Notes for the Radiation Module of Electricity and Magnetism 3
Phys 442, University of Waterloo

J. D. D. Martin

`last commit hash: ee4ffe7        last commit date: 2020-12-07, 15:43 EST`

These notes are intended to complement the course textbook: Griffiths' *Introduction to Electrodynamics, 4th ed* (GITE4).[1]

I will be grateful for your comments and/or corrections. For the most up-to-date version of these notes see:

https://www.dropbox.com/s/gbw8i1s8ugoppey/phys442_radiation_notes_generated.pdf?dl=0

(If you are viewing this file through the dropbox link, you may download it using the "..." icon on right-hand-side of screen. I plan to update these notes frequently.)

Contents

[1] Consistent with fair dealing guidelines, there is currently a small amount of copyrighted material used (and attributed) within these notes. For that reason they are not currently intended for distribution beyond the University of Waterloo.

1 Introduction

We will study four topics:

1. the Hertzian dipole. This is the simplest radiating system and of great historical importance, as it was used by Hertz to demonstrate that electromagnetic waves could be generated electrically.
2. Purcell's treatment of radiation of a point charge. It is the *acceleration* of point charges that give rise to radiation. There is a simple but intuitive picture that allows us to see this.
3. General treatment of radiation due to a point charge. Having general expressions for the fields due to arbitrary trajectories of a point charge, we may readily compute radiation in the general case.
4. Antennas. We will briefly study devices specially constructed to generate and receive electromagnetic radiation. We will focus on the so-called "dipole antenna".

In the first few following sections I tabulate pre-requisite material for the study of radiation, primarily from Chapter 10 of GITE4.

2 Background

2.1 Solving Maxwell's equations with potentials

Recall that the scalar and vector potentials are an alternate way to represent the **E** and **B** fields. There is some arbitrariness in the potential representation though. In the treatment of radiation that follows, we will exclusively use the Lorenz gauge:

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}. \tag{1}$$

When using this choice of gauge, Maxwell's equations take a nice, symmetrical form:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) V = -\frac{1}{\epsilon_0}\rho \tag{2}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \mathbf{A} = -\mu_0 \mathbf{J} \tag{3}$$

To discuss the explicit solutions for these fields, we adopt Griffiths notation, which uses $d\tau'$ as the infinitesimal volume element, as a "source" coordinate $\mathbf{r}'$ is integrated over. And for convenient compactness, we also use Griffiths' definition $\boldsymbol{\mathscr{r}} := \mathbf{r} - \mathbf{r}'$, where $\mathbf{r}$ is typically the observation point; i.e., the point at which we are observing the potentials/fields.
Given arbitrary source fields we have the solutions:

$$\begin{aligned} V(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\mathbf{r}', t_r\right)}{\mathscr{r}} d\tau' \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\left(\mathbf{r}', t_r\right)}{\mathscr{r}} d\tau' \end{aligned} \tag{GITE4 Eq. 10.26}$$

where the **retarded time** is defined as:

$$t_r \coloneqq t - \frac{\mathscr{r}}{c} \tag{GITE4 Eq. 10.25}$$

2.2 Liénard-Wiechert potentials

Given the general expressions for the potentials due to arbitrary time-dependent charge and current densities, we can specialize these results to the case of a point charge with a position vector varying in time:

$$\begin{aligned} V(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} && \text{(GITE4 Eq. 10.46)} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} \\ &= \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t) && \text{(GITE4 Eq. 10.47)} \end{aligned}$$

The quantities $\mathscr{r}$, $\boldsymbol{\mathscr{r}}$, $\mathbf{v}$ must be evaluated at the retarded time. Specifically, given the trajectory of a point particle $\mathbf{w}(t)$, to determine the potentials at a specific position $\mathbf{r}$ and time t we must solve the implicit equation:

$$t_r = t - \frac{|\mathbf{r} - \mathbf{w}(t_r)|}{c} \tag{4}$$

for t_r, and then take $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$ and $\mathbf{v} = \dot{\mathbf{w}}(t_r)$.

2.3 Fields of a point charge

As Heald and Marion [1] put it, "after some heroic algebra" the Liénard-Wiechert potentials may be used to determine the fields due to a point charge:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \left[\left(c^2 - v^2\right) \mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a}) \right] \tag{GITE4 Eq. 10.72}$$

and

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t) \tag{GITE4 Eq. 10.73}$$

where

$$\mathbf{u} \coloneqq c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}. \tag{GITE4 Eq. 10.71}$$

Again, as with the potentials it is important to realize that all of these quantities are to be evaluated at the retarded time; i.e., to determine the fields for a given point of interest, specified by $\mathbf{r}$ and t, we must first determine t_r using Eq. 4, and use it to evaluate $\boldsymbol{\mathscr{r}}$, $\mathbf{v}$ and all quantities derived from them.

3 Hertzian dipole

For now this section is just a summary of the main results.
Assuming that the wavelength is small compared to the size of the dipole, we obtain:

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega(t-r/c)\right]\hat{\mathbf{z}} \qquad \text{(GITE4 Eq. 11.17)}$$

The Lorenz gauge condition may then be integrated to determine:

$$V(r,\theta,t) = \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)] + \frac{1}{r}\cos[\omega(t-r/c)]\right\} \qquad \text{(GITE4 Eq. 11.12)}$$

When determining the $\mathbf{E}$ and $\mathbf{B}$ fields we only retain terms that give oscillating amplitudes that scale like $1/r$, ignoring $1/r^2$ and quicker fall-offs, as these do not carry energy off to infinity. To indicate this we write $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ where "rad" is for radiation:

$$\mathbf{E}_{\text{rad}} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\theta} \qquad \text{(GITE4 Eq. 11.18)}$$

$$\mathbf{B}_{\text{rad}} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\phi} \qquad \text{(GITE4 Eq. 11.19)}$$

These fields may be used to compute the time-averaged power radiated off to infinity per unit solid angle:

$$\frac{dP}{d\Omega} = r^2\left\langle \mathbf{S}\cdot\hat{r}\right\rangle_T \qquad (5)$$

where $\langle\ldots\rangle$ indicates a time average (here over a single cycle $T = 2\pi/\omega$).

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0 c}\left(\frac{\mu_0 dq\omega^2}{4\pi}\right)^2\sin^2\theta \qquad (6)$$

Integrating over all solid angles gives:

$$P = \frac{\mu_0 p_0^2\omega^4}{12\pi c}. \qquad \text{(GITE Eq. 11.22)}$$

4 Intuitive treatment of point charge radiation

The argument presented here is originally due to J. J. Thomson[2] and was popularized by Purcell's textbook.
The only thing that I add to Purcell's treatment (in the videos) is an argument for the radiation magnetic field.
The important final result is:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \qquad \text{(GITE4 Eq. 11.70)}$$

which is known as **Larmor's formula**.

[2]Thomson's presentation may be found here, but it is mainly of historical interest.

References

[1] M. A. Heald and J. B. Marion, *Classical electromagnetic radiation*, Third edition (Dover Publications, Inc, Mineola, New York, 2012).