Linear Algebra Worksheet 5

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Workshop 5

Exercise 5.1. Consider the ellipsoid defined by the equation

$$2(x^2 + y^2 + z^2 - xy - yz) = 1.$$

Write this equation in the form $v^T A v = 1$ for $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and some matrix A. Find the eigenvalues of A and check they are all positive. Find the principal directions and principal radii of this ellipsoid. (Hint: Recall from lectures that if the ellipsoid is cut out by the equation $v^T A v = 1$ then the principal directions are the eigendirections for A and the principal radii are $1/\sqrt{\lambda}$ for the corresponding eigenvalues λ.)

Exercise 5.2. Let a, b be two numbers and consider the matrix $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. What are the eigenvalues and eigenvectors of this matrix? Suppose that $a + b > 1$ and $1 > a - b > 0$. What happens to the vectors $M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $M^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as $n \to \infty$?

Exercise 5.3. Find the kernel and nullity for each of the following matrices

$$A = \begin{pmatrix} 1 & 0 & 7 \\ 4 & 2 & 1 \\ 3 & 2 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}.$$

What is the rank in each case?

Exercise 5.4. We say that a complex matrix A is *Hermitian* if $\bar{A}^T = A$ (here the bar means complex conjugation of each matrix entry). Show that any eigenvalue of a Hermitian matrix is real and that if v and w are eigenvectors for distinct eigenvalues $\lambda \neq \mu$ then $\bar{v}^T w = 0$. *These observations are important in quantum mechanics, because eigenvalues of Hermitian operators are what we measure as observable quantities like energy.*

Exercise 5.5. Find the limit $\lim_{n\to\infty} \frac{P_{n+1}}{P_n}$ where P_n is the sequence of Pell numbers (defined by the recurrence $P_{n+2} = 2P_{n+1} + P_n$, $P_1 = 0$, $P_2 = 1$).

Exercise 5.6. Find the characteristic polynomials of the matrices:

$$\begin{pmatrix} 0 & -c_0 \\ 1 & -c_1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -c_0 \\ 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \end{pmatrix}.$$

Have a guess at the characteristic polynomial of

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & 1 & 0 & -c_{n-2} \\ 0 & \cdots & 0 & 1 & -c_{n-1} \end{pmatrix}.$$

Prove your guess by induction.

Exercise 5.7. Solve Exercise 5.6 using as many different methods to compute determinants as you can.

6 Challenge problems

Exercise 6.1. Given numbers $\lambda_1, \ldots, \lambda_n$, the matrix

$$V = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots & \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

is called the *Vandermonde matrix*. Consider the quantity

$$Q = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$

(for example, if $n = 3$, $Q = (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)$). Verify that $\det(V) = Q$ for $n = 2$ and $n = 3$. Show that both $\det(V)$ and Q vanish if $\lambda_k = \lambda_\ell$ for some $k \neq \ell$.

In fact, $\det(V) = Q$ *for all* n.

Exercise 6.2. Let $C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & 1 & 0 & -c_{n-2} \\ 0 & \cdots & 0 & 1 & -c_{n-1} \end{pmatrix}$, (see Exercise 5.6 for its characteristic polynomial!).
Show that if λ is an eigenvalue of C then the row vector $w = \begin{pmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-1} \end{pmatrix}$ satisfies $wC = \lambda w$. Deduce that if C has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and V is the Vandermonde matrix from Exercise 6.1 whose ith row is $\begin{pmatrix} 1 & \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1} \end{pmatrix}$ then VCV^{-1} is the diagonal matrix $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. Hence prove that $Tr(C^m) = \sum_{i=1}^n \lambda_i^m$ for all m.

Exercise 6.3. If $t^n + c_{n-1}t^{n-1} + \cdots + c_0$ is a polynomial of degree n with leading coefficient 1 and distinct roots, show that the sum of squares of its roots is $c_{n-1}^2 - 2c_{n-2}$ and find a similar expression for the sum of cubes of the roots. (Hint: Use Exercise 6.2)

Exercise 6.4. Given a polynomial $p(z) = p_m z^m + \cdots + p_0$ of degree m and a polynomial $q(z) = q_n z^n + \cdots + q_0$ of degree n, the *resultant* of p and q is defined to be the determinant of the $(m+n)$-by-$(m+n)$ matrix

$$\begin{pmatrix} p_m & p_{m-1} & \cdots & p_1 & p_0 & 0 & 0 & \cdots & 0 \\ 0 & p_m & p_{m-1} & \cdots & p_1 & p_0 & 0 & \ddots & \vdots \\ 0 & 0 & p_m & p_{m-1} & \cdots & p_1 & p_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & p_m & p_{m-1} & \cdots & p_1 & p_0 \\ q_n & q_{n-1} & \cdots & q_0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & q_m & q_{m-1} & \cdots & q_0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & q_m & q_{m-1} & \cdots & q_0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & \ddots & \vdots \\ \vdots & & & & & \ddots & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & q_m & q_{m-1} & \cdots & q_0 \end{pmatrix}$$

It is a theorem of Sylvester that the resultant of p and q vanishes if and only if p and q have a common factor.

