Linear Algebra Worksheet 1

Jonny Evans

Here is a list $\mathcal{V}$ of vectors

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$s = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad t = \begin{pmatrix} -1/2 \\ 7 \\ i \end{pmatrix} \qquad u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$v = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \qquad \xi = \begin{pmatrix} b \\ b \\ b \\ -b \end{pmatrix}$$

Here is a list $\mathcal{M}$ of matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 1 \\ -2 & 8 \\ 1/2 & 3 \end{pmatrix} \qquad E = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -2 & 1 & 0 & 4 \\ -17 & 2 & 3 & 5 \\ 1 & -2 & 0 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Exercise 1.1. For each $V \in \mathcal{V}$ and each $M \in \mathcal{M}$, state whether the vector MV is defined and, if it is defined, compute it.

Exercise 1.2. For $N = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 4 & -2 \end{pmatrix}$ and for each $M \in \mathcal{M}$ state whether NM and/or MN is defined and calculate any products which are defined.

Exercise 1.3. Find the exponentials of the following matrices (λ is just some number, i is the square root of -1):

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad D = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Exercise 1.4. Show that if

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix} \quad \text{and} \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

then Ae_1 is the first column of A. Which vectors $e_2, \ldots, e_n$ will give the second, third,..., nth columns?

Exercise 1.5. Let X and Y denote 2-by-2 matrices. Are the following statements true or false? In each case, give a proof or a counterexample to support your claim.

- If $X^2 = I$ then $X = \pm I$.
- If $XY = 0$ then $X = 0$ or $Y = 0$.
- If X has real entries then $X^2 \neq -I$.
- If $Xe_1 = Xe_2 = 0$ then $X = 0$ (e_1, e_2 are from Exercise 1.4).

Exercise 1.6. Take the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Apply A to v. Then apply A again. Then apply A again. Continue until you spot a pattern. Can you express the pattern as a formula? Can you prove that this pattern is going to continue? (Hint: You may write F_n for the nth term in a certain famous sequence of numbers).

Exercise 1.7. Check that the matrix

$$H_\phi := \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$$

fixes the vector $v = \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}$ and sends the vector $w = \begin{pmatrix} -\sin(\phi/2) \\ \cos(\phi/2) \end{pmatrix}$ to $-w$. (Hint: Remember your trigonometric identities...)

This means that H_ϕ represents a reflection in the line containing v.

Exercise 1.8 (Special relativity velocity addition)**.** Given a number v, define the matrix $\Lambda(v) = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix}$. Check that

$$\frac{1}{\sqrt{1 - \left(\frac{u+v}{1+uv}\right)^2}} = \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}}$$

for all u, v. Deduce that

$$\Lambda(u)\Lambda(v) = \Lambda\left(\frac{u+v}{1+uv}\right).$$

2 Solutions

Solution 1.1. Here are tables of solutions for matrix/vector multiplications which are well-defined:

	A	B	D
p	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}1\\-2\\1/2\end{pmatrix}$
q	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}3\\1\end{pmatrix}$	$\begin{pmatrix}1\\8\\3\end{pmatrix}$
r	$\begin{pmatrix}1\\2\end{pmatrix}$	$\begin{pmatrix}5\\2\end{pmatrix}$	$\begin{pmatrix}2\\6\\7/2\end{pmatrix}$
s	$\begin{pmatrix}-2\\-1\end{pmatrix}$	$\begin{pmatrix}-4\\-1\end{pmatrix}$	$\begin{pmatrix}-1\\-18\\-11/2\end{pmatrix}$

	H	J
v	$\begin{pmatrix}-4\\2\\26\\-1\end{pmatrix}$	$\begin{pmatrix}-6\\2\end{pmatrix}$
w	$\begin{pmatrix}1\\-4\\-21\\5\end{pmatrix}$	$\begin{pmatrix}-2\\-1\end{pmatrix}$
ξ	$\begin{pmatrix}0\\-5b\\-17b\\-b\end{pmatrix}$	$\begin{pmatrix}-2b\\2b\end{pmatrix}$

