Notes for Electromagnetic Theory, Phys 706, Winter 2022, University of Waterloo

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I update these notes frequently. For the most recent version, please see: this link.

The symbols in the section titles are clickable links to video lectures.

Please let me know of any errors — I can easily correct and repost these notes.

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1

Phys 706, Electromagnetic Theory
Winter 2022

$\vec{F} = m\vec{a}$ ← in both systems

N kg m/s^2

lbs slugs ft/s^2

"unit systems" ←

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ SI system ← this course, Zangwill, Griffiths

$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c}\times\vec{B})$ ← Gaussian. Jackson mixture. (G)

$\vec{E}$ and $\vec{B}$ in the same units.

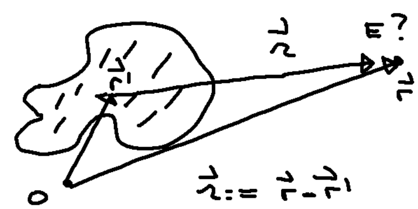
Also $\nabla\cdot\vec{E} = 4\pi\rho$ (Gaussian), $\nabla\cdot\vec{E} = \rho/\epsilon_0$ (SI)

2

Maxwell's equations in the SI system

Coulomb's law

$$\vec{E}(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0}\int \underbrace{dV'}_{d\tau'\ \text{Griffiths}} \frac{\overbrace{\rho(\vec{r}\,')}^{\text{charge density } C/m^3}}{\mathscr{r}^2}\hat{\mathscr{r}}}$$



Compute $\nabla\cdot$ and $\nabla\times$

$\nabla\cdot\vec{E} = \rho/\epsilon_0$, $\nabla\times\vec{E} = \vec{0}$ } electrostatics

Aside: point charge at origin $\rho(\vec{r}\,') = q\underbrace{\delta^3(\vec{r})}_{\delta(x)\delta(y)\delta(z)}$

③

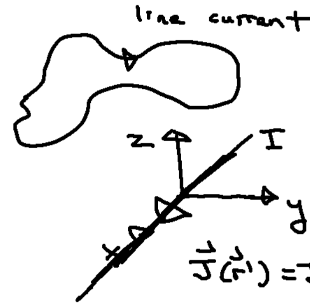
<u>Biot - Savart</u>

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int dV' \, \frac{\overbrace{\vec{J}(\vec{r}')}^{\substack{\text{volume current density} \\ A/m^2}} \times \hat{\imath}}{\imath^2}$$

$\downarrow$

gives :

$$\underbrace{\nabla \cdot \vec{B} = 0 \, , \quad \nabla \times \vec{B} = \mu_0 \vec{J}}_{\text{magnetostatics.}}$$



$$\vec{J}(\vec{r}') = I \, \delta(x) \delta(y)$$

④

Faraday discovered :

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

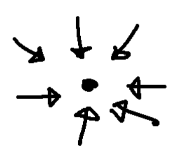
Maxwell introduced the displacement current.

For any vector field $\vec{v}$: $\quad \nabla \cdot (\nabla \times \vec{v}) = 0$

So by $\quad \nabla \times \vec{B} = \mu_0 \vec{J}$

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J}$$

$$0 = \nabla \cdot \vec{J} \quad \longleftarrow \text{unphysical constraint.}$$



$\nabla \cdot \vec{J} \neq 0$
at point.

5

What should the constraint on $\nabla\cdot\vec{J}$ be?

$$\underbrace{\oint}_{\text{over surface}} \vec{J}\cdot d\vec{a} = -\frac{dQ}{dt}$$

Q — total charge enclosed in volume.

$$\underbrace{= -\frac{d}{dt}\int dV\,\underbrace{\rho}}_{\text{always true if charge is conserved.}} \qquad *$$

For a general vector field $\vec{f}$,

$$\oint \vec{f}\cdot d\vec{a} = \int dV\ \nabla\cdot\vec{f} \quad \Big\}\ \text{divergence theorem.}$$

$*$ will always be satisfied if $\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ $(*)$

6

Maxwell took $\nabla\times\vec{B} = \mu_0\vec{J}$ and added the displacement current term,

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\cdot(\nabla\times\vec{B}) = \mu_0\nabla\cdot\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\underbrace{(\nabla\cdot\vec{E})}_{\rho/\epsilon_0}$$

$$0 = \nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t}$$

reproduces the charge continuity eqn!

$$\boxed{\begin{aligned}&\nabla\cdot\vec{E} = \rho/\epsilon_0, \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\\ &\nabla\cdot\vec{B} = 0, \quad \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\end{aligned}}$$

+ Lorentz force law.

3 2022-01-13 Lecture

Suggested reading: Appendix on Units and Dimensions in J. D. Jackson, *Classical electrodynamics*, 3rd ed (Wiley, New York, 1999).

1

Alternative unit systems in electromagnetism

- we will use the SI system (Zangwill, Griffiths)
- today look at
  1) Gaussian (Jackson, astrophysics)
  2) Heaviside–Lorentz (Peskin+ Schroeder, QFT)

- $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

Make a "change of variables"

SI new-system

$\vec{B} = \vec{B}'/\alpha$

$\vec{E} = \vec{E}'$

$q = q'$

$\rho = \rho'$ $\vec{J} = \vec{J}'$

$$\vec{F} = q'\left(\vec{E}' + \frac{\vec{v}}{\alpha}\times\vec{B}'\right)$$

2

$$\nabla\cdot\vec{E}' = \rho'/\epsilon_0, \quad \nabla\times\vec{E}' = -\frac{1}{\alpha}\frac{\partial B'}{\partial t}$$

$$\underbrace{\nabla\cdot\vec{B}' = 0, \quad \nabla\times\frac{\vec{B}'}{\alpha} = \mu_0\vec{J}' + \mu_0\epsilon_0\frac{\partial\vec{E}'}{\partial t}} \quad *$$

Now apply a 2^{nd} transformation. $F = \frac{q^2}{r^2}$

Goal: allow possibility of a simple form of Coulomb's law.

Rescale $\vec{E} = \vec{E}' = \frac{\vec{E}''}{\beta}$

If we want the Lorentz force law to remain unchanged:

electric part: $\vec{F} = q'\vec{E}' = q''\cancel{\beta}\frac{\vec{E}''}{\cancel{\beta}}$

$\nabla\cdot\vec{J} = -\partial\rho/\partial t$

$$\underbrace{\vec{E}' = \frac{\vec{E}''}{\beta}, \quad q' = q''\beta, \quad \rho' = \rho''\beta \qquad \vec{J}' = \vec{J}''\beta, \quad \vec{J}'' := \vec{J}'/\beta}$$

to preserve

③

Analogous to "preserving" the electric part of the Lorentz force law, I'd also like to keep magnetic part the same

$$\vec{F} = q' \frac{\vec{v}}{\alpha} \times \vec{B}'$$

$$= q''\beta \frac{\vec{v}}{\alpha} \times \frac{\vec{B}''}{\beta}$$

So $\vec{B}' = \vec{B}''/\beta$

What do the Lorentz force law and Maxwell's eqns look like after this 2nd change of variables.

$$\boxed{\vec{F} = q''\left(\vec{E}'' + \frac{\vec{v}}{\alpha} \times \vec{B}''\right)}$$

④

$$\nabla \cdot \vec{E}' = \rho'/\epsilon_0 \rightarrow \nabla \cdot \frac{\vec{E}''}{\beta} = \rho''\beta/\epsilon_0$$

$$\boxed{\nabla \cdot \vec{E}'' = \rho''\beta^2/\epsilon_0}$$

$$\nabla \times \vec{E}' = -\frac{1}{\alpha}\frac{\partial \vec{B}'}{\partial t} \rightarrow \boxed{\nabla \times \vec{E}'' = -\frac{1}{\alpha}\frac{\partial \vec{B}''}{\partial t}}$$

$$\nabla \cdot \vec{B}' = 0 \rightarrow \boxed{\nabla \cdot \vec{B}'' = 0}$$

$$\nabla \times \frac{\vec{B}'}{\alpha} = \mu_0 \vec{J}' + \mu_0\epsilon_0 \frac{\partial \vec{E}'}{\partial t} \rightarrow \nabla \times \frac{\vec{B}''}{\alpha\beta} = \mu_0 \beta \vec{J}'' + \mu_0\epsilon_0 \frac{1}{\beta}\frac{\partial \vec{E}''}{\partial t}$$

$$\boxed{\nabla \times \vec{B}'' = \mu_0 \alpha \beta^2 \vec{J}'' + \mu_0\epsilon_0\alpha \frac{\partial \vec{E}''}{\partial t}}$$

By choosing α and β I can "make" new unit systems.

(5)

For Gaussian system: $\alpha = c$, $\beta = \sqrt{4\pi\epsilon_0}$

$$\vec{F} = q''\left(\vec{E}'' + \frac{\vec{v}}{c}\times\vec{B}''\right)$$

$$\rightarrow \boxed{\begin{aligned} \nabla\cdot\vec{E}'' &= 4\pi\rho'' \\ \nabla\times\vec{E}'' &= -\frac{1}{c}\frac{\partial\vec{B}''}{\partial t} \\ \nabla\cdot\vec{B}'' &= 0 \end{aligned}} \rightarrow \quad F = \frac{q^2}{r^2}$$

$$\nabla\times\vec{B}'' = \mu_0\alpha 4\pi\epsilon_0\vec{J}'' + \mu_0\epsilon_0 c\frac{\partial\vec{E}''}{\partial t}$$

$$(\mu_0\epsilon_0) = \frac{1}{c^2}$$

$$\boxed{\nabla\times\vec{B}'' = \frac{4\pi\vec{J}''}{c} + \frac{1}{c}\frac{\partial\vec{E}''}{\partial t}}$$

(6)

Heaviside-Lorentz: $\alpha = c$, $\beta = \sqrt{\epsilon_0}$

$$\vec{F} = q''\left(\vec{E}'' + \frac{\vec{v}}{c}\times\vec{B}''\right)$$

$$\boxed{\begin{aligned} \nabla\cdot\vec{E}'' &= \rho'' \\ \nabla\times\vec{E}'' &= -\frac{1}{c}\frac{\partial\vec{B}''}{\partial t} \\ \nabla\cdot\vec{B}'' &= 0 \\ \nabla\times\vec{B}'' &= \frac{\vec{J}''}{c} + \frac{1}{c}\frac{\partial\vec{E}''}{\partial t} \end{aligned}} \rightarrow \quad F = \frac{q^2}{4\pi r^2}$$

(7)

Example: Larmor's formula for power radiated by accelerating charge (non-relativistic)

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} a^2 \qquad \boxed{SI}$$

a acceleration.

What is form in Gaussian units?

$q = \beta q''$ with $\beta = \sqrt{4\pi\epsilon_0}$

$$P = \frac{2 q''^2 a^2}{3c^3} \qquad \boxed{\text{Gaussian}}$$

(8)

Maxwell's equations in matter

$$\left.\begin{aligned} &\nabla\cdot\vec{D} = \rho_f \ , \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ &\nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \ , \quad \nabla\cdot\vec{B} = 0 \end{aligned}\right\}$$

$$\left\{\begin{aligned} &\vec{B} = \mu_0(\vec{H} + \vec{M}) \\ &\vec{H} := \vec{B}/\mu_0 - \vec{M} \\ &\vec{D} := \epsilon_0\vec{E} + \vec{P} \end{aligned}\right.$$

$\vec{M}$ magnetization /unit volume

$\vec{P}$ electric dipole moment /unit volume.

4 2022-01-18 Lecture

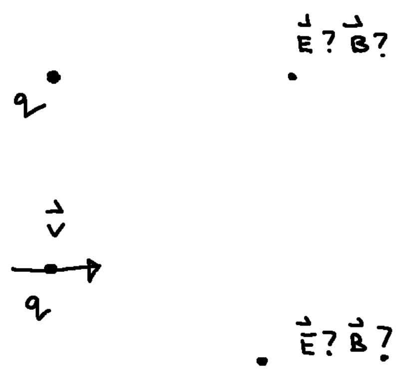
Suggested reading: Sections 22.3, 22.4, 22.5 of A. Zangwill, *Modern electrodynamics* (Cambridge University Press, Cambridge, 2013). But we will not be using the *ict* convention that Zangwill uses in his main text — that is an eccentric choice in an otherwise good book. He explains the system that we will use in his Appendix D — with the “East Coast metric”.

If you need a more thorough review of special relativity, I highly recommend Chapter 12 of D. J. Griffiths, *Introduction to electrodynamics*, Fourth edition (Cambridge University Press, Cambridge, United Kingdom ; New York, NY, 2018).

See also Phys 442 notes.

1

Special Relativity first !!!

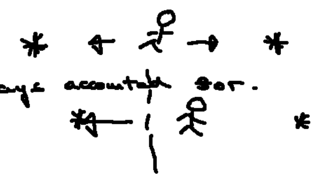


(2)

Special Relativity – Basic ideas

0) inertial reference frame.

1) relativity of simultaneity.
events : points in spacetime
observer / observation means propagation delays accounted for.
Time ordering of events may depend on inertial ref frame.



2) length contraction : in a frame in which a body is moving, its length along its direction of motion is
* observed to be contracted relative to measurements made in its rest frame.

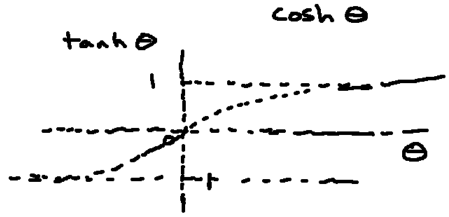
$$\ell = \ell_R \sqrt{1-(v/c)^2} \qquad v/c = \beta \,,\ \gamma = \frac{1}{\sqrt{1-\beta^2}}$$
$$= \ell_R / \gamma$$

(3)

3) In many cases I'll prefer rapidities

$$\tanh\theta = v/c$$

$$\tanh\theta := \frac{\sinh\theta}{\cosh\theta} \qquad \sinh\theta := \frac{e^{\theta}-e^{-\theta}}{2} \,,\ \cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$



$$\ell = \ell_R \sqrt{1-\tanh^2\theta}$$
$$= \ell_R \sqrt{\frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta}}$$
$$\boxed{\ell = \ell_R / \cosh\theta}$$

(4)

Advantage of rapidities: relativistic velocity addition is much simpler using rapidities. e.g.

Three inertial reference frames A, B, and C.

Relative motion all along same axis (x).

$$\underbrace{V_{AC,x}}_{\text{A wrt C}} = \frac{V_{AB,x} + V_{BC,x}}{1 + \frac{V_{AB,x} V_{BC,x}}{c^2}}$$

$\underline{\underline{\text{But}}}$: a "boost" along one axis, followed by a boost along a second <u>non-colinear</u> axis is not equivalent to a single boost.

With rapidities:

$$\Theta_{AC,x} = \Theta_{AB,x} + \Theta_{BC,x}$$

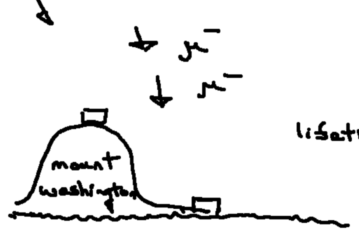
$$\left|\frac{V_{AB,x}}{c}\right| < 1 \ \wedge\ \left|\frac{V_{BC,x}}{c}\right| < 1 \quad \Rightarrow \quad \left|\frac{V_{AC,x}}{c}\right| < 1$$

(5)

4) time-dilation - "moving clocks run slow"

* $$\Delta t_R = \underbrace{\frac{\Delta t}{\cosh\Theta}}$$

time elapsed in rest frame of clock.



lifetime of muon $\simeq 2\mu s$.

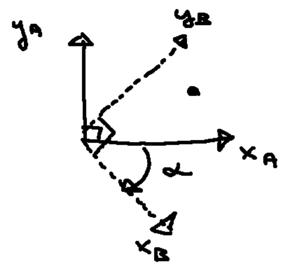
5) Lorentz transformations. frames A and B

$$\begin{bmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{bmatrix} = \begin{bmatrix} \overset{=\gamma}{\cosh\Theta} & \overset{=-\gamma\beta}{-\sinh\Theta} & & \\ -\sinh\Theta & \cosh\Theta & & \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{bmatrix}$$

where $\underset{\text{B wrt A}}{\vec{V}_{BA}} = \underbrace{\hat{x}\, c \tanh\Theta}$ "standard configuration"

⑥

Note the similarity with rotations in 2d.

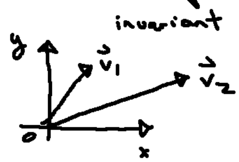


$$\begin{bmatrix} x_B \\ y_B \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{=: R(\alpha)} \begin{bmatrix} x_A \\ y_A \end{bmatrix}$$

It is obvious that the "distance" from origin is invariant under rotation of axis.

$$x_A^2 + y_A^2 = x_B^2 + y_B^2 \qquad (*)$$

Stronger statement: dot product between two "vectors" is invariant.



$$\vec{V}_1 \cdot \vec{V}_2 = v_1 v_2 \cos\Theta$$

⑦

$$\underbrace{[x_{1B} \;\; y_{1B}] \overset{I}{\,} \begin{bmatrix} x_{2B} \\ y_{2B} \end{bmatrix}} = \left(R(\alpha) \begin{bmatrix} x_{1A} \\ y_{1A} \end{bmatrix} \right)^T R(\alpha) \begin{bmatrix} x_{2A} \\ y_{2A} \end{bmatrix}$$

$$= [x_{1A} \; y_{1A}] \underbrace{R(\alpha)^T R(\alpha)}_{I} \begin{bmatrix} x_{2A} \\ y_{2A} \end{bmatrix}$$

$$= \underbrace{[x_{1A} \; y_{1A}] \begin{bmatrix} x_{2A} \\ y_{2A} \end{bmatrix}}$$

For Lorentz transformations:

$$R^T(\alpha) \, I \, R(\alpha) = I$$

(8)

Entirely analogous relationship for the Lorentz transformations:

Define $L(\theta) := \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$

$$L(\theta)^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} L(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that "Lorentz invariant scalar product" is preserved.

$$x_{1B} x_{2B} - c^2 t_{1B} t_{2B} = x_{1A} x_{2A} - c^2 t_{1A} t_{2A} \qquad *$$

$$x_B^2 - (c t_B)^2 = x_A^2 - (c t_A)^2$$

Use this as starting point for introducing 4-vectors.

5 2022-01-20 Lecture

See "Four-vectors and tensors" in Phys 442 notes.

(1)

Four-vectors and tensors

- plane electromagnetic wave solution:

$$\vec{E}(x,t) = E_0 \cos(kx - \omega t)\,\hat{y}$$
$$\vec{B}(x,t) = \frac{E_0}{c} \cos(kx-\omega t)\,\hat{z}$$

satisfy Maxwell's eqns with $\vec{J} = \vec{0}$, $\rho = 0$

- concentrate on the phase $\phi := kx - \omega t$

- since observations of phase are a "counting" procedure, phase is a Lorentz invariant scalar.

(2)

$$\phi = \overset{k_B}{k} x_B - \overset{\omega_A/c}{\frac{\omega}{c}} ct_B$$
$$= k(x_A \cosh\theta - ct_A \sinh\theta) - \frac{\omega}{c}(-x_A \sinh\theta + ct_A \cosh\theta)$$
$$= \underbrace{\left(k\cosh\theta + \frac{\omega}{c}\sinh\theta\right)}_{k_A} x_A - \underbrace{\left(k\sinh\theta + \frac{\omega}{c}\cosh\theta\right)}_{\omega_A/c} ct_A$$

To define frame specific k's and ω's:

$$k_B x_B - \frac{\omega_B}{c} ct_B = k_A x_A - \frac{\omega_A}{c} ct_A$$

$L(\theta)^{-1} = L(-\theta)$ ← verify by matrix multiplication.

Thus for phase invariance

$$\left.\begin{aligned} k_B &= k_A \cosh\theta - \frac{\omega_A}{c}\sinh\theta \\ \frac{\omega_B}{c} &= -k_A \sinh\theta + \frac{\omega}{c}\cosh\theta \end{aligned}\right\}$$

exactly like the transformation $(ct_A, x_A) \to (ct_B, x_B)$ i.e. Lorentz transformation.

③

Suggests a terminology. Anything that has the same Lorentz transformation rules as $\underbrace{(ct, \vec{r})}$ is known as a four-vector.

We have just shown $(\frac{\omega}{c}, \vec{k})$ is a four-vector.

Recall Lorentz invariant scalar product

$$(ct_1, \vec{r}_1) \diamond (ct_2, \vec{r}_2) = -ct_1\, ct_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$

Phase is of same form:

$$\underbrace{(ct, \vec{r})} \diamond \underbrace{(\frac{\omega}{c}, \vec{k})} = -\omega t + \vec{r}\cdot\vec{k} \qquad (*)$$

Introduce tensor component notation (in contrast a more "geometric" coordinate-free approach)

④

$$(\omega/c, \vec{k}) \diamond (ct, \vec{r}) = [\omega/c \;\; k_x \;\; k_y \;\; k_z] \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \underbrace{\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}}$$

covariant components of four vector $(\frac{\omega}{c}, \vec{k})$

metric tensor g

contravariant components of four-vector $(ct, \vec{r})$

$$= [-\omega/c \;\; k_x \;\; k_y \;\; k_z] \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Introduce $r^0 := ct$, $r^1 := x$, $r^2 := y$, $r^3 := z$

$k_0 := -\omega/c$, $k_1 = k_x$, $k_2 = k_y$, $k_3 = k_z$

then

$$(\omega/c, \vec{k}) \diamond (ct, \vec{r}) = \sum k_\alpha r^\alpha$$
$$= k_\alpha r^\alpha$$
$$= k^\alpha r_\alpha$$

Einstein summation convention. i.e. implicit summation understood.

(5)

The components of a four-vector don't need to be quantities. e.g. $\phi = \vec{k}\cdot\vec{r} - \omega t$

and differentiate

$$k^0 = -\frac{1}{c}\frac{\partial\phi}{\partial t},\ k^1 = \frac{\partial\phi}{\partial x},\ k^2 = \frac{\partial\phi}{\partial y},\ k^3 = \frac{\partial\phi}{\partial z}$$

or $k^\alpha = \partial^\alpha \phi$ — transform like a four-vector

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t},\ \partial_1 = \frac{\partial}{\partial x},\ \partial_2 = \frac{\partial}{\partial y},\ \partial_3 = \frac{\partial}{\partial z} \qquad *$$

Four vectors are just a special case of Lorentz tensors.

$$\partial_\beta F^\alpha \to G_\beta{}^\alpha$$

(6)

Let us write Λ for $L(\theta)$
$\quad\quad\quad\quad\ \ \propto_{4\times4} \quad \propto_{2\times2}$

$$\Lambda^T g \Lambda = g$$

By definition → the contravariant components of a four-vector transforms like:

$$\underset{\text{new system}}{\tilde{H}^\alpha} = \Lambda^\alpha{}_\beta \underset{\text{old system.}}{H^\beta}$$

How do the covariant components transform?

$$\tilde{G}_\alpha = (\Lambda^{-1})^\gamma{}_\alpha \, G_\gamma$$

$\tilde{G}_\alpha \tilde{H}^\alpha$ should be same as $G_\beta H^\beta$ (Lorentz-invariant scalar product)

$$\underbrace{G_\alpha (\Lambda^{-1})^\alpha{}_\beta}_{\tilde{G}_\beta} \underbrace{(\Lambda)^\beta{}_\gamma H^\gamma}_{\tilde{H}^\beta} \propto$$

Because, by def^n

$$(\Lambda^{-1})^\alpha{}_\beta (\Lambda)^\beta{}_\gamma = \delta^\alpha{}_\gamma$$

7

$$H_\alpha{}^\beta{}_\gamma \leftarrow \text{e.g. of rank 3 Lorentz tensor}$$

$$\tilde{H}_\sigma{}^\nu{}_w = \Lambda^\nu{}_\beta (\Lambda^{-1})_\sigma{}^\alpha (\Lambda^{-1})_w{}^\gamma H_\alpha{}^\beta{}_\gamma$$

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$A_\alpha = \underbrace{g_{\alpha\beta}}_{\begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}} A^\beta$$

$$H_\sigma{}^\sigma{}_w =: G_w$$

6 2022-01-25 Lecture

See "Proper time and the terrible twins" in Phys 442 notes.

