Department of Mathematics

15B11MA211 | Mathematics-2

Tutorial Sheet 5 | B.Tech. Core

## Partial Differential Equations and Their Applications

1. Classify the following equations :

   i) $\dfrac{\partial^2 u}{\partial x^2}+9\dfrac{\partial^2 u}{\partial x\partial t}+\dfrac{\partial^2 u}{\partial t^2}-u=0$ ii) $\dfrac{\partial^2 z}{\partial x^2}-5\dfrac{\partial z}{\partial x}+x\dfrac{\partial^2 z}{\partial y^2}+\dfrac{\partial z}{\partial y}=0$

   iii) $x\dfrac{\partial^2 u}{\partial x^2}-(x+t)\dfrac{\partial^2 u}{\partial x\partial t}+t\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial u}{\partial t}$ iv) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

2. Solve the following equations by the method of separation of variables.

   i) $\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=0$ ii) $3\dfrac{\partial u}{\partial x}+2\dfrac{\partial u}{\partial t}=0$, where $u(x,0)=4e^{-x}$

   iii) $4\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=3u$, given $u(0,y)=4e^{-3y}-e^{-5y}$

   iv) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, where $u(0,y)=u(l,y)=u(x,0)=0$ *and* $u(x,a)=\sin\left(\dfrac{n\pi x}{l}\right)$.

3. A string is stretched and fastened to two points $l$ apart. Motion is started by displacing the string in the form $u=a\sin\dfrac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a distance $x$ from one end at time $t$ is given by $u(x,t)=a\sin\dfrac{\pi x}{l}\cos\dfrac{\pi ct}{l}$.

4. A tightly stretched string with fixed end points $x=0$ and $x=l$, is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity *(a)* $\lambda x(l-x)$, *(b)* $y=y_0\sin^3\left(\dfrac{\pi x}{l}\right)$ find the displacement of the string at any distance $x$ from one end at any time t.

5. Solve the partial differential equation $\dfrac{\partial u}{\partial t}=\alpha^2\dfrac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod without radiation, subject to the following conditions:

   (i) $u$ is not infinite for $t\to\infty$ (ii) $\left.\dfrac{\partial u}{\partial x}=0\right|_{x=0,\,l}$ (iii) $u=lx-x^2$ for $t=0$, between $x=0, x=l$.

6. An insulated rod of length $l$ has its ends $A$ and $B$ maintained at $0^0$ C and $100^0$ C respectively until steady state conditions prevail. If $B$ is suddenly reduced to $0^0$ C and maintained at $0^0$ C, find the temperature at the distance $x$ from $A$ at time $t$. Solve the above problem if the change consists of raising the temperature of $A$ to $20^0$ C and reducing that of $B$ to $80^0$ C.

7. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge at $y=0$ is given by $u(x,0)=\begin{cases}5x, & 0<x\le 5\\ 5(10-x) & 5\le x<10\end{cases}$

   and the two long edges $x=0, x=10$ as well as the short edge at infinity are kept at $0^0 C$, prove that the steady state temperature distribution at any point $(x,y)$ is given by

   $$u(x,y)=\frac{200}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{10}\right)e^{-\frac{(2n-1)\pi y}{10}}.$$

*Sol:* 2. (i) $z(x,y)=\left[c_1 e^{\left(1+\sqrt{1-k}\right)x}+c_2 e^{\left(1-\sqrt{1-k}\right)x}\right].c_3 e^{-ky}$ (ii) $u(x,y)=4e^{\frac{1}{2}(3y-2x)}$ (iii) $u(x,y)=4e^{x-y}-e^{2x-5y}$

(iv) $u(x,y)=\sin\left(\frac{n\pi x}{l}\right)\left[\frac{\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi a}{l}}\right]$

4. (a) $y(x,t)=\frac{8\lambda l^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi ct}{l}\sin\frac{(2m-1)\pi x}{l}$

(b) $y(x,t)=\frac{y_0}{4}\left(3\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}-\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}\right)$

5. $u(x,t)=\frac{l^2}{6}-\frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{l}e^{\frac{-4n^2\pi^2\alpha^2 t}{l^2}}$

