Linear Algebra Worksheet 1

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Here is a list $\mathcal{V}$ of vectors

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$s = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad t = \begin{pmatrix} -1/2 \\ 7 \\ i \end{pmatrix} \qquad u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$v = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \qquad \xi = \begin{pmatrix} b \\ b \\ b \\ -b \end{pmatrix}$$

Here is a list $\mathcal{M}$ of matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 1 \\ -2 & 8 \\ 1/2 & 3 \end{pmatrix} \qquad E = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -2 & 1 & 0 & 4 \\ -17 & 2 & 3 & 5 \\ 1 & -2 & 0 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 1.1. For each $V \in \mathcal{V}$ and each $M \in \mathcal{M}$, state whether the vector MV is defined and, if it is defined, compute it.

Exercise 1.2. For $N = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 4 & -2 \end{pmatrix}$ and for each $M \in \mathcal{M}$ state whether NM and/or MN is defined and calculate any products which are defined.

Exercise 1.3. Find the exponentials of the following matrices (λ is just some number, i is the square root of -1):

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad D = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Exercise 1.4. Show that if

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} \quad \text{and} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then Ae_1 is the first column of A. Which vectors $e_2, \ldots, e_n$ will give the second, third,..., nth columns?

Exercise 1.5. Let X and Y denote 2-by-2 matrices. Are the following statements true or false? In each case, give a proof or a counterexample to support your claim.

- If $X^2 = I$ then $X = \pm I$.
- If $XY = 0$ then $X = 0$ or $Y = 0$.
- If X has real entries then $X^2 \neq -I$.
- If $Xe_1 = Xe_2 = 0$ then $X = 0$ (e_1, e_2 are from Exercise 1.4).

Exercise 1.6. Take the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Apply A to v. Then apply A again. Then apply A again. Continue until you spot a pattern. Can you express the pattern as a formula? Can you prove that this pattern is going to continue? (Hint: You may write F_n for the nth term in a certain famous sequence of numbers).

Exercise 1.7. Check that the matrix

$$H_\phi := \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$$

fixes the vector $v = \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}$ and sends the vector $w = \begin{pmatrix} -\sin(\phi/2) \\ \cos(\phi/2) \end{pmatrix}$ to $-w$. (Hint: Remember your trigonometric identities...)

This means that H_ϕ represents a reflection in the line containing v.

Exercise 1.8 (Special relativity velocity addition)**.** Given a number v, define the matrix $\Lambda(v) = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix}$. Check that

$$\frac{1}{\sqrt{1 - \left(\frac{u+v}{1+uv}\right)^2}} = \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}}$$

for all u, v. Deduce that

$$\Lambda(u)\Lambda(v) = \Lambda\left(\frac{u+v}{1+uv}\right).$$

2 Assessed questions for week 1

Question 1.1. Let $A = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}$ and let $v = \begin{pmatrix} p \\ q \end{pmatrix}$. Find Av. **2 marks**

Question 1.2. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Prove $AB \neq BA$. **4 marks**

Question 1.3. For each pair of matrices M, N from the list below, state whether the products MN and NM are well-defined and compute any which are well-defined. **6 marks**

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Question 1.4. Let $n > 0$. Define the *trace* $\mathrm{Tr}(A)$ of an n-by-n matrix A to be the sum of its diagonal entries i.e. $\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}$. Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. (Hint: What are the ii entries of AB and of BA?) Find $\mathrm{Tr}(I)$. Prove that there are no matrices A, B such that $AB - BA = I$ (Hint: Take the trace of this formula!) **3 marks**

Question 1.5. Show that if $X = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$ then $\exp(X) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$. **4 marks**

If you've never met them, cosh *and* sinh *are the hyperbolic trigonometric functions defined by their Taylor series:*

$$\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \qquad \sinh(t) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots.$$

Question 1.6. The *Pell numbers* $P_1 = 0$, $P_2 = 1$, $P_3 = 2$, $P_4 = 5$, $P_5 = 12$, $P_6 = 29$, etc are obtained by the recursion $P_{n+2} = P_n + 2P_{n+1}$. Write down a 2-by-2 matrix A such that $A \begin{pmatrix} P_n \\ P_{n+1} \end{pmatrix} = \begin{pmatrix} P_{n+1} \\ P_{n+2} \end{pmatrix}$. (Hint: Try Exercise 1.6 first.) **1 marks**

Total/20 marks