	C	E	F	G
t	$\begin{pmatrix}i+13/2\\-2i-1/2\end{pmatrix}$	$(15/2)$	$\begin{pmatrix}2i-1/2\\i-13/2\\-3/2\end{pmatrix}$	$\begin{pmatrix}-i+41/2\\7\\13/2\end{pmatrix}$
u	$\begin{pmatrix}6\\-5\end{pmatrix}$	(1)	$\begin{pmatrix}7\\2\\3\end{pmatrix}$	$\begin{pmatrix}4\\2\\3\end{pmatrix}$.

Solution 1.2. Here are the matrix multiplications which are well-defined:

$$\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}\begin{pmatrix}0&1\\1&1\end{pmatrix}=\begin{pmatrix}-1&0\\0&2\\-2&2\end{pmatrix}$$

$$\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}\begin{pmatrix}2&3\\1&1\end{pmatrix}=\begin{pmatrix}1&2\\4&6\\6&10\end{pmatrix}$$

$$\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}\begin{pmatrix}1&1&1\\1&0&-2\end{pmatrix}=\begin{pmatrix}0&1&3\\2&2&2\\2&4&8\end{pmatrix}$$

$$\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}\begin{pmatrix}0&1&-2&1\\1&1&1&1\end{pmatrix}=\begin{pmatrix}-1&0&-3&0\\0&2&-4&2\\-2&2&-10&2\end{pmatrix}$$

$$\begin{pmatrix}1&1&1\\1&0&-2\end{pmatrix}\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}=\begin{pmatrix}7&-3\\-7&3\end{pmatrix}$$

$$\begin{pmatrix}-1&1&0\end{pmatrix}\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}=\begin{pmatrix}1&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&2\\1&-1&1\\3&0&0\end{pmatrix}\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}=\begin{pmatrix}9&-5\\3&-3\\3&-3\end{pmatrix}$$

$$\begin{pmatrix}1&3&-1\\0&1&0\\1&1&0\end{pmatrix}\begin{pmatrix}1&-1\\2&0\\4&-2\end{pmatrix}=\begin{pmatrix}3&1\\2&0\\3&-1\end{pmatrix}$$

Solution 1.3. 1. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ A^3 &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ \cdots \end{aligned}$$

so $\exp(A) = I + A + \frac{1}{2}A^2 + \cdots = I + A\left(1 + \frac{1}{2} + \frac{1}{3!} + \cdots\right)$. Therefore, since $e - 1 = 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$, we have $\exp(A) = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}$.

2. If $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ then

$$\begin{aligned} B^2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ B^3 &= 0 = B^4 = \cdots, \end{aligned}$$

so $\exp(B) = I + B + \frac{1}{2}B^2 = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

3. If $C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ then $C^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^n \end{pmatrix}$, so $\exp(C) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} + \cdots = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix}$.

4. If $D = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ then $D^2 = -I$, $D^3 = -D$, $D^4 = I$ and we get

$$\exp(D) = I\left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots\right) + D\left(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots\right)$$

i.e. $\exp(D) = \begin{pmatrix} \cos(1) & i\sin(1) \\ i\sin(1) & \cos(1) \end{pmatrix}$.

Solution 1.4. When we multiply the ith row of A into e_1, we just pick up the first entry of A because only the first entry of e_1 is nonzero and it is equal to one. Therefore Ae_1 is the first column of A. To get all the columns we use vectors e_i, $i = 1, \ldots, n$ where e_i is the vector with zeros everywhere except in the ith row, where it has a 1.

Solution 1.5.

- If $X^2 = I$ then $X = \pm I$. This is false, for example $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $X^2 = I$.
- If $XY = 0$ then $X = 0$ or $Y = 0$. This is false, for example $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ satisfy $XY = 0$.
- If X has real entries then $X^2 \neq -I$. This is false, for example $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies $X^2 = -I$.
- If $Xe_1 = Xe_2 = 0$ then $X = 0$ (e_1, e_2 are from Exercise 1.4). This is true, because if $Xe_1 = Xe_2 = 0$ then the columns of X are zero.

