Phys 442 example problems
J. D. D. Martin

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Most of these questions are taken from Ref.'s [1, 2] — explicit references to the problem numbers are provided.[1]

For convenience, the "Questions" section contains links to the answers after each question, and the "Answers" section contains links to the question statement prior to the answer.

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2 Question concepts

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3 Questions

3.1 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.1

Let $\mathcal{S}$ be an inertial reference system. Use Galileo's velocity addition rule.

(a) Suppose that $\overline{\mathcal{S}}$ moves with constant velocity relative to $\mathcal{S}$. Show that $\overline{\mathcal{S}}$ is also an inertial reference system. [Hint: Use the definition in footnote 1.]

(b) Conversely, show that if $\overline{\mathcal{S}}$ is an inertial system, then it moves with respect to $\mathcal{S}$ at constant velocity.

Answer.

3.2 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.2

As an illustration of the principle of relativity in classical mechanics, consider the following generic collision: In inertial frame $\mathcal{S}$, particle A (mass m_A velocity $\mathbf{u}_A$) hits particle B (mass m_B, velocity $\mathbf{u}_B$). In the course of the collision some mass rubs off A and onto B, and we are left with particles C (mass m_C, velocity $\mathbf{u}_C$) and D (mass m_D, velocity $\mathbf{u}_D$). Assume that momentum ($\mathbf{p} \equiv m\mathbf{u}$) is conserved in $\mathcal{S}$

(a) Prove that momentum is also conserved in inertial frame $\overline{\mathcal{S}}$, which moves with velocity $\mathbf{v}$ relative to $\mathcal{S}$. [Use Galileo's velocity addition rule-this is an entirely classical calculation. What must you assume about mass?]

(b) Suppose the collision is elastic in $\mathcal{S}$; show that it is also elastic in $\overline{\mathcal{S}}$.

Answer.

3.3 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.3

(a) What's the percent error introduced when you use Galileo's rule, instead of Einstein's, with $v_{AB} = 5\text{mi/h}$ and $v_{BC} = 60\text{mi/h}$?

(b) Suppose you could run at half the speed of light down the corridor of a train going three-quarters the speed of light. What would your speed be relative to the ground?

(c) Prove, using Eq. 12.3, that if $v_{AB} < c$ and $v_{BC} < c$ then $v_{AC} < c$. Interpret this result.

Answer.

3.4 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.4

As the outlaws escape in their getaway car, which goes $\frac{3}{4}c$, the police officer fires a bullet from the pursuit car, which only goes $\frac{1}{2}c$ (Fig. 12.3). The muzzle velocity of the bullet (relative to the gun) is $\frac{1}{3}c$. Does the bullet reach its target (a) according to Galileo, (b) according to Einstein? Answer.

3.5 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.5

Synchronized clocks are stationed at regular intervals, a million km apart, along a straight line. When the clock next to you reads 12 noon:

(a) What time do you see on the 90th clock down the line?

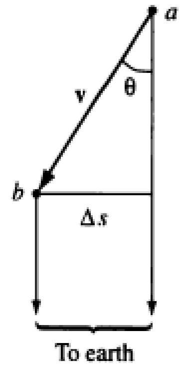
(b) What time do you observe on that clock?

Answer.

3.6 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.6

Every two years, more or less, the New York Times publishes an article in which someone claims to have found an object traveling faster than the speed of light. Many of these reports result from a failure to distinguish what is seen from what is observed — that is, from a failure to account for light travel time. Here's an example: A star is travelling with speed v at an angle θ to the line of sight (see figure below). What is its apparent speed across the sky?

(Suppose the light signal from b reaches the earth at a time Δt after the signal from a, and the star has meanwhile advanced a distance Δs across the celestial sphere; by "apparent speed" I mean $\Delta s/\Delta t$.) What angle θ gives the maximum apparent speed? Show that the apparent speed can be much greater than c, even if v itself is less than c.



Answer.

3.7 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.7

In a laboratory experiment, a muon is observed to travel 800 m before disintegrating. A graduate student looks up the lifetime of a muon (2×10^{-6} s) and concludes that its speed was

$$v = \frac{800 \text{ m}}{2 \times 10^{-6} \text{ s}} = 4 \times 10^{8} \text{ m/s}$$

Faster than light! Identify the student's error, and find the actual speed of this muon. Answer.

3.8 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.8

A rocket ship leaves earth at a speed of $\frac{3}{5}c$. When a clock on the rocket says 1 hour has elapsed, the rocket ship sends a light signal back to earth.

(a) According to earth clocks, when was the signal sent?

(b) According to earth clocks, how long after the rocket left did the signal arrive back on earth?

(c) According to the rocket observer, how long after the rocket left did the signal arrive back on earth?

Answer.

3.9 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.12

Solve Eqs. 12.18 for x, y, z, t in terms of $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, and check that you recover Eqs. 12.19. Answer.

3.10 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.13

Sophie Zabar, clairvoyante, cried out in pain at precisely the instant her twin brother, 500 km away, hit his thumb with a hammer. A skeptical scientist observed both events (brother's accident, Sophie's cry) from an airplane traveling at $\frac{12}{13}c$ to the right (Fig. 12.19). Which event occurred first, according to the scientist? How much earlier was it, in seconds? Answer.

3.11 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.14

(a) In Ex. 12.6 we found how velocities in the x direction transform when you go from $\mathcal{S}$ to $\overline{\mathcal{S}}$. Derive the analogous formulas for velocities in the y and z directions.

(b) A spotlight is mounted on a boat so that its beam makes an angle $\bar{\theta}$ with the deck (Fig. 12.20). If this boat is then set in motion at speed v, what angle θ does an individual photon trajectory make with the deck, according to an observer on the dock? What angle does the beam (illuminated, say, by a light fog) make? Compare Prob. 12.10.

Answer.

3.12 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.15

You probably did Prob. 12.4 from the point of view of an observer on the ground. Now do it from the point of view of the police car, the outlaws, and the bullet. That is, fill in the gaps in the following table:

speed of → relative to ↓	Ground	Police	Outlaws	Bullet	Do they escape?
Ground	0	$\frac{1}{2}c$	$\frac{3}{4}c$		
Police				$\frac{1}{3}c$	
Outlaws					
Bullet					

Answer.

3.13 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.16

The twin paradox revisited On their 21st birthday, one twin gets on a moving sidewalk, which carries her out to star X at speed $\frac{4}{5}c$; her twin brother stays home. When the traveling twin gets to star X, she immediately jumps onto the returning moving sidewalk and comes back to earth, again at speed $\frac{4}{5}c$. She arrives on her 39 th birthday (as determined by her watch).

(a) How old is her twin brother?

(b) How far away is star X ? (Give your answer in light years.) Call the outbound sidewalk system $\overline{\mathcal{S}}$ and the inbound one $\tilde{\mathcal{S}}$ (the earth system is $\mathcal{S}$). All three systems choose their coordinates and set their master clocks such that $x=\bar{x}=\tilde{x}=0, t=\bar{t}=\tilde{t}=0$ at the moment of departure.

(c) What are the coordinates (x, t) of the jump (from outbound to inbound sidewalk) in $\mathcal{S}$?

(d) What are the coordinates $(\bar{x}, \bar{t})$ of the jump in $\overline{\mathcal{S}}$?

(e) What are the coordinates $(\tilde{x}, \tilde{t})$ of the jump in $\tilde{\mathcal{S}}$?

(f) If the traveling twin wants her watch to agree with the clock in $\tilde{S}$, how must she reset it immediately after the jump? What does her watch then read when she gets home? (This wouldn't change her age, of course-she's still 39 -it would just make her watch agree with the standard synchronization in $\tilde{\mathcal{S}}$.)

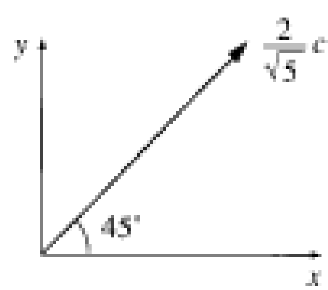
(g) If the traveling twin is asked the question, "How old is your brother right now?", what is the correct reply (i) just before she makes the jump, (ii) just after she makes the jump? (Nothing dramatic happens to her brother during the split second between (i) and (ii), of course; what does change abruptly is his sister's notion of what "right now, back home" means.)

(h) How many earth years does the return trip take? Add this to (ii) from (g) to determine how old she expects him to be at their reunion. Compare your answer to (a).

Answer.

3.14 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.25

A car is travelling along the 45° line in inertial frame $\mathcal{S}$ at (ordinary) speed $(2/\sqrt{5})c$:



(a) Find the components u_x and u_y of the (ordinary) velocity.

(b) Find the components η_x and η_y of the proper velocity.

(c) Find the zeroth component of the four-velocity (i.e. "time-part").

Inertial frame $\bar{\mathcal{S}}$ is moving in the x direction with (ordinary) speed $(\sqrt{2/5})c$ relative to $\mathcal{S}$. By using the appropriate transformation laws:

(d) Find the (ordinary) velocity components $\bar{u}_x$, $\bar{u}_y$ in $\bar{\mathcal{S}}$ using the rules for relativistic velocity addition.

(e) Find the proper velocity components $\bar{\eta}_x$ and $\bar{\eta}_y$ in $\bar{\mathcal{S}}$ using the rules for transforming four-vectors.

(f) As a consistency check, verify that:

$$\overline{\vec{\eta}} = \frac{\overline{\vec{u}}}{\sqrt{1-\overline{u}^2/c^2}}$$

Answer.

3.15 Taylor, *Classical Mechanics*, Problem 15.8

The pion (π^+ or π^-) is an unstable particle that decays with a proper half-life of 1.8×10^{-8} s. (This is the half-life measured in the pion's rest frame.)

(a) What is the pion's half-life measured in a frame $\mathcal{S}$ where it is travelling at 0.8c?

(b) If 32,000 pions are created at the same place, all travelling at this same speed, how many will remain after they have travelled down an evacuated pipe of length $d = 36$ m? Remember that after n half-lives, 2^{-n} of the original particles survive.

(c) What would the answer have been if you had ignored time dilation? (Naturally it is the answer (b) that agrees with experiment.)

Answer.

3.16 Taylor, *Classical Mechanics*, Problem 15.12

Consider the experiment of Problem 15.8 from the point of view of the pions' rest frame. What is the half-life of the pions in this frame? In part (b), how long is the pipe as "seen" by the pions and how long does it take to pass the pions? How many pions remain at the end of this time? Compare with the answer to Problem 15.8 and describe how the two different arguments led to the same result. Answer.

3.17 Taylor, *Classical Mechanics*, Problem 15.57

When a radioactive nucleus of astatine 215 decays at rest, the whole atom is torn into two in the reaction

$$^{215}\text{At} \to {}^{211}\text{Bi} +^{4}\text{He}.$$

The masses of the three atoms are (in order) 214.9986, 210.9873, and 4.0026, all in atomic mass units. (1 atomic mass unit $\approx 1.66 \times 10^{-27}$ kg ≈ 931.5 MeV/c^2 What is the total kinetic energy of the two outcoming atoms, in joules and in MeV? Answer.

3.18 Taylor, *Classical Mechanics*, Problem 15.60

A particle of mass m_a decays at rest into two identical particles each of mass m_b. Use conservation of momentum and energy to find the speed of the outgoing particles. Answer.

3.19 Taylor, *Classical Mechanics*, Problem 15.90

The first positrons to be observed were created in electron-positron pairs by high-energy cosmic-ray photons in the upper atmosphere,

(a) Show that an isolated photon cannot convert to an electron-positron pair in the process $\gamma \rightarrow \mathrm{e}^{+} + \mathrm{e}^{-}$ [Show that this process inevitably violates conservation of four-momentum.]

(b) What actually occurs is that a photon collides with a stationary nucleus with the result

$$\gamma + \text{ nucleus } \rightarrow \mathrm{e}^{+} + \mathrm{e}^{-} + \text{nucleus}.$$

Convince yourself that the formula

$$E_a^{\min} = \frac{\left(\sum m_{\text{fin}}\right)^2 - m_a^2 - m_b^2}{2m_b} c^2 \qquad \text{(15.98 of Taylor)}$$

can be used to find the minimum energy for a photon to induce this reaction. [The derivation of Taylor's (15.98) assumed that the incident particle had $m > 0$.] Show that, provided the mass of the nucleus is much greater than that of the electron, the minimum photon energy to induce this reaction is approximately $2m_e c^2$. [This is exactly the energy one would have calculated for the process $\gamma \rightarrow \mathrm{e}^{+} + \mathrm{e}^{-}$ and shows that the role of the nucleus is just as a "catalyst" that can absorb some three-momentum.]

Answer.

3.20 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.35

In the past, most experiments in particle physics involved stationary targets: one particle (usually a proton or an electron) was accelerated to a high energy E, and collided with a target particle at rest (see (a) below). Far higher *relative* energies are obtainable (with the same accelerator) if you accelerate *both* particles to energy E, and fire them at each other (see (b) below):

Classically, the energy $\bar{E}$ of one particle, relative to the other, is just $4E$ (why?) — not much of a gain (only a factor of 4). But *relativistically* the gain can be *enormous*. Assuming the two particles have the same mass, m, show that

$$\bar{E} = \frac{2E^2}{mc^2} - mc^2.$$

Suppose that you use protons ($mc^2 \approx 1\,\text{GeV}$) with $E = 30\,\text{GeV}$. What $\bar{E}$ do you get? What multiple of E does this amount to? [Because of this relativistic enhancement, most modern elementary particle experiments involve **colliding beams**, instead of fixed targets. e.g. LHC] Answer.

3.21 Taylor, *Classical Mechanics*, Problem 15.92

A positive pion decays at rest into a muon and neutrino, $\pi^+ \rightarrow \mu^+ + \nu$. The masses involved are $m_\pi = 140\,\mathrm{MeV/c^2}$, $m_\mu = 106\,\mathrm{MeV/c^2}$, and $m_\nu = 0$. (There is now convincing evidence that m_ν is not exactly zero, but it is small enough that you can take it to be zero for this problem.) Show that the speed of the outgoing muon has

$$\beta = \left(m_\pi^2 - m_\mu^2\right) / \left(m_\pi^2 + m_\mu^2\right).$$

Evaluate this numerically. Do the same for the much rarer decay mode $\pi^+ \rightarrow \mathrm{e}^+ + v$, ($m_e = 0.5\,\mathrm{MeV/c^2}$). Answer.

3.22 Taylor, *Classical Mechanics*, Problem 15.93

Consider a head-on elastic collision between a high-energy electron (energy E_0 and speed $\beta_0 c$) and a photon of energy $E_{\gamma 0}$.

(a) Show that the final energy E_γ of the photon is

$$E_\gamma = E_0 \frac{1+\beta_0}{2 + (1-\beta_\mathrm{o})\, E_\mathrm{o}/E_{\gamma 0}}.$$

[Hint: Use Taylor's Eq. (15.123).]

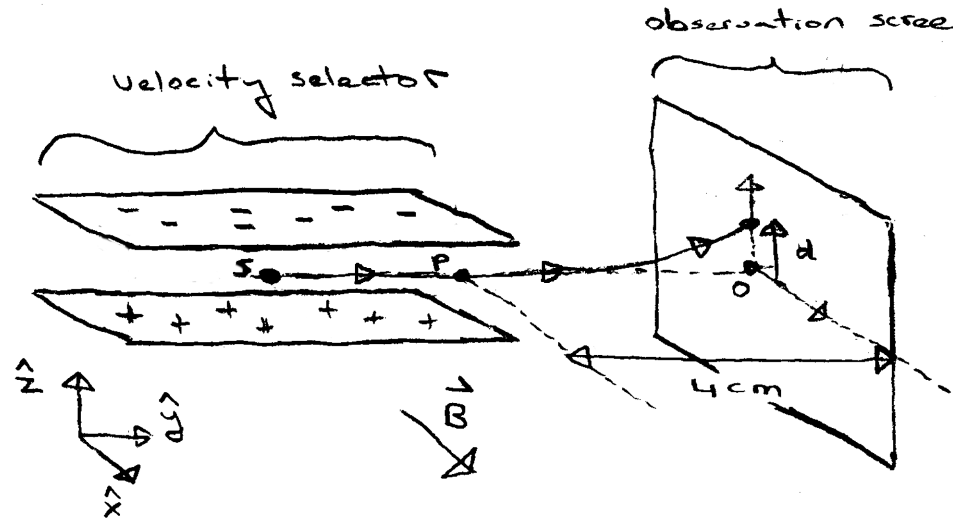
(b) Show that $E_\gamma < E_0$, but that if $\beta_0 \rightarrow 1$, then $E_\gamma / E_o \rightarrow 1$; that is, a very high-energy electron loses almost all its energy to the photon in a head-on collision. What fraction of its original energy would the electron retain if $E_0 \approx 10\,\mathrm{TeV}$ and the photon was in the visible range, $E_{\gamma 0} \approx 3\,\mathrm{eV}$? (Remember that the mass of the electron is about $0.5\,\mathrm{eV/c^2}$; $1\,\mathrm{TeV} = 1 \times 10^{12}\,\mathrm{eV}$.)