6. $u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{l}e^{\frac{c^2n^2\pi^2 t}{l^2}}$; $u(x,t)=20+\frac{60}{l}x-\frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}e^{\frac{4c^2m^2\pi^2 t}{l^2}}$

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# Tutorial sheet 5

consider $A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0$

parabolic if $B^2 - 4AC = 0$

hyperbolic if $B^2 - 4AC > 0$

elliptic if $B^2 - 4AC < 0$

1) i) $\frac{\partial^2 u}{\partial x^2} + 9\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} - u = 0$

$A = 1, \quad B = 9, \quad C = 1$

$B^2 - 4AC = 5 > 0$ hyperbolic

ii) $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = 0$

$A = 1, \quad B = 0, \quad C = x$

$$B^2 - 4AC = -4x \begin{cases} = 0, & x = 0 \quad \text{parabolic} \\ > 0, & x < 0 \quad \text{hyperbolic} \\ < 0, & x > 0 \quad \text{elliptic} \end{cases}$$

iii) $x\frac{\partial^2 u}{\partial x^2} - (x+t)\frac{\partial^2 u}{\partial x \partial t} + t\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial t}$

$A = x, \quad B = -(x+t), \quad C = t$

$B^2 - 4AC = (x+t)^2 - 4xt$

$$= (x-t)^2 \begin{cases} = 0, & x = t \quad \text{parabolic} \\ > 0, & x \neq t \quad \text{hyperbolic} \end{cases}$$

iv) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$A = 1, \quad B = 0, \quad C = 1$

$B^2 - 4AC = -4 < 0$ elliptic.

2) (i) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

$Z = X(x)\,Y(y)$

$\frac{\partial z}{\partial x} = X'Y$ $\qquad \frac{\partial^2 z}{\partial x^2} = X''Y$

$\frac{\partial z}{\partial y} = XY'$

$X''Y - 2X'Y + XY' = 0$

$(X'' - 2X')Y = -XY'$

$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = k^2$ or $-k^2$

If constt $= k^2$

$$X'' - 2X' - k^2X = 0$$

$$m^2 - 2m - k^2 = 0$$

$$m = \frac{2 \pm \sqrt{4+4k^2}}{2}$$

$$= 1 \pm \sqrt{1+k^2}$$

$$X(x) = C_1 e^{(1+\sqrt{1+k^2})x} + C_2 e^{(1-\sqrt{1+k^2})x}$$

$$Y' + Yk^2 = 0$$

$$m = -k^2$$

$$Y(y) = C_3 e^{-k^2 y}$$

$$Z = \left(C_1 e^{(1+\sqrt{1+k^2})x} + C_2 e^{(1-\sqrt{1+k^2})x}\right) C_3 e^{-k^2 y}$$

If constt $= -k^2$

$$X'' - 2X' + k^2 X = 0$$

$$m^2 - 2m + k^2 = 0$$

$$m = \frac{2 \pm \sqrt{4-4k^2}}{2}$$

$$= 1 \pm \sqrt{1-k^2}$$

$$X(x) = C_1 e^{(1+\sqrt{1-k^2})x} + C_2 e^{(1-\sqrt{1-k^2})x}$$

$$Y' - Yk^2 = 0$$

$$m = k^2$$

$$Y(y) = C_3 e^{k^2 y}$$

$$Z = \left[C_1 e^{(1+\sqrt{1-k^2})x} + C_2 e^{(1-\sqrt{1-k^2})x}\right] C_3 e^{k^2 y}$$

~~$k \neq 1, -1$.~~

~~If $k = 1, -1$~~ i.e

(ii) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial t} = 0$ $\qquad u(x,0) = 4e^{-x}$

$$3X'T + 2XT' = 0$$

$$3X'T = -2XT'$$

$$\frac{3X'}{X} = -2\frac{T'}{T} = \lambda$$

$$\frac{X'}{X} = \frac{\lambda}{3} \qquad \frac{T'}{T} = \frac{-\lambda}{2}$$

$$X = C_1 e^{\lambda/3 x} \qquad T = C_2 e^{-\lambda/2 t}$$

$$u(x,t) = C_1 C_2 e^{(\lambda/3 x - \lambda/2 t)}$$

$$4e^{-x} = u(x,0) = C_1 C_2 e^{\lambda/3 x}$$

$$\therefore -1 = \lambda/3 \Rightarrow \lambda = -3$$

$$C_1 C_2 = 4$$

$$u(x,t) = 4e^{(-x + 3/2 t)}$$

(iii) $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \qquad u(0,y) = 4e^{-y} - e^{-5y}$

