15B11MA211
Tutorial Sheet 6

Department of Mathematics

Mathematics-2
B.Tech. Core

Analytic Functions and Complex Integration

1. Evaluate the following limits: (a) $\lim_{z\to -i} \dfrac{z^2+1}{z+i}$ (b) $\lim_{z\to \frac{1+i\sqrt{3}}{2}} \dfrac{z^3+1}{z^4+z^2+1}$

2. (a) Show that $f(z) = \bar{z}$ is continuous but not differentiable at any point.
   (b) If $f(z) = x^2 + iy^2$, does $f'(z)$ exist at any point?

3. Determine whether C-R equations are satisfied for (a) $1/z$ (b) $\cosh 2z$

4. (a) Show that $f(z) = |z^2|$ is differentiable at $z = 0$ but not analytic there.
   (b) Show that $u(x,y) = 2x + y^3 - 3x^2y$ is a harmonic function. Find its harmonic conjugate and corresponding analytic function $f(z)=u+iv$.

5. Show that for the function $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$ *and* $f(0) = 0$

   C-R equations are satisfied at origin, but function is not analytic at the point.

6. Determine the analytic function $f(z) = u + iv$, where

   (a) $u - v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ (b) $u(r,\theta) = r^2\cos 2\theta$ (c) $v = (x - y)/(x^2 + y^2)$.

7. Integrate $\int_C (z + 2\bar{z})dz$ from $z = 0$ to $z = 1 + i$ along the following two paths

   (a) line joining (0,0) and (1,1) (b) the curve $x = t, y = t^2, 0 \le t \le 1$.

8. Integrate $f(z) = z$ in the positive sense around the squares with corners at (1,1), (2,1), (2,2) and (1,2).

9. Evaluate $\int_C |z| dz$, where C is the contour (a) straight line from $z = -i$ to $z = i$; (b) the unit circle $|z-1| = 1$.

10. Let m be an integer and C the circle $|z - z_0| = R$. Show that the integral of $(z - z_0)^m$ over C in the anticlockwise direction vanishes if $m \neq -1$ and is equal to $2\pi i$ if $m = -1$. Hence evaluate $\int_C [P(z)/z]dz$, where $P(z) = 2 - z + 3z^2 + z^3$ and C is the unit circle $|z| = 1$.

11. Using Cauchy theorem or otherwise show that (a) $\int_C \dfrac{dz}{z-2} = 0$, where C is the circle $|z| = 1$

    (b) $\int_C \dfrac{dz}{z} = 2\pi i$, where C is a closed contour enclosing $z = 0$. (c) $\int_C \dfrac{dz}{(z+1)^2} = 0$, where C is the circle $|z| = 2$.

12. Using the Cauchy integral formula or otherwise show that

    (a) $\int_C \dfrac{e^{-z}}{z+1} dz = 2\pi e i$, where C is the circle $|z| = 2$.

    (b) $\int_C \dfrac{e^{2z}}{(z+1)^4} dz = 8\pi i/(3e^2)$ where C is the circle $|z| = 2$.

    (c) $\int_C \dfrac{\cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i, 0, 4\pi i$ according as C is the circle $|z| = 3/2, 1/2$ or 3.

    (d) $\int_C \dfrac{e^{-z}}{z^2} dz = -2\pi i$, where C is the ellipse $2x^2 + y^2 = 2$.

Tutorial Sheet 6

a) $\lim\limits_{z \to i} \dfrac{z^2+1}{z+i} = \lim\limits_{z \to -i} \dfrac{(z+i)(z-i)}{z+i} = \lim\limits_{z \to -i} z-i = -2i$

(b) $\lim\limits_{z \to \frac{1+i\sqrt{3}}{2}} \dfrac{z^3+1}{z^4+z^2+1} = \lim\limits_{z \to \frac{1+i\sqrt{3}}{2}} \dfrac{(z+1)(z^2-z+1)}{(z^2+z+1)(z^2-z+1)}$

$= \dfrac{\frac{3+\sqrt{3}i}{2}}{1+\sqrt{3}i} \times \dfrac{1-\sqrt{3}i}{1-\sqrt{3}i}$

$= \dfrac{3-3\sqrt{3}i+\sqrt{3}i+3}{2(1+3)}$

$= \dfrac{6-2\sqrt{3}i}{8} = \dfrac{3-\sqrt{3}i}{4}.$

2) a) $f(z) = \bar{z}$

$= x - iy = u + iv.$

$\because$ $u = x$, $v = -y$ are cts

$\therefore$ $f(z)$ is cts on $\mathbb{C}$.

