex. 2 - Solve Laplace's equation in rectangle with
$u(0,y)=0$, $u(a,y)=0$, $u(x,b)=0$ and
$u(x,0)=f(x)$, $0<x<a$ and $0<y<b$.

Soln - The Laplace equation is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{——— (1)}$$

The boundary conditions are -

$$u(0,y)=0 \quad \text{——— (2)}$$
$$u(a,y)=0 \quad \text{——— (3)}$$
$$u(x,b)=0 \quad \text{——— (4)}$$
$$\& \quad u(x,0)=f(x) \quad \text{——— (5)}$$

Let $u(x,y)=X(x)\cdot Y(y)$ ——— (6)

substituting (6) in (1), we get

$$X''Y+X\dot{Y}=0$$

$$\Rightarrow \boxed{\frac{\ddot{Y}}{Y}=-\frac{X''}{X}=k \text{ (Let)}}$$ *

which gives

$$\ddot{Y}+kY=0 \quad \text{——— (7)}$$
$$\& \quad X''-kX=0 \quad \text{——— (8)}$$

Using (6) (in) (2), (3) & (4) respectively, we get

$$0=u(0,y)=X(0).Y(y) \Rightarrow X(0)=0 \quad \text{——— (9)}$$
$$0=u(a,y)=X(a).Y(y) \Rightarrow X(a)=0 \quad \text{——— (10)}$$
$$0=u(x,b)=X(x).Y(b) \Rightarrow Y(b)=0 \quad \text{——— (11)}$$

* For Type take '−' sign with X terms.

If $K \geq 0$, so (7) will have only trivial solutions, so let $K = -\lambda^2 < 0$, so (7) becomes

$$\ddot{Y} - \lambda^2 Y = 0$$

It's general soln is

$$Y(y) = Ae^{\lambda y} + Be^{-\lambda y} \quad \text{——— (12)}$$

Using (11), it gives

$$0 = Ae^{\lambda b} + Be^{-\lambda b}$$

$$\Rightarrow A = \frac{-Be^{-\lambda b}}{e^{\lambda b}}$$

So (12) becomes

$$Y(y) = -\frac{Be^{-\lambda b}}{e^{\lambda b}} \cdot e^{\lambda y} + Be^{-\lambda y}$$

$$= -\frac{B}{e^{\lambda b}}\left[e^{-\lambda b}e^{\lambda y} - e^{\lambda b}e^{-\lambda y}\right]$$

$$= \frac{2B}{e^{\lambda b}}\left[\frac{e^{\lambda(b-y)} - e^{-\lambda(b-y)}}{2}\right]$$

$$\Rightarrow Y_n(y) = M \sinh\{\lambda(b-y)\} \quad \text{——— (13)}$$

Now the general solution of (8) is

$$X(x) = C\cos\lambda x + D\sin\lambda x \quad \text{——— (14)}$$

Using (9), (14) gives

$$C = 0$$

& Using (10) in (14) with $C=0$, it gives

$$\sin\lambda a = 0 \Rightarrow \lambda a = n\pi$$

Such that

$$\lambda_n = \frac{n\pi}{a}, \text{ for } n = 1, 2, 3 \ldots$$

Thus we get infinitely many solutions

$$X_n(x) = D_n \sin\frac{n\pi x}{a} \quad \text{——— (15)}$$

From (13) & (15) the solution of (1) will be

$$u_n(x,y) = MD_n \sinh\left\{\frac{n\pi}{a}(b-y)\right\} \sin\frac{n\pi x}{a}$$

or $$u_n(x,y) = B_n \sinh\left\{\frac{n\pi}{a}(b-y)\right\} \sin\frac{n\pi x}{a}$$

We have infinitely many solutions for $n=1,2,3,\ldots$ so the complete solution will be sum of all these solutions, such that

$$u(x,y) = \sum u_n(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left\{\frac{n\pi(b-y)}{a}\right\} \sin\frac{n\pi x}{a} \quad \text{—— (16)}$$

Now to find B_n considering the boundary condition (5), we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a} \sin\frac{n\pi x}{a}$$

Here B_n's are coffs of the half Range Fourier series of $f(x)$ in $(0,a)$, such that

$$B_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$

$$\Rightarrow \quad B_n = \frac{2}{a \sinh\frac{n\pi b}{a}} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$n = 1,2,3,\ldots$ —— (17)

The eqn (16) with (17) represents the required solution.