$4X'Y + XY' = 3XY$

$(4X' - 3X)Y = -XY'$

$\frac{4X' - 3X}{X} = \frac{-Y'}{Y}$

$4\frac{X'}{X} - 3 = -\frac{Y'}{Y} = \lambda$

$\frac{X'}{X} = \frac{\lambda+3}{4} \qquad \frac{Y'}{Y} = -\lambda$

$X(x) = C_1 e^{\frac{\lambda+3}{4}x} \qquad Y(y) = C_2 e^{-\lambda y}$

$u(x,y) = C_1C_2 e^{\frac{\lambda+3}{4}x} e^{-\lambda y}$

$u(0,y) = C_1C_2 e^{-\lambda y} = 4e^{-y} - e^{-5y}$

$C_1C_2 = 4$ or $C_1C_2 = -1$

$\lambda = 1 \qquad \lambda = 5$

$u(x,y) = 4e^{x-y} - e^{2x-5y}$

(iv) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u(0,y) = u(l,y) = u(x,0) = 0$

$u(x,a) = \sin\left(\frac{n\pi x}{l}\right)$

$X''Y + XY'' = 0$

$X''Y = -XY''$

Case 1 $\frac{X''}{X} = -\frac{Y''}{Y} = k^2$

$k^2 > 0$

$X'' - k^2X = 0 \qquad Y'' + k^2Y = 0$

$m^2 - k^2 = 0 \qquad m^2 + k^2 = 0$

$m = \pm k \qquad m = \pm ki$

$X(x) = C_1e^{kx} + C_2e^{-kx}$

$Y(y) = C_3\cos kx + C_4\sin kx$

$u(x,y) = (C_1e^{kx} + C_2e^{-kx})$
$(C_3\cos ky + C_4\sin ky)$

Case 2 $\frac{X''}{X} = -\frac{Y''}{Y} = -k^2$

$k^2 > 0$

$u(x,y) = (C_5\cos kx + C_6\sin kx)$
$(C_7e^{ky} + C_8e^{-ky})$

Case 3 $k^2 = 0$

$X'' = 0$

$X(x) = C_9x + C_{10}$

$Y'' = 0 \Rightarrow Y(y) = C_{11}y + C_{12}$

$u(x,y) = (C_9x + C_{10})(C_{11}y + C_{12})$

For Case 3

$u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

$0 = u(0,y) = c_{10}(c_{11} y + c_{12})$

$\Rightarrow c_{10} = 0$.

$0 = u(l,y) = (c_9 l)(c_{11} y + c_{12})$

$c_9 = 0$.

trivial sol

For case 1

$u(x,y) = (c_1 e^{kx} + c_2 e^{-kx})(c_3 \cos ky + c_4 \sin ky)$ $k \neq 0$

$0 = u(0,y) = (c_1 + c_2)(c_3 \cos ky + c_4 \sin ky)$

$\Rightarrow c_1 = -c_2$.

$0 = u(l,y) = (c_1 e^{kl} - c_1 e^{-kl})(c_3 \cos ky + c_4 \sin ky)$

$e^{kl} = e^{-kl}$ or $c_1 = 0$

Not true trivial sol

For Case 2

$u(x,y) = (c_1 \cos kx + c_2 \sin kx)(c_3 e^{ky} + c_4 e^{-ky})$

$0 = u(0,y) = c_1(c_3 e^{ky} + c_4 e^{-ky})$

$\Rightarrow c_1 = 0$

$0 = u(l,y) = (c_2 \sin kl)(c_3 e^{ky} + c_4 e^{-ky})$

$c_2 \neq 0$.

$\sin kl = 0 = \sin n\pi$

$k = n\pi/l$

$0 = u(x,0) = \left[c_2 \sin\left(\frac{n\pi x}{l}\right)\right](c_3 + c_4)$

$c_3 = -c_4$.

