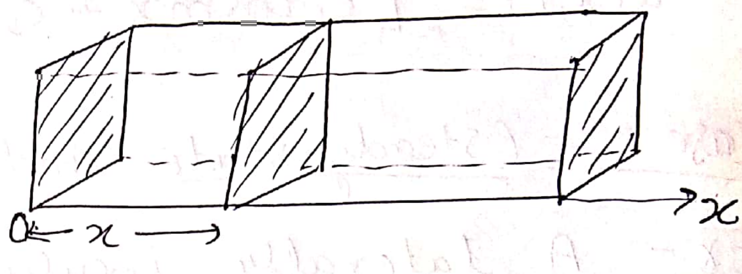
# One-dimensional heat equation

Consider a homogeneous bar of uniform cross-sectional area A and density $\rho$ placed along x-axis with one end at the origin O.




Let us assume that the bar is insulated laterally and therefore heat flows only in the x-direction.

Let $u(x,t)$ be the temperature at a distance x from O.

The one-dimensional heat flow eqn is,

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}} \quad \text{--- (1)}$$

By method of separation of variables, let

$$u(x,t) = X(x)\, T(t)$$

Using in (1), $\quad XT' = c^2 X'' T$

or, $\quad \dfrac{X''}{X} = \dfrac{T'}{c^2 T} = k \quad$ (let)

Case I – If $k = m^2$ (positive)

$\Rightarrow X(x) = C_1 e^{mx} + C_2 e^{-mx}$, $\quad T(t) = C_3 e^{c^2 m^2 t}$

Case II – If $k = -m^2$ (negative)

$\Rightarrow X(x) = C_4 \cos mx + C_5 \sin mx$, $\quad T(t) = C_6 e^{-c^2 m^2 t}$

Case III – If $k = 0$

$\Rightarrow \quad X(x) = C_7 x + C_8$, $\quad T(t) = C_9$

Out of these 3 solutions, a solution is chosen which is consistent with the physical nature of the problem. The temperature $u(x,t)$ must decrease with the increase of time. So only possible solⁿ is case II.

$$u(x,t) = [C_1 \cos mx + C_2 \sin mx] e^{-m^2c^2t}$$

Case I – (Steady State and zero boundary conditions)

Ex – A laterally insulated bar of length $l$ has its ends A and B maintained at 0°c and 100°c respectively until steady-state conditions prevail. If the temperature at B is suddenly reduced to 0°c and kept so while that of A is maintained at 0°c, find the temperature at a distance $x$ from A at any time $t$.

Solⁿ – Steady-State Condition – A condition is known as steady-state if the dependent variables are independent of time $t$.

$$\Rightarrow \quad \frac{\partial u}{\partial t} = 0 \qquad -(1) \quad [\text{At } t=0]$$

Heat ~~condition~~ Conduction eqⁿ is,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -(2)$$

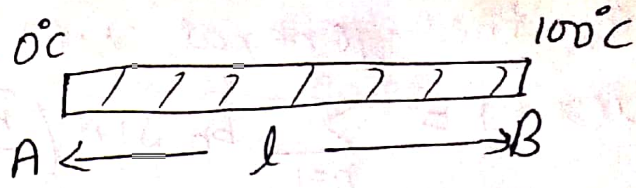
Using (1),

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \quad u = ax+b$$

where a and b are some arbitrary consts.

$$\left.\begin{aligned} &u(0,t) = 0 \\ &u(l,t) = 100 \end{aligned}\right\} \text{Boundary conditions (initialy)}$$




as $\quad u = ax + b$

$\Rightarrow \quad u(0,t) = 0 + b = 0 \quad \Rightarrow \quad b = 0$

$u(l,t) = 100 = al + b \quad$ as $b = 0 \quad \Rightarrow a = \frac{100}{l}$

so, $\quad u(x,t) = \frac{100}{l}x \quad$ at $\quad t = 0$

Hence, initial condition is

$$u(x,0) = \frac{100x}{l}$$

And the boundary conditions are

$$u(0,t) = 0$$
$$u(l,t) = 0$$

solution of ② is of the form,

$$u(x,t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2c^2t}$$

$u(0,t) = 0 = C_1\, e^{-m^2c^2t}$

$\Rightarrow$ Either $C_1 = 0$ or $e^{-m^2c^2t} = 0$

$e^{-m^2c^2t}$ can't be $0$ for a non-zero solution.

so $\quad C_1 = 0$

Also, $\quad u(l,t) = 0 = C_2 \sin ml\, e^{-m^2c^2t}$

As $C_1 = 0$ so $C_2$ can't be $0$ for a non-zero sol$^n$.