Answer.

3.23 Bucherer's test of the Lorentz force law

When special relativity was being developed it was not straightforward to accelerate charged particles to velocities close enough to the speed of light for the corrections to Newtonian mechanics to be observable. Instead, relativistic electrons due to beta decay radioactivity were used. However, these sources emit electrons over a *range* of velocities — necessitating *velocity-selection*, which we have discussed earlier in this course.

Here is a simplified version of an apparatus due to A. Bucherer that provided early experimental evidence for the laws of relativistic mechanics:



The apparatus operates according to the following principles:

(i) A radioactive source S emits electrons with a range of velocities.

(ii) The combination of an electric field $\vec{E} = E_z\hat{z}$ (created using two charged metal plates above and below the source) and magnetic field $\vec{B} = B_x\hat{x}$ act as a velocity-selector, so that electrons that arrive at P have a well-defined velocities (the plates are placed close together, so that electrons of the wrong velocities hit the plates and do not make it to the observation screen).

(iii) Once the velocity-selected electrons move outside of the region between the two plates (past P), the electric field drops to zero quickly, whereas the magnetic field $\vec{B} = B_x\hat{x}$ remains constant.

(iv) If the velocity-selected electrons were to remain undeflected past P they would arrive at point O on the observation screen, a distance 4 cm away from the exit of the selector. Instead, the electrons are deflected upwards once they leave the selector due to $\vec{B}$ and arrive a distance d upwards from O. Distances d may be measured for different values of v/c (by changing the velocity-selector fields).

The amount that the electrons are expected to travel upwards (d) differs depending on whether it is computed using relativistic or non-relativistic (Newtonian) dynamics. Thus precisely measuring d provides a test of special relativity. The test becomes more stringent the closer the electron's velocities are to c.

Questions:

(a) If $E_z = 2 \times 10^6\,\mathrm{V/m}$ what must B_x be to select electrons with $v/c \approx 0.7$?

(b) If $v/c \approx 0.7$, what is d according to non-relativistic dynamics? (Call it $d_{\text{non-rel}}$.)

(c) If $v/c \approx 0.7$, what is d according to relativistic dynamics? (Call it d_{rel}.)

(d) In practice the precise location of the point O is not straightforward to determine, contributing to uncertainty in the measurement of d. How should $\vec{E}$ and $\vec{B}$ be changed so that the electrons are deflected *downwards* from O? How would this change help to measure d?

Answer.

3.24 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 8.2

Consider the charging capacitor in Prob. 7.31(3rd ed)/7.34(4th ed).

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the direction of $\mathbf{S}$. Check that Eq. 8.14(3rd ed)/8.12(4th ed) is satisfied:

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S}.$$

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9(3rd/4th ed)— in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

Answer.

3.25 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 8.4

(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

(b) Do the same for charges that are opposite in sign.

Answer.

3.26 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 8.3

Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell, with radius R, angular velocity ω, and surface charge density σ. [This is the same as Problem 5.42(3rd ed.)/5.44(4th ed.), but use the Maxwell stress tensor and Eq. 8.22(3rd ed.)/8.21(4th ed.). Answer: $(\pi/4)\mu_0\sigma^2\omega^2R^4$.] Answer.

3.27 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.9

Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is (a) traveling in the negative x direction and polarized in the z direction; (b) traveling in the direction from the origin to thepoint $(1, 1, 1)$, with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of $\mathbf{k}$ and $\hat{\mathbf{n}}$. Answer.

3.28 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.10

The intensity of sunlight hitting the earth is about 1300 $\mathrm{W/m^2}$. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to? Answer.

3.29 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.12

In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r}, t) = A\cos(\mathbf{k}\cdot\mathbf{r} + \delta_a)$ and $g(\mathbf{r}, t) = B\cos(\mathbf{k}\cdot\mathbf{r} + \delta_b)$. Show that $\langle fg\rangle = (1/2)\Re(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same $\mathbf{k}$ and ω, but they need not have the same amplitude or phase.] For example

$$\langle u\rangle = \frac{1}{4}\Re(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*)$$

and

$$\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\Re(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*).$$

Answer.

3.30 Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.35

Suppose

$$\mathbf{E}(r, \theta, \phi, t) = A\frac{\sin\theta}{r}[\cos(kr - \omega t) - (1/kr)\sin(kr - \omega t)]\,\hat{\boldsymbol{\phi}}$$

with $\omega/k = c$. (This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.)

(a) Show that $\mathbf{E}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\mathbf{S}$ over a full cycle to get the intensity vector $\mathbf{I}$. (Does it point in the expected direction? Does it fall off like r^{-2}, as it should?)

(c) Integrate $\mathbf{I}\cdot\mathbf{a}$ over a spherical surface to determine the total power radiated. [Answer: $4\pi A^2/3\mu_0 c$]

Answer.

4 Answers

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.1 — answer

Question. Answer:

① Q833 A728

Problem 12.1 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"Use Galileo's velocity addition rule. Let S be an inertial reference system.

(a) Suppose that $\bar{S}$ moves with a constant velocity relative to S. Show that $\bar{S}$ is also an inertial reference system.

[Hint: use the definition in footnote 1]

(b) Conversely, show that if $\bar{S}$ is an inertial system, then it moves with respect to S at a constant velocity."

(a)

Footnote 1 tells us that an inertial frame is "one in which Newton's first law holds".

If frame S is inertial then an object with no forces applied should move at a constant velocity, as measured in this frame:

$$\frac{d\vec{v}_{os}}{dt} = 0$$

Call the velocity measured in the $\bar{S}$ frame $v_{o\bar{s}}$. If Galileo's velocity addition rule holds, then: $\vec{v}_{o\bar{s}} = \vec{v}_{os} + \vec{v}_{s\bar{s}}$ ← relative velocity of frames.

$$\frac{d\vec{v}_{o\bar{s}}}{dt} = \frac{d}{dt}\left(\vec{v}_{os} + \vec{v}_{s\bar{s}}\right)$$

$$= \underbrace{\cancel{\frac{d}{dt}\vec{v}_{os}}}_{=0} + \overbrace{\cancel{\frac{d}{dt}\vec{v}_{s\bar{s}}}}^{=0}$$

$$= 0$$

So $\bar{S}$ must also be an inertial frame.

(b) Start with Galileo's velocity addition rule:

$$\vec{v}_{o\bar{s}} = \vec{v}_{os} + \vec{v}_{s\bar{s}}$$

Differentiate wrt time: $\frac{d\vec{v}_{o\bar{s}}}{dt} = \frac{d\vec{v}_{os}}{dt} + \frac{d\vec{v}_{s\bar{s}}}{dt}$

If both $d\vec{v}_{os}/dt = 0$ and $d\vec{v}_{o\bar{s}}/dt = 0$ because they are inertial frames then $d\vec{v}_{s\bar{s}}/dt = 0$. ie. the two frames must be moving at a constant velocity wrt one another.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.2 — answer

Question. Answer:

① Q834 A729

Problem 12.2 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

As an illustration of the principle of relativity in classical mechanics, consider the following generic collision: In inertial frame S, particle A (mass m_A, velocity $\vec{u}_A$) hits particle B (mass m_B, velocity $\vec{u}_B$). In the course of the collision some mass rubs off A and onto B, and we are left with particles C (mass m_C, velocity $\vec{u}_C$) and D (mass m_D, velocity $\vec{u}_D$). Assume that momentum ($\vec{p} \equiv m\vec{u}$) is conserved in S.

(a) Prove that momentum is also conserved in inertial frame $\bar{S}$, which moves with velocity $\vec{v}$ relative to S. [Use Galileo's velocity addition rule — this is an entirely classical calculation. What must you assume about mass?]

(b) Suppose the collision is elastic in S; show that it is also elastic in $\bar{S}$.

(a) put bars over quantities to signify that they are measured in $\bar{S}$. Galileo's velocity addition rule tells us: $\vec{u} = \vec{\bar{u}} + \vec{v}$

~~$\vec{\bar{p}}_a + \vec{\bar{p}}_b = m_a(\bar{u}$~~

momentum before collision in $\bar{S}$ frame:

$$\vec{\bar{p}}_a + \vec{\bar{p}}_b = m_a \vec{\bar{u}}_a + m_b \vec{\bar{u}}_b \qquad (1)$$

$$= m_a(\vec{u}_a - \vec{v}) + m_b(\vec{u}_b - \vec{v}) \qquad (2)$$

$$= m_a \vec{u}_a + m_b \vec{u}_b - (m_a + m_b)\vec{v} \qquad (3)$$

likewise the momentum after the collision in the $\bar{S}$ frame is:

$$\vec{\bar{p}}_c + \vec{\bar{p}}_d = m_c \vec{u}_c + m_d \vec{u}_d - (m_c + m_d)\vec{v} \qquad (4)$$

If momentum was conserved in the S frame: $m_a\vec{u}_a + m_b\vec{u}_b = m_c\vec{u}_c + m_d\vec{u}_d$, then momentum will be conserved if the S' frame, ie. Eq (3) will equal Eq. (4), iff $m_a + m_b = m_c + m_d$. We must assume mass is conserved.

(b) KE in $\bar{S}$ before collision:

$$\frac{1}{2} m_a |\vec{\bar{u}}_a|^2 + \frac{1}{2} m_b |\vec{\bar{u}}_b|^2 = \frac{1}{2} m_a |\vec{u}_a - \vec{v}|^2 + \frac{1}{2} m_b |\vec{u}_b - \vec{v}|^2 \qquad (5)$$

$$= \frac{1}{2} m_a |\vec{u}_a|^2 + m_a \vec{u}_a \cdot \vec{v} + \frac{1}{2} m_b |\vec{u}_b|^2 + m_b \vec{u}_b \cdot \vec{v} \qquad (6)$$

$$= \frac{1}{2} m_a |\vec{u}_a|^2 + \frac{1}{2} m_b |\vec{u}_b|^2 + \vec{v} \cdot (m_a \vec{u}_a + m_b \vec{u}_b) \qquad (7)$$

Likewise the kinetic energy in $\bar{S}$ after the collision is:

$$\frac{1}{2} m_c |\vec{u}_c|^2 + \frac{1}{2} m_d |\vec{u}_d|^2 + \vec{v} \cdot (m_c \vec{u}_c + m_d \vec{u}_d) \qquad (8)$$

②

If both kinetic energy and momentum are conserved in the S frame,
i.e. $\frac{1}{2}m_a|u_a|^2 + \frac{1}{2}m_b|u_b|^2 = \frac{1}{2}m_c|u_c|^2 + \frac{1}{2}m_d|u_d|^2$ and $m_a\vec{u}_a + m_b\vec{u}_b = m_c\vec{u}_c + m_d\vec{u}_d$
we see that kinetic energy is conserved in $\bar{S}$, i.e. Eq. (7) is equal to Eq. (8).

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.3 — answer

Question. Answer:

① Q835 A730

Problem 12.3 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

(a) What's the percentage error introduced when you use Galileo's rule, instead of Einstein's, with $u_{AB} = 5$ mi/h and $u_{BC} = 60$ mi/h.

(b) Suppose you could run at half the speed of light down the corridor of a train going three quarters the speed of light. What would your speed be relative to the ground?

(c) Prove, using Eq. 12.3, that if $u_{AB} < c$ and $u_{BC} < c$ then $u_{AC} < c$. Interpret this result.

$c = 3 * 10^8$ m/s / 1609 m/mile * 3600s/hr

$\simeq 671 * 10^6 \frac{\text{miles}}{\text{hr}}$

Using Galileo's rule: $V_{AC} = u_{AB} + u_{BC}$

$= 5$ mi/h $+ 60$ mi/h

$= 65$ mi/h

Using Einstein's rule $\quad u_{AC} = \dfrac{u_{AB} + u_{BC}}{1 + \dfrac{u_{AB} u_{BC}}{c^2}}$ since $u_{AB}u_{BC}/c^2 << 1$

$$\simeq (u_{AB} + u_{BC}) \left(1 - \frac{u_{AB}u_{BC}}{c^2}\right)$$

$$\% \text{ error} = \frac{u_{AB} + u_{BC} - (u_{A} + u_{BC})(1 - u_{AB}u_{BC}/c^2)}{(u_{AB} + u_{BC})\left(1 - \dfrac{u_{AB}u_{BC}}{c^2}\right)} * 100\%$$

$$\simeq (u_{AB}u_{BC})/c^2 * 100\% \quad \simeq 6 * 10^{-14}\%$$

b) $$u_{ug} = \frac{u_{ut} + u_{tg}}{1 + \dfrac{u_{ut}u_{tg}}{c^2}}$$

$$= \frac{(1/2 + 3/4)\,c}{1 + \frac{1}{2}\frac{3}{4}}$$

$$= \frac{5}{4}\frac{8}{11}\,c$$

$$= \frac{10}{11}c$$

②

c) Eq. 12.3 of Griffiths is:

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + \frac{v_{AB} v_{BC}}{c^2}}$$

We want to show $|v_{ac}| < c$. This is equivalent to showing $\underbrace{1 - \left(\frac{v_{ac}}{c}\right)^2}_{\text{call this } \Delta} > 0$

Using $\beta_1 = v_{AB}/c$, and $\beta_2 = v_{BC}/c$,

$$\Delta = 1 - \left[\frac{(\beta_1 + \beta_2)}{1 + \beta_1\beta_2}\right]^2$$

$$= \frac{(1+\beta_1\beta_2)^2 - \beta_1^2 - \beta_2^2 - 2\beta_1\beta_2}{(1+\beta_1\beta_2)^2}$$

$$= \frac{1 + \beta_1^2\beta_2^2 + 2\beta_1\beta_2 - \beta_1^2 - \beta_2^2 - 2\beta_1\beta_2}{(1+\beta_1\beta_2)^2}$$

$$= \frac{1 + \beta_1^2\beta_2^2 - \beta_1^2 - \beta_2^2}{(1+\beta_1\beta_2)^2}$$

$$= \frac{1 + \beta_1^2(\beta_2^2 - 1) - \beta_2^2}{(1+\beta_1\beta_2)^2}$$

$$= \frac{(1-\beta_2^2) - \beta_1^2(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}$$

$$= \frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}$$

So for $|v_{AB}| < c$ $(\beta_1^2 < 1)$, and $|v_{BC}| < c$ $(\beta_2^2 < 1)$, $\Delta > 0$, and thus $|v_{AC}| < c$.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.4 — answer

Question. Answer:

Problem 12.4 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"As the outlaws escape in their getaway car, which goes $\frac{3}{4}c$, the police officer fires a bullet from the pursuit car, which only goes $\frac{1}{2}c$ (Fig 12.3). The muzzle velocity of the bullet (relative to the gun) is $\frac{1}{3}c$. Does the bullet reach its target (a) according to Galileo, (b) according to Einstein?"

Fig 12.3 of Griffiths.

Define some inertial reference frames:

o: outlaws

b: bullet

c: cops

g: ground.

(a) According to Galileo:

$$U_{ob} = V_{oc} + V_{cb}$$

$$= \frac{3}{4}c - \frac{1}{2}c - \frac{1}{3}c$$

$$= \frac{9}{12}c - \frac{6}{12}c - \frac{4}{12}c$$

$$= -\frac{1}{12}c$$

Since $U_{ob} < 0$, bullet reaches outlaws.

(b) According to Einstein:

$$V_{oc} = \frac{U_{og} + U_{gc}}{1 + \frac{U_{og}U_{gc}}{c^2}}$$

$$= \left(\frac{3}{4}c - \frac{1}{2}c\right) / \left(1 + \left(\frac{3}{4}\right)\left(\frac{-1}{2}\right)\right)$$

$$= \frac{2}{4}c\,\frac{4}{5}$$

$$= \frac{2}{5}c$$

$$v_{ob} = \frac{v_{oc} + v_{cb}}{1 + \frac{v_{oc} v_{cb}}{c^2}}$$

$$= \left(\tfrac{2}{5}c - \tfrac{1}{3}c\right) / \left(1 - \tfrac{2}{5}\tfrac{1}{3}\right)$$

$$= \frac{1}{\cancel{15}}c \; \frac{\cancel{15}}{13}$$

$$= \frac{1}{13}c$$

Since $v_{ob} > 0$, the outlaws outrun the bullet.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.5 — answer

Question. Answer:

Q837 A732

Problem 12.5 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"Synchronized clocks are stationed at regular intervals, a million km apart, along a straight line. When the clock next to you reads 12 noon:
(a) What time do you see on the 90th clock down the line?
(b) What time do you observe on that clock?"

---

a) The time it has taken for the light from the 90th clock to reach us is:

$$\frac{90*10^9\,m}{3*10^8\,m/s} \simeq 300s$$

$$300s/(60s/min) = 50min$$

Therefore the clock reads 11:10am (50 min before noon).

b) As an 'observer' we must correct for delays such as those in (a). We simply read 12 noon.