$\lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim\limits_{\Delta z \to 0} \dfrac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim\limits_{\Delta x, \Delta y \to 0} \dfrac{\overline{(x+\Delta x)+(y+\Delta y)i} - (\overline{x+iy})}{\Delta x + \Delta y i}$

$= \lim\limits_{\Delta x, \Delta y \to 0} \dfrac{\cancel{x}+\Delta x - (y+\Delta y)i - \cancel{x} + \cancel{iy}}{\Delta x + \Delta y i}$

$= \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\Delta x - \Delta y i}{\Delta x + \Delta y i}$

choose a path $y = mx$

$\Delta y = m\Delta x$

$= \lim\limits_{\Delta x \to 0} \dfrac{\cancel{\Delta x}(1-mi)}{\cancel{\Delta x}(1+mi)} = \dfrac{1-mi}{1+mi}$

which depends on m.

So limit does not exist.

$\therefore$ $f(z)$ is not differentiable at any point

(b) $f(z) = x^2 + iy^2$ $u_x = 2x$ $v_x = 0$ $\Rightarrow x = y$ $f'(z)$ exists when $x = y$

$u_y = 0$ $v_y = 2y$

$\cancel{\lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{(x+\Delta x)^2 + i(y+\Delta y)^2 - x^2 - iy^2}{(\Delta x) + i(\Delta y)}}$

$\cancel{= \lim \dfrac{(\Delta x)^2 + 2x\Delta x + i((\Delta y)^2 + 2y\Delta y)}{(\Delta x) + i(\Delta y)}}$

choose $y = mx$

$\Delta y = m\Delta x$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x + i(m^2\Delta x^2 + 2(mx)(m\Delta x))}{(\Delta x)^2 + i(m^2 \Delta x^2)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x + 2x + i(m^2 \Delta x + 2m^2 x)}{1 + i m^2 \Delta x}$$

$$= \lim_{\Delta z \to 0} \frac{2x(1+im^2)}{\Delta x(1+im^2)} + \frac{\Delta x(1+im^2)}{1+m^2 i}$$

$$= \infty \quad \cancel{2x} \quad = \cancel{f'(z)} \quad = 2x\frac{(1+im^2)}{1+im} \text{ depends on } m$$

~~lt doesnot exist~~

$\therefore$ ~~Not differentiable~~ lt doesnot exist.

3) a) $f(z) = \frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$ $\qquad \underline{z \neq 0}$

$$= \frac{x-iy}{x^2+y^2} = u + iv.$$

$$u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = \frac{-y}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_x = \frac{+y(2x)}{(x^2+y^2)^2}$$

$$v_y = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = -\frac{x(2y)}{(x^2+y^2)^2}$$

Here $u_x = v_y$

and $u_y = -v_x$.

$\therefore$ CR equations are satisfied for all $z \neq 0$.

b) $f(z) = \cosh 2z = \frac{e^{2z} + e^{-2z}}{2}$

$$= e^{2x+i2y} + e^{-2x-i2y} / 2$$

$$= \frac{e^{2x}(\cos 2y + i\sin 2y) + e^{-2x}(\cos 2y - i\sin 2y)}{2}$$

$$= \cos 2y \cosh 2x + i \sin 2y \sinh 2x.$$

$$= u(x,y) + i\, v(x,y)$$

$u_x = 2\sinh 2x \cos 2y$

$u_y = -2\sin 2y \cosh 2x$

$v_x = 2\cosh 2x \sin 2y$

$v_y = 2\cos 2y \sinh 2x$

$u_x = v_y \quad \& \quad u_y = -v_x$.

