Department of Mathematics

15B11MA211 **Mathematics-2**

Tutorial Sheet 5 **B.Tech. Core**

Partial Differential Equations and Their Applications

1. Classify the following equations :

i) $\frac{\partial^2 u}{\partial x^2}+9\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 u}{\partial t^2}-u=0$ ii) $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=0$

iii) $x\frac{\partial^2 u}{\partial x^2}-(x+t)\frac{\partial^2 u}{\partial x\partial t}+t\frac{\partial^2 u}{\partial t^2}=\frac{\partial u}{\partial t}$ iv) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

2. Solve the following equations by the method of separation of variables.

i) $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$ ii) $3\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial t}=0$, where $u(x,0)=4e^{-x}$

iii) $4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3u$, given $u(0,y)=4e^{-y}-e^{-5y}$

iv) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, where $u(0,y)=u(l,y)=u(x,0)=0$ *and* $u(x,a)=\sin\left(\frac{n\pi x}{l}\right)$.

3. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $u=a\sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by $u(x,t)=a\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$.

4. A tightly stretched string with fixed end points $x=0$ and $x=l$, is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity *(a)* $\lambda x(l-x)$, *(b)* $y=y_0\sin^3\left(\frac{\pi x}{l}\right)$ find the displacement of the string at any distance x from one end at any time t.

5. Solve the partial differential equation $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod without radiation, subject to the following conditions:

(i) u is not infinite for $t\to\infty$ (ii) $\left.\frac{\partial u}{\partial x}=0\right|_{x=0,\,l}$ (iii) $u=lx-x^2$ for t = 0, between x = 0, x = l.

6. An insulated rod of length l has its ends A and B maintained at 0^0 C and 100^0 C respectively until steady state conditions prevail. If B is suddenly reduced to 0^0 C and maintained at 0^0 C, find the temperature at the distance x from A at time t. Solve the above problem if the change consists of raising the temperature of A to 20^0 C and reducing that of B to 80^0 C.

7. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge at y = 0 is given by $u(x,0)=\begin{cases}5x, & 0<x\le 5\\ 5(10-x) & 5\le x<10\end{cases}$

and the two long edges x = 0, x = 10 as well as the short edge at infinity are kept at $0^0 C$, prove that the steady state temperature distribution at any point (x,y) is given by

$$u(x,y)=\frac{200}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{10}\right)e^{-\frac{(2n-1)\pi y}{10}}.$$

Sol: 2. (i) $z(x,y)=\left[c_1 e^{\left(1+\sqrt{1+k}\right)x}+c_2 e^{\left(1-\sqrt{1+k}\right)x}\right].c_3 e^{-ky}$ (ii) $u(x,y)=4e^{\frac{1}{2}(3y-2x)}$ (iii) $u(x,y)=3e^{x-y}-e^{2x-5y}$

(iv) $u(x,y)=\sin\left(\dfrac{n\pi x}{l}\right)\left[\dfrac{\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi a}{l}}\right]$

4. (*a*) $y(x,t)=\dfrac{8\lambda l^3}{c\pi^4}\sum_{m=1}^{\infty}\dfrac{1}{\left(2m-1\right)^4}\sin\dfrac{(2m-1)\pi ct}{l}\sin\dfrac{(2m-1)\pi x}{l}$

(*b*) $y(x,t)=\dfrac{y_0}{4}\left(3\sin\dfrac{\pi x}{l}\cos\dfrac{\pi ct}{l}-\sin\dfrac{3\pi x}{l}\cos\dfrac{3\pi ct}{l}\right)$

5. $u(x,t)=\dfrac{l^2}{6}-\dfrac{l^2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos\dfrac{2n\pi x}{l}e^{\frac{-4n^2\pi^2kt}{l^2}}$

6. $u(x,t)=\dfrac{200}{\pi}\sum_{n=1}^{\infty}\dfrac{\left(-1\right)^{n+1}}{n}\sin\dfrac{n\pi x}{l}e^{-\frac{c^2n^2\pi^2t}{l^2}}$; $u(x,t)=20+\dfrac{60}{l}x-\dfrac{40}{\pi}\sum_{m=1}^{\infty}\dfrac{1}{m}\sin\dfrac{2m\pi x}{l}e^{-\frac{4c^2m^2\pi^2t}{l^2}}$