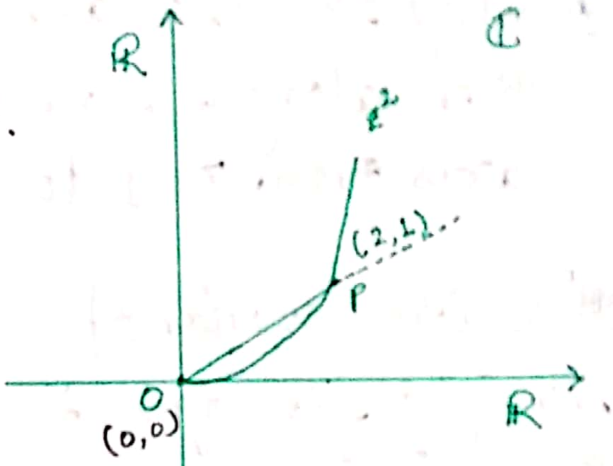
- **Example:** Evaluate $\int_C z^2 dz$, where C is the straight line joining the origin O to the point $P(2,1)$ in the complex plane.



- **Sol^n:** The equation of the line OP is: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

$$y = \frac{1-0}{2-0}(x-0)$$

$$\Rightarrow y = \frac{x}{2} \quad \text{OR} \quad 2y = x$$

$\therefore$ The line OP is

$$x = 2y \;;\; 0 \le y \le 1. \Rightarrow dx = 2dy$$

Thus, $\boxed{dz = dx + i\,dy = 2dy + i\,dy = (2+i)dy.}$

Also, $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy = (x^2-y^2) + i\,2xy.$

$$\Rightarrow z^2 = (2y)^2 - y^2 + i\,2(2y)\cdot y$$

$$\Rightarrow \boxed{z^2 = 4y^2 - y^2 + 4iy^2 = 3y^2 + 4y^2 i}$$

(Purpose of doing all this is to make our integrand in just one variable either in x or y).

Now, we have

$$\int_C z^2 dz = \int_0^1 (3y^2 + 4y^2 \cdot i)(2+i)dy = (2+11i)\int_0^1 y^2 dy$$

$$= \frac{(2+11i)}{3} \quad \#$$

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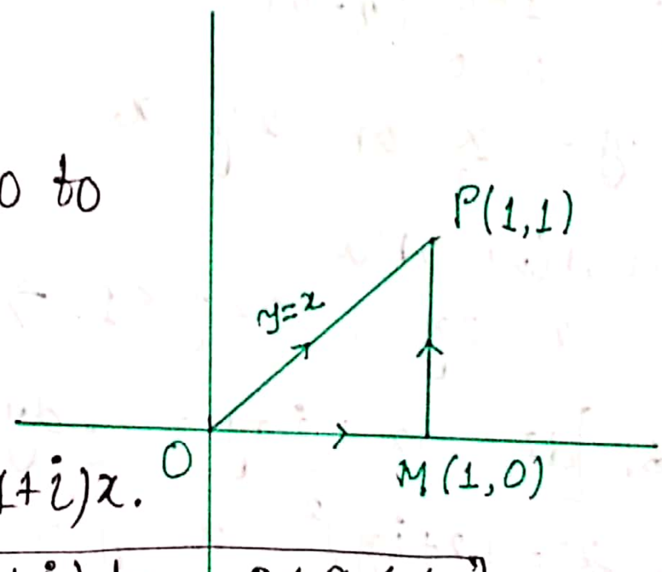
Example: Evaluate the integral $\int_0^{1+i} (x-y+ix^2)\,dz$

(a) along a straight line from $z=0$ to $z=1+i$

(b) along the real axis from $z=0$ to $z=1$ and then along a line parallel to imaginary axis from $z=1$ to $z=1+i$.

Sol^n: (a) The equation of straight line from $z=0$ to $z=1+i$ is $y=x$.



Thus along the line OP, $z = x+iy = x+ix = (1+i)x$.

which gives $\boxed{dz = (1+i)dx. \quad 0 \le x \le 1.}$

and hence

$$\int_0^{1+i} (x-y+ix^2)\,dz = \int_0^1 (x-x+ix^2)(1+i)\,dx$$

$$= (1+i)\,i \int_0^1 x^2\,dx$$

$$= \frac{i(1+i)}{3} = -\frac{1}{3}(1-i) \quad \#$$

(b) Along the path OM, we have $y=0$ and thus $z = x+iy = x+i(y=0) = x$.

