Linear Algebra Worksheet 3

Jonny Evans

Workshop 3

Exercise 1.1. For each matrix below, determine if it is invertible and, if it is, write down the inverse.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 91 & 45 \\ 2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Exercise 1.2. For which values of the parameter t are the following matrices invertible?

$$\begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}, \quad \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ t & 7 \end{pmatrix}.$$

Exercise 1.3. For each matrix below, write down the augmented matrix, put the left-hand side into echelon form using only row operations of type $R_i \mapsto R_i + \lambda R_j$ and hence compute the determinant. Then further, put the left-hand side into reduced echelon form, and hence compute the inverses.

$$A = \begin{pmatrix} 1 & 5 & -2 \\ -1 & 0 & -5 \\ 4 & -3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -5 & 1 & 3 \\ -4 & -1 & 3 \\ 5 & -3 & -4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 0 & 3 \\ -3 & -5 & 0 & 0 \\ -1 & -2 & 1 & -3 \\ 4 & 7 & 1 & -2 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 5 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$$E = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -17 & -1 & 2 & 0 \\ -4 & -1 & 1 & 0 \\ 63 & 21 & -21 & -12 \end{pmatrix} \qquad F = \begin{pmatrix} -20 & -106 & -2 & 0 \\ 0 & -3 & -1 & 0 \\ 20 & 100 & -1 & 0 \\ -100 & -500 & 10 & 1 \end{pmatrix}$$

Exercise 1.4. Write down the inverses of the 4-by-4 elementary matrices

$$E_{42}(4), \quad E_{21}(2), \quad E_4(3), \quad E_{34}(1), \quad E_{13}(2)$$

(for example the inverse of $E_{42}(4)$ is $E_{42}(-4)$, rather than as a full matrix) find the product

$$E_{42}(4)E_{21}(2)E_4(3)E_{34}(1)E_{13}(2)$$

as a 4-by-4 matrix. What is the inverse of this product?

Exercise 1.5. Let A be an m-by-n matrix. Prove the following statements:

- The number of leading indices is less than or equal to $\min(m, n)$ (where $(\min(a, b)$ denotes the minimum of a and b).
- If $m < n$, the equation $Av = 0$ has a nonzero solution (Hint: This is equivalent to showing there are some free indices).
- Suppose $m = n$. Then A is invertible if and only if $Av = 0$ has no solution other than $v = 0$.
- Suppose that $m = n$ and that B is another n-by-n matrix. If A is not invertible then BA is not invertible.

2 Solutions

Solution 2.1. The matrices have determinants:

$$\det(A) = 13 \qquad \det(B) = 2 \qquad \det(C) = 1$$
$$\det(D) = -5 \qquad \det(E) = 0 \qquad \det(F) = 0.$$

Therefore A, B, C, D are invertible, with inverses:

$$A^{-1} = \frac{1}{13}\begin{pmatrix} 8 & -3 \\ -1 & 2 \end{pmatrix} \qquad B^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$
$$C^{-1} = \begin{pmatrix} 1 & -45 \\ 2 & 91 \end{pmatrix} \qquad D^{-1} = -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}.$$

Solution 2.2. The determinants of the three matrices are:

$$t^2 - 1, \quad -\cos^2 t - \sin^2 t = -1, \quad 14 - t,$$

so the determinant is nonzero for $t \neq \pm 1$, for all t and for $t \neq 14$ respectively.

Solution 2.3.

(a) We start with the matrix

$$A = \left(\begin{array}{ccc|ccc} 1 & 5 & -2 & 1 & 0 & 0 \\ -1 & 0 & -5 & 0 & 1 & 0 \\ 4 & -3 & 3 & 0 & 0 & 1 \end{array}\right)$$

Clear row 2 using $R_2 \mapsto R_2 + (1)R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -2 & 1 & 0 & 0 \\ 0 & 5 & -7 & 1 & 1 & 0 \\ 4 & -3 & 3 & 0 & 0 & 1 \end{array}\right)$$

Clear row 3 using $R_3 \mapsto R_3 + (-4)R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -2 & 1 & 0 & 0 \\ 0 & 5 & -7 & 1 & 1 & 0 \\ 0 & -23 & 11 & -4 & 0 & 1 \end{array}\right)$$

Clear row 3 using $R_3 \mapsto R_3 + (23/5)R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -2 & 1 & 0 & 0 \\ 0 & 5 & -7 & 1 & 1 & 0 \\ 0 & 0 & -106/5 & 3/5 & 23/5 & 1 \end{array}\right)$$

At this point, we see that the determinant is -106.

