Linear Algebra, Week 3

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MATH105

Determinants

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

What about n-by-n matrices?

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What about n-by-n matrices?

Theorem

An n-by-n matrix A is invertible if and only if its determinant $\det(A)$ *is nonzero.*

Definition of determinant

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Permutations of 123.

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- and sum the resulting $n!$ terms.

$$\det(A) = \sum_{\sigma} \mathrm{sign}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}.$$

Examples: diagonal matrices

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Examples: upper triangular matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & & A_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix}$$

Examples: elementary matrices

$$E_{ij}(\lambda) = \begin{array}{cc} & \begin{array}{ccccccc} & & & & \text{col } j \downarrow & & \end{array} \\ \begin{array}{c} \\ \\ \text{row } i \rightarrow \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

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Computing determinants

Properties of the determinant

$$\det(A) = \sum_{\sigma} \mathrm{sign}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}.$$

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Corollary

Put A into echelon form using row operations of type I. Then $\det(A)$ is the product of the diagonal entries.

Example

$$A = \begin{pmatrix} 1 & 4 & -4 \\ -2 & -2 & -4 \\ 3 & -3 & 3 \end{pmatrix}$$

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$$B = \begin{pmatrix} 2 & -3 & -1 & 4 \\ 2 & -3 & 2 & 4 \\ 2 & -1 & -4 & -3 \\ 2 & -3 & 4 & 2 \end{pmatrix}$$

A useful trick

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If A' is obtained from A by swapping two rows then $\det(A') = -\det(A)$.

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Example

$$\det \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -1$$

Formulas for determinants and inverses

"Expanding along first row."

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

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$$A_{11}\det(C_{11}) - A_{12}\det(C_{12}) + A_{13}\det(C_{13})$$

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"Expanding along first row."

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \rightsquigarrow C_{14} = \begin{pmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{pmatrix}$$

$$A_{11} \det(C_{11}) - A_{12} \det(C_{12}) + A_{13} \det(C_{13})$$

"Expanding along first row."

$$\begin{pmatrix} \cancel{A_{11}} & \cancel{A_{12}} & \cancel{A_{13}} & \cancel{A_{14}} \\ A_{21} & A_{22} & A_{23} & \cancel{A_{24}} \\ A_{31} & A_{32} & A_{33} & \cancel{A_{34}} \\ A_{41} & A_{42} & A_{43} & \cancel{A_{44}} \end{pmatrix} \rightsquigarrow C_{14} = \begin{pmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{pmatrix}$$

$$A_{11} \det(C_{11}) - A_{12} \det(C_{12}) + A_{13} \det(C_{13}) - A_{14} \det(C_{14}).$$

“Expanding along first row.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \rightsquigarrow C_{14} = \begin{pmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{pmatrix}$$

$$A_{11}\det(C_{11}) - A_{12}\det(C_{12}) + A_{13}\det(C_{13}) - A_{14}\det(C_{14}).$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

"Expanding along second column."

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

"Expanding along second column."

$$\begin{pmatrix} \cancel{A_{11}} & \cancel{A_{12}} & \cancel{A_{13}} & \cancel{A_{14}} \\ A_{21} & \cancel{A_{22}} & A_{23} & A_{24} \\ A_{31} & \cancel{A_{32}} & A_{33} & A_{34} \\ A_{41} & \cancel{A_{42}} & A_{43} & A_{44} \end{pmatrix} \rightsquigarrow C_{12} = \begin{pmatrix} A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \\ A_{41} & A_{43} & A_{44} \end{pmatrix}$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \rightsquigarrow C_{12} = \begin{pmatrix} A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \\ A_{41} & A_{43} & A_{44} \end{pmatrix}$$

$$- A_{12} \det(C_{12})$$

"Expanding along second column."

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$- A_{12} \det(C_{12})$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$- A_{12} \det(C_{12})$$

"Expanding along second column."

$$\begin{pmatrix} \boxed{A_{11}} & A_{12} & \boxed{A_{13} \quad A_{14}} \\ A_{21} & A_{22} & A_{23} \quad A_{24} \\ \boxed{\begin{matrix} A_{31} \\ A_{41} \end{matrix}} & \begin{matrix} A_{32} \\ A_{42} \end{matrix} & \boxed{\begin{matrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{matrix}} \end{pmatrix}$$

$$-\,A_{12}\det(C_{12})$$

"Expanding along second column."