$u(x,y) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 e^{\frac{n\pi y}{l}} - c_3 e^{-\frac{n\pi y}{l}}\right]$

$\sin\left(\frac{n\pi x}{l}\right) = u(x,a) = c_2 c_3 \sin\frac{n\pi x}{l}\left[e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right]$

$c_2 c_3 = \frac{1}{2}\frac{2}{e^{\frac{n\pi a}{l}} - e^{-n\pi a/l}} = \frac{1}{2\sinh(n\pi a/l)}$

3)

$u(x,y)$

$$u(x,y) = \frac{1}{2\sinh\left(\frac{n\pi a}{l}\right)} \sin\left(\frac{n\pi x}{l}\right)\left[x \sinh\left(\frac{n\pi y}{l}\right)\right]$$

3) $u(0,t) = u(l,t) = \frac{\partial u}{\partial t}(x,0) = 0$

$u(x,0) = a \sin\left(\frac{\pi x}{l}\right)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \ t > 0$$

$u(x,t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ckt + c_4 \sin ckt)$

$0 = u(0,t) = c_1 (c_3 \cos ckt + c_4 \sin ckt)$

$\Rightarrow c_1 = 0$

$0 = u(l,t) = c_2 \sin kl \ (c_3 \cos ckt + c_4 \sin ckt)$

$c_2 \neq 0 \qquad \sin kl = 0 = \sin n\pi$

$k = \frac{n\pi}{l}$

$$\frac{\partial u}{\partial t} = (c_1 \cos kx + c_2 \sin kx)(-ck\, c_3 \sin ckt + ck\, c_4 \cos ckt)$$

$$0 = \frac{\partial u}{\partial t}(x,0) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[-\frac{n\pi c}{l} c_3 \underbrace{\sin\frac{n\pi ct}{l}}_{0} + \frac{n\pi c}{l} c_4 \cos\frac{n\pi ct}{l}\right]$$

$$= \underbrace{c_2 \sin\frac{n\pi x}{l}}_{\neq 0} \cdot \underbrace{\frac{n\pi c}{l}}_{\neq 0} \cdot c_4$$

$\Rightarrow c_4 = 0$

$$u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 \cos\frac{n\pi ct}{l}\right]$$

$$u(x,0) = c_2 c_3 \sin\frac{n\pi x}{l} = a \sin\left(\frac{\pi x}{l}\right)$$

$\Rightarrow c_2 c_3 = a$

$n = 1$

$$\therefore u(x,t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi ct}{l}\right)$$

4) a) $\frac{\partial u}{\partial t}(x,0) = \lambda x(l-x)$

$u(0,t) = u(l,t) = u(x,0) = 0$

$u(x,t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos kct + c_4 \sin k$

$u(0,t) = 0$

$\Rightarrow c_1 = 0$

$u(l,t) = 0$

$\Rightarrow k = n\pi/l$

$0 = u(x,0) = c_2 \sin\left(\frac{n\pi x}{l}\right)[c_3]$

$\Rightarrow c_3 = 0$

$u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right) c_4 \sin\left(\frac{n\pi ct}{l}\right)$

$= \sum b_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi ct}{l}\right)$

$\frac{\partial u}{\partial t} = \sum b_n \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi c}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$

$\frac{\partial u}{\partial t}(x,0) = \lambda x(l-x) = \sum b_n \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right)$

$b_n \frac{n\pi c}{l} = \frac{2}{l}\int_0^l \lambda x(l-x) \sin\left(\frac{n\pi x}{l}\right) dx$

$b_n \frac{n\pi c}{2\lambda} = \int_0^l \underset{I}{(lx - x^2)} \underset{II}{\sin\left(\frac{n\pi x}{l}\right)} dx$

I. $= -(lx - x^2)\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\Big|_0^l$

$+ \int_0^l (l-2x)\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right) dx$

$= \frac{l}{n\pi}\Big[(l-2x)\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\Big|_0^l$

$+ \int_0^l \frac{2l}{n\pi}\sin\left(\frac{n\pi x}{l}\right) dx\Big]$

$= -\frac{2l^2}{n^2\pi^2}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^l = \frac{-2l^3}{(n\pi)^3}\left((-1)^n - 1\right)$