Solution 1.6. We have

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad Av = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A^2v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad A^3v = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad A^4v = \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \ldots$$

so it looks like the formula $A^n v = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$ should hold, where

$$F_1, F_2, F_3, F_4, F_5, F_6 \ldots = 1, 1, 2, 3, 5, 8, \ldots$$

is the Fibonacci sequence. Indeed, by definition $F_{n+2} = F_{n+1} + F_n$, and our matrix gives

$$A \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n + F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix},$$

so since $v = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, the result follows by induction.

Solution 1.7. We have

$$\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix} = \begin{pmatrix} \cos\phi\cos(\phi/2) + \sin\phi\sin(\phi/2) \\ \sin\phi\cos(\phi/2) - \cos\phi\sin(\phi/2) \end{pmatrix},$$

and the trigonometric identities $\cos(A+B) = \cos A\cos B - \sin A \sin B$ and $\sin(A+B) = \sin A\cos B + \sin B \cos A$ imply that this is equal to $\begin{pmatrix} \cos(\phi - (\phi/2)) \\ \sin(\phi - (\phi/2)) \end{pmatrix} = \begin{pmatrix} \cos(\phi/2) \\ \sin(\phi/2) \end{pmatrix}$. Similarly, we get $H_\phi w = -w$ where $w = \begin{pmatrix} -\sin(\phi/2) \\ \cos(\phi/2) \end{pmatrix}$. We note that $v \cdot w = -\cos(\phi/2)\sin(\phi/2) + \sin(\phi/2)\cos(\phi/2) = 0$, so v and w are orthogonal. Therefore H_ϕ represents the transformation which reflects in the line spanned by v.

Solution 1.8. We have

$$\begin{aligned}
\frac{1}{\sqrt{1 - \left(\frac{u+v}{1+uv}\right)^2}} &= \frac{1}{\sqrt{\frac{(1+uv)^2 - (u+v)^2}{(1+uv)^2}}} \\
&= \frac{1+uv}{\sqrt{(1+uv)^2 - (u+v)^2}} \\
&= \frac{1+uv}{\sqrt{1 + u^2v^2 + 2uv - u^2 - v^2 - 2uv}} \\
&= \frac{1+uv}{\sqrt{1 + u^2v^2 - u^2 - v^2}} \\
&= \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}}.
\end{aligned}$$

Now

$$\begin{aligned}
\Lambda(u)\Lambda(v) &= \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1 & -u \\ -u & 1 \end{pmatrix} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1+uv & -(u+v) \\ -(u+v) & 1+uv \end{pmatrix} \\
\Lambda\left(\frac{u+v}{1+uv}\right) &= \frac{1}{\sqrt{1 - \left(\frac{u+v}{1+uv}\right)^2}} \begin{pmatrix} 1 & \frac{-(u+v)}{1+uv} \\ \frac{-(u+v)}{1+uv} & 1 \end{pmatrix} \\
&= \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1 & \frac{-(u+v)}{1+uv} \\ \frac{-(u+v)}{1+uv} & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{(1-u^2)(1-v^2)}} \begin{pmatrix} 1+uv & -(u+v) \\ -(u+v) & 1+uv \end{pmatrix}.
\end{aligned}$$

This shows $\Lambda(u)\Lambda(v) = \Lambda\left(\frac{u+v}{1+uv}\right)$.

This matrix is used in special relativity to transform from one reference frame to another which is moving with relative velocity v (working in units where the speed of light is 1, and with the simplifying assumption that space is 1-dimensional!). Naively, you would expect that if you increase the relative velocity by v and then by u, you would end up increasing it overall by $u+v$, but this formula shows that velocity addition is more subtle.