Write this matrix out in the case when $m = 2$, $n = 1$ and verify that the resultant vanishes if and only if there exists a polynomial $a + bz$ such that $p(z) = q(z)(a + bz)$).

7 Solutions

Solution 7.1. The ellipsoid equation can be written as $v^T A v = 1$ for

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The eigenvalues of A are the solutions to $\det(A - tI) = 0$, i.e. $2, 2 \pm \sqrt{2}$. The eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ \mp\sqrt{2} \\ 1 \end{pmatrix}$: these are the principal directions. The principal radii are $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2\pm\sqrt{2}}}$.

Solution 7.2. The eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are the solutions of $(a-t)^2 - b^2 = 0$, i.e. $t^2 - 2at + a^2 - b^2 = 0$. These solutions are $\frac{2a \pm \sqrt{4a^2 - 4a^2 + 4b^2}}{2} = a \pm b$. The corresponding eigenvectors are the solutions of

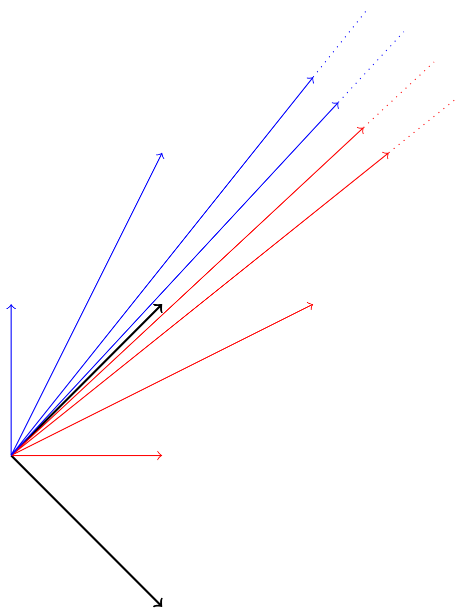
$$\begin{aligned} ax + by &= (a \pm b)x \\ bx + ay &= (a \pm b)y, \end{aligned}$$

which means $y = \pm x$. Therefore the eigenvectors are multiples of $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$.

We have $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so

$$M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} M^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} M^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}(a+b)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(a-b)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so $M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gets closer and closer to the line $y = x$ as $n \to \infty$ (because $(a+b)^n \to \infty$ as $a + b > 1$ and $(a-b)^n \to 0$ as $0 < a - b < 1$; see the red vectors in the figure below). Similarly $M^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gets closer to this line (blue vectors in the figure below).



Solution 7.3. The equation $Av = 0$ is

$$\begin{aligned} x + 7z &= 0 \\ 4x + 2y + z &= 0 \\ 3x + 2y - 6z &= 0 \end{aligned}$$

which has solution $x = -7z$, $y = 27z/2$, so the kernel consists of vectors of the form $\begin{pmatrix} -7z \\ 27z/2 \\ z \end{pmatrix}$. The nullity is 1 in this instance. By the rank-nullity theorem, the rank is 2.

The equation $Bv = 0$ is

$$\begin{aligned} 2x + 3y + w &= 0 \\ x + 2z &= 0, \end{aligned}$$

which has solution $x = -2z$, $w = 4z - 3y$, so the kernel consists of vectors of the form $\begin{pmatrix} -2z \\ y \\ z \\ 4z - 3y \end{pmatrix}$. The nullity is 2 in this instance. By the rank-nullity theorem, the rank is 2.

The equation $Cv = 0$ is

$$x - z = 0,$$

which has solution $z = x$, so the kernel consists of vectors of the form $\begin{pmatrix} x \\ y \\ x \end{pmatrix}$. The nullity in this instance is 2. By the rank-nullity theorem, the rank is 1.

Solution 7.4 (Solution to Exercise 5.4)**.** Suppose that $Av = \lambda v$. Consider the expression $\bar{v}^T Av$, where $\bar{v}$ denotes complex conjugation. Then, because $A = \bar{A}^T$, we have

$$\bar{\lambda}\bar{v}^T v = (\overline{Av})^T v = \bar{v}^T Av = \lambda \bar{v}^T v.$$

Note that if $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then $\bar{v}^T v = \sum |x_1|^2 + \cdots + |x_n|^2 > 0$ if $v \neq 0$, so dividing through by $\bar{v}^T v$ we get $\bar{\lambda} = \lambda$ and deduce that λ is real.