$b_n \dfrac{n\pi c}{2\lambda} = \dfrac{2l^3}{(n\pi)^3}(1-(-1)^n)$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

$b_n = \dfrac{4\lambda l^3}{(n\pi)^4 c}(1-(-1)^n)$

$= \begin{cases} 0, & n \text{ is even} \\ \dfrac{8\lambda l^3}{(n\pi)^4 c}, & n \text{ is odd} \end{cases}$

$u(x,t) = \dfrac{8\lambda l^3}{\pi^4 c} \sum_{m=1}^{\infty} \dfrac{1}{(2m-1)^4} \sin\left(\dfrac{(2m-1)\pi x}{l}\right)\sin\left(\dfrac{(2m-1)\pi ct}{l}\right)$

b) $y(x,0) = y_0 \sin^3\left(\dfrac{\pi x}{l}\right) = y_0\left[\dfrac{3}{4}\sin\left(\dfrac{\pi x}{l}\right) - \dfrac{1}{4}\sin\left(\dfrac{3\pi x}{l}\right)\right]$

$y(0,t) = y(l,t) = \dfrac{\partial y}{\partial t}(x,0) = 0$

$y(x,t) = (c_1\cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$y(0,t) = 0 \Rightarrow c_1 = 0$

$y(l,t) = 0 \Rightarrow p = \dfrac{n\pi}{l}$

$\dfrac{\partial y}{\partial t} = c_2 \sin\left(\dfrac{n\pi x}{l}\right)\left[-\dfrac{n\pi c}{l} c_3 \sin\left(\dfrac{n\pi ct}{l}\right) + \dfrac{n\pi c}{l} c_4 \cos\left(\dfrac{n\pi ct}{l}\right)\right]$

$0 = \dfrac{\partial y}{\partial t}(x,0) = c_2 \sin\left(\dfrac{n\pi x}{l}\right)\left(\dfrac{n\pi c}{l}\right)(-c_4)$

$\Rightarrow c_4 = 0$

$y(x,t) = c_2 \sin\left(\dfrac{n\pi x}{l}\right)\left[c_3 \cos \dfrac{n\pi ct}{l}\right]$

$= \sum b_n \sin\left(\dfrac{n\pi x}{l}\right)\cos\left(\dfrac{n\pi ct}{l}\right)$

$y_0\left[\dfrac{3}{4}\sin\left(\dfrac{\pi x}{l}\right) - \dfrac{1}{4}\sin\left(\dfrac{3\pi x}{l}\right)\right] = \sum b_n \sin\left(\dfrac{n\pi x}{l}\right)$

$n=1 \Rightarrow b_1 = \dfrac{3y_0}{4}$, $n=3 \Rightarrow b_3 = -\dfrac{y_0}{4}$, $b_n = 0 \; \forall n$ except 1 & 3

$$y(x,t) = \frac{3y_0}{4}\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right) - \frac{y_0}{4}\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{3\pi c}{l}\right.$$

5)
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

i) $u \not\to \infty$ when $t \to \infty$

ii) $\left.\frac{\partial u}{\partial x} = 0\right|_{x=0,l}$

iii) $u = lx - x^2$ for $t=0$, $0<x<l$

$$XT' = \alpha^2 X''T$$

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = \lambda$$

$T' - \alpha^2\lambda T = 0$ $\qquad\qquad$ $X'' - \lambda X = 0$.

$$T(t) = C_3 e^{\alpha^2\lambda t}$$

Case 1. $\lambda = 0$

$T' = 0$ $\qquad\qquad$ $X'' = 0$

$T(t) = C_3$ $\qquad\qquad$ $X(x) = C_1 x + C_2$.

$$u(x,t) = C_1C_3 x + C_2C_3.$$

It satisfies condition (i)

$$\frac{\partial u}{\partial x} = C_1C_3.$$

$$\frac{\partial u}{\partial x}(0,t) = C_1C_3 = 0$$

$\Rightarrow C_1 = 0$ or $C_3 = 0$

$\Rightarrow C_1 = 0$

$$\frac{\partial u}{\partial x}(l,t) = 0 = C_1C_3 l + C_2C_3.$$

$\Rightarrow C_2C_3 = 0$

$C_3 = 0$ or $C_2 = 0$.

$u(x,t) = 0$.