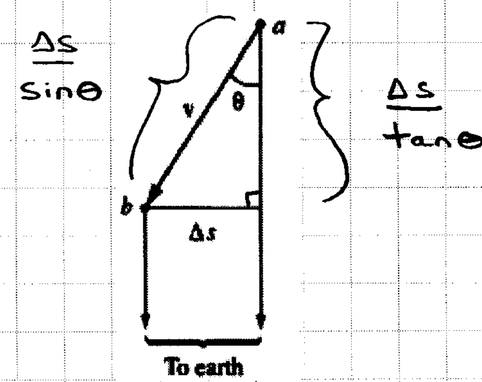
Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.6 — answer

Question. Answer:

Q655 A561

Problem 12.6 from Griffiths, *Introduction to Electrodynamics*, 3rd ed.
Problem 12.6 from Griffiths, *Introduction to Electrodynamics*, 4th ed.

Every two years, more or less, the New York Times publishes an article in which someone claims to have found an object traveling faster than the speed of light. Many of these reports result from a failure to distinguish what is seen from what is observed — that is, from a failure to account for light travel time. Here's an example: A star is travelling with speed v at an angle θ to the line of sight (see figure below). What is its apparent speed across the sky?
(Suppose the light signal from b reaches the earth at a time Δt after the signal from a, and the star has meanwhile advanced a distance Δs across the celestial sphere; by "apparent speed" I mean $\Delta s/\Delta t$.) What angle θ gives the maximum apparent speed? Show that the apparent speed can be much greater than c, even if v itself is less than c.



In what follows we limit θ, such that $0 \le \theta \le \pi/2$, so that $\tan\theta$, $\sin\theta$, and $\cos\theta$ are all ≥ 0.

The time Δt must account for the fact that the star is closer to earth at b, and thus the light takes less time to travel to earth than the light from point a:

$$\Delta t = \underbrace{\frac{\Delta s}{\sin\theta}\frac{1}{v}}_{\text{time req'd for star to get from a to b}} - \underbrace{\frac{\Delta s}{\tan\theta}\frac{1}{c}}_{\text{difference in time required for light to travel to earth.}} \qquad (1)$$

Thus

$$\frac{\Delta s}{\Delta t} = \frac{1}{\dfrac{1}{v\sin\theta} - \dfrac{1}{c\tan\theta}} \qquad (2)$$

We are interested in comparing $\Delta s/\Delta t$ to c, so we divide both sides by c, and multiply the numerator and denominator on the RHS by $v \sin\theta$, to obtain:

$$\frac{1}{c}\frac{\Delta s}{\Delta t} = \frac{v}{c}\frac{\sin\theta}{\left(1 - \frac{v}{c}\cos\theta\right)} \tag{3}$$

Introduce $\beta \equiv v/c$, so that:

$$\frac{1}{c}\frac{\Delta s}{\Delta t} = \beta\frac{\sin\theta}{(1-\beta\cos\theta)} \tag{4}$$

For a given β, let's compute the θ that corresponds to a maximum apparent velocity. Differentiate Eq. (4) wrt θ:

$$\frac{d}{d\theta}\left(\frac{1}{c}\frac{\Delta s}{\Delta t}\right) = \beta\frac{[\cos\theta(1-\beta\cos\theta) - \sin\theta\,\beta\sin\theta]}{(1-\beta\cos\theta)^2} \tag{5}$$

$$= \beta\frac{[\cos\theta - \beta\cos^2\theta - \beta\sin^2\theta]}{(1-\beta\cos\theta)^2} \tag{6}$$

$$= \beta\frac{(\cos\theta - \beta)}{(1-\beta\cos\theta)^2} \tag{7}$$

This derivative will be zero if $\cos\theta = \beta$. Is this a local minima or maxima? To answer this differentiate again wrt θ:

$$\frac{d^2}{d\theta}\left(\frac{1}{c}\frac{\Delta s}{\Delta t}\right) =$$

$$\beta\frac{[-\sin\theta(1-\beta\cos\theta)^2 - (\cos\theta-\beta)2(1-\beta\cos\theta)\beta\sin\theta]}{(1-\beta\cos\theta)^4} \tag{8}$$

Substituting $\cos\theta = \beta$:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{c}\frac{\Delta s}{\Delta t}\right)_{\beta=\cos\theta} = -\beta\left[\frac{\sin\theta\,(1-\beta^2)^2}{(1-\beta^2)^4}\right] \qquad (9)$$

$$= -\beta\,\frac{\sin\theta}{(1-\beta^2)^2} \qquad (10)$$

$$= -\beta\,\frac{(1-\beta^2)^{1/2}}{(1-\beta^2)^2} \qquad (11)$$

$$= -\frac{\beta}{(1-\beta^2)^{3/2}} \qquad (12)$$

Since this second derivative is < 0 we conclude that $\beta = \cos\theta$ corresponds to a local maxima in the apparent speed $\frac{\Delta s}{\Delta t}$.

Substituting $\beta = \cos\theta$ into Eq. (4) we obtain the maximum apparent speed for a given β:

$$\left(\frac{1}{c}\frac{\Delta s}{\Delta t}\right)_{max} = \frac{\beta\sin\theta}{1-\beta^2}$$

$$= \beta\,\frac{\sqrt{1-\beta^2}}{1-\beta^2}$$

$$= \frac{\beta}{\sqrt{1-\beta^2}}$$

As β is varied from 0 to 1, this apparent speed increases from 0 to ∞. Thus we expect that at there is some "threshold" β_T for which larger β's will give maximum apparent speeds $> c$.

Let's compute this threshold β_+ :

$$1 = \frac{\beta_+}{\sqrt{1-\beta_+^2}}$$

Rearrange to solve for β_+ :

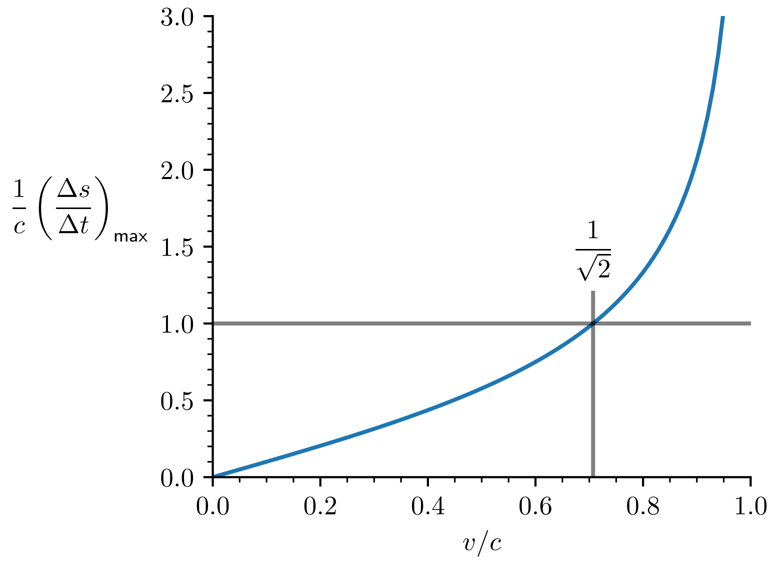
$$1-\beta_+^2 = \beta_+^2$$

$$\beta_+ = \frac{1}{\sqrt{2}}$$

Therefore once the speed of the star exceeds $\frac{1}{\sqrt{2}}c$, then for $\cos\theta = \frac{v}{c}$ the apparent speed will exceed c.

Please see the following page for a summary, and a plot of the maximum apparent speed as a function of the real speed.

Shown below is maximum *apparent* speed as a fraction of c, as a function of the actual speed of the star v. Once v/c exceeds $1/\sqrt{2}$, the apparent speed can exceed c.



Summarizing:

The apparent speed of the star across the sky is

$$\frac{\Delta s}{\Delta t} = \frac{v \sin\theta}{1 - (v/c)\cos\theta}.$$

The angle θ that gives the maximum apparent speed satisfies:

$$v/c = \cos\theta.$$

Once v/c exceeds $1/\sqrt{2}$, the maximum apparent speed is greater than c.

This problem illustrates the special use of the terms *observer* and *observed* in special relativity. The speed of the star is always *observed* to travel less than the speed of light. *Observers* always correct for time it takes for light to propagate to their location. Observations are best made (at least conceptually) by a network of evenly spaced, synchronized clocks within inertial frames, so that no correction is necessary.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.7 — answer

Question. Answer:

Q838 A733

Problem 12.7 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"In a laboratory experiment a muon is observed to travel 800m before disintegrating. A graduate student looks up the lifetime of a muon ($2*10^{-6}$ s) and concludes its speed was

$$v = \frac{800m}{2*10^{-6}s} = 4*10^{8} m/s$$

Faster than light! Identify the student's error and find the actual speed of the muon."

The lifetime in the lab-frame is $\frac{800m}{v}$. ← velocity

The time dilation result is $t_L = \frac{t_R}{\sqrt{1-(v/c)^2}}$ ← muon lifetime in rest frame ($2*10^{-6}$s)

Equating these: $\frac{800m}{v} = \frac{2*10^{-6}}{\sqrt{1-(v/c)^2}}$

We can solve for v. For simplicity, lets rewrite in terms of β ($\equiv v/c$):

$$\frac{1}{c}\frac{8*10^{2} m}{2*10^{-6}s} = \frac{1}{c}\frac{v}{\sqrt{1-(v/c)^2}}$$

$$\frac{1}{c} 4*10^{8}\frac{m}{s} = \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\frac{4}{3} = \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\frac{16}{9} = \frac{\beta^2}{1-\beta^2}$$

$$\frac{16}{9} - \frac{16}{9}\beta^2 = \beta^2$$

$$\frac{16}{9} = \beta^2\left(\frac{16}{9} + \frac{9}{9}\right)$$

$$\beta^2 = \frac{16/9}{25/9}$$

$$\beta = 4/5$$

So $v = \frac{4}{5}c$

slower than the speed of light.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.8 — answer

Question. Answer:

Q839 A734

Problem 12.9 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"A Lincoln Continental is twice as long as a VW Beetle, when they are at rest. As the Continental overtakes the VW, going through a speed trap, a (stationary) policeman observes that they both have the same length. The VW is going at half the speed of light. How fast is the Lincoln going? (Leave your answer as a multiple of c)"

Use formula for length contraction:

$$\ell = \underbrace{\ell_0}_{\text{rest length}} \sqrt{1-(v/c)^2}$$

Lets call the rest length of the Continental ℓ_c. Conditions of the problem give:

$$\underbrace{\ell_c \sqrt{1-\left(\frac{v}{c}\right)^2}}_{\text{policeman's observation of length of Lincoln}} = \underbrace{\frac{\ell_c}{2} \sqrt{1-\left(\frac{1}{2}\right)^2}}_{\text{policeman's observation of length of VW.}} \leftarrow 1/2\,c$$

Square each side and simplify:

$$\cancel{\ell_c^2}\left(1-\left(\frac{v}{c}\right)^2\right) = \frac{\cancel{\ell_c^2}}{4}(1-1/4)$$

$$1-\left(\frac{v}{c}\right)^2 = \frac{3}{16}$$

$$\left(\frac{v}{c}\right)^2 = \frac{13}{16}$$

$$\frac{v}{c} = \sqrt{\frac{13}{16}}$$

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.12 — answer

Question. Answer:

① Q840 A735

Problem 12.12 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"Solve Eqs. 12.18: $\bar{x} = \gamma(x - vt)$

$$\bar{y} = y$$
$$\bar{z} = z$$
$$\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

for x, y, z, t in terms of $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, and check that you recover Eqs. 12.19:

$$x = \gamma(\bar{x} + v\bar{t})$$
$$y = \bar{y}$$
$$z = \bar{z}$$
$$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)$$"

Write transformation laws in a slightly different form:

$$\frac{\bar{x}}{\gamma} = x - vt \qquad (1)$$

and

$$\frac{\bar{t}}{\gamma} = t - \frac{v}{c^2}x \qquad (2)$$

Multiply Eq. (2) by v:

$$\frac{v\bar{t}}{\gamma} = vt + \frac{v^2}{c^2}x \qquad (3)$$

Add to Eq. (1):

$$\frac{\bar{x}}{\gamma} + \frac{v\bar{t}}{\gamma} = x - \frac{v^2}{c^2}x$$

Rearrange to solve for x:

$$x = \overbrace{\frac{1}{1 - \frac{v^2}{c^2}}}^{\gamma^2} \frac{1}{\gamma}(\bar{x} + v\bar{t})$$

$$x = \gamma(\bar{x} + v\bar{t})$$

as req'd. //

Now lets look at expression for t. Multiply Eq (1) by v/c^2:

$$\frac{v}{c^2}\frac{\bar{x}}{\gamma} = \frac{v}{c^2}x - \frac{v^2}{c^2}t$$

Add to Eq. (2):

$$\frac{v}{c^2}\frac{\bar{x}}{\gamma} + \frac{\bar{t}}{\gamma} = t - \frac{v^2}{c^2}t$$

Rearrange to solve for t:

$$t = \overbrace{\frac{1}{1-\frac{v^2}{c^2}}}^{\gamma^2}\frac{1}{\gamma}\left(\frac{v}{c^2}\bar{x} + \bar{t}\right)$$

$$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)$$

as req'd. //

The case of $\bar{y}$ and $\bar{z}$ (orthogonal to relative frame motion) are trivial.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.13 — answer

Question. Answer:

Q841 A736

Problem 12.13 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"Sophie Zabar, clairvoyante, cried out in pain at precisely the instant her twin brother, 500 km away, hit his thumb with a hammer. A skeptical scientist observed both events (brother's accident, Sophie's cry) from an airplane travelling at $\frac{12}{13}c$ to the right (see Fig. 12.19). Which event occurred first, according to the scientist? How much earlier was it, in seconds?"

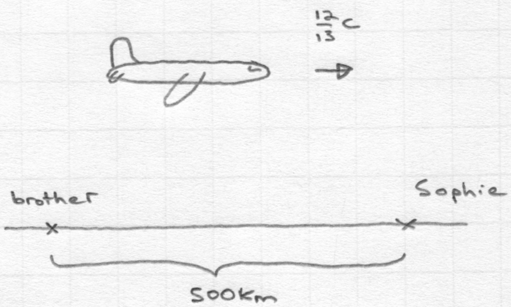
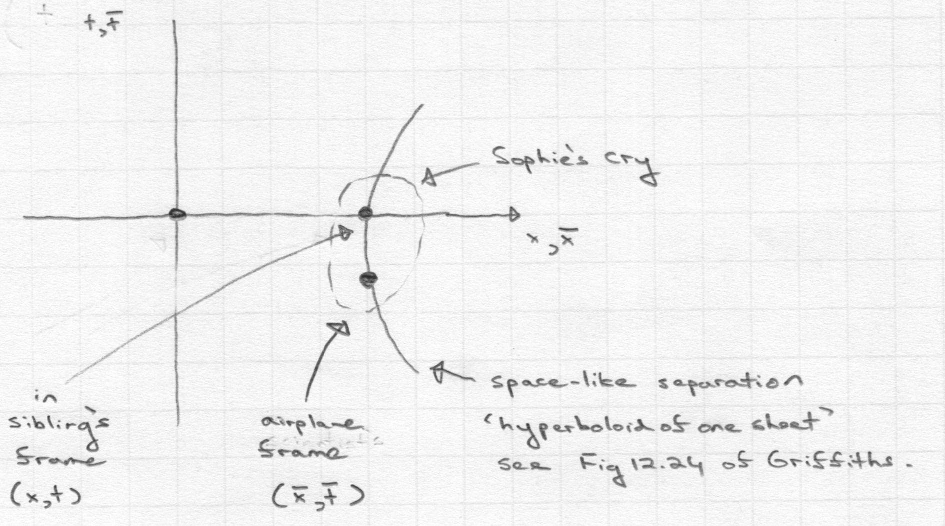


Fig 12.19 of Griffiths

For this problem we can use the Lorentz transformation. (Eq. 12.18 of Griffiths)
Plane frame will be 'bar' $(\bar{x}, \bar{t})$ and Sophie and brother's frame will be without the 'bar'.

Graphically:



$$\bar{t} = \gamma \left(t - \frac{v}{c^2} x \right)$$

$$= \frac{1}{\sqrt{1 - \left(\frac{12}{13}\right)^2}} \left(\frac{500 * 10^3 \, m}{3 * 10^8 \, m/s} * \frac{12}{13} \right)$$

$$\simeq 4 * 10^{-3}$$

$\boxed{4 \, ms}$

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.14 — answer

Question. Answer:

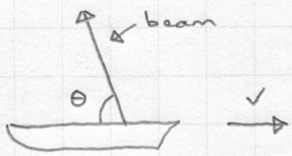
Q842 A737

Problem 12.14 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

(a) In Ex. 12.6 we found how velocities 'in the x direction' transform when you go from S to $\bar{S}$. Derive the analogous formulas for velocities in the y and z directions.

(b) A spotlight is mounted on a boat so that its beam makes an angle $\bar{\theta}$ with the deck. (Fig. 12.20). If this boat is then set in motion at speed v, what angle θ does an observer on the dock say the beam makes with the deck? Compare Prob. 12.10, and explain the difference."

