Q-1 :- Find the fourier series for the function $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \infty = \frac{\pi^2}{8}$

Soln → Here, $2l = 2$

$\Rightarrow \boxed{l = 1}$

Now, fourier series expansion of the function is -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Now, $a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx$

$$\Rightarrow a_0 = \frac{1}{1}\int_0^2 f(x)\, dx$$

$$= \int_0^1 \pi x\, dx + \int_1^2 \pi(2-x)\, dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{\pi}{2}[1-0] + \pi\left[4 - \frac{4}{2} - 2 + \frac{1}{2}\right]$$

$$= \frac{\pi}{2} + \pi\left[4 - 2 - \cancel{4} + \frac{1}{2}\right]$$

$$= \pi/2 + \frac{\pi}{2} = \pi$$

and $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$$= 1\int_0^1 \underset{I}{\pi x}\,\underset{II}{\cos n\pi x}\,dx + \int_1^2 \underset{I}{\pi(2-x)}\,\underset{II}{\cos n\pi x}\,dx$$

$$= \pi\left[x\frac{\sin n\pi x}{n\pi} - \int_0^1 1\times\frac{\sin n\pi x}{n\pi}\,dx + (2-x)\frac{\sin n\pi x}{n\pi} - \int_1^2 (-1)\times\frac{\sin n\pi x}{n\pi}\,dx\right]$$

$$= \pi\left[x\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\right]_0^1 + \pi\left[(2-x)\frac{\sin n\pi x}{(n\pi)} - \frac{\cos n\pi x}{(n\pi)^2}\right]_1^2$$

$$= \pi\left[1\times\frac{\cancelto{0}{\sin n\pi}}{(n\pi)} + \frac{\cos n\pi}{(n\pi)^2} - 0 + \frac{\cos 0}{(n\pi)^2}\right] + \pi\left[0 - \frac{\cos 2n\pi}{(n\pi)^2} - (-1)\frac{\cancelto{0}{\sin n\pi}}{(n\pi)} + \frac{\cos n\pi}{(n\pi)^2}\right]$$

$$= \frac{\pi}{(n\pi)^2}(-1)^n - \frac{\pi}{(n\pi)^2} - \frac{\pi}{(n\pi)^2} + \frac{\pi(-1)^n}{(n\pi)^2}$$

$$= \frac{2\pi}{(n\pi)^2}\left[(-1)^n - 1\right] = \begin{cases} \dfrac{-4\pi}{(n\pi)^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

and $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$= 1\int_0^1 \underset{I}{\pi x}\,\underset{II}{\sin n\pi x}\,dx + \int_1^2 \underset{I}{\pi(2-x)}\,\underset{II}{\sin n\pi x}\,dx$$

$$= \pi\left[-x\frac{\cos n\pi x}{n\pi} - \int 1\times\left(-\frac{\cos n\pi x}{n\pi}\right)dx\right]_0^1 + \pi\left[(2-x) - \frac{\cos n\pi x}{n\pi} - \int(-1)\,\frac{-\cos n\pi x}{(n\pi)}\,dx\right.$$

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$$= \pi\left[-x\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{(n\pi)^2}\right]_0^1 + \pi\left[-(2-x)\frac{\cos n\pi x}{(n\pi)} - \frac{\sin n\pi x}{(n\pi)^2}\right]_1^2$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} + \cancel{\frac{\sin n\pi}{(n\pi)^2}}^{0} + 0 - \cancel{\frac{\sin 0}{(n\pi)^2}}^{0}\right] + \pi\left[0 + \cancel{\frac{\sin 2n\pi}{(n\pi)^2}}^{0} - \frac{1\times\cos n\pi}{n\pi} - \cancel{\frac{\sin n\pi}{(n\pi)^2}}^{0}\right]$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi}\right] + \frac{\pi\cos n\pi}{n\pi} = 0$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1,3,-}^{\infty} \frac{-4\pi}{(n\pi)^2}\cos n\pi x \qquad \text{Putting, } n = 2m+1$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos(2m+1)\pi x$$

at $x=0$

$f(0) = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{———} \infty\right]$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{- — —} \infty.$$

Q.2→ Find the half Range sine series for the function $x\cos x$ in the Range $(0, \pi)$

Sol^n→ fourier series Expansion for half Range sine series can be written as→

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} x \cos x \sin nx \, dx \qquad [\text{Given } f(x) = x\cos x]$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi} x[\sin(x+nx) - \sin(x-nx)]\,dx$$

$$[\text{as } 2\cos A \sin B = \sin(A+B) - \sin(A-B)]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_0^{\pi} \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\sin(x+nx)}\,dx - \int_0^{\pi} \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\sin(x-nx)}\,dx\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\left[-x\frac{\cos(x+nx)}{(1+n)} - \int 1 \times -\frac{\cos(x+nx)}{(1+n)}\,dx\right] - \left[-x \times \frac{\cos(x-nx)}{(1-n)} - \int 1 \times -\frac{\cos(x-nx)}{(1-n)}\,dx\right]^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-x\cos(x+nx)}{1+n} + \frac{\sin(x+nx)}{(1+n)^2} + \frac{x\cos(x-nx)}{1-n} - \frac{\sin(x-nx)}{(1-n)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(\pi+n\pi)}{(1+n)} + \frac{\overset{\to 0}{\sin(\pi+n\pi)}}{(1+n)^2} + \frac{\pi\cos(\pi-n\pi)}{(1-n)} - \frac{\overset{\to 0}{\sin(\pi-n\pi)}}{(1-n)^2} + 0 - \underbrace{\frac{\sin 0}{(1+n)^2}}_{=0} - 0 + \underbrace{\frac{\sin(0)}{(1-n)^2}}_{=0}\right]$$

$$= \frac{1}{\pi}\left[+\pi\frac{\cos n\pi}{(1+n)} - \pi\frac{\cos(n\pi)}{1-n}\right]$$

$$= \frac{(-1)^n}{(1+n)} - \frac{(-1)^n}{(1-n)} = (-1)^n\left[\frac{1-n-1-n}{1-n^2}\right] = \frac{-2n(-1)^n}{1-n^2}$$

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but $n \neq 1$

$\therefore$ we will find b_1 alone

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin x \, dx$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \underset{\mathrm{I}}{x}\ \underset{\mathrm{II}}{\sin 2x}\, dx$$

$$\therefore b_1 = \frac{1}{\pi}\left[x \times \left[-\frac{\cos 2x}{2}\right] - \int 1 \times \left[-\frac{\cos 2x}{2}\right]dx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{1}{4}\sin 2x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2} + \cancel{\frac{1}{4}\sin 2\pi}^{\,0} + 0 - \cancel{\frac{1}{4}\sin 0}^{\,0}\right]$$

$$= -\frac{\cos 2\pi}{2} = \frac{-1}{2}$$

Now, $b_n = \dfrac{-2n(-1)^n}{(1-n^2)}$

$$\therefore f(x) = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{-2n(-1)^n}{(1-n)^2}\sin nx$$

$$= -\frac{1}{2}\sin x - 4\sin 2x + \frac{3}{2}\sin 3x - \frac{8}{9}\sin 4x + \text{----}$$

-: The End :-