Case 2 $\lambda = k^2$

$T(t) = C_3 e^{\alpha^2k^2t}$ $\qquad\qquad$ $X(x) = C_1 e^{kx} + C_3 e^{-kx}$

$$u(x,t) = (C_1 e^{kx} + C_2 e^{-kx})(C_3 e^{\alpha^2k^2t})$$

At $t \to \infty$

$u \to \infty$

$\left(\frac{3\pi x}{l}\right) \cos\left(\frac{3\pi x}{l}\right)$

Case 3 $\quad \lambda = -k^2$

$T(t) = c_3 e^{-\alpha^2 k^2 t}$

$X(x) = c_1 \cos kx + c_2 \sin kx$

$u(x,t) = (c_1 \cos kx + c_2 \sin kx)(c_3 e^{-\alpha^2 k^2 t})$

At $t \to \infty$

$u \to 0 \neq \infty$. cond (i) is satisfied

$$\frac{\partial u}{\partial x} = k(-c_1 \sin kx + c_2 \cos kx)(c_3 e^{-\alpha^2 k^2 t})$$

$$\frac{\partial u}{\partial x}(0,t) = k(c_2)(c_3 e^{-\alpha^2 k^2 t}) \qquad k \neq 0, \; c_3 \neq 0$$

$$\Rightarrow c_2 = 0.$$

$$\frac{\partial u}{\partial x}(l,t) = (\underbrace{-k}_{\neq 0} \underbrace{c_1}_{\neq 0} \sin kl)(\underbrace{c_3 e^{-\alpha^2 k^2 t}}_{\neq 0})$$

$$\sin kl = 0 = \sin n\pi$$

$$k = \frac{n\pi}{l}$$

$$\therefore u(x,t) = c_1 c_3 \cos \frac{n\pi x}{l} e^{-(\alpha^2 n^2 \pi^2 / l^2) t}$$

$$lx - x^2 = u(x,0) = c_1 c_3 \cos\left(\frac{n\pi x}{l}\right)$$

$$= \sum b_n \cos\left(\frac{n\pi x}{l}\right), \quad 0 < x < l.$$

$$b_0 = \frac{2}{l}\int_0^l (lx - x^2)\,dx$$

$$= \frac{2}{l}\left[\frac{l}{2}x^2 - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right] = \frac{2}{l} l^3 \left[\frac{1}{6}\right] = \frac{l^2}{3}. \quad \Rightarrow \frac{b_0}{2} = \frac{l^2}{6}.$$

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi x}{l}\right) dx.$$

$$= \frac{2}{l}\left[(lx - x^2)\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)\Big|_0^l - \int_0^l \frac{l}{n\pi}(l - 2x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{-l}{n\pi}\right]\left[-(l-2x)\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\Big|_0^l + \int_0^l \frac{l}{n\pi}(-2)\cos\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{-2}{n\pi}\left[\frac{l^2}{n\pi}(-1)^n + \frac{l^2}{n\pi} - \frac{2l^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{l}\right)\Big|_0^l\right]$$

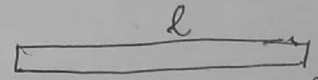
$$b_n = \frac{-2l^2}{(n\pi)^2}\left((-1)^n + 1\right)$$

$$= \begin{cases} 0 & , \quad n \text{ is odd} \\ -\dfrac{4l^2}{n^2\pi^2} & \quad n \text{ is even} \\ & \quad \underline{\underline{n=2m}} \end{cases}$$

$$u(x,t) = \frac{l^2}{6} - \frac{4l^2}{\pi^2} \sum \frac{1}{4n^2} \cos\frac{2n\pi x}{l} \, e^{-\left(\frac{4\alpha^2 n^2\pi^2}{l^2}\right)t}$$

$$= \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{l}\right) e^{-\left(\frac{4\alpha^2 n^2\pi^2}{l^2}\right)t}$$

6) until steady state cond.




$0°c$ $\quad$ $100°C$

steady state

$\downarrow$ $\quad$ $\downarrow$

$20°c$ $\quad$ $80°C$.

transient + steady.