Make leading entry in row 1 equal to 1 using $R_1 \mapsto (1/5)R_1$.

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -2 & 1 & 0 & 0 \\ 0 & 1 & -7/5 & 1/5 & 1/5 & 0 \\ 0 & 0 & -106/5 & 3/5 & 23/5 & 1 \end{array}\right)$$

Clear column 2 using $R_1 \mapsto R_1 + (-5)R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & -1 & 0 \\ 0 & 1 & -7/5 & 1/5 & 1/5 & 0 \\ 0 & 0 & -106/5 & 3/5 & 23/5 & 1 \end{array}\right)$$

Make leading entry in row 2 equal to 1 using $R_2 \mapsto (-5/106)R_2$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & -1 & 0 \\ 0 & 1 & -7/5 & 1/5 & 1/5 & 0 \\ 0 & 0 & 1 & -3/106 & -23/106 & -5/106 \end{array}\right)$$

Clear column 3 using $R_1 \mapsto R_1 + (-5)R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 15/106 & 9/106 & 25/106 \\ 0 & 1 & -7/5 & 1/5 & 1/5 & 0 \\ 0 & 0 & 1 & -3/106 & -23/106 & -5/106 \end{array}\right)$$

Clear column 3 using $R_2 \mapsto R_2 + (7/5)R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 15/106 & 9/106 & 25/106 \\ 0 & 1 & 0 & 17/106 & -11/106 & -7/106 \\ 0 & 0 & 1 & -3/106 & -23/106 & -5/106 \end{array}\right)$$

Now the right-hand block is the inverse of the matrix we started with.

(b) We start with the matrix

$$B = \left(\begin{array}{ccc|ccc} -5 & 1 & 3 & 1 & 0 & 0 \\ -4 & -1 & 3 & 0 & 1 & 0 \\ 5 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

Clear row 2 using $R_2 \mapsto R_2 + (-4/5)R_1$

$$\left(\begin{array}{ccc|ccc} -5 & 1 & 3 & 1 & 0 & 0 \\ 0 & -9/5 & 3/5 & -4/5 & 1 & 0 \\ 5 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

Clear row 3 using $R_3 \mapsto R_3 + (1)R_1$

$$\left(\begin{array}{ccc|ccc} -5 & 1 & 3 & 1 & 0 & 0 \\ 0 & -9/5 & 3/5 & -4/5 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 & 1 \end{array}\right)$$

Clear row 3 using $R_3 \mapsto R_3 + (-10/9)R_2$

$$\left(\begin{array}{ccc|ccc} -5 & 1 & 3 & 1 & 0 & 0 \\ 0 & -9/5 & 3/5 & -4/5 & 1 & 0 \\ 0 & 0 & -5/3 & 17/9 & -10/9 & 1 \end{array}\right)$$

At this point, we see that the determinant is -3.

Make leading entry in row 0 equal to 1 using $R_0 \mapsto (-1/5)R_0$.

$$\left(\begin{array}{ccc|ccc} 1 & -1/5 & -3/5 & -1/5 & 0 & 0 \\ 0 & -9/5 & 3/5 & -4/5 & 1 & 0 \\ 0 & 0 & -5/3 & 17/9 & -10/9 & 1 \end{array}\right)$$

Make leading entry in row 1 equal to 1 using $R_1 \mapsto (-5/9)R_1$.

$$\left(\begin{array}{ccc|ccc} 1 & -1/5 & -3/5 & -1/5 & 0 & 0 \\ 0 & 1 & -1/3 & 4/9 & -5/9 & 0 \\ 0 & 0 & -5/3 & 17/9 & -10/9 & 1 \end{array}\right)$$

