Linear Algebra, Week 4

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MATH105

Eigenvalues, eigenvectors

Definition

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Theorem

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$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{3}{2} & \frac{5}{2} & 3 \\ -\frac{1}{2} & -\frac{3}{2} & -3 \\ 1 & 1 & 2 \end{pmatrix}$$

Applications, I: Differential equations

Differential equations

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General solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ce^{\lambda_- t} v_- + De^{\lambda_+ t} v_+.$$

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$$y = Ce^t, \quad x = (Ct + D)e^t.$$

Applications, II: Ellipsoids

Definition

We say that an n-by-n matrix A is *positive definite* if $v^T Av > 0$ for any nonzero vector $v \in \mathbb{R}^n$.

Definition

An *ellipsoid* is a subset in $\mathbb{R}^n$ of the form

$$\left\{ v \in \mathbb{R}^n \ : \ v^T Av = c \right\}$$

for some positive definite symmetric matrix A and constant $c > 0$.

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- *If $Au = \lambda u$ and $Av = \mu v$ with $\lambda \neq \mu$ then $u \cdot v = 0$.*

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is the result of rotating the ellipsoid

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Example

$A = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

Applications, III: Dynamics

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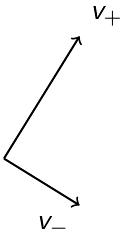

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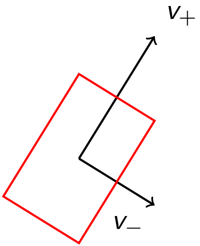

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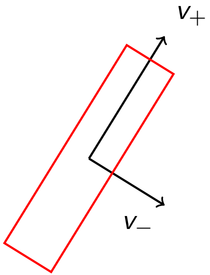

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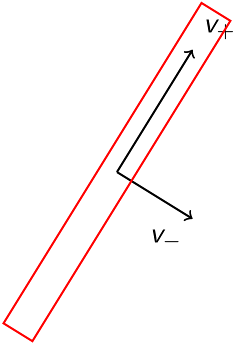

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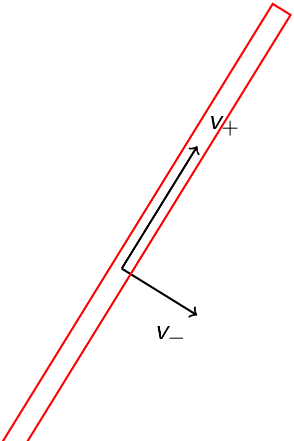

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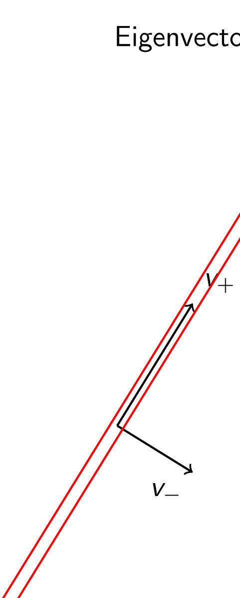

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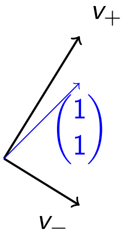

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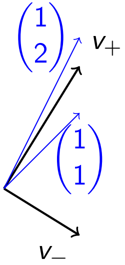

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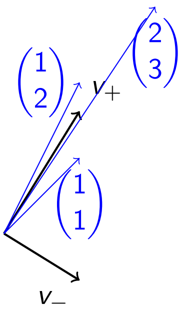

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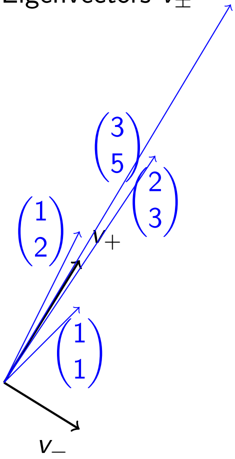

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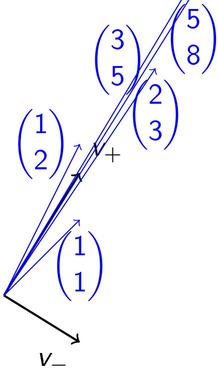

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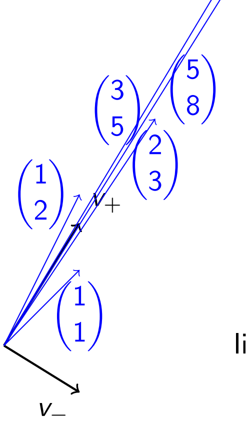

$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1.618\cdots$