Notes for Electromagnetic Theory, Phys 706, Winter 2024, University of Waterloo

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I update these notes frequently. For the most recent version, please see: this link.

The symbols in the section titles are clickable links to video lectures.

Please let me know of any errors — I can easily correct and repost these notes.

Contents

1 Correspondence between lectures and topics

1) 2024-01-09 Lecture: "philosophy" of course, basic ideas of special relativity

2) 2024-01-12 Lecture: transformation of k, ω for an electromagnetic wave; four-vectors

3) 2024-01-16 Lecture: general definition of a Lorentz tensor; invariance of the interval; proper time

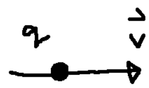
2 Lectures

2.1 2024-01-09 Lecture

1

Phys 706 - Electromagnetic theory

- one of the simplest questions we can ask in electromagnetism: what are the $\vec{E}$ and $\vec{B}$ fields due to a uniformly moving point charge?



$E?$ $B?$

Two approaches:

1) Solve Maxwell's equations directly. (awkward and messy)

alternately

2) Take the fields due to a stationary point charge, and then use the rules for transforming $\vec{E}$ and $\vec{B}$ between inertial ref. frames.

more elegant !!!
philosophy of course: use SR wherever possible, a la Landau + Lifschitz.

2

Special relativity - basic ideas

0) inertial reference frame.

1) relativity of simultaneity.
events: locations in spacetime.
two events can be "observed" at the same time in one inertial reference frame, but at different times in another inertial reference frame.

2) length contraction: in a frame in which a body is moving, its length along the direction of motion is observed to be contracted relative to measurements made in its rest frame.

$$\ell = \underset{\substack{\uparrow \\ \text{length in rest frame}}}{\ell_R} \sqrt{1-\left(\frac{v}{c}\right)^2} = \frac{\ell_R}{\gamma} \qquad \gamma := \frac{1}{\sqrt{1-(v/c)^2}}$$

$$\beta := v/c$$

③

3) Instead of γ and β, I'll in general prefer to use "rapidities"

$$\tanh\theta = v/c$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}, \quad \sinh\theta := \frac{e^{\theta}-e^{-\theta}}{2}, \quad \cosh\theta := \frac{e^{\theta}+e^{-\theta}}{2}$$

Going back to length contraction

$$\ell = \ell_R\sqrt{1-(\tanh\theta)^2}$$

$$= \ell_R\sqrt{\frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta}}$$

$$= \ell_R/\cosh\theta$$

Advantage of rapidities: relativistic velocity addition is "trivial" with rapidities.

④

Suppose we have 3 inertial ref. frames: A, B, C.

$$\underbrace{v_{AC,x}}_{\text{A wrt C}} = \frac{v_{AB,x} + v_{BC,x}}{1 + \frac{v_{AB,x}\, v_{BC,x}}{c^2}}$$

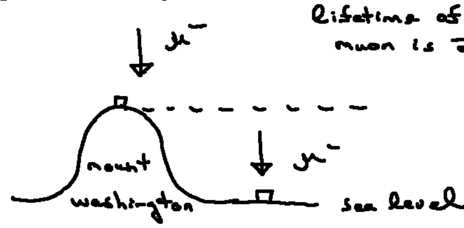
With rapidities:

$$\theta_{AC,x} = \theta_{AB,x} + \theta_{BC,x}$$

4) time-dilation - "moving clocks run slow"

$$\Delta t_R = \frac{\Delta t}{\cosh\theta}$$

change in time observed in rest frame of clock



lifetime of muon is $2\mu s$.

(5)

5) Length contraction, time dilation, velocity addition, simultaneity, ... can all be understood / explained in terms of the Lorentz transformations.

Inertial reference frames A, B

$$\begin{bmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} \overset{=\gamma}{\cosh\theta} & \overset{=-\gamma\beta}{-\sinh\theta} & 0 & 0 \\ \underset{=-\gamma\beta}{-\sinh\theta} & \overset{=\gamma}{\cosh\theta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{bmatrix}$$

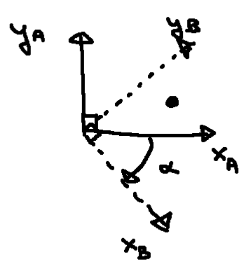
motion of B wrt to A

$$\underbrace{\vec{V}_{BA}}_{B \text{ wrt } A} = \hat{x}\, c \tanh\theta$$

"standard configuration"

(6)

Note the similarity with rotations in 2d:

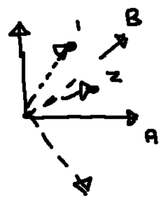


$$\begin{bmatrix} x_B \\ y_B \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{=:R(\alpha)} \begin{bmatrix} x_A \\ y_A \end{bmatrix}$$

"rotation matrix"

It is obvious that the "distance" from origin is invariant under rotation of axis:

$$x_A^2 + y_A^2 = x_B^2 + y_B^2$$



Stronger statement $\quad x_{1B}\,x_{2B} + y_{1B}\,y_{2B} = x_{1A}\,x_{2A} + y_{1A}\,y_{2A}$

dot product between $\vec{v}_1$ and $\vec{v}_2$ is invariant.

7

$$\underbrace{[x_{1B}\ y_{1B}]\begin{bmatrix}x_{2B}\\y_{2B}\end{bmatrix}}_{x_{1B}x_{2B}+y_{1B}y_{2B}} = \left(R(\alpha)\begin{bmatrix}x_{1A}\\y_{1A}\end{bmatrix}\right)^T R(\alpha)\begin{bmatrix}x_{2A}\\y_{2A}\end{bmatrix}$$

$$= [x_{1A}\ y_{1A}]\ \underbrace{R(\alpha)^T R(\alpha)}_{I}\begin{bmatrix}x_{2A}\\y_{2A}\end{bmatrix}$$

$$= [x_{1A}\ y_{1A}]\begin{bmatrix}x_{2A}\\y_{2A}\end{bmatrix}$$

Thus this invariance is a consequence of $\underbrace{R^T(\alpha) I R(\alpha) = I}$

8

Entirely analogous relationship holds for the Lorentz transformations.

Define $L(\theta) := \begin{bmatrix}\cosh\theta & -\sinh\theta\\ -\sinh\theta & \cosh\theta\end{bmatrix}$

Can show:

$$L(\theta)^T\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}L(\theta) = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

thus: $\underbrace{x_{1B}x_{2B} - c^2 t_{1B}t_{2B}} = \underbrace{x_{1A}x_{2A} - c^2 t_{1A}t_{2A}}$

"Lorentz invariant scalar product"

$$x_A^2 - ct_A^2 = x_B^2 - ct_B^2$$

Next lecture: use this a starting point to define "four-vectors"

2.2 2024-01-12 Lecture

1

Four-vectors and tensors

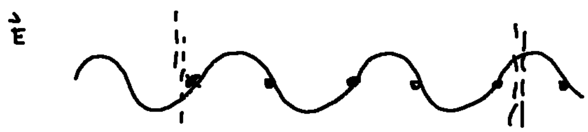
- plane electromagnetic wave solution:

$$\left.\begin{aligned}\vec{E}(x,t) &= E_0 \cos(kx-\omega t)\,\hat{y}\\ \vec{B}(x,t) &= \frac{E_0}{c}\cos(kx-\omega t)\,\hat{z}\end{aligned}\right\} \quad \frac{\omega}{k} = c$$

satisfy Maxwell's equations with $\vec{J} = \vec{0}$, $\rho = 0$

$$\left(\begin{aligned}&\nabla\cdot\vec{E} = \rho/\epsilon_0 && \nabla\cdot\vec{B} = 0\\ &\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} && \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\end{aligned}\right)$$

- concentrate on phase $\phi := kx - \omega t$



- since observations of phase correspond to "counting", phase is a Lorentz invariant scalar.

2

Let's discuss two different inertial reference frames: A and B,

$$\phi = kx_B - \frac{\omega}{c}ct_B$$

For ϕ to be invariant, k and ω must be frame specific.

$$\begin{aligned}\phi &= k_B x_B - \frac{\omega_B}{c}ct_B\\ &= k_B\left(x_A\cosh\theta - ct_A\sinh\theta\right) - \frac{\omega}{c}\left(-x_A\sinh\theta + ct_A\cosh\theta\right)\\ &= \underbrace{\left(k_B\cosh\theta + \frac{\omega}{c}\sinh\theta\right)}_{=:\,k_A}x_A - \underbrace{\left(k_B\sinh\theta + \frac{\omega}{c}\cosh\theta\right)}_{=:\,\frac{\omega_A}{c}}ct_A\end{aligned}$$

③

$$\begin{bmatrix} k_A \\ \omega_A/c \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} k_B \\ \omega_B/c \end{bmatrix}$$

$$L^{-1}(\theta) = L(\theta)$$

$$\left[\tanh\theta = \frac{\vec{v}_{BA}\cdot\hat{x}}{c} \right]$$

$$\begin{bmatrix} k_B \\ \omega_B/c \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} k_A \\ \omega_A/c \end{bmatrix}$$

Exactly the same as for $x_A, ct_A \rightarrow x_B, ct_B$.

The similarity in the transformation rules suggests a terminology: anything that has the same Lorentz transformation rules will be called a <u>four-vector</u>.
e.g. $(ct, \vec{r})$ is a four-vector as is $(\frac{\omega}{c}, \vec{k})$.

④

Recall Lorentz invariant scalar product

$$(ct_1, \vec{r}_1) \diamond (ct_2, \vec{r}_2) = -ct_1 ct_2 + x_1x_2 + y_1y_2 + z_1z_2$$

$$(ct, \vec{r}) \diamond (\tfrac{\omega}{c}, \vec{k}) = -\omega t + \vec{r}\cdot\vec{k} \qquad (*)$$

We just used invariance of this to reason out transformation rules for $(\omega/c, \vec{k})$.

Introduce tensor component notation.

$$(\omega/c, \vec{k}) \diamond (ct, \vec{r}) = [\omega/c \;\; k_x \;\; k_y \;\; k_z] \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

"metric"

(5)

$$= \underbrace{\begin{bmatrix} -\omega/c & k_x & k_y & k_z \end{bmatrix}}_{\text{"covariant components"}} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \Big\} \text{"contravariant components"}$$

Introduce $r^0 := ct$, $r^1 := x$, $r^2 := y$, $r^3 := z$

$k_0 := -\omega/c$, $k_1 := k_x$, $k_2 := k_y$, $k_3 := k_z$

$$(\omega/c, \vec{k}) \diamond (ct, \vec{r}) = \sum k_\alpha r^\alpha$$

repeated up/down indices are summed over

$$= k_\alpha r^\alpha$$

(Einstein summation convention)

$$= r_\alpha k^\alpha \quad \text{d math.}$$

(6)

The components of four vectors don't need to be physical quantities.

e.g. $\phi = \vec{k}\cdot\vec{r} - \omega t$

define $k^0 = -\frac{1}{c}\frac{\partial\phi}{\partial t}$, $k^1 = \frac{\partial\phi}{\partial x}$, $k^2 = \frac{\partial\phi}{\partial y}$, $k^3 = \frac{\partial\phi}{\partial z}$

$$k^\alpha = \partial^\alpha \phi$$

transforms like the contravariant components of a four-vector.

$$\partial_0 := \frac{1}{c}\frac{\partial}{\partial t}, \quad \partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z}.$$

$$k^\alpha \qquad G^{\alpha\beta\gamma} \leftrightarrow G^{\alpha}{}_{\beta}{}^{\gamma} \leftrightarrow G_{\alpha\beta\gamma}$$

$$F^{\alpha\gamma}$$

2.3 2024-01-16 Lecture

①

Lorentz tensors

- Lorentz invariant scalars and four-vectors are just special cases of Lorentz tensors.

The components of Lorentz tensors are specified using an arbitrary number of indices.

$$T^{\alpha\beta} \quad \beta \text{ 0 to 3}, \ \alpha \text{ 0 to 3} \quad \Rightarrow 16 \text{ components.}$$

(remember 4 components for a four-vector; i.e. one index
1 "component" for a Lorentz invariant scalar)

The components may be written in a variety of ways

$$T^{\alpha\beta} \qquad T_{\alpha\beta} \qquad T_{\alpha}{}^{\beta} \qquad T^{\alpha}{}_{\beta}$$

fully contra. fully cov. mixed.

②

- the # of indices is known as the rank
(four vectors are rank-1, scalars are rank 0)

The arrangement of the indices is known as the valence.

To change valence, generalize rule for 4-vectors

$$\underbrace{g_{\alpha\beta}}_{\text{metric}} x^{\alpha} = x_{\beta}$$

e.g.

$$T_{\alpha\beta} = g_{\gamma\alpha} g_{\epsilon\beta} T^{\gamma\epsilon}$$

What is a Lorentz tensor of arb. rank?

③

I have been using $L(\theta)$ for a specific form of the Lorentz transformation.

Let's generalize and use the 4x4 matrix Λ to represent arb. Lorentz transformations (not just the "standard configuration")

In the standard configuration

$$\Lambda = \begin{bmatrix} \cosh\theta & -\sinh\theta & \multicolumn{2}{c}{0} \\ -\sinh\theta & \cosh\theta & & \\ \multicolumn{2}{c}{0} & 1 & 0 \\ & & 0 & 1 \end{bmatrix}$$

$$G_\alpha H^\alpha = \tilde{G}_\alpha \tilde{H}^\alpha \quad \Rightarrow \quad \tilde{G}_\alpha = (\Lambda^{-1})^\gamma{}_\alpha G_\gamma$$

two different inertial ref. frames

$$\tilde{G}_\alpha \tilde{H}^\alpha = (\Lambda^{-1})^\gamma{}_\alpha G_\gamma \Lambda^\alpha{}_\beta H_\beta$$

$$= \delta^\gamma_\beta \quad \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$$

④

Generalizing: $\tilde{T}^{\alpha\beta} = \Lambda^\alpha{}_\nu \Lambda^\beta{}_\epsilon T^{\nu\epsilon}$

$$\tilde{T}_{\alpha\beta} = (\Lambda^{-1})^\nu{}_\alpha (\Lambda^{-1})^\epsilon{}_\beta T_{\nu\epsilon}$$

$$\tilde{T}_\beta{}^\alpha = \Lambda^\alpha{}_\nu (\Lambda^{-1})^\epsilon{}_\beta T_\epsilon{}^\nu$$

Λ's are not Lorentz tensors!

We've written things a bit backwards here. These rules are what we use to define tensors.

Usually we can can deduce that something is a Lorentz tensor from its construction using other Lorentz tensors.

5

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$A^\alpha B_\alpha$ $\qquad\qquad$ $F_\mu{}^\mu$

6

Proper time

$r^\alpha r_\alpha$ is a Lorentz invariant scalar.

(ct, x, y, z) $\qquad$ $(-ct, x, y, z)$

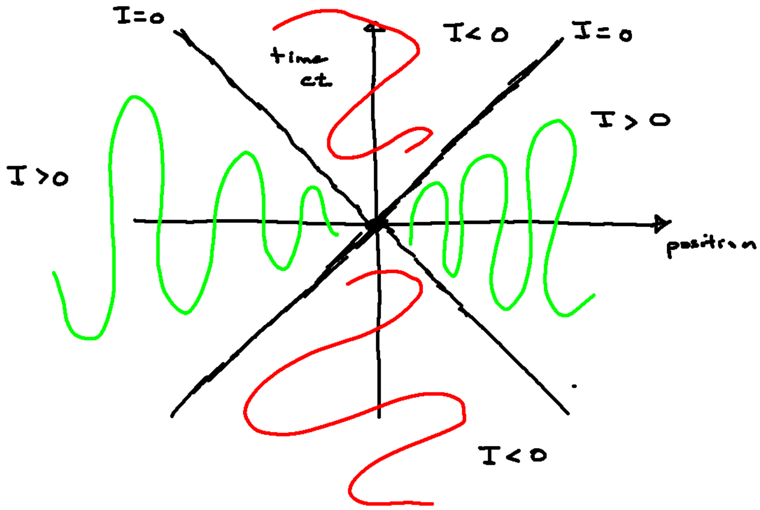
But it is not especially useful as it depends on origin of coordinate system.

But suppose I have two space time events p, q

$$(p_\alpha - q_\alpha)(p^\alpha - q^\alpha) = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

We call this quantity the *interval*.

⑦

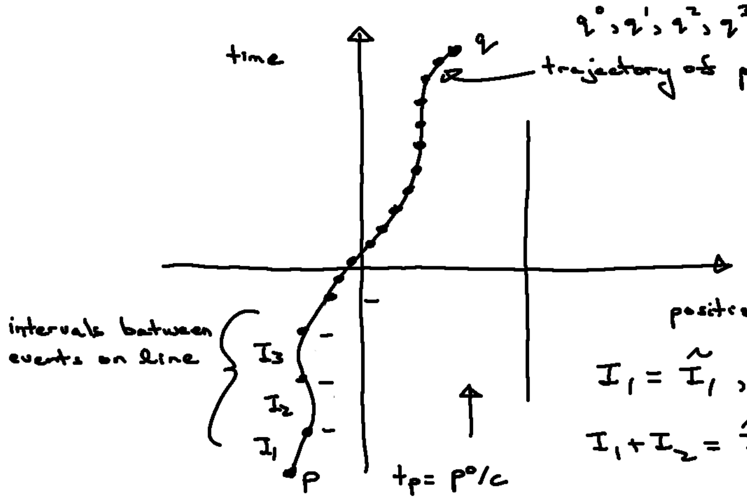


$I > 0$ is called "space-like" separation (event ordering depends on inertial ref. frame)

$I < 0$ is called "time-like" separation. (event ordering fixed)

$I = 0$ is "light-like" separation.

