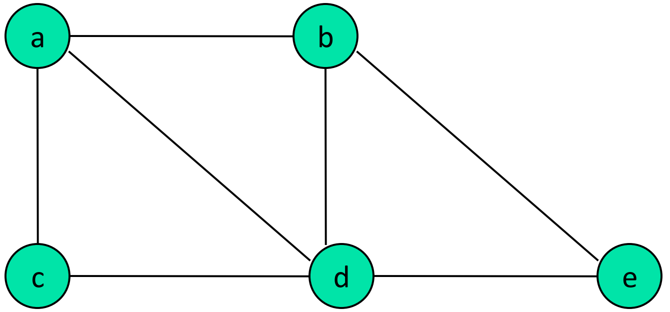
Euler Paths and Circuits

- An *Euler path* is a path using every edge of the graph *G* exactly once.
- An *Euler circuit* is an Euler path that returns to its start.

- G1 has Euler path a, c, d, e, b, d, a, b



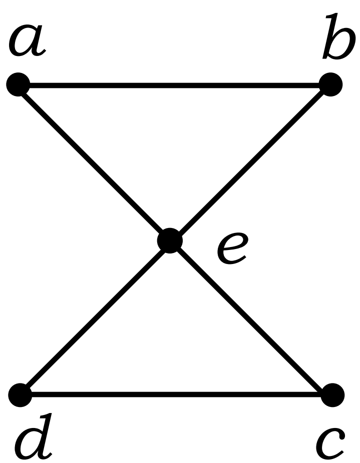
NOTE: The definition applies both to undirected as well as directed graphs of all types.

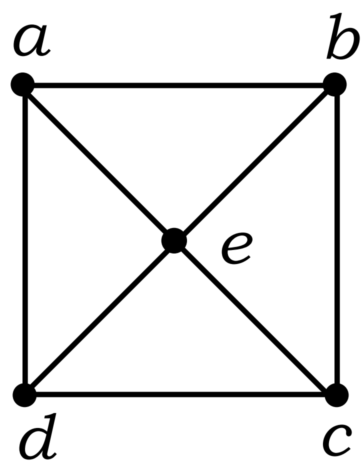
Necessary and Sufficient Conditions

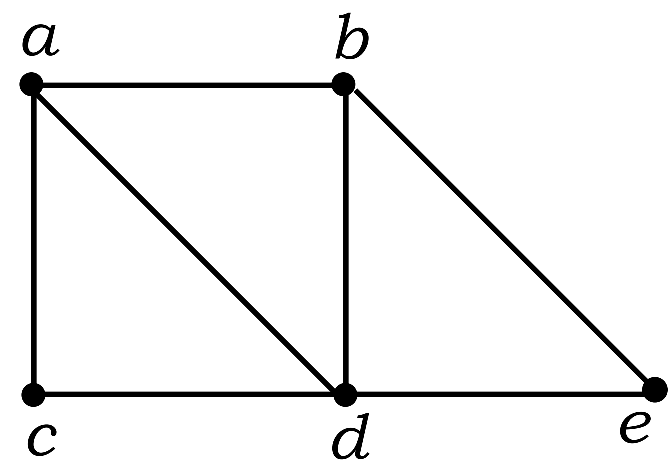
- A connected graph has a Euler circuit iff *each of its vertices has an even degree*.
- A connected graph has a Euler path but not an Euler circuit iff *it has exactly two vertices of odd degree*.

Example

- Which of the following graphs has an Euler *circuit*?







yes

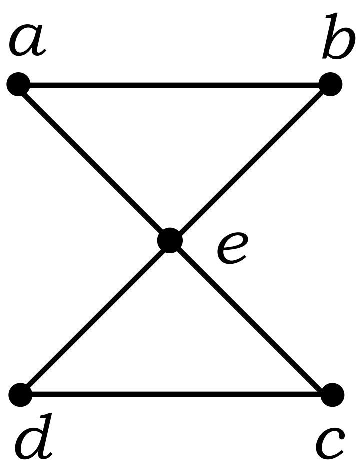
(a, e, c, d, e, b, a)

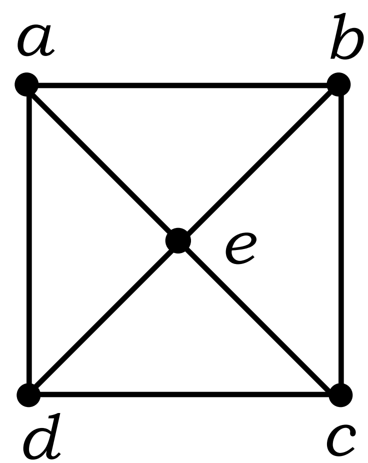
no

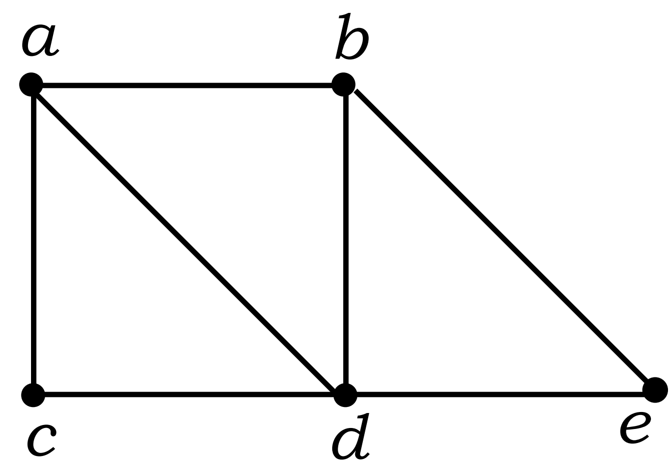
no

Example

- Which of the following graphs has an Euler *path*?