Ex 3. Find the steady-state temperature in a rectangular plate $0<x<a,\ 0<y<b$ satisfying the following boundary conditions

$$\frac{\partial u}{\partial x}=0 \text{ for } x=0,\quad \frac{\partial u}{\partial x}=0 \text{ for } x=a$$

$$\frac{\partial u}{\partial y}=0 \text{ for } y=b \text{ and } u(x,0)=K\cos\left(\frac{\pi x}{a}\right)$$

Solⁿ - Consider the Laplace's equation in the steady state

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{——— (1)}$$

The solution $u(x,y)$ of (1) satisfies the following condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0, \quad \text{——— (2)}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \quad \text{——— (3)}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=b}=0 \quad \text{——— (4)}$$

$$u(x,0)=K\cos\left(\frac{\pi x}{a}\right) \quad \text{——— (5)}$$

Let $u(x,y)=X(x).Y(y)$ ——— (6)

Using (2) in (6)

$$X'(0)=0 \quad \text{——— (7)}$$

Using (3) in (6)

$$X'(a)=0 \quad \text{——— (8)}$$

Using (4) in (6)

$$\dot{Y}(b)=0 \quad \text{——— (9)}$$

substituting (6) in (1), we get

$$X''Y + X\dot{Y} = 0$$

$$\Rightarrow \frac{\ddot{Y}}{Y} = -\frac{X''}{X} = K \text{ (Let)} \qquad \text{(Type II Case)}$$

which gives

$$\ddot{Y} + KY = 0 \quad \text{——— (10)}$$

$$\& \; X'' - KX = 0 \quad \text{——— (11)}$$

If $K \geq 0$, then (10) will have only trivial sol^{ns}

so let $K = -\lambda^2 < 0$, so (10) becomes

$$\ddot{Y} - \lambda^2 Y = 0$$

It's general sol^n is

$$Y(y) = A e^{\lambda y} + B e^{-\lambda y} \quad \text{——— (12)}$$

$$\Rightarrow \dot{Y}(y) = A\lambda e^{\lambda y} + B(-\lambda) e^{-\lambda y}$$

Using (9) in it, it gives

$$0 = \lambda (A e^{\lambda b} - B e^{-\lambda b})$$

$$\Rightarrow A = \frac{B e^{-\lambda b}}{e^{\lambda b}}$$

So (12) gives

$$Y(y) = \frac{B e^{-\lambda b} e^{\lambda y}}{e^{\lambda b}} + B e^{-\lambda y}$$

$$= \frac{2B}{e^{\lambda b}} \left(\frac{e^{-\lambda(b-y)} + e^{\lambda(b-y)}}{2} \right)$$

$$\Rightarrow Y(y) = E \cosh\{\lambda(b-y)\} \quad \text{——— (13)}$$

Now the general soln of (11) is

$$X(x) = C \cos \lambda x + D \sin \lambda x \quad \text{——— (14)}$$

Diff it w.r.t. to x

$$X'(x) = -C\lambda \sin \lambda x + D\lambda \cos \lambda x$$

Using (7) and (8) in it, we get

$$D = 0 \quad \& \quad \sin \lambda a = 0 \Rightarrow \lambda a = n\pi$$

such that

$$\lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, 3 \ldots$$

Thus we get infintely many solutions

$$X_n(x) = C_n \cos \frac{n\pi x}{a} \qquad \text{———} (15)$$

From (13) & (15), the solution of (1) will be

$$u_n(x,y) = EC_n \cosh\left\{\frac{n\pi}{a}(b-y)\right\} \cos \frac{n\pi x}{a}$$

Or $u_n(x,y) = B_n \cosh\left\{\frac{n\pi}{a}(b-y)\right\} \cos \frac{n\pi x}{a}$

We have infintely many solutions for $n = 1, 2, 3, \ldots$ so the complete solution will be sum of all these solutions, such that

$$u(x,y) = \sum u_n(x,y) = \sum_{n=1}^{\infty} B_n \cosh\left\{\frac{n\pi}{a}(b-y)\right\} \cos \frac{n\pi x}{a} \qquad \text{———} (16)$$

Now to find B_n considering the boundary condition (5), we get

For n

$$B_n = \frac{2}{a \cosh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \cos \frac{n\pi x}{a}\, dx$$

From (5) for n = 1

$$B_1 = \frac{2}{a \cosh\left(\frac{\pi b}{a}\right)} \int_0^a K . \cos \frac{\pi x}{a} . \cos \frac{n\pi x}{a}\, dx$$

$$\Rightarrow B_1 = \frac{\cancel{2}}{\cancel{a}\cosh\left(\frac{\pi b}{a}\right)} k \cdot \frac{\cancel{\pi}}{\cancel{2}} \cdot \frac{\cancel{a}}{\cancel{\pi}}$$

$$= k \operatorname{sech}\left(\frac{\pi b}{a}\right)$$

For $n \geq 2$, $B_n = 0$

So, the required solⁿ is

$$u(x, y) = k \operatorname{sech}\left(\frac{\pi b}{a}\right) \cosh\left\{\frac{\pi}{a}(b-y)\right\} \cos\frac{\pi x}{a}$$

[As $n = 1$ only].

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