⑧



q^0, q^1, q^2, q^3, $t_q = \frac{q^0}{c}$

$I_1 = \tilde{I}_1$, $I_2 = \tilde{I}_2$

$I_1 + I_2 = \tilde{I}_1 + \tilde{I}_2$

$$
\begin{aligned}
J &:= \sum_i \sqrt{-I_i} \\
&= \sum_i \sqrt{(c\Delta t_i)^2 - (\Delta d_i)^2} \\
&= \sum_i c\Delta t_i \sqrt{1 - \left(\frac{\Delta d_i}{c\Delta t_i}\right)^2}
\end{aligned}
$$

⑨

Convert to integral:

$$J = \lim_{\Delta t_i \to 0} \sum c \Delta t_i \sqrt{1 - \left(\frac{\Delta d_i}{c \Delta t_i}\right)^2}$$

$$= c \int_{p^0/c}^{q^0/c} dt \sqrt{1-(v/c)^2}$$

We have constructed a Lorentz invariant scalar characterizing the trajectory.

$$\int_{p^0/c}^{q^0/c} dt \sqrt{1-\left(\frac{v}{c}\right)^2} = \int_{\tilde{p}^0/c}^{\tilde{q}^0/c} d\tilde{t} \sqrt{1-\left(\frac{\tilde{v}}{c}\right)^2}$$

$$\boxed{\tau = \int_{p^0/c}^{q^0/c} dt \sqrt{1-\left(\frac{v}{c}\right)^2}} \leftarrow \text{proper time.}$$

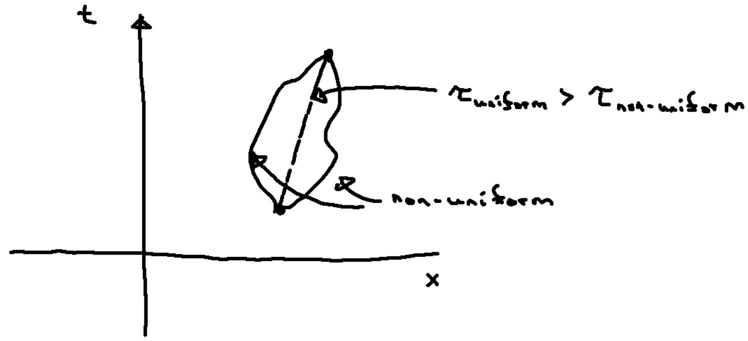
2.4 2024-01-18 Lecture

Unfortunately, I made a mistake and deleted my recording of the lecture. But the corresponding 2022 lectures are available here (2022-01-25 and 2022-01-27).

(1)

Proper time - cont'd

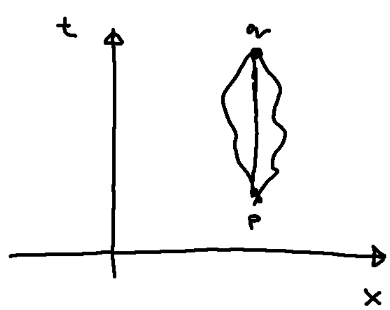
proper time is maximal for the constant velocity trajectory connecting two points in space-time.



(2)

Why?

I can always transform into an inertial reference frame where the uniform motion trajectory has $v = 0$.



$$\tau_{uniform} = \int_p^q dt$$

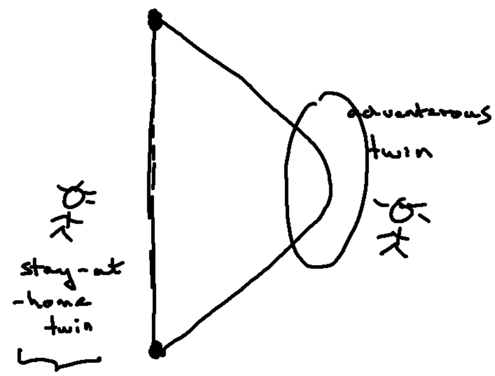
$$\tau_{non\text{-}uniform} = \int_p^q dt \underbrace{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

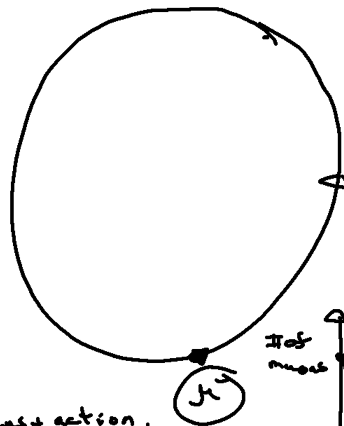
since $\sqrt{1-\left(\frac{v}{c}\right)^2} \leq 1$

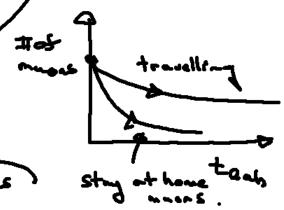
$$\Rightarrow \quad \tau_{non\text{-}uniform} < \tau_{uniform}.$$

③

Travelling twins







$\tau_{\text{stay-at-home}} > \tau_{\text{adv.}}$

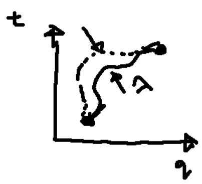
using Lagrangian + action.

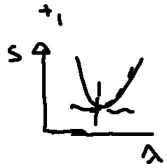
Proper time will be starting point for relativistic mechanics.

④

Brief review of Lagrangian Mechanics

General concept	1-d example (conservative force)
Start with Lagrangian $\mathcal{L}(q,\dot{q},t)$; q generalized coordinates, $\dot{q}$ generalized velocities	$\mathcal{L} = T - U = \underbrace{\frac{1}{2}mv^2}_{} - V(x)$
Action: $S := \int_{t_1}^{t_2} dt\, \mathcal{L}(q,\dot{q},t)$ (*)	$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2}mv^2 - V(x)\right)$





$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}}$$

Euler-Lagrange equations

$$\underbrace{p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i}}$$

$p = mv$

EL: $\boxed{\frac{dp}{dt} = -\frac{\partial V(x)}{\partial x}}$

(5)

"Momenergy" ≈ four-vector containing relativistic momentum and energy

from a Lorentz invariant action

- start with principle of "most proper time".

$$S = \underset{\text{"most"} \to \text{"least"}}{-}\underset{>0 \text{ TBD}}{\alpha} \int \underbrace{d\tau}_{\tau}$$

$$= -\alpha \int dt \sqrt{1-\left(\frac{v}{c}\right)^2}$$

By comparison with (*) above $\left(S = \int dt\, \mathcal{L} \right)$

$$\mathcal{L} = -\alpha \sqrt{1-\left(\frac{v}{c}\right)^2}$$

Remember:

$$\mathcal{L} = \frac{1}{2} m v^2 \quad , \text{"normally"}$$

(6)

Use $\mathcal{L} = \frac{1}{2} m v^2$ to get α :

$$\mathcal{L} = -\alpha \left(1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots \right)$$

$$= -\alpha + \frac{\alpha}{2}\left(\frac{v}{c}\right)^2 + \cdots$$

$$\frac{1}{\cancel{2}} m v^2 = \frac{\alpha}{\cancel{2} c^2} v^2$$

Thus $\alpha = mc^2$

and $\boxed{\mathcal{L} = -mc^2 \sqrt{1-\left(\frac{v}{c}\right)^2}}$ relativistic Lagrangian for a free particle.

Let's look at corresponding momenta:

$$p_x := \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial v_x} = \frac{\partial}{\partial v_x}\left(-mc^2 \sqrt{1-\left(\frac{v}{c}\right)^2} \right)$$

7

$$p_x = \frac{-mc^2}{2\sqrt{1-\left(\frac{v}{c}\right)^2}} * -\frac{1}{c^2}\frac{\partial}{\partial v_x}\left(v_x^2+\cancel{v_y^2}+\cancel{v_z^2}\right)$$

$$p_x = \frac{m v_x}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$\boxed{\vec{p} = \frac{m\vec{v}}{\sqrt{1-\left(\frac{v}{c}\right)^2}}}$$

Let's consider energy. (time indep. Lagrangians)

$$\frac{d\mathcal{L}}{dt} = \sum_i \underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i$$

8

$$= \sum_i \left(\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right]\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i \right) \quad \text{(using EL)}$$

$$= \sum_i \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right]$$

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right]$$

Rearrange

$$0 = \frac{d}{dt}\Big[\underbrace{\overbrace{\mathcal{L}}^{①} - \sum_i \overbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}^{②\ p_i}\dot{q}_i}_{=:E\ ,\ \text{"energy"}}\Big]$$

Constant of motion when $\mathcal{L}$ has no explicit time dependence.

9

Specialize for our $\mathcal{L}$:

$$E = \overset{①}{mc^2\sqrt{1-\left(\frac{v}{c}\right)^2}} + \overset{②}{\frac{m}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\left(v_x^2+v_y^2+v_z^2\right)}$$

$$= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\left[mc^2\left(1-\left(\frac{v}{c}\right)^2\right) + mv^2\right]$$

$$= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\left(mc^2 - \cancel{mv^2} + \cancel{mv^2}\right)$$

$$\boxed{E = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}}}$$

Next: combine with relativistic $\vec{p}$ to get a four-vector.

2.5 2024-01-23 Lecture

1

Combining our expressions for $\vec{p}$ and E to get "monenergy" (a four-vector)

Last lecture: $\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}$, $E = \frac{mc^2}{\sqrt{1-(v/c)^2}}$

To deal with "awkward" $1/\sqrt{1-(v/c)^2}$ factor remember that

$\tau = \int dt \sqrt{1-(v/c)^2}$ $\qquad \frac{d\tau}{dt} = \sqrt{1-(v/c)^2}$, $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-(v/c)^2}}$

$$E = mc^2 \frac{dt}{d\tau}$$
$$= mc \frac{d(ct)}{d\tau} \qquad (*)$$

2

$$\vec{p} = m\vec{v} \frac{dt}{d\tau}$$
$$= m \frac{d\vec{r}}{dt} \frac{dt}{d\tau}$$
$$= m \frac{d\vec{r}}{d\tau} \qquad (\ddagger)$$

Define a four-vector:

$$\underbrace{p^\alpha := m \frac{dx^\alpha}{d\tau}}_{\text{combines } (*) \text{ and } (\ddagger)} \quad \text{where} \quad x^\alpha \text{ "=" } (ct, \vec{r})$$

where p^α "=" $\left(\frac{E}{c}, \vec{p}\right)$ "monenergy"

(3)

Griffiths (and others), discuss "proper velocity"

$$\eta^{\alpha} = \frac{dx^{\alpha}}{d\tau} \quad \text{"="} \quad \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right)$$

What the Lorentz invariant $\eta^{\alpha}\eta_{\alpha}$

$$\eta^{\alpha}\eta_{\alpha} = \underbrace{-c^2\left(\frac{dt}{d\tau}\right)^2}_{\alpha=0} + \underbrace{\left(\frac{dx}{d\tau}\right)^2}_{\alpha=1} + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2$$

$$= -c^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dt}{dt}\right)^2 v^2$$

$$= -c^2 \frac{\cancel{(1-(v/c)^2)}}{\cancel{(1-(v/c)^2)}}$$

$$\boxed{\eta^{\alpha}\eta_{\alpha} = -c^2}$$

(4)

Using this result we can compute the Lorentz invariant $p^{\alpha}p_{\alpha}$

$$p^{\alpha}p_{\alpha} = m^2\eta^{\alpha}\eta_{\alpha}$$

$$-\left(\frac{E}{c}\right)^2 + p^2 = -m^2c^2$$

$$E^2 = p^2c^2 + m^2c^4$$

$$\boxed{E = \sqrt{(pc)^2 + (mc^2)^2}}$$

⑤

Lorentz force law from an invariant action "S"

$$S = \underbrace{-mc^2 \int d\tau}_{\text{free particle}} + \underbrace{q \int A_\alpha dx^\alpha}_{\text{Lorentz invariant.}}$$

Try to get Lagrangian. (plan: get Lagrangian, the EL eqns to get equations of motion)

$$\frac{d\tau}{dt} = \sqrt{1-(v/c)^2}$$

$$S = -mc^2 \int dt \sqrt{1-(v/c)^2} \quad \Big\} \text{ free-particle (already did this)}$$

$$\int A_\alpha dx^\alpha = \underbrace{\int -A^0 c\, dt}_{0\text{th}} + \underbrace{\int \vec{A}\cdot d\vec{r}}_{1,2,3} \qquad \left(A_\alpha \text{ "=" } (-A^0, \vec{A}) \right)$$

⑥

For the last term: $q \int \vec{A}\cdot d\vec{r} = q \int \vec{A}\cdot\underbrace{\vec{v}\, dt}$

Put this all together: $\left(S = \int \mathcal{L}\, dt \right)$

$$\boxed{\mathcal{L} = -mc^2 \sqrt{1-(v/c)^2} - qcA^0 + q\vec{A}\cdot\vec{v}} \qquad *$$

Crank through Lagrangian procedure:

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$P_x = \frac{m v_x}{\sqrt{1-(v/c)^2}} + qA_x \quad \text{, likewise for } y \text{ and } z$$

"canonical momentum" $\vec{P} = \vec{p} + q\vec{A}$ "momentum"

⑦

$$\left(\text{Recall EL: } \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \right)$$

$$\frac{\partial \mathcal{L}}{\partial x} = -qc\,\frac{\partial A^0}{\partial x} + q\frac{\partial}{\partial x}(\vec{A}\cdot\vec{v})$$ likewise for y and z

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-(v/c)^2}} + q\vec{A}\right) = -qc\,\nabla A^0 + q\nabla(\vec{A}\cdot\vec{v}) \qquad (*)$$

For $\nabla(\vec{A}\cdot\vec{v})$ term, "recall" the identity:

$$\nabla(\vec{a}\cdot\vec{b}) = \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a}) + (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a}$$

Since ∇ is "partial"

$$\nabla(\vec{A}\cdot\vec{v}) = \vec{v}\times(\nabla\times\vec{A}) + (\vec{v}\cdot\nabla)\vec{A}$$

Substitute into $(*)$

⑧

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}\right) = -q\underbrace{\frac{d\vec{A}}{dt}} - qc\,\nabla A^0 + q\vec{v}\times(\nabla\times\vec{A}) + q(\vec{v}\cdot\nabla)\vec{A} \qquad (*)$$

$\frac{d\vec{A}}{dt}$ is a total derivative.

From chain-rule : $\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \underbrace{(\vec{v}\cdot\nabla)\vec{A}}$

Put into $(*)$; with some cancellation

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}\right) = q\left[-c\nabla A^0 - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times(\nabla\times\vec{A})\right]$$

If we say $\underbrace{\vec{E} = -c\nabla A^0 - \frac{\partial\vec{A}}{\partial t} \text{ and } \vec{B} = \nabla\times\vec{A}}$, then

(9)

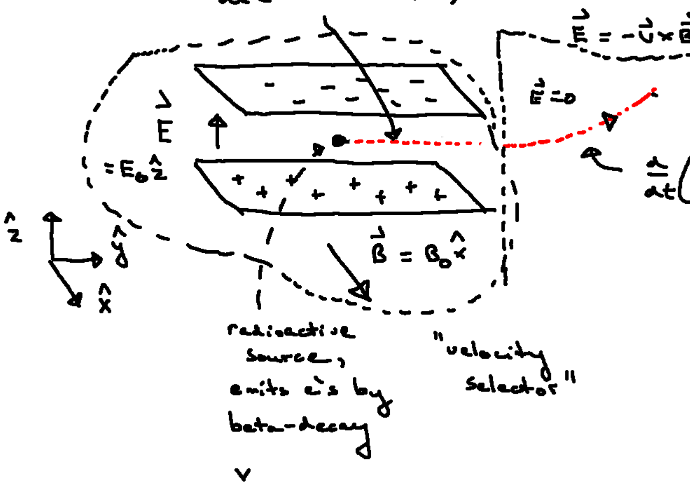
$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}\right) = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

which looks like a generalization of the Lorentz force law.

2.6 2024-01-25 Lecture

① **How to test generalized Lorentz force law?**

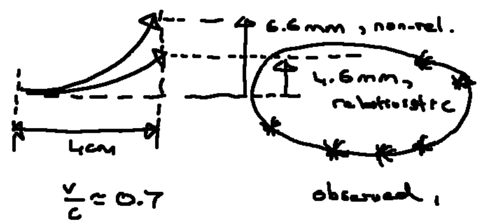
$$\frac{d}{dt}\left(m\vec{v}/\sqrt{1-(v/c)^2}\right) = q(\vec{E}+\vec{v}\times\vec{B}) = \vec{0} \qquad (*)$$



Bucherer's test of generalized Lorentz force law

$$\frac{d}{dt}\left(m\vec{v}/\sqrt{1-(v/c)^2}\right) = q\vec{v}\times\vec{B}$$

Leads to circular motion



② **Minkowski force** (gives a manifestly covariant version of Lorentz force law)

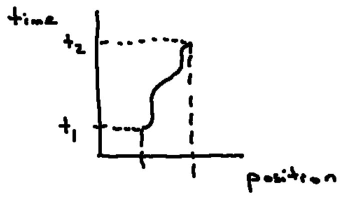
$$\frac{dp_\beta}{d\tau} = q\underbrace{\eta^\alpha}_{\text{four-velocity}} F_{\beta\alpha}$$

"Faraday tensor" contains $\vec{E}$ and $\vec{B}$ field components

$F_{\beta\alpha} := (\partial_\beta A_\alpha - \partial_\alpha A_\beta)$

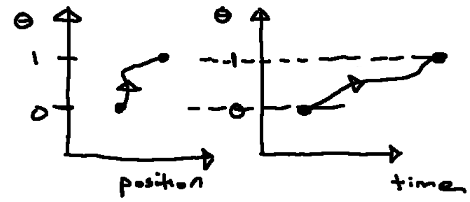
$$\left(\quad \frac{dp^\beta}{d\tau} = q\eta_\alpha F^{\beta\alpha} \quad \text{equally valid !!!} \right)$$

(3)



Strategy:
don't treat time preferentially!

$$S = \int_{t_1}^{t_2} dt \ \mathcal{L}$$



$\vec{r}(t)$ vs. $x^i(\theta)$

First consider a free particle: $\quad S_F = -mc^2 \int d\tau \quad \leftarrow$

$$= -mc \int_0^1 d\theta \sqrt{-\frac{dx^\alpha}{d\theta}\frac{dx_\alpha}{d\theta}} \quad \leftarrow$$

$$(\underbrace{c\, dt \sqrt{1-(v/c)^2}}_{d\tau} = d\theta \sqrt{\left(\frac{dct}{d\theta}\right)^2 - \left(\frac{dx}{d\theta}\right)^2 - \left(\frac{dy}{d\theta}\right)^2 - \left(\frac{dz}{d\theta}\right)^2}$$

(4)

$$= d\theta \left(\frac{dt}{d\theta}\right) \sqrt{c^2 - v^2}$$

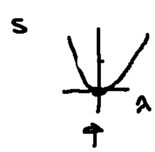
$$= c\, dt \sqrt{1-(v/c)^2}$$

For a world-line with stationary action we should "perturb" the world-line:

$$x^\alpha(\theta) \rightarrow x^{(\alpha)}(\theta) + \lambda\, \epsilon^\alpha(\theta)$$

arb. deviation with $\epsilon^\alpha(0) = \epsilon^\alpha(1) = 0$

and set $\left.\dfrac{dS_F}{d\lambda}\right|_{\lambda=0} = 0$



want this trajectory.