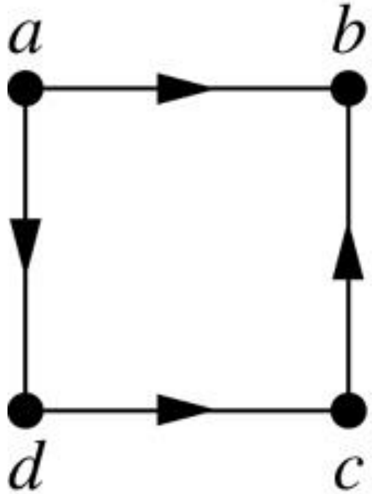




yes **no** **yes**

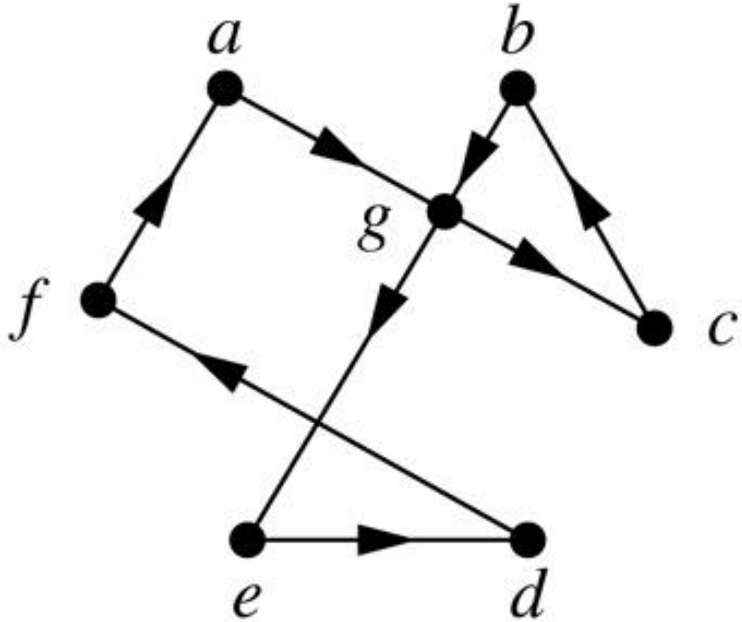
(a, e, c, d, e, b, a) (a, c, d, e, b, d, a, b)

Euler Circuit in Directed Graphs



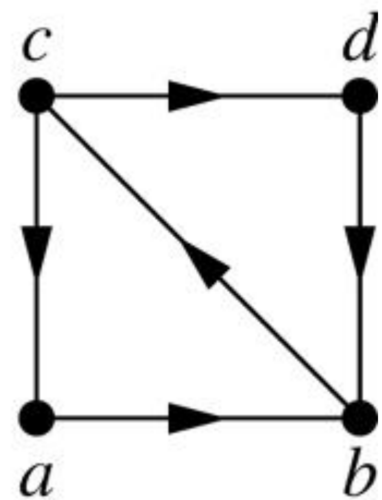
H_1

NO



H_2

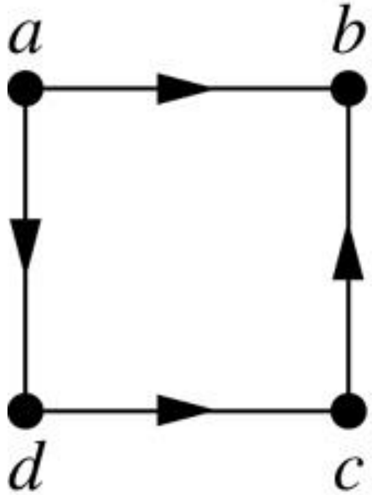
(a, g, c, b, g, e, d, f, a)



