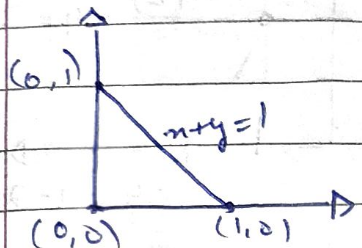
TUTORIAL - 3

Q1 a)



$$I = \int_0^1 \int_0^{1-x} (x^2 + y^2)\, dy\, dx$$

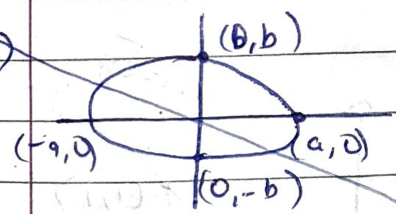
$$= \int_0^1 x^2(1-x) + \frac{(1-x)^3}{3}\, dx$$

$$I = \left| \frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12} \right|_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{4-3+1}{12} = \frac{2}{12}$$

$$\boxed{I = \frac{1}{6}}$$

b)



$$I = 4\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} (x+y)^2\, dy\, dx$$

$$= 4\int_0^a \left| \frac{(x+y)^3}{3} \right|_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

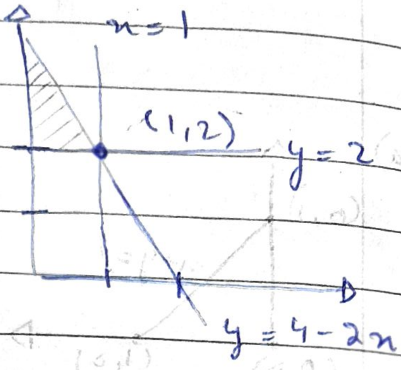
$$= \frac{4}{3}\int_0^a \left(x + b\sqrt{1-\frac{x^2}{a^2}} \right)^3 dx$$

Let $x = a\cos\phi$

$dx = -a\sin\phi\, d\phi$

$$I = \frac{4a}{3}\int_0^{\pi/2} \left[\frac{a}{2}\sin 2\phi + ab\left(\frac{1+\cos\phi}{2}\right) \right]^3 d\phi$$

Q2 a) $\int_0^1 \int_2^{4-2x} dy\, dx$



$$I = \int_0^1 (4-2x-2)\,dx$$

$$= 2\left| x - \frac{x^2}{2} \right|_0^1$$

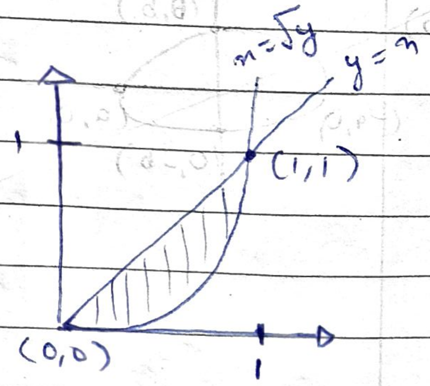
$$= 2\left[1 - \frac{1}{2}\right] = 1$$

$$I = \int_0^2 \int_1^{(4-y)/2} dx\, dy$$

$$= \int_0^2 \left(\frac{4-y}{2} - 1\right) dy$$

$$= \left| 2y - \frac{y^2}{4} - y \right|_0^2 = 2 - 1 = 1$$

b) $\int_0^1 \int_y^{\sqrt{y}} dx\, dy$



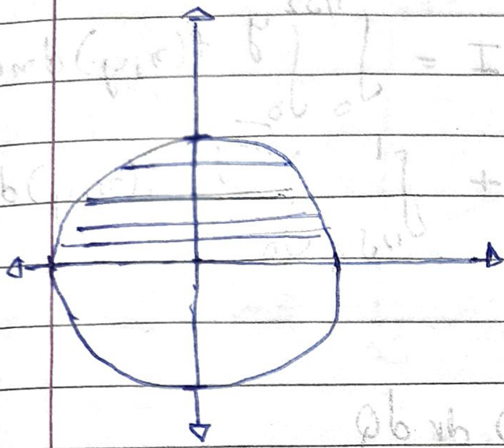
$$I = \int_0^1 \sqrt{y} - y\; dy$$

$$= \left| \frac{2}{3} y^{3/2} - \frac{y^2}{2} \right|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$I = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 x - x^2\, dx$$

