Linear Algebra Worksheet 2

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Workshop 2

Here is a list $\mathcal{V}$ of vectors

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad w = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Exercise 2.1. For every vector in $\mathcal{V}$, find its length and write down a vector orthogonal to it.

Exercise 2.2. Find the angle between u and v. Find the angle between w and ξ.

Here is a list $\mathcal{M}$ of matrices.

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2}\sqrt{\frac{3}{2}} \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} + \frac{1}{\sqrt{3}} & -\frac{1}{3} + \frac{1}{\sqrt{3}} \\ \frac{1}{3} - \frac{1}{\sqrt{3}} & \frac{1}{3} & -\frac{1}{3} - \frac{1}{\sqrt{3}} \\ -\frac{1}{3} - \frac{1}{\sqrt{3}} & -\frac{1}{3} + \frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}.$$

Exercise 2.3. Which matrices $M \in \mathcal{M}$ are orthogonal matrices? (Hint: There should be two!)

Exercise 2.4. The orthogonal matrices from $\mathcal{M}$ are actually rotation matrices. In each case, find the axis and angle of rotation.

Here is a list $\mathcal{N}$ of matrices

$$D = \begin{pmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 2.5. Which of the matrices $N \in \mathcal{N}$ are in echelon form? Which are in reduced echelon form?

Exercise 2.6. For each $N \in \mathcal{N}$ which is in reduced echelon form, state (a) for which vectors b the equation $Nv = b$ has a solution and (b) the dimension of the space of solutions to $Nv = b$, assuming that b is chosen so that there is a solution.

Exercise 2.7. For each system of simultaneous equations below, write it in matrix form, put the augmented matrix into reduced echelon form using row operations. Determine if the system has a solution and, if it does, give the general solution.

$$\begin{aligned} x + y + 2z + 3w &= 0 \\ y + 4z - w &= 1 \end{aligned}$$

$$\begin{aligned} x &= y - 3 \\ 2x + y &= 6 \\ y - 3x &= 1 \end{aligned}$$

$$\begin{aligned} 4x - w &= 0 \\ 3y - 2z + w &= 4 \\ 4x - 2y + 4z - 3w &= 0 \\ 3x + y - z &= 2 \end{aligned}$$

Exercise 2.8. Put the following matrices into reduced echelon form using row operations. In each case, what is the number of free indices?

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

Exercise 2.9. Let A, B, C be m-by-n, n-by-p and p-by-q matrices respectively. Write out the matrix products $A(BC)$ and $(AB)C$ in index notation and check that they give the same answer (this shows that matrix multiplication is associative).

Exercise 2.10. Suppose that A is an n-by-n matrix whose columns are the vectors $v_1, \ldots, v_n$. Show that A is an orthogonal matrix (i.e. $A^T A = I$) if and only if

$$v_i \cdot v_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{for all } i, j.$$

In other words, the columns of A are orthogonal to one another (this is where the name "orthogonal matrix" comes from).

Exercise 2.11. We say that a matrix M is *symmetric* if $M^T = M$ and *antisymmetric* if $M^T = -M$.

1. Show that if N is an m-by-n matrix then MM^T is a symmetric m-by-m matrix and $M^T M$ is a symmetric n-by-n matrix.
2. Show that, given any n-by-n matrix C, the matrix $A = C + C^T$ is symmetric and the matrix $B = C - C^T$ is antisymmetric. Deduce that C can be written as the sum of a symmetric and an antisymmetric matrix (called the *symmetric* and *antisymmetric* parts of C respectively).

Exercise 2.12. A system of m equations in n unknowns is called *underdetermined* if $m < n$ and overdetermined if $m > n$. As rules of thumb, underdetermined equations tend to have general solutions with $m - n$ free parameters, and overdetermined equations tend to have no solutions. Give counterexamples to these rules of thumb (e.g. an underdetermined system with no solutions and an overdetermined system with a solution).

3 Solutions

Solution 3.1. Any of the following are correct, but there are many possible answers (just check that the dot product with the original vector is zero):

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We have $|u| = \sqrt{2}$, $|v| = \sqrt{5}$, $|w| = \sqrt{14}$, $|\xi| = \sqrt{2}$.

Solution 3.2. We have $u \cdot v = 3$, $|u| = \sqrt{2}$, $|v| = \sqrt{5}$, so the angle between u and v is $\cos^{-1}(3/\sqrt{10}) \approx 0.321750554$ radians.

We have $w \cdot \xi = 1$, $|w| = \sqrt{14}$ and $|\xi| = \sqrt{2}$, so the angle between w and ξ is $\cos^{-1}(1/\sqrt{28}) \approx 1.38067072$ radians.