(5)

$$\frac{1}{-mc}\frac{dS_F}{d\lambda} = \frac{d}{d\lambda}\int_0^1 d\theta \sqrt{-\frac{d(x^\alpha+\lambda\epsilon^\alpha)}{d\theta}\frac{d(x_\alpha+\lambda\epsilon_\alpha)}{d\theta}}$$

$$= \int_0^1 d\theta \frac{-1}{2\sqrt{\cdots}}\left[\frac{d\epsilon^\alpha}{d\theta}\frac{d(x_\alpha+\lambda\epsilon_\alpha)}{d\theta} + \frac{d(x^\alpha+\lambda\epsilon^\alpha)}{d\theta}\frac{d\epsilon_\alpha}{d\theta}\right]$$

$$-\frac{1}{mc}\frac{dS_F}{d\lambda}\bigg|_{\lambda=0} = \int_0^1 d\theta \frac{-1}{2\sqrt{-\frac{dx^\alpha}{d\theta}\frac{dx_\alpha}{d\theta}}}\cdot\left[\frac{d\epsilon^\alpha}{d\theta}\frac{dx_\alpha}{d\theta} + \frac{dx^\alpha}{d\theta}\frac{d\epsilon_\alpha}{d\theta}\right]$$

$$\int_0^1 d\theta \frac{-1}{\underbrace{\sqrt{\frac{dx^\alpha}{d\theta}\frac{dx_\alpha}{d\theta}}}_{= c\frac{d\tau}{d\theta}}}\frac{d\epsilon^\alpha}{d\theta}\frac{dx_\alpha}{d\theta}$$

$$-\frac{1}{mc}\frac{dS_F}{d\lambda}\bigg|_{\lambda=0} = -\int_0^1 d\theta \underbrace{\frac{dx_\alpha}{d\tau}}_{\text{covariant components of proper velocity } \eta_\alpha}\frac{d\epsilon^\alpha}{d\theta}$$

(6)

Use integration by parts

$$= \underbrace{-\eta_\alpha\epsilon^\alpha\Big|_0^1}_{\substack{=0 \text{ because} \\ \epsilon^\alpha(0)=\epsilon^\alpha(1)=0}} + \int_0^1 d\theta\, \underbrace{\epsilon^\alpha}_{\text{arb.}}\frac{d\eta_\alpha}{d\theta} \qquad (*)$$

Since ϵ^α arb. if we want $\frac{dS_F}{d\lambda}\Big|_{\lambda=0} = 0$, for all ϵ^α

then $(*)$ tells us $\frac{d\eta_\alpha}{d\theta} = 0$.

Or, the proper velocity is constant.

Trivial result !!! But let's apply same procedure with field.

(7)

Now $S = S_F + S_A$
$$:= q \int A_\alpha dx^\alpha$$

$$\frac{1}{q}\frac{dS_A}{d\lambda} = \frac{d}{d\lambda}\int_0^1 d\theta \; A_\alpha(x^0+\lambda\epsilon^0, x^1+\lambda\epsilon^1, x^2+\lambda\epsilon^2, x^3+\lambda\epsilon^3)\,\frac{d(x^\alpha+\lambda\epsilon^\alpha)}{d\theta}$$

$$\frac{1}{q}\frac{dS_A}{d\lambda}\Big|_{\lambda=0} = \int_0^1 d\theta \, \frac{dx^\alpha}{d\theta}\epsilon^\beta\,[-\partial_\alpha A_\beta + \partial_\beta A_\alpha]$$

Combine with $dS_F/d\lambda|_{\lambda=0}$

$$\frac{dS}{d\lambda}\Big|_{\lambda=0} = \int_0^1 d\theta \;\epsilon^\beta\Big[\underbrace{-m\frac{d\eta_\beta}{d\theta}}_{\text{due to } S_F} + q\frac{dx^\alpha}{d\theta}(\partial_\beta A_\alpha - \partial_\alpha A_\beta)\Big]$$

Since ϵ^β is arb.

(8)

$$m\frac{d\eta_\beta}{d\theta} = q\frac{dx^\alpha}{d\theta}(\partial_\beta A_\alpha - \partial_\alpha A_\beta)$$

$$\underbrace{m\frac{d\eta_\beta}{d\tau}}_{\frac{dp_\beta}{d\tau}}\cancel{\frac{d\tau}{d\theta}} = q\underbrace{\frac{dx^\alpha}{d\tau}}_{\eta^\alpha}\cancel{\frac{d\tau}{d\theta}}\underbrace{(\partial_\beta A_\alpha - \partial_\alpha A_\beta)}_{=: F_{\beta\alpha} \text{ "Faraday tensor"}}$$

$$\boxed{\frac{dp_\alpha}{d\tau} = \underbrace{q\eta^\alpha F_{\beta\alpha}}_{\text{Minkowski - force.}}}$$

What is significance of "time-like" component.

(9)

$$\frac{dp_0}{d\tau} = q\eta^0 F_{00} + q\eta^1 F_{01} + \cdots$$

$$\eta^1 = \frac{v_x}{\sqrt{1-(v/c)^2}} \qquad F_{01} = \partial_0 A_1 - \partial_1 A_0$$

$$\underbrace{\frac{1}{c}\frac{\partial A^1}{\partial t}}_{= -E_x/c} \qquad \vec{E} = \nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\frac{dp_0}{d\tau} = -q \frac{1}{\sqrt{1-(v/c)^2}} \vec{v}\cdot\frac{\vec{E}}{c}$$

$$\boxed{\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-(v/c)^2}}\right) = q\vec{v}\cdot\vec{E}}$$

work-energy theorem.

2.7 2024-01-30 Lecture

①

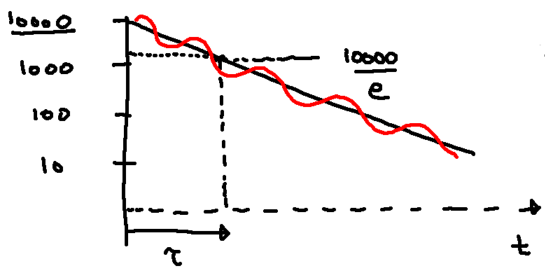
$$N = N_0 e^{-t/\tau} \quad \text{lifetime.} \qquad \frac{N}{N_0} = \frac{1}{e}$$

$$\frac{dN}{dt} = -\frac{1}{\tau} N$$

$$10^{-4} \simeq e^{-t/\tau}$$

$$\ln 10^{-4} = -t/\tau$$

$$t = \tau \ln 10^{4}$$



②

Transformation of $\vec{E}$ and $\vec{B}$ between different inertial reference frames

$$\frac{dp_\alpha}{d\tau} = q\eta^{\alpha} \underbrace{F_{\beta\alpha}}_{\text{Faraday tensor}} \qquad \underbrace{F_{\beta\alpha} = \partial_\beta A_\alpha - \partial_\alpha A_\beta}$$

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A} \quad (*)$$

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}\right) = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

where
$$\underbrace{\vec{E} = -c\nabla A^0 - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}}$$

(3)

$$F_{\beta\alpha} \text{ "="} \begin{array}{c} \overbrace{\hphantom{0 \quad F_{01} \quad F_{02} \quad F_{03}}}^{\alpha} \\ \left[\begin{array}{cccc} 0 & F_{01} & F_{02} & F_{03} \\ -F_{01} & 0 & F_{12} & F_{13} \\ -F_{02} & -F_{12} & 0 & F_{23} \\ -F_{03} & -F_{13} & -F_{23} & 0 \end{array}\right] \end{array} \Bigg\} \beta$$

only 6 indep. components, which matches nicely with 3 components of $\vec{E}$, and 3 of $\vec{B}$.

$$F_{01} = \partial_0 A_1 - \partial_1 A_0$$
$$= \frac{\partial}{\partial(ct)} A_x - \frac{\partial}{\partial x}\left(-\Phi/c\right)$$
$$= \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\frac{\partial}{\partial x}\Phi$$
$$= -\frac{1}{c}E_x$$

$$E_x = -\frac{\partial}{\partial x}\Phi - \frac{\partial}{\partial t}A_x$$

By similar argument: $F_{02} = -E_y/c$, $F_{03} = -E_z/c$

(4)

$$F_{12} = \partial_1 A_2 - \partial_2 A_x$$
$$= \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x \quad \ni \hat{z}\cdot(\nabla\times\vec{A})$$
$$= B_z$$

By similar argument $F_{23} = B_x$, $F_{31} = B_y$

$$F_{\mu\nu} \text{ "="} \begin{array}{c} \left\{\begin{array}{cccc} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{array}\right\} \downarrow \mu \\ \xrightarrow{\nu} \end{array}$$

⑤

$$F_{\alpha\nu} = g_{\alpha\mu}\, g_{\nu\beta}\, F^{\mu\beta} \qquad \text{where } g_{\alpha\mu} \text{ "="} \begin{Bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{Bmatrix}$$

Using ↑ you may determine the components of $F^{\mu\beta}$

$$F^{\mu\beta} \text{ "="} \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix}$$

Now determine how the fields transform.

$$\tilde{F}^{\mu\beta} = \Lambda^{\mu}_{\ \alpha}\, \Lambda^{\beta}_{\ \kappa}\, F^{\alpha\kappa}$$

⑥

Remember in "standard configuration"

$$\Lambda = \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ -s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

cosh θ — sinh θ

Start with $\tilde{E}_x = c\tilde{F}^{01}$

$$= c\,\Lambda^{0}_{\ \alpha}\Lambda^{1}_{\ \kappa}\, F^{\alpha\kappa}$$

$$= c\,\Lambda^{0}_{\ \alpha}\,(\Lambda^{1}_{\ 0}F^{\alpha 0} + \Lambda^{1}_{\ 1}F^{\alpha 1} + \underbrace{\Lambda^{1}_{\ 2}}_{=0}F^{\alpha 2} + \underbrace{\Lambda^{1}_{\ 3}}_{=0}F^{\alpha 3})$$

$$\frac{\tilde{E}_x}{c} = \Lambda^{0}_{\ 0}\cdot(\Lambda^{1}_{\ 0}F^{00} + \Lambda^{1}_{\ 1}F^{01}) + \Lambda^{0}_{\ 1}(\Lambda^{1}_{\ 0}F^{10} + \Lambda^{1}_{\ 1}F^{11})$$

$$= c\theta\,(-s\theta\, \underbrace{\cancel{F^{00}}}_{=0} + c\theta\, F^{01}) - s\theta\,(-s\theta\, F^{10} + c\theta\, \underbrace{\cancel{F^{11}}}_{=0})$$

$$= (c\theta)^2 F^{01} + (s\theta)^2 F^{10}$$

$$= F^{01}\,\underbrace{(\cosh^2\theta - \sinh^2\theta)}_{=1}$$

(7)

$$\frac{\tilde{E}_x}{c} = \frac{E_x}{c}$$

You can also find $\tilde{B}_x = B_x$ by similar approach.

So components of $\vec{E}$ and $\vec{B}$ along direction of relative motion are unchanged between different inertial ref. frames.

Let's look at transverse fields.

$$\frac{\tilde{E}_y}{c} = \tilde{F}^{02}$$

$$= \Lambda^0_{\ \alpha} \Lambda^2_{\ \kappa} F^{\alpha\kappa}$$

$$= \Lambda^0_{\ \alpha} \left(\underbrace{\Lambda^2_{\ 0}}_{=0} F^{\alpha 0} + \underbrace{\Lambda^2_{\ 1}}_{=0} F^{\alpha 1} + \underbrace{\Lambda^2_{\ 2}}_{=1} F^{\alpha 2} + \underbrace{\Lambda^2_{\ 3}}_{=0} F^{\alpha 3} \right)$$

$$= \Lambda^0_{\ \alpha} F^{\alpha 2}$$

(8)

$$\frac{\tilde{E}_y}{c} = \underbrace{\Lambda^0_{\ 0}}_{c\theta} \underbrace{F^{02}}_{E_y/c} + \underbrace{\Lambda^0_{\ 1}}_{-s\theta} \underbrace{F^{12}}_{B_z} + \cancel{\Lambda^0_{\ 2}} F^{22} + \cancel{\Lambda^0_{\ 3}} F^{32}$$

$$\boxed{\frac{\tilde{E}_y}{c} = \cosh\theta \, \frac{E_y}{c} - \sinh\theta \, B_z}$$

By similar manipulations

$$\boxed{\begin{aligned} \frac{\tilde{E}_z}{c} &= \cosh\theta \, \frac{E_z}{c} + \sinh\theta \, B_y \\ \tilde{B}_y &= \cosh\theta \, B_y + \sinh\theta \, E_z/c \\ \tilde{B}_z &= \cosh\theta \, B_z - \sinh\theta \, E_y/c \end{aligned}}$$

9

Write results in frame-indep. manner: $\vec{v}$ relative frame velocity

$$\vec{E} = \underbrace{(\hat{v}\cdot\vec{E})\hat{v}}_{\vec{E}_\parallel} + \underbrace{(\vec{E} - (\hat{v}\cdot\vec{E})\hat{v})}_{\vec{E}_\perp}$$

$$\tilde{\vec{E}}_\parallel = \vec{E}_\parallel$$

$$\tilde{\vec{E}}_\perp = c\theta\,(\vec{E}_\perp + \vec{v}\times\vec{B})$$

$$\tilde{\vec{B}}_\parallel = \vec{B}_\parallel$$

$$\tilde{\vec{B}}_\perp = c\theta\,\left(\vec{B}_\perp - \frac{\vec{v}}{c}\times\frac{\vec{E}}{c}\right)$$

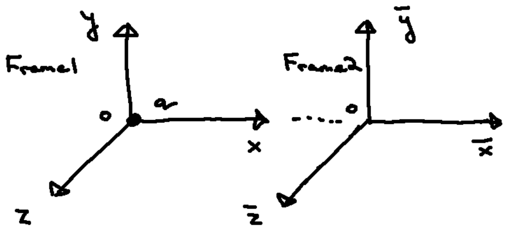
Remember

$$\vec{v} = \hat{v}\,c\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$$

2.8 2024-02-01 Lecture

①

Fields due to a uniformly moving point charge



point charge q is stationary

point charge is in uniform motion

$$\underbrace{\vec{V}_{21}}_{\text{velocity of frame 2 wrt 1}} = \hat{x} c \beta$$

②

In frame 1 (charge stationary)

$$\vec{B} = 0$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(x\hat{x} + y\hat{y} + z\hat{z})}{r^3} \quad \vec{r} = r\hat{r}$$

where $r^3 = (x^2 + y^2 + z^2)^{3/2}$

Apply field transformation rules.

$$\bar{\vec{E}}_{\parallel} = \vec{E}_{\parallel}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{x}{r^3} \hat{x}$$

③

Express in terms of coordinates in frame 2
$(c\bar{t}, \bar{x}, \bar{y}, \bar{z})$

$$x = \gamma(\bar{x} + \beta c\bar{t})$$
$$ct = \gamma(c\bar{t} + \beta\bar{x})$$
$$y = \bar{y}$$
$$z = \bar{z}$$

$$\vec{E}_{\parallel} = \frac{q}{4\pi\epsilon_0} \frac{\overbrace{\gamma(\bar{x}+\beta c\bar{t})}^{x}\hat{x}}{[\underbrace{(\gamma(\bar{x}+\beta c\bar{t}))}_{x}^2 + \underbrace{\bar{y}}_{y}^2 + \underbrace{\bar{z}}_{z}^2]^{3/2}} \quad \Big\} \qquad *$$

④

$$\vec{E}_{\perp} = \gamma(\vec{E}_{\perp} + \cancel{\vec{\beta}\times\vec{B}c}^{\,=0})$$

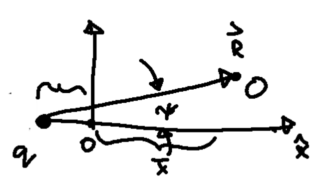
$$= \frac{q}{4\pi\epsilon_0}\gamma\frac{(y\hat{y} + z\hat{z})}{r^3}$$

Write in terms of coordinates in frame 2

$$\vec{E}_{\perp} = \frac{q}{4\pi\epsilon_0}\gamma\frac{(\bar{y}\hat{y} + \bar{z}\hat{z})}{\cdots} \qquad *$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\gamma\frac{[(\bar{x}+\beta c\bar{t})\hat{x} + \bar{y}\hat{y} + \bar{z}\hat{z}]}{[(\gamma(\bar{x}+\beta c\bar{t}))^2 + \bar{y}^2 + \bar{z}^2]^{3/2}}$$

in frame 2 define ψ



$$\vec{R} = (\bar{x} + \beta c\bar{t})\hat{x} + \bar{y}\hat{y} + \bar{z}\hat{z}$$
$$\hat{x}\cdot\vec{R} = \bar{x} + \beta c\bar{t}$$

(5)

$$\hat{x}\cdot\vec{R} = R\cos\psi \qquad\qquad \bar{x} + \beta c\bar{t} = R\cos\psi$$

$$y^2 + z^2 = R^2 \sin^2\psi$$

$$\vec{\bar{E}} = \frac{q}{4\pi\epsilon_0}\,\gamma\,\frac{\vec{R}}{\left[(\gamma\cos\psi)^2 + \sin^2\psi\right]^{3/2} R^3}$$

Can be cleaned up using $\gamma^2\cos^2\psi + \sin^2\psi = \gamma^2(1-\beta^2\sin^2\psi)$ (*)

Show this identity.

