Tutorial-1

1\. (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x-y}{x^2+y^2}$

$x = r\cos\theta$
$y = r\sin\theta$

$$\lim_{r\to 0} \frac{1-r(\cos\theta+\sin\theta)}{r^2} = \lim_{r\to 0} \frac{1}{r^2} - \frac{(\cos\theta+\sin\theta)}{r}$$

As limiting value depends on θ so limit DNE.

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{|xy|}$

$$\lim_{(x,y)\to(0,0^-)} \frac{xy}{-xy} = -1 \qquad \lim_{(x,y)\to(0^-,0)} \frac{xy}{-xy} = -1$$

$$\lim_{(x,y)\to(0^-,0^-)} \frac{xy}{xy} = 1 \qquad \lim_{(x,y)\to(0^+,0)} \frac{xy}{xy} = 1$$

If we approach $(0,0)$ via different path we get different value of limit hence limit DNE.

(c) $\lim\limits_{(x,y)\to(0,0)} \cot^{-1}\dfrac{1}{\sqrt{x^2+y^2}}$

$x = r\cos\theta,\ y = r\sin\theta$

$$\lim_{r\to 0} \cot^{-1}\frac{1}{r} = \lim_{r\to 0} \tan^{-1}|r|$$

$= 0 =$ exists

2\. $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$\frac{x^2+y^2}{|x|+|y|} < \frac{(|x|+|y|)^2}{|x|+|y|} < \frac{4\delta^2}{2\delta} < 2\delta$$

As $\sqrt{x^2+y^2} < \delta$

$x^2+y^2 < \delta^2$

$\Rightarrow |x| < \delta$

or $|y| < \delta$

$2\delta < \varepsilon$

$\delta < \varepsilon/2$

Limit exist

continuous

$$f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - 0}{x}$$

$$= \lim_{x \to 0} \frac{f(x,0)}{x} = \lim_{x \to 0} \frac{x^2}{|x|\,x} = \frac{x}{|x|}$$

$$\lim_{x \to 0^-} \frac{x}{|x|} = -1 \quad \& \quad \lim_{x \to 0^+} \frac{x}{|x|} = 1$$

Limit doesn't exist $\Rightarrow f_x$ doesn't exist.

Similarly

$$f_y(0,0) = \lim_{y \to 0} \frac{y}{|y|}$$

f_y DNE.

5. $$f(x,y) = \begin{cases} \dfrac{xy}{x^2+5y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$y = mx$

$$\lim_{x \to 0} \frac{x(mx)}{x^2 + 5m^2x^2} = \frac{m}{1+5m^2}$$

As limiting value depends on chosen path so limit doesn't exist.

Discontinuous.

$$f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0)}{\Delta x}$$

$$= \lim_{x \to 0} \frac{f(x,0)}{x} = \lim_{x \to 0} \frac{0}{x} = 0 \quad \underline{\text{exists}}$$

Similarly

$$f_y(0,0) = \lim_{y \to 0} \frac{f(0,y)}{y}$$

$= 0$ exists

4. $f(x,y) = \begin{cases} \frac{x^3y}{x^2+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$f_{xy} = \frac{\partial}{\partial x} f_y = \lim_{\Delta x \to 0} \frac{f_y(x+\Delta x, y) - f_y(x,y)}{\Delta x}$ $\quad f_{yx} = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_x(0,0)}{\Delta y}$

$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$

~~$\frac{\partial f_y}{\partial x} = \lim_{\Delta x \to 0} f(x+$~~

$f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{x} = 0$

$f_x(0,\Delta y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, \Delta y) - f(0,\Delta y)}{\Delta x} = \lim_{x \to 0} \frac{f(x,y) - f(0,y)}{x} = \lim_{x \to 0} \frac{\frac{x^3y}{x^2+y} - 0}{x} = 0$

~~$f_y(0,0) = 0 \; \& \; f_y(\Delta x, 0) = 0$~~

~~$f_y(\Delta x$~~ $f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$

$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{y \to 0} \frac{f(0,y)}{y} = 0$

$f_y(\Delta x, 0) = \lim_{\Delta y \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0)}{\Delta y} = \lim_{y \to 0} \frac{f(x,y) - f(x,0)}{y} = \lim_{y \to 0} \frac{\frac{x^3y}{x^2+y} - 0}{y} = x$

$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{x - 0}{x}$

$= 1$

$f_{yx}(0,0) = \lim_{y \to 0} \frac{0 - 0}{y} = 0$

Since $f_{xy} \neq f_{yx}$

so discontinuous

7. (i) $u = \tan^{-1}\frac{x^3+y^3}{x-y}$

$z = f(x,y) = \frac{x^3+y^3}{x-y}$

$f(\lambda x, \lambda y) = \frac{\lambda^3(x^3+y^3)}{\lambda(x-y)}$

degree = 2

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$x\frac{\partial \tan u}{\partial x} + y\frac{\partial \tan u}{\partial y} = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u$$

$$= \sin 2u$$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u - 1)\sin 2u$

as in (i) part $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

diff w.r.t. x

$$x \times \left(\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x}\right) \quad \text{—(1)}$$

diff w.r.t. y

$$y\left(x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u\frac{\partial u}{\partial y}\right) \quad (2)$$

Add (1) + (2)

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\underbrace{\qquad\qquad\qquad}_{} = 2\cos 2u\sin 2u - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}$$

$$= (2\cos 2u - 1)\sin 2u$$

8. $u = x^2 - y^2 + \sin yz \quad y = e^x \quad z = \log x$

M-1

$u = x^2 - e^{2x} + \sin e^x \log x$

$\frac{du}{dx} = 2x - 2e^{2x} + \cos e^x \log x \left(\frac{e^x}{x} + \log x \, e^x \right)$

M-2

$u \to (x, y, z) \to (x, ~~\ldots~~)$

$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}$

$\frac{du}{dx} = 2x + e^x\left(-2y + (\cos yz) z\right) + (\cos yz) y \frac{1}{x}$

9. $u = f(x-y, y-z, z-x)$

$x - y = p$
$y - z = q$
$z - x = r$

$u \to (p, q, r) \to (x, y, z)$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \times 0 + \frac{\partial u}{\partial r}(-1) = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad -①$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$

$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q} + 0 = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \quad -②$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \quad -③$

Add ① & ② & ③

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

$$u+v = x+y$$
$$u^2-v^2 = -(3x+y)$$

$$2u^2 = -2x \qquad 2v^2 = 4x+2y$$
$$u^2 = -x \qquad v^2 = 2x+y$$

$z = \cancel{-x(2x+y)}$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$

$= \cancel{-2x^2-xy}$

$\frac{\partial z}{\partial x} = \cancel{-4x-y}$ $\frac{2u^2-v^2}{2uv}$

11- notes

12. $u = x^2-y^2,\ v = 2xy$

$f \to (u,v) \to (x,y)$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial x} = 2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}(-2y) + \frac{\partial f}{\partial v}\,2x$$

$$\frac{\partial f}{\partial y} = 2x\frac{\partial f}{\partial v} - 2y\frac{\partial f}{\partial u}$$

$\cancel{\phi \to (x,y) \to}$ $\phi \to (x,y) \to (u,v)$

—— 2 —— 13 ——