Solution 3.3. The matrices B, C are orthogonal; A is not. You can check this explicitly: $A^T A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, is not too hard; more of a pain is:

$$B^T B = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2}\sqrt{\frac{3}{2}} \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2}\sqrt{\frac{3}{2}} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{16} + \frac{1}{16} + \frac{3}{8} & \frac{3}{16} + \frac{3}{16} - \frac{3}{8} & \left(\frac{3}{8} - \frac{1}{8} + \frac{1}{4}\right)\sqrt{\frac{3}{2}} \\ \frac{3}{16} + \frac{3}{16} - \frac{3}{8} & \frac{1}{16} + \frac{9}{16} + \frac{3}{8} & \left(\frac{1}{8} - \frac{3}{8} + \frac{1}{4}\right)\sqrt{\frac{3}{2}} \\ \left(\frac{3}{8} - \frac{1}{8} + \frac{1}{4}\right)\sqrt{\frac{3}{2}} & \left(\frac{1}{8} - \frac{3}{8} + \frac{1}{4}\right)\sqrt{\frac{3}{2}} & \frac{3}{8} + \frac{3}{8} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly $C^T C = I$.

Solution 3.4. For B: A vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ pointing along the axis solves $Bv = v$. This means

$$\begin{aligned} \frac{3x}{4} + \frac{y}{4} + \frac{1}{2}\sqrt{\frac{3}{2}}z &= x \\ \frac{x}{4} + \frac{3y}{4} - \frac{1}{2}\sqrt{\frac{3}{2}}z &= y \\ \frac{1}{2}\left(z - \sqrt{\frac{3}{2}}x + \sqrt{\frac{3}{2}}y\right) &= z, \end{aligned}$$

so $x - y = 2z\sqrt{\frac{3}{2}}$ and $x - y = z\sqrt{\frac{2}{3}}$. This implies that $z = 0$ and $x = y$. Therefore the axis is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. If we pick $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (orthogonal to the axis), it goes to $Bv = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{3}{2}} \\ -\frac{1}{2}\sqrt{\frac{3}{2}} \\ \frac{1}{2} \end{pmatrix}$, and $v \cdot Bv = \frac{1}{2}$, so the angle of rotation is $\pi/3 = \cos^{-1}(1/2)$.

For C: Similar arguments give axis $u = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and angle 90 degrees (e.g. if we pick $v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ orthogonal to the axis u, it goes to $w = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \end{pmatrix}$, and $v \cdot w = 0$).

Solution 3.5. D, F, G are in echelon form, D, G are in reduced echelon form, E is in neither.

Solution 3.6. For D: there are no zero-rows, so there are no constraints on b for a solution to $Dv = b$ to exist; are three free indices, so the space of solutions is 3-dimensional.

For G: the last row is zero, so we need $b_3 = 0$. There is one free index, so the space of solutions (assuming there are some solutions) is 1-dimensional.

Solution 3.7. The first system is

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & -1 & 1 \end{array}\right),$$

which is almost in reduced echelon form already. The row operation $R_1 \mapsto R_1 - R_2$ gives

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 4 & -1 \\ 0 & 1 & 4 & -1 & 1 \end{array}\right).$$

The general solution is therefore $x = -1 + 2z - 4w$, $y = 1 + w - 4z$, i.e. $\begin{pmatrix} -1 + 2z - 4w \\ 1 + w - 4z \\ z \\ w \end{pmatrix}$.

The second system is

$$\left(\begin{array}{cc|c} 1 & -1 & -3 \\ 2 & 1 & 6 \\ -3 & 1 & 1 \end{array}\right)$$

(be careful because I mixed up the order of the letters a little to trick you; systems of equations in real life rarely come in the nice form we've been studying them without any rearrangement). We perform the row operations $R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 + 3R_1$ to clear the first column:

$$\left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 3 & 12 \\ 0 & -2 & -8 \end{array}\right).$$

Now $R_2 \mapsto \frac{1}{3}R_2$, $R_3 \mapsto -\frac{1}{2}R_3$ and $R_3 \mapsto R_3 - R_2$ gives

$$\left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right).$$

Finally, $R_1 \mapsto R_1 + R_2$ gives

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right),$$

so the unique solution is $x = 1$, $y = 4$.