Also $e^{-m^2c^2t} \neq 0 \quad \Rightarrow \quad \sin ml = 0 = \sin n\pi$

$$\Rightarrow \quad m = \frac{n\pi}{l}$$

$$\Rightarrow \quad u(x,t) = C_2 \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2c^2}{l^2}t}$$

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So, the general $sol^n$ is,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

Now, $u(x,0) = 100\frac{x}{l}$

$$\Rightarrow \frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

which is Fourier Half-range Sine series.

So the $sol^n$ is,

$$b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{200}{l^2}\left[\left\{-\frac{xl}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right\}_0^l + \int_0^l \frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}\cos(n\pi) + \frac{l^2}{n^2\pi^2}\underbrace{\left\{\sin\left(\frac{n\pi x}{l}\right)\right\}_0^l}_{0}\right]$$

$$= \frac{-200}{n\pi}(-1)^n$$

$$= \frac{200}{n\pi}(-1)^{n+1}$$

So, $$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

Case II - Steady-state and non-zero boundary Conditions -

Ex. A bar AB of length 10 cm has its ends A and B kept at 30° and 100°c resp. until steady-state condition is reached. Then the temperature of A is lowered to 20°c and that of B to 40°c and these temperatures are maintained. find the subsequent temp. distribution in the bar.

Sol$^n$ - In the steady-state condition at $t=0$, u is independent of t

$$\Rightarrow \quad \frac{\partial u}{\partial t} = 0$$

So $\quad \frac{\partial^2 u}{\partial x^2} = 0 \quad \Rightarrow \quad u = ax + b$

Also, $\quad u(0,t) = 30 \quad , \quad u(10,t) = 100$

$\Rightarrow \quad 30 = b \quad \Rightarrow \quad b = 30$

and $\quad 100 = 10a + 30 \quad \Rightarrow \quad a = 7$

So $\quad u(x,t) = 7x + 30 \quad$ at $t=0$

So, initial condition is,

$$u(x,0) = 7x + 30$$

Boundary conditions are, $\quad u(0,t) = 20$

$$u(10,t) = 40$$

To, find the temperature distribution in the bar, assume the solution as

$$u(x,t) = \underbrace{u_s(x)}_{\text{Steady-state sol}^n} + \underbrace{u_{tr}(x,t)}_{\text{Transient sol}^n}$$

(Transient $sol^n$ - If $u(x,t)$ decreases as $t$ increases)

To find steady-state $sol^n$ $u_s$, solve $\frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \quad u_s = a_1 x + b_1$$

Now, $u_s(0,t) = 20$, $u_s(10,t) = 40$

$$\Rightarrow \quad 20 = b_1 \quad \text{and} \quad 40 = 10a_1 + 20$$

$$\Rightarrow \quad a_1 = 2$$

$$\Rightarrow \quad u_s = 2x + 20$$

As transient $sol^n$ $u_{tr}(x,t)$ satisfies one-dimen-sional heat $eq^n$ so,

$$u_{tr}(x,t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-c^2 m^2 t}$$

Hence, $u(x,t) = u_s + u_{tr}$

$$= 2x + 20 + [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2 c^2 t}$$

Now, $u(0,t) = 20$

$$\Rightarrow \quad 20 = 20 + [C_1 e^{-m^2 c^2 t}]$$

$$\Rightarrow \quad C_1 = 0 \quad \text{as} \quad e^{-m^2 c^2 t} \neq 0$$

and $u(10,t) = 40$

$$\Rightarrow \quad 40 = 20 + 20 + C_2 (\sin 10m)\, e^{-m^2 c^2 t}$$

$$\Rightarrow \quad C_2 (\sin 10m)\, e^{-m^2 c^2 t} = 0$$

$C_2 \neq 0$, $e^{-m^2 c^2 t} \neq 0$ $\Rightarrow$ $\sin 10m = 0 = \sin n\pi$

$$\Rightarrow \quad m = \frac{n\pi}{10}$$

$$\Rightarrow u(x,t) = 2x + 20 + c_2 \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2 c^2}{100} t}$$