Fig. 12.20 of Griffiths



(a)

$$d\bar{y} = dy$$

$$d\bar{t} = \gamma \left(dt - \frac{u}{c^2}\, dx\right)$$

$$\frac{d\bar{y}}{d\bar{t}} = \frac{dy}{\gamma\left(dt - \frac{u}{c^2}\, dx\right)}$$

$$\boxed{\frac{d\bar{y}}{d\bar{t}} = \frac{dy}{dt}\,\frac{1}{\gamma}\,\frac{1}{\left(1 - \frac{u}{c^2}\frac{dx}{dt}\right)}}$$

(b)

$$\tan\theta = \frac{s_y}{s_x} = \underbrace{c\sin\theta\,\frac{1}{\gamma}\,\frac{1}{\left(1+\frac{u}{c^2}c\cos\theta\right)}}_{s_x}\;\underbrace{\frac{1+\frac{u}{c^2}c\cos\theta}{(c\cos\theta + u)}}_{1/s_y}$$

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta + \frac{u}{c}}\,\frac{1}{\gamma}}$$

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.15 — answer

Question. Answer:

(1) Q843 PC738

Problem 12.15 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

"You probably did Prob. 12.4 from the point of view of an observer on the ground. Now do it from the point of view of the police car, the outlaws, and the bullet. That is, fill in the gaps in the following table:

speed of → relative to ↓	ground	police	outlaws	bullet	do they escape?
ground					
police					
outlaws					
bullet					

"

Use the shorthand g, p, o, b for ground, police, outlaws and bullet.

Some of the entries in the table we can write down immediately from the speeds given in the statement of the problem.

ie. v_{pg}, v_{og}, v_{bp}

Note, also, that $v_{xy} = -v_{yx}$, ~~and~~ that all of the diagonal entries in the table are zero.

We are left to calculate v_{bg}, v_{op}, and v_{bo}.

(1) Calculation of v_{bg}:

$$v_{bg} = \frac{v_{bp} + v_{pg}}{1 + \frac{v_{bp}v_{pg}}{c^2}}$$

$$= \left(\tfrac{1}{3}c + \tfrac{1}{2}c\right) / \left(1 + \tfrac{1}{3}c\,\tfrac{1}{2}c\,\tfrac{1}{c^2}\right)$$

$$= \left(\tfrac{2}{6}c + \tfrac{3}{6}c\right) / \left(1 + 1/6\right)$$

$$= \frac{6}{7}\left(\frac{5}{6}c\right)$$

$$= \frac{5}{7}c$$

(2) Calculation of v_{op}:

$$v_{op} = \frac{v_{og} + v_{gp}}{1 + \frac{v_{og}v_{gp}}{c^2}}$$

②

$$u_{op} = \frac{\frac{3}{4}c + \left(-\frac{1}{2}c\right)}{1 + \frac{3}{4}\left(\frac{-1}{2}\right)}$$

$$= \frac{1}{4}c \; \frac{1}{5/8}$$

$$= \frac{2}{5}c$$

③ Calculation of u_{bo} :

$$u_{bo} = \frac{u_{bp} + u_{po}}{1 + \frac{u_{bp}u_{po}}{c^2}}$$

$$= \frac{\frac{1}{3}c - \frac{2}{5}c}{1 - \frac{1}{3}\frac{2}{5}}$$

$$= \frac{5-6}{15} \; \frac{1}{\frac{13}{15}} \; c$$

$$= -\frac{1}{13}c$$

We can now complete the table :

speed of

relative to	ground	police	outlaws	bullet	do they escape?
ground	0	1/2 c	3/4 c	5/7 c	
police	−1/2 c	0	2/5 c	1/3 c	yes
outlaws	−3/4 c	−2/5 c	0	−1/13 c	
bullet	−5/7 c	−1/3 c	1/13 c	0	

this has to be the same in all cases !

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.16 — answer

Question. Answer:

Problem 12.16 of Griffiths, "Introduction to Electrodynamics", 3rd ed.

" The Twin Paradox Revisited   On their 21st birthday, one twin gets gets on a moving sidewalk, which carries her out to star X at $\frac{4}{5}c$; her twin brother stays home. When the travelling twin gets to star X, she immediately jumps onto the returning moving sidewalk and comes back to earth, again at speed $\frac{4}{5}c$. She arrives on her 39th birthday (as determined by 'her' watch).

(a) How old is her twin brother (who stayed at home)?
(b) How far away is star X? (Give your answer in light years.)

Call the outbound sidewalk system $\bar{S}$ and the inbound one $\tilde{S}$ (the earth system is S). All three systems set their master clocks, and choose their origins, so that $x = \bar{x} = \tilde{x} = 0$, $t = \bar{t} = \tilde{t} = 0$ at the moment of departure.

(c) What are the coordinates (x, t) of the jump (from outbound to inbound sidewalk) in S?
(d) What are the coordinates $(\bar{x}, \bar{t})$ of the jump in $\bar{S}$?
(e) What are the coordinates $(\tilde{x}, \tilde{t})$ of the jump in $\tilde{S}$?
(f) If the travelling twin wanted her watch to agree with the clock in $\tilde{S}$, how would she have to reset it immediately after the jump? If she did this, what would her watch read when she got home? (This wouldn't change her age, of course – she's still 39 – it would just make her watch agree with the standard synchronization in $\tilde{S}$.)
(g) If the travelling twin is asked the question, "How old is your brother 'right now'", what is the correct reply (i) just 'before' she makes the jump, (ii) just after she makes the jump? (Nothing dramatic happens to her brother during the split second between (i) and (ii), of course; what does change abruptly is his sister's notion of what "right now, back home", means.
(h) How many earth years does the return trip take? Add this to (ii) from (g) to determine how old 'she' expects him to be at their reunion. Compare your answer to (a).

a) $\text{brothers age} = 21 \text{ yrs} + \underbrace{(39-21) * \frac{1}{\sqrt{1-\left(\frac{4}{5}\right)^2}}}_{\text{standard time dilation}} \text{ yrs}$

$= 21 \text{ yrs} + 18 * \frac{5}{3} \text{ yrs}$

$= 51 \text{ yrs}$

b) distance = time of trip for brother * speed / 2

$= \frac{(39-21) \text{ yrs}}{\sqrt{1-\left(\frac{4}{5}\right)^2}} * \frac{4}{5} c \ \frac{1}{2}$

$= 30 \text{ yrs} * \frac{2}{5} c$

$= 12 \ c \text{ yrs}$

c) x = distance from b

$= 12 c \text{ yrs}$

$t = \frac{\text{time of trip for brother}}{2}$

$= 15 \text{ yrs}$ (from a)

$\boxed{x = 12 c\text{yrs}, \ t = 15 \text{ yrs}}$

d) Use Lorentz transformations:

$\bar{x} = \gamma (x - v_{\text{bar},\text{unbar}} t) = 0$

$\bar{t} = \gamma \left(t - \frac{v_{\text{bar},\text{unbar}}\, x}{c^2}\right)$

$= \frac{5}{3} \left(15 \text{ yrs} - \frac{4}{5} * 12 \text{ yrs}\right)$

$= \frac{5}{3} * \frac{(75-48)}{5}$

$= 27/3 \text{ yrs}$

$= 9 \text{ yrs}$

$\boxed{\bar{x} = 0, \ \bar{t} = 9 \text{yrs}}$

e) Use Lorentz transformations again

$$\tilde{x} = \gamma \left(x - v_{\sim,\text{unbar}}\, t \right)$$

$$= \frac{5}{3} \left(12c\,\text{yrs} + \frac{4}{5} c * 15\,\text{yrs} \right)$$

$$= \frac{5}{3}\, 24c\,\text{yrs}$$

$$= 40\, c\,\text{yrs}$$

$$\tilde{t} = \gamma \left(t - \frac{v_{\sim,\text{unbar}}\, x}{c^2} \right)$$

$$= \frac{5}{4} \left(15\,\text{yrs} + \frac{4}{5} * 12\,\text{yrs} \right)$$

$$= \frac{5}{4}\, \frac{(75+48)}{5}\,\text{yrs}$$

$$= \frac{123}{4}\,\text{yrs}$$

$$= 30.75\,\text{yrs}$$

$$\boxed{\tilde{x} = 40\, c\,\text{yrs}\ ,\ \tilde{t} = 30.75\,\text{yrs}}$$

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.25 — answer

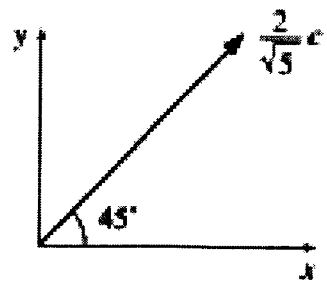
Question. Answer:

(1)

Q680 A586

Problem 12.25 from Griffiths, *Introduction to Electrodynamics*, 3rd ed.
Problem 12.25 from Griffiths, *Introduction to Electrodynamics*, 4th ed.

A car is travelling along the 45° line in inertial frame $\mathcal{S}$ at (ordinary) speed $(2/\sqrt{5})c$:



(a) Find the components u_x and u_y of the (ordinary) velocity.

(b) Find the components η_x and η_y of the proper velocity.

(c) Find the zeroth component of the four-velocity (i.e. "time-part").

Inertial frame $\bar{\mathcal{S}}$ is moving in the x direction with (ordinary) speed $(\sqrt{2/5})c$ relative to $\mathcal{S}$. By using the appropriate transformation laws:

(d) Find the (ordinary) velocity components $\bar{u}_x$, $\bar{u}_y$ in $\bar{\mathcal{S}}$.

(e) Find the proper velocity components $\bar{\eta}_x$ and $\bar{\eta}_y$ in $\bar{\mathcal{S}}$. using the rules for transforming four-vectors.

(f) As a consistency check, verify that:

$$\vec{\bar{\eta}} = \frac{\vec{\bar{u}}}{\sqrt{1 - \bar{u}^2/c^2}}$$

(a)

$$u_x = \cos\theta \ u$$

$$= \frac{1}{\sqrt{2}} \frac{2}{\sqrt{5}} c$$

$$= \sqrt{\frac{2}{5}} c$$

$$u_y = \sin\theta \ u$$

$$= \sqrt{\frac{2}{5}} c$$

$$\boxed{u_x = u_y = \sqrt{\frac{2}{5}} c}$$

By definition, the proper velocity is

$$\vec{\eta} \equiv \gamma(v)\,\vec{u}$$

$$= \frac{1}{\sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2}}\,\vec{u}$$

$$= \frac{1}{\sqrt{1-\frac{4}{5}}}\,\vec{u}$$

$$= \sqrt{5}\,\vec{u}$$

Since from part (a) $u_x = u_y = \sqrt{\frac{2}{5}}\,c$, we have

$$\boxed{\eta_x = \eta_y = \sqrt{2}\,c}$$

(c) By definition, the "time-part" of the four-velocity is:

$$\boxed{\gamma(v)\,c = \sqrt{5}\,c}$$

(d) In frame $\bar{S}$ moving in the x-direction with speed $\sqrt{\frac{2}{5}}\,c$ wrt S:

$$\bar{u}_x = \frac{u_x - \sqrt{\frac{2}{5}}\,c}{1 - \frac{u_x}{c}\sqrt{\frac{2}{5}}}$$

$$= 0$$

Note to Chuchong Ni: I distributed problem set with relative frame velocity $= \frac{2}{\sqrt{5}}c$ initially. Please accept these answers as correct as well.

$$\bar{u}_y = \frac{u_y}{\gamma\left(\sqrt{\frac{2}{5}}\right)\left[1 - \frac{u_x}{c}\sqrt{\frac{2}{5}}\right]}$$

$$= \frac{\sqrt{2/5}\, c}{1 - \sqrt{\frac{2}{5}}\sqrt{\frac{2}{5}}} \sqrt{1 - \left(\sqrt{\frac{2}{5}}\right)^2}$$

$$= \frac{\sqrt{2/5}\, c}{3/5} \sqrt{1 - 2/5}$$

$$= \sqrt{\frac{2}{3}}\, c$$

$$\boxed{\bar{u}_x = 0 \;,\; \bar{u}_y = \sqrt{\frac{2}{3}}\, c}$$

(e)

$\bar{\eta}_y = \eta_y = \sqrt{2}\, c$ since relative frame velocity is along x. (the "standard configuration")

$$\bar{\eta}_x = \cosh\theta\, \eta_x - \sinh\theta \underbrace{\eta_+}_{\text{“time part”}} \qquad (*)$$

In this case $\cosh\theta = \dfrac{1}{\sqrt{1 - 2/5}} = \sqrt{\dfrac{5}{3}}$

$$\sinh\theta = \sqrt{\cosh^2\theta - 1}$$

$$= \sqrt{\frac{2}{3}}$$

Substituting into (*)

$$\bar{\eta}_x = \underbrace{\sqrt{\frac{5}{3}}}_{\cosh\theta} \underbrace{\sqrt{2}\, c}_{\eta_x} - \underbrace{\sqrt{\frac{2}{3}}}_{\sinh\theta} \underbrace{\sqrt{5}\, c}_{\text{“}\eta_+\text{”}}$$

$= 0$ as expected.

$$\boxed{\bar{\eta}_x = 0 \;,\; \bar{\eta}_y = \sqrt{2}\, c}$$

④ Q680 AS86

f) By definition

$$\bar{\vec{\eta}} = \gamma(\bar{v})\,\bar{\vec{u}}$$

$$\gamma(\bar{v}) = \frac{1}{\sqrt{1-\frac{2}{3}}}$$

$$= \sqrt{3}$$

Thus $\bar{\eta}_x = 0$ (since $\bar{u}_x = 0$)

$$\bar{\eta}_y = \sqrt{3}\,\underbrace{\sqrt{\frac{2}{3}}\,c}_{\bar{u}_y}$$

$$= \sqrt{2}\,c$$

Thus $\boxed{\bar{\eta}_x = 0\,,\ \bar{\eta}_y = \sqrt{2}\,c}$

in agreement with part (e).

Taylor, *Classical Mechanics*, Problem 15.8 — answer

Question. Answer:

1

Q659 A565

Problem 15.8 from Taylor, *Classical Mechanics*

The pion (π^+ or π^-) is an unstable particle that decays with a proper half-life of 1.8×10^{-8} s. (This is the half-life measured in the pion's rest frame.)

(a) What is the pion's half-life measured in a frame S where it is travelling at 0.8c?

(b) If 32,000 pions are created at the same place, all travelling at this same speed, how many will remain after they have travelled down an evacuated pipe of length $d = 36$ m? Remember that after n half-lives, 2^{-n} of the original particles survive.

(c) What would the answer have been if you had ignored time dilation? (Naturally it is the answer (b) that agrees with experiment.)

P : pion frame
E : earth frame (this is the frame in which the pion is moving at 0.8c)

(a) "Moving clocks run slow"

$$\Delta t_{1/2,E} = \Delta t_{1/2,P} \cosh \theta_{PE,x} \qquad (1)$$

where $\theta_{PE} = \tanh^{-1}(v_{PE}/c)$ (2)

In this case:

$$\Delta t_{1/2,E} = 1.8 * 10^{-8}\,s * \cosh(\tanh^{-1}(0.8)) \qquad (3)$$
$$\simeq 3 * 10^{-8}\,s \qquad (4)$$

$\boxed{\text{The pion's half-life in the frame where it is travelling at 0.8c is } 3*10^{-8}\,s.}$

The time elapsed in the earth frame is:

$$\Delta t_E = \frac{d}{v} \qquad (5)$$
$$= \frac{36m}{0.8 * 3 * 10^8\,m/s} \qquad (6)$$
$$\simeq 1.5 * 10^{-7}\,s \qquad (7)$$

Thus, since:

$$N_{pions}(t) = N_{pions}(0)\, 2^{-\frac{\Delta t_E}{\Delta t_{1/2,E}}} \qquad (8)$$

in this case:

$$\boxed{\underbrace{N_{pions}}_{\text{surviving trip}} \simeq 32000\, 2^{-\frac{1.5*10^{-7}}{3*10^{-8}}} \qquad (9)}$$

$$\boxed{\simeq 1000} \qquad (10)$$

(c)

To ignore time dilation we compute N_{pions} using their rest-frame lifetime:

$$\boxed{N_{pions} \simeq 32000\, 2^{-\frac{1.5*10^{-7}}{1.8*10^{-8}}} \qquad (11)}$$

$$\boxed{\simeq 100} \leftarrow \text{erroneous as time dilation has been ignored.} \qquad (12)$$

Taylor, *Classical Mechanics*, Problem 15.12 — answer

Question. Answer:

① Q660 A566

Problem 15.12 from Taylor, *Classical Mechanics*

Consider the experiment of Problem 15.8 from the point of view of the pions' rest frame. What is the half-life of the pions in this frame? In part (b), how long is the pipe as "seen" by the pions and how long does it take to pass the pions? How many pions remain at the end of this time? Compare with the answer to Problem 15.8 and describe how the two different arguments led to the same result.

The half-life of the pions in their rest frame is $1.8 * 10^{-8}$ s, as stated in Problem 15.8.