If v and w are two eigenvectors for distinct eigenvalues λ, μ then

$$\begin{aligned} \lambda \bar{w}^T v &= \bar{w}^T (Av) \\ &= \overline{(Aw)}^T v \\ &= \mu \bar{w}^T v \end{aligned}$$

(since $\bar{\mu} = \mu$ by the first part) so, since $\lambda \neq \mu$, we must have $\bar{w}^T v = 0$.

Solution 7.5. The Pell numbers satisfy

$$\begin{pmatrix} P_{n+2} \\ P_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix}.$$

(Let's write M for the matrix in this expression). The eigenvalues of M are the solutions of $0 = \det(M - tI)$, i.e.

$$0 = -t(2 - t) - 1 = t^2 - 2t - 1,$$

which are $\frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$. The eigenvectors are $\begin{pmatrix} 1 \\ 1 \pm \sqrt{2} \end{pmatrix}$. This means that $\lim_{n \to \infty} \frac{P_{n+1}}{P_n} = 1 + \sqrt{2}$ (by the same argument that we used for the Fibonacci numbers in lectures).

Solution 7.6. We have

$$\det\begin{pmatrix}-t & -c_0\\ 1 & -c_1-t\end{pmatrix} = t^2+c_1t+c_0$$

$$\det\begin{pmatrix}-t & 0 & -c_0\\ 1 & -t & -c_1\\ 0 & 1 & -c_2-t\end{pmatrix} = -(t^3+c_2t^2+c_1t+c_0)$$

$$\det\begin{pmatrix}-t & 0 & 0 & -c_0\\ 1 & -t & 0 & -c_1\\ 0 & 1 & -t & -c_2\\ 0 & 0 & 1 & -c_3-t\end{pmatrix} = t^4+c_3t^3+c_2t^2+c_1t+c_0.$$

More generally, we have

$$\det\begin{pmatrix}-t & 0 & \cdots & 0 & -c_0\\ 1 & \ddots & \ddots & \vdots & \vdots\\ 0 & \ddots & \ddots & 0 & -c_{n-3}\\ \vdots & \ddots & 1 & -t & -c_{n-2}\\ 0 & \cdots & 0 & 1 & -c_{n-1}-t\end{pmatrix} = (-1)^n(t^n+c_{n-1}t^{n-1}+\cdots+c_0).$$

To prove this by induction, assume it's true for $n-1$ and let's evaluate the determinant

$$D_n := \det\begin{pmatrix}-t & 0 & \cdots & 0 & -c_0\\ 1 & \ddots & \ddots & \vdots & \vdots\\ 0 & \ddots & \ddots & 0 & -c_{n-3}\\ \vdots & \ddots & 1 & -t & -c_{n-2}\\ 0 & \cdots & 0 & 1 & -c_{n-1}-t\end{pmatrix}.$$

Expanding along the first column, we get

$$-tD_{n-1} - \det\begin{pmatrix}0 & \cdots & \cdots & 0 & -c_0\\ 1 & -t & & 0 & -c_1\\ 0 & 1 & \ddots & & \vdots\\ \vdots & \ddots & \ddots & -t & -c_{n-2}\\ 0 & \cdots & 0 & 1 & -c_{n-1}-t\end{pmatrix}$$

We evaluate this final determinant as follows: pick c_0 from the top row (everything else is zero); pick either 1 or $-t$ from the second row; if you picked 1 you can pick either 1 or $-t$ from the third row whereas if you picked $-t$ you have to pick $-t$ from the third row; continue in this manner and you either pick all the 1s or else you pick $-t$ until you reach the last row and you're forced to pick a zero. Therefore the only term we get is c_0 (with a sign). The sign is $(-1)^{n+1}$ as we can see because this permutation is $(123\cdots n)$. Therefore

$$D_n = -tD_{n-1} - (-1)^{n+1}c_0 = (-1)^n(t^n+c_{n-1}t^{n-1}+\cdots+c_0)$$

by inductive hypothesis (i.e. substituting in what we assumed was the formula for D_{n-1}). We already checked the base case D_2, so we're done.

