Linear Algebra Worksheet 4

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Workshop 4

Exercise 4.1. Find the determinants of the following matrices by using the inductive formula.

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 7 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 5 & 13 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 4.2. Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & t & 1 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

For which value of t does this matrix fail to be invertible? For this value of t, find an element of $\ker(A)$.

Exercise 4.3. For each matrix below, find its characteristic polynomial, its eigenvalues and its eigenvectors.

$$A = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} -1 & -2 & 2 \\ -1 & 1 & 1 \\ -5 & -4 & 6 \end{pmatrix}, \quad H = \begin{pmatrix} -2 & 3 & -3 \\ -6 & 7 & -6 \\ -6 & 6 & -5 \end{pmatrix}, \quad J = \begin{pmatrix} 18 & -5 & -6 \\ 81 & -20 & -18 \\ -22 & 6 & 7 \end{pmatrix}$$

Exercise 4.4. Write down two 2-by-2 matrices A, B with $\det(A) = \det(B) = 1$. Find $\det(A + B)$. Repeat twice more with different matrices. Can you get any value for $\det(A + B)$?

Exercise 4.5. Suppose that A has an eigenvector v with eigenvalue λ. Show that $\exp(A)$ has v as an eigenvector and find the eigenvalue.

Exercise 4.6. The *Jacobian* of a differentiable map $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is the m-by-n matrix

$$Jac(F) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_{n-1}} & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_{n-1}} & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_{n-1}} & \frac{\partial F_m}{\partial x_n} \end{pmatrix},$$

where $F(x_1, \ldots, x_n) = \begin{pmatrix} F_1(x_1, \ldots, x_n) \\ \vdots \\ F_m(x_1, \ldots, x_n) \end{pmatrix}$.

Find $\det(Jac(F))$ in the following examples:

1. $m = n = 2$, $F(r, \phi) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$.

2. $m = n = 3$, $F(r, \theta, \phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$

If $m = n$ and F is a change of coordinates $\mathbb{R}^n \to \mathbb{R}^n$ then the determinant of the Jacobian matrix is an important quantity: if $y = F(x)$ then the volume element $dy_1 \cdots dy_n$ is equal to $\det(Jac(F))dx_1 \cdots dx_n$. In the examples we've just computed the volume element in polar and spherical coordinates.

Exercise 4.7. Suppose that A is an n-by-n orthogonal matrix ($AA^T = I$).

1. Show that $\det(A) = \pm 1$.
2. If $\det(A) = 1$, show that $\det(A - I) = (-1)^n \det(A - I)$. (Hint: Use the fact that $A - I = A(I - A^T)$.)
3. Deduce that if n is odd then any orthogonal matrix with determinant one has a fixed vector.

Exercise 4.8. Let A be an n-by-n matrix with characteristic polynomial $\chi_A(t)$; suppose that $\chi_A(t)$ has n distinct roots. By considering $\chi_A(0)$, prove that $\det(A)$ is the product of the eigenvalues of A. One of the coefficients in the polynomial $\chi_A(t)$ is equal to minus the sum of the eigenvalues of A: which coefficient? *(Hint: Recall that if a polynomial $p(t)$ of degree n has roots $\lambda_1, \ldots, \lambda_n$ and the coefficient of t^n is 1 then $p(t) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n)$).*

4 Assessed problems for Week 4

Total marks available: 20.

Question 4.1. Using the inductive formula, find the determinant of $M_t = \begin{pmatrix} 1 & 0 & t \\ 15 & 13 & 2 \\ 8 & 7 & 1 \end{pmatrix}$. For which value of t is this matrix not invertible? For this value of t, find a vector v such that $M_t v = 0$. **4 marks**

Question 4.2. Let a, b, c be numbers. Using the inductive formula, find the determinant of $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$. **4 marks**

Question 4.3. Find the eigenvalues and any eigenvectors of the following matrices **4+8 marks**

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$