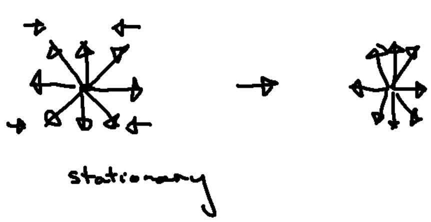
$$\gamma^2(1-\sin^2\psi) + \sin^2\psi = \gamma^2 + (1-\gamma^2)\sin^2\psi$$
$$= (c\theta)^2 + (1-(c\theta)^2)\sin^2\psi$$
$$= (c\theta)^2 + -(s\theta)^2\sin^2\psi$$
$$= (c\theta)^2(1-(t\theta)^2\sin^2\psi)$$
$$= \gamma^2(1-\beta^2\sin^2\psi) \quad ✓$$

$$(c\theta)^2 - (s\theta)^2 = 1$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

(6)

$$\boxed{\vec{\bar{E}} = \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\psi)^{3/2}}}$$



$\psi = 90^\circ$

$$\vec{\bar{E}}_{90} = \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2}\frac{1}{\sqrt{1-\beta^2}}$$
$$= \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2}\gamma$$

Now consider the magnetic field.

$$\vec{\bar{B}}_\perp = \gamma\left[\vec{B}_\perp - \vec{\beta}\times\frac{\vec{E}}{c}\right]$$ for could use this expression.

Instead lets us inverse result.

⑦

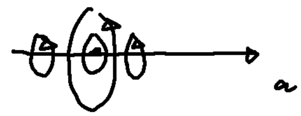
$$\vec{B}_\perp = \gamma\left[\bar{\vec{B}}_\perp + \vec{\beta}\times\frac{\bar{\vec{E}}}{c}\right]$$

$$\vec{B}_\perp = \vec{0}$$

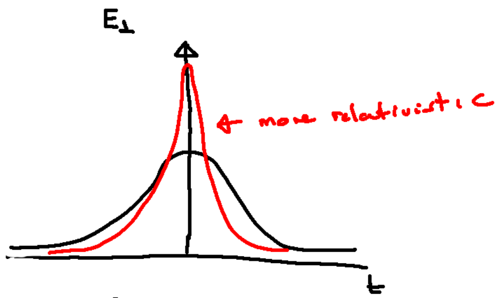
Rearrange to give:

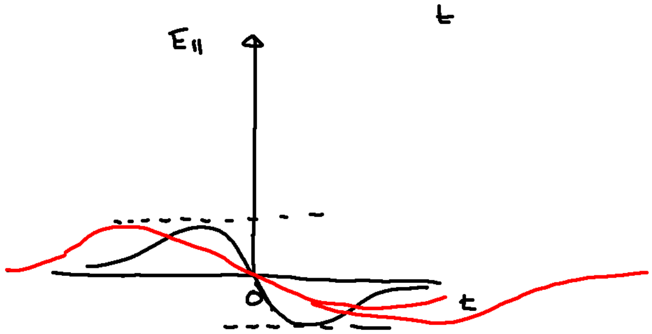
$$\bar{\vec{B}}_\perp = -\vec{\beta}\times\frac{\bar{\vec{E}}}{c}$$

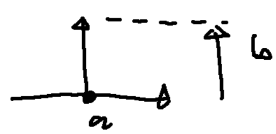
$$\boxed{\bar{\vec{B}} = -\frac{\vec{v}}{c}\times\frac{\bar{\vec{E}}}{c}}$$



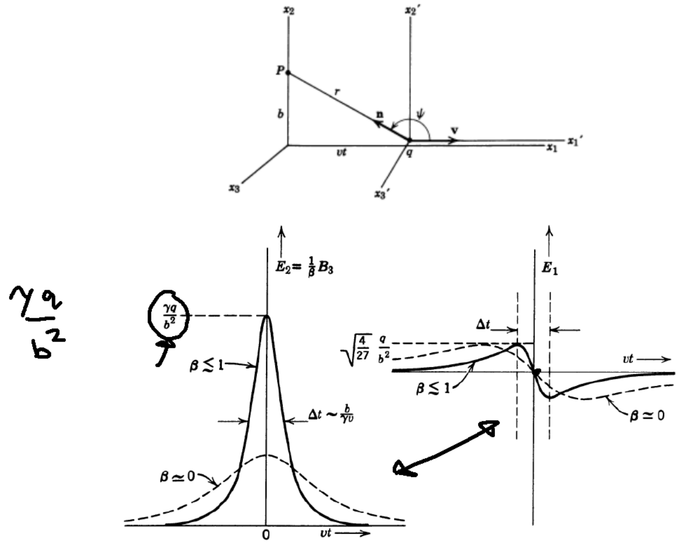
⑧

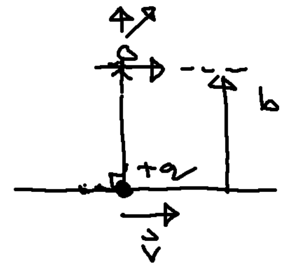






Electric field due to a moving point charge





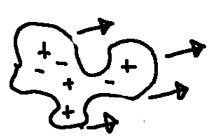
Source: Fig.'s 11.8 and 11.9 from J. D. Jackson, *Classical electrodynamics*, 3rd ed (Wiley, New York, 1999).

2.9 2024-02-06 Lecture

1

Maxwell's equations in a manifestly covariant form

First: justify that $(\rho c, \vec{J})$ is a four-vector.



$\rho \neq 0$, $\vec{J} \neq \vec{0}$

move into inertial reference frame where charge is stationary

$\rho_0 \neq 0$, $\vec{J}_0 = \vec{0}$

charge is conserved between inertial reference frames.
(total charge, point charge value) but charge density is not. i.e. length contraction along direction of motion.

②

$$\left\{\begin{aligned} \rho &= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\,\rho_0 \\ \vec{J} &= \rho\,\vec{v} \end{aligned}\right.$$

frame in which charge is moving

$$= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\,\rho_0\,\vec{v}$$

$$(\rho c, \vec{J}) = (\rho_0 c\gamma, \vec{v}\gamma\rho_0)$$

$$= \underbrace{\rho_0}_{\text{Lorentz-invariant}} \underbrace{(c\gamma, \gamma\vec{v})}_{\text{proper-velocity four-vector } \eta^\alpha}$$

Thus we see that $(\rho c, \vec{J})$ is a four-vector.

③

Charge continuity can be expressed in a manifestly covariant form.

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \qquad (*)$$

$$\partial_\beta J^\beta = 0 \qquad (\ddagger)$$

Justification that $(*)$ and $(\ddagger)$ are equivalent.

$$\frac{\partial}{\partial(ct)}(\rho c) + \frac{\partial}{\partial x}J_x + \frac{\partial}{\partial y}J_y + \frac{\partial J_z}{\partial z} = 0$$

$$\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0\right) \quad *$$

Thus $(*)$ and $(\ddagger)$ are equivalent.

Another approach:

(4)

∂_α form a four vector.

Since $\partial_\alpha J^\alpha = 0$ with $J^\alpha = (\rho c, \vec{J})$ is equivalent to charge continuity eqn, which we will take as true for all $\rho, \vec{J}$ and inertial reference frames

$\Rightarrow$ J^α must be components of four vector (by argument of Zangwill 22.5 "Quotient Theorem")

(5)

$$\left[\begin{array}{ll} \nabla\cdot\vec{E} = \rho/\epsilon_0, & \cancel{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \\ \cancel{\nabla\cdot\vec{B} = 0}, & \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \end{array}\right]$$

$\nabla\cdot\vec{B} = 0$ means we can write $\boxed{\vec{B} = \nabla\times\vec{A}}$

$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ becomes $\nabla\times\vec{E} = -\frac{\partial}{\partial t}\nabla\times\vec{A}$

$\nabla\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = \vec{0}$ means we can write $\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\nabla\varphi$

$$\boxed{\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}}$$

⑥

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\nabla \cdot \left(-\nabla\varphi - \frac{\partial \vec{A}}{\partial t}\right) = \rho/\epsilon_0$$

$$\boxed{-\nabla^2\varphi - \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = \rho/\epsilon_0} \qquad (*)$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times(\nabla\times\vec{A}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\boxed{\underset{①}{\nabla(\nabla\cdot\vec{A})} - \underset{②}{\nabla^2\vec{A}} + \underset{③}{\frac{1}{c^2}\nabla\frac{\partial\varphi}{\partial t}} + \underset{④}{\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}} = \mu_0\vec{J}} \qquad (\#)$$

Recall the concept of "gauge".

⑦

$\vec{A}$ and φ are not uniquely determined by $\vec{E}$ and $\vec{B}$.

e.g.

$$\left.\begin{aligned}\vec{A}' &:= \vec{A} + \nabla g \leftarrow \text{arb. scalar field} \\ \varphi' &:= \varphi - \frac{\partial g}{\partial t}\end{aligned}\right\} \text{gauge transformation.}$$

Assuming A and φ give $\vec{E}$ and $\vec{B}$.

$$\nabla\times\vec{A}' = \nabla\times(\vec{A}) + \underbrace{\nabla\times(\nabla g)}_{=\vec{0}}$$
$$= \vec{B}$$

$$-\nabla\varphi' - \frac{\partial\vec{A}'}{\partial t} = -\nabla\varphi + \cancel{\frac{\partial}{\partial t}\nabla g} - \frac{\partial\vec{A}}{\partial t} - \cancel{\frac{\partial}{\partial t}\nabla g}$$
$$= \vec{E}$$

The gauge transformation allows us to set $\nabla\cdot\vec{A}$ to our will.

(8)

$$\nabla \cdot \vec{A}' = \nabla \cdot (\vec{A} + \nabla g)$$

$$\underbrace{\nabla \cdot \vec{A}' - \nabla \cdot \vec{A}}_{} = \nabla^2 g$$

Poisson's eqn with source term (LHS) that can be solved to give gauge transformation.

"Lorenz gauge"

Choose gauge $\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t}$ so that terms (1) and (3) in (‡) cancel.

(*) becomes $\boxed{-\nabla^2 \varphi + \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \rho/\epsilon_0}$

(‡) becomes $\boxed{-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}}$

(xx)

both expressions involve $-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ which can be written as $-\partial_\alpha \partial^\alpha$

(9)

(Recall ∂_α "=" $\left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$)

If $A^\alpha := (\varphi/c, \vec{A})$ and J^β "=" $(\rho c, \vec{J})$

then (xx) on previous page can be written as

$$\boxed{-\partial_\alpha \partial^\alpha A^\beta = \mu_0 J^\beta}$$

Maxwell's equations in a manifestly covariant form

[with the choice of Lorenz gauge]

$$-\bar{\partial}_\alpha \bar{\partial}^\alpha \bar{A}^\beta = \mu_0 \bar{J}^\beta$$

2.10 2024-02-08 Lecture

1

Maxwell's equations from a Lorentz invariant action (following Landau ...)

Recall that for a single particle:

$$S = \underbrace{-mc^2 \int d\tau}_{1} + \underbrace{q \int A_\alpha dx^\alpha}_{2} + \underbrace{\overset{?}{\cdots}}_{3}$$

1 and 2: gave us Lorentz force law

3: will give us Maxwell's equations

First write 2 using J^α.

Involves generalization of 1 and 2 to multiple particles.

$$S = \sum_{\substack{\text{all}\\ \text{particles}}} \left(-mc^2 \int d\tau + q \int A_\alpha dx^\alpha \right)$$

2

Define $S_A :=$ 2 accounting for multiple particles.

$$S_A = \sum q \int A_\alpha dx^\alpha$$
$$= \int dV \rho \int A_\alpha dx^\alpha$$
$$= \int dV \rho \int A_\alpha \frac{dx^\alpha}{dt} dt$$

Recall $J^\alpha := \rho \frac{dx^\alpha}{dt}$, so that we can write:

$$S_A = \int dV dt \, A_\alpha J^\alpha$$

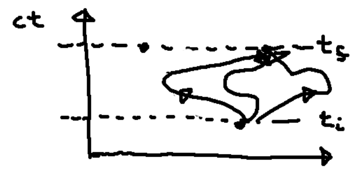
For 3:

$$S_{EM} = \underset{\text{TBD}}{a} \underbrace{\int dV dt \, F_{\alpha\beta} F^{\alpha\beta}}_{\text{Lorentz invariant}}$$

Satisfies two properties
1. Lorentz invariant
2. will lead to principle of superposition (to be seen!)

(3)

Regarding integration:



fix the field configurations at t_i and t_f.

all possible values of A_α throughout 3-space.

(remember $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$)

Vary "trajectory" of the fields : $A_\alpha \rightarrow A_\alpha + \lambda \epsilon_\alpha$

arb. fcn of space and time, but $= 0$ for all space @ t_i and @ t_f.

Apply principle of stationary action:

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = 0$$

where

$$S = S_A + S_{EM}$$

(4)

$$\frac{dS_A}{d\lambda} = \frac{d}{d\lambda} \int dV\, dt\, (A_\alpha + \lambda \epsilon_\alpha) J^\alpha$$

$$= \int dV\, dt\, \epsilon_\alpha J^\alpha$$

$\frac{dS_{EM}}{d\lambda}$ now.

$$F_{\alpha\beta} F^{\alpha\beta} = [\partial_\alpha(A_\beta + \lambda\epsilon_\beta) - \partial_\beta(A_\alpha + \lambda\epsilon_\alpha)] \times [\partial^\alpha(A^\beta + \lambda\epsilon^\beta) - \partial^\beta(A^\alpha + \lambda\epsilon^\alpha)]$$

Differentiate wrt to λ and set $\lambda = 0$

$$\left.\frac{d}{d\lambda}(F_{\alpha\beta}F^{\alpha\beta})\right|_{\lambda=0} = \underbrace{4(\partial_\alpha\epsilon_\beta)F^{\alpha\beta}}_{\text{try integration by parts}}$$

(by algebra)

(5)

$$\partial_\alpha(\epsilon_\beta F^{\alpha\beta}) = (\partial_\alpha \epsilon_\beta) F^{\alpha\beta} + \epsilon_\beta \partial_\alpha F^{\alpha\beta}$$

Rearrange to give:

$$(\partial_\alpha \epsilon_\beta) F^{\alpha\beta} = \partial_\alpha(\epsilon_\beta F^{\alpha\beta}) - \epsilon_\beta \partial_\alpha F^{\alpha\beta}$$

$$\frac{1}{4a} \frac{dS_{EM}}{d\lambda}\bigg|_{\lambda=0} = \int dV dt \left[\underbrace{\partial_\alpha(\epsilon_\beta F^{\alpha\beta})}_{①} - \epsilon_\beta \partial_\alpha F^{\alpha\beta} \right]$$

to deal with ① recall divergence theorem.

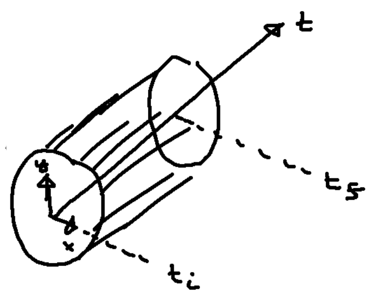
in 3-d : $\int dV \; \nabla \cdot \vec{w} = \oint d\vec{a} \cdot \vec{w}$

in 4-d : $\int dV dt \; \partial_\alpha(\epsilon_\beta F^{\alpha\beta}) = \oint dx_\alpha \, \epsilon_\beta F^{\alpha\beta}$

"surface"

(6)

To visualize "surface" integral, go to two spatial dimensions



Assume fields go to zero as $r \to \infty$.

That means curved part of surface doesn't contribute to integral.

On end-caps $\epsilon_\beta = 0$, so no contribution to integral.

Thus $\oint dx_\alpha \, \epsilon_\beta F^{\alpha\beta} = 0$.

⑦

Combining $dS_A/d\lambda|_{\lambda=0}$ and $dS_{EM}/d\lambda|_{\lambda=0}$ gives

$$\frac{dS}{d\lambda}\Big|_{\lambda=0} = \int dV\,dt\,[\epsilon_\alpha J^\alpha - 4a\,\underbrace{\epsilon_\beta \partial_\alpha F^{\alpha\beta}}_{\epsilon_\alpha \partial_\beta F^{\beta\alpha}}]$$

$$= \int dV\,dt\,\epsilon_\alpha [J^\alpha - 4a\,\partial_\beta F^{\beta\alpha}]$$

Since this is arb. we must have

$$0 = J^\alpha - 4a\,\partial_\beta F^{\alpha\alpha}$$

$$\partial_\beta F^{\alpha\beta} = \frac{1}{4a} J^\alpha$$ ← Maxwell's Equations!!!

?

also need to determine a.

⑧

$\alpha=0$

$$\partial_0 \overset{0}{\cancel{F^{00}}} + \partial_1 \overset{E_x/c}{F^{01}} + \partial_2 \overset{E_y/c}{F^{02}} + \partial_3 \overset{E_z/c}{F^{03}} = \frac{1}{4a} J^0$$

$$\frac{\partial}{\partial x}\left(\frac{E_x}{c}\right) + \frac{\partial}{\partial y}\left(\frac{E_y}{c}\right) + \frac{\partial}{\partial z}\left(\frac{E_z}{c}\right) = \frac{1}{4a}\rho c$$

$$\nabla\cdot\vec{E} = \frac{1}{4a}\rho c^2$$

In SI system $\quad \dfrac{c^2}{4a} = \dfrac{1}{\epsilon_0} \qquad a = \dfrac{\epsilon_0 c^2}{4}$

If we look at $\alpha=1,2,3$ we get

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}.$$

9

And $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

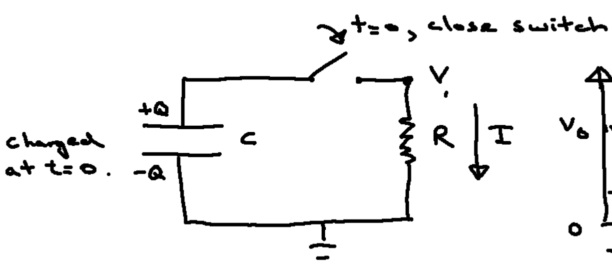
automatically are satisfied by representing $\vec{E}$ and $\vec{B}$ by $\vec{A}$ and Φ.

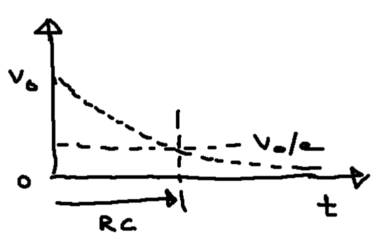
2.11 2024-02-13 Lecture

1

Aside: next Problem Set due 28th of Feb.
no lectures next week (Reading week).

Energy conservation in electromagnetism
Poynting's Theorem





$V = IR$, $V = Q/C$

$$\frac{Q}{RC} = -\frac{dQ}{dt} \Rightarrow Q = Q_0 e^{-t/(RC)} \quad \text{equivalently} \quad V = V_0 e^{-t/(RC)}$$

② What about power dissipated in resistor?