$\therefore$ CR eq are satisfied.

4) a) $f(z) = |z^2|$

$= x^2 + y^2$

is differentiable at $(0,0)$

but not analytic anywhere

$z = x + iy$

$z^2 = x^2 - y^2 + i(2xy)$

$|z^2| = \sqrt{(x^2-y^2)^2 + 4x^2y^2}$

$= \sqrt{x^4 + y^4 + 2x^2y^2}$

$= x^2 + y^2$.

since $u(x,y) = x^2 + y^2$

$v(x,y) = 0$.

$u_x = 2x, \quad u_y = 2y$

$v_x = v_y = 0$.

$u_x = v_y \Rightarrow 2x = 0$

$u_y = -v_x \Rightarrow 2y = 0$

$$\lim_{\Delta z \to 0} \frac{|(z+\Delta z)^2| - |z^2|}{\Delta z}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - x^2 - y^2}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x)^2 + (\Delta y)^2 + 2x\Delta x + 2y\Delta y}{\Delta x + i\Delta y}$$

$= \begin{cases} \frac{2y}{i} & \text{If } \Delta x \to 0 \\ 2x & \text{If } \Delta y \to 0 \end{cases}$

$\frac{2y}{i} = 2x \Rightarrow y = ix$

$\Rightarrow x = y = 0$.

diff only for $z = 0$.

b) $u(x,y) = 2x + y^3 - 3x^2y$

$u_x = 2 - 6xy$.

$u_y = 3y^2 - 3x^2$

$u_{xx} = -6y$

$u_{yy} = 6y$.

$u_{xx} + u_{yy} = -6y + 6y = 0$.

$\therefore u$ is harmonic function

From CR equations

$u_x = v_y$.

$v_y = 2 - 6xy$.

$\Rightarrow v(x,y) = 2y - 3xy^2 + g(x)$

Also $u_y = -v_x$

$\Rightarrow 3y^2 - 3x^2 = -(-3y^2) + g'(x)) \Rightarrow g'(x) = 3x^2$

$$v(x,y) = 2y - 3xy^2 + x^3 + c$$

$$f(z) = 2x + y^3 - 3x^2y + i(2y - 3xy^2 + x^3 + c)$$

$$= 2(x+iy) + y^3 - 3x^2y + i(-3xy^2 + x^3) + ic$$

$$= 2(x+iy) + i(x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3) + ic$$

$$= 2(x+iy) + i(x+iy)^3 + ic$$

$$= 2z + iz^3 + ic$$

5) $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$, $f(0) = 0$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right) = u + iv$$

$$u_x = \frac{(x^2+y^2)(3x^2) - (x^3-y^3)(2x)}{(x^2+y^2)^2}$$

$$= \frac{3x^4 + 3x^2y^2 - 2x^4 + 2xy^3}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 2xy^3 + 3x^2y^2}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(-3y^2) - (x^3-y^3)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-3x^2y^2 - 3y^4 - 2x^3y + 2y^4}{(x^2+y^2)^2}$$

$$= -\left[\frac{y^4 + 2x^3y + 3x^2y^2}{(x^2+y^2)^2}\right]$$

$$v_x = \frac{(x^2+y^2)(3x^2) - (x^3+y^3)(2x)}{(x^2+y^2)^2}$$

$$= \frac{3x^4 + 3x^2y^2 - 2x^4 - 2xy^3}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2+y^2)^2}$$

$$v_y = \frac{y^4 + 3x^2y^2 - 2yx^3}{(x^2+y^2)^2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{\Delta y \to 0} \frac{u(0,\Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{-\Delta y}{\Delta y} = -1$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ CR eq are satisfie[d] at origin

Now $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(\Delta z) - f(0)}{\Delta z}$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x^3 - \Delta y^3 + i(\Delta x^3 + \Delta y^3)}{(\Delta x^2 + \Delta y^2)(\Delta x + i\Delta y)}$$