$$\Rightarrow \boxed{dz = dx.; \quad 0 \le x \le 1.}$$

- Also along the path MP, we have $x=1$, and thus

$$z = x+iy = 1+iy$$

$$\Rightarrow \boxed{dz = i\,dy\ ;\quad 0 \le y \le 1.}$$

$\therefore$ The line integral

$$\int_0^{1+i} (x-y+ix^2)\,dz = \underbrace{\int_0^1 (x+ix^2)\,dx}_{\substack{\text{Line OM} \\ \text{Put } y=0 \text{ in } (x-y+ix^2) \\ =(x+ix^2)}} + \underbrace{\int_0^1 (1-y+i)(i\,dy)}_{\substack{\text{Line MP} \\ \text{Similarly.}}}.$$

$$\Rightarrow \int_0^{1+i} (x-y+ix^2)\,dz = \int_0^1 (x+ix^2)\,dx + \int_0^1 (1-y+i)(i\,dy).$$

$$= \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1 + \left[(i-1)y - \frac{iy^2}{2}\right]_0^1$$

$$= \frac{1}{2} + \frac{i}{3} + (i-1) - \frac{i}{2}$$

$$= -\frac{1}{2} + \frac{5}{6}i = \left(\frac{5i}{6} - \frac{1}{2}\right) \#$$

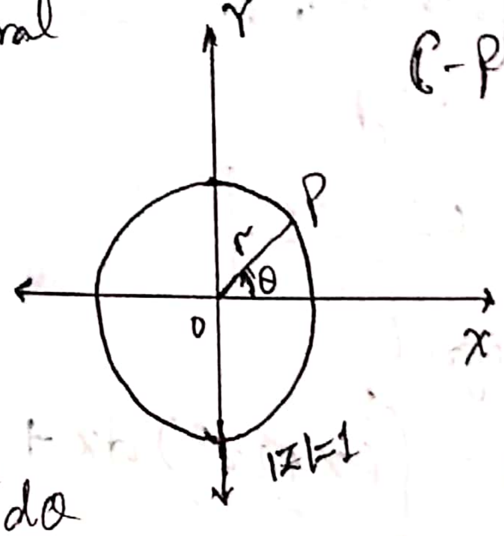
Example: Evaluate $\oint_C \ln z\,dz$, where C is the unit circle $|z|=1$ taken in counter clockwise sense.

Solution: $|z|=1$ represents a circle in complex plane with center at $(0,0)$ and radius as 1 unit.

In parametric form $z = re^{i\theta}$; $0 \le \theta \le 2\pi$

$$\Rightarrow dz = ie^{i\theta}\,d\theta.$$

Thus, the line integral becomes



ℂ-plane.

$$\oint_C \ln z\, dz$$

$$= \int_0^{2\pi} \ln e^{i\theta} \cdot dz$$

$$= \int_0^{2\pi} \ln e^{i\theta} \cdot i e^{i\theta}\, d\theta$$

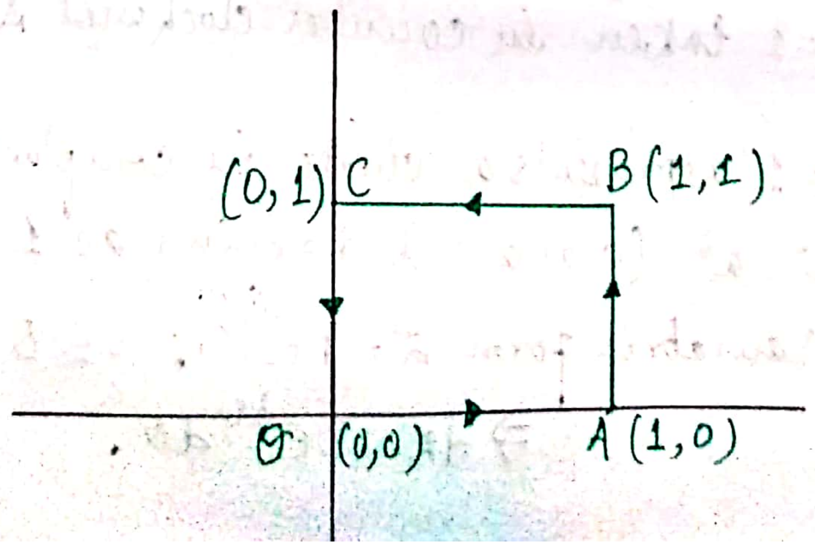
$$= \int_0^{2\pi} i\theta \cdot i e^{i\theta}\, d\theta = -\int_0^{2\pi} \theta e^{i\theta}\, d\theta$$ (By parts Integrating)

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} - \frac{1}{i} \cdot \frac{e^{i\theta}}{i}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{2\pi i}}{i} + e^{2\pi i} - 1\right] = -\frac{2\pi}{i} \times \frac{i}{i}$$ Multiplying / Divided by i

$$= 2\pi i \quad \#$$

Example: Evalute $\oint_C |z|^2 dz$ around the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.