①

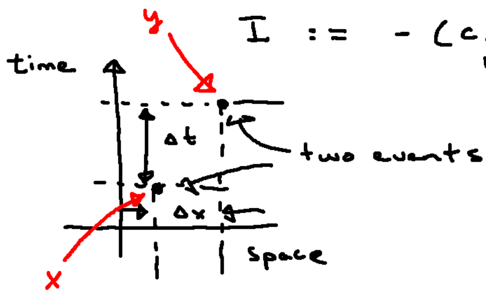
Proper time

$r_\alpha r^\alpha$ Lorentz invariant

By itself, not particularly useful as it depends on origin of our coordinate system.

Instead, the interval is more useful.

$$I := -\underbrace{(c\Delta t)^2}_{} + \underbrace{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}_{(\Delta d)^2} \quad **$$



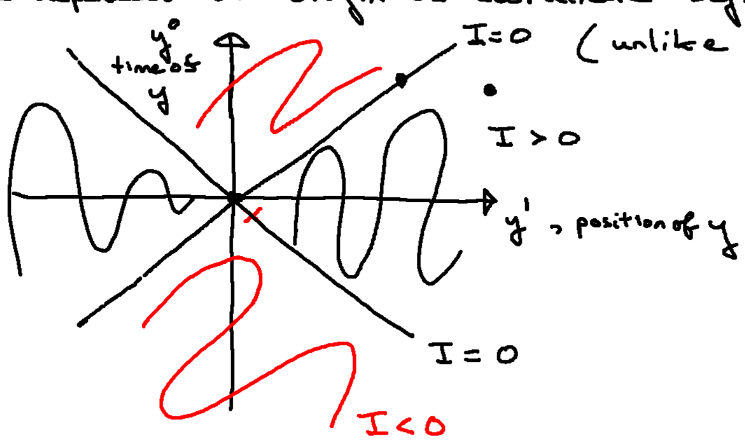
The interval is defined for two points in space-time.

②

The interval is Lorentz invariant. (by construction)

$$I = \underbrace{(x^\alpha - y^\alpha)}_{}\underbrace{(x_\alpha - y_\alpha)}_{}$$

Independent of origin of coordinate system (unlike $x^\alpha x_\alpha$ or $y^\alpha y_\alpha$)

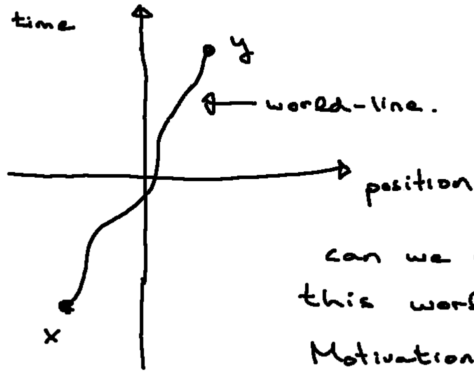


$I=0$: light-like separation

$I<0$: time-like

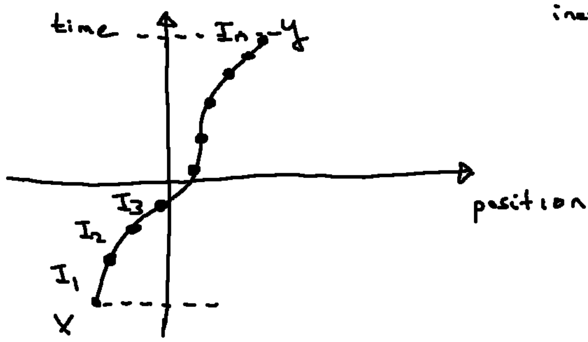
$I>0$: space-like

(3)



Can we construct Lorentz invariants for this world-line?

Motivation: equations of motion indep. of inertial ref. frame.



Start with something that is not so useful.

$\sum_i I_i$ is a Lorentz invariant.

(4)

Instead $\quad J := \sum_i \sqrt{-I_i}$

$$= \sum_i \sqrt{(c\Delta t_i)^2 - (\Delta d_i)^2}$$

$$= \sum_i c\Delta t_i \sqrt{1 - \left(\frac{\Delta d_i}{c\Delta t_i}\right)^2}$$

Take limit as $\Delta t_i \to 0$

$$J = c \int_{x/c}^{y/c} dt \, \underbrace{\sqrt{1-(v/c)^2}}$$

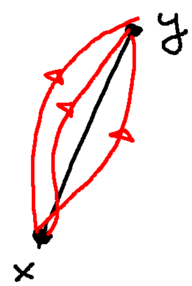
by contraction this is a Lorentz-invariant

Remember time dilation. $\quad \Delta t_R = \Delta t_L \sqrt{1-(v/c)^2}$

$$\tau = \int dt \sqrt{1-(v/c)^2}$$

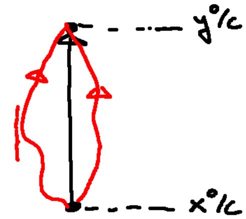
"proper time" Lorentz invariant. (time elapsed in instantaneous rest frames)

(5) For all possible trajectories between two points in space-time, the uniform motion trajectory has maximal proper time.



$\tau < \tau_{\text{uniform}}$.

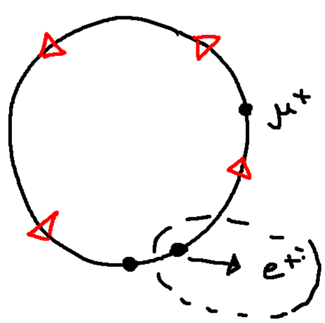
transform to the inertial ref frame where particle is stationary

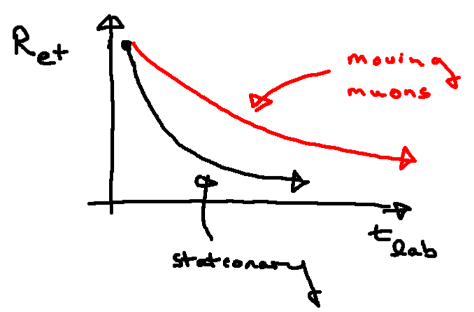


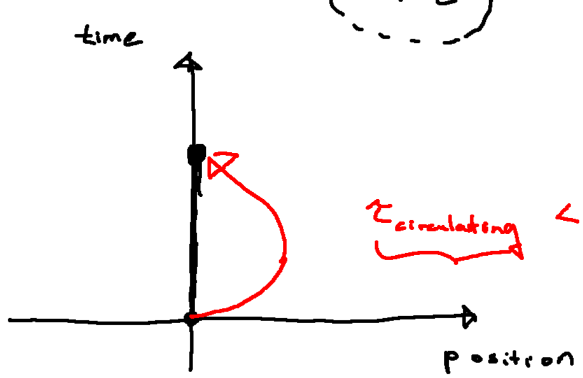
$$\tau_{\text{uniform}} = \int_{x^0/c}^{y^0/c} dt \underbrace{\sqrt{1-(v/c)^2}}_{=1}$$

$$\tau_{\text{non-uniform}} = \int_{x^0/c}^{y^0/c} dt \underbrace{\sqrt{1-(v/c)^2}}_{<1 \text{ for part of traj.}}$$

(6) Muons in a storage ring are the travelling twins







$\underbrace{\tau_{\text{circulating}}} < \tau_{\text{stationary}}$

7 2022-01-27 Lecture

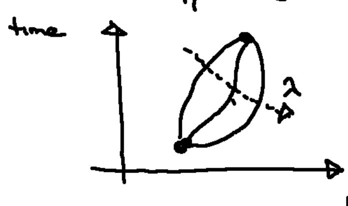
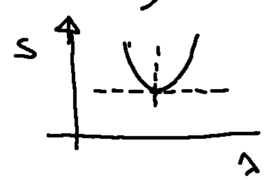
Please review the basics of Lagrangian mechanics from undergrad.

See “Momenergy from an invariant action” in Phys 442 notes.

See also Sections 25 and 26 of L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, 4. rev. Engl. ed., repr (Elsevier Butterworth Heinemann, Amsterdam Heidelberg, 2010).

①

Brief review of Lagrangian and Hamiltonian Dynamics

General concepts	1-d example (conservative force)
Start with Lagrangian $\mathcal{L}(q, \dot{q}, t)$ q generalized coordinates, $\dot{q}$ generalized velocities	$\mathcal{L} = T - U = \frac{1}{2}mv^2 - V(x)$
Action $S := \int_{t_1}^{t_2} dt\, \mathcal{L}(q, \dot{q}, t)$ $\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}}_{\text{reproduce Newton's 2nd law}}$ Euler-Lagrange eqn's.	$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2}mv^2 - V(x)\right)$  

②

Define canonical momenta: $p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$	$p_x = m\dot{x}$ (*)
Hamiltonian $H := -\mathcal{L} + \sum_i \dot{q}_i p_i$	$H = -\frac{1}{2}m\dot{x}^2 + V(x) + \dot{x}m\dot{x}$ $= \frac{1}{2}\frac{p_x^2}{m} + V(x)$
For any quantity $A(p, q, t)$ $\frac{dA}{dt} = \underbrace{\{A, H\}}_{\text{Poisson Bracket}} + \frac{\partial A}{\partial t}$ $\{A,B\} = \sum_i \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$	$\frac{dx}{dt} = \{x, H\} = \frac{\partial x}{\partial x}\frac{\partial H}{\partial p} + \cancel{\frac{\partial x}{\partial p}}\frac{\partial H}{\partial x} = \frac{p_x}{m}$ ($\frac{\partial x}{\partial p} = 0$) similarly $\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$

③

"Momenergy" from an invariant action

(four-vector)

- start with principle of "most proper time":

$$S = -\alpha \int d\tau \qquad (*)$$

"most" → "least" TBD ($\alpha > 0$)

$$= -\alpha \int dt \sqrt{1-(v/c)^2}$$

$$S = \int dt\, \mathcal{L} \quad \Rightarrow \quad \mathcal{L} = -\alpha \sqrt{1-(v/c)^2}$$

For a non-relativistic motion of a free particle

$$\mathcal{L} = \frac{1}{2} m v^2$$

Get α from this limit.

④

$$\mathcal{L} = -\alpha \left(1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots\right)$$

$$= -\alpha + \underbrace{\frac{\alpha}{2}\frac{v^2}{c^2}}_{\text{to agree with } \frac{mv^2}{2}} + \cdots \qquad \alpha = mc^2$$

Thus $\mathcal{L} = -mc^2 \sqrt{1-(v/c)^2}$

Crank the handle on the Lagrangian / Hamiltonian machine!

$$p_x := \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial v_x} = \frac{\partial}{\partial v_x}\left(-mc^2\sqrt{1-(v/c)^2}\right)$$

$$= \frac{-m\cancel{c^2}}{2\sqrt{1-(v/c)^2}} * -\frac{1}{\cancel{c^2}} \frac{\partial}{\partial v_x}\left(v_x^2 + \underbrace{\cancel{v_y^2} + \cancel{v_z^2}}_{=0}\right)$$

$$p_x = m v_x / \sqrt{1-(v/c)^2}$$

⑤

$$\boxed{\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}}$$

Now consider energy.

$$\frac{d\mathcal{L}}{dt} = \sum_i \underset{*}{\frac{\partial \mathcal{L}}{\partial q_i}} \dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i$$

Aside: $\left[\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] = \underset{*}{\frac{\partial \mathcal{L}}{\partial q_i}} \right]$

$$= \sum_i \left(\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right)$$

$E := -\mathcal{L} + \sum_i p_i \dot{q}_i$

$$= \sum_i \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i\right]$$

Rearrange:

$$\frac{d}{dt}\left[\mathcal{L} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i\right] = 0$$

conserved in time!!!

$=: -E$

⑥

Specialize for our $\mathcal{L}$.

$$E = +mc^2\sqrt{1-(v/c)^2} + \frac{m}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \underbrace{(v_x^2+v_y^2+v_z^2)}_{v^2}$$

$$= \frac{+mc^2\left(1-\left(\frac{v}{c}\right)^2\right) + mv^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$+mc^2 - mv^2 + mv^2$

$$\boxed{E = \frac{mc^2}{\sqrt{1-(v/c)^2}}}$$

Try to combine with expression for $\vec{p}$.

(7)

$$E = mc^2 \frac{dt}{d\tau}$$

$$\int d\tau = \int dt \sqrt{1-(v/c)^2}$$

$$= mc \frac{d(ct)}{d\tau}$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}$$

$$= m \frac{d\vec{r}}{dt} \frac{1}{\sqrt{1-(v/c)^2}}$$

$$= m \frac{d\vec{r}}{d\tau}$$

four-vector.

$$\underbrace{\left(\frac{E}{c}, \vec{p}\right)}_{\text{"momenergy"}} = m \underbrace{\frac{d}{d\tau} \overbrace{(ct, \vec{r})}}_{}$$

four vector by construction

"proper velocity", η^α

(8)

$$\eta^\alpha := \frac{dx^\alpha}{d\tau}, \quad p^\alpha = m\eta^\alpha$$

Lorentz invariant

$$\eta_\alpha \eta^\alpha = -c\frac{dt}{d\tau}c\frac{dt}{d\tau} + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2$$

$$= -c^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dt}{d\tau}\right)^2 v^2$$

$$= \frac{v^2 - c^2}{1-(v/c)^2}$$

$$= -c^2$$

$$\Rightarrow \quad p_\alpha p^\alpha = -m^2c^2$$

go back to non-tensor notation

$$-\left(\frac{E}{c}\right)^2 + p^2 = -m^2c^2$$

9

$$\frac{E^2}{c^2} = m^2c^2 + p^2$$

$$E^2 = m^2c^4 + (pc)^2$$

$$\boxed{E = \sqrt{(mc^2)^2 + (pc)^2}}$$

when $p = 0 \Rightarrow E = mc^2$

8 2022-02-01 Lecture

See "The Lorentz force law from an invariant action" in Phys 442 notes.

See also Section 17 of L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, 4. rev. Engl. ed., repr (Elsevier Butterworth Heinemann, Amsterdam Heidelberg, 2010).

①

The Lorentz force law from an invariant action

$$S = \underbrace{-mc^2 \int d\tau}_{\substack{\text{free particle} \\ \text{(last lecture)}}} + \overbrace{\underset{\substack{\uparrow \\ \text{"coupling"}}}{q} \int \underbrace{A_\alpha}_{\substack{\text{covariant components of} \\ \text{"Some" four-vector field} \\ \text{defined throughout space-time}}} dx^\alpha}^{*} \quad \leftarrow (c\,dt, d\vec{r})$$

Look at consequences of adding this additional term.

Remember how we got $\mathcal{L}$: $\quad \frac{d\tau}{dt} = \sqrt{1-(v/c)^2}$

$S = -mc^2 \int dt \sqrt{1-(v/c)^2}$ and since $S := \int \mathcal{L}\, dt$

$\Rightarrow \mathcal{L} = -mc^2\sqrt{1-(v/c)^2}$. Apply some procedure to (*).

②

$$A^\alpha \text{"="} (A^0, \vec{A}) \; ; \; A_\alpha \text{"="} (-A^0, \vec{A})$$

$$\int A_\alpha dx^\alpha = -\int A^0 c\, dt + \underbrace{\int \vec{A}\cdot d\vec{r}}_{\int \vec{A}\cdot\vec{v}\, dt}$$

Since $S = \int \mathcal{L}\, dt$, we now have:

$$\boxed{\mathcal{L} = -mc^2\sqrt{1-(v/c)^2} - qcA^0 + q\vec{A}\cdot\vec{v}}$$

$$\frac{d}{dt}\Bigg(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{=:\,P_i \text{ the momenta}}\Bigg) = \frac{\partial \mathcal{L}}{\partial q_i}$$

"regular" (not tensor) notation.

③

$$P_x = \frac{\partial \mathcal{L}}{\partial v_x}$$

$$= \frac{m v_x}{\sqrt{1-(v/c)^2}} + q A_x$$

$$\vec{P} = \underbrace{\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}}_{=: \vec{p}} + q\vec{A}$$

$$\vec{P} = \vec{p} + q\vec{A}$$

canonical / generalized momenta

ordinary / momenta

④

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$ } while holding $\dot{q}_i$ fixed !!!

$$\frac{d}{dt}\left(\underbrace{\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}}_{\vec{p}} + q\vec{A}\right) = -qc\nabla A^0 + q\nabla(\vec{A}\cdot\vec{v})$$

$$\frac{d\vec{p}}{dt} = -q\frac{d\vec{A}}{dt} - qc\nabla A^0 + \underbrace{q\nabla(\vec{A}\cdot\vec{v})}_{③}$$

For term ③ remember, for arb. fields $\vec{a}, \vec{b}$

$$\nabla(\vec{a}\cdot\vec{b}) = \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a}) + (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a}$$

$$\nabla(\vec{A}\cdot\vec{v}) = \vec{v}\times(\nabla\times\vec{A}) + \underbrace{(\vec{v}\cdot\nabla)}\vec{A}$$

(5)

$$\underbrace{\frac{d\vec{A}}{dt}}_{\text{"convective" derivative}} = \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A} \qquad \text{by chain rule.}$$

$$\frac{d\vec{p}}{dt} = -q\left(\frac{\partial \vec{A}}{\partial t} + \cancel{(\vec{v}\cdot\nabla)\vec{A}}\right) - qc\nabla A^0 + q\left(\vec{v}\times(\nabla\times\vec{A})\right) + \cancel{q(\vec{v}\cdot\nabla)\vec{A}}$$

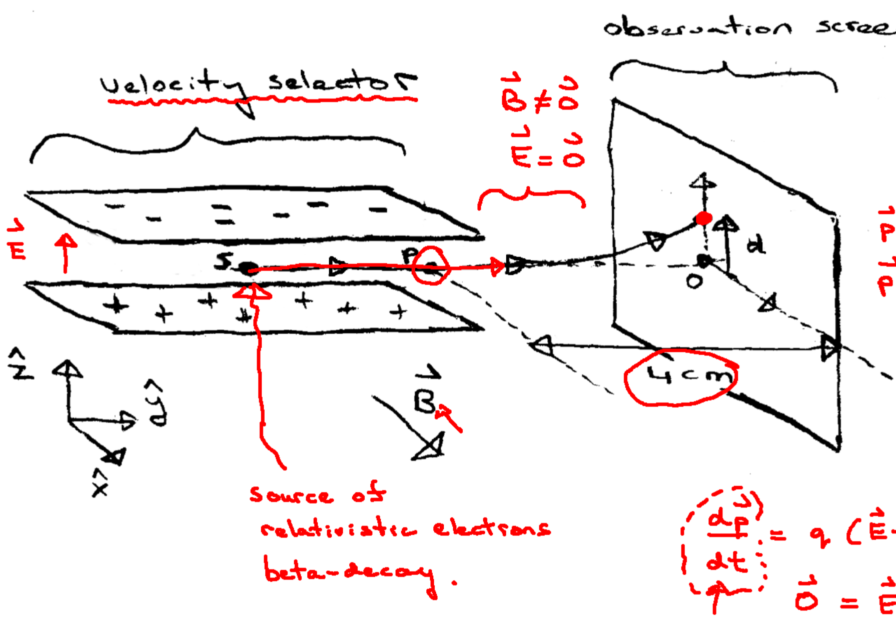
$$\frac{d\vec{p}}{dt} = q\Big[\underbrace{\left(-c\nabla A^0 - \frac{\partial \vec{A}}{\partial t}\right)}_{\vec{E}\,(?)} + \vec{v}\times\underbrace{(\nabla\times\vec{A})}_{\vec{B}\,(?)}\Big]$$

suggested by non-relat. Lorentz force law.

$$\boxed{\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}\right) = q\left(\vec{E}+\vec{v}\times\vec{B}\right)}$$

Are these the same $\vec{E}$ and $\vec{B}$ that Max Eq give?

Bucherer's test of Lorentz force law



Some numbers:

$v/c \simeq 0.7$, $\vec{B}\simeq 10^{-2}\,T$, $\vec{E}\simeq 10^{6}\,V/m$

$\vec{p} = m\vec{v} \Rightarrow d = 6.6\,mm$ ✗

$\vec{p} = \gamma m\vec{v} \Rightarrow d = 4.6\,mm$ ✓

for a fixed $\vec{E}$ and $\vec{B}$, only certain velocities are unaccelerated

$$\frac{d\vec{p}}{dt} = q\left(\vec{E}+\vec{v}\times\vec{B}\right)$$

$$\vec{0} = \vec{E} + \vec{v}\times\vec{B}$$

(6)

We have treated space and time very differently here. Our "new" Lorentz force law is not manifestly covariant.

Next lecture:

$$\frac{dp_\beta}{d\tau} = q\eta^\alpha \underbrace{F_{\beta\alpha}}_{:= (\partial_\beta A_\alpha - \partial_\alpha A_\beta)}$$

Faraday tensor.

proper velocity

9 2022-02-03 Lecture

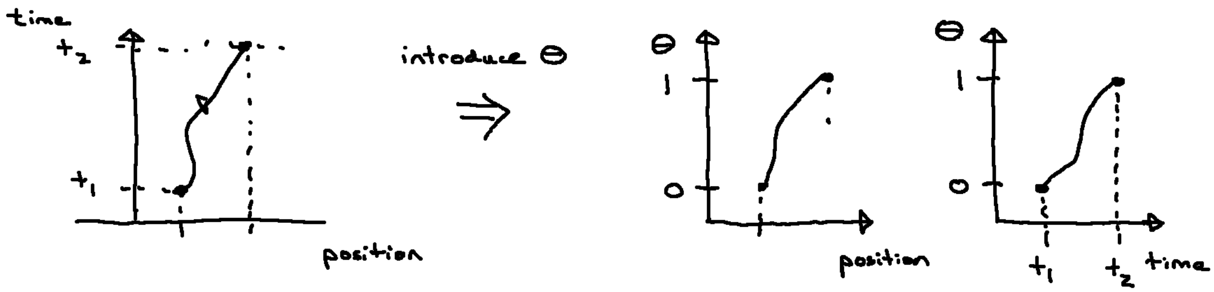
See "Minkowski force" in Phys 442 notes.

See Section 23 of L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, 4. rev. Engl. ed., repr (Elsevier Butterworth Heinemann, Amsterdam Heidelberg, 2010).

(1)

Minkowski force

strategy : don't treat time coordinate of world-line preferentially.