Clear column 2 using $R_1 \mapsto R_1 + (1/5)R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2/3 & -1/9 & -1/9 & 0 \\ 0 & 1 & -1/3 & 4/9 & -5/9 & 0 \\ 0 & 0 & -5/3 & 17/9 & -10/9 & 1 \end{array}\right)$$

Make leading entry in row 2 equal to 1 using $R_2 \mapsto (-3/5)R_2$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2/3 & -1/9 & -1/9 & 0 \\ 0 & 1 & -1/3 & 4/9 & -5/9 & 0 \\ 0 & 0 & 1 & -17/15 & 2/3 & -3/5 \end{array}\right)$$

Clear column 3 using $R_1 \mapsto R_1 + (2/3)R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -13/15 & 1/3 & -2/5 \\ 0 & 1 & -1/3 & 4/9 & -5/9 & 0 \\ 0 & 0 & 1 & -17/15 & 2/3 & -3/5 \end{array}\right)$$

Clear column 3 using $R_2 \mapsto R_2 + (1/3)R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -13/15 & 1/3 & -2/5 \\ 0 & 1 & 0 & 1/15 & -1/3 & -1/5 \\ 0 & 0 & 1 & -17/15 & 2/3 & -3/5 \end{array}\right)$$

Now the right-hand block is the inverse of the matrix we started with.

(c) We start with the matrix

$$C = \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ -3 & -5 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & -3 & 0 & 0 & 1 & 0 \\ 4 & 7 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear row 2 using $R_2 \mapsto R_2 + (3)R_1$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ -1 & -2 & 1 & -3 & 0 & 0 & 1 & 0 \\ 4 & 7 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear row 3 using $R_3 \mapsto R_3 + (1)R_1$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 7 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear row 4 using $R_4 \mapsto R_4 + (-4)R_1$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -10 & -4 & 0 & 0 & 1 \end{array}\right)$$

Clear row 4 using $R_4 \mapsto R_4 + (1)R_2$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \end{array}\right)$$

Clear row 4 using $R_4 \mapsto R_4 + (-1)R_3$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 & -1 & 1 \end{array}\right)$$

At this point, we see that the determinant is -1.

Clear column 2 using $R_1 \mapsto R_1 + (-2)R_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -15 & -5 & -2 & 0 & 0 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 & -1 & 1 \end{array}\right)$$

Make leading entry in row 3 equal to 1 using $R_3 \mapsto (-1)R_3$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -15 & -5 & -2 & 0 & 0 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 1 & -1 \end{array}\right)$$

Clear column 4 using $R_1 \mapsto R_1 + (15)R_4$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 25 & -17 & 15 & -15 \\ 0 & 1 & 0 & 9 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 1 & -1 \end{array}\right)$$

Clear column 4 using $R_2 \mapsto R_2 + (-9)R_4$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 25 & -17 & 15 & -15 \\ 0 & 1 & 0 & 0 & -15 & 10 & -9 & 9 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 1 & -1 \end{array}\right)$$

Now the right-hand block is the inverse of the matrix we started with.

(d) We start with the matrix

$$D = \left(\begin{array}{cccc|cccc} 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear row 3 using $R_3 \mapsto R_3 + (-1)R_1$

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear row 4 using $R_4 \mapsto R_4 + (-1/2)R_1$

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1/2 & 0 & 0 & 1 \end{array}\right)$$

Clear row 4 using $R_4 \mapsto R_4 + (-1)R_2$

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1/2 & -1 & 0 & 1 \end{array}\right)$$

Clear row 4 using $R_4 \mapsto R_4 + (1/3)R_3$

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -5/6 & -1 & 1/3 & 1 \end{array}\right)$$

At this point, we see that the determinant is 6.

Make leading entry in row 0 equal to 1 using $R_0 \mapsto (1/2)R_0$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -5/6 & -1 & 1/3 & 1 \end{array}\right)$$

Make leading entry in row 2 equal to 1 using $R_2 \mapsto (1/3)R_2$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & -5/6 & -1 & 1/3 & 1 \end{array}\right)$$

Clear column 3 using $R_1 \mapsto R_1 + (-1)R_3$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5/6 & 0 & -1/3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & -5/6 & -1 & 1/3 & 1 \end{array}\right)$$

Clear column 3 using $R_2 \mapsto R_2 + (-1)R_3$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5/6 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & 1 & 1/3 & 1 & -1/3 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & -5/6 & -1 & 1/3 & 1 \end{array}\right)$$

Clear column 4 using $R_2 \mapsto R_2 + (-1)R_4$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5/6 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & 0 & 7/6 & 2 & -2/3 & -1 \\ 0 & 0 & 1 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & -5/6 & -1 & 1/3 & 1 \end{array}\right)$$

Now the right-hand block is the inverse of the matrix we started with.