In the pion frame, the length of the evacuated tube, which is stationary in the earth frame is observed to be length contracted:

$$\Delta L_P = \frac{\Delta L_E}{\cosh \Theta_{PE,x}} \qquad (1)$$

ΔL_E ← length of tube in earth frame

ΔL_P ← length of tube in pion frame

where $\Theta_{PE,x} = \tanh^{-1}(V_{PE,x}/c)$

Evaluating the numbers in this case:

$$\Delta L_P \simeq \frac{36\text{ m}}{\cosh(\tanh^{-1}(0.8))}$$

$$\simeq 21.6\text{ m}$$

The time it takes the tube to pass the pions (as observed in pion frame)

$$\Delta t_P = \frac{\Delta L_P}{V_{PE,x}}$$

Evaluating the numbers in this case

$$\Delta t_P \simeq \frac{21.6\text{m}}{0.8 * 3 * 10^8 \text{ m/s}}$$

$$\simeq 9.0 * 10^{-8} \text{ s}$$

Thus the number of pions that remain after the tube has passed is:

$$N_{pions}(\Delta t_P) = N_{pions}(0)\, 2^{-\frac{\Delta t_P}{\Delta t_{1/2,P}}}$$

In this case

$$N_{pions} \simeq 32000 * 2^{-\frac{9.0*10^{-8}\,s}{1.8*10^{-8}\,s}}$$

$$\simeq 1000$$

which is the same as was computed in Problem 15.8.

Summarizing,

$$\boxed{\text{The pipe is seen to be 21.6 m in the pion frame, and it takes } 9.0*10^{-8}\,s \text{ to pass the pions. The number of pions remaining after the passage of the pipe is 1000.}}$$

The number of surviving pions is the same as we computed in Problem 15.8.

Comparing:

Problem 15.8, computation in Earth frame $\left\{ \dfrac{N_{pions,final}}{N_{pions,initial}} = 2^{-\frac{\ell}{V_{PE}} \frac{1}{(t_{1/2,P}\cosh\Theta_{PE})}} \right.$ ← $t_{1/2,E}$ time dilation

This problem computation in pion frame $\left\{ \quad '' \quad = 2^{-\frac{\ell}{\cosh\Theta_{PE}} \frac{1}{V_{PE}} \frac{1}{t_{1/2,P}}} \right.$ ← ℓ_P, length contraction

the same!

In the pion frame the tube length is contracted. In the earth frame the "pion clock" runs slow. In both cases the effect is the same on the number of surviving pions.

Taylor, *Classical Mechanics*, Problem 15.57 — answer

Question. Answer:

Problem 15.57 from Taylor, *Classical Mechanics*

When a radioactive nucleus of astatine 215 decays at rest, the whole atom is torn into two in the reaction

$$^{215}\text{At} \rightarrow {}^{211}\text{Bi} + {}^{4}\text{He}.$$

The masses of the three atoms are (in order) 214.9986, 210.9873, and 4.0026, all in atomic mass units. (1 atomic mass unit $\approx 1.66 \times 10^{-27}$ kg $\approx 931.5\,\text{MeV}/c^2$ What is the total kinetic energy of the two outcoming atoms, in joules and in MeV?

Before disintegration:

$$E_{tot} = m_{At} c^2 \tag{1}$$

After disintegration:

$$E_{tot} = \underbrace{T_{tot}}_{\text{total kinetic energy}} + m_{Bi} c^2 + m_{^4He} c^2 \tag{2}$$

By conservation of energy we equate Eq (1) and (2):

$$m_{At} c^2 = T_{tot} + m_{Bi} c^2 + m_{He} c^2$$

and thus

$$T_{tot} = (m_{At} - m_{Bi} - m_{He}) c^2$$

Substituting numbers

$$T_{tot} \simeq (\overbrace{214.9986 - 210.9873 - 4.0026}^{\Delta m}) * 1.66 * 10^{-27}\,\text{kg} * (3 * 10^8\,\text{m/s})^2$$

$$\simeq 1.30 * 10^{-12}\,\text{J}$$

$$T_{tot} \simeq \Delta m * c^2 * 931.5\,\frac{\text{MeV}}{c^2}$$

$$\simeq 8.10\,\text{MeV}$$

$$\boxed{\begin{aligned} T_{tot} &\simeq 1.30 * 10^{-12}\,\text{J} \\ &\simeq 8.10\,\text{MeV} \end{aligned}}$$

Taylor, *Classical Mechanics*, Problem 15.60 — answer

Question. Answer:

Q682 A588

Problem 15.60 from Taylor, *Classical Mechanics*

A particle of mass m_a decays at rest into two identical particles each of mass m_b. Use conservation of momentum and energy to find the speed of the outgoing particles.

Conservation of "momenergy":

$$\underbrace{(m_a c, \vec{0})}_{\text{before}} = \underbrace{\left(\frac{E}{c}, \vec{p}\right) + \left(\frac{E}{c}, -\vec{p}\right)}_{\text{after}} \tag{1}$$

The final particles have the same masses. By conservation of momentum they must have equal and opposite momenta. Thus they will both have the same speed.

Applying conservation of energy:

$$m_a c = \frac{2E}{c} \tag{2}$$

$$= \frac{2 m_b c}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{3}$$

Rearranging to solve for v/c:

$$\boxed{\frac{v}{c} = \sqrt{1-\left(\frac{2m_b}{m_a}\right)^2}} \tag{4}$$

Taylor, *Classical Mechanics*, Problem 15.90 — answer

Question. Answer:

(1)

Q683 A589

Problem 15.90 from Taylor, *Classical Mechanics*

The first positrons to be observed were created in electron-positron pairs by high-energy cosmic-ray photons in the upper atmosphere,

(a) Show that an isolated photon cannot convert to an electron-positron pair in the process $\gamma \to e^+ + e^-$ [Show that this process inevitably violates conservation of four-momentum.]

(b) What actually occurs is that a photon collides with a stationary nucleus with the result

$$\gamma + \text{ nucleus } \to e^+ + e^- + \text{nucleus}.$$

Convince yourself that the formula

$$E_a^{\min} = \frac{(\sum m_{\text{fin}})^2 - m_a^2 - m_b^2}{2m_b} c^2 \qquad \text{(15.98 of Taylor)}$$

can be used to find the minimum energy for a photon to induce this reaction. [The derivation of Taylor's (15.98) assumed that the incident particle had $m > 0$.] Show that, provided the mass of the nucleus is much greater than that of the electron, the minimum photon energy to induce this reaction is approximately $2m_ec^2$. [This is exactly the energy one would have calculated for the process $\gamma \to e^+ + e^-$ and shows that the role of the nucleus is just as a "catalyst" that can absorb some three-momentum.]

(a) Conservation of momenergy:

$$\underbrace{(p_\gamma, \vec{p}_\gamma)}_{\text{before}} = \underbrace{\left(\frac{E_{e^+}}{c}, \vec{p}_{e^+}\right) + \left(\frac{E_{e^-}}{c}, \vec{p}_{e^-}\right)}_{\text{after}} \qquad (1)$$

For the space-part:

$$\vec{p}_\gamma = \vec{p}_{e^+} + \vec{p}_{e^-} \qquad (2)$$

the triangle inequality must be respected: (a purely mathematical idea)

$$p_\gamma \leq p_{e^+} + p_{e^-} \qquad (3)$$

By energy conservation (time part of Eq. (1))

$$p_\gamma = \frac{E_{e^+}}{c} + \frac{E_{e^-}}{c} \qquad (4)$$

where

$$\frac{E_{e^+}}{c} = \sqrt{(p_{e^+})^2 + (m_ec)^2} \qquad (5)$$

and similarly for $\frac{E_{e^-}}{c}$.

We may now write Eq. (3) in terms of E_{e^+} and E_{e^-} using (4) and (5):

$$\frac{E_{e^+}}{c} + \frac{E_{e^-}}{c} \leq \sqrt{\left(\frac{E_{e^+}}{c}\right)^2 - (m_e c)^2} + \sqrt{\left(\frac{E_{e^-}}{c}\right)^2 - (m_e c)^2} \quad (6)$$

which is impossible to satisfy for $m_e > 0$. Thus Eq. (3) cannot be true, and thus nor can Eq. (2). It is impossible to satisfy momentum conservation for the process: $\gamma \rightarrow e^+ + e^-$. //

(b) Consider the frame where $\vec{p} = 0$ (this was not possible for part (a)). Write the momenergy in this frame as:

$$\left(\frac{E_{com}}{c}, \vec{0}\right) \quad (7)$$

The momenergy in the lab-frame:

$$(p_\gamma, \vec{p}_\gamma) + (m_n c, \vec{0}) \quad (8)$$

Evaluating the Lorentz <u>invariant</u> scalar product in both frames gives:

$$-\left(\frac{E_{com}}{c}\right)^2 = p_\gamma^2 - (p_\gamma + m_n c)^2 \quad (9)$$

$$= -(m_n c)^2 - 2 p_\gamma m_n c \quad (10)$$

E_{com} must be <u>at least</u> $2m_e c^2 + m_n c^2$ and thus

$$c\sqrt{(m_n c)^2 + 2 p_\gamma m_n c} \geq 2 m_e c^2 + m_n c^2 \quad (11)$$

$$(m_n c)^2 + 2p_\gamma m_n c \geq (2m_e c + m_n c)^2 \qquad (12)$$

$$p_\gamma \geq \frac{(2m_e c + m_n c)^2 - (m_n c)^2}{2m_n c} \qquad (13)$$

$$\geq c \left(\frac{4m_e^2 + \cancel{m_n^2} + 4m_e m_n - \cancel{m_n^2}}{2m_n} \right) \qquad (14)$$

$$p_\gamma \geq 2c\,m_e \left(1 + \frac{m_e}{m_n} \right) \qquad (15)$$

or

$$\boxed{c p_\gamma \geq 2m_e c^2 \left(1 + \frac{m_e}{m_n} \right)} \qquad (16)$$

Note that $\frac{m_e}{m_n} \ll 1$, and so the required photon energy cp_γ is just slightly more than the rest mass of an electron and positron $(2m_e c^2)$.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 12.35 — answer

Question. Answer:

Q684 A590

Problem 12.34 from Griffiths, *Introduction to Electrodynamics*, 3rd ed.
Problem 12.35 from Griffiths, *Introduction to Electrodynamics*, 4th ed.

In the past, most experiments in particle physics involved stationary targets: one particle (usually a proton or an electron) was accelerated to a high energy E, and collided with a target particle at rest (see (a) below). Far higher *relative* energies are obtainable (with the same accelerator) if you accelerate *both* particles to energy E, and fire them at each other (see (b) below):

(a) (b)

Classically, the energy $\bar{E}$ of one particle, relative to the other, is just $4E$ (why?) — not much of a gain (only a factor of 4). But *relativistically* the gain can be *enormous*. Assuming the two particles have the same mass, m, show that

$$\bar{E} = \frac{2E^2}{mc^2} - mc^2.$$

Suppose that you use protons ($mc^2 \approx 1\,\text{GeV}$) with $E = 30\,\text{GeV}$. What $\bar{E}$ do you get? What multiple of E does this amount to? [Because of this relativistic enhancement, most modern elementary particle experiments involve **colliding beams**, instead of fixed targets. e.g. LHC]

Without special relativity:

$$\bar{E} = \frac{1}{2} m \underbrace{(v_{rel})^2}_{\text{relative speed}} \qquad (1)$$

$$= \frac{1}{2} m \underbrace{(2v_e)^2}_{\text{laboratory speed}} \qquad (2)$$

$$= 4\,\frac{1}{2} m v_e^2 \qquad (3)$$

$$= 4E \quad \text{as req'd.} // \qquad (4)$$

With special relativity:

This problem requires velocity addition, for which rapidities are ideal!

$$\bar{E} = mc^2 \cosh(\overbrace{2\theta}^{\theta+\theta}) \qquad (5)$$

with the identity $\cosh 2\theta = 2\cosh^2\theta - 1$ (DLMF 4.35.27)

$$\bar{E} = mc^2 (2\cosh^2\theta - 1) \qquad (6)$$

$$= mc^2 \left(2\left(\frac{E}{mc^2}\right)^2 - 1\right) \qquad (7)$$

Q684 A590

$$\bar{E} = \frac{2E^2}{mc^2} - mc^2 \tag{8}$$

as req'd.//

With $E = 30\,GeV$, $mc^2 \simeq 1\,GeV$

$$\bar{E} \simeq \frac{2*(30)^2}{1\,GeV}\,GeV^2 - 1\,GeV \tag{9}$$

$$\simeq 1800\,GeV \tag{10}$$

$$\boxed{\bar{E} \simeq 60\,E} \tag{11}$$

Taylor, *Classical Mechanics*, Problem 15.92 — answer

Question. Answer:

Q685 A591

Problem 15.92 from Taylor, *Classical Mechanics*

A positive pion decays at rest into a muon and neutrino, $\pi^+ \to \mu^+ + \nu$. The masses involved are $m_\pi = 140\,\mathrm{MeV/c^2}$, $m_\mu = 106\,\mathrm{MeV/c^2}$, and $m_\nu = 0$. (There is now convincing evidence that m_ν is not exactly zero, but it is small enough that you can take it to be zero for this problem.) Show that the speed of the outgoing muon has

$$\beta = \left(m_\pi^2 - m_\mu^2\right) / \left(m_\pi^2 + m_\mu^2\right).$$

Evaluate this numerically. Do the same for the much rarer decay mode $\pi^+ \to e^+ + v$, ($m_e = 0.5\,\mathrm{MeV/c^2}$).

Apply conservation of momenergy:

$$\underbrace{(m_\pi c, \vec{0})}_{\text{before}} = \underbrace{\left(\frac{E_\mu}{c}, \vec{p}_\mu\right) + \underbrace{(p_\nu, \vec{p}_\nu)}_{\text{note simplicity for a massless particle}}}_{\text{after}} \qquad (1)$$

Conservation of momentum (the "space" component of Eq. (1)) tells us:

$$\vec{p}_\mu = -\vec{p}_\nu \qquad (2)$$

and thus

$$p_\mu = p_\nu \qquad (3)$$

allowing energy conservation ("time" part of Eq. (1)) to be written as:

$$m_\pi c = \frac{E_\mu}{c} + p_\mu \qquad (4)$$

Introduce rapidity θ to describe μ particle:

$$p_\mu = m_\mu \underbrace{\cosh\theta}_{\gamma} \underbrace{\tanh\theta}_{v/c} c \qquad (5)$$

$$= m_\mu c \sinh\theta \qquad (6)$$

and

$$\frac{E_\mu}{c} = \gamma m_\mu c \qquad (7)$$

$$= m_\mu c \cosh\theta \qquad (8)$$

Thus energy conservation (Eq. 4) may be rewritten as:

$$m_\pi c = m_\mu c \sinh\Theta + m_\mu c \cosh\Theta \qquad (9)$$

$$m_\pi = m_\mu (\sinh\Theta + \cosh\Theta) \qquad (10)$$

$$m_\pi = m_\mu e^{\Theta} \qquad (11)$$

and thus

$$e^{\Theta} = m_\pi / m_\mu \qquad (12)$$

We may now determine v/c using e^{Θ}:

$$\frac{v}{c} = \tanh\Theta \qquad (13)$$

$$= \frac{e^{\Theta} - e^{-\Theta}}{e^{\Theta} + e^{-\Theta}} \qquad (14)$$

$$= \frac{\left(\frac{m_\pi}{m_\mu} - \frac{m_\mu}{m_\pi}\right)}{\left(\frac{m_\pi}{m_\mu} + \frac{m_\pi}{m_\mu}\right)} \qquad (15)$$

$$\boxed{\frac{v}{c} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}} \qquad (16)$$

as req'd.

Substituting $m_\pi = 140\ \mathrm{MeV}/c^2$, $m_\mu = 106\ \mathrm{MeV}/c^2$ gives

$$\boxed{\frac{v_\mu}{c} \simeq 0.27 \quad \text{for } \pi^+ \rightarrow \mu^+ + \nu} \qquad (17)$$

Replacing m_μ by $m_e \simeq 0.5\ \frac{\mathrm{MeV}}{c^2}$ gives

$$\boxed{\frac{v_e}{c} \simeq 0.99997 \quad \text{for } \pi^+ \rightarrow e^+ + \nu} \qquad (18)$$

Taylor, *Classical Mechanics*, Problem 15.93 — answer

Question. Answer:

Q692 A598

Problem 15.93 from Taylor, *Classical Mechanics*

Consider a head-on elastic collision between a high-energy electron (energy E_0 and speed $\beta_0 c$) and a photon of energy $E_{\gamma 0}$.

(a) Show that the final energy E_γ of the photon is

$$E_\gamma = E_0 \frac{1+\beta_0}{2 + (1-\beta_o)\, E_o / E_{\gamma 0}}.$$

[Hint: Use Taylor's Eq. (15.123).]

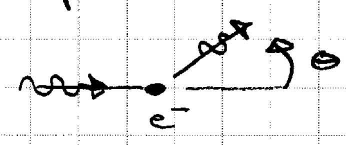
(b) Show that $E_\gamma < E_0$, but that if $\beta_0 \to 1$, then $E_\gamma / E_o \to 1$; that is, a very high-energy electron loses almost all its energy to the photon in a head-on collision. What fraction of its original energy would the electron retain if $E_0 \approx 10\,\text{TeV}$ and the photon was in the visible range, $E_{\gamma 0} \approx 3\,\text{eV}$? (Remember that the mass of the electron is about $0.5\,\text{eV}/\text{c}^2$; $1\,\text{TeV} = 1 \times 10^{12}\,\text{eV}$.)