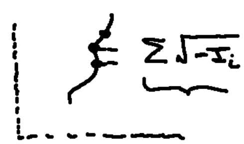


$$S = \int dt \, \mathcal{L}$$

(2)

$$S_F = -mc^2 \int d\tau$$

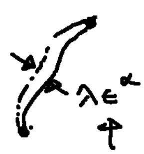
$$= -mc \int_0^1 d\Theta \sqrt{-\frac{dx^\alpha}{d\Theta}\frac{dx_\alpha}{d\Theta}}$$



For a world-line with stationary action we should "perturb" the world-line :

$$x^\alpha \rightarrow x^\alpha + \lambda \epsilon^\alpha_{\text{arb.}}$$

stationary action $\quad \left.\frac{dS_F}{d\lambda}\right|_{\lambda=0} = 0$ for any ϵ^α where $\epsilon^\alpha = 0$ at $\Theta = 0$ and $\Theta = 1$ (endpoints)



(3)

$$\frac{1}{-mc}\frac{dS_F}{d\lambda} = \frac{d}{d\lambda}\int_0^1 d\theta \sqrt{-\frac{d(x^\alpha+\lambda\epsilon^\alpha)}{d\theta}\frac{d(x_\alpha+\lambda\epsilon_\alpha)}{d\theta}}$$

$$= \int_0^1 d\theta \frac{-1}{2\sqrt{\cdots}}\left[\frac{d\epsilon^\alpha}{d\theta}\frac{d(x_\alpha+\lambda\epsilon_\alpha)}{d\theta} + \frac{d(x^\alpha+\lambda\epsilon^\alpha)}{d\theta}\frac{d\epsilon_\alpha}{d\theta}\right]$$

$$-\frac{1}{mc}\frac{dS_F}{d\lambda}\bigg|_{\lambda=0} = \int_0^1 d\theta \frac{-1}{2\sqrt{\cdots}}\left[\underbrace{\frac{d\epsilon^\alpha}{d\theta}\frac{dx_\alpha}{d\theta}} + \underbrace{\frac{dx^\alpha}{d\theta}\frac{d\epsilon_\alpha}{d\theta}}\right]$$

$$= \int_0^1 d\theta \frac{-1}{\sqrt{-\frac{dx^\alpha}{d\theta}\frac{dx_\alpha}{d\theta}}}\frac{d\epsilon^\alpha}{d\theta}\underbrace{\frac{dx_\alpha}{d\theta}}$$

Recall proper time derivation : $\sqrt{-\frac{dx^\alpha}{d\theta}\frac{dx_\alpha}{d\theta}} = c\frac{d\tau}{d\theta}$

(4)

$$-\frac{1}{m}\frac{dS_F}{d\lambda}\bigg|_{\lambda=0} = -\int_0^1 d\theta\, \underbrace{\frac{dx_\alpha}{d\tau}}_{\eta_\alpha \text{ proper velocity}}\frac{d\epsilon^\alpha}{d\theta}$$

integration by parts

$$= \underbrace{-\eta_\alpha\epsilon^\alpha\Big|_0^1}_{=0} + \int_0^1 d\theta\, \underset{\text{arb.}}{\epsilon^\alpha}\frac{d\eta_\alpha}{d\theta}$$

For stationary action $\left(\frac{dS_F}{d\lambda}\Big|_{\lambda=0} = 0 \text{ for all } \epsilon^\alpha\right)$,

$\frac{d\eta_\alpha}{d\theta} = 0$; i.e., constant proper velocity.

lots of work for something we knew !!!

(5)

$$S = \underbrace{S_F}_{\text{just looked at}} + \underbrace{S_A}_{:= q \int A_\alpha dx^\alpha}$$

$$\frac{1}{q}\frac{dS_A}{d\lambda} = \frac{d}{d\lambda}\int_0^1 d\theta \; \underbrace{A_\alpha(x^0+\lambda\epsilon^0, x^1+\lambda\epsilon^1, x^2+\lambda\epsilon^2, x^3+\lambda\epsilon^3)}\; \frac{d(x^\alpha+\lambda\epsilon^\alpha)}{d\theta}$$

$$\left.\frac{dA_\alpha}{d\lambda}\right|_{\lambda=0} = \underbrace{(\partial_\beta A_\alpha)\,\epsilon^\beta} \qquad \text{analogous to: } \frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \underbrace{(\vec{v}\cdot\nabla)\vec{A}}$$

$$\left.\frac{1}{q}\frac{dS_A}{d\lambda}\right|_{\lambda=0} = \int_0^1 d\theta \left[\underbrace{A_\alpha \frac{d\epsilon^\alpha}{d\theta}}_{\text{integration by parts !!!}} + (\partial_\beta A_\alpha)\,\epsilon^\beta \frac{dx^\alpha}{d\theta}\right]$$

$$= \underbrace{A_\alpha \epsilon^\alpha \Big|_0^1}_{=0} + \int_0^1 d\theta \left[-\underbrace{\frac{dA_\alpha}{d\theta}}\,\epsilon^\alpha + (\partial_\beta A_\alpha)\,\epsilon^\beta \frac{dx^\alpha}{d\theta}\right]$$

(6)

$$\left.\frac{1}{q}\frac{dS_A}{d\lambda}\right|_{\lambda=0} = \int_0^1 d\theta \left[-\underbrace{(\partial_\beta A_\alpha)\frac{dx^\beta}{d\theta}}_{dA_\alpha/d\theta}\,\epsilon^\alpha + (\partial_\beta A_\alpha)\,\epsilon^\beta \frac{dx^\alpha}{d\theta}\right]$$

$$= \int_0^1 d\theta \left[-(\partial_\alpha A_\beta)\frac{dx^\alpha}{d\theta}\,\epsilon^\beta + (\partial_\beta A_\alpha)\,\epsilon^\beta \frac{dx^\alpha}{d\theta}\right]$$

$$= \int_0^1 d\theta \; \frac{dx^\alpha}{d\theta}\,\epsilon^\beta \left[-\partial_\alpha A_\beta + \partial_\beta A_\alpha\right]$$

Combine with results for S_F (remember $S = S_F + S_A$):

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = \int_0^1 d\theta \; \underset{\text{arb.}}{\epsilon^\beta} \underbrace{\left[-m\frac{d\eta_\beta}{d\theta} + q\frac{dx^\alpha}{d\theta}(\partial_\beta A_\alpha - \partial_\alpha A_\beta)\right]}_{=0 \text{ for stationary action.}}$$

(7)

$$m \frac{d\eta_\beta}{d\theta} = q \frac{dx^\alpha}{d\theta} (\partial_\beta A_\alpha - \partial_\alpha A_\beta)$$

$$m \cancel{\frac{d\tau}{d\theta}} \frac{d\eta_\beta}{d\tau} = q \cancel{\frac{d\tau}{d\theta}} \frac{dx^\alpha}{d\tau} (\partial_\beta A_\alpha - \partial_\alpha A_\beta)$$

$$\frac{dp_\beta}{d\tau} = q\eta^\alpha \underbrace{(\partial_\beta A_\alpha - \partial_\alpha A_\beta)}_{=: F_{\beta\alpha}}$$

"Faraday tensor"

The four-vector version of the Lorentz force law:

$$** \quad \boxed{\frac{dp_\beta}{d\tau} = q\eta^\alpha F_{\beta\alpha}} \quad **$$

"contains" 3 components of $\vec{E}$ and 3 " $\vec{B}$

Space part reproduces the Lorentz force law.

(8)

What about time part?

$$\frac{dp_0}{d\tau} = q\eta^0 \underbrace{F_{00}}_{=0} + q\eta^1 F_{01} + \cdots$$

$$\partial_0 A_1 - \partial_1 A_0 = \frac{1}{c}\frac{\partial A^1}{\partial t} + \frac{\partial \Phi / c}{\partial x_1}$$

$$= \underbrace{- E_x / c}$$

$$\left(\vec{E} = -\nabla\Phi - \frac{\partial \vec{B}}{\partial t}\right)$$

energy

$$\frac{dp_0}{d\tau} = -q \left(\frac{1}{\sqrt{1-(v/c)^2}} v_i\right) \cdot \vec{E}/c$$

$$\boxed{\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-(v/c)^2}}\right) = q\vec{v}\cdot\vec{E}}$$

work-energy theorem !!!

$$\left(\int \vec{E}\cdot d\vec{l}\right)$$

10 2022-02-08 Lecture

See "Transformation of electric and magnetic fields between inertial frames" in Phys 442 notes.

①

Transformation of $\vec{E}$ and $\vec{B}$ fields between inertial reference frames

$$\frac{dp_\beta}{d\tau} = \underbrace{q\eta^\alpha F_{\beta\alpha}}, \quad \text{where } F_{\beta\alpha} := \partial_\beta A_\alpha - \partial_\alpha A_\beta . \quad *$$

$$A^\alpha \text{ "=" } (\Phi/c, \vec{A}), \qquad \vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \quad *$$

$$\vec{B} = \nabla \times \vec{A}$$

e.g. $E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t}$

$$F_{\beta\alpha} = \begin{array}{c} \overbrace{\hphantom{F_{01}\ F_{02}\ F_{03}\ F_{04}}}^{\alpha} \\ \begin{bmatrix} 0 & F_{01} & F_{02} & F_{03} \\ -F_{01} & 0 & F_{12} & F_{13} \\ -F_{02} & -F_{12} & 0 & F_{23} \\ -F_{03} & -F_{13} & -F_{23} & 0 \end{bmatrix} \end{array} \left.\vphantom{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}\right\} \beta$$

only 6 independent components of the Faraday tensor.

matches nicely with
$\vec{E}$: 3 components
$\vec{B}$: 3 components.

②

$$\begin{aligned} F_{01} &= \partial_0 A_1 - \partial_1 A_0 \\ &= \frac{\partial A_x}{\partial(ct)} - \frac{\partial(-\Phi/c)}{\partial x} \\ &= \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\frac{\partial \Phi}{c} \\ &= -E_x/c \end{aligned}$$

By same argument: $F_{02} = -E_y/c$
$F_{03} = -E_z/c$

$$\begin{aligned} F_{12} &= \partial_1 A_2 - \partial_2 A_1 \\ &= \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \\ &= B_z \end{aligned}$$

Recall $\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$

By same argument $F_{23} = B_x$, $F_{31} = B_y$

3

$$F_{\mu\nu} \text{ "=" } \begin{array}{cc} \nu & \\ \begin{Bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{Bmatrix} & \mu \end{array}$$

$$\downarrow$$

$$F^{\mu\nu} \text{ "=" } \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix}$$

$$F_{\alpha\nu} = g_{\alpha\mu} \, g_{\nu\beta} \, F^{\mu\beta} \qquad \text{where} \qquad g_{\alpha\mu} \text{ "=" } \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

4

$$\tilde{F}^{\mu\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\beta}{}_{\kappa} \, F^{\alpha\kappa} \qquad (*)$$

Remember in "standard configuration" special case:

$$\Lambda = \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ -s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (\ddagger)$$

Now determine transformation rules for $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$:

$$\tilde{E}_x = c \, \tilde{F}^{01}$$
$$= c \, \Lambda^{0}{}_{\alpha} \Lambda^{1}{}_{\kappa} \, F^{\alpha\kappa} \qquad (*)$$

5

$$\tilde{E}_x = c\Lambda^0{}_\alpha\left(\Lambda^1{}_0 F^{\alpha 0} + \Lambda^1{}_1 F^{\alpha 1} + \cancel{\Lambda^1{}_2} F^{\alpha 2} + \cancel{\Lambda^1{}_3} F^{\alpha 3}\right)$$

With Λ of form $(\ddagger)$: $\Lambda^1{}_2 = \Lambda^1{}_3 = 0$

$$\begin{aligned}
\frac{\tilde{E}_x}{c} &= \Lambda^0{}_0\left(\Lambda^1{}_0 F^{00} + \Lambda^1{}_1 F^{01}\right) + \Lambda^0{}_1\left(\Lambda^1{}_0 F^{10} + \Lambda^1{}_1 F^{11}\right) \\
&\quad + \cancel{\Lambda^0{}_2}(\ldots) + \cancel{\Lambda^0{}_3}(\ldots) \\
&= c\theta\left(-s\theta\, \cancel{F^{00}} + c\theta\, F^{01}\right) - s\theta\left(-s\theta\, F^{10} + \cancel{c\theta\, F^{11}}\right) \\
&= (c\theta)^2 F^{01} + (s\theta)^2 F^{10} \\
&= (c\theta)^2 F^{01} - (s\theta)^2 F^{01} \\
&= \left[(c\theta)^2 - (s\theta)^2\right] F^{01} \qquad \text{Recall } \cosh^2\theta - \sinh^2\theta = 1 \\
&= E_x/c \quad *
\end{aligned}$$

6

By same argument $\tilde{B}_x = B_x$

$$\begin{aligned}
\frac{\tilde{E}_y}{c} &= \tilde{F}^{02} \\
&= \Lambda^0{}_\alpha \Lambda^2{}_\kappa F^{\alpha\kappa} \\
&= \Lambda^0{}_\alpha\left(\cancel{\Lambda^2{}_0} F^{\alpha 0} + \cancel{\Lambda^2{}_1} F^{\alpha 1} + \Lambda^2{}_2 F^{\alpha 2} + \cancel{\Lambda^2{}_3} F^{\alpha 3}\right) \qquad \Lambda^2{}_2 = 1
\end{aligned}$$

With Λ in $(\ddagger)$ form : $\Lambda^2{}_0 = \Lambda^2{}_3 = \Lambda^2{}_1 = 0)$

$$\begin{aligned}
\tilde{E}_y/c &= \Lambda^0{}_\alpha F^{\alpha 2} \\
&= \Lambda^0{}_0 F^{02} + \Lambda^0{}_1 F^{12} + \Lambda^0{}_2 \cancel{F^{22}} + \cancel{\Lambda^0{}_3} F^{32} \\
&= c\theta\, E_y/c - s\theta\, B_z \\
\tilde{E}_y &= \cosh\theta\, E_y - \sinh\theta\, cB_z \qquad (*)
\end{aligned}$$

By same manipulations :

$$\begin{aligned}
\tilde{E}_z &= \cosh\theta\, E_z + \sinh\theta\, cB_y \qquad (* \\
\tilde{B}_y &= \cosh\theta\, B_y + \sinh\theta\, E_z/c \\
\tilde{B}_z &= \cosh\theta\, B_z - \sinh\theta\, E_y/c\,.
\end{aligned}$$

7

Write results in a frame-indep. manner.

$$\vec{E} = \underbrace{(\hat{v}\cdot\vec{E})\hat{v}} + (\vec{E} - (\hat{v}\cdot\vec{E})\hat{v})$$

$$= \vec{E}_{\parallel} + \vec{E}_{\perp}$$

$$\tilde{\vec{E}}_{\perp} = \hat{y}\,(c\theta\, E_y - s\theta\, cB_z) + \hat{z}\,(c\theta\, E_z + s\theta\, cB_y)$$

$$= c\theta\,(\hat{z}E_z + \hat{y}E_y) + s\theta\, c(\hat{z}B_y - \hat{y}B_z)$$

$$= c\theta\,\vec{E}_{\perp} + s\theta\, c\,(\hat{z}(\vec{B}\cdot\hat{y}) - \hat{y}(\vec{B}\cdot\hat{z}))$$

Remember $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$= c\theta\,\vec{E}_{\perp} + s\theta\,\hat{x}\times c\vec{B}$$

$$\tilde{\vec{E}}_{\perp} = c\theta\left(\vec{E}_{\perp} + \vec{v}\times\vec{B}\right), \quad \tilde{\vec{B}}_{\perp} = c\theta\left(\vec{B}_{\perp} - \frac{\vec{v}}{c}\times\frac{\vec{E}}{c}\right)$$

$= c\tanh\theta$

11 2022-02-10 Lecture

See section 22.6.4 in Zangwill [2].

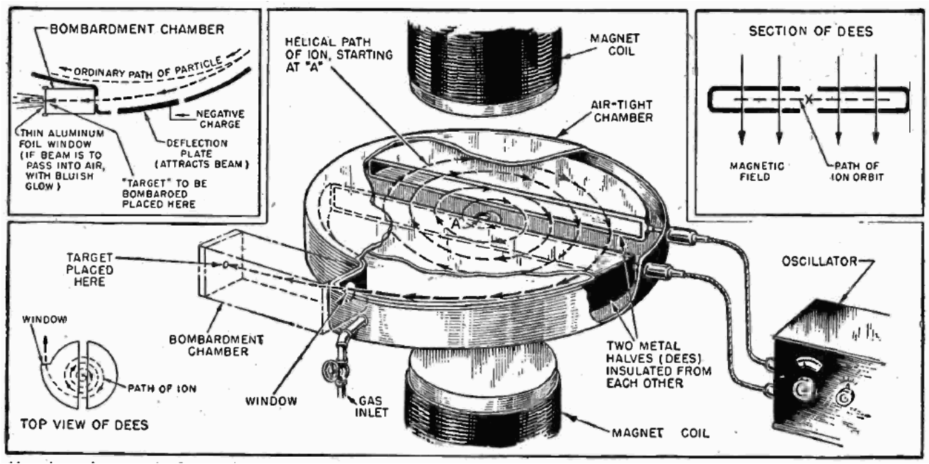
Some logistics:

no lectures during reading week

22nd and 24th of February

no problem set due on 25th of February

Cyclotron

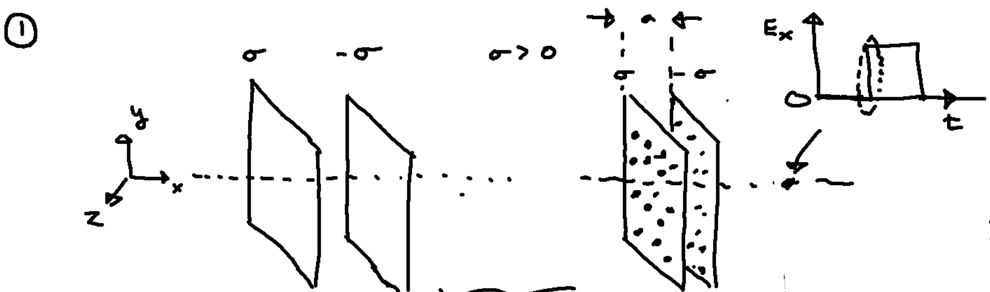


from https://en.wikipedia.org/wiki/File:Cyclotron_diagram.png

$e^{-z/(\lambda)}$

e^{-2T}

①



$\vec{E} = \vec{0}$ $\qquad$ $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{x}$ $\qquad$ $\vec{E} = 0$

stationary

moving

length contraction.

J^{μ} "=" $(\rho c, \vec{J})$ $\quad \frac{C}{m^2 s}$

non-zero $\vec{J}$

$\rho = \sigma\, \delta(x - vt) - \sigma\, \delta(x - vt - a)$

$\vec{J} = \sigma v \hat{x}\, \delta(x - vt) - \sigma v \hat{x}\, \delta(x - vt - a)$

to satisfy $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$.

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

cancel at jump.

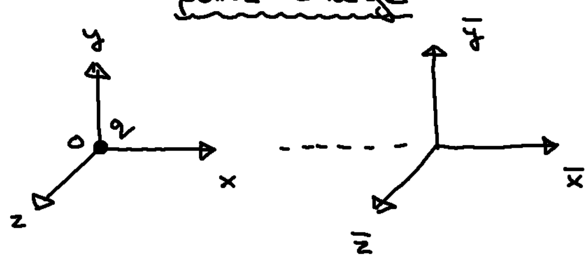
(2)

From last lecture:

$$\tilde{\vec{E}}_\perp = \cosh\theta\left(\vec{E}_\perp + \vec{v}\times\vec{B}\right), \quad \tilde{\vec{E}}_\parallel = \vec{E}_\parallel$$
$$\tilde{\vec{B}}_\perp = \cosh\theta\left(\vec{B}_\perp - \frac{\vec{v}}{c}\times\frac{\vec{E}}{c}\right), \quad \tilde{\vec{B}}_\parallel = \vec{B}_\parallel$$

(3)

Fields due to a uniformly moving point charge



Frame 1: point is stationary

Frame 2: point charge in uniform motion.

$$\vec{v}_{21} = \hat{x}\, v_{21,x}$$
$$= \hat{x}\, c\beta$$

④

In frame 1 (charge stationary)

$$\vec{B} = 0$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{(x\hat{x} + y\hat{y} + z\hat{z})}{r^3} \Big\} \approx$$

$$r^3 = (x^2 + y^2 + z^2)^{3/2}$$

Apply field transformation rules.