H_3

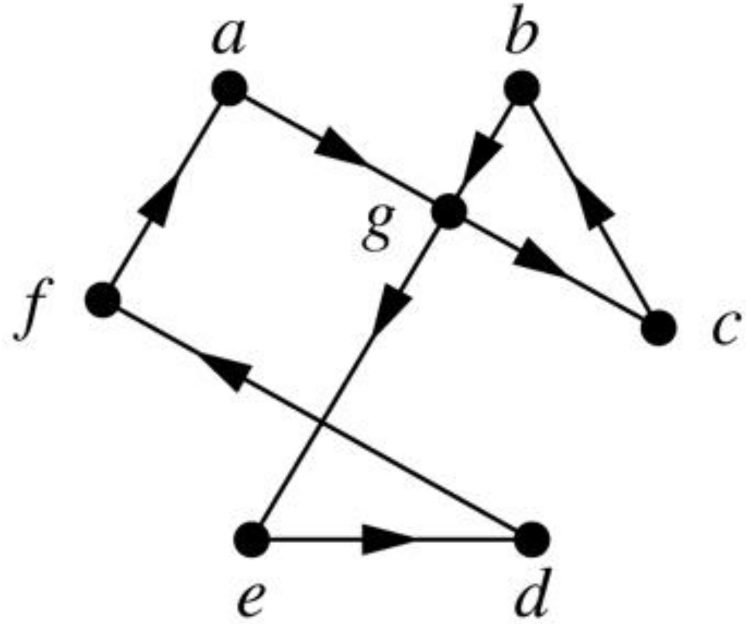
NO

Euler Path in Directed Graphs



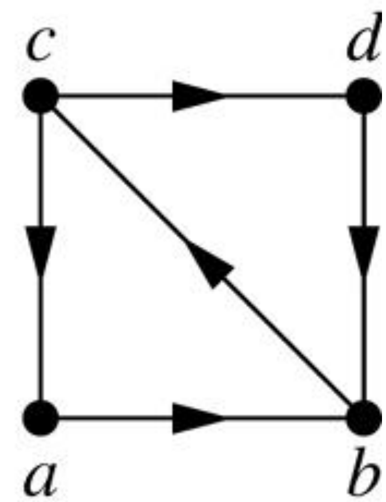
H_1

NO



H_2

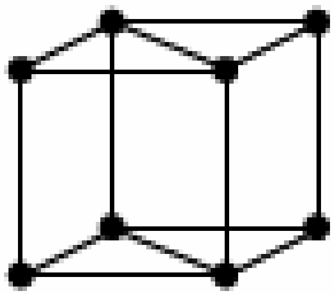
(a, g, c, b, g, e, d, f, a)



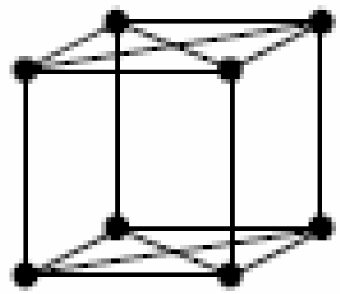
H_3

(c, a, b, c, d, b)

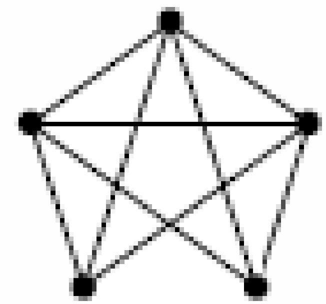
For each of the following graphs, decide if it has an Eulerain path , an Eulerian circuit, both, or neither.



ii.



iii.



Since all graphs are connected, we need to check the parity of degrees

i) Four vertices of odd degree , so neither an euler path nor euler circuit

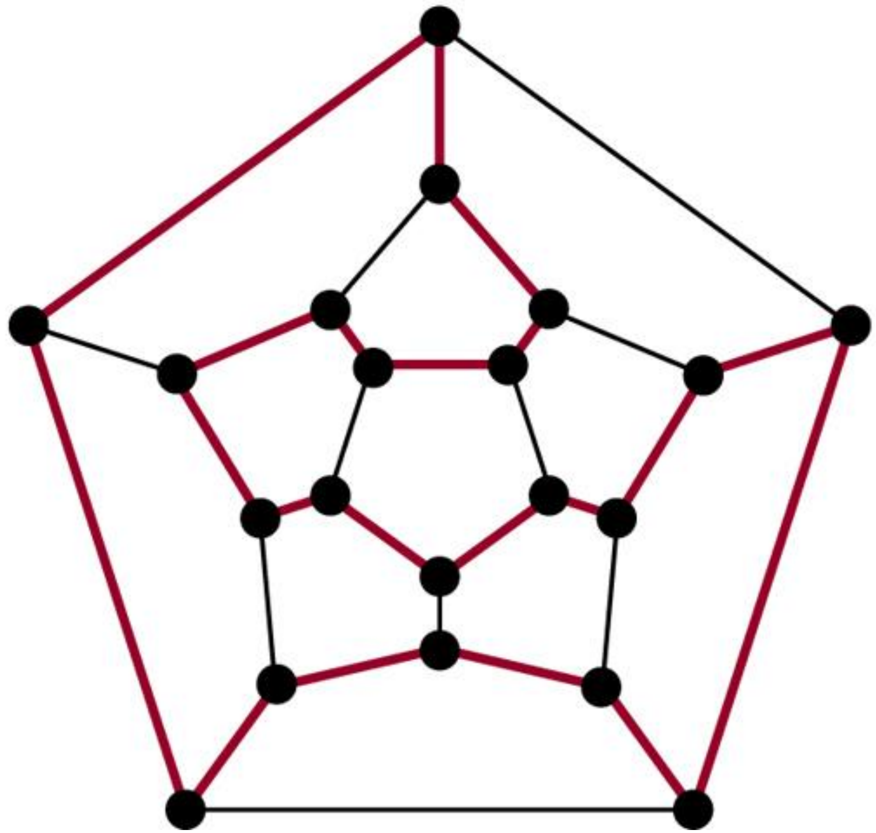
ii) All vertices have even degree, so there is an euler circuit.