(a) This is variation on Compton scattering.

In the frame where the e^- is initially stationary (S):

$$\frac{1}{\tilde{E}_{\gamma S}} = \frac{1}{E_{\gamma S}} + \frac{(1-\cos\theta)}{m_e c^2} \qquad (1)$$

where $E_{\gamma S}$ and $\hat{E}_{\gamma S}$ are the initial and final photon energies, and θ is the angle that the photon is scattered by



In this problem $\theta = \pi$ (a "head-on" collision) and thus

$$\frac{1}{\tilde{E}_{\gamma S}} = \frac{1}{E_{\gamma S}} + \frac{2}{m_e c^2} \qquad (2)$$

To obtain the desired result we Lorentz transform the photon into the e^- stationary frame (S), use the Compton result (Eq. 2), and then transfer back to the laboratory frame.

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$$E_{\nu S} = E_{\nu L} \cosh \theta_{SL} + \underbrace{E_{\nu L} \sinh \theta_{SL,x}} \tag{3}$$

assume photon is travelling in -ve x direction, and electron in +ve x direction.

$$= E_{\nu L} \left(\cosh \theta_{SL} + \sinh \theta_{SL} \right) \tag{4}$$

After collision, photon is travelling in +ve x-direction. Thus

$$\tilde{E}_{\nu L} = \tilde{E}_{\nu S} \cosh \theta_{SL} + \tilde{E}_{\nu S} \sinh \theta_{SL} \tag{5}$$

$$= \tilde{E}_{\nu S} \left(\cosh \theta_{SL} + \sinh \theta_{SL} \right) \tag{6}$$

Combining Eq.'s (6), (4) and (2) gives:

$$\tilde{E}_{\nu L} = \frac{\cosh \theta_{SL} + \sinh \theta_{SL}}{\dfrac{1}{E_{\nu L} (\cosh \theta_{SL} + \sinh \theta_{SL})} + \dfrac{2}{m_e c^2}} \tag{7}$$

$$= \frac{\cosh \theta_{SL} + \sinh \theta_{SL}}{\dfrac{1}{E_{SL}\, e^{\theta_{SL}}} + \dfrac{2}{m_e c^2}} \tag{8}$$

To write in the desired form, note that

$$\cosh \theta + \sinh \theta = \cosh \theta \left(1 + \tanh \theta \right) \tag{9}$$

$$= \cosh \theta \left(1 + \beta \right) \tag{10}$$

and also $e^{-\theta_{SL}} = \cosh \theta \left(1 - \beta \right)$ (11)

giving

$$\tilde{E}_{\nu L} = \frac{\cosh \theta_{SL} \left(1 + \beta_{SL} \right)}{\dfrac{\cosh \theta_{SL} \left(1 - \beta_{SL} \right)}{E_{SL}} + \dfrac{2}{m_e c^2}} \tag{12}$$

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$$\tilde{E}_{\gamma L} = \frac{m_e c^2 \cosh\theta_{SL} (1+\beta_{SL})}{\frac{m_e c^2 \cosh\theta_{SL}(1-\beta_{SL})}{E_{SL}} + 2} \tag{13}$$

$$\boxed{\underbrace{\tilde{E}_{\gamma L}}_{\text{final photon energy}} = \frac{E_{e^-}(1+\beta_{e^-})}{\frac{\overbrace{E_{e^-}}^{\text{initial electron energy}}(1-\overbrace{\beta_{e^-}}^{\text{initial electron } v/c})}{\underbrace{E_{SL}}_{\text{initial photon energy}}} + 2}} \tag{14}$$

as req'd.//

(b) To show $E_\gamma < E_0$ we need to show

$$\frac{1+\beta_{e^-}}{2 + \frac{E_0(1-\beta_{e^-})}{E_{\gamma 0}}} < 1 \tag{15}$$

This is true because the numerator is bounded from above by 2, whereas the denominator is bounded from below by 2.

If $\beta_0 \to 1$ then (14) shows $\tilde{E}_{\gamma L} \to E_{e^-}$. i.e. most of the electron energy is transferred to the photon.

For $E_0 \simeq 10\,\text{TeV}$, $E_{\gamma 0} \simeq 3\,\text{eV}$

$$\underbrace{\frac{E_0 - E_\gamma + E_{\gamma 0}}{E_0}}_{\substack{\text{Fraction of energy} \\ e^- \text{ retains.} \\ E_{\gamma 0} \text{ negligible.}}} = 1 - \frac{(1+\beta)}{2 + (1-\beta)\frac{E_0}{E_{\gamma 0}}}$$

$$\simeq 1 - \frac{1}{1 + \frac{(1-\beta)}{2}\frac{E_0}{E_{\gamma 0}}}$$

(4) Q692 A598

We should avoid computing β for highly relativistic situations $(\beta \approx 1)$
Instead consider:

$$E_o = \frac{mc^2}{\sqrt{1-\beta^2}}$$

$$1-\beta^2 = \left(\frac{mc^2}{E_o}\right)^2$$

$$(1-\beta)(1+\beta) = \left(\frac{mc^2}{E_o}\right)^2$$

$$(1-\beta) \approx \frac{1}{2}\left(\frac{mc^2}{E_o}\right)^2$$

and thus

$$\frac{E_o - E_M}{E_o} \approx 1 - \frac{1}{1 + \frac{1}{4}\frac{mc^2}{E_o}\frac{mc^2}{E_{MO}}}$$

$$\approx 1 - \frac{1}{1 + \frac{1}{4} * \frac{(0.5 * 10^6)^2}{3 * 10^{13}}}$$

$$\approx 1 - \left(1 - \frac{1}{4}\frac{(0.5 * 10^6)^2}{3 * 10^{13}}\right)$$

$$\approx \frac{1}{4} * \frac{0.25}{3} * \frac{10^{12}}{10^{13}}$$

$$\approx \frac{1}{48} 10^{-1}$$

$$\approx 2 * 10^{-3}$$

$$\boxed{\text{The electron retains only} \approx 2*10^{-3} \left(\tfrac{2}{1000}\right) \text{ of its original energy after scattering the photon.}}$$

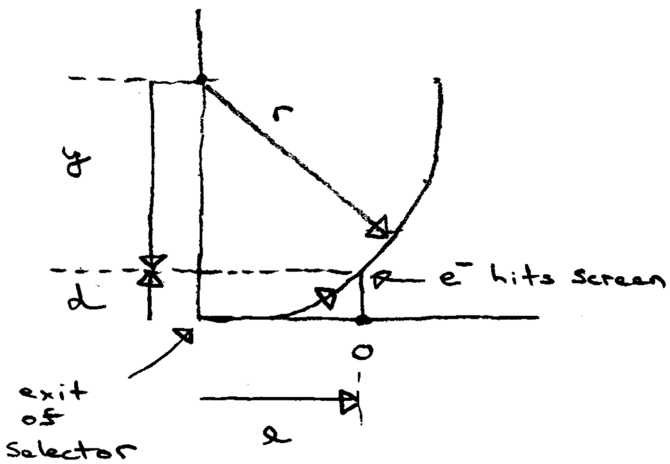
Bucherer's test of the Lorentz force law — answer

Question. Answer:

(a) For $d\vec{p}/dt = 0$, we have $-\vec{E} = \vec{v} \times \vec{B}$. In this case, for $v/c = 0.7$ we have $\boxed{B = 9.53 \times 10^{-3}\,\mathrm{T}}$.

(b) The electron will travel in a circular orbit once leaving the selector with radius of curvature:

$$r = \frac{p}{qB}$$



Recall the equation of a circle

$$r^2 = \ell^2 + y^2 .$$

Rearranging gives:

$$y^2 = \ell^2 - r^2$$

$$y = \sqrt{\ell^2 - r^2}$$

and thus

$$d = r - \sqrt{\ell^2 - r^2} .$$

where ℓ is the distance from the exit of the velocity selector to the screen. Using B from part (a), and $p = mv$ gives $\boxed{d_{\text{non-rel}} \approx 6.6\,\mathrm{mm}}$.

(c) The same formula for r holds, only now we use $p = \gamma(v)mv$ giving $\boxed{d_{\text{rel}} \approx 4.6\,\mathrm{mm}}$.

(d) Reversing the direction of *both* $\vec{E}$ and $\vec{B}$ will select the same velocity, but once the electron leaves the selector it will now be deflected *downwards*. The *difference* in the location of the electrons before and after the field reversal will be $2d$.

Measuring the difference $2d$ avoids the need to precisely locate the point O on the screen.

The numbers for this problem are taken from a description of Bucherer's experiment in Ref. [3].

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 8.2 — answer

Question. Answer:

1

Problem 8.2 in third edition / Problem 8.2 in fourth edition
Consider the charging capacitor in Prob. 7.31. (3rd ed) / 7.34 (4th ed)

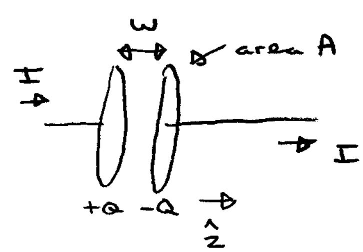
(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\vec{S}$ in the gap. Note especially the direction of $\vec{S}$. Check that Eq. 8.14 is satisfied:

$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \vec{S}.$$

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9 — in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

(a) From the solution to 7.31 (3rd ed) / 7.34 (4th ed)



$$\vec{E} = \frac{Q}{\epsilon_0 A}\,\hat{z}$$

With $Q = 0$ @ $t = 0$, and steady current I, $Q = It$ =

$$\vec{E} = \frac{It}{\epsilon_0 A}\,\hat{z}$$

$$\vec{B} = \frac{\mu_0 s I}{2A}\,\hat{\phi}$$

(b) $$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

$$= \frac{1}{2}\epsilon_0 \frac{I^2 t^2}{\epsilon_0^2 A^2} + \frac{1}{2\mu_0}\frac{\mu_0^2 s^2 I^2}{4A^2}$$

$$u_{em} = \frac{1}{2}\frac{I^2 t^2}{\epsilon_0 A^2} + \frac{\mu_0}{8}\frac{s^2 I^2}{A^2} = \frac{\mu_0 I^2}{A^2}\left[\frac{(ct)^2}{2} + \frac{s^2}{8}\right]$$

$$\frac{\partial u_{em}}{\partial t} = \frac{I^2 t}{\epsilon_0 A^2}$$

②

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$$= \frac{It}{\epsilon_0 A} \frac{\mu_0 s I}{2A} \frac{\hat{z} \times \hat{\phi}}{\mu_0}$$

$$= \frac{I^2 t s}{2 \epsilon_0 A^2} (-\hat{s})$$

$$\nabla \cdot \vec{S} = -\frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{I^2 t s}{2 \epsilon_0 A^2} \right)$$ (using expression for $\nabla \cdot$ in cylindrical coordinates)

$$= -\frac{1}{s} 2s \frac{I^2 t}{2 \epsilon_0 A^2}$$

$$\nabla \cdot \vec{S} = -\frac{I^2 t}{\epsilon_0 A^2}$$

Comparing to $\frac{\partial u_{em}}{\partial t}$ from the previous page, we see that $\nabla \cdot \vec{S} = -\frac{\partial u_{em}}{\partial t}$ is satisfied. ($u_{mech} = 0$).

(c) $W_{em} = \int d\tau \, u_{em}$

$$= \frac{1}{2} \frac{I^2 t^2}{\epsilon_0 A^2} * \underbrace{wA}_{\text{total volume}} + \int_0^a 2\pi s \, ds \frac{\mu_0}{8} \frac{s^2 I^2}{A^2} w$$

$$= \frac{1}{2} \frac{I^2 t^2}{\epsilon_0 A} w + \frac{a^4}{4} 2\pi \frac{\mu_0}{8} \frac{I^2}{A^2} w$$

$$\boxed{W_{em} = \frac{1}{2} \frac{I^2 t^2}{\epsilon_0 A} w + \frac{1}{16} \frac{\mu_0 I^2 a^2 w}{A}}$$

using $\epsilon_0 \mu_0 = \frac{1}{c^2}$ we can also write:

$$W_{em} = \frac{I^2 w}{\epsilon_0 \pi a^2} \left(\frac{t^2}{2} + \frac{1}{16} \frac{a^2}{c^2} \right)$$

$$\frac{d}{dt} W_{em} = \frac{I^2 t w}{\epsilon_0 A}$$

Now let's find total electromagnetic energy flowing into volume:

$$= - \int \vec{S} \cdot d\vec{A}$$

$$= \frac{I^2 t a}{2\epsilon_0 A^2} * w \, 2\pi a \qquad \& \quad A = \pi a^2$$

$$= \frac{I^2 t w}{\epsilon_0 A}$$

which is consistent with $\frac{d}{dt} W_{em}$ from previous page.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 8.4 — answer

Question. Answer:

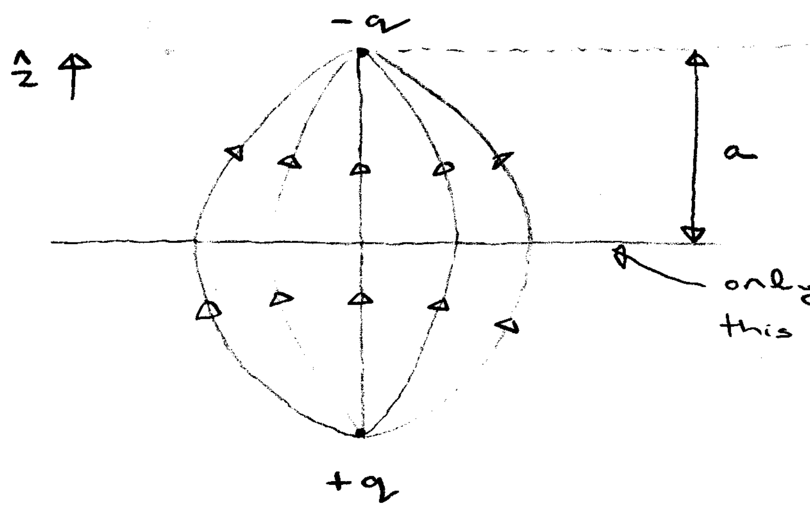
1

Problem 8.4 in third edition / Problem 8.4 in fourth edition of Griffiths', "Introduction to Electrodynamics".

(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

(b) Do the same for charges that are opposite in sign.

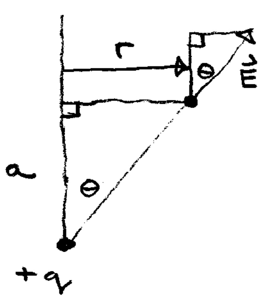
(a)



To determine $\overleftrightarrow{T}$ in the plane we need to determine $\vec{E}$ in the plane (there is no magnetic field in this problem).

$$E_z = 2 \, \frac{q}{4\pi\epsilon_0} \Big(\underbrace{\frac{1}{a/\cos\theta}}_{\text{distance to charge}} \Big)^2 \underbrace{\cos\theta}_{\text{component in z direction}} \tag{1}$$

each charge makes an identical contribution



With $E_x = E_y = 0$ in the plane, the only non-zero element of $\overleftrightarrow{T}$ is T_{zz}. $\left[T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E_i E_j \right) \right]$ (2)

$$T_{zz} = \frac{\epsilon_0}{2} \Big(E_z^2 \underbrace{- E_x^2 - E_y^2}_{=0} \Big) \tag{3}$$

In a time-independent problem:

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a} \tag{4}$$

which in this case (Force on bottom charge: $\hat{n} = \hat{z}$)

$$= \hat{z} \int_{\text{plane}} T_{zz}\, da$$

note that the contribution of the other surface req'd to make this closed will be zero.

$$= \hat{z} \int_0^\infty dr\, 2\pi r\, T_{zz} \tag{5}$$

see Eq. (3)

$$= \hat{z} \int_0^\infty dr\, 2\pi r\, \frac{\epsilon_0}{2} E_z^2 \tag{6}$$

It will be more convenient to write the integral in terms of θ where $r = a\tan\theta$, $dr = \frac{a}{\cos^2\theta}\, d\theta$ (see diagram on previous page)

$$\vec{F} = \hat{z} \int_0^{\pi/2} d\theta\, \frac{a}{\cos^2\theta}\, 2\pi\, a\tan\theta\, \frac{\epsilon_0}{2} E_z^2 \tag{7}$$

Substituting from Eq. (1):

$$= \hat{z} \int_0^{\pi/2} d\theta\, \frac{a}{\cos^2\theta}\, 2\pi a\tan\theta\, \frac{\epsilon_0}{2} \left[\frac{2q}{4\pi\epsilon_0} \frac{\cos^3\theta}{a^2}\right]^2 \tag{8}$$

$$= \hat{z} \frac{q^2}{4\pi\epsilon_0 a^2} \int_0^{\pi/2} d\theta\, \tan\theta \cos^4\theta \tag{9}$$

$$= \hat{z} \frac{q^2}{4\pi\epsilon_0 a^2} \int_0^{\pi/2} d\theta\, \sin\theta \cos^3\theta \tag{10}$$

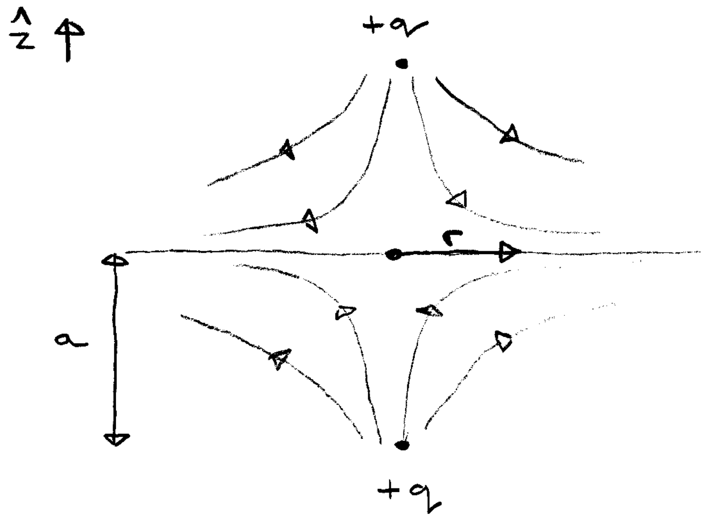
Make substitution $u = \cos\theta$, $d\theta = -\frac{du}{\sin\theta}$

$$= \hat{z} \frac{q^2}{4\pi\epsilon_0 a^2} (-1) \int_1^0 du\, u^3 \tag{11}$$

$$\boxed{\vec{F} = \hat{z}\,\frac{q^2}{4\pi\epsilon_0 a^2}\,\frac{1}{4}} \qquad (12)$$

This is the force that we expect for 2 point charges placed 2a apart. We computed force on the lower charge, which points upwards.