$$\begin{pmatrix} \boxed{A_{11}} & A_{12} & \boxed{A_{13} \quad A_{14}} \\ \cancel{A_{21}} & \cancel{A_{22}} & \cancel{A_{23} \quad A_{24}} \\ \boxed{\begin{matrix} A_{31} \\ A_{41} \end{matrix}} & \begin{matrix} A_{32} \\ A_{42} \end{matrix} & \boxed{\begin{matrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{matrix}} \end{pmatrix} \rightsquigarrow C_{22} = \begin{pmatrix} \boxed{\begin{matrix} A_{11} & A_{13} & A_{14} \\ A_{31} & A_{33} & A_{34} \\ A_{41} & A_{43} & A_{44} \end{matrix}} \end{pmatrix}$$

$$- A_{12} \det(C_{12})$$

"Expanding along second column."

$$\begin{pmatrix} \boxed{A_{11}} & A_{12} & \boxed{A_{13} \quad A_{14}} \\ \cancel{A_{21}} & \cancel{A_{22}} & \cancel{A_{23} \quad A_{24}} \\ \boxed{\begin{matrix} A_{31} \\ A_{41} \end{matrix}} & \begin{matrix} A_{32} \\ A_{42} \end{matrix} & \boxed{\begin{matrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{matrix}} \end{pmatrix} \rightsquigarrow C_{22} = \begin{pmatrix} \boxed{\begin{matrix} A_{11} & A_{13} & A_{14} \\ A_{31} & A_{33} & A_{34} \\ A_{41} & A_{43} & A_{44} \end{matrix}} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22})$$

"Expanding along second column."

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22})$$

"Expanding along second column."

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$- A_{12} \det(C_{12}) + A_{22} \det(C_{22})$$

“Expanding along second column.”

$$\begin{pmatrix} \boxed{A_{11}} & A_{12} & \boxed{A_{13} \quad A_{14}} \\ \boxed{A_{21}} & A_{22} & \boxed{A_{23} \quad A_{24}} \\ \cancel{A_{31}} & \cancel{A_{32}} & \cancel{A_{33} \quad A_{34}} \\ \boxed{A_{41}} & A_{42} & \boxed{A_{43} \quad A_{44}} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22})$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \rightsquigarrow C_{32} = \begin{pmatrix} A_{11} & A_{13} & A_{14} \\ A_{21} & A_{23} & A_{24} \\ A_{41} & A_{43} & A_{44} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22})$$

“Expanding along second column.”

$$\begin{pmatrix} \boxed{\begin{matrix} A_{11} \\ A_{21} \end{matrix}} & A_{12} & \boxed{\begin{matrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{matrix}} \\ \cancel{A_{31}} & \cancel{A_{32}} & \cancel{A_{33} \quad A_{34}} \\ \boxed{A_{41}} & A_{42} & \boxed{A_{43} \quad A_{44}} \end{pmatrix} \rightsquigarrow C_{32} = \begin{pmatrix} A_{11} & A_{13} & A_{14} \\ A_{21} & A_{23} & A_{24} \\ A_{41} & A_{43} & A_{44} \end{pmatrix}$$

$$- A_{12} \det(C_{12}) + A_{22} \det(C_{22}) - A_{32} \det(C_{32})$$

"Expanding along second column."

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$- A_{12} \det(C_{12}) + A_{22} \det(C_{22}) - A_{32} \det(C_{32})$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22}) - A_{32}\det(C_{32})$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22}) - A_{32}\det(C_{32})$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \rightsquigarrow C_{42} = \begin{pmatrix} A_{11} & A_{13} & A_{14} \\ A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22}) - A_{32}\det(C_{32})$$

"Expanding along second column."

$$\begin{pmatrix} \boxed{\begin{matrix} A_{11} \\ A_{21} \\ A_{31} \end{matrix}} & \begin{matrix} A_{12} \\ A_{22} \\ A_{32} \end{matrix} & \boxed{\begin{matrix} A_{13} & A_{14} \\ A_{23} & A_{24} \\ A_{33} & A_{34} \end{matrix}} \\ A_{41} & A_{42} & \begin{matrix} A_{43} & A_{44} \end{matrix} \end{pmatrix} \rightsquigarrow C_{42} = \begin{pmatrix} \boxed{\begin{matrix} A_{11} & A_{13} & A_{14} \\ A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \end{matrix}} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22}) - A_{32}\det(C_{32}) + A_{42}\det(C_{42}).$$

“Expanding along second column.”