8 Solutions to challenge problems

Solution 8.1. We have

$$\det\begin{pmatrix}1 & \lambda_1\\ 1 & \lambda_2\end{pmatrix} = \lambda_2-\lambda_2.$$

Next:

$$\det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} = \lambda_2\lambda_3^2 - \lambda_2^2\lambda_3 - \lambda_1(\lambda_3^2 - \lambda_2^2) + \lambda_1^2(\lambda_3 - \lambda_2)$$

i.e. $\lambda_2\lambda_3(\lambda_3 - \lambda_2) - \lambda_1(\lambda_3 - \lambda_2)(\lambda_3 + \lambda_2) + \lambda_1^2(\lambda_3 - \lambda_2)$, so we can pull out a factor of $\lambda_3 - \lambda_2$, leaving

$$\lambda_2\lambda_3 - \lambda_1\lambda_3 - \lambda_1\lambda_2 + \lambda_1^2 = (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)$$

and again we get $\det(V) = Q$ as required.

If $\lambda_k = \lambda_\ell$ then the Vandermonde determinant vanishes (two rows coincide) and the expression Q vanishes (because $\lambda_k - \lambda_\ell$ is a factor.

Solution 8.2. We saw that the characteristic polynomial of C is $\pm(t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$, so the eigenvalues of C are the roots $\lambda_1, \ldots, \lambda_n$ of this polynomial. If we form the corresponding Vandermonde matrix whose ith row is given by powers of λ_i then we see that

$$\begin{aligned} \begin{pmatrix} 1 & \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1} \end{pmatrix} C &= \begin{pmatrix} \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1} & -c_0 - c_1\lambda_i - \cdots - c_{n-1}\lambda_i^{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_i & \lambda_i^2 & \cdots & \lambda_i^{n-1} & \lambda_i^n \end{pmatrix} \\ &= \lambda \begin{pmatrix} 1 & \lambda & \cdots & \lambda^{n-1} \end{pmatrix}. \end{aligned}$$

Therefore the ith row of VC is λ_i times the ith row of V. Let R_i be the ith row of V and C_i be the ith columns of V^{-1} (so $R_iC_j = \delta_{ij}$). Then VC has rows $\lambda_i R_i$, so the ij entry of VCV^{-1} is $\lambda_i R_i C_j = \lambda_i \delta_{ij}$, and we see that VCV^{-1} is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ down the diagonal.

We know that $\mathrm{Tr}((VCV^{-1})^m) = \mathrm{Tr}(VC^mV^{-1}) = \mathrm{Tr}(V^{-1}VC^m) = \mathrm{Tr}(C^m)$ because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ from sheet 1. Therefore $\mathrm{Tr}(C^m) = \mathrm{Tr}(diag(\lambda_1^m, \ldots, \lambda_n^m)) = \sum \lambda_i^m$.

Solution 8.3. By the previous question, we know that the sum of squares of the roots of this polynomial is $\mathrm{Tr}(C^2)$. The only nonzero diagonal entries of C^2 are the final two, which are $-c_{n-2}$ and $c_{n-1}^2 - c_{n-2}$, so we get $c_{n-1}^2 - 2c_{n-2}$ as required. For the sum of cubes, we compute $\mathrm{Tr}(C^3)$ and we get $-c_{n-1}^3 + 3c_{n-2}c_{n-1} - 3c_{n-3}$.

Solution 8.4. In the case $m = 2, n = 1$ the Sylvester matrix is

$$\begin{pmatrix} p_2 & p_1 & p_0 \\ q_1 & q_0 & 0 \\ 0 & q_1 & q_0 \end{pmatrix},$$

whose determinant is $p_2q_0^2 - p_1q_1q_0 + p_0q_1^2$.

If there exist a, b such that $p = (a + bz)q$ then

$$p_2 = bq_1, \quad p_1 = aq_1 + bq_0, \quad p_0 = aq_0,$$

so the determinant is

$$bq_1q_0^2 - (aq_1 + bq_0)q_1q_0 + aq_0q_1^2 = 0.$$

Conversely, suppose that $p_2q_0^2 - p_1q_1q_0 + p_0q_1^2 = 0$. Note that $q_1 \neq 0$ or else q has degree less than 1.

- If $q_0 = 0$ then $p_0q_1^2 = 0$, so $p_0 = 0$ and $z|p(z)$. Since $q_0 = 0$, $q(z) = q_1z$, so $z|q(z)$ and we have found a common factor.
- if $q_0 \neq 0$ then consider the polynomial $(p_2/q_1)z + (p_0/q_0)$. We have

$$(p_2/q_1)z + (p_0/q_0)(q_1z + q_0) = p_2z^2 + \left(\frac{p_2q_0}{q_1} + \frac{p_0q_1}{q_0}\right) + p_0$$

  and the term in brackets is equal to p_1 because $p_2q_0^2 + p_0q_1^2 = p_1q_0q_1$. Therefore q is a common factor of p and q.