(b) Now let us consider the situation where the 2 charges are of opposite sign.



now $E_z = 0$ in this plane, and E_x and $E_y \neq 0$.

From symmetry we have only a z component to the force. Thus we need only worry about T_{zz}, T_{zx} and T_{zy}.
But since

$$T_{zx} = \epsilon_0 E_z E_x \qquad (13)$$

and $E_z = 0$ in the plane, we have $T_{zx} = 0$
Likewise $T_{zy} = 0$.
For T_{zz}:

$$T_{zz} = \frac{\epsilon_0}{2}\left(\underbrace{E_z^2}_{=0} - E_x^2 - E_y^2\right) \qquad (14)$$

we have $$T_{zz} = -\frac{\epsilon_0}{2}\left(E_x^2 + E_y^2\right) \qquad (15)$$

Although E_x and E_y change as we move about in the plane, the value of $E_x^2 + E_y^2$ remains constant ~~for~~ a given r. (see diagram above)

Specifically,

$$E_x^2 + E_y^2 = \left(2 \frac{q}{4\pi\epsilon_0} \left(\frac{1}{a/\cos\theta} \right)^2 \sin\theta \right)^2 \qquad (16)$$

We can use this to evaluate the integral of the stress tensor.

$$\vec{F} = \oint \overleftrightarrow{T} \cdot d\vec{a}$$

Remembering that the force will only have a z-component (from symmetry) and that $T_{zx} = T_{zy} = 0$ (since $E_z = 0$ in plane), we have (for lower charge)

$$\vec{F} = \hat{z} \oint T_{zz}\, da$$

Same as part (a). See Eq. 7

$$= \hat{z} \int_0^{\pi/2} \overbrace{\frac{a}{\cos^2\theta}\, 2\pi a \tan\theta}\; T_{zz}$$

Using Eq.'(s) (15) and (16):

$$\vec{F} = \hat{z} \int_0^{\pi/2} \frac{a}{\cos^2\theta}\, 2\pi a \tan\theta \left(\frac{-\epsilon_0}{2} \right) \left(\frac{2q}{4\pi\epsilon_0} \frac{\cos^2\theta \sin\theta}{a^2} \right)^2$$

$$= -\hat{z} \frac{q^2}{4\pi\epsilon_0 a^2} \int_0^{\pi/2} d\theta\, \cos\theta \sin^3\theta$$

$u = \sin\theta$

$$= -\hat{z} \frac{q^2}{4\pi\epsilon_0 a^2} \int_0^1 du\, \frac{1}{\cos\theta} \cos\theta\, u^3$$

$$\boxed{\vec{F} = -\hat{z} \frac{q^2}{4\pi\epsilon_0 a^2} \frac{1}{4}}$$

again this agrees with what we would expect for the force on the lower charge, when the charges are 2a apart. It is repelled from the upper charge.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 8.3 — answer

Question. Answer:

Griffiths, Problem 8.3 (3rd/4th ed.):
Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell, with radius R, angular velocity ω, and surface charge density σ. [This is the same as Problem 5.42(3rd ed.)/5.44(4th ed.), but use the Maxwell stress tensor and Eq. 8.22(3rd ed.)/8.21(4th ed.). Answer: $(\pi/4)\mu_0\sigma^2\omega^2R^4$.]

First we determine $\vec{B}$ everywhere (both inside and outside the sphere).

In a simply connected region with no current sources, we can write:

$$\vec{B} = -\nabla U \qquad (1)$$

where U satisfies Laplace's Eqn. We will use U_{in} inside the sphere and U_{out} outside the sphere.

One set of solutions of Laplace's eqn is

$$U(r,\theta) = \sum_{\ell} \Big[\underbrace{A_\ell}_{\text{constants}} r^\ell + \underbrace{B_\ell}_{\text{constants}} \frac{1}{r^{\ell+1}} \Big] \underbrace{P_\ell(\cos\theta)}_{\text{the Legendre Polynomials.}} \qquad (2)$$

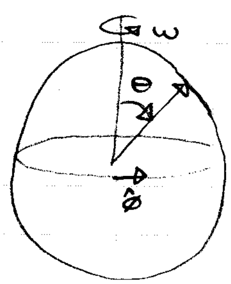
To avoid unphysical $\vec{B}$'s, we constrain $A_\ell = 0$ outside the sphere and $B_\ell = 0$ inside the sphere.

To determine the remaining expansion coefficients we can apply boundary conditions at the surface of the sphere:

$$\vec{B}_{in}\cdot\hat{r} = \vec{B}_{out}\cdot\hat{r} \qquad (\text{from } \nabla\cdot\vec{B} = 0) \qquad (3)$$

and
$$\vec{B}_{out} - \vec{B}_{in} = \mu_0 \underbrace{\vec{K}}_{\text{surface current density}} \times \hat{r} \qquad (\text{from } \nabla\times\vec{B} = \mu_0\vec{J}) \qquad (4)$$

The surface current density in given by:



$$\vec{K} = \hat{\phi}\,\sigma\omega R\sin\theta \qquad (5)$$

Since $\hat{\phi}\times\hat{r} = \hat{\theta}$, the discontinuity in $\vec{B}$ at the surface must be in θ direction.

The appearance of $\sin\theta$ in Eq. (5) suggests that only the $l=1$ terms from Eq. (2) are necessary to match u_{out} and u_{in} at the sphere's surface (we rely on uniqueness once we satisfy all boundary conditions).

Let's use the trial solutions:

$$u_{in} = A r \cos\theta \tag{6}$$

$$u_{out} = B \frac{1}{r^2} \cos\theta \tag{7}$$

First let's apply the boundary condition of Eq. (3):

$$\vec{B}_{in} \cdot \hat{r} \Big|_{r=R} = \hat{r} \cdot -\nabla u_{in} \Big|_{r=R} \tag{8}$$

$$= -A \cos\theta \tag{9}$$

$$\vec{B}_{out} \cdot \hat{r} \Big|_{r=R} = \hat{r} \cdot -\nabla u_{out} \Big|_{r=R} \tag{10}$$

$$= +\frac{2B}{R^3} \cos\theta \tag{11}$$

Equating the RHS' of Eq's (9) and (11), as per Eq. (3), gives:

$$A = -\frac{2B}{R^3} \tag{12}$$

Now let's apply the boundary condition of Eq. (4).

$$\vec{B}_{in} \cdot \hat{\theta} \Big|_{r=R} = \hat{\theta} \cdot -\nabla u_{in} \Big|_{r=R} \tag{13}$$

$$= A \sin\theta \tag{14}$$

$$\vec{B}_{out} \cdot \hat{\theta} \Big|_{r=R} = \hat{\theta} \cdot \nabla u_{out} \Big|_{r=R} \tag{15}$$

$$= \frac{B}{R^3} \sin\theta \tag{16}$$

Substituting the RHS of Eq.'s (14) and (16) into the boundary conditions given by Eq. (4):

$$\frac{B}{R^3}\sin\theta - A\sin\theta = \mu_0 \sigma \omega R \sin\theta \qquad (17)$$

$$\frac{B}{R^3} - A = \mu_0 \sigma \omega R \qquad (18)$$

Using Eq. (12) to write A in terms of B:

$$\frac{B}{R^3} + \frac{2B}{R^3} = \mu_0 \sigma \omega R \qquad (19)$$

$$B = \frac{\mu_0 \sigma \omega R^4}{3} \qquad (20)$$

So, we now have the appropriate magnetic potentials:

$$U_{in} = -\frac{2}{3}\mu_0 \sigma \omega R \; r\cos\theta \qquad (21)$$

$$U_{out} = \frac{1}{3}\mu_0 \sigma \omega R^4 \; \frac{1}{r^2}\cos\theta \qquad (22)$$

From which we can determine $\vec{B}_{in}$ and $\vec{B}_{out}$:

$$\vec{B}_{in} = -\nabla U_{in} \qquad (23)$$

$$= -\nabla\left(-\frac{2}{3}\mu_0 \sigma \omega R \underbrace{r\cos\theta}_{= z}\right) \qquad (24)$$

$$= \frac{2}{3}\mu_0 \sigma \omega R \; \nabla z \qquad (25)$$

$$\vec{B}_{in} = \frac{2}{3}\mu_0 \sigma \omega R \; \hat{z} \qquad (26)$$

$$\vec{B}_{out} = -\nabla U_{out} \qquad (27)$$

$$= -\nabla\left(\frac{1}{3}\mu_0 \sigma \omega R^4 \; \frac{1}{r^2}\cos\theta\right) \qquad (28)$$

$$= -\frac{1}{3}\mu_0 \sigma \omega R^4 \; \nabla\left(\frac{1}{r^2}\cos\theta\right) \qquad (29)$$

(4)

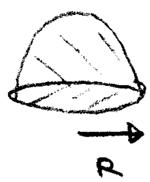
$$\vec{B}_{out} = -\frac{1}{3}\mu_0\sigma\omega R^4\left[-\frac{2}{r^3}\cos\theta\,\hat{r} + \frac{1}{r^3}(-\sin\theta)\,\hat{\theta}\right] \tag{30}$$

$$\boxed{\vec{B}_{out} = \frac{1}{3}\mu_0\sigma\omega R^4\frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]} \tag{31}$$

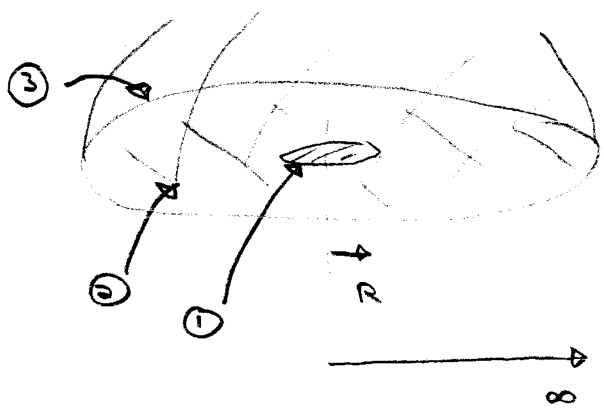
Now that we have the magnetic fields inside and outside the sphere, we can evaluate the total force on the top hemisphere, using the Maxwell Stress tensor:

$$\vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a} \tag{32}$$

One obvious surface to use would be the "half-hemisphere"



This works, but instead let's employ a different surface (discussed in the electrostatic example of Griffiths)



i.e. an infinite hemisphere enclosing the upper hemisphere.

Breaking up the required integral:

$$\vec{F} = \oint \overleftrightarrow{T}\cdot d\vec{a} \tag{33}$$

$$= \underbrace{\int^{①} \overleftrightarrow{T}\cdot d\vec{a}}_{\text{disk of radius } R} + \underbrace{\int^{②} \overleftrightarrow{T}\cdot d\vec{a}}_{\text{infinite disk with annular hole}} + \underbrace{\int^{③} \overleftrightarrow{T}\cdot d\vec{a}}_{\text{curved infinite hemisphere surface.}} \tag{34}$$

The third contribution vanishes because

$$B_{out} \propto \frac{1}{r^3}$$

so $\overleftrightarrow{T}_{out} \propto \frac{1}{r^6}$.

Since surface area $\propto r^2$, $\vec{F}_3$ must vanish as $r \to \infty$.

For surfaces ① and ②, the surface normal $\hat{n} = -\hat{z}$.
Thus we need only determine T_{xz}, T_{yz}, T_{zz}.

If we argue from symmetry that the net force will be parallel to $\hat{z}$, we need only compute T_{zz}.

In general (with no electric fields):

$$T_{ij} = \frac{1}{\mu_0}\left[B_i B_j - \frac{1}{2}B^2 \delta_{ij}\right] \tag{35}$$

So $$T_{zz} = \frac{1}{\mu_0}\left[B_z^2 - \frac{1}{2}B^2\right] \tag{36}$$

On the xy plane, inside the sphere (surface 1):
(from Eq. 26)

$$B_z = \frac{2}{3}\mu_0 \sigma \omega R \tag{37}$$

$$B_x = B_y = 0 \tag{38}$$

So $$T_{zz} = \frac{1}{2\mu_0}\left(\frac{2}{3}\mu_0 \sigma \omega R\right)^2 \tag{39}$$

and thus

$\hat{n}$ points in opposite direction to $\hat{z}$

$$\vec{F}_1 \cdot \hat{z} = -\int_{\substack{\text{surface}\\1}} T_{zz}\, da \tag{40}$$

an integral that even I can do!

$$= \frac{-1}{2\mu_0}\left(\frac{2}{3}\mu_0 \sigma \omega R\right)^2 \pi R^2 \tag{41}$$

On the xy plane, outside the sphere (surface 2), from Eq. (31):

on xy plane $\hat{z} = -\hat{\theta}$

$$B_z = -\frac{1}{3}\mu_0 \sigma \omega R^4 \frac{1}{r^3} \tag{42}$$

$$B_x = B_y = 0 \tag{43}$$

$$\text{So} \quad T_{zz} = \frac{1}{2\mu_0}\left(\frac{1}{3}\mu_0 \sigma \omega R^4\right)^2 \frac{1}{r^6} \tag{44}$$

$$\vec{F}_2 \cdot \hat{z} = -\int_{\substack{\text{surface}\\ 2}} T_{zz}\, da \tag{45}$$

$$= \frac{-1}{2\mu_0}\left(\frac{1}{3}\mu_0 \sigma \omega R^4\right)^2 \int_r^\infty dr\, 2\pi r \frac{1}{r^6} \tag{46}$$

$$= -\frac{2\pi}{2\mu_0}\left(\frac{1}{3}\mu_0 \sigma \omega R^4\right)^2 \left[\frac{1}{-4} \cdot \frac{1}{r^4}\right]_r^\infty \tag{47}$$

$$= -\frac{2\pi}{8\mu_0}\left(\frac{1}{3}\mu_0 \sigma \omega R^4\right)^2 \frac{1}{R^4} \tag{48}$$

Combining results (41) and (48):

$$\vec{F}_{tot} \cdot \hat{z} = -\pi \mu_0 \sigma^2 \omega^2 R^4 \left(\frac{2}{8*9} + \frac{1}{2}\frac{4}{9}\right) \tag{49}$$

$$\left(\frac{2}{8*9} + \frac{16}{8*9}\right)$$

$$\frac{18}{8*9}$$

$$\frac{1}{4}$$

$$\boxed{\text{So} \quad \vec{F}_{tot} = \underbrace{-\hat{z}}_{\substack{\text{top sphere}\\ \text{pulled towards}\\ \text{bottom sphere}}} \frac{\pi \mu_0 \sigma^2 \omega^2 R^4}{4}} \tag{50}$$

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.9 — answer

Question. Answer:

1

Problem 9.9 in third edition / Problem 9.9 in fourth edition
Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is (a) traveling in the negative x direction and polarized in the z direction; (b) traveling in the direction from the origin to the point $(1,1,1)$, with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of $\vec{k}$ and $\hat{n}$.