Along $y = mx \Rightarrow \Delta y = m\Delta x$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^3 - m^3\Delta x^3 + i(\Delta x)^3(1+m^3)}{\Delta x^2(1+m^2)\,\Delta x(1+im)}$$

$= \lim$ $\underline{1 - m^3 + i(1+m^3)}$ depende[nt]

a) $u+v = \dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$

$f(z) = u+iv$

$(1+i)f(z) = F(z) = (u-v)+i(u+v)$

$F(z) = U+iV$

$V, \to V_x, V_y$

$= \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x+v_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} \quad -①$$

$$u_y+v_y = \frac{-\sin 2x(2\sinh 2y)}{(\cosh 2y-\cos 2x)^2} \quad -②$$

Now $u_x = v_y$ and $u_y = -v_x$ $\quad -③$

$$\therefore -v_x+u_x = -\frac{\sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2} \quad -④ \quad \text{(using ② \& ③)}$$

using ① & ④,

$$\cancel{2}u_x = \frac{(\cosh 2y - \cos 2x)(\cancel{2}\cos 2x) - \cancel{2}\sin^2 2x - \sin 2x(\cancel{2}\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$u_x = v_y = \frac{\cosh 2y\cos 2x - \cos^2 2x - \sin^2 2x - \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$v_x = \frac{(\cosh 2y - \cos 2x)2\cos 2x - 2\sin^2 2x - \cosh 2y\cos 2x + \cos^2 2x + \sin^2 2x + \sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{\cosh 2y\cos 2x - 1 + \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$f'(z) = u_x + iv_x$

$$= \frac{\cosh 2y\cos 2x - \sin 2x\sinh 2y - 1 + i(\cosh 2y\cos 2x + \sin 2x\sinh 2y - 1)}{(\cosh 2y - \cos 2x)^2}$$

(using Thomson Milne)

$x = z$, $y = 0$

$$f'(z) = \frac{\cos 2z - 1 + i(\cos 2z - 1)}{(1-\cos 2z)^2}$$

$$= \frac{1+i}{\cos 2z - 1}$$

$$= \frac{1+i}{-2\sin^2 z} = \frac{-1}{2}\left(\operatorname{cosec}^2 z + i \operatorname{cosec}^2 z\right)$$

$$f(z) = \frac{\cot z}{2}(1+i) + C$$

OR

$$f(z) = u + iv$$

$$F(z) = (1+i) f(z) = u - v + i(u+v)$$
$$= U + iV$$

$$u+v = V = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$V_x = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\underset{u-v}{\downarrow} \quad V_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Put $x = z$
$y = 0$

$$V_x = \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$V_y = 0 \qquad \because \sinh 0 = 0$$

Now,

$$F'(z) = V_y + V_x i \quad \text{\& Thomson Milne } x = z,\ y = 0$$

$$= i\left(\frac{2\cos 2z - 2}{(1-\cos 2z)^2}\right) = \frac{2i}{\cos 2z - 1} = \frac{-i}{\sin^2 z} = -\operatorname{cosec}^2 z\ i$$

$$F(z) = +\cot z\ i + c$$

$$f(z) = \frac{1}{1+i} F(z) = \frac{1-i}{2} \times i\cot z + c$$

$$= \frac{(1+i)}{2}\cot z + c.$$

$u(r,\theta) = r^2 \cos 2\theta$

$\underline{u_r = \frac{1}{r} v_\theta}$

$u_r = 2r\cos 2\theta$

$= \frac{1}{r} v_\theta$

$\Rightarrow v_\theta = 2r^2 \cos 2\theta$

$v(r,\theta) = r^2 \sin 2\theta + g(r)$

$\underline{u_\theta = -r v_r}$

$u_\theta = -2r^2 \sin 2\theta$

$= -r v_r$

$v_r = 2r \sin 2\theta$

$v(r,\theta) = r^2 \sin 2\theta + h(\theta)$

$\therefore \quad g(r) = h(\theta) = C$

$v(r,\theta) = r^2 \sin 2\theta + C$

$f(z) = r^2 \cos 2\theta + i r^2 \sin 2\theta + C$

$= r^2 e^{i2\theta} + C$

$= z^2 + C$ ✓ $\quad z = re^{i\theta}$

c) $v = \frac{x-y}{x^2+y^2}$

$$v_x = \frac{x^2+y^2-(x-y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2} = \frac{y^2-x^2+2xy}{(x^2+y^2)^2}$$