Solution: The contour of integration C is OABCO as in figure.

Here, we have $|z|^2 = (x^2+y^2)$ and also along:

OA: $y=0$; $0 \le x \le 1$; $dz = dx$, $|z|^2 = x^2$

AB: $x=1$; $0 \le y \le 1$; $dz = i\,dy$, $|z|^2 = 1+y^2$

BC: $y=1$; x goes from 1 to 0; $dz = dx$; $|z|^2 = 1+\cancel{y^2}\,x^2$

CO: $x=0$; y goes from 1 to 0; $dz = i\,dy$; $|z|^2 = y^2$

Thus,

$$\oint_C |z|^2 dz = \int_{OA} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{CO} |z|^2 dz$$

$$= \int_0^1 x^2\,dx + i\int_0^1 (1+y^2)\,dy + \int_1^0 \cancel{(1+y^2)}\,(1+x^2)\,dx + i\int_1^0 y^2\,dy$$

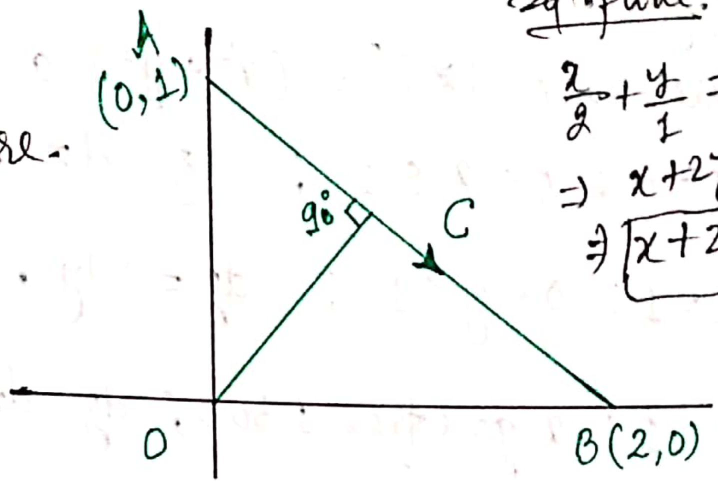
$$= \left.\frac{x^3}{3}\right|_0^1 + i\left.\left(y+\frac{y^3}{3}\right)\right|_0^1 + \left.\left(x+\frac{x^3}{3}\right)\right|_1^0 + i\left.\frac{y^3}{3}\right|_1^0$$

$$= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = (-1+i) \text{ or } (i-1) \quad \#$$

Example: Find an upper bound to the integral $I = \int_C \frac{e^z}{z^2}\,dz$, where C is the straight line from (0, 1) to (2, 0) in the complex plane.

Solution:

The path C is the line segment AB as shown in figure.



Eqn of line:

$$\frac{x}{2}+\frac{y}{1}=1$$
$$\Rightarrow x+2y=2$$
$$\Rightarrow \boxed{x+2y-2=0} \quad 'C'$$

Now, Consider

$$|f(z)| = \left|\frac{e^z}{z^2}\right| = \left|\cancel{\frac{e^{iy+x}}{(x+iy)^2}}\right| = \cancel{e^{(x+iy)}}$$

$$\le \frac{|e^z|}{|z|^2} \le \frac{|e^x|\cdot|e^{iy}|}{x^2+y^2}$$

$$|f(z)| \le \frac{e^x}{x^2+y^2} \quad \text{——} \quad (*)$$

On C, e^x is maximum at $x=2$, so max $e^x = e^2$.

Next the minimum value of x^2+y^2 on C is the square OP, the perpendicular distance from O to the line AB. given by $x+2y-2=0$. This is $(2/\sqrt{5})^2 = 4/5$.

Thus, from (*) we have,

$$\cancel{|f(z)| \le \frac{4e^2}{(5/4)} = 4e^2}$$

$$|f(z)| \le \frac{e^2}{(4/5)} = \frac{5e^2}{4}.$$

The length of C is $|AB| = \sqrt{5}$.

Using ML-inequality, we have

$$\left| \int_C \frac{e^z}{z^2}\, dz \right| \leq \frac{5e^2}{4}(\sqrt{5}) = 20.65. \#$$