Linear Algebra Worksheet 3

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Workshop 3

Exercise 1.1. For each matrix below, determine if it is invertible and, if it is, write down the inverse.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 91 & 45 \\ 2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Exercise 1.2. For which values of the parameter t are the following matrices invertible?

$$\begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix}, \quad \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ t & 7 \end{pmatrix}.$$

Exercise 1.3. For each matrix below, write down the augmented matrix, put the left-hand side into echelon form using only row operations of type $R_i \mapsto R_i + \lambda R_j$ and hence compute the determinant. Then further, put the left-hand side into reduced echelon form, and hence compute the inverses.

$$A = \begin{pmatrix} 1 & 5 & -2 \\ -1 & 0 & -5 \\ 4 & -3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -5 & 1 & 3 \\ -4 & -1 & 3 \\ 5 & -3 & -4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 & 0 & 3 \\ -3 & -5 & 0 & 0 \\ -1 & -2 & 1 & -3 \\ 4 & 7 & 1 & -2 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 5 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$$E = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -17 & -1 & 2 & 0 \\ -4 & -1 & 1 & 0 \\ 63 & 21 & -21 & -12 \end{pmatrix} \qquad F = \begin{pmatrix} -20 & -106 & -2 & 0 \\ 0 & -3 & -1 & 0 \\ 20 & 100 & -1 & 0 \\ -100 & -500 & 10 & 1 \end{pmatrix}$$

Exercise 1.4. Write down the inverses of the 4-by-4 elementary matrices

$$E_{42}(4), \quad E_{21}(2), \quad E_4(3), \quad E_{34}(1), \quad E_{13}(2)$$

(for example the inverse of $E_{42}(4)$ is $E_{42}(-4)$, rather than as a full matrix) find the product

$$E_{42}(4)E_{21}(2)E_4(3)E_{34}(1)E_{13}(2)$$

as a 4-by-4 matrix. What is the inverse of this product?

Exercise 1.5. Let A be an m-by-n matrix. Prove the following statements:

- The number of leading indices is less than or equal to $\min(m, n)$ (where $(\min(a, b)$ denotes the minimum of a and b).
- If $m < n$, the equation $Av = 0$ has a nonzero solution (Hint: This is equivalent to showing there are some free indices).
- Suppose $m = n$. Then A is invertible if and only if $Av = 0$ has no solution other than $v = 0$.
- Suppose that $m = n$ and that B is another n-by-n matrix. If A is not invertible then BA is not invertible.

2 Assessed problems for Week 3

Exercise 2.1. Find the determinant of the matrix **4 marks**

$$\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}.$$

Can you give an example of four real numbers t, x, y, z (not all zero!) such that this determinant vanishes?

Exercise 2.2. Find the determinant of the following matrix by reducing to echelon form using only row operations of the form $R_i \mapsto R_i + \lambda R_j$, $i \neq j$. **4 marks**

$$A = \begin{pmatrix} -4 & -2 & 3 & 2 \\ 5 & 1 & 0 & 4 \\ -2 & -3 & -1 & -1 \\ -1 & -3 & 2 & 3 \end{pmatrix}$$

Exercise 2.3. Find the inverse of the following matrices: **4+8 marks**

$$\begin{pmatrix} 4 & -25 & -13 \\ 0 & 5 & 1 \\ -20 & 150 & 75 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -2 & -1 \\ 0 & 85 & -6 & 0 \\ 0 & -70 & 5 & 0 \\ 1 & 0 & 2 & 1 \end{pmatrix}.$$