(e) We start with the matrix

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -17 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\ -4 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 63 & 21 & -21 & -12 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear column 1, row 2 using $R_2 \mapsto R_2 + (17/3)R_1$

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 17/3 & 1 & 0 & 0 \\ -4 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 63 & 21 & -21 & -12 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear column 1, row 3 using $R_3 \mapsto R_3 + (4/3)R_1$

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 17/3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 4/3 & 0 & 1 & 0 \\ 63 & 21 & -21 & -12 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear column 1, row 4 using $R_4 \mapsto R_4 + (-21)R_1$

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 17/3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 4/3 & 0 & 1 & 0 \\ 0 & 21 & -21 & -12 & -21 & 0 & 0 & 1 \end{array}\right)$$

Clear column 2, row 3 using $R_3 \mapsto R_3 + (-1)R_2$

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 17/3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -13/3 & -1 & 1 & 0 \\ 0 & 21 & -21 & -12 & -21 & 0 & 0 & 1 \end{array}\right)$$

Clear column 2, row 4 using $R_4 \mapsto R_4 + (21)R_2$

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 17/3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -13/3 & -1 & 1 & 0 \\ 0 & 0 & 21 & -12 & 98 & 21 & 0 & 1 \end{array}\right)$$

Clear column 3, row 4 using $R_4 \mapsto R_4 + (21)R_3$

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 17/3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -13/3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -12 & 7 & 0 & 21 & 1 \end{array}\right)$$

At this point, we see that the determinant is -36.

Make leading entry in row 1 equal to 1 using $R_1 \mapsto (1/3)R_1$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 17/3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -13/3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -12 & 7 & 0 & 21 & 1 \end{array}\right)$$

Make leading entry in row 2 equal to 1 using $R_2 \mapsto (-1)R_2$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -17/3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -13/3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -12 & 7 & 0 & 21 & 1 \end{array}\right)$$

Make leading entry in row 3 equal to 1 using $R_3 \mapsto (-1)R_3$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -17/3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 13/3 & 1 & -1 & 0 \\ 0 & 0 & 0 & -12 & 7 & 0 & 21 & 1 \end{array}\right)$$

Clear column 3 using $R_2 \mapsto R_2 + (2)R_3$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 13/3 & 1 & -1 & 0 \\ 0 & 0 & 0 & -12 & 7 & 0 & 21 & 1 \end{array}\right)$$

Make leading entry in row 4 equal to 1 using $R_4 \mapsto (-1/12)R_4$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 13/3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -7/12 & 0 & -7/4 & -1/12 \end{array}\right)$$

Now the right-hand block is the inverse of the matrix we started with.

(f) We start with the matrix

$$\left(\begin{array}{cccc|cccc} -20 & -106 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ 20 & 100 & -1 & 0 & 0 & 0 & 1 & 0 \\ -100 & -500 & 10 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear column 1, row 3 using $R_3 \mapsto R_3 + (1)R_1$

$$\left(\begin{array}{cccc|cccc} -20 & -106 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -6 & -3 & 0 & 1 & 0 & 1 & 0 \\ -100 & -500 & 10 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Clear column 1, row 4 using $R_4 \mapsto R_4 + (-5)R_1$

$$\left(\begin{array}{cccc|cccc} -20 & -106 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -6 & -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 30 & 20 & 1 & -5 & 0 & 0 & 1 \end{array}\right)$$

Clear column 2, row 3 using $R_3 \mapsto R_3 + (-2)R_2$

$$\left(\begin{array}{cccc|cccc} -20 & -106 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 30 & 20 & 1 & -5 & 0 & 0 & 1 \end{array}\right)$$

Clear column 2, row 4 using $R_4 \mapsto R_4 + (10)R_2$

$$\left(\begin{array}{cccc|cccc} -20 & -106 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 10 & 1 & -5 & 10 & 0 & 1 \end{array}\right)$$

Clear column 3, row 4 using $R_4 \mapsto R_4 + (10)R_3$

$$\left(\begin{array}{cccc|cccc} -20 & -106 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -10 & 10 & 1 \end{array}\right)$$

At this point, we see that the determinant is -60.