⑤

$$\bar{\vec{E}}_{\parallel} = \vec{E}_{\parallel}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{x}{r^3}\hat{x}$$

Express in terms of coordinates in frame 2. $(\bar{x}, \bar{y}, \bar{z}, c\bar{t})$

$$x = \gamma(\bar{x} + \beta c\bar{t})$$
$$ct = \gamma(c\bar{t} + \beta\bar{x})$$
$$y = \bar{y}$$
$$z = \bar{z}$$

$$\bar{\vec{E}}_{\parallel} = \frac{q}{4\pi\epsilon_0}\frac{\gamma(\bar{x} + \beta c\bar{t})\hat{x}}{[(\gamma(\bar{x}+\beta c\bar{t}))^2 + \bar{y}^2 + \bar{z}^2]^{3/2}}$$

⑥

Now consider transverse component.

$$\bar{\vec{E}}_\perp = \gamma\left(\vec{E}_\perp + \vec{\beta}\times \cancel{\vec{B}}c\right) \qquad \vec{B} = \vec{0}$$

$$= \frac{q}{4\pi\epsilon_0}\,\gamma\,\frac{(y\hat{y} + z\hat{z})}{r^3} \qquad \text{use frame 2 coordinates}$$

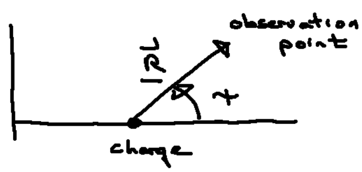
$$= \frac{q}{4\pi\epsilon_0}\,\gamma\,\frac{(\bar{y}\hat{y} + \bar{z}\hat{z})}{\cdots}$$

$$\bar{\vec{E}} = \bar{\vec{E}}_\parallel + \bar{\vec{E}}_\perp$$

$$= \frac{q}{4\pi\epsilon_0}\,\gamma\,\frac{[(\bar{x}+\beta c\bar{t})\hat{x} + \bar{y}\hat{y} + \bar{z}\hat{z}]}{[\underbrace{(\gamma(\bar{x}+\beta c\bar{t}))}_{\gamma R\cos\psi}{}^2 + \underbrace{\bar{y}^2 + \bar{z}^2}_{R^2\sin^2\psi}]^{3/2}}$$

⑦

Frame 2:



Define: $\vec{R} := (\bar{x} + \beta c\bar{t})\hat{x} + \bar{y}\hat{y} + \bar{z}\hat{z})$

$\hat{x}\cdot\vec{R} = R\cos\psi$

$\dfrac{y^2+z^2}{R^2} = \sin^2\psi$

$$\bar{\vec{E}} = \frac{q}{4\pi\epsilon_0}\,\gamma\,\frac{\hat{R}}{[(\gamma\cos\psi)^2 + \sin^2\psi]^{3/2}\,R^2}$$

$$\gamma^2\cos^2\psi + \sin^2\psi = \gamma^2(1-\sin^2\psi) + \sin^2\psi$$
$$= \gamma^2 + (1-\gamma^2)\sin^2\psi$$
$$= (c\theta)^2 - (s\theta)^2\sin^2\psi$$
$$= (c\theta)^2\,(1-(t\theta)^2\sin^2\psi)$$

$1-(c\theta)^2$

$(c\theta)^2-(s\theta)^2 = 1$

$1-(c\theta)^2 = 1-(1+(s\theta)^2)$

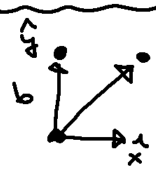
⑧

$$\gamma^2 \cos^2\psi + \sin^2\psi = \gamma^2 (1 - \beta^2 \sin^2\psi)$$

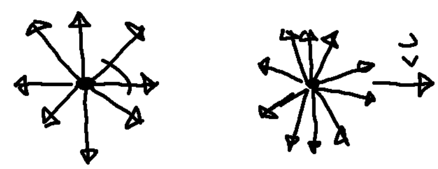
$$\vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{\hat{R}}{R^2} \frac{\gamma}{\gamma^3} \frac{1}{(1-\beta^2\sin^2\psi)^{3/2}}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

$$\boxed{\vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{\hat{R}}{R^2} \frac{(1-\beta^2)}{(1-\beta^2\sin^2\psi)^{3/2}}}$$



$$\psi = 90^\circ, \quad \vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{\hat{y}}{b^2} \frac{1-\beta^2}{(1-\beta^2)^{3/2}} = \frac{q}{4\pi\epsilon_0} \frac{\hat{y}}{b^2} \gamma$$



stationary

⑨

Now consider $\vec{B}$. $\quad \underbrace{\vec{B}'_{\parallel} = \vec{B}_{\parallel} = \vec{0}}$

$$\vec{B}'_\perp = \gamma \left[\vec{B}_\perp - \vec{\beta} \times \frac{\vec{E}}{c} \right]$$

What about inverse transform?

$$\underset{\vec{0}\uparrow}{\vec{B}_\perp} = \gamma \left[\underset{\uparrow}{\vec{B}'_\perp} + \vec{\beta} \times \frac{\vec{E}'}{c} \right]$$

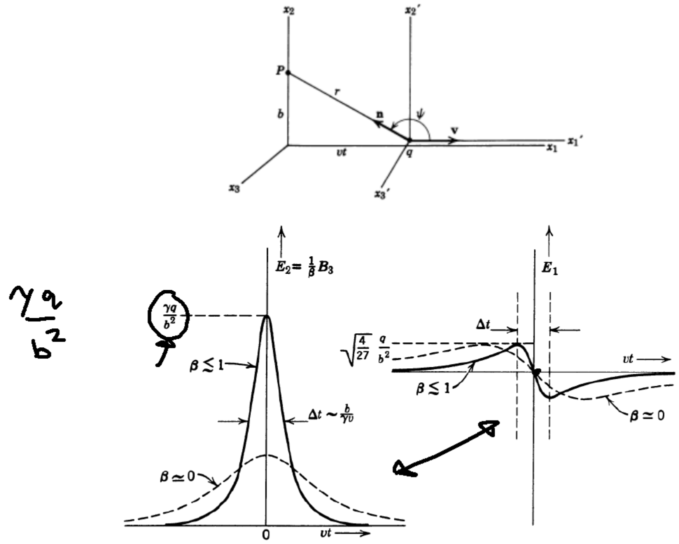
$$\vec{B}'_\perp = -\vec{\beta} \times \frac{\vec{E}'}{c}$$

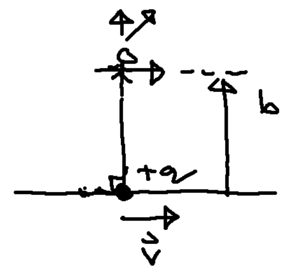
frame 2 wrt frame 1

velocity of point charge in frame 2

$$\boxed{\vec{B}' = \frac{\vec{v}}{c} \times \frac{\vec{E}'}{c}}$$

Electric field due to a moving point charge





Source: Fig.'s 11.8 and 11.9 from J. D. Jackson, *Classical electrodynamics*, 3rd ed (Wiley, New York, 1999).

12 2022-02-15 Lecture

See "Maxwell's equations in a manifestly covariant form" in Phys 442 notes.

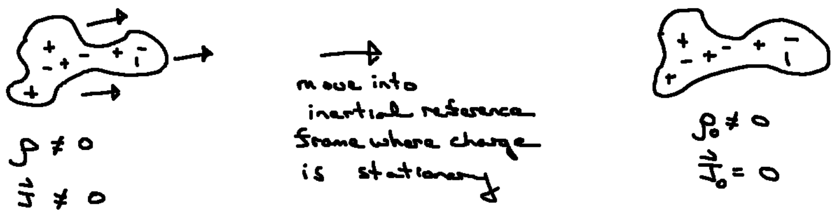
See Section 22.7.2 of A. Zangwill, *Modern electrodynamics* (Cambridge University Press, Cambridge, 2013).

As an aside at the start of the lecture I mentioned a modern application of stellar aberration: the detection of acceleration of the solar system by the Gaia astrometry satellite mission: Gaia Collaboration et al., "Gaia Early Data Release 3: Acceleration of the solar system from Gaia astrometry", A&A **649**, A9 (2021).

1

Maxwell's equations in manifestly covariant form

First : justify that $(\rho c, \vec{J})$ is a four-vector.



charge is conserved between inertial reference frames (total charge, point charge, etc...) but charge density is not. i.e. length contraction along direction of relative frame motion.

2

$$\rho = \frac{1}{\sqrt{1-(v/c)^2}}\,\rho_0$$

$$\underbrace{\vec{J}}_{\text{frame in which charge is moving.}} = \frac{1}{\sqrt{1-(v/c)^2}}\,\vec{v}\rho_0$$

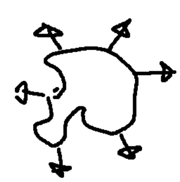
$$(\rho c, \vec{J}) = (\rho_0 c\gamma, \vec{v}\rho_0)$$

$$\underbrace{\phantom{(\rho c, \vec{J})}}_{\text{four-vector.}} = \rho_0\,\underbrace{(c\gamma, \gamma\vec{v})}_{\text{proper-velocity four-vector.}}$$

Aside : charge continuity is Lorentz covariant.

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \quad *$$

$$\boxed{\partial_\beta J^\beta = 0} \qquad \frac{\partial}{\partial(\cancel{c}t)}\rho\cancel{c} + \overbrace{\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}}^{\nabla\cdot\vec{J}} = 0$$



③

$$\left.\begin{array}{l}\nabla\cdot\vec{E} = \rho/\epsilon_0 \;,\quad \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \;\;*\\[2ex] \nabla\cdot\vec{B} = 0 \;\;* \;,\quad \nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\dfrac{\partial\vec{E}}{\partial t}\end{array}\right\}$$

$$\underbrace{\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}}_{①} \;,\quad \underbrace{\vec{B} = \nabla\times\vec{A}}_{②} \quad \Big\}$$ from Lorentz force law.

② $\Rightarrow$ $\nabla\cdot\vec{B} = 0$ satisfied automatically $\left(\nabla\cdot(\nabla\times\vec{A}) = 0\right)$ (any vector field)

① $\Rightarrow$

$$\nabla\times(\vec{E}) = \nabla\times\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\right)$$
$$= \underbrace{-\nabla\times(\nabla\varphi)}_{=0} - \frac{\partial}{\partial t}\underbrace{(\nabla\times\vec{A})}_{\vec{B}} \text{ by ②}$$
$$= -\frac{\partial\vec{B}}{\partial t}$$

④

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$
$$\nabla\cdot\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\right) = \rho/\epsilon_0$$
$$\boxed{-\nabla^2\varphi - \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = \rho/\epsilon_0} \quad 1 \qquad (*)$$
$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$
$$\underbrace{\nabla\times(\nabla\times\vec{A})} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\right)$$
$$\boxed{\underset{①}{\nabla(\nabla\cdot\vec{A})} - \underset{②}{\nabla^2\vec{A}} + \underset{③}{\frac{1}{c^2}\nabla\frac{\partial}{\partial t}\varphi} + \underset{④}{\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}} = \mu_0\vec{J}} \quad 3$$

(5)

Recall the concept of gauge.

$\vec{A}$ and φ are not uniquely determined by $\vec{E}$ and $\vec{B}$.

$$\left.\begin{aligned} \vec{A}' &:= \vec{A} + \nabla g \quad \text{any field} \\ \varphi' &:= \varphi - \frac{\partial g}{\partial t} \end{aligned}\right\} \text{gauge transformation.}$$

$$\vec{B}' = \nabla \times (\vec{A} + \nabla g) = \nabla \times \vec{A} = \vec{B}$$

$$\vec{E}' = -\nabla\varphi' - \frac{\partial \vec{A}'}{\partial t} = -\nabla\left(\varphi - \frac{\partial g}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \nabla g\right)$$

$$= -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} = \vec{E}$$

The gauge transformation allows us to set $\underbrace{\nabla \cdot \vec{A}}$ arbitrarily.

(6)

$$\nabla \cdot (\vec{A} + \nabla g) = t$$

$$\nabla^2 g = \underbrace{t - \nabla \cdot \vec{A}}_{\text{source term}}$$

"Just" need to solve for gauge transformation.

Choose $\nabla \cdot \vec{A}$ so that terms (1) and (3) cancel.

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t} \quad \Big\} \text{ "Lorenz gauge"}$$

So that $\boxed{-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}}$ *

Also (*) becomes:

$$\boxed{-\nabla^2 \varphi + \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \rho/\epsilon_0} \quad *$$

Both expressions involve $-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$

7

$$\partial_\alpha \equiv \left(\frac{\partial}{\partial(ct)}, \vec{\nabla}\right)$$

$$\partial_\alpha \partial^\alpha = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$A^\alpha := (\varphi/c, \vec{A}), \quad J^\beta := (\rho c, \vec{J})$$

$$\boxed{-\partial_\alpha \partial^\alpha A^\beta = \mu_0 J^\beta}$$ Maxwell's equations in Lorentz gauge

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\varphi}{c} = \mu_0 \rho c$$

$$-\left(\nabla^2 - \frac{1}{c}\frac{\partial^2}{\partial t^2}\right)\varphi = \cancel{\mu_0}\rho \frac{1}{\cancel{\mu_0}\epsilon_0}$$

13 2022-02-17 Lecture

Pre-lecture: We will discuss how Maxwell's equations can be derived from an action. I will follow Sections 27 and 30 of L&L [4] (adopted to SI units and the other conventions of our course). Also see Section 24.3 of Zangwill [2].

Post-lecture: At the start of the lecture I showed a paper by Hughes *et al.* [6] that put experimental bounds on the difference between the magnitude of the charge on an electron and a proton. This experiment provides indirect evidence of the Lorentz invariance of charge.

①

Maxwell's equations from an invariant action (following LBL)

For a single particle: $S = -mc^2 \int d\tau + q \int A_\alpha dx^\alpha + ?$

$$S = \underbrace{-mc^2 \int d\tau + q \int A_\alpha dx^\alpha}_{\text{gave Lorentz force law}} + \underbrace{q\int A_\alpha dx^\alpha + ?}_{\text{will give Maxwell's equations.}}$$

First write ② using J^α four-vector.

For multiple particles we generalize:

$$S = \sum_{\substack{\text{all}\\ \text{particles}}} \left(-mc^2 \int d\tau + q \int A_\alpha dx^\alpha \right)$$

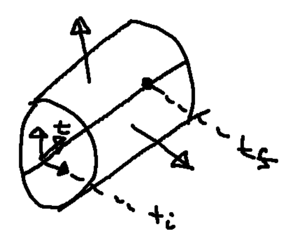
common.

②

Define $S_A :=$ ② in the case of multiple particles.

$$S_A = \sum q \int A_\alpha dx^\alpha$$
$$= \int dV \rho \int A_\alpha dx^\alpha$$
$$= \int dV \rho \int A_\alpha \frac{dx^\alpha}{dt} dt$$

$J^\alpha := \rho \frac{dx^\alpha}{dt}$, so



$$\underbrace{S_A}_{} = \int \underbrace{dV dt}_{\text{Lorentz invariant}} A_\alpha J^\alpha$$

For ③: $\underbrace{S_{EM}}_{\text{TBD.}} = a \int \underbrace{dV dt}_{\text{Lorentz invariant.}} \overbrace{F_{\alpha\beta} F^{\alpha\beta}}^{\text{Lorentz invariant}}$

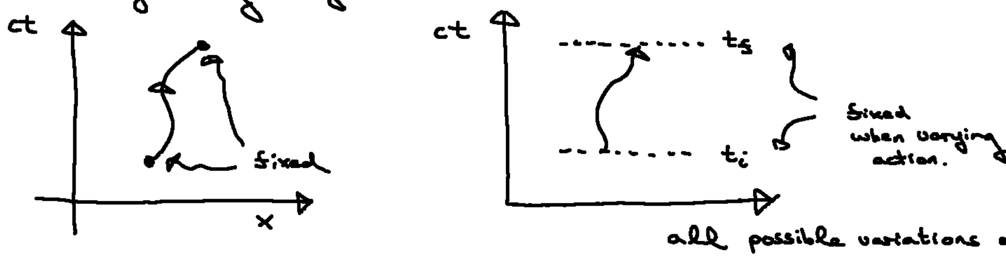
Satisfies two properties:

1) Lorentz invariant.

2) gives superposition.

③

Regarding integrations:



single particle

all possible variations of A_α throughout 3d space.

Vary "trajectory" of the fields: $A_\alpha \to A_\alpha + \lambda \epsilon_\alpha$
(remember $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$)

arb. fcn of space and time but $= 0$ for all space @ t_i and t_f.

Apply principle of stationary action:

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = 0$$

where $S = S_A + S_{EM}$

④

$$\frac{dS_A}{d\lambda} = \frac{d}{d\lambda}\int dV\,dt\,(A_\alpha + \lambda\epsilon_\alpha) J^\alpha$$

$$= \int dV\,dt\,\epsilon_\alpha J^\alpha \qquad (*)$$

$\frac{dS_{EM}}{d\lambda}$ now. First some intermediate results.

$$F_{\alpha\beta}F^{\alpha\beta} = [\partial_\alpha(A_\beta + \lambda\epsilon_\beta) - \partial_\beta(A_\alpha + \lambda\epsilon_\alpha)] \times [\partial^\alpha(A^\beta + \lambda\epsilon^\beta) - \partial^\beta(A^\alpha + \lambda\epsilon^\alpha)]$$

differentiate wrt λ, and then set $\lambda = 0$

$$\left.\frac{d}{d\lambda}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right|_{\lambda=0} = 4\,(\partial_\alpha\epsilon_\beta)F^{\alpha\beta} \qquad (*)$$

Recalling how the Euler-Lagrange eqns arise, we want to factor ϵ_β out

(5)

Use "integration by parts":

$$\partial_\alpha(\epsilon_\beta F^{\alpha\beta}) = (\partial_\alpha \epsilon_\beta)F^{\alpha\beta} + \epsilon_\beta \partial_\alpha F^{\alpha\beta}$$

Rearrange:

$$(\partial_\alpha \epsilon_\beta)F^{\alpha\beta} = \partial_\alpha(\epsilon_\beta F^{\alpha\beta}) - \epsilon_\beta \partial_\alpha F^{\alpha\beta}$$

$$\frac{1}{4a}\frac{dS_{EM}}{d\lambda}\bigg|_{\lambda=0} = \int dV dt\, [\underbrace{\cancel{\partial_\alpha(\epsilon_\beta F^{\alpha\beta})}}_{=0} - \overbrace{\epsilon_\beta \partial_\alpha F^{\alpha\beta}}^{*}]$$

Apply "divergence theorem"

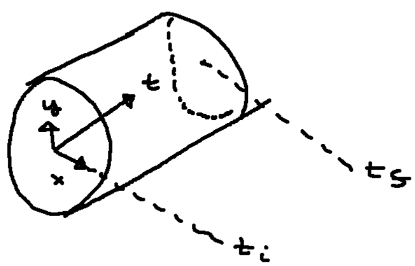
in 3-d $\quad \int_{\mathcal{V}_{3d}} dV\, \nabla\cdot\vec{w} = \oint_{\tau} d\vec{a}\cdot\vec{w}$

in 4-d $\quad \int dV dt\; \partial_\alpha(\epsilon_\beta F^{\alpha\beta}) = \oint dx_\alpha\, \epsilon_\beta F^{\alpha\beta}$

"volume" $\qquad$ "surface"

(6)

To visualize surface integral go to two spatial dimensions.



Assume fields go to zero as $r \to \infty$.

That means curved surface doesn't contribute to integral.

On end-caps $\epsilon_\beta = 0$ over all space.

(7)

$$\left.\frac{dS}{d\lambda}\right|_{\lambda=0} = \int dV dt \left[\epsilon_\alpha J^\alpha - 4a\,\epsilon_\beta \partial_\alpha F^{\alpha\beta}\right]$$

$$\epsilon_\alpha \partial_\beta F^{\beta\alpha}$$

$$= \int dV dt \;\underset{\text{arb.}\Rightarrow}{\epsilon_\alpha} \Big[\underbrace{J^\alpha - 4a\,\partial_\alpha F^{\alpha\beta}}_{=0}\Big]$$

$$\partial_\beta F^{\alpha\beta} = \frac{1}{4a} J^\alpha$$

$\alpha = 0:$

$$\partial_0 \cancel{F^{00}} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} = \frac{1}{4a} J^0$$

$$\frac{\partial}{\partial x}\left(\frac{E_x}{c}\right) + \frac{\partial}{\partial y}\left(\frac{E_y}{c}\right) + \frac{\partial}{\partial z}\left(\frac{E_z}{c}\right) = \frac{1}{4a}\rho c$$

$$\boxed{\nabla\cdot\vec{E} = \frac{1}{4a}\rho c^2} \quad *$$

For our SI system $\frac{c^2}{4a} = \frac{1}{\epsilon_0}$, $a = \frac{\epsilon_0 c^2}{4} = \frac{1}{4\mu_0}$

(8)

$\alpha = 1:$ $\partial_0 F^{10} + \partial_1 \cancel{F^{11}} + \partial_2 F^{12} + \partial_3 F^{13} = J^1/(4a)$

$$\frac{\partial}{\partial(ct)}\left(-\frac{E_x}{c}\right) + \underbrace{\frac{\partial}{\partial y}B_z + \frac{\partial}{\partial z}(-B_y)}_{} = J_x/(4a)$$

Rearrange: $\hat{x}\cdot(\nabla\times\vec{B}) = \frac{J_x}{4a} + \frac{1}{c^2}\frac{\partial E_x}{\partial t}$

$\alpha=2$ and $\alpha=3$ are similar

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad *$$

Remember that $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\nabla\cdot\vec{B} = 0$ $(*)$
just follow from

$$\underbrace{\vec{B} = \nabla\times\vec{A}}_{} \quad \text{and} \quad \underbrace{\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t}}_{}$$

Reading week!

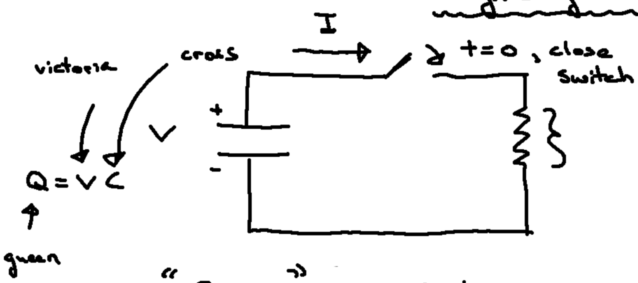
14 2022-03-01 Lecture

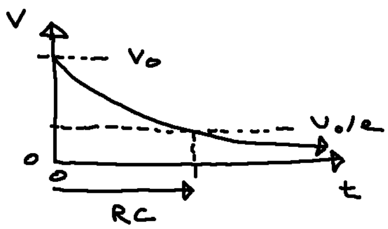
Pre-lecture: Conservation laws will be discussed. See Sections 15.4 and 15.5 of Zangwill [2].

Post-lecture: At the start of the lecture I discussed the updated guidelines on the project that are posted to Learn.

1

Energy conservation in electromagnetism
Poynting's theorem





"Circuit" point of view:

$$C \frac{dV}{dt} = -I = -\frac{V}{R}$$

$$\frac{dV}{dt} = -\frac{1}{RC} V$$

$$V = V_0 e^{-t/(RC)}$$

$$P = VI = \frac{V^2}{R}$$

$$P = \frac{V_0^2}{R} e^{-2t/(RC)}$$

②

Integrate power wrt time to get energy dissipated in resistor (heating).

$$E = \int_0^\infty dt\, P(t)$$

$$= \int_0^\infty dt\, \frac{V_0^2}{R} e^{-2t/(RC)}$$

$$= \frac{V_0^2}{\cancel{R}} \frac{\cancel{R}C}{(-2)} e^{-2t/(RC)} \Big|_{t=0}^{\infty}$$

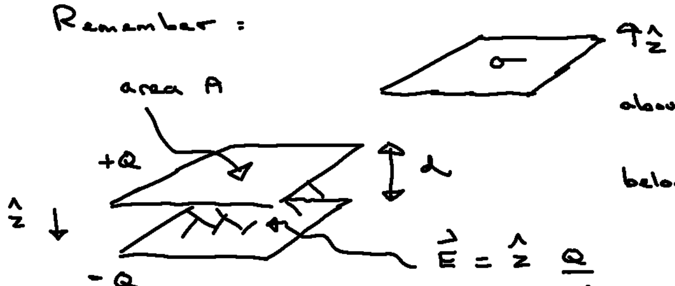
$$\underbrace{E}_{\text{energy}} = \frac{CV_0^2}{2}$$

is there a way to view this energy as being stored in the fields of capacitor (E-field)

③

What is $\vec{E}$ between plates?

Remember:



above: $\vec{E} = \hat{z} \frac{\sigma}{2\epsilon_0}$

below: $\vec{E} = -\hat{z} \frac{\sigma}{2\epsilon_0}$

$$\vec{E} = \hat{z} \frac{Q}{A\epsilon_0}$$

$$V = \int_b^t \vec{E} \cdot d\vec{l} = \frac{Q}{A\epsilon_0} d$$

$$\Rightarrow Q = \frac{A\epsilon_0}{d} V$$

$$\Rightarrow C = \frac{A\epsilon_0}{d}$$

What is $\vec{E}$ in terms of V_0?

$$\vec{E} = \hat{z} \frac{V_0}{d}$$

$$V_0 = d E_z$$

(4)

$$\underset{\text{capacitor}}{\underset{\text{in}}{\text{Energy}}} = \frac{1}{2} C V^2$$

$$= \frac{1}{2} \frac{\epsilon_0 A}{d} (E_z d)^2$$

$$= \frac{1}{2} \epsilon_0 \underbrace{Ad}_{\text{volume between plates}} E_z^2$$

Maybe (?) : E field has an energy density of

$$\frac{1}{2} \epsilon_0 E^2 \leftarrow *$$

Similar argument gives : energy density of $\vec{B}$ field $\frac{1}{2} \frac{B^2}{\mu_0} \leftarrow$

Show these results rigorously.

(5)

Field version of $P = VI$:

$$P = q \vec{v} \cdot \vec{E}$$

earlier we derived $\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \right) = q \vec{v} \cdot \vec{E}$ work-energy theorem.