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \rightsquigarrow C_{42} = \begin{pmatrix} A_{11} & A_{13} & A_{14} \\ A_{21} & A_{23} & A_{24} \\ A_{31} & A_{33} & A_{34} \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22}) - A_{32}\det(C_{32}) + A_{42}\det(C_{42}).$$

Example

$$B = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ -1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

Signs

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$A_{11}\det(C_{11}) - A_{12}\det(C_{12}) + A_{13}\det(C_{13}) - A_{14}\det(C_{14}).$$

Signs

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$-A_{12}\det(C_{12}) + A_{22}\det(C_{22}) - A_{32}\det(C_{32}) + A_{42}\det(C_{42}).$$

Formula for inverses (Cramer's rule)

Definition

The *adjugate matrix* of A is the matrix

$$\operatorname{adj}(A) := \begin{pmatrix} +\det(C_{11}) & -\det(C_{12}) & +\det(C_{13}) & \cdots \\ \det(C_{21}) & +\det(C_{22}) & -\det(C_{23}) & \cdots \\ +\det(C_{31}) & -\det(C_{32}) & +\det(C_{33}) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}^T.$$

Theorem

If $\det(A) \neq 0$ *then* $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.

More about determinants

Lemma

If A' is obtained from A by $R_i \mapsto \lambda R_i$ then $\det(A') = \lambda \det(A)$.

Corollary

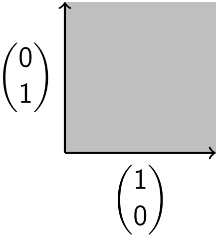
An n-by-n matrix is invertible if and only if its determinant is nonzero.

Theorem

$$\det(AB) = \det(A)\det(B).$$

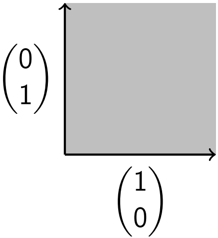
Geometric interpretation of determinants

Theorem



Geometric interpretation of determinants

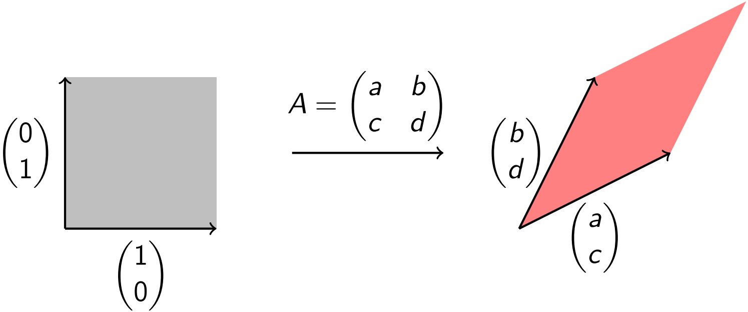
Theorem



$$\xrightarrow{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

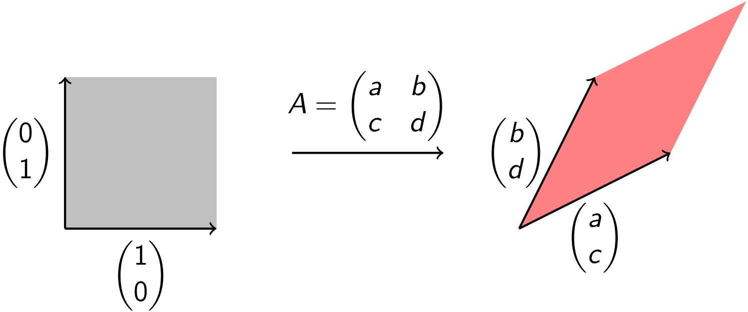
Geometric interpretation of determinants

Theorem



Geometric interpretation of determinants

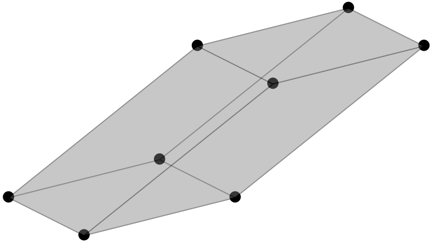
Theorem



The area of the red parallelogram is $|\det(A)|$.

Theorem

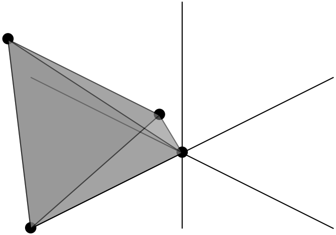
$|\det(A)|$ *is the volume of* $A(C)$, *where* C *is the unit n-cube.*



"Parallelopiped"

Theorem

Let $a_1, \ldots, a_n$ be column vectors of height n.
Let A be the matrix whose columns are $a_1, \ldots, a_n$.
Let Δ be the simplex whose vertices are at $0, a_1, \ldots, a_n$.
Then $\mathrm{Vol}(\Delta) = \frac{1}{n!}\det(A)$.



$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \sqrt{2} & \sqrt{2} \end{pmatrix}$$