iii) There are two vertices of odd degree, so an euler path but no euler circuit

Hamilton Paths and Circuits

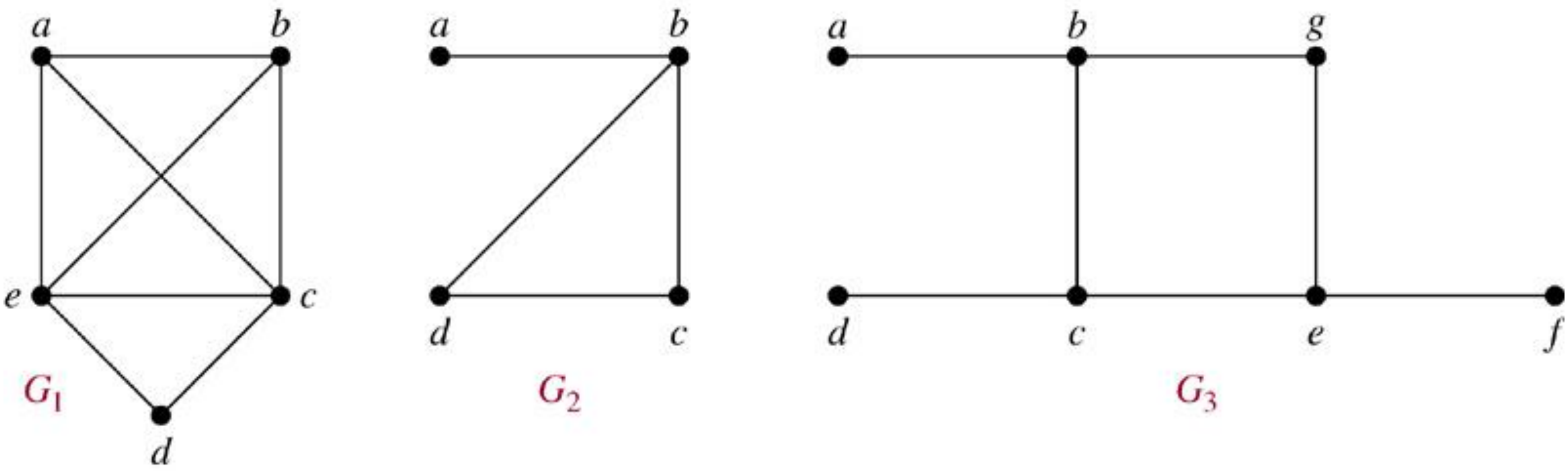
- A *Hamilton path* in a graph G is a path which visits every vertex in G exactly once.
- A *Hamilton circuit* is a Hamilton path that returns to its start.

Hamilton Circuits



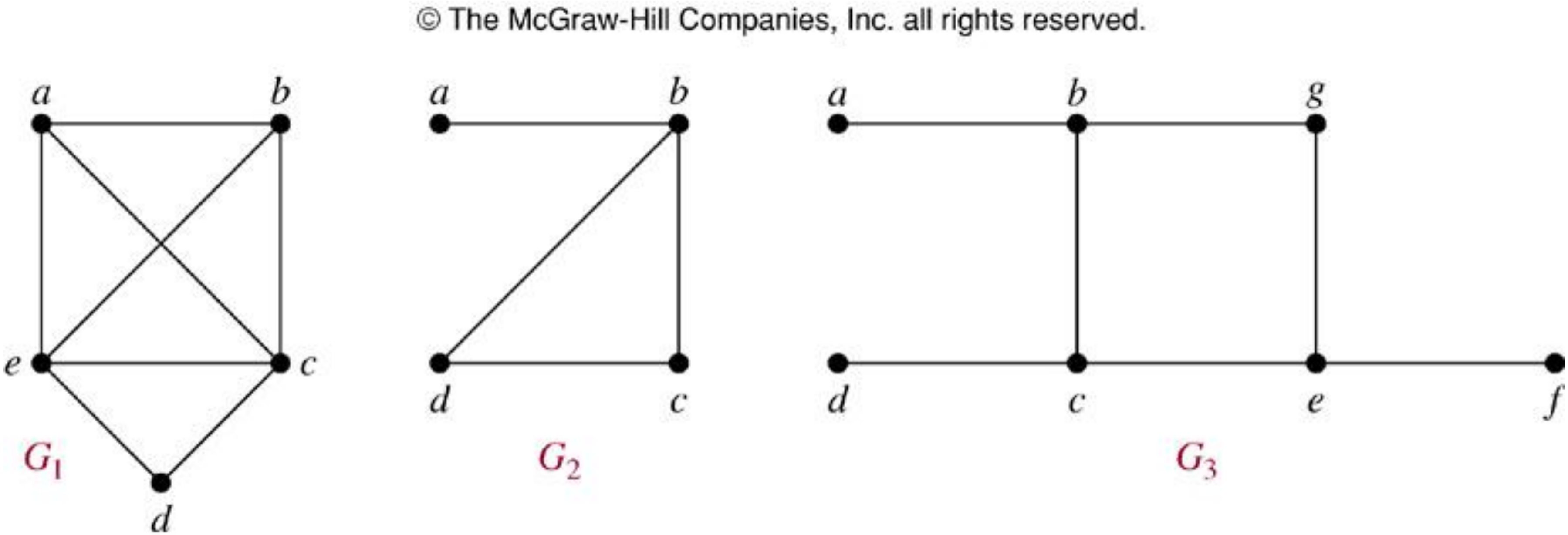
Yes; this is a circuit that passes through each vertex exactly once.

Finding Hamilton Circuits



Which of these three figures has a Hamilton circuit?
Of, if no Hamilton circuit, a Hamilton path?

