Linear Algebra Worksheet 2

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Here is a list $\mathcal{V}$ of vectors

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad w = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Exercise 2.1. For every vector in $\mathcal{V}$, find its length and write down a vector orthogonal to it.

Exercise 2.2. Find the angle between u and v. Find the angle between w and ξ.

Here is a list $\mathcal{M}$ of matrices.

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2}\sqrt{\frac{3}{2}} \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} + \frac{1}{\sqrt{3}} & -\frac{1}{3} + \frac{1}{\sqrt{3}} \\ \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} & -\frac{1}{3} - \frac{1}{\sqrt{3}} \\ -\frac{1}{3} - \frac{1}{\sqrt{3}} & -\frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}.$$

Exercise 2.3. Which matrices $M \in \mathcal{M}$ are orthogonal matrices? (Hint: There should be two!)

Exercise 2.4. The orthogonal matrices from $\mathcal{M}$ are actually rotation matrices. In each case, find the axis and angle of rotation.

Here is a list $\mathcal{N}$ of matrices

$$D = \begin{pmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 2.5. Which of the matrices $N \in \mathcal{N}$ are in echelon form? Which are in reduced echelon form?

Exercise 2.6. For each $N \in \mathcal{N}$ which is in reduced echelon form, state (a) for which vectors b the equation $Nv = b$ has a solution and (b) the dimension of the space of solutions to $Nv = b$, assuming that b is chosen so that there is a solution.

Exercise 2.7. For each system of simultaneous equations below, write it in matrix form, put the augmented matrix into reduced echelon form using row operations. Determine if the system has a solution and, if it does, give the general solution.

$$\begin{aligned} x + y + 2z + 3w &= 0 \\ y + 4z - w &= 1 \end{aligned}$$

$$\begin{aligned} x &= y - 3 \\ 2x + y &= 6 \\ y - 3x &= 1 \end{aligned}$$

$$\begin{aligned} 4x - w &= 0 \\ 3y - 2z + w &= 4 \\ 4x - 2y + 4z - 3w &= 0 \\ 3x + y - z &= 2 \end{aligned}$$

Exercise 2.8. Put the following matrices into reduced echelon form using row operations. In each case, what is the number of free indices?

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

Exercise 2.9. Let A, B, C be m-by-n, n-by-p and p-by-q matrices respectively. Write out the matrix products $A(BC)$ and $(AB)C$ in index notation and check that they give the same answer (this shows that matrix multiplication is associative).

Exercise 2.10. Suppose that A is an n-by-n matrix whose columns are the vectors $v_1, \ldots, v_n$. Show that A is an orthogonal matrix (i.e. $A^T A = I$) if and only if

$$v_i \cdot v_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{for all } i, j.$$

In other words, the columns of A are orthogonal to one another (this is where the name "orthogonal matrix" comes from).

Exercise 2.11. We say that a matrix M is *symmetric* if $M^T = M$ and *antisymmetric* if $M^T = -M$.

1. Show that if N is an m-by-n matrix then MM^T is a symmetric m-by-m matrix and $M^T M$ is a symmetric n-by-n matrix.
2. Show that, given any n-by-n matrix C, the matrix $A = C + C^T$ is symmetric and the matrix $B = C - C^T$ is antisymmetric. Deduce that C can be written as the sum of a symmetric and an antisymmetric matrix (called the *symmetric* and *antisymmetric* parts of C respectively).

Exercise 2.12. A system of m equations in n unknowns is called *underdetermined* if $m < n$ and overdetermined if $m > n$. As rules of thumb, underdetermined equations tend to have general solutions with $m - n$ free parameters, and overdetermined equations tend to have no solutions. Give counterexamples to these rules of thumb (e.g. an underdetermined system with no solutions and an overdetermined system with a solution).

3 Assessed problems for Week 2

Total marks available: 20.

Question 3.1. Find the angle between the vectors $v = \begin{pmatrix} \frac{1}{2\sqrt{6}}(3+\sqrt{2}) \\ \frac{1}{2\sqrt{6}}(-3+\sqrt{2}) \\ \frac{1}{2\sqrt{3}} \end{pmatrix}$ and $w = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$. Find a vector u which is orthogonal to both of them. **4 marks**

Question 3.2. The matrix $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$ is a 3-d rotation matrix. Check that A is orthogonal. Find the axis and angle of rotation. **8 marks**

Question 3.3. Write the system of simultaneous equations

$$\begin{aligned} 3x + 2y + z &= 0 \\ 3x + y + 2z &= 3 \\ -x - y &= 1 \end{aligned}$$

as a matrix equation. Using row operations, put the augmented matrix in reduced echelon form, and hence find the general solution of the system of equations. **8 marks**

Total/20 marks