Generalize to cont. charge distributions :

$$P = \int \underbrace{d\tau}_{\text{volume}} \; \vec{J} \cdot \vec{E}$$

write in terms of $\vec{E}$ and $\vec{B}$ using Maxwell's equations.

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{J} = \frac{\nabla \times \vec{B}}{\mu_0} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$P = \int d\tau \left[\frac{\nabla \times \vec{B}}{\mu_0} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \cdot \vec{E}$$

⑥

$$P = \int d\tau \left[\frac{1}{\mu_0}(\nabla\times\vec{B})\cdot\vec{E} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\cdot\vec{E} \right]$$

Remember general result:

$$\nabla\cdot(\vec{v}\times\vec{w}) = \vec{w}\cdot(\nabla\times\vec{v}) - \vec{v}\cdot(\nabla\times\vec{w})$$

$$-\nabla\cdot(\vec{E}\times\vec{B}) = \vec{E}\cdot(\nabla\times\vec{B}) - \vec{B}\cdot(\nabla\times\vec{E})$$

$$\Rightarrow \quad \vec{E}\cdot(\nabla\times\vec{B}) = -\nabla\cdot(\vec{E}\times\vec{B}) + \vec{B}\cdot(\nabla\times\vec{E})$$

$$= -\nabla\cdot(\vec{E}\times\vec{B}) - \vec{B}\cdot\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday's law}$$

$$P = \int d\tau \left[\underbrace{-\frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})}_{\text{apply div. theorem}} - \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} - \epsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} \right]$$

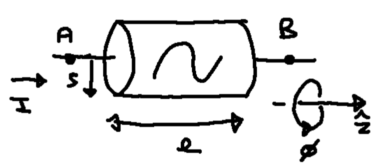
$$= -\oint \underbrace{\frac{1}{\mu_0}(\vec{E}\times\vec{B})}_{=:\ \vec{S}\text{, Poynting vector}}\cdot d\vec{a} - \frac{d}{dt}\int d\tau \underbrace{\left[\frac{B^2}{2\mu_0} + \frac{\epsilon_0 E^2}{2} \right]}_{=:\ u_{EM}}$$

⑦

Poynting's theorem: $\frac{W}{m^2}$ $\quad$ $\frac{J}{m^3}$

$$\underbrace{P}_{①\ \text{power into motion of particles.}} = \underbrace{-\oint \vec{S}\cdot d\vec{a}}_{②\ \text{power flow out of volume}} \underbrace{- \frac{d}{dt}\int d\tau\, u_{EM}}_{③\ \text{change in stored energy of fields.}}$$

Example of Poynting vector usage (③ = 0)



$$U_{AB} = IR$$
$$P = I^2 R$$

Let's interpret from "field" point of view. Steady state $\frac{du_{EM}}{dt} = 0$

8

What are fields on surface of resistor?

Ampere's law : $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$ $\rightarrow$ $2\pi s B_\phi = \mu_0 I$

$$\vec{B} = \hat{\phi}\,\frac{\mu_0 I}{2\pi s}$$

$$\vec{E} = \frac{V}{l}\,\hat{z}$$

$$\frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{V}{l}\,\hat{z}\times\hat{\phi}\,\frac{\mu_0 I}{2\pi s}$$

$$= \frac{V}{l}\,(-\hat{s})\,\frac{\mu_0 I}{2\pi s}$$

Only non-zero contribution to $\oint \vec{S}\cdot d\vec{a}$ is curved outer surface.

$$-\oint \frac{\vec{E}\times\vec{B}}{\mu_0}\cdot d\vec{a} = \frac{V}{\cancel{l}}\,\frac{\cancel{\mu_0} I}{\cancel{2\pi s}}\,\frac{1}{\cancel{\mu_0}}\,\underbrace{\cancel{2\pi s}\,\cancel{l}}_{\text{area}}$$

$$P = VI$$ $\leadsto$ consistent with circuit rule !!!

15 2022-03-03 Lecture

Pre-lecture: Conservation laws will be discussed. See Sections 15.4 and 15.5 of Zangwill [2].

Post-lecture:

①

Momentum conservation and Maxwell's equations

briefly review momentum as it appears in elementary classical mechanics:

$$\vec{P} = \sum \vec{p}_i \quad \leftarrow \text{many particles}$$

$$\frac{d\vec{P}}{dt} = \sum \frac{d}{dt}\vec{p}_i \quad \text{Newton's } 2^{nd} \text{ law}$$

$$= \sum \vec{F}_i$$

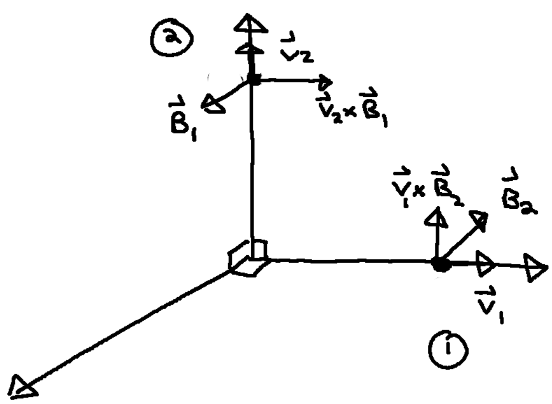
$$= \sum_{ij} \vec{F}_{ij} \quad \text{force on } i^{th} \text{ particle due to } j^{th} \text{ particle}$$

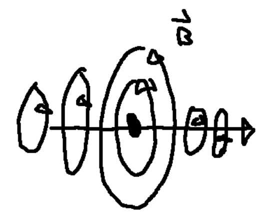
Newton's 3^{rd} law $\quad \vec{F}_{ij} = -\vec{F}_{ji}$ *

$$\frac{d\vec{P}}{dt} = \vec{0} \quad \text{momentum conservation.}$$

This changes with EM fields.

② two point particles in motion





Newton's 3^{rd} law is violated.

Is there a problem with momentum conservation?

No! We must account for momentum in EM field.

③

$$\frac{d\vec{p}}{dt} = \underbrace{q}_{d\tau\,\rho}(\vec{E}+\vec{v}\times\vec{B})$$

generalize to continuous distrib.

$$\frac{d\vec{p}}{dt} = \int d\tau\,[\rho\vec{E}+\vec{J}\times\vec{B}] \qquad (*)$$

Rewrite ρ and $\vec{J}$ using the $\vec{E}$ and $\vec{B}$ fields (like Poynting's theorem).

$$\rho = \epsilon_0\nabla\cdot\vec{E}\ , \quad \vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\frac{d\vec{p}}{dt} = \int d\tau\left[\epsilon_0(\nabla\cdot\vec{E})\vec{E} + \left(\frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]$$

$$(\nabla\cdot\vec{B})\vec{B} = 0 \qquad\qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

④

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \underbrace{\vec{E}\times\frac{\partial\vec{B}}{\partial t}}_{-\vec{E}\times(\nabla\times\vec{E})}$$

$$\frac{d\vec{p}}{dt} = \int d\tau\Big[\epsilon_0(\nabla\cdot\vec{E})\vec{E} - \epsilon_0\vec{E}\times(\nabla\times\vec{E}) + \frac{1}{\mu_0}(\nabla\cdot\vec{B})\vec{B} - \frac{1}{\mu_0}\vec{B}\times(\nabla\times\vec{B}) - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\Big]$$

For arbitrary $\vec{A}$ and $\vec{B}$:

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} \quad *$$

$\vec{v} = \vec{A} = \vec{B}$ $\quad \nabla(v^2) = 2\vec{v}\times(\nabla\times\vec{v}) + 2(\vec{v}\cdot\nabla)\vec{v}$

$$\Rightarrow \vec{v}\times(\nabla\times\vec{v}) = \frac{1}{2}\nabla(v^2) - (\vec{v}\cdot\nabla)\vec{v}$$

(5)

$$\int d\tau \left[(\nabla\cdot\vec{v})\vec{v} + (\vec{v}\cdot\nabla)\vec{v} - \frac{1}{2}\nabla(v^2)\right] =: ?$$

Consider components :

$$\hat{x}\cdot ? = \int d\tau \left[\underbrace{(\nabla\cdot\vec{v})v_x}_{①} + \underbrace{(\vec{v}\cdot\nabla)v_x}_{\substack{(v_x\partial_x + v_y\partial_y + v_z\partial_z)v_x \\ \vec{v}\cdot\nabla v_x}} - \underbrace{\frac{1}{2}\hat{x}\cdot\nabla(v^2)}_{-\frac{1}{2}\nabla\cdot(\hat{x}v^2)}\right]$$

For general f and $\vec{A}$: $\underbrace{\nabla\cdot(f\vec{A}) = \overset{②}{\nabla f\cdot\vec{A}} + f\overset{*}{\nabla\cdot\vec{A}}}$

$$\hat{x}\cdot ? = \int d\tau \left[\nabla\cdot(v_x\vec{v}) - \frac{1}{2}\nabla\cdot(\hat{x}v^2)\right]$$
$$= \int d\tau\ \nabla\cdot\left(v_x\vec{v} - \frac{1}{2}\hat{x}v^2\right) \quad *$$

Apply divergence theorem : $\int d\tau\ \nabla\cdot\vec{A} = \oint \vec{A}\cdot d\vec{a}$ $\quad *$

(6)

$$\left.\begin{aligned}\hat{x}\cdot\frac{d\vec{p}}{dt} = \epsilon_0 \oint \left(E_x\vec{E} - \frac{1}{2}E^2\hat{x}\right)\cdot d\vec{a} \\ + \frac{1}{\mu_0}\oint\left(B_x\vec{B} - \frac{1}{2}B^2\hat{x}\right)\cdot d\vec{a}\end{aligned}\right\}$$
$$- \int d\tau\ \hat{x}\cdot\epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\frac{d\vec{p}}{dt} = \underbrace{\hat{x}_i \oint T_{ij}\, da_j}_{\text{implicit summation}} - \int d\tau\ \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \qquad (*)$$

Maxwell's Stress tensor. $\rightarrow$ $T_{ij} := \epsilon_0\left(E_iE_j - \frac{1}{2}E^2\delta_{ij}\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}B^2\delta_{ij}\right)$

$$\vec{0} = \underbrace{\frac{d\vec{p}}{dt}}_{\text{changing momentum of particles}} + \underbrace{\int d\tau\ \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})}_{\text{changing momentum of field}} - \underbrace{\oint \overleftrightarrow{T}\cdot d\vec{a}}_{\text{momentum flow out of region.}} \qquad (*)$$

16 2022-03-08 Lecture

Pre-lecture:

Post-lecture:

①

Electromagnetic Plane Waves

$$\nabla\cdot\vec{E} = 0 \quad (1) \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (2)$$

$$\nabla\cdot\vec{B} = 0 \quad (3) \qquad \nabla\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \quad (4)$$

Take $\nabla\times$ of both sides of Eq. (2):

$$\nabla\times(\nabla\times\vec{E}) = \nabla\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

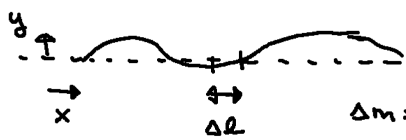
$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\nabla\times\vec{B})$$

$$\boxed{\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{E}}$$

also

$$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{B}$$

② string under tension : T



linear mass density: λ

$\Delta m = \lambda\,\Delta l$

$$\lambda\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}$$

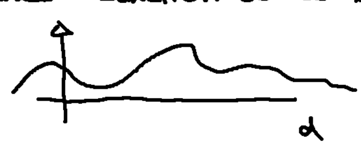
also a wave equation!

From Griffiths' Intro to Electrodynamics:

"a wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity"

Consider any twice differentiable function of a single variable : $f(d)$

$x - vt$



(3)

$$y = f(x - vt) \quad ?$$ is a solution to wave eqn.

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial t}$$

$$= \frac{\partial f}{\partial \alpha}(-v)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 f}{\partial \alpha^2}(-v)^2$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial \alpha^2}$$

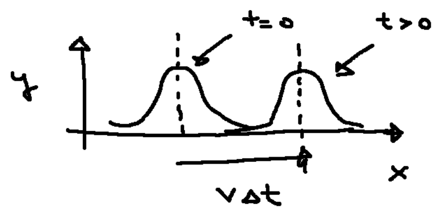
$$\frac{\partial^2 y}{\partial t^2}\frac{1}{v^2} = \frac{\partial^2 y}{\partial x^2}$$

$$\lambda \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

Compare to wave eqn. It will be satisfied if

$$v = \pm\sqrt{\frac{T}{\lambda}}$$

(4)



Another way to view solving the wave equation (Landau + Lifschitz)

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$$

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) y = 0 \qquad (*)$$

define two new coordinates:

$$\xi := t - \frac{x}{v}, \quad \eta := t + \frac{x}{v}$$

Rewrite (*), in terms of new coordinates.

⑤

$$f(t,x) = f\left(\tfrac{1}{2}(\xi+\eta), \tfrac{1}{2}(\eta-\xi)\right)$$

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial t}\frac{1}{2} + \frac{\partial f}{\partial x}\left(-\frac{1}{2}\right)$$

$$\frac{\partial}{\partial \xi} = \frac{1}{2}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial \eta} = \frac{1}{2}\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}$$

(*) can be written as

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} y(\xi,\eta) = 0$$

There must be solutions of the form

$$y = y_1(\xi) + y_2(\eta)$$
$$= y_1\left(t - \frac{x}{v}\right) + y_2\left(t - \frac{x}{v}\right)$$

arb. apart from being differentiable.

⑥

return to electromagnetic waves.

$$\vec{E} = \vec{E}_0 \cos(kz - \omega t) \quad \text{satisfies}$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \underbrace{\frac{\partial^2 \vec{E}}{\partial z^2}}_{\nabla^2 \vec{E}}$$

$$\vec{E} = \vec{E}_0 \cos\left(k\left(z - \frac{\omega}{k} t\right)\right)$$

Solution of wave eqn. if $\frac{\omega}{k} = c$

$$= \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

(7)

Use complex numbers to simplify algebra when time dependence is sinusoidal.

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(kz-\omega t)} \qquad (*)$$

complex number.

similarly for $\vec{B}$.

Meaning? We should interpret the real part of $\vec{E}(\vec{r},t)$ as being the actual field.

Recall for any complex number z:

$$Re(z) = (z + z^*)/2$$

Thus suppose $(*)$ and its complex conjugate satisfy Maxwell's eqns, then so does real part.

Simplifies algebra!!!

(8)

Double check that $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$ satisfies wave equation $\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

$$\vec{E}_0 (ik)^2 \cancel{e^{i(kz-\omega t)}} = \mu_0 \epsilon_0 (-i\omega)^2 \cancel{e^{i(kz-\omega t)}}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad (\text{"c"})$$

Apply $\nabla \cdot \vec{E} = 0$

$$\nabla \cdot \left(\vec{E}_0 e^{i(kz-\omega t)}\right) = 0$$

Remember identity $\nabla \cdot (f\vec{v}) = \nabla f \cdot \vec{v} + f\, \nabla \cdot \vec{v}$

(9)

$$\nabla \cdot \left(\vec{E}_0 e^{i(kz-\omega t)} \right) = \vec{E}_0 \cdot \nabla e^{i(kz-\omega t)}$$

$$= \vec{E}_0 \cdot \hat{z}\,(ik)\, e^{i(kz-\omega t)}$$

$\boxed{\vec{E}_0 \cdot \hat{z} = 0}$. Similarly $\boxed{\vec{B}_0 \cdot \hat{z} = 0}$

What is relationship between $\vec{E}$ and $\vec{B}$?

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (*)$$

$$\nabla \times \left(\vec{E}_0 e^{i(kz-\omega t)} \right) = -\frac{\partial}{\partial t}\left(\vec{B}_0 e^{i(kz-\omega t)} \right)$$

$$\nabla \times (f\vec{v}) = \nabla f \times \vec{v} + f\,\nabla \times \vec{v}$$

$$\cancel{i} k \hat{z}\, \cancel{e^{i(kz-\omega t)}} \times \vec{E}_0 = \vec{B}_0 (\cancel{i}\omega t)\, \cancel{e^{i(kz-\omega t)}}$$

$$\vec{B}_0 = \frac{1}{c}\,\hat{z} \times \vec{E}_0 \qquad (*)$$

(10)

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{E}_0 = -\hat{z} \times c\vec{B}_0 \qquad (\ddagger)$$

So are $(*)$ and $(\ddagger)$ consistant ?

$$\hat{z} \times \vec{E}_0 = \hat{z} \times (-\hat{z} \times c\vec{B}_0) \qquad \vec{a} \times (\vec{b} \times \vec{c})$$

$$= -\left[\hat{z}\,\underbrace{(\hat{z} \cdot c\vec{B}_0)}_{=0 \text{ from } \nabla \cdot \vec{B} = 0} - c\vec{B}_0\,\overbrace{(\hat{z} \cdot \hat{z})}^{=1} \right] \qquad = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$= c\vec{B}_0$$

I've just shown $(*)$ from $(\ddagger)$. Vice-Versa also holds.

11

Summarize plane EM solutions

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{B} = \frac{\vec{k} \times \vec{E}_0}{c} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{E}_0 \cdot \hat{k} = 0 \qquad \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

17 2022-03-10 Lecture

Pre-lecture:

Will be discussing energy and momentum conservation in the context of plane electromagnetic waves. See Section 16.3 of Zangwill [2].

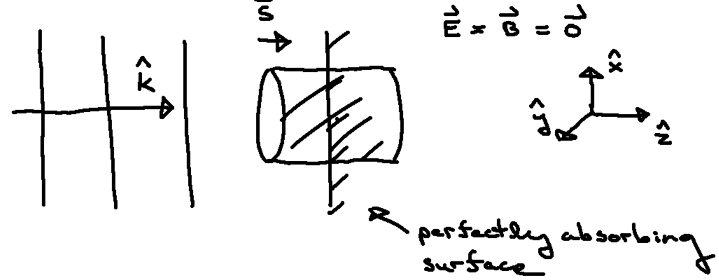
Post-lecture:

At the start of lecture (before video recording starts), I discussed the similarity between the use of $\nabla \cdot \mathbf{E} = 0$ on Question 1 of Problem Set 7 (Exercise 2.10 of Brau [7]) and the usage of $\nabla \cdot \mathbf{B} = 0$ to deduce the "transverse" gradients at the centre of a magneto-optical trap (using Fig. 9.8 of Foot [8]).

(1)

Energy and momentum transport by electromagnetic waves

$\vec{E} \times \vec{B} = \vec{0}$



$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$

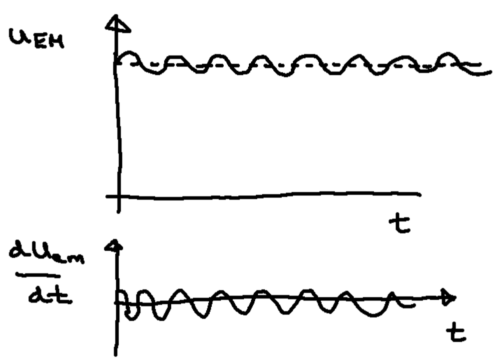
$$\vec{B} = \hat{k} \times \frac{\vec{E}_0}{c} \cos(\vec{k}\cdot\vec{r} - \omega t)$$

linearly polarized

(2)

Apply Poynting's theorem:

$$\underset{\text{heating}}{P} = -\frac{d}{dt}\int dV \underbrace{\left[\frac{\epsilon_0 E^2}{2} + \frac{1}{\mu_0}\frac{B^2}{2}\right]}_{=:\, u_{EM}} - \frac{1}{\mu_0}\oint (\vec{E}\times\vec{B})\cdot d\vec{a}$$



$$\left\langle \frac{dU_{EM}}{dt} \right\rangle_t = 0$$

time average

Only need to consider the Poynting vector term.

(3)

$$\vec S = \frac{1}{\mu_0}\,\vec E\times\vec B$$

$$= \frac{1}{\mu_0}\,\vec E_0\cos(\vec k\cdot\vec r-\omega t)\times\left(\hat k\times\frac{\vec E_0}{c}\right)\cos(\vec k\cdot\vec r-\omega t)$$

$$\langle\vec S\rangle_t = \frac{1}{\mu_0 c}\,\vec E_0\times(\hat k\times\vec E_0)\,\langle\cos^2(\vec k\cdot\vec r-\omega t)\rangle_t = \frac{1}{2}$$

$\vec a\times(\vec b\times\vec c) = \vec b(\vec a\cdot\vec c)-\vec c(\vec a\cdot\vec b)$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$$= \frac{1}{2\mu_0 c}\,\hat k\,(E_0^2-\underbrace{\vec E_0\cdot\hat k}_{\nabla\cdot\vec E=0\ \Rightarrow\ \vec k\cdot\vec E_0=0})$$

$$\langle S\rangle_t = \frac{E_0^2}{2\mu_0 c}\,\hat k$$

(4)

Putting into Poynting's theorem:

$$\langle P\rangle_t = A\,\frac{E_0^2}{2\mu_0 c}$$

For a plane electromagnetic wave, linearly polarized, with amplitude E_0, we associate a power / unit area:
time averaged.

$$\boxed{\frac{P}{A} = \frac{E_0^2}{2\mu_0 c}} \quad (*)$$

intensity

What units does $\mu_0 c$ have?

$$\frac{N}{A^2}\frac{m}{s} = \frac{J}{\cancel{m}AC}\frac{\cancel{m}}{\cancel{s}} = \frac{V}{A} = \Omega$$

$\mu_0 c \simeq 377\,\Omega$ → "impedance of free space"

⑤

Now consider momentum conservation.

$$\frac{d\vec{p}}{dt} = -\oint \overleftrightarrow{T}\cdot d\vec{a} + \frac{d}{dt}\int d\tau\ \epsilon_0 (\vec{E}\times\vec{B})$$

By similar argument to the energy case, $=0$ when time averaged.

$$T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}E^2\delta_{ij}\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}B^2\delta_{ij}\right)$$

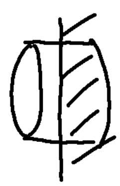
$$\vec{E} = \hat{x}\ E_0\cos(kz-\omega t)$$
$$\vec{B} = \hat{y}\ \frac{E_0}{c}\cos(kz-\omega t)$$

$$\langle\overleftrightarrow{T}\rangle = \frac{1}{2}\left\{\frac{\epsilon_0 E_0^2}{2}\begin{bmatrix}1 & & \\ & -1 & \\ & & -1\end{bmatrix} + \frac{1}{2\mu_0}\left(\frac{E_0}{c}\right)^2\begin{bmatrix}-1 & & \\ & 1 & \\ & & -1\end{bmatrix}\right\}$$

time averaging $\cos^2$ $\qquad\qquad$ $\frac{1}{\mu_0\epsilon_0} = c^2$

⑥

$$\langle\overleftrightarrow{T}\rangle = \frac{1}{2}\epsilon_0 E_0^2\begin{bmatrix}0 & & \\ & 0 & \\ & & -1\end{bmatrix}$$



$$-\oint \overleftrightarrow{T}\cdot d\vec{a}$$

of

$$\boxed{\frac{1}{A}\left\langle\frac{d\vec{p}}{dt}\right\rangle = \hat{z}\ \frac{E_0^2}{2\mu_0 c^2}} \quad (*)$$

$$\left\langle\frac{d\vec{p}}{dt}\right\rangle = -\oint\overleftrightarrow{T}\cdot d\vec{a}$$
$$= A\ \frac{1}{2}\epsilon_0 E_0^2\ \hat{z} \qquad \epsilon_0 = \frac{1}{\mu_0 c^2}$$

It is natural identify a time-averaged momentum / unit area of a plane, linearly polarized EM wave

(7)

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

$m=0$

$$E = pc$$

$$\hbar\omega = \hbar k c$$

$$c = \frac{\omega}{k}$$

quantum mechanical picture of photons.