Finding Hamilton Circuits



- G_1 has a Hamilton circuit: a, b, c, d, e, a
- G_2 does not have a Hamilton circuit, but does have a Hamilton path: a, b, c, d
- G_3 has neither.

Finding Hamilton Circuits

- Unlike the Euler circuit problem, finding Hamilton circuits is hard.
- There is no simple set of necessary and sufficient conditions, and no simple algorithm.

Properties to look for ...

- No vertex of degree 1
- If a node has degree 2, then both edges incident to it must be in any Hamilton circuit.
- No smaller circuits contained in any Hamilton circuit (the start/endpoint of any smaller circuit would have to be visited twice).

A Sufficient Condition

- **DIRAC'S Theorem:** if G is a simple graph with n vertices with n ≥ 3 such that the degree of every vertex $\geq n/2$, then G has a Hamilton circuit.

- **ORE'S Theorem:** if G is a simple graph with n vertices with n ≥ 3 such that deg (u) + deg (v) ≥ n for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.
- **Sufficient means –** both the conditions are not satisfied even then a graph has a hamilton circuit.

Travelling Salesman Problem

A Hamilton circuit or path may be used to solve practical problems that require visiting "vertices", such as:

- road intersections
- pipeline crossings
- communication network nodes

A classic example is the Travelling Salesman Problem – finding a Hamilton circuit in a complete graph such that the total weight of its edges is minimal.

Summary

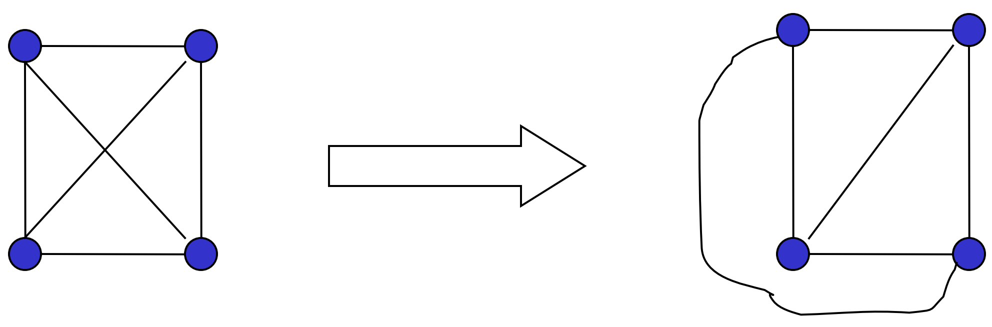
Property	Euler	Hamilton
Repeated visits to a given node allowed?	Yes	No
Repeated traversals of a given edge allowed?	No	No
Omitted nodes allowed?	No	No
Omitted edges allowed?	No	Yes

Planar Graphs

Planar graphs are graphs that can be drawn in the plane without edges having to cross.

Application Example: VLSI design (overlapping edges requires extra layers), Circuit design (cannot overlap wires on board)

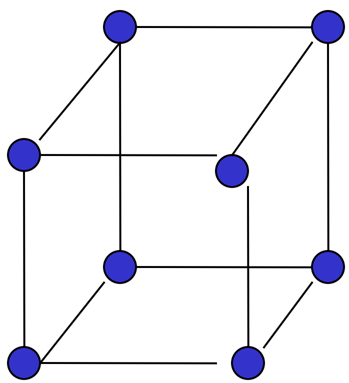
Representation examples: $K1$,$K2$,$K3$,$K4$ are planar, Kn for n>4 are non-planar

