Linear Algebra, Week 2

Jonny Evans

MATH105

Simultaneous equations

Simultaneous equations...

$$\begin{aligned} x - y &= -1 \\ x + y &= 3 \end{aligned}$$

Simultaneous equations...

$$x - y = -1$$
$$x + y = 3$$

...in matrix form

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Simultaneous equations...

$$x - y = -1$$
$$x + y = 3$$

...in matrix form

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The *augmented matrix*

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & 3 \end{array}\right)$$

Row operations

Definition (Row operation of type I)

Add λ times row j to row i.

$$R_i \mapsto R_i + \lambda R_j$$

Row operations

Definition (Row operation of type I)

Add λ times row j to row i.

$$R_i \mapsto R_i + \lambda R_j$$

Definition (Row operation of type II)

Multiply row i by λ.

$$R_i \mapsto \lambda R_i$$

Row operations

Definition (Row operation of type I)

Add λ times row j to row i.

$$R_i \mapsto R_i + \lambda R_j$$

Definition (Row operation of type II)

Multiply row i by λ.

$$R_i \mapsto \lambda R_i$$

$\lambda \neq 0$

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Leading indices:

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Leading indices: 1,

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Leading indices: 1, 3,

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Leading indices: 1, 3, 6.

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Leading indices: 1, 3, 6.

Definition (Echelon form)

A matrix is in *echelon form* if the sequence of leading indices is *strictly increasing* (and all zero rows are at the bottom).

Echelon form

$$\begin{pmatrix} 1 & 7 & 2 & 0 & 1 & 13 \\ 0 & 0 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Leading indices: 1, 3, 6.

Definition (Echelon form)

A matrix is in *echelon form* if the sequence of leading indices is *strictly increasing* (and all zero rows are at the bottom).

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1
- in a column containing a leading entry, everything but the leading entry is zero.

Reduced echelon form

$$\begin{pmatrix} \boxed{1} & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1
- in a column containing a leading entry, everything but the leading entry is zero.

Reduced echelon form

$$\begin{pmatrix} \boxed{1} & \boxed{0} & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1
- in a column containing a leading entry, everything but the leading entry is zero.

Reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 13 \\ 0 & 1 & 15 & 221 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition (Reduced echelon form)

A matrix A is in *reduced echelon form* if

- A is in echelon form
- all leading entries are equal to 1
- in a column containing a leading entry, everything but the leading entry is zero.

Definition (Free indices)

An index $i \in \{1, \ldots, n\}$ is called *free* if it is not a leading index. Write $F \subset \{1, \ldots, n\}$ for the set of free indices.

Theorem

If A is m-by-n in reduced echelon form with k nonzero rows and leading indices $j_1, \ldots, j_k$ then the general solution to

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

exists if and only if $b_{k+1} = \cdots = b_m = 0$, has $n - k$ free variables x_p, $p \in F$, and k dependent variables

$$x_{j_m} = b_i - \sum_{p \in F} A_{ip} x_p.$$

Echelon form theorems

Type I $R_i \mapsto R_i + \lambda R_j$

Type II $R_i \mapsto \lambda R_i$

Theorem (Echelon form theorem)

Any matrix can be put into echelon form using only row operations of type I.

Theorem (Reduced echelon form theorem)

Any matrix can be put into reduced echelon form using row operations of types I and II.

Example

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 5 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Example

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & b_1 \\ 1 & 1 & -1 & b_2 \\ 4 & 0 & -2 & b_3 \\ 0 & 2 & -1 & b_4 \end{array}\right)$$

Type I $R_i \mapsto R_i + \lambda R_j$

Type II $R_i \mapsto \lambda R_i$

Theorem (Echelon form theorem)

Any matrix can be put into echelon form using only row operations of type I.

Theorem (Reduced echelon form theorem)

Any matrix can be put into reduced echelon form using row operations of types I and II.

Inverses, I

Can we "divide" by a matrix?

Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*. If* $ad - bc \neq 0$*, define*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then $AA^{-1} = A^{-1}A = I$*.*

Can we “divide” by a matrix?

Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *If* $ad - bc \neq 0$, *define*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then $AA^{-1} = A^{-1}A = I$.

Definition

An n-by-n matrix A is *invertible* if there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

Can we "divide" by a matrix?

Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *If* $ad - bc \neq 0$, *define*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then $AA^{-1} = A^{-1}A = I$.

Definition

An n-by-n matrix A is *invertible* if there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Unique if it exists.

Can we "divide" by a matrix?

Theorem

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *If* $ad - bc \neq 0$, *define*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then $AA^{-1} = A^{-1}A = I$.

Definition

An n-by-n matrix A is *invertible* if there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Unique if it exists.

Lemma

$$(AB)^{-1} = B^{-1}A^{-1}.$$

How to find the inverse?

Theorem (Invertibility theorem)

Let A be an n-by-n matrix.

How to find the inverse?

Theorem (Invertibility theorem)

Let A be an n-by-n matrix.

- *A is invertible if and only if its reduced echelon form is the identity matrix.*

How to find the inverse?

Theorem (Invertibility theorem)

Let A be an n-by-n matrix.

- *A is invertible if and only if its reduced echelon form is the identity matrix.*
- *Suppose A is invertible. The reduced echelon form of the augmented matrix $(A|I_n)$ is $\left(I_n|A^{-1}\right)$.*

How to find the inverse?

Theorem (Invertibility theorem)

Let A be an n-by-n matrix.

- *A is invertible if and only if its reduced echelon form is the identity matrix.*
- *Suppose A is invertible. The reduced echelon form of the augmented matrix $(A|I_n)$ is $\left(I_n|A^{-1}\right)$.*

Example

$$\begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 1 & -3 \\ 1 & 0 & 1 & 7 \end{pmatrix}$$

Example

$$\begin{pmatrix} -3 & -2 & -4 \\ 2 & 3 & 3 \\ -1 & 4 & -4 \end{pmatrix}$$

Inverses, II

Goal

Theorem (Invertibility theorem)

Let A be an n-by-n matrix.

- *A is invertible if and only if its reduced echelon form is the identity matrix.*
- *Suppose A is invertible. The reduced echelon form of the augmented matrix $(A|I_n)$ is $\left(I_n|A^{-1}\right)$.*

Elementary matrices

Definition

$$E_{ij}(\lambda) = \begin{array}{rc}
 & \begin{array}{ccccccc} & & & & \text{col } j & & \\ & & & & \downarrow & & \end{array} \\
\text{row } i \rightarrow & \begin{pmatrix}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & \lambda & & \\
 & & & \ddots & & & \\
 & & & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix}
\end{array}$$

Elementary matrices

Definition

$$E_{ij}(\lambda) = \begin{array}{cc} & \begin{array}{ccccccc} & & & & \text{col } j & & \\ & & & & \downarrow & & \end{array} \\ \begin{array}{c} \\ \\ \text{row } i \rightarrow \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \lambda & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Lemma

$E_{ij}(\lambda)A$ is obtained from A by $R_i \mapsto R_i + \lambda R_j$.

Elementary matrices

Definition

$$E_i(\lambda) = \begin{array}{cc} & \text{col } i \\ & \downarrow \\ \text{row } i \rightarrow & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Elementary matrices

Definition

$$E_i(\lambda) = \begin{array}{cc} & \begin{array}{ccccccc} & & \text{col } i & & & & \\ & & \downarrow & & & & \end{array} \\ \text{row } i \rightarrow & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Lemma

$E_i(\lambda)A$ is obtained from A by $R_i \mapsto \lambda R_i$.

Proof of invertibility theorem

Theorem (Invertibility theorem)

Let A be an n-by-n matrix.

- *A is invertible if and only if its reduced echelon form is the identity matrix.*
- *Suppose A is invertible. The reduced echelon form of the augmented matrix $(A|I_n)$ is $\left(I_n|A^{-1}\right)$.*

Lemma

Any invertible matrix is a product of elementary matrices.

Lemma

Any invertible matrix is a product of elementary matrices.

Lemma

Conversely, any product of elementary matrices is invertible.

Lemma

Any invertible matrix is a product of elementary matrices.

Lemma

Conversely, any product of elementary matrices is invertible.

Proof.

$$(E_{ij}(\lambda))^{-1} = E_{ij}(-\lambda)$$
$$(E_i(\lambda))^{-1} = E_i(1/\lambda)$$

and $(M_1 \cdots M_k)^{-1} = M_k^{-1} \cdots M_1^{-1}$. $\square$