The general forms for a linearly polarized plane electromagnetic wave are :

$$\vec{E} = E_0 \, \hat{n} \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

$$\vec{B} = \frac{E_0}{c} (\hat{k}\times\hat{n}) \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

(a) For a wave travelling in the negative x direction and polarized in the z direction :

$$\vec{k} = k\,(-1,0,0) \,, \quad \hat{n} = (0,0,1) \,, \quad \hat{k}\times\hat{n} = (0,1,0)$$

and

$$\vec{E} = E_0 \, \hat{z} \cos(-kx - \omega t)$$

$$\vec{B} = \frac{E_0}{c} \, \hat{y} \cos(-kx - \omega t)$$

(b) For a wave travelling in the direction from the origin to the point $(1,1,1)$, with polarization parallel to the xz plane :

$$\vec{k} = k \frac{1}{\sqrt{3}} (1,1,1)$$

The constraints on $\hat{n}$ are : $\hat{n}\cdot(1,1,1) = 0$ (transverse)
$\hat{n}\cdot(0,1,0) = 0$ lies in xz plane

There are 2 possible solutions for $\hat{n} = \pm\frac{1}{\sqrt{2}}(1,0,-1)$

$$\hat{k}\times\hat{n} = \pm \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}} (1,1,1)\times(1,0,-1)$$

$$= \pm \frac{1}{\sqrt{6}} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

(2)

$$\hat{k}\times\hat{n} = \pm\frac{1}{\sqrt{6}}\left[\hat{x}(-1)+\hat{y}(1+1)+\hat{z}(-1)\right]$$

$$= \pm\frac{1}{\sqrt{6}}(-1,2,-1)$$

Summarizing:

$$\vec{k} = k\frac{1}{\sqrt{3}}(1,1,1)\ ,\quad \hat{n} = \pm\frac{1}{\sqrt{2}}(1,0,-1)\ ,\quad \hat{k}\times\hat{n} = \pm\frac{1}{\sqrt{6}}(-1,2,-1)$$

take same sign.

$$\vec{E} = E_0\left(\pm\frac{1}{\sqrt{2}}(\hat{x}-\hat{z})\right)\cos\left(k\frac{1}{\sqrt{3}}(x+y+z)-\omega t\right)$$

$$\vec{B} = \frac{E_0}{c}\left(\pm\frac{1}{\sqrt{6}}(-\hat{x}+2\hat{y}-\hat{z})\right)\cos\left(k\frac{1}{\sqrt{3}}(x+y+z)-\omega t\right)$$

In all of the above $k = \frac{\omega}{c}$.

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.10 — answer

Question. Answer:

Problem 9.10 in third edition / Problem 9.10 in fourth edition
The intensity of sunlight hitting the earth is about 1300 W/m^2. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

From Eq. 9.64 of Griffiths (third edition), the radiation pressure for a perfect absorber is:

$$P = I/c$$

$$= 1300 \, w/m^2 \,/\, 3 * 10^8 \, m/s$$

$$= 1300 \, \frac{N \cdot m}{s \cdot m^2} \, \frac{1}{3 * 10^8 \, m/s}$$

$$= 4.3 * 10^{-6} \, \frac{N}{m^2}$$

For a perfect reflector, the pressure will be twice this value:

$$P = 8.6 * 10^{-6} \, \frac{N}{m^2}$$

Standard sea level atmospheric pressure is: $101325 \, \frac{N}{m^2}$

which is approximately 10^{10} times larger!

Written as a fraction:

$$\frac{P_{reflected\,sunlight}}{P_{1atm}} = \frac{8.6 * 10^{-6} \, N/m^2}{101325 \, N/m^2}$$

$$\simeq 10^{-10}$$

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.12 — answer

Question. Answer:

Problem 9.11 in third edition / Problem 9.12 in fourth edition
In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r},t) = A\cos(\mathbf{k}\cdot\mathbf{r}+\delta_a)$ and $g(\mathbf{r},t) = B\cos(\mathbf{k}\cdot\mathbf{r}+\delta_b)$. Show that $\langle fg\rangle = (1/2)\Re(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same $\mathbf{k}$ and ω, but they need not have the same amplitude or phase.] For example

$$\langle u\rangle = \frac{1}{4}\Re(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*)$$

and

$$\langle \mathbf{S}\rangle = \frac{1}{2\mu_0}\Re(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*).$$

First let's calculate the time average using the explicit time dependent forms of f and g.

$$\langle f(\vec{r},t)\, g(\vec{r},t)\rangle_T = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\; A\cos(\vec{k}\cdot\vec{r}-\omega t+\delta_a)\, B\cos(\vec{k}\cdot\vec{r}-\omega t+\delta_b)$$

Using $\cos(\alpha)\cos(\beta) = \dfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$ (for all α and β):

$$\langle f(\vec{r},t)\, g(\vec{r},t)\rangle_T = \lim_{T\to\infty}\frac{AB}{T}\int_0^T dt\left[\frac{\cos(2(\vec{k}\cdot\vec{r}-\omega t)+\delta_a+\delta_b)+\cos(\delta_a-\delta_b)}{2}\right]$$

The first term will average to zero, and the 2nd term is time-independent, so

$$\langle f(\vec{r},t)\, g(\vec{r},t)\rangle_T = \frac{AB}{2}\cos(\delta_a-\delta_b) \qquad (\ddagger)$$

Now let's look at proposed phasor formula.
The phasor representations of f and g are:

$$\tilde{f} = Ae^{i\delta_a}, \quad \tilde{g} = Be^{i\delta_b}$$

So
$$\frac{1}{2}\mathrm{Re}(\tilde{f}\tilde{g}^*) = \frac{AB}{2}\mathrm{Re}(e^{i\delta_a}e^{-i\delta_b})$$
$$= \frac{AB}{2}\mathrm{Re}(e^{i(\delta_a-\delta_b)})$$
$$= \frac{AB}{2}\mathrm{Re}\left[\cos(\delta_a-\delta_b)+i\sin(\delta_a-\delta_b)\right]$$

$$\frac{1}{2} \mathrm{Re} (\tilde{f} \tilde{g}^*) = \frac{AB}{2} \cos (\delta_a - \delta_b)$$

We see that this agrees with (#) above ; so

$$\langle f(\vec{r},t) g(\vec{r},t) \rangle_T = \frac{1}{2} \mathrm{Re} (\tilde{f} \tilde{g}^*) \quad \text{as req'd.}$$

Griffiths, *Introduction to Electrodynamics*, 4th ed., Problem 9.35 — answer

Question. Answer:

(1)

Problem 9.33 in third edition / Problem 9.35 in fourth edition
Suppose

$$\mathbf{E}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}[\cos(kr-\omega t) - (1/kr)\sin(kr-\omega t)]\hat{\phi}$$

with $\omega/k = c$. (This is, incidentally, the simplest possible spherical wave. For notational convenience, let $(kr - \omega t) \equiv u$ in your calculations.)

(a) Show that $\mathbf{E}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\mathbf{S}$ over a full cycle to get the intensity vector $\mathbf{I}$. (Does it point in the expected direction? Does it fall off like r^{-2}, as it should?)

(c) Integrate $\mathbf{I}\cdot\mathbf{a}$ over a spherical surface to determine the total power radiated. [Answer: $4\pi A^2/3\mu_0 c$]

Use the abbreviations $S_u \to \sin(u)$, $C_u \to \cos(u)$

We will use the expressions for $\nabla\cdot$ and $\nabla\times$ in spherical coordinates given on the inside cover of Griffiths.

(a) $$\nabla\cdot\vec{E} = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)}_{=0} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \underbrace{E_\theta}_{=0}) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}}_{=0}$$

$= 0$ as req'd in free space.

We can use $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ to determine $\vec{B}$.

$$\nabla\times\vec{E} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi) - \underbrace{\frac{\partial E_\theta}{\partial\phi}}_{=0}\right]\hat{r} + \frac{1}{r}\left[\underbrace{\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}}_{=0} - \frac{\partial}{\partial r}(rE_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\underbrace{\frac{\partial}{\partial r}(rE_\theta)}_{=0} - \underbrace{\frac{\partial E_r}{\partial\theta}}_{=0}\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{A\sin\theta}{r}\left[C_u - \frac{1}{kr}S_u\right]\right)\right]\hat{r}$$

$$- \frac{1}{r}\frac{\partial}{\partial r}\left(A\sin\theta\left[C_u - \frac{1}{kr}S_u\right]\right)\hat{\theta}$$

②

$$\nabla\times\vec{E} = \frac{1}{r\sin\theta}\, 2\sin\theta\cos\theta\, \frac{A}{r}\left[Cu - \frac{1}{kr} Su\right]\hat{r}$$

$$- \frac{1}{r} A\sin\theta \left[-Su\,k + \frac{1}{kr^2} Su - \frac{1}{kr} Cu\,k\right]\hat{\theta}$$

$$= \frac{2A\cos\theta}{r^2}\left[Cu - \frac{1}{kr} Su\right]\hat{r} - \frac{A\sin\theta}{r} k \left[-Su + \frac{Su}{(kr)^2} - \frac{1}{kr} Cu\right]\hat{\theta}$$

Now use $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ to determine $\vec{B}$:

$$\vec{B} = -\frac{2A\cos\theta}{r^2}\left[\frac{Su}{-\omega} - \frac{1}{kr}\frac{(-Cu)}{-\omega}\right]\hat{r}$$

$$+ \frac{A\sin\theta\, k}{r}\left[\frac{Cu}{-\omega} + \frac{-Cu}{(kr)^2}\frac{1}{-\omega} - \frac{1}{kr}\frac{Su}{-\omega}\right]\hat{\theta}$$

$$\boxed{\vec{B} = \frac{2A\cos\theta}{r^2\omega}\left[Su + \frac{Cu}{(kr)}\right]\hat{r} + \frac{A\sin\theta}{r}\frac{k}{\omega}\left[-Cu + \frac{Cu}{(kr)^2} + \frac{Su}{(kr)}\right]\hat{\theta}}$$

Now we should check $\nabla\cdot\vec{B} = 0$.

$$\nabla\cdot\vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, B_\theta) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}}_{=0}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{2A\cos\theta}{\omega}\left[Su + \frac{Cu}{(kr)}\right]\right)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{A\sin^2\theta}{r}\frac{k}{\omega}\left[-Cu + \frac{Cu}{(kr)^2} + \frac{Su}{(kr)}\right]\right)$$

$$= \frac{1}{r^2}\frac{2A\cos\theta}{\omega}\left[Cu\,k - \frac{1}{kr^2}Cu - \frac{Su}{kr}k\right]$$

$$+ \frac{2A\cos\theta}{r^2}\frac{k}{\omega}\left[-Cu + \frac{Cu}{(kr)^2} + \frac{Su}{kr}\right]$$

$$\nabla\cdot\vec{B} = \frac{2A\cos\theta}{r^2}\,\frac{k}{\omega}\left[Cu - \frac{1}{(kr)^2}Cu - \frac{Su}{(kr)}\right]$$

$$+\,\frac{2A\cos\theta}{r^2}\,\frac{k}{\omega}\left[-Cu + \frac{Cu}{(kr)^2} + \frac{Su}{(kr)}\right]$$

$$= 0 \quad \text{as req'd.} \;//$$

Now let's check $\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times\vec{B} = \frac{1}{r\sin\theta}\left[\underbrace{\frac{\partial}{\partial\theta}(\sin\theta B_\phi)}_{=0} - \underbrace{\frac{\partial B_\theta}{\partial\phi}}_{=0}\right]\hat{r}$$

$$+\,\frac{1}{r}\left[\frac{1}{\sin\theta}\underbrace{\frac{\partial B_r}{\partial\phi}}_{=0} - \underbrace{\frac{\partial}{\partial r}(rB_\phi)}_{=0}\right]\hat{\theta}$$

$$+\,\frac{1}{r}\left[\underbrace{\frac{\partial}{\partial r}(rB_\theta)}_{①} - \underbrace{\frac{\partial B_r}{\partial\theta}}_{②}\right]\hat{\phi} \qquad (\neq)$$

$$① = \frac{\partial}{\partial r}\left(A\sin\theta\,\frac{k}{\omega}\left[-Cu + \frac{Cu}{(kr)^2} + \frac{Su}{kr}\right]\right)$$

$$= A\sin\theta\,\frac{k}{\omega}\left[Su\,k + \frac{-2Cu}{k^2r^3} + \frac{-Su}{(kr)^2}k - \frac{Su}{kr^2} + \frac{Cu}{kr}k\right]$$

$$= A\sin\theta\,\frac{k^2}{\omega}\left[Su - \frac{2Cu}{(kr)^3} - \frac{2Su}{(kr)^2} + \frac{Cu}{(kr)}\right]$$

$$② = -\frac{2A\sin\theta}{r^2\omega}\left[Su + \frac{Cu}{kr}\right]$$

Now combine ① and ② into $(\neq)$:

$$\nabla\times\vec{B} = \frac{1}{r}A\sin\theta\,\frac{k^2}{\omega}\left[Su - \cancel{\frac{2Cu}{(kr)^3}} - \cancel{\frac{2Su}{(kr)^2}} + \frac{Cu}{kr}\right.$$

$$\left. + \cancel{\frac{2Su}{(kr)^2}} + \cancel{\frac{2Cu}{(kr)^3}}\right]$$

④

$$\nabla \times \vec{B} = \frac{A\sin\theta}{r} \frac{k^2}{\omega} \left[Su + \frac{Cu}{kr} \right] \qquad (\times)$$

What is $c^2 \frac{\partial \vec{E}}{\partial t}$?

$$c^2 \frac{\partial \vec{E}}{\partial t} = c^2 \frac{A\sin\theta}{r} \left[-Su(-\omega) - \frac{1}{kr} Cu(-\omega) \right]$$

$$= c^2 \frac{A\sin\theta}{r} \omega \left[Su + \frac{1}{kr} Cu \right]$$

Or, recognizing that $c^2 = (\omega/k)^2$

$$c^2 \frac{\partial \vec{E}}{\partial t} = \frac{A\sin\theta}{r} \frac{k^2}{\omega} \left[Su + \frac{1}{kr} Cu \right]$$

Comparing this with the expression for $\nabla \times \vec{B}$ given above $(\times)$ we see that $\nabla \times \vec{B} = c^2 \frac{\partial \vec{E}}{\partial t}$

is satisfied.

We have shown that all 4 of Maxwell's equations are satisfied by this solution.

(b)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$= \frac{1}{\mu_0} \frac{A\sin\theta}{r} \left[Cu - \frac{1}{kr} Su \right] \frac{2A\cos\theta}{r^2 \omega} \left[Su + \frac{Cu}{(kr)} \right] \hat{\phi} \times \hat{r}$$

$$+ \frac{1}{\mu_0} \frac{A\sin\theta}{r} \left[Cu - \frac{1}{kr} Su \right] \frac{A\sin\theta}{r} \frac{k}{\omega} \left[-Cu + \frac{Cu}{(kr)^2} + \frac{Su}{(kr)} \right] \hat{\phi} \times \hat{\theta}$$

$$\vec{S} = \frac{A^2 \sin\theta}{\mu_0 r} \left\{ \frac{2\cos\theta}{r^2 \omega} \left[C_u S_u + \frac{(C_u)^2}{kr} - \frac{1}{kr}(S_u)^2 - \frac{1}{(kr)^2} S_u C_u \right] \hat{\theta} \right.$$

$$+ \frac{\sin\theta}{r} \frac{k}{\omega} \left[-(C_u)^2 + \frac{(C_u)^2}{(kr)^2} + \frac{C_u S_u}{kr} + \frac{1}{kr} C_u S_u \right.$$

$$\left.\left. - \frac{S_u C_u}{(kr)^3} - \frac{(S_u)^2}{(kr)^2} \right] (-\hat{r}) \right\}$$

$$\vec{S} = \frac{A^2 \sin\theta}{\mu_0 r^2 \omega} \left\{ \frac{2\cos\theta}{r} \left[C_u S_u + \frac{1}{kr}(C_u^2 - S_u^2) - \frac{1}{(kr)^2} S_u C_u \right] \right.$$

$$+ \sin\theta\, k \left[+ C_u^2 + \frac{1}{(kr)^2}(S_u^2 - C_u^2) \right.$$

$$\left.\left. - \frac{2 C_u S_u}{kr} + \frac{S_u C_u}{(kr)^3} \right] \hat{r} \right\}$$

The time averages of C_u^2 and S_u^2 are $1/2$, and the time average of $S_u C_u$ is zero.

$$\vec{I} = \langle \vec{S} \rangle = \frac{A^2 \sin\theta}{\mu_0 r^2 \omega} \sin\theta\, k \left[\frac{1}{2} \right] \hat{r}$$

$$\boxed{\vec{I} = \frac{A^2 \sin^2\theta}{2 \mu_0 r^2 c} \hat{r}}$$

points outwards from origin, which makes sense. Also falls off like $1/r^2$ which we expect from energy conservation.

(c) $$\vec{P} = \int \vec{I} \cdot d\vec{a}$$

$$= r^2 \, 2\pi \int_0^{\pi} d\theta \sin\theta \; \vec{I} \cdot \hat{r}$$

$$= \frac{A^2 \pi}{\mu_0 c} \int_0^{\pi} d\theta \sin^3\theta$$

(6)

$$\boxed{\vec{P} = \frac{A^2\pi}{\mu_0 c}\frac{4}{3}}$$

Whew!

References

[1] D. J. Griffiths, *Introduction to electrodynamics*, Fourth edition (Cambridge University Press, Cambridge, United Kingdom ; New York, NY, 2018).

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[3] W. S. C. Williams, *Introducing special relativity* (Taylor & Francis, London, 2002).