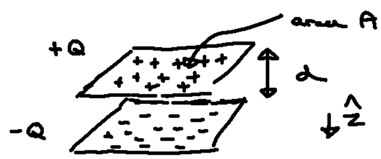
$$P = VI$$
$$= V^2/R$$
$$= \frac{V_0^2}{R} e^{-2t/(RC)}$$

Integrate P wrt time to get energy dissipated.

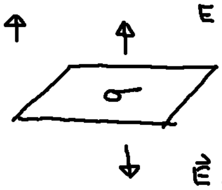
$$E = \int_0^\infty dt\, P(t)$$
$$= \int_0^\infty dt\, \frac{V_0^2}{R} e^{-2t/(RC)}$$
$$= \frac{V_0^2}{R} \frac{RC}{-2} e^{-2t/(RC)} \Big|_{t=0}^{\infty} = \frac{CV_0^2}{2}$$

energy stored in a capacitor.

③ What is $\vec{E}$ between the plates?



$\hat{z}$ ↑



$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z}$$

$$\vec{E} = -\frac{\sigma}{2\epsilon_0}\hat{z}$$

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z} \quad \text{(pointing from + to − plate)}$$

between plates

$$= \frac{Q}{A\epsilon_0}\hat{z}$$

$$V = -\int_b^t \vec{E}\cdot d\vec{l}$$
$$= \frac{Q}{A\epsilon_0} d$$

Rearrange:

$$V\left(\underbrace{\frac{\epsilon_0 A}{d}}_{C}\right) = Q$$

$$\text{Energy} = \frac{1}{2}CV^2$$
$$= \frac{1}{2}\frac{\epsilon_0 A}{d} E^2 d^2$$

electric field

$$\text{Energy} = \frac{1}{2}\epsilon_0 \underbrace{Ad}_{\text{volume}} E^2$$

Suggests that electric field has energy density $\left(\frac{1}{2}\epsilon_0 E^2\right)$

④

Similar argument using an inductor gives energy density for magnetic field $\frac{B^2}{2\mu_0}$

Show these results rigorously.

What is field version of $P = VI$?

$P = q\,\vec{v}\cdot\vec{E}$ Seems feasible. Justification:

Earlier (in an earlier lecture): $\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-(v/c)^2}}\right) = \underbrace{q\vec{v}\cdot\vec{E}}$

Generalize to current density $\vec{J}$:

$$P = \int_{\text{volume}} d\tau\, \vec{J}\cdot\vec{E}$$

write in terms of $\vec{E}$ and $\vec{B}$ using Maxwell's equations.

⑤

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \quad \vec{J} = \frac{\nabla\times\vec{B}}{\mu_0} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Substitute into $P = \int d\tau\, \vec{J}\cdot\vec{E}$

$$P = \int d\tau \left(\frac{\nabla\times\vec{B}}{\mu_0} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot\vec{E} \qquad (*)$$

This form suggests integration by parts // divergence theorem ?

General vector calculus result:

$$\nabla\cdot(\vec{v}\times\vec{w}) = \vec{w}\cdot(\nabla\times\vec{v}) - \vec{v}\cdot(\nabla\times\vec{w})$$

$$-\nabla\cdot(\vec{E}\times\vec{B}) = \vec{E}\cdot(\nabla\times\vec{B}) - \vec{B}\cdot\underbrace{(\nabla\times\vec{E})}_{-\frac{\partial\vec{B}}{\partial t}}$$

Use this result in $(*)$:

(6)

$$P = \int d\tau \left[\underbrace{-\frac{1}{\mu_0} \nabla\cdot(\vec{E}\times\vec{B})}_{} - \underbrace{\frac{1}{\mu_0}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t}}_{} - \epsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t}\right]$$

apply divergence theorem

$$\left[\int d\tau\, \nabla\cdot\vec{v} = \oint \vec{v}\cdot d\vec{a}\right]$$

$$P = \oint \underbrace{-\frac{1}{\mu_0}(\vec{E}\times\vec{B})}_{=:\vec{S}}\cdot d\vec{a} - \frac{d}{dt}\int d\tau \underbrace{\left[\frac{B^2}{2\mu_0}+\frac{1}{2}\epsilon_0 E^2\right]}_{=:\, u_{EM}}$$

"Poynting vector" W/m^2 "energy density of EM field"

Poynting's Theorem:

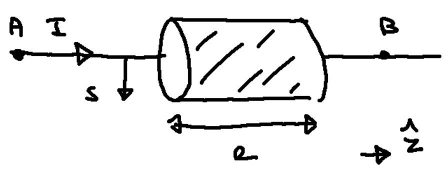
$$P = -\underbrace{\oint \vec{S}\cdot d\vec{a}}_{\text{power flow out of volume}} - \underbrace{\frac{d}{dt}\int d\tau\, u_{EM}}_{\text{decrease in energy stored in field in volume}} \quad \frac{J}{m^3}$$

P: rate of change of energy of particles in a volume

(7)

Example of Poynting vector usage.

idealized resistor



From circuit point of view:

$$V_{AB} = RI$$
$$P = I^2 R$$
$$= VI$$

Let's look at from field point of view using Poynting's theorem:

Steady state $\Rightarrow \frac{d}{dt} u_{EM} = 0$

$\vec{B}$ on surface: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$, $B_\phi 2\pi s = \mu_0 I$

$$B_\phi = \mu_0 I/(2\pi s)$$

8

$$\vec{E} = \frac{V}{\ell}\hat{z}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$= \frac{1}{\cancel{\mu_0}}\frac{V}{\ell}\frac{\cancel{\mu_0} I}{2\pi s}(-\hat{s})$$

$$-\oint \vec{s}\cdot d\vec{a} = \frac{V}{\cancel{\ell}}\frac{I}{\cancel{2\pi s}}\overbrace{\cancel{2\pi s}\,\cancel{\ell}}^{\text{area}}$$

$$= \underbrace{V I}_{\text{same as LHS of Poynting's theorem!!!}}$$

2.12 2024-02-15 Lecture

Lecture follows treatment in Brau C. A. Brau, *Modern problems in classical electrodynamics* (Oxford University Press, New York, 2004).

①

Relativistically covariant momentum/energy conservation

(following "Modern Problems in Classical Electrodynamics" Brau)

$$S_{EM} = \frac{1}{4\mu_0}\int d^3x\, dt \;\; F_{\alpha\beta}F^{\alpha\beta}$$

Today we will write $\frac{1}{4\mu_0} = \frac{1}{2}k_1$ (defines k_1)

$$D := \frac{1}{2}k_1 F_{\alpha\beta}F^{\alpha\beta} \qquad \text{"Lagrangian density"}$$

On the "hunch" that we will be reproducing Poynting's law, focus on

$$\partial_\gamma D$$

②

$$\partial_\gamma D = \frac{1}{2}k_1\Big[\underbrace{(\partial_\gamma F_{\alpha\beta})F^{\alpha\beta} + F_{\alpha\beta}(\partial_\gamma F^{\alpha\beta})}_{\text{the same!}}\Big]$$

$$= k_1(\partial_\gamma F_{\alpha\beta})F^{\alpha\beta} \qquad (*)$$

$$= k_1 F^{\alpha\beta}\big(\partial_\gamma(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\big)$$

$$= k_1 F^{\alpha\beta}(\partial_\gamma\partial_\alpha A_\beta - \partial_\gamma\partial_\beta A_\alpha)$$

$$= k_1 F^{\alpha\beta}\partial_\gamma\partial_\alpha A_\beta - \underbrace{k_1 F^{\alpha\beta}\partial_\gamma\partial_\beta A_\alpha}_{\text{swap } \alpha \text{ and } \beta}$$

$$\partial_\gamma D = k_1 F^{\alpha\beta}\partial_\alpha\partial_\gamma A_\beta - k_1 \underbrace{F^{\beta\alpha}}_{-F^{\alpha\beta}}\partial_\alpha\partial_\gamma A_\beta$$

$$= 2k_1 F^{\alpha\beta}\partial_\alpha\partial_\gamma A_\beta \qquad (*)$$

3

To make a symmetric "stress tensor" we will add the term:

$$F^{\alpha\beta}\partial_\alpha\partial_\beta A_\gamma$$ which is allowed, since it is zero.

Justification:

$$\begin{aligned} F^{\alpha\beta}\partial_\alpha\partial_\beta A_\gamma &= -F^{\beta\alpha}\partial_\alpha\partial_\beta A_\gamma \quad \text{switch } \alpha,\beta \\ &= -F^{\alpha\beta}\partial_\beta\partial_\alpha A_\gamma \\ &= -F^{\alpha\beta}\partial_\alpha\partial_\beta A_\gamma \end{aligned}$$

Anything equal to its negative is zero. //

$$\partial_\gamma D = \underbrace{2k_1 F^{\alpha\beta}\partial_\alpha\partial_\gamma A_\beta}_{\text{from } (*)} - \underbrace{2k_1 F^{\alpha\beta}\partial_\alpha\partial_\beta A_\gamma}_{=0}$$

4

$$\begin{aligned} \partial_\gamma D &= 2k_1 F^{\alpha\beta}\left(\partial_\alpha\partial_\gamma A_\beta - \partial_\alpha\partial_\beta A_\gamma\right) \\ &= 2k_1 F^{\alpha\beta}\partial_\alpha F_{\gamma\beta} \qquad (\ddagger) \end{aligned}$$

From

$$\partial_\alpha\left(F^{\alpha\beta}F_{\gamma\beta}\right) = \left(\partial_\alpha F^{\alpha\beta}\right)F_{\gamma\beta} + F^{\alpha\beta}\partial_\alpha\left(F_{\gamma\beta}\right)$$

we can write

$$F^{\alpha\beta}\partial_\alpha F_{\gamma\beta} = \partial_\alpha\left(F^{\alpha\beta}F_{\gamma\beta}\right) - \left(\partial_\alpha F^{\alpha\beta}\right)F_{\gamma\beta}$$

Substitute into $(\ddagger)$

$$\partial_\gamma D = 2k_1\left[\partial_\alpha\left(F^{\alpha\beta}F_{\gamma\beta}\right) - \underbrace{\left(\partial_\alpha F^{\alpha\beta}\right)}F_{\gamma\beta}\right]$$

can be replaced using Maxwell's equations. Specialize to no sources initially $\partial_\alpha F^{\alpha\beta} = 0$

$$\partial_\gamma D = 2k_1\partial_\alpha\left(F^{\alpha\beta}F_{\gamma\beta}\right)$$

(5)

$$\partial_\gamma D = (\partial_\alpha D)\, \underbrace{\delta^\alpha{}_\gamma}_{\text{Kronecker delta.}}$$

$$0 = \underbrace{\partial_\gamma D}_{\text{just derived}} - \underbrace{(\partial_\alpha D)}_{\text{from } \partial_\alpha (F^{\mu\nu} F_{\mu\nu})} \delta^\alpha{}_\gamma$$

$$0 = 2k_1 \partial_\alpha \left(F^{\alpha\beta} F_{\gamma\beta} \right) - \partial_\alpha \left(\tfrac{1}{2} k_1 F^{\mu\nu} F_{\mu\nu} \right) \delta^\alpha{}_\gamma$$

$$= \partial_\alpha \left(2k_1 \left\{ F^{\alpha\beta} F_{\gamma\beta} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \delta^\alpha{}_\gamma \right\} \right)$$

$$\boxed{0 = \partial_\alpha T^\alpha{}_\gamma}$$

without sources ! (sources are next lecture).

where $T^\alpha{}_\gamma := -2k_1 \left\{ F^{\alpha\beta} F_{\gamma\beta} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \delta^\alpha{}_\gamma \right\}$

(6)

This Lorentz stress tensor contains things that are known to us :

$$T^\alpha{}_\gamma \text{ “=” } \begin{bmatrix} \frac{\epsilon_0 E^2}{2} + \frac{1}{\mu_0} B^2 & -cg_x & -cg_y & -cg_z \\ cg_x & & & \\ cg_y & & \ddots & \\ cg_z & & & \end{bmatrix}$$

where $\vec{g} = \epsilon_0 (\vec{E} \times \vec{B})$

From this form we see that Poynting's theorem arises from $\partial_\alpha T^\alpha{}_0 = 0$

2.13 2024-02-27 Lecture

①

Before break: $\partial_\alpha T^{\alpha}{}_{0} = 0$ (fields without charges)

Gives Poynting's theorem.

For momentum conservation:

$$\vec{0} = \underbrace{\frac{d\vec{p}}{dt}}_{\text{changing momentum of the particles}} + \underbrace{\underbrace{\int d\tau}_{\text{volume integral}} \epsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B})}_{\text{changing momentum of fields within volume}} - \underbrace{\underbrace{\oint}_{\text{surface integral.}} \overset{\leftrightarrow}{T} \cdot d\vec{a}}_{\text{momentum flow in/out of volume}}$$

②

where

$$T_{ij} := \epsilon_0 \left(E_i E_j - \frac{1}{2}E^2 \delta_{ij}\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2\delta_{ij}\right)$$

is "Maxwell's stress tensor".

Let's look at these in the context of:

Electromagnetic Plane Waves

$\nabla \cdot \vec{E} = 0$ (1) $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (2)

$\nabla \cdot \vec{B} = 0$ (3) $\quad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ (4)

(3)

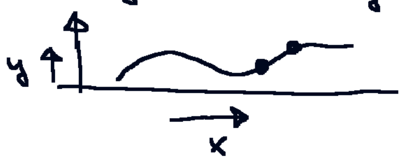
$$\underbrace{\nabla\times(\nabla\times\vec{E})} = \nabla\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

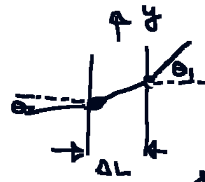
$$\nabla(\underbrace{\nabla\cdot\vec{E}}_{=0}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\underbrace{(\nabla\times\vec{B})}_{}$$ make use of (4)

$$\boxed{\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

Can also find that $\nabla^2\vec{B} = \mu_0\epsilon_0\dfrac{\partial^2\vec{B}}{\partial t^2}$

analogous to string under tension.





$$F_y = T\tan\theta - T\tan\theta_2$$

$$\underbrace{\lambda\Delta L}_{m}\frac{d^2y}{dt^2} = T\left(\frac{dy_1}{dx} - \frac{dy_2}{dx}\right)$$

mass density

(4)

$$\boxed{\lambda\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}} \quad (*) \qquad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Griffiths: "a wave is a disturbance of a continuous medium that propagates with a fixed shape at a constant velocity"

So then why do we call (*) a "wave equation"?

Suppose we have any function of a single variable:

twice differentiable.

(5)

Assert: $y = f(x-vt)$ $(\pm)$
is a solution to wave equation
(with a constraint on v).

Substitute $(\pm)$ into $(*)$:

$$\lambda f''(x-vt)v^2 = T f''(x-vt)$$

Thus $(*)$ is satisfied is $v = \pm\sqrt{\frac{T}{\lambda}}$

$$\sqrt{\frac{N}{kg/m}} = \sqrt{\frac{\cancel{kg}\cdot m/s^2}{\cancel{kg}/m}} = \frac{m}{s} \quad \checkmark$$

(6)

By analogy $\nabla^2 E = \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2}$ $(*)$

and $v^2 = \frac{1}{\mu_0\epsilon_0}$ $\Rightarrow$ $v \triangleq 3 * 10^8$ m/s
speed of light.
$=: $"c"

e.g. $\vec{E} = \vec{E}_0 \cos(kz-\omega t)$ satisfies $(*)$

where $\frac{\omega}{k} = c$ $\left(= \frac{1}{\sqrt{\mu_0\epsilon_0}}\right)$

Let's use complex numbers to simplify some of the algebra.

7

Complex "phasor" representation:

$$\vec{E}(\vec{r},t) = E_0 e^{i(kz-\omega t)}$$

where we interpret $\mathrm{Re}(\vec{E}(r,t))$ as being the actual field.

Why does this work?

Recall $\mathrm{Re}(z) = (z+z^*)/2$

The linear nature of Maxwell's equations (without sources) allow superposition. So if $\vec{E}$ is a solution and so is $\vec{E}^*$ then $\mathrm{Re}(\vec{E})$ is a solution.

8

Next lecture: discuss polarization of electromagnetic waves (using phasor representation).

2.14 2024-02-29 Lecture

Lecture cancelled.

2.15 2024-03-05 Lecture

1

Plane wave solutions to Maxwell's equations

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(kz-\omega t)} \qquad (\times)$$

Double check that this satisfies the wave equation:

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Substitute into

$$\nabla^2 \vec{E} = \vec{E}_0 (ik)^2 e^{i(kz-\omega t)}$$

$$\frac{\partial^2}{\partial t^2} \vec{E} = \vec{E}_0 (i\omega)^2 e^{i(kz-\omega t)}$$

$$k^2 = \mu_0 \epsilon_0 \omega^2$$

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}\ c \simeq 3*10^8 \text{ m/s}$$

2

Apply $\nabla \cdot \vec{E} = 0$ to $(\times)$

$$\nabla \cdot \left(\vec{E}_0\, e^{i(kz-\omega t)}\right) = 0 \qquad (*)$$

Remember vector identity

$$\nabla \cdot (\underbrace{f}_{\text{scalar field}}\ \underbrace{\vec{v}}_{\text{vector field}}) = f\, \nabla \cdot \vec{v} + \nabla f \cdot \vec{v}$$

$$\nabla \cdot \left(\vec{E}_0\, e^{i(kz-\omega t)}\right) = \vec{E}_0 \cdot ik\hat{z}\, e^{i(kz-\omega t)}$$

Together with $(*)$, this tells us that

$\vec{E}_0 \cdot \hat{z} = 0$ i.e. a transverse wave.

③

What is the relationship between $\vec{E}$ and $\vec{B}$?

Use $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$.

$$\nabla\times\left(\vec{E}_0 e^{i(kz-\omega t)}\right) = -\frac{\partial}{\partial t}\left(\vec{B}_0 e^{i(kz-\omega t)}\right)$$

Remember the identity: $\nabla\times(f\vec{v}) = f\,\nabla\times\vec{v} + \nabla f\times\vec{v}$

$$ik\hat{z}\,\cancel{e^{i(kz-\omega t)}}\times\vec{E}_0 = B_0(i\omega)\,\cancel{e^{i(kz-\omega t)}}$$

$$\vec{B}_0 = \frac{k}{\omega}\,\hat{z}\times\vec{E}_0$$

$$\vec{B}_0 = \frac{1}{c}\,\hat{z}\times\vec{E}_0 \qquad (\mathrm{xx})$$

④

Is this result consistent with $\nabla\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$?