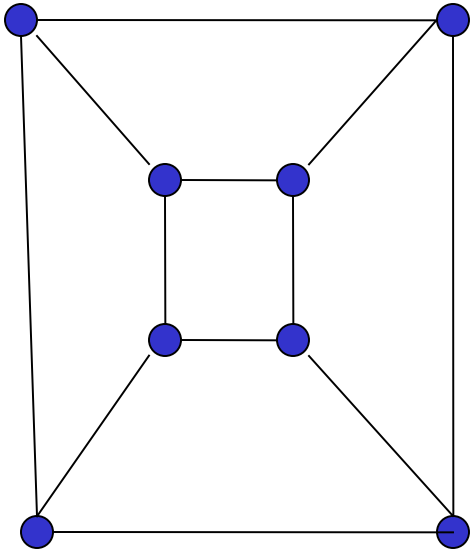


K_4

Planar Graphs

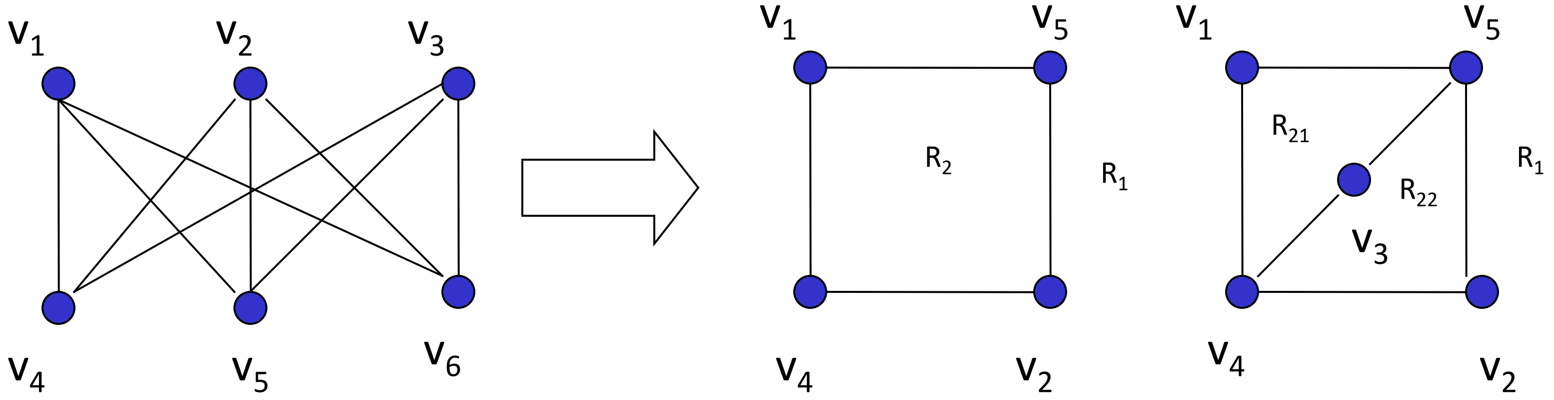
- Representation examples: Q_3





Planar Graphs

- Representation examples: $K_{3,3}$ is Nonplanar



Planar Graphs

Theorem : *Euler's planar graph theorem*

For a **connected** planar graph or multigraph:

$$v - e + r = 2$$

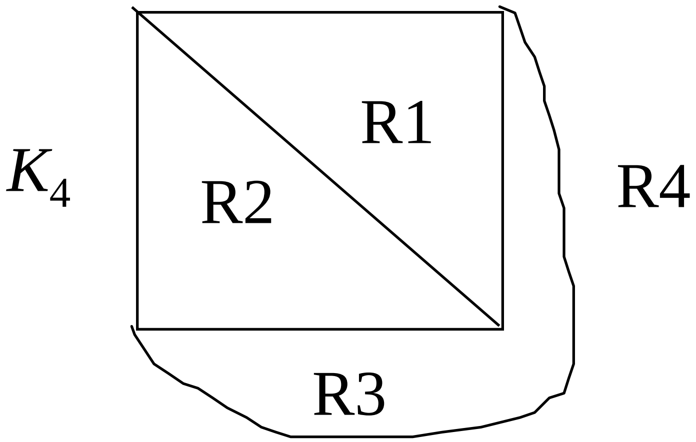
number of vertices

number of edges

number of regions

Planar Graphs

Example of Euler's theorem



A planar graph divides the plane into several regions (faces), one of them is the infinite region.

v=4,e=6,r=4, v-e+r=2

Planar Graphs

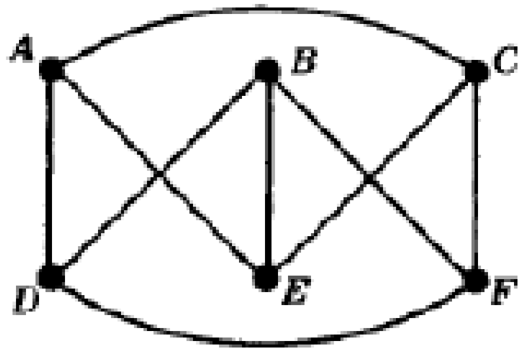
Some inequalities that must be satisfied by planar graphs

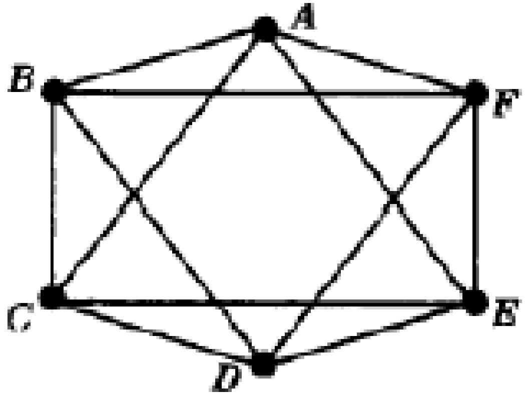
Corollary 1: Let G = (V, E) be a connected simple planar graph with $|V| = v$, $|E|$, and $v \geq 3$. Then $e \leq 3v - 6$

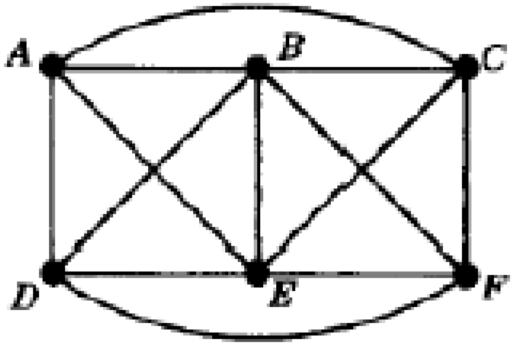
Corollary 2: Let G = (V, E) be a connected simple planar graph then G has a vertex of degree not exceeding 5

Corollary 3: Let G = (V, E) be a connected simple planar graph with v vertices ($v \geq 3$) , e edges, and no circuits of length 3 then $e \leq 2v - 4$

Eg. Planar or not ?