$$= \left| \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

c) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx\, dy$



$I = 6 \int_0^1 y\sqrt{1-y^2}\, dy$

let $1-y^2 = t$

$-2y\, dy = dt$

$= \frac{6}{2}\int_0^1 \cancel{\sqrt{1-y^2}\, dy}$

$= \frac{6}{2}\int_0^1 \sqrt{t}\, dt$

$I = 3\, \frac{2(1)^{3/2}}{3} = 2$

$I = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 3y \, dy\, dx = \int_{-1}^{1} \left| \frac{3y^2}{2} \right|_0^{\sqrt{1-x^2}} dx$

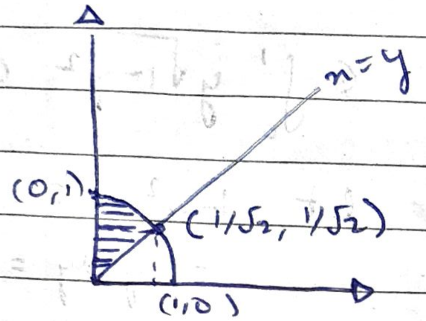
$= \int_{-1}^{1} \frac{3(1-x^2)dx}{2} = \cancel{2}\int_0^1 3\,\frac{(1-x^2)dx}{\cancel{2}}$

$= 3\int_0^1 1-x^2\, dx = 3\left| x - \frac{x^3}{3}\right|_0^1$

$= \left(1-\frac{1}{3}\right)3 = 2$

Page No.
Date

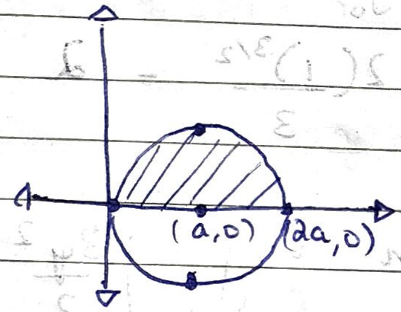
Q3 1) $I = \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$



$I = \int_0^{1/\sqrt{2}} \int_0^{y} f(x,y)\,dx\,dy$

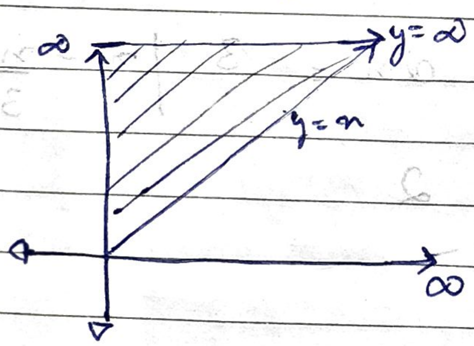
$+ \int_{1/\sqrt{2}}^{1} \int_0^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$