By the same manipulations that we used with $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$, we get:

$$\vec{E}_0 = -\hat{z}\times\vec{B}_0 c \qquad (\#\#)$$

Is this result consistent with (xx)?

$$\hat{z}\times\vec{E}_0 = \hat{z}\times(-\hat{z}\times\vec{B}_0 c) \qquad \left[\begin{aligned}&\vec{a}\times(\vec{b}\times\vec{c})\\ &= \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})\end{aligned}\right]$$

"BAC–CAB"

$$= -\left[\hat{z}(\hat{z}\cdot\vec{B}_0 c) - c\vec{B}_0(\hat{z}\cdot\hat{z})\right]$$

Remember that $\hat{z}\cdot\vec{B}_0 = 0$ (from $\nabla\cdot\vec{B} = 0$)

$\hat{z}\times\vec{E}_0 = c\vec{B}_0$, or $\boxed{\vec{B}_0 = \frac{1}{c}\hat{z}\times\vec{E}_0.}$ We have shown (##) implies (xx). Vice versa also works.

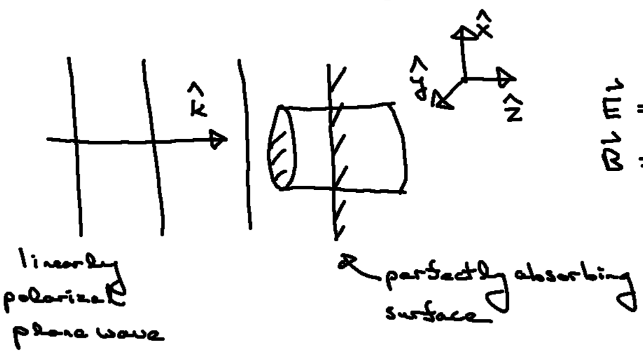
(5)

Generalize to arb. direction of propagation and summarize:

$$\boxed{\begin{aligned} \vec{E} &= \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{B} &= \frac{\hat{k}\;\vec{E}_0}{c}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{E}_0\cdot\hat{k} &= 0\,, \quad \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c \end{aligned}}$$

(6)

<u>Energy and momentum transport by plane electromagnetic waves</u>

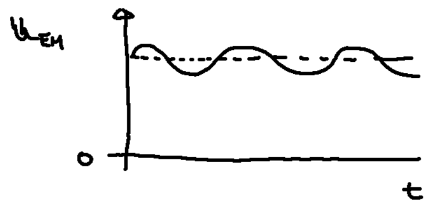


$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r}-\omega t)$$
$$\vec{B} = \hat{k}\times \frac{\vec{E}_0}{c} \cos(\vec{k}\cdot\vec{r}-\omega t)$$

Apply Poynting's theorem:

$$\underset{\text{heating of surface}}{P} = -\frac{d}{dt}\overbrace{\int dV \left[\frac{\epsilon_0 E^2}{2} + \frac{1}{\mu_0}\frac{B^2}{2}\right]}^{=:\,U_{EM}} - \frac{1}{\mu_0}\underbrace{\oint(\vec{E}\times\vec{B})\cdot d\vec{a}}_{\text{"Poynting"}}$$

(7)



$$\left\langle \frac{d}{dt} u_{EM} \right\rangle_t = 0$$

time-average.

Only the Poynting vector term contributes to average heating.

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$= \frac{1}{\mu_0} \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t) \times (\hat{k} \times \vec{E}_0/c) \cos(\vec{k}\cdot\vec{r} - \omega t)$$

$$\langle \vec{S} \rangle_t = \frac{1}{\mu_0} \vec{E}_0 \times (\hat{k} \times \vec{E}_0/c) \underbrace{\langle \cos^2(\vec{k}\cdot\vec{r} - \omega t) \rangle_t}_{1/2}$$

$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$= \frac{1}{\mu_0}\left[\hat{k} \left(\vec{E}_0 \cdot \frac{\vec{E}_0}{c}\right) - \frac{\vec{E}_0}{c} \left(\vec{E}_0 \cdot \hat{k}\right) \right]$$

$= 0$ because $\nabla \cdot \vec{E} = 0$

(8)

$$\boxed{\langle \vec{S} \rangle_t = \frac{E_0^2}{2\mu_0 c} \hat{k}}$$

$\oint \vec{S} \cdot d\vec{a}$ over surface, only need to worry about endcap contribution,

Subing into Poynting's theorem gives:

$$P = \frac{E_0^2}{2\mu_0 c} A$$

For a plane electromagnetic wave, linearly polarized, with amplitude E_0, we associate a time averaged power per unit area of:

⑨

$$\boxed{\frac{P}{A} = \frac{E_0^2}{2\mu_0 c}}$$ "analogous to" $$P = \frac{V^2}{R}$$

What units does $\mu_0 c$ have?

$$\left(\frac{V}{m}\right)^2 \frac{m^2}{W} \to \frac{V^2}{W} \to \frac{V^2}{V \cdot A} \to \frac{V}{A} \to \Omega$$

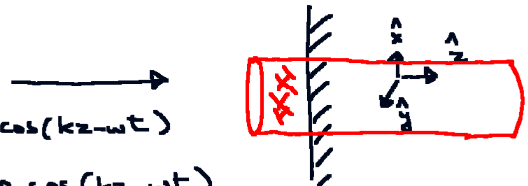
$\mu_0 c \simeq 377\Omega$, "impedance of free space"

Next Lecture: Look at momentum transport.

2.16 2024-03-12 Lecture

(1)

Momentum conservation for a plane, linearly polarized electromagnetic wave being fully absorbed



$\vec{E} = \hat{x}\, E_0 \cos(kz - \omega t)$

$\vec{B} = \hat{y}\, \frac{E_0}{c} \cos(kz - \omega t)$

General result for momentum conservation: (for a specific volume)

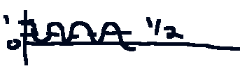
$$\underbrace{\frac{d\vec{p}}{dt}}_{\text{momentum change of particles in volume}} = \underbrace{-\oint \overleftrightarrow{T} \cdot d\vec{a}}_{\text{momentum flow into volume}} + \underbrace{\frac{d}{dt}\int dV\, \epsilon_0 \vec{E} \times \vec{B}}_{\text{change in momentum stored in EM field within volume}}$$

$$T_{ij} := \epsilon_0 \left(E_i E_j - \tfrac{1}{2} E^2 \delta_{ij}\right) + \frac{1}{\mu_0}\left(B_i B_j - \tfrac{1}{2} B^2 \delta_{ij}\right)$$

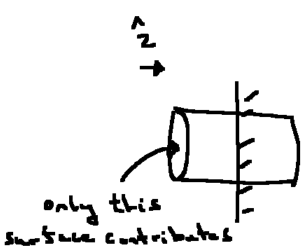
(2)

$$\langle \overleftrightarrow{T} \rangle_t = \frac{1}{2}\left\{ E_0^2 \frac{\epsilon_0}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \frac{1}{2} \frac{E_0^2}{c^2 \mu_0} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

$\langle \cos^2(kx - \omega t) \rangle_t = \frac{1}{2}$



$$\langle \overleftrightarrow{T} \rangle_t = \frac{1}{2} E_0^2 \epsilon_0 \begin{bmatrix} 0 & & \\ & 0 & \\ & & -1 \end{bmatrix}$$



$$\left\langle \frac{d\vec{p}}{dt} \right\rangle_t = -\int \langle \overleftrightarrow{T} \rangle \cdot d\vec{a}$$

$$= -\hat{z}\, \cdot -\frac{1}{2} \epsilon_0 E^2 A\, (-1)$$

$$= -\hat{z}\, \tfrac{1}{2} \epsilon_0 E^2 A$$

Sign doesn't make sense! Fix later.

③

It is natural to associate a momentum per unit area per unit time with a linearly polarized EM plane wave :

$$\frac{1}{A}\left\langle \frac{dp}{dt} \right\rangle_t = \underbrace{\hat{k}}_{\text{direction of propagation}} \; \frac{1}{2}\epsilon_0 E^2$$

$$\frac{1}{c^2} = \mu_0 \epsilon_0$$

$$\epsilon_0 = \frac{1}{\mu_0 c^2}$$

$$\boxed{\frac{1}{A}\left\langle \frac{dp}{dt} \right\rangle_t = \hat{k}\,\frac{1}{2}\,\frac{1}{\mu_0 c^2}\,E^2}$$

$$\frac{1}{m^2}\;\; kg\cdot\frac{m}{s}\;\;\frac{1}{s} \rightarrow \left(\frac{J}{kg\cdot\frac{m^2}{s^2}}\right)\frac{1}{m}\,\cancel{kg}\,\frac{1}{\cancel{s^2}} \rightarrow \frac{J}{m^3}$$

$$\left(\frac{V}{m}\right)^2 \frac{1}{V/A}\;\frac{s}{m} \rightarrow \frac{VA}{m^3}\,s \rightarrow \frac{J/\cancel{s}}{m^3}\cdot\cancel{s} \rightarrow \frac{J}{m^3}$$

④

Solutions of Maxwell's equations with sources i.e., ρ and $\vec{J}$

3-vector notation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\rho/\epsilon_0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = f$$

Lorentz tensor notation

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\Phi}{c}\right) + \nabla\cdot\vec{A} = 0$$

$$\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$\partial_\beta F^{\alpha\beta} = \partial_\beta \partial^\alpha A^\beta - \partial_\beta \partial^\beta A^\alpha$$

$$\left(\frac{\Phi}{c}, \vec{A}\right) \qquad = \partial^\alpha \underbrace{\partial_\beta A^\beta}_{\text{can set } = 0} - \partial_\beta \partial^\beta A^\alpha$$

, i.e. Lorenz gauge

$$\underbrace{\partial_\beta \partial^\beta}_{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2} A^\alpha = -\mu_0 J^\alpha \qquad (\rho c, \vec{J})$$

(5) Aside

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$

units of φ must be
units A times $\frac{m}{s}$

Physical motivation for form of solutions:
Consider scalar potential:

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)\Phi = -\rho/\epsilon_0 \rightarrow \text{time indep.} \qquad \nabla^2\Phi = -\rho/\epsilon_0$$

very familiar!

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int dv' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

(6)

Guess (?) :

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int dv' \frac{\rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \qquad *$$

Is this really the solution?

Two equally valid approaches:

1) verify by substitution into wave eqn.
2) generate constructively. ← do this next lecture.

2.17 2024-03-14 Lecture

1

Constructive approach

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \qquad *$$

Use Fourier transforms.

$$\vec{J}(\vec{r},t) = \vec{J}_\omega(\vec{r})\, e^{-i\omega t}$$
$$\vec{A}(\vec{r},t) = \vec{A}_\omega(\vec{r})\, e^{-i\omega t}$$

(Fourier decomposition will handle arb. time-dependance)

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{A}_\omega(\vec{r}) = -\mu_0 \vec{J}_\omega(\vec{r})$$

2

Now use spatial Fourier transforms.

$$\vec{A}_\omega(\vec{r}) = \int d^3k\, e^{i\vec{k}\cdot\vec{r}}\, \vec{A}_{\omega,\vec{k}} \qquad (*)$$

Inverse:

$$\vec{A}_{\omega,\vec{k}} = \frac{1}{(2\pi)^3}\int d^3r\, e^{-i\vec{k}\cdot\vec{r}'}\, \vec{A}_\omega(\vec{r}) \qquad (\ddagger)$$

Also use Fourier transform for current density.

Check: Evaluate RHS of $(*)$ using $(\ddagger)$

$$\text{RHS} = \frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot\vec{r}} \int d^3r'\, e^{-i\vec{k}\cdot\vec{r}'}\, \vec{A}_\omega(\vec{r}')$$
$$= \frac{1}{(2\pi)^3}\int d^3r'\, \vec{A}_\omega(\vec{r}') \int d^3k\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

③

$$\text{RHS} = \frac{1}{(2\pi)^3}\int d^3r'\ \vec{A}_\omega(\vec{r}')\,(2\pi)^3\ \underbrace{\delta^3(\vec{r}-\vec{r}')}_{\delta(x-x')\delta(y-y')\delta(z-z')}$$

$$= \vec{A}_\omega(\vec{r}) \quad \checkmark$$

$$\int dk\ e^{ikx} = 2\pi\,\delta(x)$$

$$\left(-k^2+\frac{\omega^2}{c^2}\right)\vec{A}_{\omega,\vec{k}} = -\mu_0\vec{J}_{\omega,\vec{k}}$$

$$\vec{A}_{\omega,\vec{k}} = \frac{\mu_0\vec{J}_{\omega,\vec{k}}}{k^2-\left(\frac{\omega}{c}\right)^2} \qquad \frac{1}{(2\pi)^3}\int dV'\,e^{-i\vec{k}\cdot\vec{r}'}\vec{J}_\omega(\vec{r}')$$

inverse transform

$$\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{(2\pi)^3}\int d^3k\ \frac{1}{k^2-\left(\frac{\omega}{c}\right)^2}\int dV'\ e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\ \vec{J}_\omega(\vec{r}')$$

④

Switch orders of integration, to get:

$$\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{(2\pi)^3}\int dV'\ \vec{J}_\omega(\vec{r}')\underbrace{\int d^3k\ \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{k^2-\left(\frac{\omega}{c}\right)^2}}_{=:\,G_\omega(\vec{r}-\vec{r}')} \qquad (\times)$$

Use spherical coordinates in k-space with $\vec{r}-\vec{r}'$ defining the direction of the z-axis.

$$\int d^3k \to \int_0^{2\pi} d\phi\int_0^{\pi} d\theta\sin\theta\int_0^{\infty} dk\ k^2\ ,\quad \vec{k}\cdot(\vec{r}-\vec{r}') \to k\,|\vec{r}-\vec{r}'|\cos\theta$$

⑤

$$G_\omega(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^2}\int_0^\pi d\theta \sin\theta \int_0^\infty dk\, k^2 \, \frac{e^{ik|\vec{r}-\vec{r}\,'|\cos\theta}}{k^2-(\omega/c)^2}$$

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta \,, \quad d\theta = \frac{du}{-\sin\theta}$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dk \frac{k^2}{k^2-\left(\frac{\omega}{c}\right)^2} \underbrace{\int_1^{-1} du \frac{\sin\theta}{-\sin\theta}}_{\int_{-1}^{1} du} e^{ik|\vec{r}-\vec{r}\,'|u}$$

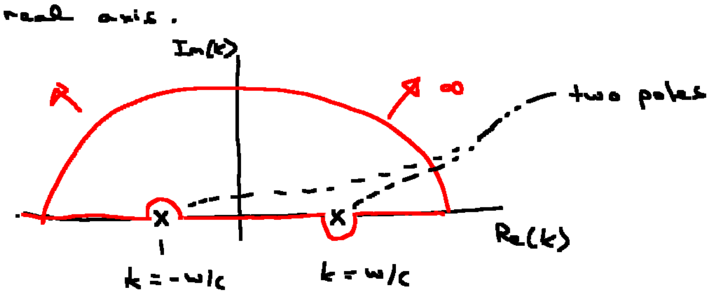
$$= \frac{1}{(2\pi)^2}\int_0^\infty dk \frac{k^2}{k^2-\left(\frac{\omega}{c}\right)^2} \left.\frac{e^{ik|\vec{r}-\vec{r}\,'|u}}{ik|\vec{r}-\vec{r}\,'|}\right|_{-1}^{1}$$

⑥

$$G_\omega(r-\vec{r}\,') = \frac{1}{(2\pi)^2}\int_0^\infty dk \frac{k^2}{k^2-\left(\frac{\omega}{c}\right)^2} \frac{e^{ik|\vec{r}-\vec{r}\,'|}-e^{-ik|\vec{r}-\vec{r}\,'|}}{ik|\vec{r}-\vec{r}\,'|}$$

$$= \frac{2}{(2\pi)^2}\frac{1}{|\vec{r}-\vec{r}\,'|}\int_0^\infty dk \frac{k}{k^2-\left(\frac{\omega}{c}\right)^2}\sin(k|\vec{r}-\vec{r}\,'|)$$

classic problem in Complex variables.
poles on the real axis.



(7)

$$G_\omega(\vec{r}-\vec{r}\,') = \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}\,'|}\, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}$$

$$\vec{A}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\; e^{-i\omega t}\,\underbrace{\vec{A}_\omega(\vec{r})}_{\text{from } (*) \text{ on pg 4}}$$

$$= \int_{-\infty}^{\infty} d\omega\; e^{-i\omega t}\,\frac{\mu_0}{4\pi}\int dv'\,\frac{1}{|\vec{r}-\vec{r}\,'|}\, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}\;\frac{1}{2\pi}\int dt'\,\vec{J}(\vec{r}\,',t')\,e^{+i\omega t'}$$

$$= \frac{1}{\cancel{2\pi}}\frac{\mu_0}{4\pi}\int dv'\,\frac{1}{|\vec{r}-\vec{r}\,'|}\int dt'\,\vec{J}(\vec{r}\,',t')\,\underbrace{\int d\omega\; e^{i\omega\left(t'-t+\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}}_{\cancel{2}\pi\,\delta\left(t'-t+\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}$$

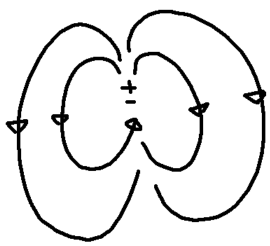
$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int dv'\,\frac{\vec{J}\left(\vec{r}\,',\overbrace{t-\frac{|\vec{r}-\vec{r}\,'|}{c}}^{\text{"retarded time"}}\right)}{|\vec{r}-\vec{r}\,'|}}$$

2.18 2024-03-19 Lecture

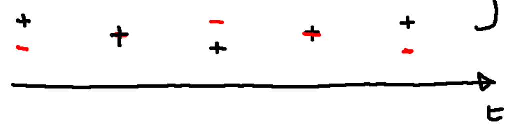
① Electric dipole radiation

• $\vec{E}$?