Graph Coloring

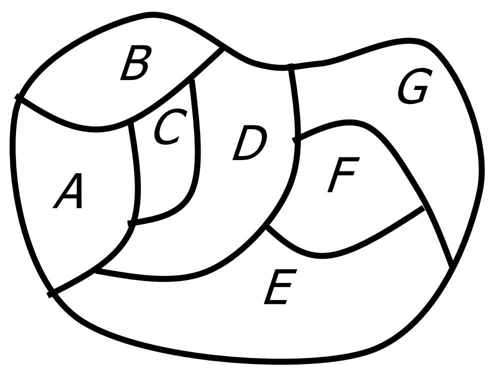
Introduction

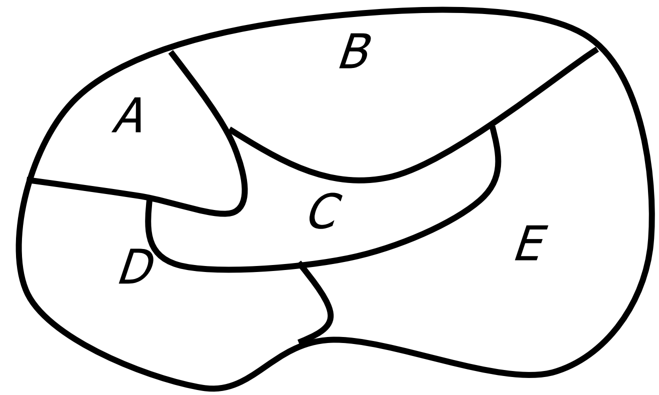
- When a map is colored, two regions with a common border are customarily assigned different colors.
- We want to use a small amount of colors instead of just assigning every region its own color.

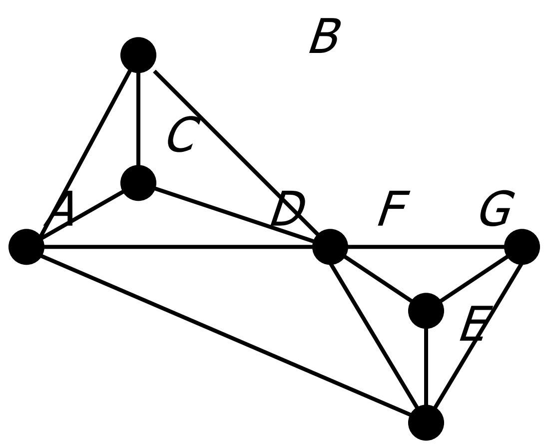
Graph Coloring

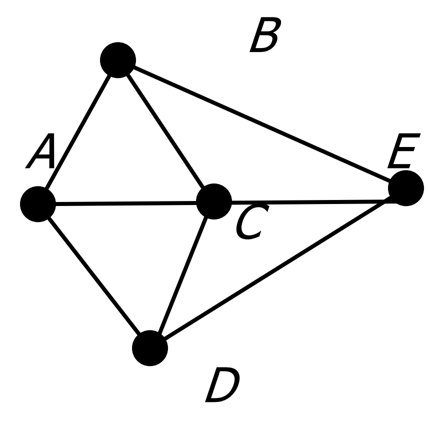
- Each map in a plane can be represented by a graph.
  - Each region is represented by a vertex.
  - Edges connect to vertices if the regions represented by these vertices have a common border.
  - Two regions that touch at only one point are not considered adjacent.
- The resulting graph is called the *dual graph* of the map.

Dual Graph Examples







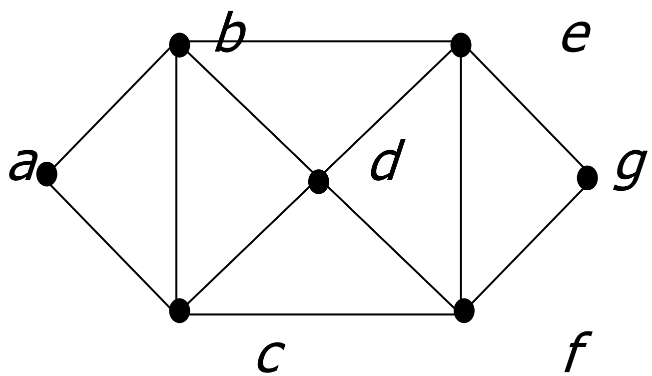


Graph Coloring

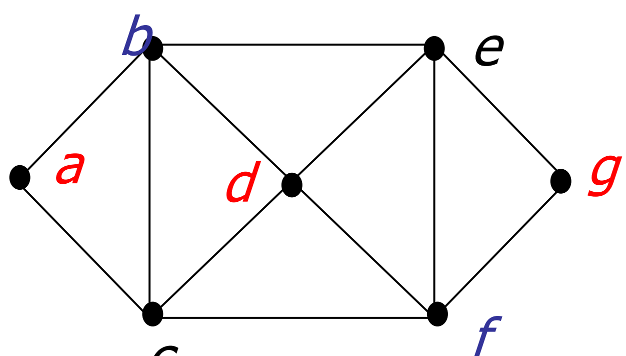
- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
- The *chromatic number* of a graph is the least number of colors needed for a coloring of the graph.
- The Four Color Theorem: *The chromatic number of a planar graph is no greater than four.*

Example

- What is the chromatic number of the graph shown below?



The chromatic number must be at least 3 since *a*, *b*, and *c* must be assigned different colors. So lets try 3 colors first.