$$v_y = \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

$$u_x = v_y = \frac{y^2-x^2-2xy}{(x^2+y^2)^2} = u_x$$

$$u_y = -v_x = \frac{x^2-y^2-2xy}{(x^2+y^2)^2} = u_y \xrightarrow{\text{exact}} \text{diff}$$

$$\therefore\ u(x,y) = \frac{x+y}{x^2+y^2}$$

$$f(z) = \frac{x+y+i(x-y)}{x^2+y^2}$$

$$= \frac{x-iy+i(x-iy)}{(x+iy)(x-iy)}$$

$$= \frac{1+i}{z}$$

$$\frac{1+i}{x+iy} \times \frac{x-iy}{x-iy}$$

$$\frac{x+y+i(x-y)}{x^2+y^2}$$

$$u = \frac{x+y}{x^2+y^2}$$

$$u_x = \frac{x^2+y^2-(x+y)(2x)}{(x^2+y^2)^2}$$

$$= y^2-2xy-x^2$$

$$u_y = \frac{x^2+y^2-(x+y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2-2xy}{(x^2+y^2)^2}$$

OR.

$$f'(z) = v_y + v_x i$$

$$= \frac{y^2-x^2-2xy}{(x^2+y^2)^2} + \frac{y^2-x^2+2xy}{(x^2+y^2)^2}\, i$$

Put $x = z$, $y = 0$ (Thomson Milne)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= M\,dx + N\,dy$$

$$= \frac{-z^2}{z^4} + \frac{-z^2}{z^4}\, i$$

$= -\frac{1}{[illegible]}(1+i)$

$\int_C (z+2\bar{z})\,dz$

$= \int_C (x+iy)+(2x-2iy)\,(dx+idy)$

line joining $(0,0)$ & $(1,1)$

$\frac{x-0}{1-0} = \frac{y-0}{1-0} = t$

$\Rightarrow x=t,\ y=t \qquad 0 \le t \le 1$.

$dz = dx + idy$
$= dt + idt$

$= \int_0^1 [t(1+i) + 2t(1-i)](1+i)\,dt$

$= \int_0^1 (3t - ti)(1+i)\,dt$

$= \int_0^1 (3t + 3ti - ti + t)\,dt$

$= \frac{4}{2}t^2 + \frac{2}{2}t^2 i \Big|_0^1$

$= 2 + i$

(b) $\int_0^1 [(t+it^2)+(2t-2it^2)](dt+2it\,dt)$ $\quad C: x=t,\ y=t^2,\ 0 \le t \le 1$.

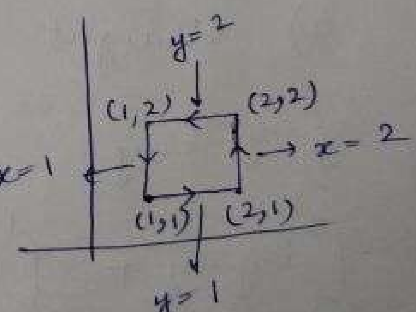
$= \int_0^1 (3t - it^2)(1+2it)\,dt$

$= \int_0^1 (3t + 6it^2 - it^2 + 2t^3)\,dt$

$= \frac{3}{2}t^2 + \frac{2}{4}t^4 + \frac{5}{3}t^3 i \Big|_0^1$

$= \frac{3}{2} + \frac{1}{2} + \frac{5}{3}i = 2 + \frac{5}{3}i$.