2) $I = \int_0^{\pi/2} \int_0^{2a\cos\theta} f(r,\theta)\,dr\,d\theta$



$I = \int_0^{2a} \int_0^{\cos^{-1}(r/2a)} f(r,\theta)\,d\theta\,dr$

3) $I = \int_0^{\infty} \int_x^{\infty} (e^{-y}/y)\,dy\,dx$



$I = \int_0^{\infty} \int_0^{y} e^{-y}/y\,dx\,dy$

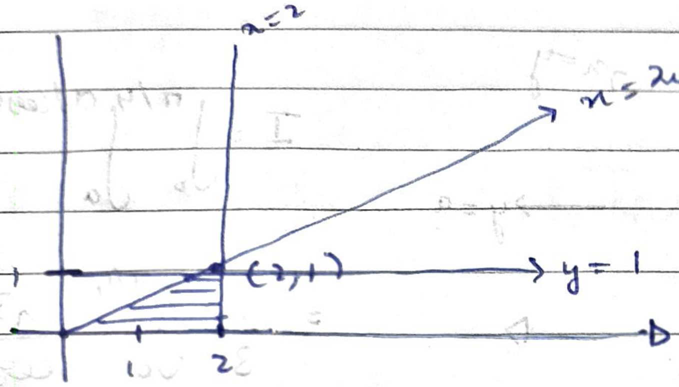
$= \int_0^{\infty} e^{-y}\,dy$

$= \left| -e^{-y} \right|_0^{\infty}$

$= -e^{-\infty} + e^{-0}$

$= e^{0} = 1$

4) $$\int_0^1 \int_{2y}^{2} e^{-x^2}\, dx\, dy$$



$$I = \int_0^2 \int_0^{x/2} e^{-x^2}\, dy\, dx$$

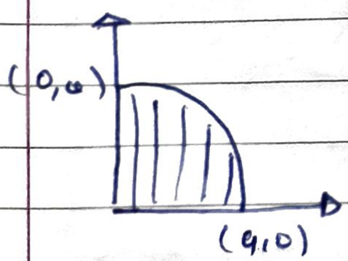
$$= \int_0^2 \frac{x}{2} e^{-x^2}\, dx$$

$$= \frac{1}{4} \int_0^4 e^{-t}\, dt$$

$$= \frac{1}{4} \left| -e^{-t} \right|_0^4 = \frac{1}{4}\left[1 - e^{-4}\right]$$

Q4 a) $$\int_0^a \int_0^{\sqrt{a^2-y^2}} x^2+y^2\, dx\, dy$$

y $\quad x = r\cos\theta \qquad y = r\sin\theta \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = r$

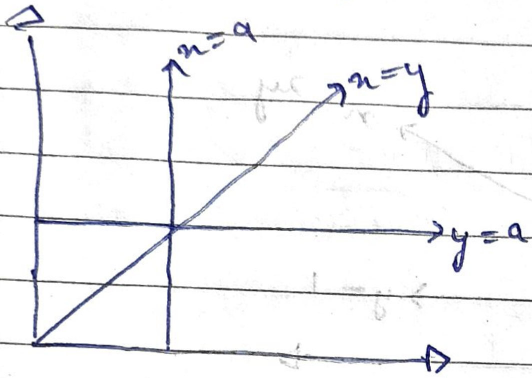


$$I = \int_0^{\pi/2} \int_0^a r^3\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{a^4}{4}\, d\theta$$

$$= \frac{\pi a^4}{8}$$

Q4 b) $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy$



$$I = \int_0^{\pi/4} \int_a^{a/\cos\theta} r^2 \cos^2\theta \, dr\, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/4} \frac{a^3}{\cos\theta}\, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/4} \sec\theta \, d\theta$$

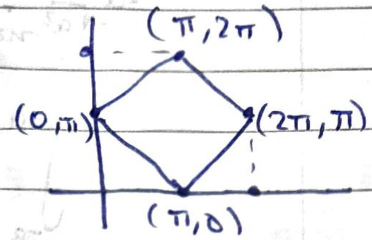
$$= \frac{a^3}{3} \left[\ln\left(1 + \frac{\sqrt{2}}{\cancel{2}}\right) - \overset{0}{\ln 1} \right]$$

$$= \frac{a^3}{3} \times \ln\left(1 + \frac{\sqrt{2}}{\cancel{2}}\right)$$

Q7 $\iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy$

$x-y = u$

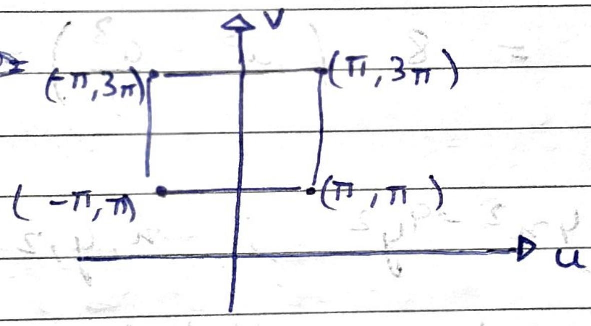
$x+y = v$



$$\iint_{R^*} u^2 \cos^2 v \, |J| \, du\, dv$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$J = \frac{1}{J'} = \frac{1}{2}$$