18 2022-03-15 Lecture

Pre-lecture:

We will discuss the general solutions of the potentials in the Lorenz gauge (in preparation for discussing radiation). See section 20.2 of Zangwill [2].

Post-lecture:

I followed the approach in Franklin [9].

If you need to review Cauchy's Residue Theorem, see Section 14.6 of Nearing [10] (available for download at: http://www.physics.miami.edu/~nearing/mathmethods/).

(1)

Solutions of the wave equations with sources

Maxwell's equations

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\nabla\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t} \quad \text{Lorenz gauge} \quad *$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -\rho/\epsilon_0$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

$$\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$$

$$F^{\alpha\beta} = \underbrace{\partial^\alpha A^\beta - \partial^\beta A^\alpha}$$

$$\partial_\beta A^\beta = 0$$

$$\partial_\beta \partial^\beta A^\alpha = -\mu_0 J^\alpha$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = f$$

(2)

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\Phi = -\rho/\epsilon_0 \qquad *$$

time indep. $\quad \nabla^2\Phi = -\rho/\epsilon_0$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int dv' \, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \leftarrow r$$

+ advanced time

retarded time

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int dv' \, \frac{\rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|} \leftarrow \text{guess?}$$

similar expression for $\vec{A}$ (consider $\vec{J}$ at retarded time)

Retarded solutions can be verified by substitution into wave equation.

③

Take constructive approach to generating solutions using Fourier transforms.

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\vec{J}(\vec{r},t) = \vec{J}_\omega(\vec{r})\, e^{-i\omega t}$$
$$\vec{A}(\vec{r},t) = \vec{A}_\omega(\vec{r})\, e^{-i\omega t}$$

Fourier decomposition will handle arbitrary time-dependence.

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{A}_\omega(\vec{r}) = -\mu_0 \vec{J}_\omega(\vec{r})$$

Now use spatial Fourier transforms.

$$\vec{A}_\omega(\vec{r}) = \int d^3k\; e^{i\vec{k}\cdot\vec{r}}\,\vec{A}_{\omega,\vec{k}}\,, \qquad \vec{A}_{\omega,\vec{k}} = \frac{1}{(2\pi)^3}\int d^3r\; e^{-i\vec{k}\cdot\vec{r}}\,\vec{A}_\omega(\vec{r})$$

Similarly for $\vec{J}$.

④

$$\left(-k^2 + \frac{\omega^2}{c^2}\right)\vec{A}_{\omega,k} = -\mu_0 \vec{J}_{\omega,k}$$

$$\vec{A}_{\omega,k} = \frac{\mu_0 \vec{J}_{\omega,k}}{k^2 - \left(\frac{\omega}{c}\right)^2} \leftarrow \frac{1}{(2\pi)^3}\int dV\, e^{-i\vec{k}\cdot\vec{r}}\,\vec{J}_\omega(\vec{r})$$

move back to real space.

$$\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{(2\pi)^3}\int d^3k\,\frac{1}{k^2-\left(\frac{\omega}{c}\right)^2}\int dV'\; e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\,\vec{J}_\omega(\vec{r}\,')$$

Rearrange orders of integration :

$$\vec{A}_\omega(\vec{r}) = \frac{\mu_0}{(2\pi)^3}\int dV'\int d^3k\;\frac{\vec{J}_\omega(\vec{r}\,')}{k^2-\left(\frac{\omega}{c}\right)^2}\; e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}$$

$$\vec{A}_\omega(\vec{r}) = \mu_0\int dV'\; G_\omega(\vec{r}-\vec{r}\,')\,\vec{J}_\omega(\vec{r}\,') \qquad (*)$$

(5)

$$G(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3} \underbrace{\int d^3k} \; \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{k^2-\left(\frac{\omega}{c}\right)^2}$$

Use spherical coordinates in k-space with $\vec{r}-\vec{r}\,'$ being the "z" axis.

$$\vec{k}\cdot(\vec{r}-\vec{r}\,') = k\,|\vec{r}-\vec{r}\,'|\cos\theta$$

$$G(\vec{r}-\vec{r}\,') = \frac{1}{(2\pi)^3} \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta}_{2\pi} \int_0^{\infty} dk\, k^2 \; \frac{e^{ik|\vec{r}-\vec{r}\,'|\cos\theta}}{k^2-\left(\frac{\omega}{c}\right)^2}$$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} dk\, k^2 \; \frac{e^{ik|\vec{r}-\vec{r}\,'|\cos\theta}}{k^2-\left(\frac{\omega}{c}\right)^2} \; \frac{-1}{ik\,|\vec{r}-\vec{r}\,'|}\Bigg|_0^{\pi}$$

$$\int_0^{\infty} dk \downarrow$$

$$= \frac{1}{(2\pi)^2} \frac{k}{\left[k^2-\left(\frac{\omega}{c}\right)^2\right]} \left(\frac{-1}{i}\right) \left[e^{-ik|\vec{r}-\vec{r}\,'|} - e^{ik|\vec{r}-\vec{r}\,'|} \right]$$

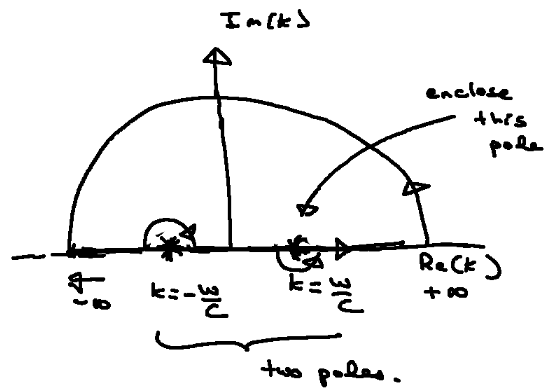
(6)

$$G(\vec{r}-\vec{r}\,') = \frac{2}{(2\pi)^2} \frac{1}{|\vec{r}-\vec{r}\,'|} \underbrace{\int_0^{\infty} dk \; \frac{k \sin(k|\vec{r}-\vec{r}\,'|)}{\underbrace{k^2-\left(\frac{\omega}{c}\right)^2}}}_{I} \quad *$$

$$I = \frac{1}{2} \int_{-\infty}^{\infty} dk \; \frac{k \sin(k|\vec{r}-\vec{r}\,'|)}{k^2-\left(\frac{\omega}{c}\right)^2}$$

$$= \frac{1}{2i} \int_{-\infty}^{\infty} dk \; k \; \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{k^2-\left(\frac{\omega}{c}\right)^2}$$

$$= \frac{1}{2i} \int_{-\infty}^{\infty} dk \; \frac{k\, e^{ik|\vec{r}-\vec{r}\,'|}}{\left(k-\frac{\omega}{c}\right)\left(k+\frac{\omega}{c}\right)}$$



(7)

Pole avoidance based on getting retarded time solution. Use Cauchy's residue theorem.

$$\frac{1}{2i}\int_{-\infty}^{\infty} dk \, \frac{k \, e^{ik|\vec{r}-\vec{r}'|}}{(k-\frac{\omega}{c})\underbrace{(k+\frac{\omega}{c})}} = \frac{2\pi i}{2i} \, \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{2\cancel{\frac{\omega}{c}}} \, \cancel{\frac{\omega}{c}}$$

$$= \frac{\pi}{2} \, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}$$

$$G(\vec{r}-\vec{r}') = \frac{1}{4\pi} \, \frac{1}{|\vec{r}-\vec{r}'|} \, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \, \vec{A}_\omega(\vec{r})$$

$$\vec{J}_\omega(\vec{r}') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \, \vec{J}(\vec{r},t') \, e^{i\omega t}$$

(8)

$$\vec{A}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \, \frac{\mu_0}{4\pi}\int dV' \, \frac{1}{|\vec{r}-\vec{r}'|} \, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}$$

$$\frac{1}{2\pi}\int dt \, J(\vec{r}',t') \, e^{i\omega t'}$$

$$= \frac{1}{2\pi} \, \frac{\mu_0}{4\pi}\int dV' \, \frac{1}{|\vec{r}-\vec{r}'|}\int dt' \, J(\vec{r}',t') \underbrace{\int d\omega \, e^{i\omega(t'-t+|\frac{\vec{r}-\vec{r}'}{c}|)}}$$

$$* \int dk \, e^{ikx} = 2\pi\,\delta(x)$$

$$\rightarrow \underline{2\pi} \, \delta\left(t'-t+|\frac{\vec{r}-\vec{r}'}{c}|\right)$$

$$\int dx \, \delta(x-a) f(x) = f(a)$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int dV' \, \frac{\vec{J}\left(\vec{r}', t-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}}$$

19 2022-03-17 Lecture

Pre-lecture:

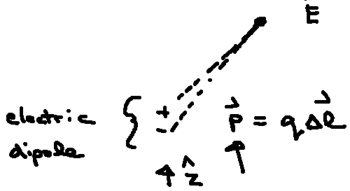
I will start to discuss electric and magnetic dipole radiation.

For electric dipole radiation, see Section 20.4 and 20.5 of Zangwill [2].

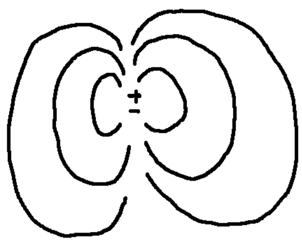
Post-lecture:

1

Electric dipole radiation

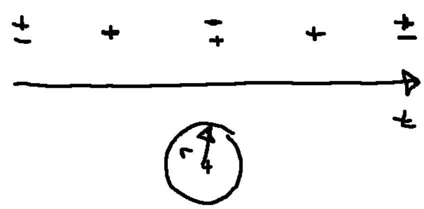


$$\vec E_{dipole} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat r + \sin\theta\,\hat\theta\right)$$



What about time-dependent electric-dipole moments?

$$\to \nabla\cdot\vec J = -\frac{\partial\rho}{\partial t}$$



$E \propto \frac{1}{r}$, $B \propto \frac{1}{r}$ } radiation fields carry energy off to ∞.

2

$$\underbrace{\vec J(\vec r,t)}_{\frac{C}{m^2\cdot s}} = \underbrace{\vec j_0(t)}_{\frac{C\cdot m}{s}\ =\ \dot{\vec p}(t)}\ \underbrace{\delta^3(\vec r)}_{\frac{1}{m^3}} \qquad \frac{C}{m^2\cdot s}\,m^3 = \frac{C\cdot m}{s}$$

$$\rho = -\nabla\cdot(\vec p\,\delta^3(\vec r))$$

dipole moment (not charge density)

gives consistency with $\nabla\cdot\vec J = -\frac{\partial\rho}{\partial t}$

$$\vec A(\vec r,t) = \frac{\mu_0}{4\pi}\int dv'\,\frac{\dot{\vec p}(t-\mathscr{r}/c)\,\delta^3(\vec r')}{\mathscr{r}} \qquad |\vec r-\vec r'|$$

$$= \frac{\mu_0}{4\pi}\,\frac{\dot{\vec p}(t-r/c)}{r}$$

③

To get the scalar potential we could use general solution. Instead use Lorenz gauge condition:

$$\frac{\partial \Phi}{\partial t} = -c^2 \nabla \cdot \vec{A} \qquad c^2 = \frac{1}{\mu_0 \epsilon_0}$$

and integrate.

$$\frac{\partial \Phi}{\partial t} = -c^2 \frac{\mu_0}{4\pi} \nabla \cdot \left[\dot{\vec{p}}\left(t - \frac{r}{c}\right)\frac{1}{r}\right] \qquad \nabla\cdot(f\vec{v}) = \nabla f \cdot \vec{v} + f \nabla \cdot \vec{v}$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\nabla \cdot \dot{\vec{p}}(t - r/c)\frac{1}{r} + \dot{\vec{p}}\left(t - \frac{r}{c}\right) \cdot \nabla \frac{1}{r}\right]$$

$$= \frac{-1}{4\pi\epsilon_0}\left[\ddot{\vec{p}}(t - r/c) \cdot -\frac{1}{c}\frac{\hat{r}}{r} + \dot{\vec{p}}\left(t - \frac{r}{c}\right) \cdot \frac{-\hat{r}}{r^2}\right]$$

④

$$\frac{\partial \varphi}{\partial t} = \frac{\hat{r}\cdot}{4\pi\epsilon_0}\left[\frac{\ddot{\vec{p}}(t - r/c)}{cr} + \frac{\dot{\vec{p}}(t - r/c)}{r^2}\right]$$

$$\varphi = \frac{\hat{r}}{4\pi\epsilon_0} \cdot \left[\frac{\overbrace{\dot{\vec{p}}(t - r/c)}^{\dot{\vec{p}}_{ret}}}{cr} + \underbrace{\frac{\vec{p}(t - r/c)}{r^2}}_{\text{static electric dipole term}}\right] + \text{static indep. field.}$$

Compute $\vec{E}$ and $\vec{B}$ by $\vec{B} = \nabla \times \vec{A}$, $\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$

by messy algebra

$$\vec{B} = -\frac{\mu_0}{4\pi}\, \hat{r} \times \left[\frac{\dot{\vec{p}}_{ret}}{r^2} + \frac{\ddot{\vec{p}}_{ret}}{cr}\right]$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{r}(\hat{r}\cdot\vec{p}_{ret}) - \vec{p}_{ret}}{r^3} + \frac{3\hat{r}(\hat{r}\cdot\dot{\vec{p}}_{ret}) - \dot{\vec{p}}_{ret}}{cr^2} + \frac{\hat{r}(\hat{r}\cdot\ddot{\vec{p}}_{ret}) - \ddot{\vec{p}}_{ret}}{c^2 r}\right]$$

Static dipole

radiation fields

⑤

The $\vec{p}_{ret}$ terms are "displacement like"
" $\dot{\vec{p}}_{ret}$ " "velocity like"
$\ddot{\vec{p}}_{ret}$ " "acceleration like"

It is the acceleration-like terms that correspond to radiation since they scale like $\frac{1}{r}$.

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$P_{radiated} = \oint \vec{S} \cdot d\vec{a} \propto \underset{E}{\frac{1}{r}} \; \underset{B}{\frac{1}{r}} \; \underset{A}{r^2}$$

} power radiated away

Evaluate $P_{radiated}$ and $\frac{dP_{radiated}}{d\Omega}$ next lecture.

20 2022-03-22 Lecture

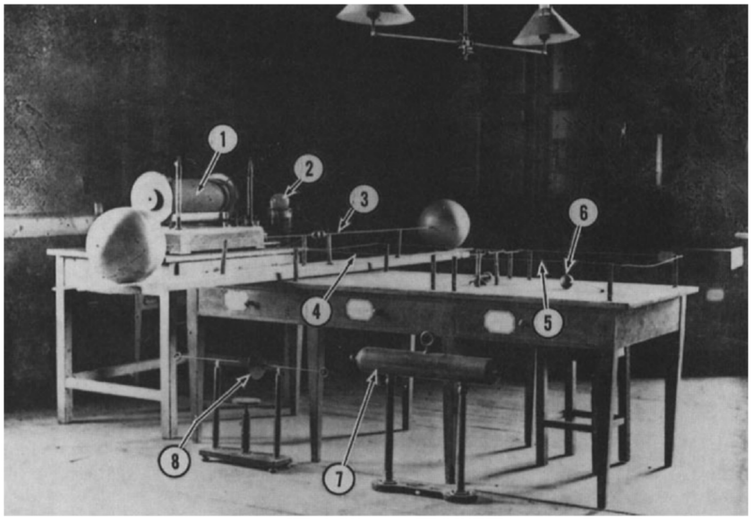
Pre-lecture:

We will determine radiated power from electric dipole. See 2022-03-17 reference to Zangwill.

Post-lecture:

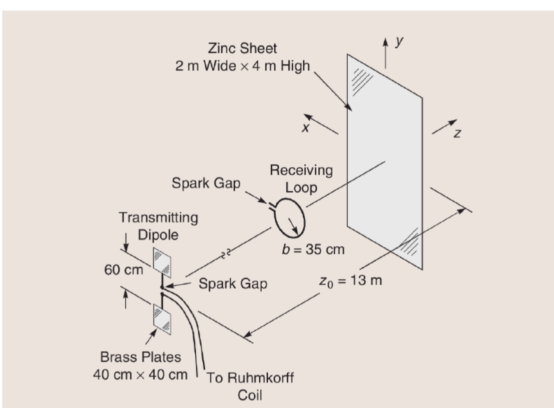
The figures at the start of the lecture are from Ref.'s [11] and [12].

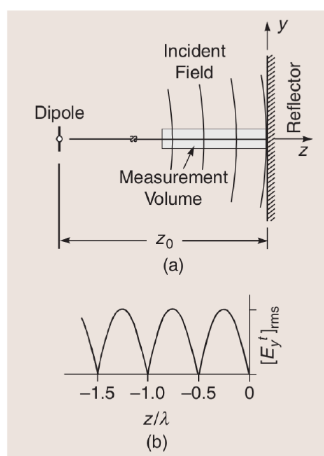
Hertz's apparatus



From pg 44 of D. Baird et al., *Heinrich Hertz: classical physicist, modern philosopher*, (Springer, Dordrecht; London, 2011).

Hertz's measurement of the speed of light





Figures 2 and 3 from G. S. Smith, "Analysis of Hertz's Experimentum Crucis on Electromagnetic Waves [Historical Corner]", IEEE Antennas and Propagation Magazine **58**, 96–108 (2016).

(1)

Radiated power from an electric dipole

Point dipole: $\left.\begin{aligned} \rho &= -\nabla\cdot\left(\vec{p}(t)\,\delta^3(\vec{r})\right) \\ \vec{J} &= \dot{\vec{p}}(t)\,\delta^3(\vec{r}) \end{aligned}\right\}$ satisfy $\nabla\cdot\vec{J} = -\dfrac{\partial\rho}{\partial t}$

Working in Lorenz gauge, we found $\vec{A}$ and Φ, and thus $\vec{E}$ and $\vec{B}$

$$\vec{B}(\vec{r},t) = -\frac{\mu_0}{4\pi}\,\hat{r}\times\left\{\frac{\dot{\vec{p}}_{ret}}{r^2} + \underbrace{\frac{\ddot{\vec{p}}_{ret}}{cr}}\right\}$$

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{3\hat{r}(\hat{r}\cdot\vec{p}_{ret}) - \vec{p}_{ret}}{r^3} + \frac{3\hat{r}(\hat{r}\cdot\dot{\vec{p}}_{ret}) - \dot{\vec{p}}_{ret}}{cr^2} + \underbrace{\frac{\hat{r}(\hat{r}\cdot\ddot{\vec{p}}_{ret}) - \ddot{\vec{p}}_{ret}}{c^2 r}}\right\}$$

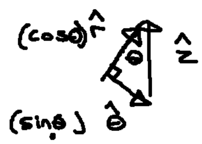
$$\vec{p}_{ret} = \vec{p}\left(t-\frac{r}{c}\right)$$

(2)

For concreteness, take $\vec{p}(t) = p_0\cos(\omega t)\,\hat{z}$.
Just consider $\frac{1}{r}$ "radiation fields".

$$\vec{B}_{rad} = -\frac{\mu_0}{4\pi}\,\frac{\hat{r}\times\ddot{\vec{p}}_{ret}}{cr}$$

$$= -\frac{\mu_0}{4\pi}(-\omega^2)\,\frac{p_0\cos(\omega t_{ret})}{cr}\;\hat{r}\times\hat{z}$$



$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$
$$\hat{r}\times\hat{\theta} = \hat{\phi}$$

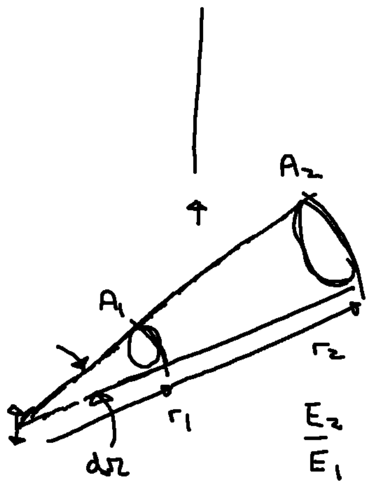
$$\boxed{\vec{B}_{rad} = -\frac{\mu_0}{4\pi}\,\frac{\omega^2 p_0}{cr}\,\sin\theta\,\cos\left(\omega(t-r/c)\right)\,\hat{\phi}}$$

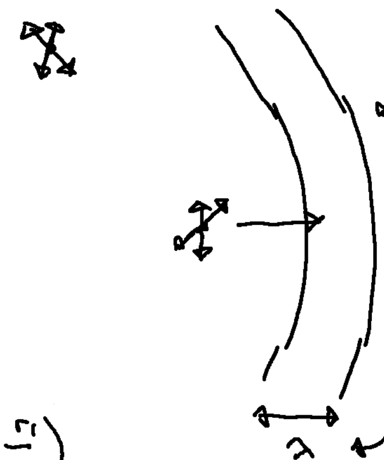
$$\vec{E}_{rad} = \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{r}(\hat{r}\cdot\ddot{\vec{p}}_{ret}) - \ddot{\vec{p}}_{ret}}{c^2 r}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{(-\omega^2)}{c^2 r}\cos(\omega t_{ret})\,\underbrace{\left[\hat{r}(\hat{r}\cdot\hat{z}) - \hat{z}\right]}_{\hat{r}\cancel{\cos\theta} - \hat{r}\cancel{\cos\theta} + \sin\theta\,\hat{\theta}}$$

③

$$\boxed{\vec{E}_{rad} = -\frac{\omega^2 p_0}{4\pi\epsilon_0}\,\frac{1}{c^2 r}\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\sin\theta\;\hat{\theta}}$$





over length scales small compared to r, looks like linearly polarized EM wave

from $\frac{2\pi}{\lambda} = \frac{\omega}{c}$

$$\frac{E_2}{E_1} = \left(\frac{r_1}{r_2}\right) \qquad \frac{A_2}{A_1} = \left(\frac{r_2}{r_1}\right)^2 \qquad \frac{B_2}{B_1} = \left(\frac{r_1}{r_2}\right)$$

$$\Rightarrow \frac{E_2 B_2 A_2}{E_1 B_1 A_1} = \text{a constant}$$

④

magnitude of Poynting vector * area of sphere defined by $d\Omega$ = constant

$$\Rightarrow \frac{dP}{d\Omega} \text{ is independent of distance to radiator.}$$

time averaged power

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} r^2 \langle \underbrace{\vec{S}}_{\frac{1}{\mu_0}\vec{E}\times\vec{B}} \cdot \hat{r} \rangle_t \quad \text{a time average}$$

true for any radiator

For electric dipole

$$\frac{dP}{d\Omega} = \frac{\omega^2 p_0}{4\pi\epsilon_0 c^2}\,\frac{\mu_0}{4\pi}\,\frac{\omega^2 p_0}{c}\,\frac{1}{\mu_0}\,\sin^2\theta\,\underbrace{\langle\cos^2(\omega t)\rangle_t}_{1/2}\,\underbrace{(\hat{\theta}\times\hat{\phi})\cdot\hat{r}}_{1}$$

$$\frac{dP}{d\Omega} = \frac{\omega^4 p_0^2}{2(4\pi)^2\epsilon_0 c^3}\sin^2\theta \qquad c^2 = \frac{1}{\mu_0\epsilon_0}$$

5

$$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0}{2c}\left(\frac{\omega^2 p_0}{4\pi}\right)^2 \sin^2\theta}$$

$$P = \int d\Omega \frac{dP}{d\Omega}$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \quad \sin^2\theta = \frac{8\pi}{3}$$

$$\boxed{P = \frac{1}{12\pi} \frac{\mu_0}{c} (\omega^2 p_0)^2}$$

$\mu_0 c \to \Omega$

$$\frac{\Omega}{\cancel{m}/s} \frac{1}{\cancel{m}/s} \frac{1}{s^4} c^2 \cdot \cancel{m}^2$$

$$= \Omega \cdot A^2 \checkmark$$

$P = VI$

$P = I^2 R$

21 2022-03-24 Lecture

Pre-lecture:

We will discuss the E&M aspects of thermal blackbody radiation. It would be useful to review what you covered in Stat. Mech. on this subject. See, for example, Section 7.4 of D. V. Schroeder, *An introduction to thermal physics* (Addison Wesley, San Francisco, CA, 2000).