Make leading entry in row 1 equal to 1 using $R_1 \mapsto (-1/20)R_1$.

$$\left(\begin{array}{cccc|cccc} 1 & 53/10 & 1/10 & 0 & -1/20 & 0 & 0 & 0 \\ 0 & -3 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -10 & 10 & 1 \end{array}\right)$$

Make leading entry in row 2 equal to 1 using $R_2 \mapsto (-1/3)R_2$.

$$\left(\begin{array}{cccc|cccc} 1 & 53/10 & 1/10 & 0 & -1/20 & 0 & 0 & 0 \\ 0 & 1 & 1/3 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -10 & 10 & 1 \end{array}\right)$$

Clear column 2 using $R_1 \mapsto R_1 + (-53/10)R_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -5/3 & 0 & -1/20 & 53/30 & 0 & 0 \\ 0 & 1 & 1/3 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -10 & 10 & 1 \end{array}\right)$$

Make leading entry in row 3 equal to 1 using $R_3 \mapsto (-1)R_3$.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -5/3 & 0 & -1/20 & 53/30 & 0 & 0 \\ 0 & 1 & 1/3 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -10 & 10 & 1 \end{array}\right)$$

Clear column 3 using $R_1 \mapsto R_1 + (5/3)R_3$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -103/60 & 51/10 & -5/3 & 0 \\ 0 & 1 & 1/3 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -10 & 10 & 1 \end{array}\right)$$

Clear column 3 using $R_2 \mapsto R_2 + (-1/3)R_3$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -103/60 & 51/10 & -5/3 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & -1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -10 & 10 & 1 \end{array}\right)$$

Now the right-hand block is the inverse of the matrix we started with.

Solution 2.4. We have

$$E_{42}(4)^{-1} = E_{42}(-4) \qquad E_{21}(2)^{-1} = E_{21}(-2) \qquad E_4(3)^{-1} = E_4(1/3)$$
$$E_{34}(1)^{-1} = E_{34}(-1) \qquad E_{13}(2)^{-1} = E_{13}(-2)$$

We can calculate the product

$$E_{42}(4)E_{21}(2)E_4(3)E_{34}(1)E_{13}(2)$$

by performing row operations to the identity (starting with $R_1 \mapsto R_1+2R_3$, ending with $R_4 \mapsto R_4+4R_2$), yielding:

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 8 & 4 & 0 & 3 \end{pmatrix}$$

The inverse of this product is the product

$$E_{13}(2)^{-1}E_{34}(1)^{-1}E_4(3)^{-1}E_{21}(2)^{-1}E_{42}(4)^{-1},$$

which is (again, applying row operations to the identity)

$$\begin{pmatrix} 1 & -8/3 & -2 & 2/3 \\ -2 & 1 & 0 & 0 \\ 0 & 4/3 & 1 & -1/3 \\ 0 & -4/3 & 0 & 1/3 \end{pmatrix}.$$

Solution 2.5.

- Each leading index refers to a row and there are m rows, so there are at most m leading indices.
- If $m < n$ then there must be some free indices because there are n columns and at most m leading indices. The space of solutions to $Av = 0$ has dimension equal to the number of free indices providing it's nonempty, so if it's nonempty it contains a nonzero solution. It's nonempty because 0 is always a solution.
- If A is invertible then its reduced echelon form is the identity, so $Av = 0$ has a unique solution, which is $v = 0$. Conversely, if A is not invertible then its reduced echelon form is not the identity. Since $m = n$, this means there has to be a free index, so again $Av = 0$ has a positive-dimensional solution space.
- If A is not invertible then there is a nonzero v with $Av = 0$ (using the previous part of the question). Then $BAv = 0$. Therefore BA is not invertible (using the previous part of the question).