$$I = \frac{1}{2}\int_{\pi}^{3\pi}\int_{-\pi}^{\pi} u^2 \cos^2 v \, du\, dv$$

$$= \frac{1}{2}\int_{\pi}^{3\pi} \left[\frac{u^3 \cos^2 v}{3}\right]_{-\pi}^{\pi} dv$$

$$= \frac{1}{6}\int_{\pi}^{3\pi} 2\pi^3 \cos^2 v \, dv$$

$$= \frac{\pi^3}{6} \times \frac{2}{2}\int_{\pi}^{3\pi} (1-\cos 2v)\, dv$$

$$= \frac{\pi^3}{6}\left[v - \frac{\sin 2v}{2}\right]_{\pi}^{3\pi} = \frac{\pi^3}{6}\left[2\pi - 0\right]$$

$$I = \frac{\pi^4}{3}$$

Page No.
Date

Q10 b) $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\ dy\,dx$$

$$= 4\int_{-a}^{a} a^2-x^2\ dx$$

$$= 2\times 4\int_{0}^{a} a^2-x^2\ dx$$

$$= 8\left(a^3-\frac{a^3}{3}\right) = \frac{16a^3}{3}$$

a) $z = 36-4x^2-9y^2 \quad ; \quad x,y,z \geq 0$

Converting into cylindrical form

$$\iiint_D r\,f(r,Q,z)\,dr\,dQ\,dz$$

$$\int_0^{\pi/2}\int_0^{\sqrt{\frac{36}{4+5\sin^2 Q}}}\int_0^{36-(4+5\sin^2Q)r^2} r\ dz\,dr\,dQ$$

$$\int_0^{\pi/2}\int_0^{\sqrt{\frac{36}{4+5\sin^2Q}}} 36r - r^3(4+5\sin^2Q)\ dr\,dQ$$

$$\int_0^{\pi/2}\left|\frac{36r^2}{2}-\frac{r^4}{4}(4+5\sin^2Q)\right|_0^{\sqrt{\frac{36}{4+5\sin^2Q}}} dQ$$

$$\int_0^{\pi/2}\frac{18\times 36}{4+5\sin^2Q}-\frac{\overset{9}{\cancel{36}}\times 36}{\cancel{4}(4+5\sin^2Q)}\ dQ$$

$(3^2 - x^2)^{3/2} dx$ $\quad 9(1-\sec^2 Q)$

$x = 3\sec Q$
$dx = 3\sec Q \tan Q\, dQ$

Page No.
Date

$$= 36 \int_0^{\pi/2} \frac{9}{4 + 5\sin^2 Q}\, dQ$$

Let $Q =$

$$= 36 \times 9 \int_0^{\pi/2} \frac{\sec^2 Q\, dQ}{4\sec^2 Q + 5\tan^2 Q}$$

$$= 36 \times 9 \int_0^{\tan^{-1}(\pi/2)} \frac{dt}{4(1+t^2) + 5t^2}$$

$$= 36 \times 9 \int_0^{\tan^{-1}\frac{\pi}{2}} \frac{dt}{4 + 9t^2}$$

$$= \frac{36 \times 9}{3} \times \frac{1}{2} \left| \tan^{-1}\left(\frac{3t}{2}\right) \right|_0^{\pi/2}$$

$$= \frac{\cancel{36}^{6} \times 9}{\cancel{6}} \times \frac{\pi}{2}$$

$$= 27\pi$$

Q6 $\quad r = a(1 - \cos Q)$ rotated around axis by converting to sperical integral

$$\iiint_D dV_a = \iiint_{D^*} f(r, Q, \phi)\, r^2 \sin\phi\, dr\, dQ\, d\phi$$

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^{a(1-\cos\phi)} r^2 \sin\phi\, d\phi\, dQ$$

