Boolean Algebra

Boolean Algebra

•Boolean algebra provides the operations and the rules for working with the set **{0, 1}.**

•These are the rules that underlie **electronic circuits**, and the methods we will discuss are fundamental to **VLSI design**.

•We are going to focus on three operations:

- Boolean complementation,
- Boolean sum (denoted by + , same as disjunction), and
- Boolean product (denoted by . , same as conjunction)

Boolean Algebra

$x \vee 0 = x$ $x \wedge 1 = x$	Identity laws
$x \vee \neg x = 1$ $x \wedge \neg x = 0$	Complement laws
$(x \vee y) \vee z = x \vee (y \vee z)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$	Associative laws
$x \vee y = y \vee x$ $x \wedge y = y \wedge x$	Commutative laws
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	Distributive laws

Examples

- Find the value of

$1.0 + \overline{(0+1)}$ Ans : 0

- Translate $1.0 + \overline{(0+1)} = 0$ into a logical equivalence Ans : $(T \wedge F) \vee \neg(F \vee T) \equiv F$
- Translate the logical equivalence

$(T \wedge T) \vee \neg F \equiv T$ into an identity in Boolean algebra Ans : $(1.1) + \overline{0} = 1$

Boolean Functions and Expressions

- **Definition:** Let B = {0, 1}. The variable x is called a **Boolean variable** if it assumes values only from B.
- A function from B^n, the set $\{(x_1, x_2, \ldots, x_n) \mid x_i \in B, 1 \leq i \leq n\}$, to B is called a **Boolean function of degree n**.
- Boolean functions can be represented using expressions made up from the variables and Boolean operations.

Boolean Functions and Expressions

- The **Boolean expressions** in the variables $x_1, x_2, \ldots, x_n$ are defined **recursively** as follows:

- $0, 1, x_1, x_2, \ldots, x_n$ are Boolean expressions.
- If E_1 and E_2 are Boolean expressions, then $(\sim E_1)$, (E_1E_2), and $(E_1 + E_2)$ are Boolean expressions.

- Each Boolean expression represents a Boolean function.

- The **values** of this function are obtained by substituting 0 and 1 for the variables in the expression.

Boolean Functions and Expressions

- **Example:** Give a Boolean expression for the Boolean function F(x, y) as defined by the following table:

x	y	F(x, y)
0	0	0
0	1	1
1	0	0
1	1	0

Possible solution: F(x, y) = (~x).y

Boolean Functions and Expressions

x	y	z	F(x, y, z)
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Possible solution I:

F(x, y, z) = -(xz + y)

Possible solution II:

F(x, y, z) = (-(xz))(-y)

Boolean Functions and Expressions

- **Definition:** The Boolean functions F and G of n variables are **equal** if and only if $F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$ whenever $b_1, b_2, \ldots, b_n$ belong to B.
- Two different Boolean expressions that represent the same function are called **equivalent**.
- For example, the Boolean expressions xy, $xy + 0$, and $xy \cdot 1$ are equivalent.

Boolean Functions and Expressions

- The **complement** of the Boolean function F is the function –F, where $-F(b_1, b_2, \ldots, b_n) =$ $-(F(b_1, b_2, \ldots, b_n))$.
- Let F and G be Boolean functions of degree n. The **Boolean sum F+G** and **Boolean product FG** are then defined by
- $(F + G)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n) + G(b_1, b_2, \ldots, b_n)$
- $(FG)(b_1, b_2, \ldots, b_n) = F(b_1, b_2, \ldots, b_n)\, G(b_1, b_2, \ldots, b_n)$

Boolean Functions

Example 1:

Evaluate the following expression when $\mathbf{A} = \mathbf{1}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \mathbf{1}$

$$\mathbf{F} = \mathbf{C} + \overline{\mathbf{C}}\mathbf{B} + \mathbf{B}\overline{\mathbf{A}}$$

- Solution

$$\mathbf{F} = \mathbf{1} + \overline{\mathbf{1}} \cdot \mathbf{0} + \mathbf{0} \cdot \overline{\mathbf{1}} = \mathbf{1} + \mathbf{0} + \mathbf{0} = \mathbf{1}$$

Example 2:

Evaluate the following expression when $\mathbf{A} = \mathbf{0}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \mathbf{1}, \mathbf{D} = \mathbf{1}$

$$\mathbf{F} = \mathbf{D}(\mathbf{B}\overline{\mathbf{C}}\mathbf{A} + \overline{(\mathbf{A}\overline{\mathbf{B}} + \mathbf{C})} + \mathbf{C})$$

Solution

$$\mathbf{F} = \mathbf{1} \cdot (\mathbf{0} \cdot \overline{\mathbf{1}} \cdot \mathbf{0} + \overline{(\mathbf{0} \cdot \overline{\mathbf{0}} + \mathbf{1})} + \mathbf{1}) = \mathbf{1} \cdot (\mathbf{0} + \overline{\mathbf{1}} + \mathbf{1}) = \mathbf{1} \cdot \mathbf{1} = \mathbf{1}$$

Boolean Expressions and Boolean Functions

Question: How many different Boolean functions of degree 2 are there? 16

A	B	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		0 Null	AB	Inhibition	A	Inhibition	B	$A \oplus B$	$A + B$	$\overline{A + B}$	$\overline{A \oplus B}$	$\overline{B}$	Implication	$\overline{A}$	Implication	$\overline{AB}$	1 Identity

Boolean Functions and Expressions

- **Question:** How many different Boolean functions of degree 1 are there?
- **Solution:** There are four of them, F_1, F_2, F_3, and F_4:

x	F_1	F_2	F_3	F_4
0	0	0	1	1
1	0	1	0	1

Boolean Functions and Expressions

- **Question:** How many different Boolean functions of degree n are there?
- **Solution:**
- There are 2^n different n-tuples of 0s and 1s.
- A Boolean function is an assignment of 0 or 1 to each of these 2^n different n-tuples.
- Therefore, there are $\mathbf{2^{2^n}}$ different Boolean functions.
- Since the number of rows of a truth table with *n* Boolean variables is 2^n, and each row can get one of two values (true or false), the number of such truth tables is $\mathbf{2^{2^n}}$.

Representing Boolean Functions

Any Boolean function can be represented as a :

- **Sum of products (SOP) of variables and their complements.**

Disjunctive normal form (DNF)

Sum-of-products Expansions

$$F(A, B, C, D) = AB + \bar{B}C\bar{D} + AD$$

Or

- **Product of sums (POS) of variables and their complements.**

Conjunctive normal form (CNF)

Product-of-sums Expansions

$$F(A, B, C, D) = (A + B)(\bar{B} + C + \bar{D})(A + D)$$