Assignment - 2

1.) $3.\frac{\partial u}{\partial x} + 2.\frac{\partial u}{\partial y} = 0$ & $u(x,0) = 4.e^{(-x)}$

$\Rightarrow$ Assuming $u(x,y) = X(x).Y(y)$

$$\frac{\partial u}{\partial x} = Y.\frac{dX}{dx}$$

$$\frac{\partial u}{\partial y} = X.\frac{dY}{dy}$$

i.e., $3Y.\frac{dX}{dx} + 2X.\frac{dY}{dy} = 0$

$$3Y.X' + 2X.Y' = 0$$

$$\frac{3X'}{X} = \frac{-2Y'}{Y} = k \quad \text{(Say)}$$

$$\frac{X'}{X} = \frac{k}{3} \quad \& \quad \frac{Y'}{Y} = \frac{(-k)}{2}$$

$$\log X = \frac{kx}{3} + \log c_1 \quad \& \quad \log Y = \frac{-ky}{2} + \log C_2$$

$$X = c_1.e^{kx/3} \quad \& \quad Y = C_2.e^{-ky/2}$$

$$\Rightarrow u(x,y) = c_1.c_2.e^{k(\frac{x}{3}-\frac{y}{2})}$$

As, $u(x,0) = 4.e^{-x}$

$$c_1.c_2.e^{kx/3} = 4.e^{-x}$$

$\Rightarrow \boxed{c_1.c_2 = 4}$ & $\frac{k}{3} = (-1) \Rightarrow \boxed{k = (-3)}$

$$\therefore \boxed{u(x,y) = 4.e^{(\frac{3y}{2}-x)}}$$

2.) Eq^n of Displacement of any point is ;

$$\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2} \quad ; \quad c \rightarrow \text{Constant}$$

Now,

Using Method of Variable Separation,

Let $y(x,t) = X(x) \cdot T(t)$

$$\frac{\partial^2 y}{\partial t^2} = X \cdot \frac{d^2 T}{dt^2}$$

& $$\frac{\partial^2 y}{\partial x^2} = T \cdot \frac{d^2 X}{dx^2}$$

$$\Rightarrow X \cdot \frac{d^2 T}{dt^2} = c^2 \cdot T \cdot \frac{d^2 X}{dx^2}$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = k$$

Case-1:- If $k = 0$,

$$\frac{1}{X} \cdot \frac{d^2 X}{dx^2} = 0 \Rightarrow X(x) = c_1 x + c_2$$

& $$\frac{1}{c^2 \cdot T} \cdot \frac{d^2 T}{dt^2} = 0 \Rightarrow T(t) = c_3 t + c_4$$

$$\Rightarrow y(x,t) = (c_1 x + c_2)(c_3 t + c_4)$$

Case-2:- If $k = p^2$ (i.e. Positive)

$$\frac{1}{X} \cdot \frac{d^2 X}{dx^2} = p^2 \Rightarrow \underbrace{\frac{d^2 X}{dx^2} - p^2 X = 0}_{(p,\,-p)} \Rightarrow X(x) = c_1 \cdot e^{px} + c_2 e^{-px}$$

$$\frac{1}{c^2 \cdot T}\frac{d^2T}{dt^2} = p^2 \Rightarrow \frac{d^2T}{dt^2} - p^2c^2 \cdot T = 0 \Rightarrow T(t) = c_3 \cdot e^{pct} + c_4 \cdot e^{-pct}$$

$$\hookrightarrow (pc, -pc)$$

$$\Rightarrow y(x,t) = (c_1 \cdot e^{px} + c_2 \cdot e^{-px})(c_3 \cdot e^{pct} + c_4 \cdot e^{-pct})$$

Case-3: If $k = -p^2$ (i.e., Negative)

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0 \rightarrow \text{Imaginary Roots}$$

$$\Rightarrow X(x) = c_1 \cos px + c_2 \sin px$$

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = -p^2 \Rightarrow \frac{d^2T}{dt^2} + p^2c^2T = 0 \rightarrow \text{Imaginary Roots}$$

$$\Rightarrow T(t) = c_3 \cos pct + c_4 \sin pct$$

$$\Rightarrow y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$$

As, at $t=0$, $y(x,0) = a \sin(\pi x/10)$

So, among the above 3 cases, Case-3 is consistent with the Equation given at $t=0$. Also, the motion of the string will be periodic.

$$\therefore y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$$

At $t=0$,

$$y(x,0) = (c_1 \cos px + c_2 \sin px)(c_3 \times 1 + 0)$$
$$= c_1c_3 \cos px + c_2c_3 \sin px$$

Let the fastened pt.'s be $x = 0$ & $x = 10\,cm$

$\Rightarrow \quad y(0,t) = 0$

& $y(10,t) = 0$

$$y(0,t) = c_1(c_3 \cos pct + c_4 \sin pct) = 0$$

$$\Rightarrow \boxed{c_1 = 0}$$

$$y(10,t) = c_2 \sin(10p)\left(c_3 \cos pct + c_4 \sin pct\right) = 0$$

$$y(x,0) = c_3(c_1 \cos px + c_2 \sin px) = a \cdot \sin\left(\frac{\pi x}{10}\right)$$

$$\Rightarrow c_2 \cdot c_3 \cdot \sin px = a \sin\left(\frac{\pi x}{10}\right)$$

$$\therefore \boxed{p = \frac{\pi}{10}} \quad \& \quad \boxed{c_2 c_3 = a}$$

$$\therefore y(x,t) = c_2 \sin\left(\frac{\pi x}{10}\right)\left[\frac{a}{c_2} \cos \frac{\pi ct}{10} + c_4 \sin \frac{\pi ct}{10}\right]$$

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