8) $\int z\,dz$



Along $y=1$

$\int z\,dz = \int_1^2 (x+i)\,dx$

$= \frac{x^2}{2} + ix\Big|_1^2$

$= 2+2i - (\frac{1}{2}+i)$

$= \frac{3}{2} + i$

Along $x = 2$

$$\int z\,dz = \int_1^2 (2 + yi)\,i\,dy$$

$$= 2iy + \frac{i^2}{2} y^2 \Big|_1^2$$

$$= 4i - 2i - \frac{1}{2}(4-1)$$

$$= -\frac{3}{2} + 2i$$

Along $y = 2$

$$\int z\,dz = \int_2^1 (x + 2i)\,dx$$

$$= \frac{x^2}{2} + 2ix \Big|_2^1$$

$$= \frac{1}{2} - 2 + 2i(1-2)$$

$$= -\frac{3}{2} - 2i$$

Along $x = 1$

$$\int z\,dz = \int_2^1 (1 + iy)\,i\,dy$$

$$= i(1-2) - \frac{1}{2}(1-4)$$

$$= -i + \frac{3}{2}$$

$\therefore$ Adding all

$$\int z\,dz = 0.$$

9) $\int_C |z|\,dz$

a) straight line from $z = -i$ to $z = i$

$\Rightarrow x = 0,\ -1 \le y \le 1$

$$\int_{-1}^{1} |y|\,i\,dy$$

$z = +iy$

$-1 \le y \le 1$

$dz = i\,dy$

$|z| = |y|$

$$= -\int_{-1}^{0} iy\,dy + \int_0^1 iy\,dy$$

$$= -\frac{i}{2} y^2 \Big|_{-1}^{0} + \frac{i}{2} y^2 \Big|_0^1 = -\frac{i}{2}(0-1) + \frac{i}{2}(1) = i.$$

b) the unit circle $|z-1|=1$

$z = 1+e^{i\theta} \quad 0 \le \theta \le 2\pi$.

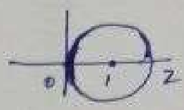
$= 1+\cos\theta + i\sin\theta$.

$-1 \le z-1 \le 1$

$0 \le z \le 2$.

$|z| = \sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}$

$= \sqrt{2+2\cos\theta}$



$= \sqrt{2(2\cos^2\frac{\theta}{2})}$

$= 2\left|\cos\frac{\theta}{2}\right|$

$= \begin{cases} 2\cos\frac{\theta}{2}, & 0 \le \theta \le \pi \\ -2\cos\frac{\theta}{2}, & \pi \le \theta \le 2\pi \end{cases}$

$\int_c |z|\,dz = \int_0^{\pi} 2\cos\frac{\theta}{2}\, i e^{i\theta}\, d\theta$.

$2i\, I = 2i\left[\frac{\cos\frac{\theta}{2}\, e^{i\theta}}{i} + \int \frac{\frac{1}{2}\sin\frac{\theta}{2}\, e^{i\theta}}{i}\, d\theta\right]$

where $I = \int_0^{\pi} \cos\frac{\theta}{2}\, e^{i\theta}\, d\theta$

$= 2\left[\cos\frac{\theta}{2}\, e^{i\theta} + \frac{1}{2}\left[\frac{\sin\frac{\theta}{2}\, e^{i\theta}}{i} - \int \frac{\frac{1}{2}\cos\frac{\theta}{2}\, e^{i\theta}}{i}\, d\theta\right]\right]$

$= 2\cos\frac{\theta}{2}\, e^{i\theta}\Big|_0^{\pi} + \frac{1}{i}\sin\frac{\theta}{2}\, e^{i\theta}\Big|_0^{\pi} \frac{1}{2i} I$.

$\left(2i + \frac{1}{2i}\right) I = -2 + \frac{1}{i}(-1)$

$= -2 - \frac{1}{i}$

$e^{i\pi}$
$= \cos\pi + i\sin\pi$
$= -1$.

$I = \frac{-2i-1}{i} \times \frac{2i}{1-4}$.

$= \frac{2}{3}(1+2i)$.