$$\int_0^{2\pi} \int_0^{\pi} \frac{a^3}{3} (1 - \cos\phi)^3 \sin\phi\, dQ$$

$$\frac{a^3}{3} \int_0^{2\pi} \left. \frac{(1 - \cos\phi)^4}{4} \right|_0^{\pi} dQ$$

$$= \frac{a^3}{3} \times 2\pi \times \frac{2^4}{4} = \frac{8a^3}{3}$$

Q9a) $x^2+y^2+z^2 = 2a^2$ ← upper bound

$az = x^2+y^2$ ← lower bound

$$\int_0^{2\pi}\int_0^{a}\int_{r^2/a}^{\sqrt{2a^2-r^2}} r\, dz\, dr\, dQ$$

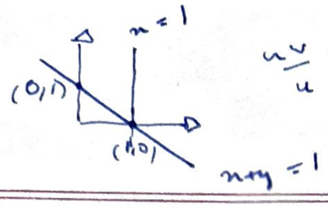
$$\int_0^{2\pi}\int_0^{\sqrt{2}a}\left(r\sqrt{2a^2-r^2} - \frac{r^3}{a}\right) dr\, dQ$$

$$\int_0^{2\pi}\left| -\frac{(2a^2-r^2)^{3/2}}{3} - \frac{r^4}{4a}\right|_0^{a} dQ$$

$$2\pi\left[\frac{2\sqrt{2}a^3}{3} - \frac{a^4}{4a} - \frac{a^3}{3}\right]$$

$$\pi a^3\left[\frac{4\sqrt{2}}{3} - \frac{7}{6}\right]$$

$\frac{v}{0} = \frac{x}{x+y}$ $\frac{uv}{u}$ $\frac{\partial(x,y)}{\partial(u,v)}$ $1-x = uv$ $u=1$ $v = \frac{xy}{x+y}$



Page No.
Date $-\frac{y0}{(x+y)^2}$

Q8 $x + y = u \qquad y = uv$

$$I = \int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} \, dy \, dx$$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$J^{-1} = \begin{vmatrix} 1 & -y/(x+y)^2 \\ 1 & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

$$J = (x+y)$$

$$I = \int_0^1 \int_0^1 e^{v} \times u \, dv \, du$$

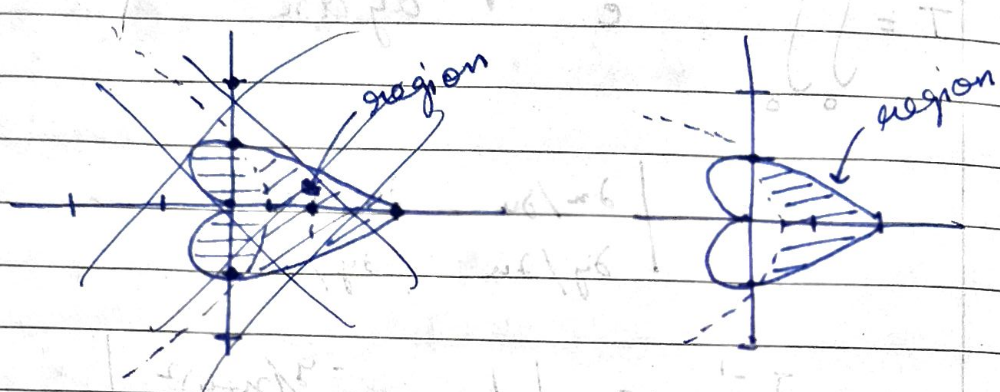
$$= \int_0^1 \left| u e^{v} \right|_0^1 du$$

$$= \int_0^1 u \, (e-1) \, du$$

$$= \left| \frac{u^2}{2} (e-1) \right|_0^1$$

$$= \frac{1}{2} (e-1)$$

Q5 i) $r(1+\cos\theta) = 1$ outside
$r = 1+\cos\theta$ inside



Area of region $= 2$[Area of cardoid − Area of parabola] in first quadrant

$$A = 2\left[\int_0^{\pi/2}\int_0^{1+\cos\theta} r\,dr\,d\theta - \int_0^1\int_0^{\frac{1-y^2}{2}} dx\,dy\right]$$