$\pm \qquad \vec{p} = q\Delta\vec{\ell} \qquad\qquad \vec{E}_{dipole} = \dfrac{p}{4\pi\epsilon_0 r^3}\underbrace{\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}_{\hat{z}\text{ is in direction of }\vec{p}}$



What happens if the dipole moment is time-dependent?



We will find

$E \propto \frac{1}{r}$

$B \propto \frac{1}{r}$

radiation fields that carry energy ~~dss~~ to ∞.

②

$$\vec{J}(\vec{r},t) = \underbrace{\vec{j}_0(t)}_{\underbrace{\dot{\vec{p}}(t)}_{\frac{C\cdot m}{s}}}\ \underbrace{\delta^3(\vec{r})}_{\frac{1}{m^3}} \rightarrow \frac{C}{s\cdot m^2}$$

$$\rho = -\nabla\cdot\left(\vec{p}\,\delta^3(\vec{r})\right)$$

$$\frac{1}{m}\quad C\cdot m\quad \frac{1}{m^3} \rightarrow \frac{C}{m^3}$$

Together, $\vec{J}(\vec{r},t) = \dot{\vec{p}}(t)\,\delta^3(\vec{r})$ and $\rho = \nabla\cdot(\vec{p}\,\delta^3(\vec{r}))$ satisfy the charge-continuity equation:

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \iff \oint \vec{J}\cdot d\vec{a} = -\frac{d}{dt}\overbrace{\int dV\,\rho}^{Q_{enclosed}}$$

③

Use expressions for potentials from the last lecture:

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int dV' \; \frac{\vec{J}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{\underbrace{|\vec{r}-\vec{r}'|}_{\mathscr{r}}}$$

In this case:

$$= \frac{\mu_0}{4\pi}\int dV' \; \frac{\dot{\vec{p}}(t-\mathscr{r}/c)\,\delta^3(\vec{r}')}{\mathscr{r}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{\dot{\vec{p}}(t-r/c)}{r}$$

(in my notes, I get Φ from the Lorenz gauge condition $\frac{\partial \Phi}{\partial t} = -c^2\,\nabla\cdot\vec{A}$; let's try direct approach instead.)

④

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int dV' \; \frac{\rho(\vec{r}', t-\mathscr{r}/c)}{\mathscr{r}}$$

$$= \frac{-1}{4\pi\epsilon_0}\int dV' \; \frac{\nabla'\cdot\left[\vec{p}(t-\mathscr{r}/c)\,\delta^3(\vec{r}')\right]}{\mathscr{r}}$$

$$= \frac{-1}{4\pi\epsilon_0}\int dV' \; \frac{(\nabla'\cdot\vec{p}(t-\mathscr{r}/c))\,\delta^3(\vec{r}') + \vec{p}(t-\mathscr{r}/c)\cdot\nabla'\delta^3(\vec{r}')}{\mathscr{r}}$$

Looks awkward!!! Let's go from $\vec{A}$ instead, using the Lorenz gauge condition: $\partial\Phi/\partial t = -c^2\,\nabla\cdot\vec{A}$

$$\frac{\partial \Phi}{\partial t} = -c^2\,\frac{\mu_0}{4\pi}\,\nabla\cdot\left(\dot{\vec{p}}(t-r/c)\,\frac{1}{r}\right)$$

using $\nabla\cdot(f\vec{v}) = \nabla f\cdot\vec{v} + f\,\nabla\cdot\vec{v}$

$$= -\frac{1}{4\pi\epsilon_0}\left[\underbrace{(\nabla\cdot\dot{\vec{p}}(t-r/c))\frac{1}{r}} + \dot{\vec{p}}(t-r/c)\cdot\nabla\frac{1}{r}\right]$$

(5)

$$\frac{\partial \Phi}{\partial t} = -\frac{1}{4\pi\epsilon_0}\Big[\underbrace{\ddot{\vec{p}}(t-r/c)\cdot -\frac{1}{c}\hat{r}}_{\text{using chain rule}}\ \frac{1}{r} + \dot{\vec{p}}(t-r/c)\cdot\left(-\frac{\hat{r}}{r^2}\right)\Big]$$

$$\frac{\partial \Phi}{\partial t} = \frac{\hat{r}\cdot}{4\pi\epsilon_0}\left[\frac{\ddot{\vec{p}}(t-r/c)}{cr} + \frac{\dot{\vec{p}}(t-r/c)}{r^2}\right]$$

Integrate wrt time, to get:

$$\Phi = \frac{\hat{r}\cdot}{4\pi\epsilon_0}\left[\frac{\overbrace{\dot{\vec{p}}(t-r/c)}^{\dot{\vec{p}}_{ret}}}{cr} + \frac{\overbrace{\vec{p}(t-r/c)}^{\vec{p}_{ret}}}{r^2}\right] + \text{static time-indep field ?}$$

So we have $\vec{A}$ and Φ due to a time-dependent point dipole.

(6)

Compute $\vec{E}$ and $\vec{B}$ by $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla\times\vec{A}$

messy algebra.

$$\vec{B} = -\frac{\mu_0}{4\pi}\hat{r}\times\left[\frac{\ddot{\vec{p}}_{ret}}{r^2} + \frac{\dddot{\vec{p}}_{ret}}{cr}\right]$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{r}(\hat{r}\cdot\vec{p}_{ret}) - \vec{p}_{ret}}{r^3} + \frac{3\hat{r}(\hat{r}\cdot\dot{\vec{p}}_{ret}) - \dot{\vec{p}}_{ret}}{cr^2} + \frac{\hat{r}(\hat{r}\cdot\ddot{\vec{p}}_{ret}) - \ddot{\vec{p}}_{ret}}{c^2 r}\right]$$

radiation fields

static dipole term

2.19 2024-03-21 Lecture

1

Radiated power from an electric dipole.

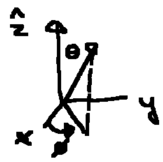
Use $1/r$ terms in $\vec{E}$ and $\vec{B}$.

$$\vec{B}_{rad} = -\frac{\mu_0}{4\pi}\, \hat{r} \times \frac{\ddot{\vec{p}}_{ret}}{cr} \qquad \ddot{p}(t - r/c)$$

$$\vec{E}_{rad} = \frac{1}{4\pi\epsilon_0}\left(\frac{\hat{r}\,(\hat{r}\cdot\ddot{\vec{p}}_{ret}) - \ddot{\vec{p}}_{ret}}{c^2 r}\right)$$

For concreteness, take $\vec{p}(t) = p_0 \cos(\omega t)\, \hat{z}$.

Make use of spherical coordinates:



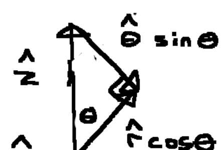
2

To get $\vec{B}_{rad}$ in a more convenient form:

$$\hat{r} \times \hat{z} = \hat{r} \times (\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta})$$

$$= -\sin\theta\, \underbrace{\hat{r} \times \hat{\theta}}_{\hat{\phi}}$$

$$= -\sin\theta\, \hat{\phi}$$



$$\vec{B}_{rad} = -\frac{\mu_0}{4\pi}\, \underbrace{p_0\,(-\cos(\omega(t - r/c))\,\omega^2}_{\ddot{p}(t_{ret})}\, \frac{1}{cr}(-\sin\theta)\hat{\theta}$$

$$\underbrace{\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta}$$

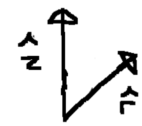
$$\boxed{\vec{B}_{rad} = -\frac{\mu_0}{4\pi}\, \frac{\omega^2 p_0}{cr}\, \sin\theta\, \cos(\omega(t - r/c))\, \hat{\phi}}$$

✓

③

$$\vec{E}_{rad} = \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{r}(\hat{r}\cdot\ddot{\vec{p}}_{ret}) - \ddot{\vec{p}}_{ret}}{c^2 r}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{p_0(-\omega^2)\cos(\omega(t-r/c))}{c^2 r}\,\underbrace{\hat{r}\,\underbrace{(\hat{r}\cdot\hat{z})}_{\cos\theta} - \hat{z}}_{\sin\theta\,\hat{\theta}}$$



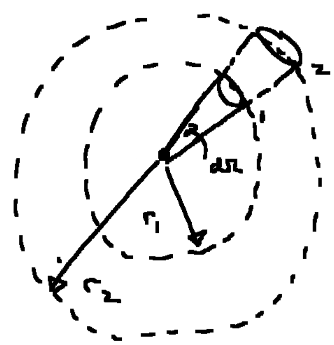
$$\underbrace{\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}$$

$$\boxed{\vec{E}_{rad} = \frac{1}{4\pi\epsilon_0}\frac{-\omega^2 p_0}{c^2 r}\cos(\omega(t-r/c))\sin\theta\,\hat{\theta}}$$

④

Now consider the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$



$$\frac{A_2}{A_1} = \frac{r_2^2}{r_1^2}$$

$$\frac{E_2}{E_1} = \frac{r_1}{r_2}$$

$$\frac{B_2}{B_1} = \frac{r_1}{r_2}$$

$$\int_{d\Omega} \vec{S}\cdot d\vec{a} \sim \frac{1}{\mu_0}\underbrace{\frac{1}{r}\frac{1}{r}}_{\text{fields}}\underset{\text{area}}{r^2} \rightarrow \text{constant with } r$$

$\frac{dP}{d\Omega}$ ← time-averaged power radiated per unit solid angle is independent of r !

(5)

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} r^2 \langle \vec{S}\cdot\hat{r}\rangle_t \quad \leftarrow \text{time average}$$

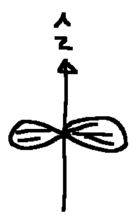
$$\frac{dP}{d\Omega} = \frac{\omega^2 p_0}{4\pi\epsilon_0 c^2}\;\frac{\cancel{\mu_0}}{4\pi}\;\frac{\omega^2 p_0}{c}\;\frac{1}{\cancel{\mu_0}}\;\sin^2\theta\;\underbrace{\langle\cos^2(\omega t)\rangle_t}_{1/2}\;\underbrace{(\hat{\theta}\times\hat{\phi})}_{\hat{r}}\cdot\hat{r}$$

$$\frac{dP}{d\Omega} = \frac{\omega^4 p_0^2}{2(4\pi)^2\epsilon_0 c^3}\sin^2\theta$$

$c^2 = \frac{1}{\mu_0\epsilon_0}$, $c^3\epsilon_0$

$= c/\mu_0$

$= c^2/Z_0$



$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0}{2}\left(\frac{\omega^2 p_0}{c\,4\pi}\right)^2\sin^2\theta}$$

377Ω

$\frac{s^{-2}\,C\cdot m}{m/s} \to \frac{C}{s} \to A$ ✓

$Z_0 = \mu_0 c$

(6)

total power radiated : integrate over all solid angles

$$\int d\Omega\,\sin^2\theta = 2\pi\int_0^{\pi} d\theta\,\sin\theta\,\sin^2\theta$$

$$= \frac{8\pi}{3}$$

$B = \mu_0 H$

$$P = \frac{8\pi}{3}\,\frac{Z_0}{2}\left(\frac{\omega^2 p_0}{c\,4\pi}\right)^2$$

$E = Bc$

$E = \mu_0 H c$

$E = \underbrace{\mu_0 c}_{377\Omega}\,H$

$$= \frac{8\pi}{4\pi}\,\frac{Z_0}{4\pi\,6}\left(\frac{\omega^2 p_0}{c}\right)^2$$

$$\boxed{P = \frac{Z_0}{12\pi}\left(\frac{\omega^2 p_0}{c}\right)^2}$$

$\omega q\; d/\lambda$

$\mu_0, \epsilon_0 \to Z_0, c$

2.20 2024-03-26 Lecture

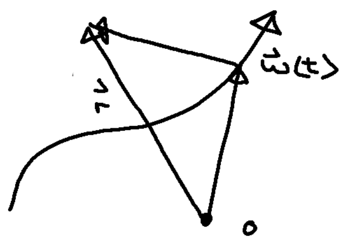
1

The potentials due to a moving point charge in the Lorenz gauge

We have already derived that in the Lorenz gauge:

$$\Phi(r,t) = \frac{1}{4\pi\epsilon_0}\int dV' \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \qquad t_r := t - \frac{|\vec{r}-\vec{r}'|}{c}$$

For a moving point charge: $\rho(\vec{r},t) = q\,\delta^3(\vec{r}-\underbrace{\vec{w}(t)}_{\text{describes trajectory.}}))$



$$\vec{J}(\vec{r},t) = q\vec{v}\,\delta^3(\vec{r}-\vec{w}(t))$$

$$\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int dV' \frac{\delta^3(\vec{r}'-\vec{w}(t-\frac{|\vec{r}-\vec{r}'|}{c}))}{|\vec{r}-\vec{r}'|} \qquad (\times)$$

2

Make a (wrong) guess.

Solve for $\vec{r}'$ in:

$$\vec{0} = \vec{r}' - \vec{w}(t - \frac{|\vec{r}-\vec{r}'|}{c})$$

Once I get $\vec{r}'$, then (?)

$$\cancel{\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}}$$

because of location, we can't use standard result i.e. $\int dx\, \delta(x-a) f(x) = f(a)$

Introduce a time integration:

$$\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int dt' dV' \frac{1}{|\vec{r}-\vec{r}'|}\,\delta^3(\vec{r}'-\vec{w}(t'))\,\delta(t'-(t-\frac{|\vec{r}-\vec{r}'|}{c})) \qquad (*)$$

If we do time integration first, we just get $(\times)$.

Instead do space integration first.

(3)

$$\int dx\ \delta(x) = 1 \quad \checkmark \qquad \int dx\ f(x)\,\delta(x) = f(0)$$

$$\int dx\ \delta(f(x)) = \ ?$$

Suppose $f(x)$ has a single zero @ $x = x_0$; i.e. $f(x_0) = 0$ ✓

then $$\int dx\ \delta(f(x)) = \frac{1}{|f'(x_0)|}$$ ←

Generalization:

$$\int dx\ g(x)\,\delta(f(x)) = g(x_0)\,\frac{1}{|f'(x_0)|}$$

Using this result in (*) on pg 2 gives:

(4)

$$\Phi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0}\ \underset{\uparrow}{\frac{1}{|\vec{r} - \vec{w}(t_r)|}}\ \left.\frac{1}{\left|\frac{d}{dt'}\left(t' - \left(t - \frac{|\vec{r} - \vec{w}(t')|}{c}\right)\right)\right|}\right|_{t' = t_r}$$

where t_r satisfies $t_r = t - \dfrac{|\vec{r} - \vec{w}(t_r)|}{c}$

$$\frac{d}{dt'}|\vec{r} - \vec{w}(t')| = \frac{d}{dt'}\sqrt{|\vec{r} - \vec{w}(t')|^2}$$

$$= \frac{1}{2|\vec{r} - \vec{w}(t')|}\ 2(\vec{r} - \vec{w}(t')) \cdot (\underbrace{-\vec{w}'(t_r)}_{=: \vec{v}(t_r)})$$

$$= -\underbrace{\frac{(\vec{r} - \vec{w}(t'))}{|\vec{r} - \vec{w}(t')|}}_{\to\ \hat{\imath}(t')} \cdot \vec{v}(t')$$

Define $\underset{\sim}{\vec{\imath}(t')} := \vec{r} - \vec{w}(t')$

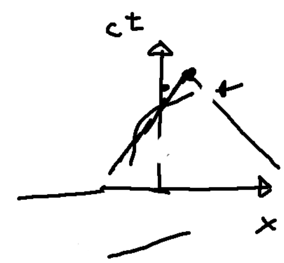
(5)

$$\boxed{\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{\imath(t_r)}\frac{1}{\left(1-\frac{\hat{\imath}\cdot\vec{v}(t_r)}{c}\right)}}$$

$\vec{v}=0$ just reproduces electrostatic result.

The derivation for $\vec{A}$ given $\vec{w}(t)$ is virtually identical

$$\boxed{\vec{A}(\vec{r},t) = \frac{1}{c^2}\vec{v}(t_r)\,\Phi(\vec{r},t)}$$



t_r has to satisfy $\quad t_r = t - \frac{|\vec{r}-\vec{w}(t_r)|}{c}$

Next step: $\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}$

$\vec{B} = \nabla\times\vec{A}$

(6)

Both results are contained in the 4-vector expression:

$$A^\mu = -\frac{q}{4\pi\epsilon_0 c}\frac{\eta^\mu}{\eta^\nu \imath_\nu}$$

$\imath_\nu = x_\nu - w_\nu(t_r)$

$\eta^\mu = (\gamma c, \gamma\vec{v})$

i.e. $\eta^\nu \imath_\nu = (\gamma c, \gamma\vec{v}) \diamond (ct - ct_r, \vec{\imath})$

what is A^0? you get Φ/c as given on previous page.

$A^1, A^2, A^3 \Rightarrow \vec{A}$

2.21 2024-03-28 Lecture

①

The fields due to a point charge in arbitrary motion

- start from an intermediate point in derivation of potentials.

$$\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \int dt' \frac{1}{|\vec{r}-\vec{w}(t')|} \delta\left(t' - \left(t - \frac{|\vec{r}-\vec{w}(t')|}{c}\right)\right)$$

$$\vec{A}(\vec{r},t) = q\frac{\mu_0}{4\pi} \int dt' \frac{\vec{v}(t')}{|\vec{r}-\vec{w}(t')|} \delta\left(t' - \left(t - \frac{|\vec{r}-w(t')|}{c}\right)\right)$$

Determine $\vec{E}(\vec{r},t)$ from $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$ and
$\vec{B}(\vec{r},t)$ from $\vec{B} = \nabla\times\vec{A}$.

②

Define $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t')$

$\nabla \equiv \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$

Start by computing $\nabla \mathscr{r}$. Also $\nabla\left(\frac{1}{\mathscr{r}}\right)$.

Why? To get $\vec{E}$, I need $\nabla\Phi$:

$$\nabla\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \nabla \int dt' \frac{1}{\mathscr{r}} \delta(t' - (t - \mathscr{r}/c))$$

$$= \frac{q}{4\pi\epsilon_0} \int dt' \left[\nabla\left(\frac{1}{\mathscr{r}}\right) \delta(t' - (t-\mathscr{r}/c)) + \frac{1}{\mathscr{r}} \underbrace{\nabla\delta(t' - (t - \mathscr{r}/c))}_{=:*} \right]$$

no problem $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

Problems!