$u_t(x,t) + u_s(x)$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow u = ax + b.$$

$$u(0) = 0, \quad u(l) = 100.$$

$$0 = u(0) = b$$

$$\Rightarrow b = 0$$

$$100 = u(l) = al.$$

$$a = \frac{100}{l}$$

$$\therefore u = \frac{100x}{l}.$$

$$\boxed{\Rightarrow u(x,0) = \frac{100x}{l}}$$

6) Now $\quad u_t(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 e^{-c^2p^2t})$

$$\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u_s(x) = ax + b$$

$$20 = u_s(0) = b.$$

$$80 = u_s(l) = al + 20$$

$$al = 60 \quad \Rightarrow a = \frac{60}{l}.$$

$$\Rightarrow u_s(x) = \frac{60x}{l} + 20.$$

Now $\quad u(x,t) = u_t(x,t) + u_s(x)$

$$20 = u(0,t) = u_t(0,t) + 20 \Rightarrow u_t(0,t) = 0$$

$$80 = u(l,t) = u_t(l,t) + u_s(l)$$
$$= u_t(l,t) + 80 .$$
$$u_t(l,t) = 0 .$$

$$u_t(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 e^{-c^2p^2t})$$

$$0 = u_t(0,t) = c_1 (c_3 e^{-c^2p^2t})$$
$$\Rightarrow c_1 = 0 .$$

$$0 = u_t(l,t) = (c_2 \sin pl)(c_3 e^{-c^2p^2t}).$$
$$\Rightarrow \sin pL = 0 = \sin n\pi$$
$$\Rightarrow p = \frac{n\pi}{l}$$

$$u_t(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right) e^{-c^2n^2\pi^2/l^2 t}$$
$$= \Sigma a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2c^2\pi^2}{l^2}\right)t}$$

$$u(x,t) = u_s(x) + u_t(x,t)$$
$$= \frac{60}{l}x + 20 + \Sigma a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2c^2\pi^2}{l^2}t}$$

Now $u(x,0) = \frac{100x}{l} = \frac{60}{l}x + 20 + \Sigma a_n \sin\left(\frac{n\pi x}{l}\right)$

$$\Rightarrow \Sigma a_n \sin\left(\frac{n\pi x}{l}\right) = \frac{40x}{l} - 20 .$$

$$\Rightarrow a_n = \frac{2}{l}\int_0^l \left(\frac{40}{l}x - 20\right) \sin\left(\frac{n\pi x}{l}\right) dx .$$

$$= \frac{2}{l}\left[-\left(\frac{40x}{l} - 20\right) \frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right)\Big|_0^l + \int_0^l \left(\frac{40}{l}\right)\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right) dx\right]$$

$$= \frac{2}{l}\left[-\frac{20l}{n\pi}(-1)^n - \frac{20l}{n\pi} + \frac{40l}{(n\pi)^2} \sin\frac{n\pi x}{l}\Big|_0^l\right]$$

$$= -\frac{40}{n\pi}\left[(-1)^n + 1\right]$$

$$= \begin{cases} 0 & , n \text{ is odd} \\ -\frac{80}{n\pi} & , n \text{ is even} \end{cases}$$

$$u(x,t) = 20 + \frac{60x}{l} + \sum_{m=1}^{\infty} \frac{-80}{2m\pi} \sin\left(\frac{2m\pi x}{l}\right) e^{-\frac{4m^2c^2\pi^2 t}{l^2}}$$

$$= 20 + \frac{60x}{l} - \frac{40}{\pi}\sum_{m=1}^{\infty} \frac{1}{m}\sin\left(\frac{2m\pi x}{l}\right) e^{-\frac{4m^2c^2\pi^2 t}{l^2}}$$

a) $u(x,0) = \frac{100x}{l}$

Also $u(0,t) = 0$
$u(l,t) = 0$ $\quad \forall t$

$$u(x,t) = (c_1\cos px + c_2 \sin px)\, e^{-c^2p^2t}\, c_3$$

$0 = u(0,t) \Rightarrow c_1 = 0$.