The third system is

$$\begin{pmatrix} 4 & 0 & 0 & -1 \\ 0 & 3 & -2 & 1 \\ 4 & -2 & 4 & -3 \\ 3 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 2 \end{pmatrix}.$$

The augmented matrix is

$$\left(\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 4 \\ 4 & -2 & 4 & -3 & 0 \\ 3 & 1 & -1 & 0 & 2 \end{array}\right).$$

We use the row operations $R_3 \mapsto R_3 - R_1$, $R_4 \mapsto R_4 - \frac{3}{4}R_1$ to get

$$\left(\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 4 \\ 0 & -2 & 4 & -2 & 0 \\ 0 & 1 & -1 & 3/4 & 2 \end{array}\right).$$

Now we use $R_3 \mapsto R_3 + \frac{2}{3}R_2$ and $R_4 \mapsto R_4 - \frac{1}{3}R_2$ to get

$$\left(\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 4 \\ 0 & 0 & 8/3 & -4/3 & 8/3 \\ 0 & 0 & -1/3 & 5/12 & 2/3 \end{array}\right).$$

Now $R_4 \mapsto R_4 + \frac{1}{8}R_3$ gives

$$\left(\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 4 \\ 0 & 0 & 8/3 & -4/3 & 8/3 \\ 0 & 0 & 0 & 1/4 & 1 \end{array}\right).$$

Let's tidy up a bit with $R_4 \mapsto 4R_4$, $R_3 \mapsto \frac{3}{8}R_3$, which gives

$$\left(\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 4 \\ 0 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right).$$

Now do $R_3 \mapsto R_3 + \frac{1}{2}R_4$, $R_2 \mapsto R_2 + 2R_3$ $R_2 \mapsto R_2 - R_4$, $R_1 \mapsto R_1 + R_4$ to get

$$\left(\begin{array}{cccc|c} 4 & 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

We finish with $R_1 \mapsto \frac{1}{4}R_1$ and $R_2 \mapsto \frac{1}{3}R_2$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right),$$

so $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ is the unique solution.

Solution 3.8. First, take

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We can clear the first column just by subtracting R_1 from all the other rows:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Now clear column three by subtracting R_2 from R_3 and R_4, and column four by further subtracting R_3 from R_4:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This has one free index (the index 2).

Second, take

$$Y = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{pmatrix}.$$

Subtract $2R_1$ from R_3 and $3R_2$ from R_3. This gives

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Now $R_1 \mapsto \frac{1}{2}R_1$, $R_3 \mapsto -\frac{1}{3}R_3$ and $R_1 \mapsto R_1 - \frac{1}{2}R_3$ gives the identity matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

as the echelon form, so there are no free indices.

Finally, take

$$Z = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

$R_2 \mapsto R_2 - 5R_1$, $R_3 \mapsto R_3 - 9R_1$ gives

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{pmatrix}.$$

$R_2 \mapsto -\frac{1}{4}R_2$, $R_3 \mapsto -\frac{1}{8}R_3$ gives

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

$R_3 \mapsto R_3 - R_2$, $R_1 \mapsto R_1 - 2R_2$ gives

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are therefore two free indices (3 and 4).

Solution 3.9. The two expressions are:

$$\sum_{j=1}^{n} A_{ij} \sum_{k=1}^{p} B_{jk} C_{k\ell} \qquad \sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{k\ell},$$

and these are both equal to $\sum_{j=1}^{n} \sum_{k=1}^{p} A_{ij} B_{jk} C_{k\ell}$.

Solution 3.10. If the columns of A are $v_1, \ldots, v_n$ then the rows of A^T are $v_1^T, \ldots, v_n^T$. The product $A^T A$ has ij entry equal to $v_i^T v_j$ (multiplying the ith row of A^T into the jth column of A) which is precisely $v_i \cdot v_j$. Since $A^T A = I$, this means $v_i \cdot v_j = \delta_{ij}$, as required.

Solution 3.11.

1. We have $(M^T M)^T = M^T (M^T)^T = M^T M$, so $M^T M$ is symmetric. Similarly $(MM^T)^T = (M^T)^T M^T = MM^T$.
2. We have $(C + C^T)^T = C^T + (C^T)^T = C^T + C$, so $C + C^T$ is symmetric. Similarly, $(C - C^T)^T = C^T - (C^T)^T = C^T - C$, so $C - C^T$ is antisymmetric. Since

$$C = \frac{1}{2}(C + C^T) + \frac{1}{2}(C - C^T),$$

we see that C can be written as the sum of a symmetric and an antisymmetric matrix.

Solution 3.12.

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \end{aligned}$$

is an underdetermined system (three variables, two equations) with no solutions.

$$\begin{aligned} x &= 1 \\ 2x &= 2 \end{aligned}$$

is an overdetermined system (one variable, two equations) with a solution.