③

For (*) use chain rule in the form:

$$\nabla f(g(\vec{r})) = f'(g(\vec{r}))\, \nabla g(\vec{r})$$

In this case:

$$\nabla \delta(t' - (t - r/c)) = \delta'(t' - (t - r/c)) \left(\frac{\nabla r}{c}\right)$$

$$= -\frac{\partial}{\partial t} \delta(t' - (t - r/c))\, \frac{\hat{r}}{c}$$

$$\nabla \Phi(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int dt' \left[\left(-\frac{\hat{r}}{r^2}\right) \delta(t' - (t - r/c)) - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{r}}{r}\, \delta\left(t' - \frac{r}{c}\right)\right)\right]$$

$$\frac{\partial \vec{A}(\vec{r}, t)}{\partial t} = \frac{\mu_0}{4\pi} q \frac{d}{dt} \int dt' \; \frac{\vec{v}(t')}{r}\; \delta(t' - (t - r/c))$$

④

Skipping a few steps
↓

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left\{ \int dt' \,\frac{\hat{r}}{r^2}\, \delta(t' - (t - r/c)) + \frac{d}{dt} \int dt' \;\delta(t' - (t - r/c)) \left(\frac{\hat{r}}{cr} - \frac{\vec{v}(t')}{c^2 r}\right)\right\}$$

Use result from last lecture $\displaystyle\int dx\; g(x)\, \delta(\underbrace{f(x)}_{f(x_0)=0}) = \frac{g(x_0)}{|f'(x_0)|}$

Also from last lecture:

$$\frac{d}{dt'}(t' - (t - r/c)) = 1 - \frac{\hat{r}\cdot\vec{v}}{c} := g$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{\hat{r}}{r^2}\frac{1}{g} + \frac{d}{dt}\left[\frac{1}{g}\left(\frac{\hat{r} - \vec{v}/c}{cr}\right)\right]\right\}$$ Zangwill Eq. 23.24.

(5)

Some steps : convert $\frac{d}{dt}$ into $\frac{d}{dt_r}$

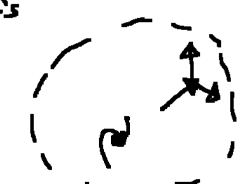
Need to find $\frac{dt}{dt_r}$. You find $\frac{d}{dt} = \frac{1}{g}\frac{d}{dt_r}$.

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{\imath}-\vec{\beta})(1-\beta^2)}{g^3\imath^2} + \frac{\hat{\imath}\times\{(\hat{\imath}-\vec{\beta})\times\dot{\vec{\beta}}\}}{cg^3\imath}\right]}$$ Zangwill's 23.31

where $\vec{\beta} = \vec{v}/c$

$$\boxed{c\vec{B} = \hat{\imath}\times\vec{E}}$$ Zangwill's 23.34

$$g = 1 - \frac{\hat{\imath}\cdot\vec{v}}{c}$$



2.22 2024-04-02 Lecture

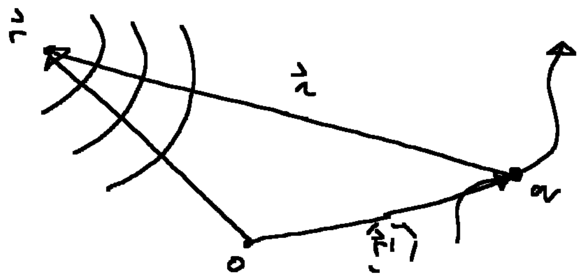
1

Radiation from a point particle
(non-relativistic)

Recall $$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{\mathscr{r}}-\vec{\beta})(1-\beta^2)}{g^3 \mathscr{r}^2} + \frac{\hat{\mathscr{r}} \times \{(\hat{\mathscr{r}}-\vec{\beta})\times \dot{\vec{\beta}}\}}{c g^3 \mathscr{r}}\right] \qquad (*)$$

$$\frac{\vec{B}}{c} = \hat{\mathscr{r}} \times \vec{E}$$

$g := 1 - \hat{\mathscr{r}}\cdot\vec{\beta}$, $\beta = \vec{v}/c$



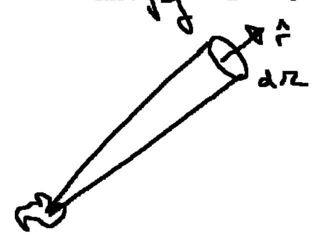
Remember to use retarded time.

2

Non-relativistic Limit: $g \simeq 1$,
$\hat{\mathscr{r}} - \beta \simeq \hat{\mathscr{r}}$

Far field: $\vec{\mathscr{r}} = \vec{r}$

Only second term in $(*)$ contributes to energy carried away to ∞.



$$\frac{dP}{d\Omega} = \lim_{r\to\infty} r^2\, \vec{S}\cdot\hat{r}$$

i.e. to get non-zero $\frac{dP}{d\Omega}$,
$\vec{E}$ and $\vec{B}$ have to scale $\frac{1}{r}$

③

$$\mu_0 \epsilon_0 = \frac{1}{c^2} \qquad \mu_0 c = Z_0 \qquad , \mu_0 =$$

thus $$\epsilon_0 = \frac{1}{\mu_0 c\, c} = \frac{1}{Z_0 c}$$

$$k := \frac{q}{4\pi\epsilon_0 c} = \frac{q}{4\pi \frac{1}{Z_0 c} c} = \frac{q Z_0}{4\pi}$$

So $$\frac{\vec{E}_{rad}}{k} = \frac{\overset{A}{\hat{r}} \times (\overset{B}{\hat{r}} \times \overset{C}{\dot{\vec{\beta}}})}{r} \qquad A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

$$= \frac{(\hat{r}(\hat{r}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}})}{r}$$

$$c\vec{B}_{rad} = \hat{r}\times\vec{E}_{rad}$$

④

$$c\vec{B}_{rad} = \hat{r}\times(\hat{r}(\hat{r}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}})\, k/r$$

$$= k\,\frac{\dot{\vec{\beta}}\times\hat{r}}{r}$$

$$r^2\vec{S} = r^2 \frac{1}{\mu_0} \vec{E}'_{rad}\times\vec{B}'_{rad} \quad , k^2$$

$$\frac{1}{\mu_0} = \frac{1}{c}\left(\frac{c}{Z_0}\right)\left(\frac{qZ_0}{4\pi}\right)^2 [\hat{r}(\hat{r}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}]\times[\dot{\vec{\beta}}\times\hat{r}]$$

$$= \frac{q^2 Z_0}{(4\pi)^2}\left[(\hat{r}\cdot\dot{\vec{\beta}})\,\overset{A}{\hat{r}}\times(\overset{B}{\dot{\vec{\beta}}}\times\overset{C}{\hat{r}}) - \overset{A}{\dot{\vec{\beta}}}\times(\overset{B}{\dot{\vec{\beta}}}\times\overset{C}{\hat{r}})\right]$$

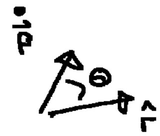
$$= \frac{q^2 Z_0}{(4\pi)^2}\left[(\hat{r}\cdot\dot{\vec{\beta}})(\dot{\vec{\beta}}\underbrace{(\hat{r}\cdot\hat{r})}_{=1} - \hat{r}(\hat{r}\cdot\dot{\vec{\beta}})) - \dot{\vec{\beta}}(\dot{\vec{\beta}}\cdot\hat{r}) + \hat{r}(\dot{\vec{\beta}}\cdot\dot{\vec{\beta}})\right]$$

⑤

$$\frac{r^2 \vec{S}}{\left\{ \frac{c q^2 Z_0}{(4\pi)^2} \right\}} = -\hat{r}(\hat{r}\cdot\dot{\vec{\beta}})^2 + \hat{r}\,\dot{\vec{\beta}}^2$$

$$= \hat{r}\left(\dot{\vec{\beta}}^2 - (\hat{r}\cdot\dot{\vec{\beta}})^2\right)$$

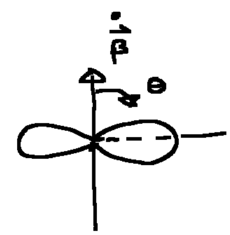
Define θ so that $\hat{r}\cdot\dot{\vec{\beta}} = \dot{\beta}\cos\theta$



$$= \dot{\beta}^2(1-\cos^2\theta)$$

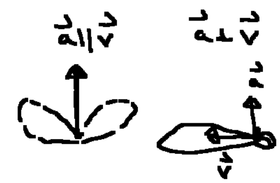
$$= \dot{\beta}^2 \sin^2\theta$$

$$\frac{dP}{d\Omega} = \frac{q^2 Z_0}{(4\pi)^2}\,\dot{\beta}^2 \sin^2\theta$$



non-relativistic

Aside: relativistic results



⑥

$$P = \int d\Omega \, \frac{dP}{d\Omega}$$

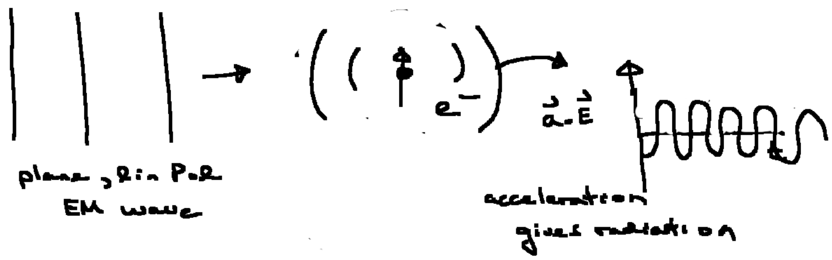
$$= \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \sin^2\theta}_{\frac{8\pi}{3} = \frac{2(4\pi)}{3}} \; \frac{q^2 Z_0}{(4\pi)^2}$$

$$\boxed{P = \frac{1}{6\pi}\,\frac{q^2 Z_0}{c^2}\,a^2}$$

Larmor's formula

(7)

An application of Larmor's formula: Thomson scattering



Calculate cross-section.

$$m\ddot{x} = qE \qquad \text{(aside: ignore } \vec{v}\times\vec{B} \text{ as its second-order)}$$

$$= q E_0 \cos(\omega t)$$

& put $\ddot{x}$ into Larmor's formula.

(8)

$$P = \frac{1}{6\pi} \frac{q^2 Z_0}{c^2} \frac{q^2 E_0^2}{m^2} \underbrace{\cos^2(\omega t)}_{\langle\ \rangle_{time} = 1/2}$$

$$\underline{\langle P \rangle_{time} = \frac{1}{2}\frac{1}{6\pi}\frac{q^2 Z_0}{c^2}} \frac{q^2 E_0^2}{m^2}$$

What power/unit area in incident wave.

$$\vec{S}\cdot\hat{z} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})\cdot\hat{z}$$

$$= \frac{E_0^2}{Z_0}\cos^2(\omega t)$$

$$\langle \vec{S}\cdot\hat{z}\rangle_{time} = \frac{E_0^2}{Z_0}\frac{1}{2}$$

9

Characterize scattering by "cross-section"

$$\sigma = \frac{\langle P \rangle_t}{\langle \vec{S} \cdot \hat{z} \rangle_t}$$

$$= \frac{\cancel{\frac{1}{2}} \frac{1}{6\pi} \frac{q^2 Z_0}{c^2} \frac{q^2 \cancel{E_0^2}}{m^2}}{\frac{\cancel{E_0^2}}{Z_0} \cancel{\frac{1}{2}}}$$

$$\boxed{\sigma = \frac{1}{6\pi} \frac{q^4 Z_0^2}{c^2 m^2}}$$

Thomson scattering cross-section

For e^-'s : $\sigma \simeq 6.6 \times 10^{-29}\ m^2$ $\qquad (1b \simeq 10^{-28}\ m^2)$

1

Alternative unit systems in electromagnetism

- throughout the course we used the "SI system". (Zangwill, Griffiths)
- today, consider the alternatives:
    1) Gaussian (Jackson, astrophysics)
    2) Heaviside-Lorentz (Peskin + Schroeder, QFT)

- Lorentz force law (SI):

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Based on goal of using the same units for $\vec{E}$ and $\vec{B}$, make a change of variables:

$$\underset{SI}{\vec{B}} = \vec{B}'/\alpha \quad \text{new system}$$

new system: $q' = q$, $\vec{E}' = \vec{E}$

2

$$\vec{F} = q'\left(\vec{E}' + \frac{\vec{v}}{\alpha} \times \vec{B}'\right) \qquad (*)$$

Maxwell's equations now look like:

$$\nabla \cdot \vec{E} = \rho/\epsilon_0, \quad \nabla \times \vec{E}' = -\frac{1}{\alpha}\frac{\partial \vec{B}'}{\partial t}$$

$$\nabla \cdot \vec{B}' = 0, \quad \nabla \times \frac{\vec{B}'}{\alpha} = \mu_0 \vec{J}' + \mu_0 \epsilon_0 \frac{\partial \vec{E}'}{\partial t} \qquad (\times)$$

Now apply a second transformation:
Goal: allow possibility of simpler forms of Coulomb's law

$$F = \frac{q^2}{4\pi\epsilon_0 r^2} \xrightarrow[?]{\text{instead}} F = \frac{q^2}{r^2}$$

$$\longrightarrow F = \frac{q^2}{4\pi r^2}$$

③

Rescale $\vec{E} = \vec{E}' = \dfrac{\vec{E}''}{\beta}$

If we want the Lorentz force law to remain unchanged

$$\vec{F} = q' \vec{E}'$$

$$= \underbrace{q''\beta}_{} \frac{\vec{E}''}{\beta} \qquad (*)$$

make sense because than

$$\vec{F} = q'' \vec{E}'' \leftarrow$$

$$\vec{E}' = \frac{\vec{E}''}{\beta}, \quad q' = q''\beta, \quad \rho' = \rho''\beta$$

④

We want to know $\vec{J}''$ in terms of $\vec{J}'$.

We have the continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad \underset{\text{divergence}}{\Leftrightarrow} \quad \oint \vec{J} \cdot da = -\frac{d}{dt} \underbrace{\int \rho \, d\tau}_{Q_{tot}}$$

$$\nabla \cdot \underbrace{\vec{J}''\beta}_{\vec{J}'} = -\frac{\partial \rho''}{\partial t}\beta$$

Thus $\vec{J}' = \vec{J}''\beta$ preserves form of continuity eqn.

What about $\vec{B}''$ in terms of $\vec{B}'$?

(5)

Again try to preserve the Lorentz force law.

$$\vec{F} = q' \frac{\vec{v}}{\alpha} \vec{B}'$$

$$= q'' \beta \frac{\vec{v}}{\alpha} \frac{\vec{B}''}{\beta}$$

$$= q'' \frac{\vec{v}}{\alpha} \vec{B}''$$

Thus we choose $\dfrac{\vec{B}''}{\beta} = \vec{B}'$.

Finally write Maxwell's eqn's in " system.

(6)

$$\nabla \cdot \vec{E}' = \rho' / \epsilon_0 \rightarrow \nabla \cdot \frac{\vec{E}''}{\beta} = \rho'' \beta / \epsilon_0$$

$$\boxed{\nabla \cdot \vec{E}'' = \rho'' \beta^2 / \epsilon_0}$$

$$\nabla \times \vec{E}' = -\frac{1}{\alpha} \frac{\partial \vec{B}'}{\partial t} \qquad \boxed{\nabla \times \vec{E}'' = -\frac{1}{\alpha} \frac{\partial \vec{B}''}{\partial t}}$$

$$\boxed{\nabla \cdot \vec{B}'' = 0}$$

Start from (x) on pg 2:

$$\nabla \times \frac{\vec{B}'}{\alpha} = \mu_0 \vec{J}' + \mu_0 \epsilon_0 \frac{\partial \vec{E}'}{\partial t} \rightarrow \nabla \times \frac{\vec{B}}{\alpha \beta} = \mu_0 \beta \vec{J}'' + \mu_0 \epsilon_0 \frac{1}{\beta} \frac{\partial \vec{E}''}{\partial t}$$

$$\boxed{\nabla \times \vec{B}'' = \mu_0 \alpha \beta^2 \vec{J}'' + \mu_0 \epsilon_0 \alpha \frac{\partial \vec{E}''}{\partial t}}$$

⑦

By choosing α and β appropriately, I can "make" new unit systems.

For Gaussian system: $\alpha = c$, $\beta = \sqrt{4\pi\epsilon_0}$

Now Maxwell's eqn's look like:

$$\boxed{\nabla \cdot \vec{E}'' = 4\pi\rho''} \Rightarrow \left(F = \frac{q^2}{r^2} \right)$$

$$\boxed{\nabla \times \vec{E}'' = -\frac{1}{c}\frac{\partial \vec{B}''}{\partial t}}$$

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

$$\boxed{\nabla \cdot \vec{B}'' = 0}$$

$$\nabla \times \vec{B}'' = \underbrace{\mu_0 \alpha \beta^2}_{\mu_0 c 4\pi\epsilon_0} \vec{J}'' + \mu_0\epsilon_0 c \frac{\partial \vec{E}''}{\partial t}$$

Remember $\frac{1}{\mu_0\epsilon_0} = c^2$

$$\boxed{\nabla \times \vec{B}'' = \frac{4\pi}{c}\vec{J}'' + \frac{1}{c}\frac{\partial \vec{E}''}{\partial t}}$$

⑧

For Heaviside-Lorentz, $\alpha = c$, $\beta = \sqrt{\epsilon_0}$

$$\vec{F} = q''\left(\vec{E}'' + \frac{\vec{v}}{c} \times \vec{B}''\right)$$

$$\boxed{\begin{aligned} &\nabla \cdot \vec{E}'' = \rho \\ &\nabla \times \vec{E}'' = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ &\nabla \cdot \vec{B}'' = 0 \\ &\nabla \times \vec{B}'' = \frac{\vec{J}}{c} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \end{aligned}} \longrightarrow F = \frac{q^2}{4\pi r^2}$$

Maybe this too simple/explicit, but we definitely now have good rules for transforming formulae between systems!

(9)

Example: Larmor's formula.

$$\boxed{P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} a^2}$$ SI

What is form in Gaussian system?

$q = \beta q''$ with $\beta = \sqrt{4\pi\epsilon_0}$

$$\boxed{P = \frac{2}{3} \frac{q''^2 a^2}{c^3}}$$ gaussian.

3 Supplemental notes

References

[1] C. A. Brau, *Modern problems in classical electrodynamics* (Oxford University Press, New York, 2004).