$0 = u(l,t) \Rightarrow p = \frac{n\pi}{l}$.

$$u(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

$$\frac{100x}{l} = u(x,0) = \sum b_n \sin\frac{n\pi x}{l}.$$

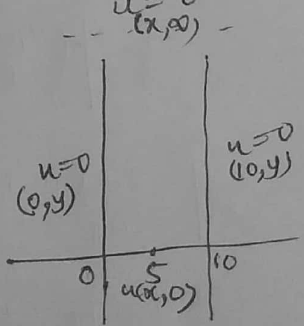
$$b_n = \frac{2}{l}\int_0^l \frac{100}{l} x \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{200}{l^2}\left[-x\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l.$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi}(-1)^n\right] = (-1)^{n+1}\frac{200}{n\pi}.$$

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{cn\pi}{l}\right)^2 t}.$$

7) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$




Case 1

$u(x,y) = (c_9 x + c_{10})(c_{11}y + c_{12})$

$0 = u(0,y) = c_{10}(c_{11}y + c_{12})$
$\Rightarrow c_{10} = 0$.

$0 = u(10,y) = 10c_9(c_{11}y + c_{12}) \Rightarrow c_9 = 0$

trivial sol.

Case 2 $u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

$0 = u(0,y) = (c_1 + c_2)(c_3 \cos py + c_4 \sin py)$

$\Rightarrow c_1 = -c_2$

$0 = u(10,y) = (c_1 e^{10p} + c_2 e^{-10p})(c_3 \cos py + c_4 \sin py)$

$\Rightarrow c_1 e^{10p} = c_1 e^{-10p}$

$\Rightarrow c_1 = 0 = c_2$.

trivial sol.

Case 3 $u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

$0 = u(0,y) \Rightarrow c_1 = 0$.

$0 = u(10,y) \Rightarrow 10p = n\pi$

$p = \frac{n\pi}{10}$.

$0 = u(x,\infty) = \lim_{y\to\infty} c_2 \sin\left(\frac{n\pi x}{10}\right) c_3 e^{py}$

$\Rightarrow c_3 = 0$.

$u(x,y) = c_2 c_4 \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi y}{10}}$

$= \Sigma b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi y}{10}}$.

$f(x) = u(x,0) = \Sigma b_n \sin\left(\frac{n\pi x}{10}\right)$

$b_n = \frac{2}{\cancel{10}\ 5} \int_0^5 \cancel{5}x \sin\left(\frac{n\pi x}{10}\right) dx$

$+ \frac{2\times 5}{10} \int_5^{10} (10-x) \sin\left(\frac{n\pi x}{10}\right) dx$.

$\int x \sin\left(\frac{n\pi x}{10}\right) dx = -x \times \frac{10}{n\pi} \cos\left(\frac{n\pi x}{10}\right) + \frac{100}{(n\pi)^2} \sin\frac{n\pi x}{10}$

$b_n = \left[-\frac{10x}{n\pi} \cos\left(\frac{n\pi x}{10}\right) + \frac{100}{n^2\pi^2} \sin\left(\frac{n\pi x}{10}\right)\right]_0^5$

$+ \left[\frac{10\times 10}{n\pi} \cos \frac{n\pi x}{10}\right]\Big|_5^{10} - \left[-\frac{10x}{n\pi} \cos\left(\frac{n\pi x}{10}\right) + \frac{100}{n^2\pi^2} \sin\left(\frac{n\pi x}{10}\right)\right]_5^{10}$

$$= \frac{-50}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{-100}{n\pi}\left[\cos \cancel{n}\pi - \cos\frac{n\pi}{2}\right] - \left[\frac{-100}{n\pi}\cancel{\cos n\pi}\right]$$

$$+ \left(\frac{-50}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \cos\left(\frac{n\pi}{2}\right)\left[\cancel{\frac{-50}{n\pi}} + \cancel{\frac{100}{n\pi}} - \cancel{\frac{50}{n\pi}}\right]$$

$$+ \frac{200}{n^2\pi^2} \sin\frac{n\pi}{2}.$$

$\cos\frac{n\pi}{2} = 0$ n odd

$n=$

$$= \begin{cases} 0 & n = \text{even} \ (\cancel{\text{odd}}) \\ \frac{200}{n^2\pi^2} & n = 1, 5, 9, \ldots \\ -\frac{200}{n^2\pi^2} & n = 3, 7, 11, \ldots \end{cases}$$

$$\frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{10}\right) e^{-\frac{(2n-1)\pi y}{10}}$$