Post-lecture:

A derivation of Lorentz invariance of phase space densities is given in Section 10 of L&L [4].

The U-2 observation of the so-called "dipole anisotropy" of the CMB is described in Smoot *et al.* [14]. A description for the non-specialist is given by Muller [15], and many interesting photos and background information are available at: https://aether.lbl.gov/www/projects/u2/

①

Lorentz invariance of phase space density

Show $\dfrac{dN}{d^3p\, d^3x}$ is a Lorentz invariant.

If we view N as a Lorentz invariant, then we just need to show that the "volume" $d^3p\, d^3x$ is a Lorentz invariant.

For photons, N invariant supported by Lorentz invariance of U_{EM}/ω.

Steps: ① show $\dfrac{d^3p}{E}$ is Lorentz invariant

② $E\, d^3x$ is Lorentz invariant

$\Big\}\Rightarrow d^3p\, d^3x$ is a Lorentz scalar.

②

① Assume standard config.

$$p_x' = \gamma p_x - \gamma\beta \frac{E}{c}$$

$$dp_x' = \gamma\, dp_x - \gamma\beta \frac{1}{c}\, dE \qquad (*)$$

?

$$E = \sqrt{(mc^2)^2 + (pc)^2} \Rightarrow \frac{dE}{dp_x} = \frac{\cancel{2} p_x c^2}{\cancel{2} E} \Rightarrow dE = \frac{p_x}{E}\, dp_x\, c^2$$

$$(*) \Rightarrow \quad dp_x' = dp_x\, \gamma \left(1 - \beta \frac{1}{c} \frac{p_x}{E} c^2\right)$$

$$\frac{E'}{c} = \gamma \frac{E}{c} - \gamma\beta p_x = \gamma \frac{E}{c}\left(1 - \beta p_x \frac{c}{E}\right)$$

$$\frac{dp_x'}{E'} = \frac{dp_x}{E} \Rightarrow \frac{d^3p}{E} \text{ is Lorentz invariant.}$$

$$\frac{d^3k}{k}$$

③

② To show invariance of $E\,d^3x$, again consider standard configuration. Imagine transforming point in phase space to frame in which particles are stationary. By length contraction:

$$\underset{\text{particles moving}}{dx'} = \frac{dx}{\gamma} \quad \leftarrow \text{stationary}$$

$$dx'\gamma = dx$$

$$dx'\frac{E'}{mc^2} = dx$$

$$dx'E' = dx\,E \quad \Rightarrow \quad E\,d^3x \text{ is Lorentz invariant.} \;//$$

① + ② $\dfrac{d^3p}{E}$ is invariant, $E\,d^3x$ is Lorentz invariant $\Rightarrow$ $d^3p\,d^3x$ Lorentz invariant.

④

$$n = \frac{1}{\exp\left(\frac{h\nu}{kT}\right)-1}$$

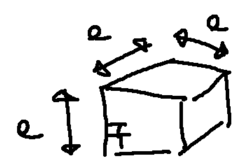
stat. mech. independent Bosons.

Consider modes in k-space.

$$N = \frac{4}{3}\pi k^3 \frac{1}{\Delta k_x \Delta k_y \Delta k_z}$$

enclosed within volume of radius k

$$= \frac{4}{3}\pi k^3 \frac{1}{\left(\frac{2\pi}{L}\right)^3}$$



quantization box, periodic BC's.

$$\frac{dN}{dk} = \frac{4\pi k^2 L^3}{(2\pi)^3}$$

$$\frac{du}{dk} = \frac{dN}{dk}h\nu, \qquad \frac{du}{d\nu} = \frac{2\pi}{c}\frac{dN}{dk}h\nu$$

5

two polarizations

$$\frac{du}{d\nu} = \frac{2\pi}{c} \frac{4\pi k^2}{(2\pi)^3} L^3 \frac{2}{\exp\left(\frac{h\nu}{kT}\right)-1} h\nu$$

$$k = \frac{\omega}{c}$$

$$\frac{du}{dV\,d\nu} = 8\pi \frac{h\nu^3}{c^3} \left(\frac{1}{\left(\exp\left(\frac{h\nu}{kT}\right)-1\right)}\right) \qquad \frac{J}{m^3} \cdot s \quad \checkmark$$

$(*)$

d^3x

n is a Lorentz invariant. Why?

$$\frac{dn}{d^3k} = \frac{n}{\left(\frac{2\pi}{L}\right)^3}$$

$$\frac{dn}{d^3k\,dV} = \frac{n}{(2\pi)^3}$$

Lorentz invariant.

6

Observe I_ν (specific intensity)

$$= \frac{dP}{dA\,d\Omega\,d\nu}$$

$$= \frac{c}{4\pi} \frac{du}{dV\,d\nu} \Big\} \text{ from } (*)$$

$$I_\nu = \frac{2h\nu^3}{c^2} \left(\frac{1}{\left(\exp\left(\frac{h\nu}{kT}\right)-1\right)}\right)$$

Lorentz invariant.

$\frac{I_\nu}{\nu^3}$ must be a Lorentz invariant.

(7)

Suppose we are observing at α' wrt direction of motion

What is relationship between ν and ν'.

$$\nu = (\gamma - \gamma\beta \cos\alpha')\,\nu' \qquad \text{using Lorentz transformations.}$$

$$I_{\nu'} = \frac{(\nu')^3}{(\nu)^3} I_{\nu}.$$

$$= \frac{2h}{c^3} \frac{\nu'^3}{\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]}$$

$$I_{\nu'} = \frac{2h}{c^3} \nu'^3 \frac{1}{\left[\exp\left(\frac{h}{kT}(\gamma - \gamma\beta\cos\alpha')\nu'\right) - 1\right]}$$

(8)

$$I_{\nu'} = \frac{2h}{c^3} \frac{\nu'^3}{\left[\exp\left(\frac{h\nu'}{kT'}\right) - 1\right]}$$

$$T' = \frac{T}{(\gamma - \gamma\beta\cos\alpha')}$$

$\alpha' = 0$

$$\frac{T'}{T} = \sqrt{\frac{1+\beta}{1-\beta}} \qquad \text{increase } T$$

$\alpha' = \pi$

$$\frac{T'}{T} = \sqrt{\frac{1-\beta}{1+\beta}}$$

$$\beta c \simeq \underbrace{400 \text{ km/s}}$$

The cosmic microwave background

- satellites (COBE, WMAP, Planck) can measure the background thermal radiation in space ($c/(3 \times 10^{11}\,\mathrm{s}^{-1}) \approx 1\,\mathrm{mm}$):

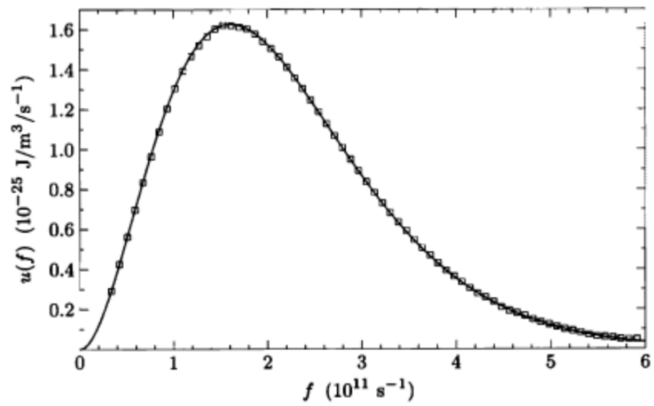
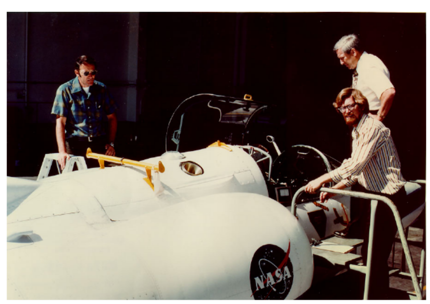


Figure 7.20 of Schroeder, *Introduction to Thermal Physics*

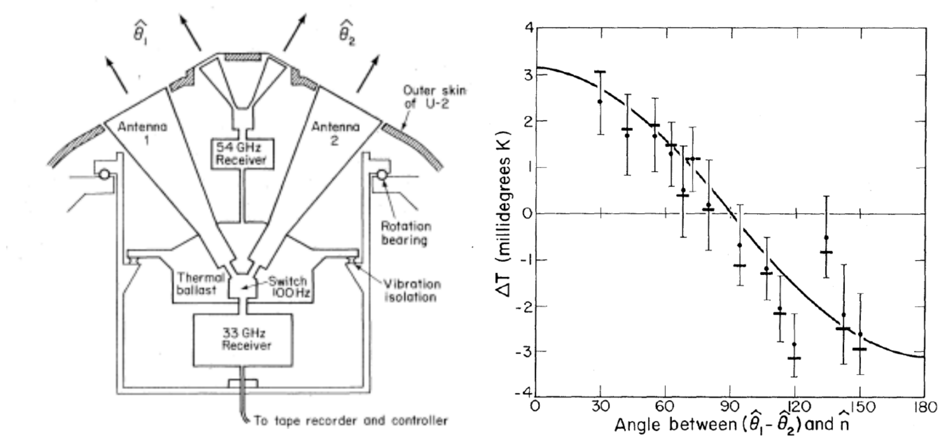
- follows Planck distribution with a temperature of 2.728(2) K.
- of relevance to cosmology (esp. small anisotropies); Nobel Prize in Physics, 2006, Mather and Smoot

NASA's U-2 allowed observation of CMB at high altitude



Photos from: `https://aether.lbl.gov/www/projects/u2/`

Two antennas looking in different directions measure
"temperature" difference



Figures 1 and 3 from G. F. Smoot et al., "Detection of Anisotropy in the Cosmic Blackbody Radiation", Phys. Rev. Lett. **39**, 898–901 (1977)

22 2022-03-29 Lecture

Pre-lecture:

We will discuss the potentials due to a moving point charge in the Lorenz gauge: the so-called Liénard-Wiechert potentials. See Section 23.2.1 of Zangwill [2].

Post-lecture:

(1)

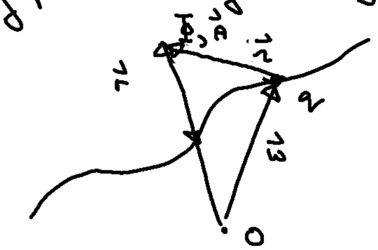
The potentials due to a moving point charge in the Lorenz gauge

In the Lorenz gauge:

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int dv' \, \frac{\rho(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|}$$

where $t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$ and similarly for $\vec{A}(\vec{r},t)$

For moving point charge : $\rho(\vec{r},t) = q\,\delta^3(\vec{r}-\underbrace{\vec{w}(t)}_{\text{trajectory}}))$



$$\vec{J}(\vec{r},t) = q\vec{v}\,\delta^3(\vec{r}-\vec{w}(t))$$

(2)

$$\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int dv' \, \frac{\delta^3(\vec{r}'-\vec{w}(t-\frac{|\vec{r}-\vec{r}'|}{c}))}{|\vec{r}-\vec{r}'|} \quad (*)$$

Might guess that just solve:

$$\vec{r}' = \vec{w}(t - \frac{|\vec{r}-\vec{r}'|}{c})$$

for $\vec{r}'$ given $\vec{r},t$ and $\vec{w}$ and then

$$\cancel{\Phi(\vec{r},t) \stackrel{?}{=} \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-\vec{r}'|}}$$

wrong because of how $\vec{r}'$ shows up in $(*)$

Introduce t'

$$\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int dt' dv' \, \frac{1}{|\vec{r}-\vec{r}'|}\,\delta^3(\vec{r}'-\vec{w}(t'))\,\delta(t' - (\underbrace{t - \frac{|\vec{r}-\vec{r}'|}{c}}_{t_r}))$$

Do space integration.

$$= \frac{q}{4\pi\epsilon_0}\int dt' \, \frac{1}{|\vec{r}-\vec{w}(t')|}\,\delta(t' - (\underbrace{t - \frac{|\vec{r}-\vec{w}(t')|}{c}}_{t_r})) \quad (\times)$$

③

Some δ-fcn integration results:

$$\int \delta(x)\,dx = 1$$

$$\int \delta(f(x))\,dx = ?$$

Suppose $f(x)$ has a single zero @ $x = x_0$: $f(x_0) = 0$

$$\int \delta(f(x))\,dx = \frac{1}{|f'(x_0)|}$$

Generalization:

$$\int g(x)\,\delta(f(x))\,dx = g(x_0)\,\frac{1}{|f'(x_0)|}$$

④

Using this result $(*)$ on pg 2 becomes:

$$\Phi(\vec r, t) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec r - \vec w(t_r)|}\,\frac{1}{\left|\frac{d}{dt'}\left(t' - \left(t - \frac{|\vec r - \vec w(t')|}{c}\right)\right)\right|_{t'=t_r}} \quad *$$

where t_r satisfies $t_r = t - \frac{|\vec r - \vec w(t_r)|}{c}$ i.e. a zero of fcn within the δ-fcn.

$$\frac{d}{dt'}\,|\vec r - \vec w(t')| = \frac{d}{dt'}\sqrt{|\vec r - \vec w(t')|^2}$$

$$= \frac{1}{\cancel{2}\,|\vec r - \vec w(t')|}\,\cancel{2}\,(\vec r - \vec w(t')) \cdot \underbrace{-\vec w'(t')}_{=:\,\vec v(t')}$$

$$= -\frac{(\vec r - \vec w(t'))}{|\vec r - \vec w(t')|} \cdot \vec v(t')$$

Call $\vec{\mathscr{r}}(t') := \vec r - \vec w(t')$

(5)

$$\boxed{\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\imath(t_r)} \frac{1}{\left(1 - \frac{\hat{\imath}\cdot\vec{v}(t_r)}{c}\right)}} \quad (*)$$

Note: $\vec{v}=0$, just reduces to electrostatics case.

Derivation for $\vec{A}$ is virtually identical.

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} q \frac{\vec{v}(t_r)}{\imath(t_r)} \frac{1}{\left(1 - \frac{\hat{\imath}(t_r)\cdot\vec{v}(t_r)}{c}\right)} \quad (*)$$

"Liénard - Wiechert potentials"

$$= \epsilon_0\mu_0 \vec{v}(t_r)\Phi(\vec{r},t)$$

$$\boxed{A(\vec{r},t) = \frac{1}{c^2}\vec{v}(t_r)\Phi(\vec{r},t)}$$

t_r has to satisfy : $t_r = t - \frac{|\vec{r}-\vec{w}(t_r)|}{c}$. *

(6)

From SR point of view: (GIFFEY, Problem 12.58)

$$\boxed{A^\mu = -\frac{q}{4\pi\epsilon_0 c} \frac{\eta^\mu}{\eta^\nu \imath_\nu}} \leftarrow$$

where

$$\imath_\nu = x_\nu - w_\nu(t_r), \quad \eta^\mu \equiv (\gamma c, \gamma\vec{v})$$

four-velocity

this is equivalent to what we have shown.

$$\eta^\nu \imath_\nu = (\gamma c, \gamma\vec{v}) \diamond (ct - ct_r, \vec{\imath})$$

Remember that $t_r = t - \frac{\imath}{c} \Rightarrow ct - ct_r = \imath$

$$\eta^\nu \imath_\nu = \gamma(\vec{v}\cdot\vec{\imath} - \imath c)$$

Look at A^0 $(= \Phi/c)$

(7)

$$A^0 = -\frac{q}{4\pi\epsilon_0 c} \frac{\gamma c}{\gamma(\vec{v}\cdot\vec{r} - cr)}$$

$$= \frac{1}{c}\frac{1}{4\pi\epsilon_0}\frac{1}{(r - \vec{v}\cdot\vec{r}/c)}$$

$$= \frac{1}{c}\frac{1}{4\pi\epsilon_0}\frac{1}{r}\frac{1}{\left(1 - \frac{\vec{v}\cdot\hat{r}}{c}\right)}$$

$$= \frac{\Phi}{c}$$ from top of pg 5.

23 2022-03-31 Lecture

Pre-lecture:

We will discuss the **E** and **B** fields due to a moving point charge. See Section 23.2.4 of Zangwill [2].

Post-lecture:

1

The fields due to a point charge in arbitrary motion

- start from intermediate point in derivation of the LW potentials : before integration over t' : trajectory

$$\Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int dt' \frac{1}{|\vec{r}-\vec{w}(t')|}\,\delta\left(t'-\left(t-\frac{|\vec{r}-\vec{w}(t')|}{c}\right)\right)$$

$$\vec{A}(\vec{r},t) = \frac{q\mu_0}{4\pi}\int dt' \frac{\vec{v}(t')}{|\vec{r}-\vec{w}(t')|}\,\delta\left(t'-\left(t-\frac{|\vec{r}-\vec{w}(t')|}{c}\right)\right)$$

Determine $\vec{E}(\vec{r},t)$ from $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$ last lecture.

$\vec{\imath} = \vec{r} - \vec{w}(t')$. Some derivatives req'd. $\frac{d\imath}{dt}$, $\nabla\imath$ look at now.

2

$$\nabla = \hat{x}\,\partial_x + \hat{y}\,\partial_y + \hat{z}\,\partial_z$$

$$\partial_x \imath = \partial_x\sqrt{(\vec{r}-\vec{w}(t'))\cdot(\vec{r}-\vec{w}(t'))}$$

$$= \frac{1}{2\imath}\,\partial_x\left(\left[(\vec{r}-\vec{w}(t'))\cdot(\vec{r}-\vec{w}(t'))\right]\right)$$

$$= \frac{1}{\cancel{2}\imath}\,\cancel{2}\,\hat{x}\cdot[\underbrace{\vec{r}-\vec{w}(t')}_{\vec{\imath}}]$$

$$= \hat{x}\cdot\hat{\imath}$$

Similarly, $\partial_y \imath = \hat{y}\cdot\hat{\imath}$ and $\partial_z \imath = \hat{z}\cdot\hat{\imath}$

$$\nabla\imath = \hat{x}(\hat{x}\cdot\hat{\imath}) + \hat{y}(\hat{y}\cdot\hat{\imath}) + \hat{z}(\hat{z}\cdot\hat{\imath})$$

$$= \hat{\imath}$$

Also $\nabla\left(\frac{1}{\imath}\right) = -\frac{\hat{\imath}}{\imath^2}$ Coulomb's law from potential.

③

$$\nabla \Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \nabla \int dt' \, \frac{1}{r} \, \delta\left(t' - \left(t - \frac{r}{c}\right)\right)$$

no problem.

$$= \frac{q}{4\pi\epsilon_0} \int dt' \left[\nabla\left(\frac{1}{r}\right) \delta\left(t' - \left(t - \frac{r}{c}\right)\right) + \frac{1}{r} \underbrace{\nabla \delta\left(t' - \left(t - \frac{r}{c}\right)\right)}_{(*)} \right]$$

For (*) use chain rule in the form:

$$\nabla \delta(g(\vec{r})) = \delta'(g(\vec{r})) \nabla g(\vec{r})$$

In this case:

$$\nabla \delta\left(t' - \left(t - \frac{r}{c}\right)\right) = \delta'\left(t' - \left(t - \frac{r}{c}\right)\right) \frac{\nabla r}{c}$$

from previous page

$$= \underbrace{-\frac{\partial}{\partial t} \delta\left(t' - \left(t - \frac{r}{c}\right)\right)} \frac{\hat{r}}{c}$$

④

$$\nabla \Phi(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \int dt' \left[\left(-\frac{\hat{r}}{r^2}\right) \delta\left(t' - \left(t - \frac{r}{c}\right)\right) - \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{\hat{r}}{r} \, \delta\left(t' - \left(t - \frac{r}{c}\right)\right) \right) \right]$$

fcn of t'

$$= \frac{q}{4\pi\epsilon_0} \left\{ \int dt' \left[-\frac{\hat{r}}{r^2} \delta(t' - (t - r/c)) \right] - \underbrace{\frac{d}{dt} \int dt' \, \frac{\hat{r}}{cr} \, \delta\left(t' - \left(t - \frac{r}{c}\right)\right)} \right\}$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi} q \underbrace{\frac{d}{dt} \int dt' \, \frac{\vec{v}(t')}{r} \, \delta(t' - (t - r/c))}$$

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}$$

$$= \frac{q}{4\pi\epsilon_0} \left\{ \int dt' \, \frac{\hat{r}}{r^2} \, \delta\left(t' - \left(t - \frac{r}{c}\right)\right) + \frac{d}{dt} \int dt' \, \delta\left(t' - \left(t - \frac{r}{c}\right)\right) \left(\frac{\hat{r}}{cr} - \frac{\vec{v}(t')}{c^2 r} \right) \right\}$$

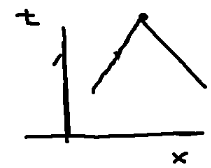
⑤

Use result from last lecture:

$$\int dx\; g(x)\,\delta(f(x)) = g(x_0)\,/\,|f'(x_0)|$$

Last lecture:

$$\frac{d}{dt'}\left(t' - (t - \mathscr{r}/c)\right) = 1 - \frac{\hat{\mathscr{r}}\cdot\vec{v}}{c} := g$$



$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left\{ \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\frac{1}{g} + \frac{d}{dt}\left[\frac{1}{g}\left(\frac{\hat{\mathscr{r}} - \vec{v}/c}{c\,\mathscr{r}}\right)\right]\right\}$$

Zangwill's Eq. 23.24

now $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$ where $\underbrace{t_r = t - \frac{|\vec{r} - \vec{w}(t_r)|}{c}}$

Convert $\frac{d}{dt}$ into $\frac{d}{dt_r}$. What is $\frac{dt}{dt_r}$?

$$1 = \frac{dt}{dt_r} - \frac{-(\hat{\mathscr{r}}\cdot\vec{v})}{c} \Rightarrow \frac{dt}{dt_r} = 1 - \frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}$$

Zangwill's Eq. 23.27

$$= g. \qquad \frac{d}{dt} = \frac{1}{g}\frac{d}{dt_r}$$

⑥

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\underbrace{\frac{(\hat{\mathscr{r}} - \vec{\beta})(1-\beta^2)}{g^3\mathscr{r}^2}} + \underbrace{\frac{\hat{\mathscr{r}}\times\{(\hat{\mathscr{r}}-\vec{\beta})\times\dot{\vec{\beta}}\}}{c\,g^3\mathscr{r}}}\right]$$

$$\vec{\beta} = \vec{v}/c$$

Zangwill's Eq. 23.31

$$c\vec{B} = \hat{\mathscr{r}}\times\vec{E}$$

Zangwill's Eq. 23.34

24 2022-04-05 Lecture

Pre-lecture:

I will discuss the radiation due to an accelerating point particle. See section 23.3 of Zangwill [2].