$$= \cancel{2}\left[\int_0^{\pi/2}\frac{(1+\cos\theta)^2}{\cancel{2}}\,d\theta - \int_0^1 \frac{1-y^2}{\cancel{2}}\,dy\right]$$

$$= \int_0^{\pi/2}(1+\cos\theta)^2\,d\theta - \int_0^1 1-y^2\,dy\Big]$$

$$= \int_0^{\pi/2} 1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta$$

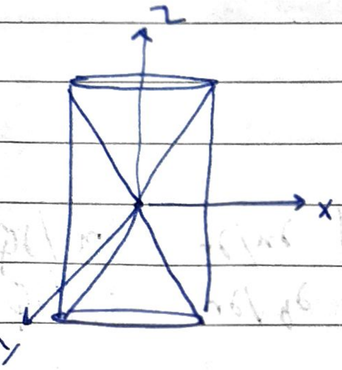
$$- \int_0^1 1 - y^2\,dy$$

$$= \left|\frac{3}{2}\theta + \frac{\sin 2\theta}{2} + 2\sin\theta\right|_0^{\pi/2} - \left|y - \frac{y^3}{3}\right|_0^1$$

Page No.
Date

$$A = \frac{3\pi}{4} + 2 - \frac{2}{3}$$

$$\boxed{A = \frac{3\pi}{4} - \frac{4}{3}}$$

ii) 

$$V = \iiint_D dV$$

$$x^2 + y^2 = a^2$$

$$x^2 + y^2 = z^2$$

Converting to cylindrical cordinates

$$\int_0^{2\pi}\int_0^a\int_{-r}^{r} r \, dz\, dr\, d\phi = \int_0^{2\pi}\int_0^a 2r^2 \, dr\, d\phi$$

$$= \int_0^{2\pi} \frac{2a^3}{3} \, d\phi = \frac{4a^3\pi}{3}$$

$$\boxed{V = \frac{4\pi a^3}{3}}$$

Page No.
Date

Q1 b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ← Domain

$$I = \iint_D (x+y)^2 \, dx \, dy$$

let $x = ar\cos\varphi$
$y = br\sin\varphi$

$$|J| = \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\varphi \\ \partial y/\partial r & \partial y/\partial\varphi \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\varphi & ar(-\sin\varphi) \\ b\sin\varphi & br\cos\varphi \end{vmatrix}$$

$$|J| = abr$$

$$I = \int_0^{2\pi}\int_0^1 (ar\cos\varphi + br\sin\varphi)^2 \, abr \, dr \, d\varphi$$

$$= \int_0^{2\pi}\int_0^1 (a^2r^2\cos^2\varphi + b^2r^2\sin^2\varphi + abr^2\sin 2\varphi) \, abr \, dr \, d\varphi$$

Considering periodicity of 0 to 2π
for $\int \sin 2\varphi = 0$

$$= ab\int_0^{2\pi}\int_0^1 (a^2r^3\cos^2\varphi + b^2r^3\sin^2\varphi) \, dr \, d\varphi$$

$$= ab\int_0^{2\pi} \left|\frac{r^4}{4}\right|_0^1 (a^2\cos^2\varphi + b^2\sin^2\varphi) \, d\varphi$$

Page No.
Date

$$I = \frac{ab}{4}\int_0^{2\pi}\left(\frac{a^2}{2} + \frac{a^2\cos 2\varphi}{2} + \frac{b^2}{2} - \frac{b^2\cos 2\varphi}{2}\right)$$

considering periodicity of $\sin 2\varphi$ between 0 to 2π

$$I = \frac{ab}{4}\int_0^{2\pi}\left(\frac{a^2}{2} + \frac{b^2}{2}\right)d\varphi$$

$$= \frac{ab}{4} \times 2\pi \times \frac{a^2+b^2}{2}$$

$$\boxed{I = \frac{\pi ab\,(a^2+b^2)}{4}}$$