$\int_0^{\pi} |z|\,dz = \frac{2}{3}(1+2i)(2i)$

$= \frac{4i}{3} + \frac{8}{3}i^2$

$= -\frac{8}{3} + \frac{4i}{3}$. — 1

$z = x + iy$
$\bar{z} = x - iy$ $\Big\}$ $w = f(z) = u(x,y) + iv(x,y)$

$x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv(\quad)$$

$$\int_{\pi}^{2\pi} -2\cos\frac{\theta}{2} i e^{i\theta}\, d\theta$$

$$-2iI = -2\cos\frac{\theta}{2} e^{i\theta} \mp \frac{1}{i}\sin\frac{\theta}{2} e^{i\theta} + \frac{1}{2i} I$$

$$\left(2i + \frac{1}{2i}\right) I = 2\cos\frac{\theta}{2} e^{i\theta} + \frac{1}{i}\sin\frac{\theta}{2} e^{i\theta} \Big|_{\pi}^{2\pi}$$

$$= 2(-1) - 0 + \frac{1}{i}\left(-(1)(-1)\right)$$

$$= -2 + \frac{1}{i}$$

$$\int_{\pi}^{2\pi} \cos\frac{\theta}{2} e^{i\theta}\, d\theta = \left(-2 + \frac{1}{i}\right)\left(\frac{2i}{1-4}\right)$$

$$= (1-2i)\frac{2}{-3} = -\frac{2}{3} + \frac{4}{3}i$$

$$-2i\int_{\pi}^{2\pi} \cos\frac{\theta}{2} e^{i\theta}\, d\theta = -2i\left(-\frac{2}{3} + \frac{4}{3}i\right)$$

$$= \frac{4i}{3} + \frac{8}{3} \quad -(2)$$

(1) + (2)

$$\int_C |z|\, dz = \frac{8i}{3}$$

) $\int_C \frac{1}{z-z_0}\,dz$. $C: |z-z_0| = R$.

$z - z_0 = Re^{i\theta}$ $0 \le \theta \le 2\pi$.

$= \int_0^{2\pi} \frac{1}{\cancel{Re^{i\theta}}} \cancel{R}ie^{\cancel{i\theta}}\,d\theta$.

$= i\theta\Big|_0^{2\pi} = 2\pi i$

$\int_C (z-z_0)^m\,dz$, $m \neq -1$

$= \int_0^{2\pi} R^m e^{im\theta}\, Rie^{i\theta}\,d\theta$

$= \frac{\cancel{i}R^{m+1}\, e^{i(m+1)\theta}}{\cancel{i}(m+1)}\Bigg|_0^{2\pi}$

$= \frac{R^{m+1}}{m+1}(1-1) = 0$.

$e^{i(m+1)2\pi}$
$= \cos(m+1)2\pi$
$+ i\sin(m+1)2\pi$
$= 1$.

Now $\int_C \frac{P(z)}{z}\,dz$ $C: |z| = 1$

$= \int_C \left(\frac{2}{z} - 1 + 3z + z^2\right) dz$

$= \int_C \frac{2}{z}\,dz - \int_C dz + 3\int_C z\,dz + \int_C z^2\,dz$.

$= 2 \times 2\pi i$

$= 4\pi i$

(From Above proof)
for $z_0 = 0$
$R = 1$.

11) a) $\int_C \frac{dz}{z-2}$ $C: |z| = 1$

$\frac{1}{z-2}$ is analytic in $|z| = 1$. $f'(z) = \frac{-1}{(z-2)^2}$ is cts within & on C .

so by Cauchy theorem .

$\int \frac{dz}{z-2} = 0$.

b) $\int_c \frac{dz}{z} = 2\pi i$ C is closed contour enclosing $z = 0$.

Here $a = 0$.

$f(z) = 1$

$f(a=0) = 1$

$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz$

Cauchy integral formula

$\therefore f(0) = 1 = \frac{1}{2\pi i} \int \frac{1}{z} dz$

$2\pi i = \int \frac{dz}{z}$

c) $\int_c \frac{dz}{(z+1)^2}$ $C : |z| = 2$.