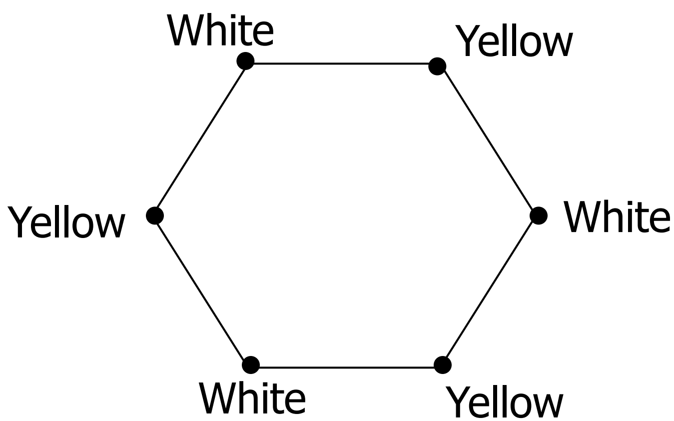


3 colors work, so the chromatic number of this graph is 3.

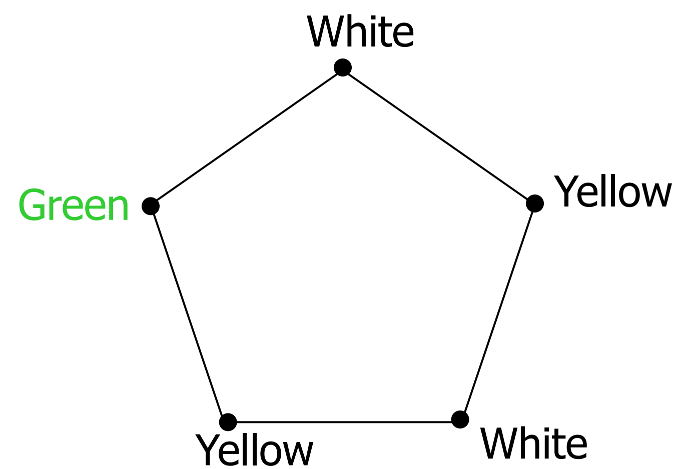
Example

- What is the chromatic number for each graph?

The chromatic number for C_n = 3 (n is odd and greater than 1) or 2 (n is even)



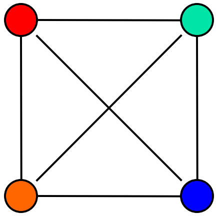
Chromatic number: 2



Chromatic number: 3

Example

- The chromatic number for $K_n = n$,
- Kn: fully connected graph with n vertices

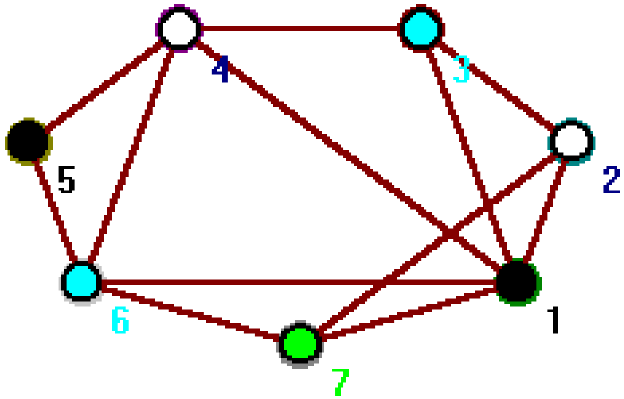


K_4

Application

- e.g. Scheduling Final Exams
- Suppose you want to schedule final exams and, being very considerate, you want to avoid having a student do more than one exam a day. We shall call the courses 1,2,3,4,5,6,7. In the table below a star in entry ij means that course i and j have at least one student in common so you can't have them on the same day. What is the least number of days you need to schedule all the exams? Show how you would schedule the exams.

.	1	2	3	4	5	6	7
1	.	*	*	*	-	*	*
2	*	.	*	-	-	-	*
3	*	*	.	*	-	-	-
4	*	-	*	.	*	*	-
5	-	-	-	*	.	*	-
6	*	-	-	*	*	.	*
7	*	*	-	-	-	*	.

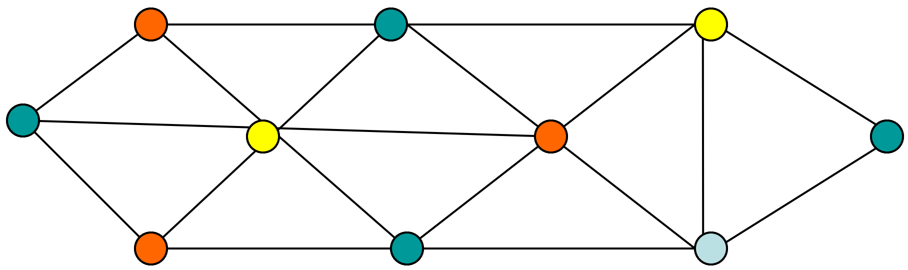


Day	Exam
1	1, 5
2	2, 4
3	3, 6
4	7

Example

- Problem: A state legislature has a number of committees that meet each week for one hour. How can we schedule the committee meetings times such that the least amount of time is used but such that two committees with overlapping membership do not meet at the same time.

Example (cont)



An vertex represents a meeting

An edge represents a conflict between to meetings

The chromatic number of this graph is four. Thus four hours suffice to schedule committee meetings without conflict.