Post-lecture:

At the start of the lecture I showed a video illustrating the generation of radiation by a particle that decelerates to zero velocity: `https://youtu.be/pTadi-5aw9Q`.

There is a very nice argument for Larmor's formula based on this deceleration scenario. See, for example Purcell's textbook [16].[1]

The first experimental observation of Thomson scattering [17] was just three years after the demonstration of the laser. A famous early use of Thomson scattering the measurements of temperatures in a tokamak; see `https://en.wikipedia.org/wiki/Tokamak#Culham_Five`.

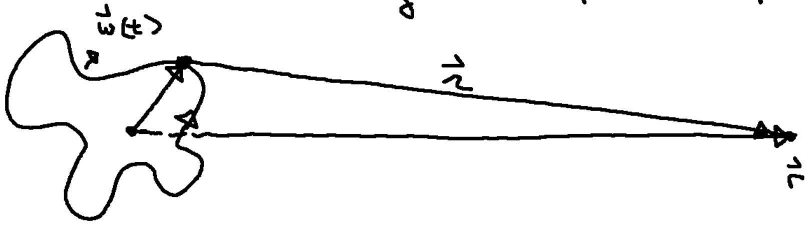
1

Radiation from a point particle
(non-relativistic)

Recall $$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{\mathscr{r}}-\vec{\beta})(1-\beta^2)}{g^3\mathscr{r}^2} + \frac{\hat{\mathscr{r}}\times\{(\hat{\mathscr{r}}-\vec{\beta})\times\dot{\vec{\beta}}\}}{c\,g^3\mathscr{r}}\right]$$

$$\frac{\vec{B}}{c} = \hat{\mathscr{r}}\times\vec{E}$$

where $g := 1-\hat{\mathscr{r}}\cdot\vec{\beta}$ and $\vec{\beta} = \vec{v}/c$.



Look at non-relativistic limit. $g = 1$

$\hat{\mathscr{r}} - \vec{\beta} = \hat{\mathscr{r}}$

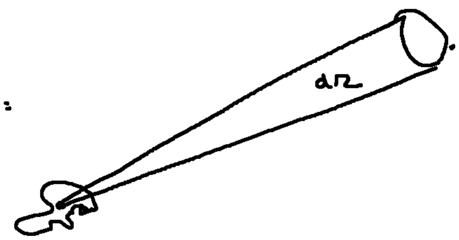
Far field: $\vec{\mathscr{r}} = \vec{r}$

[1]The argument presented is originally due Thomson here, but it is primarily of historical interest.

(2)

$$\frac{\vec{E}_{rad}}{k} = \frac{\overset{a}{\hat{r}} \times (\overset{b}{\hat{r}} \times \overset{c}{\dot{\vec{\beta}}})}{r} \quad , \quad k = \frac{q}{4\pi\epsilon_0 c}$$

$$= \frac{\hat{r}(\hat{r}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}}{r}$$



As with electric dipole radiation :

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} r^2 \underset{\underset{\frac{1}{\mu_0}\vec{E}\times\vec{B}}{\uparrow}}{\vec{S}}\cdot\hat{r}$$

$$c\vec{B}_{rad} = \hat{r}\times\vec{E}_{rad}$$

$$= \hat{r}\times\left(\hat{r}(\hat{r}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}\right) k/r$$

$$= k\,\frac{\dot{\vec{\beta}}\times\hat{r}}{r}$$

(3)

$$\vec{S} = \frac{1}{\mu_0}\vec{E}_{rad}\times\vec{B}_{rad}$$

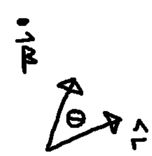
$$r^2\vec{S} = \frac{k^2}{\mu_0 c}\left[\hat{r}(\hat{r}\cdot\dot{\vec{\beta}}) - \dot{\vec{\beta}}\right]\times\left[\dot{\vec{\beta}}\times\hat{r}\right]$$

$$= \frac{k^2}{\mu_0 c}\left[(\hat{r}\cdot\dot{\vec{\beta}})\,\overset{a}{\hat{r}}\times(\overset{b}{\dot{\vec{\beta}}}\times\overset{c}{\hat{r}}) - \overset{a}{\dot{\vec{\beta}}}\times(\overset{b}{\dot{\vec{\beta}}}\times\overset{c}{\hat{r}})\right]$$

$$= \frac{k^2}{\mu_0 c}\left[(\hat{r}\cdot\dot{\vec{\beta}})(\dot{\vec{\beta}}\underbrace{(\hat{r}\cdot\hat{r})}_{=1} - \hat{r}(\hat{r}\cdot\dot{\vec{\beta}})) - \dot{\vec{\beta}}(\dot{\vec{\beta}}\cdot\hat{r}) + \hat{r}\,\dot{\beta}^2\right]$$

$$= \frac{k^2}{\mu_0 c}\left[-\hat{r}(\hat{r}\cdot\dot{\vec{\beta}})^2 + \hat{r}\,\dot{\beta}^2\right]$$

$$= \frac{k^2}{\mu_0 c}\,\hat{r}\left(\dot{\beta}^2 - (\hat{r}\cdot\dot{\vec{\beta}})^2\right)$$



Define θ so $\hat{r}\cdot\dot{\vec{\beta}} = \dot{\beta}\cos\theta$

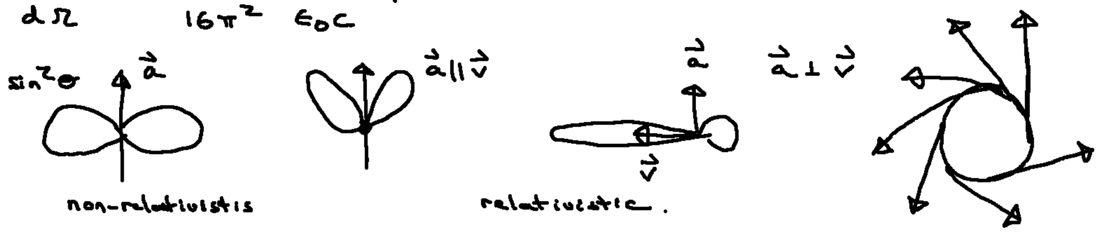
④

$$r^2 \vec{S} = \frac{k^2}{\mu_0 c} \dot{\beta}^2 \underbrace{(1-\cos^2\theta)}_{=\sin^2\theta}$$

$$\frac{dP}{d\Omega} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{\mu_0 c^3} \dot{\beta}^2 \sin^2\theta$$

$$\mu_0 = \frac{1}{\epsilon_0 c^2}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2} \frac{1}{\epsilon_0 c} \dot{\beta}^2 \sin^2\theta$$



What is total power radiated?

⑤

$$P = \int d\Omega \frac{dP}{d\Omega}$$

$$= \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \sin^2\theta}_{\frac{8\pi}{3}} \frac{q^2}{16\pi^2 \epsilon_0} \dot{\beta}^2$$

$$\boxed{P = \frac{1}{6\pi} \frac{q^2}{\epsilon_0 c^3} a^2}$$

← Larmor's formula.

$\mu_0 c \simeq 377\Omega \qquad \epsilon_0 c = \frac{1}{\mu_0 c}$

$$= \frac{1}{6\pi} \frac{\mu_0 c}{c^2} a^2 q^2$$

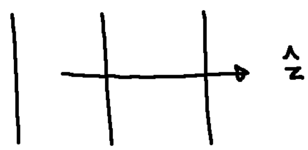
$$\frac{V}{A} \frac{s^2}{\cancel{m^2}} \frac{\cancel{m^2}}{s^4} c^2 = \frac{J}{\cancel{C}} \frac{S}{\cancel{C}} \frac{1}{s^2} \cancel{c^2}$$

$= J/s$ ✓

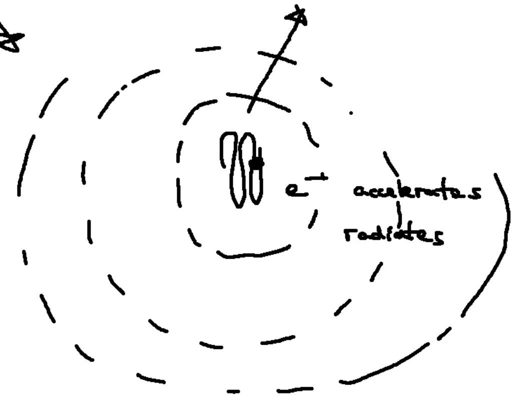
⑥

<u>Application of Larmor's formula</u>:

<u>Thomson scattering</u>



plane EM wave



$$m\ddot{x} = qE$$
$$= qE_0 \cos(\omega t)$$

$$P = \frac{\mu_0 q^2}{6\pi c}\ddot{x}^2 \quad , \quad P(t) = \frac{\mu_0 q^2}{6\pi c}\frac{q^2 E_0^2}{m^2}\underbrace{\cos^2(\omega t)}_{\langle\ \rangle_{time} = 1/2}$$

$$\langle P \rangle_t = \frac{1}{2}\frac{\mu_0 q^2}{6\pi c}\frac{q^2 E_0^2}{m^2}$$

⑦

Incident plane wave

$$\vec{S}\cdot\hat{z} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})\cdot\hat{z}$$
$$= \frac{E_0^2}{\mu_0 c}\cos^2(\omega t)$$

$$\langle \vec{S}\cdot\hat{z}\rangle_t = \frac{1}{2}\frac{E_0^2}{\mu_0 c}$$

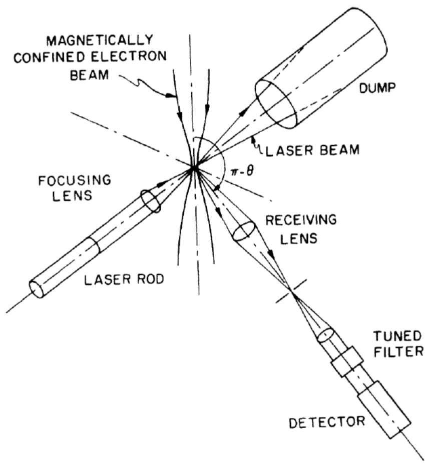
Characterize scattering by "cross-section"

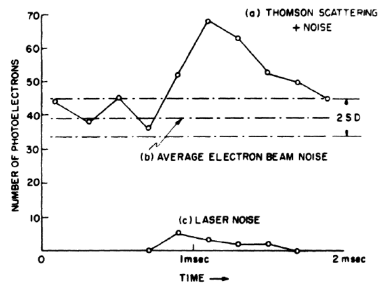
$$\sigma = \frac{\langle P \rangle_t}{\langle \vec{S}\cdot\hat{z}\rangle_t} = \frac{\mu_0 q^4}{6\pi c}\frac{\cancel{E_0^2}}{m^2}\frac{\mu_0 c}{\cancel{E_0^2}}$$

$$\sigma = \frac{\mu_0^2 q^4}{6\pi m^2}$$ Thomson scattering cross-section

For e^-'s $\quad \sigma \simeq 6.6 * 10^{-29}\ m^2 \qquad 1\,b \simeq 10^{-28}\ m^2$

First observation of Thomson scattering





Number of trials	Average Thomson photoelectrons	Standard deviation
15	3.4	1.14
21	1.85	0.90
22	5.8	1.85

Figures 1 and 2 from G. Fiocco and E. Thompson, "Thomson Scattering of Optical Radiation from an Electron Beam", Phys. Rev. Lett. **10**, 89–91 (1963).

25 2022-04-07 Lecture

Pre-lecture: I will discuss bremsstrahlung due to collisions between electrons and ions in plasmas. See, for example, Ref. [18].

Post-lecture:

For more information on astrophysical bremsstrahlung see: Ref. 's [19–21]. I used all of these sources to prepare this lecture. (I recommend Ref. [21] for many topics concerning EM radiation in astrophysics.)

A "cleaner" astrophysical bremsstrahlung spectrum with a theory comparison is in given in Ref. [22].

Longair [23] (Section 10.5.1) gives a short overview of the bremsstrahlung assisted cooling of the gas within galaxy clusters.

①

Bremsstrahlung (non-relativistic)

Start from Larmor's formula:

$$\frac{dE}{dt} = \frac{1}{6\pi} \frac{q^2}{\epsilon_0 c^3} a_q^2$$

Fourier decompose acceleration:

$$a(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \ \tilde{a}(\omega) \exp(-i\omega t)$$

$$\tilde{a}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \ a(t) \exp(i\omega t)$$

$$\int_{-\infty}^{\infty} dt \ |a(t)|^2 = \int_{-\infty}^{\infty} d\omega \ |\tilde{a}(\omega)|^2$$ Parseval's theorem.

Since $a(t)$ is real $|\tilde{a}(\omega)|^2 = |\tilde{a}(-\omega)|^2$

②

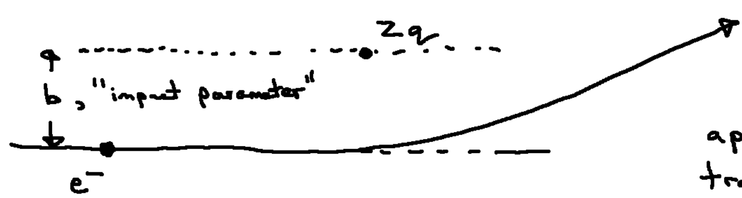
from entire collision

$$E = \int_{-\infty}^{\infty} dt \ \frac{dE}{dt}$$ from Larmor's Theorem

$$= \frac{1}{6\pi\epsilon_0 c^3} q^2 \int_{-\infty}^{\infty} dt \ |a(t)|^2$$

$$= \frac{1}{6\pi\epsilon_0 c^3} q^2 \, 2 \int_0^{\infty} d\omega \ |\tilde{a}(\omega)|^2$$

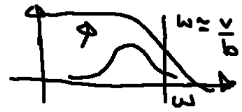
$$E = \int_0^{\infty} d\omega \ E_\omega \quad \text{where} \quad E_\omega = \frac{q^2}{3\pi\epsilon_0 c^3} \underbrace{|\tilde{a}(\omega)|^2}$$



approximate with straight-line trajectory.

③

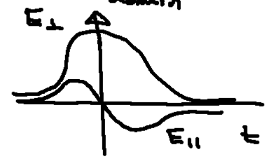
Recall (with $\gamma = 1$)



to spectral domain

$$E_\perp = \frac{Z q^2 b}{4\pi\epsilon_0 \left[b^2 + (vt)^2 \right]^{3/2}}$$

$$E_\parallel = \frac{Z q^2 vt}{4\pi\epsilon_0 \left[b^2 + (vt)^2 \right]^{3/2}}$$

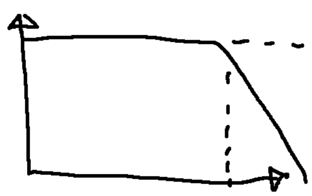


$$\underset{\text{energy}}{E_\omega} = \frac{q^2}{4\pi\epsilon_0 c^3} \frac{Z^2 q^2}{m^2} \left(|\underset{\text{electric field}}{\tilde{E}_\perp(\omega)}|^2 + |\tilde{E}_\parallel(\omega)|^2 \right)$$

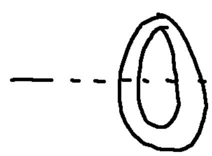
from problem set.

$$= \frac{q^6 Z^2}{24\pi^4 \epsilon_0^3 c^3 m_e^2 V^2} \frac{\omega^2}{V^2} \left[\left(K_0\left(\frac{\omega b}{V}\right)\right)^2 + \left(K_1\left(\frac{\omega b}{V}\right)\right)^2 \right]$$

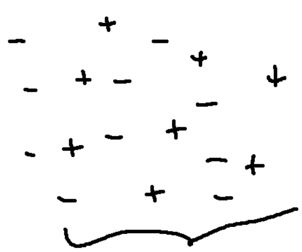
④



$$\frac{Z^2 q^6}{24\pi^4 \epsilon_0^3 c^3 m_e^2 V^2 b^2}$$



Consider a mixture.



nuclei stationary
electrons moving at V

number densities

n_Z : nuclei / unit volume
n_e : electrons / unit volume

(5)

$$\frac{dE}{dt\,dV\,d\omega} = n_Z n_e \, 2\pi v \int_{b_{min}}^{b_{max}} db \; b \; \underbrace{E_\omega(b)}_{}$$

$$\underbrace{b_{min}}_{} \quad \underbrace{b \;\; 1/b^2}_{1/b}$$

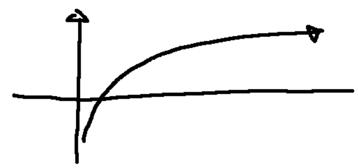
$$\underbrace{\frac{1}{m^3}\;\frac{1}{m^3}\;\frac{m}{s}\;\; m^2 \;\; \frac{J}{s^{-1}}}_{\frac{J}{m^3 s \; s^{-1}}}$$

$$\int db \, \frac{1}{b} = \underbrace{\ln(b)}_{}$$

$$b_{max} \simeq v/\omega$$

$$b_{min} \simeq \frac{\hbar}{2 m_e v} \qquad (\text{from } \Delta x \Delta p \approx \hbar)$$

$$\frac{b_{max}}{b_{min}} \simeq \frac{v/\omega}{\hbar/(2 m_e v)} \simeq \frac{2 m_e v^2}{\hbar \omega}$$



(6)

$$\frac{dP}{dV\,d\omega} = \frac{\int_{v_{min}}^{\infty} dv \, \frac{dE}{dt\,d\omega\,dV} \, v^2 \, e^{-\frac{mv^2}{2kT}}}{\int_0^{\infty} dv \; v^2 \, e^{-\frac{mv^2}{2kT}}}$$

$$\hbar\omega = \frac{1}{2} m v_{min}^2 \quad \}$$

i.e. enough energy has to be available to make photon.

$$\frac{dP}{dV\,d\nu} \propto \frac{1}{\sqrt{T}} \; Z^2 \, n_Z n_e \; \underbrace{e^{-\frac{h\nu}{kT}}}_{} \qquad (*)$$

Integrate over all ν

$$\frac{dP}{dV} \propto Z^2 n_Z n_e \sqrt{T} \qquad (*)$$

Plasmas radiate due to acceleration during collisions between electrons and ions

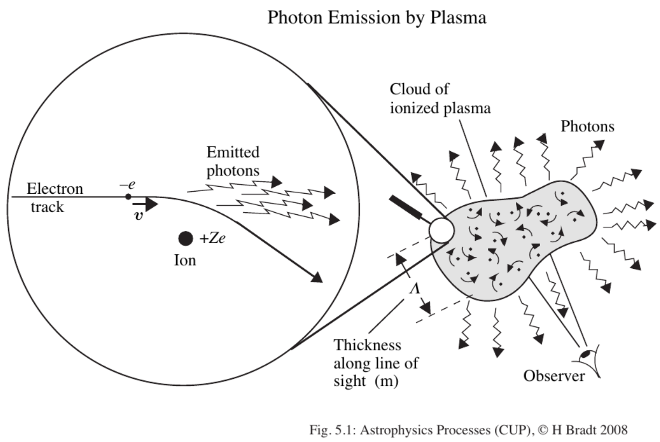
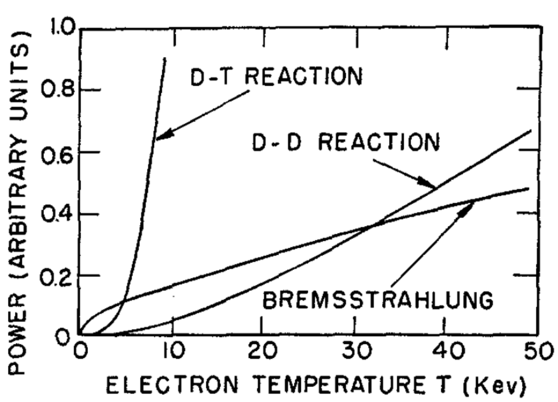


Figure 5.1 from H. Bradt, *Astrophysics processes: the physics of astronomical phenomena*, (Cambridge Univ. Press, Cambridge, 2014), 504 pp..

Bremsstrahlung losses are significant in fusion plasmas



*A comparison between the production of energy by thermonuclear reactions and the loss of energy by bremsstrahlung, as a function of the electron temperature (*1 keV $\approx 1.2 \times 10^7$ K*). Figure 4 from G. Bekefi and S. C. Brown, "Emission of Radio-Frequency Waves from Plasmas", American Journal of Physics* ***29****, 404–428 (1961).*

The X-ray spectrum of the Perseus Cluster of galaxies

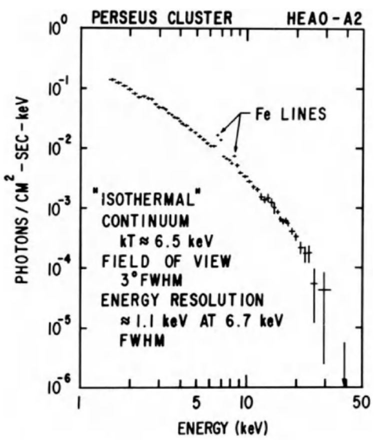


Figure 1 from pg 174 of R. Giacconi and G. Setti, X-Ray Astronomy: Proceedings of the NATO Advanced Study Institute held at Erice, Sicily, July 1-14, 1979, (Springer Netherlands, Dordrecht, 1980).

This bremsstrahlung radiation spectrum has frequency dependence consistent with temperature of $T \approx 7.5 \times 10^7$ K. Intensity can be used to estimate mass of intergalactic gas in cluster M. S. Longair, *High energy astrophysics*, 3rd ed (Cambridge University Press, Cambridge ; New York, 2011), 861 pp..

Maximum photon energy based on initial electron energy — the Duane-Hunt law, a quantum effect

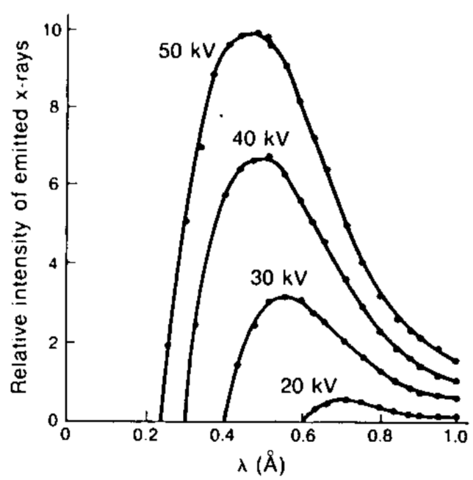


Figure 1-9a from pg 22 of A. P. French and E. F. Taylor, An introduction to quantum physics, Reprint, The M.I.T. Introductory Physics Series (Nelson Thornes, Cheltenham, 2001), 670 pp..

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