$\int_c \frac{1/z+1}{z+1} dz$

$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^2}$

$a = -1$ lies inside C.

And $f(z) = 1$.

$f'(z) = 0$

$f'(a=-1) = 0$.

$\therefore \int \frac{dz}{(z+1)^2} = 0 \times 2\pi i = 0$.

12) a) $\int_c \frac{e^{-z}}{z+1} dz$ $C: |z| = 2$.

$a = -1$ lies inside C

$f(z) = e^{-z}$ is analytic in C.

$= 2\pi i\, f(a=-1)$

$= 2\pi i\, e^{-(-1)} = 2\pi e i$

b) $\int_C \frac{e^{2z}}{(z+1)^4} dz$. $C : |z| = 2$.

$f^{(n)}(a) = \frac{n!}{2\pi i} \int \frac{f(z) dz}{(z-a)^{n+1}}$

$a = -1$ lies inside C.

$n = 3$

$f(z) = e^{2z}$ is analytic in C

$f'''(z) = 8e^{2z}$

$f^{(3)}(a) = 8e^{2(-1)} = \frac{8}{e^2}$

$\frac{8}{e^2} \times \frac{2\pi i}{3!} = \int \frac{e^{2z}}{(z+1)^4} dz$

$\frac{8\pi i}{3e^2} = \int \frac{e^{2z}}{(z+1)^4} dz$

c) $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$. $C: |z| = 3/2$

$a = 1$ lies inside C .

$f(z) = \frac{\cos \pi z^2}{z-2}$ is analytic in C

$= f(1)\, 2\pi i$

$= \frac{\cos \pi}{1-2} \times 2\pi i = 2\pi i$

$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ $C: |z| = 1/2$.

$a = 1, 2$ lie outside C

So $f(z) = \frac{\cos \pi z^2}{(z-1)(z-2)}$ is analytic in C

$f'(z)$ is cts in C .

$\therefore \int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz = 0$.

$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ $C: |z| = 3$.

$a = 1$ & 2 both lie inside C .

$= \int_C \frac{\cos \pi z^2}{z-2} dz - \int_C \frac{\cos \pi z^2}{z-1} dz$ $\frac{1}{(z-1)(z-2)} = \frac{(z-1)-(z-2)}{(z-1)(z-2)}$

$= \frac{-1}{z-1} + \frac{1}{z-2}$

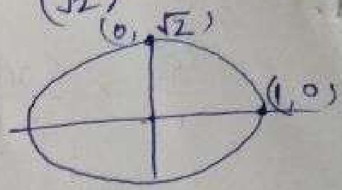
$z-1-(z-2)$

$f(z) = \cos \pi z^2$

$= f(2)\, 2\pi i - f(1)\, 2\pi i$

a) $\int_C \frac{e^{-z}}{z^2}\, dz$ $\quad$ C is ellipse $2x^2+y^2=2$

$x^2+\left(\frac{y}{\sqrt{2}}\right)^2 = 1$



$a=0$ lies in C.

$f(z) = e^{-z}$ analytic in C

$f'(z) = -e^{-z}$

$2\pi i f'(a) = \int_C \frac{f(z)}{(z-a)^2}\, dz$

$= 2\pi i f'(0)$

$= 2\pi i(-e^0) = -2\pi i$

$\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz$

$|z-z_0| = r$

$z-z_0 = re^{i\theta}$

$z = z_0 + re^{i\theta}$

As $r \to 0$

$\int_{\theta=0}^{2\pi} \frac{f(z_0+re^{i\theta})}{\cancel{re^{i\theta}}} i\cancel{re^{i\theta}}\, d\theta$

$i\int_{\theta=0}^{2\pi} f(z_0+re^{i\theta})\, d\theta$

$i\int_{\theta=0}^{2\pi} f(z_0)\, d\theta = i f(z_0)\, 2\pi$

$\oint_{C_1} \frac{|dz|}{\cdot}$

$z-z_0 =$

$dz = ire^{i\theta} d\theta$

$|dz| = r\, d\theta$