General solⁿ is,

$$u(x,t) = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2 c^2}{100} t}$$

Now, $u(x,0) = 7x + 30$

$$\Rightarrow 7x + 30 = 2x + 20 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$$

$$\Rightarrow 5x + 10 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$$

which is Fourier - Half range sine series,

$$\Rightarrow b_n = \frac{2}{10} \int_0^{10} (5x+10) \sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{5}\left[\left\{-(5x+10)\frac{10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)\right\}_0^{10} + \underbrace{\int_0^{10} 5 \times \frac{10}{n\pi} \cos\left(\frac{n\pi x}{10}\right) dx}_{\text{becomes 0 after integration}}\right]$$

$$= \frac{2}{n\pi}\left[-60\cos(n\pi) + 10\right]$$

$$= \frac{20}{n\pi}\left[1 - 6(-1)^n\right]$$

So,

$$u(x,t) = 2x + 20 + \sum_{n=1}^{\infty} \frac{20}{n\pi}\left(1 + 6(-1)^{n+1}\right) \sin\left(\frac{n\pi x}{10}\right) \times e^{-\frac{n^2\pi^2 c^2}{100} t}.$$

## Case III – Both ends insulated

Ex – The temperature at one end of a 50 cm long bar with insulated sides, is kept at 0°c and that the other end is kept at 100°c until steady-state condition prevails. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

Soln – Eqn of temp. distribution is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

For steady-state at $t=0$, $\frac{\partial u}{\partial t} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = ax+b$$

$u(0,t)=0 \Rightarrow 0=b$

$u(50,t)=100 \Rightarrow 100 = 50a \Rightarrow a=2$

So, initial condition is, $u(x,0) = 2x$

When both the ends $x=0$ and $x=50$ of the bar are insulated, no heat can flow through them so boundary conditions are

At $x=0$, $\frac{\partial u}{\partial x} = 0 \quad \forall t$ i.e. $\frac{\partial u(0,t)}{\partial x} = 0$

At $x=50$, $\frac{\partial u}{\partial x} = 0 \quad \forall t$ i.e. $\frac{\partial u}{\partial x}(50,t) = 0$

Now, $u(x,t) = [C_1 \cos mx + C_2 \sin mx]\, e^{-m^2c^2t}$

$$\Rightarrow \frac{\partial u}{\partial x} = [-C_1 m \sin mx + C_2 m \cos mx]\, e^{-m^2c^2t}$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \quad \Rightarrow \quad 0 = C_2\, m\, e^{-m^2c^2t}$$

$$\Rightarrow \quad C_2 = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -C_1 m (\sin mx)\, e^{-m^2c^2t}$$

$$\frac{\partial u(50,t)}{\partial x} = 0 = -C_1 m \sin(50m)\, e^{-m^2c^2t}$$

as $C_1 \neq 0$, $e^{-m^2c^2t} \neq 0 \Rightarrow \sin(50m) = 0 = \sin n\pi$

$$\Rightarrow \quad m = \frac{n\pi}{50}$$

$$\Rightarrow \quad u(x,t) = C_1 \cos\left(\frac{n\pi x}{50}\right) e^{-m^2c^2t}$$

So, the general soln is,

$$u(x,t) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{50}\right) e^{-m^2c^2t}$$

$$u(x,0) = \frac{b_0 x^2}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{50}\right) = 2x$$

which is Fourier Half-range cosine series,

$$\Rightarrow 2b_0 = \frac{\cancel{1}2}{50}\int_0^{50} 2x\, dx = 100 \Rightarrow b_0 = 50$$

$$b_n = \frac{2}{50}\int_0^{50} 2x \cos\left(\frac{n\pi x}{50}\right) dx$$

$$= \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow u(x,t) = \frac{200}{\pi^2} \sum_{n=0}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos\left(\frac{n\pi x}{50}\right) e^{-m^2c^2t}$$