Linear Algebra, Week 5

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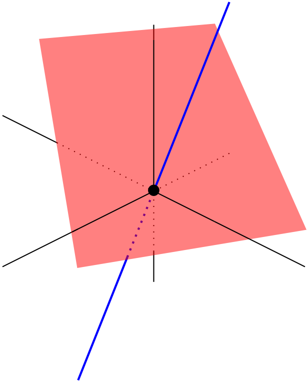
MATH105

Subspaces, I

Definition

A subset $V \subset \mathbb{R}^n$ is a *linear subspace* if

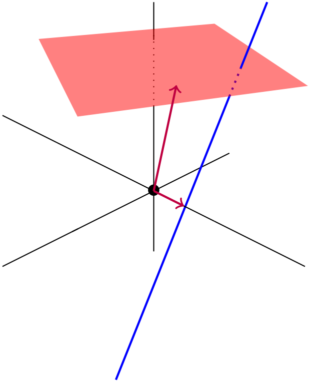
- $v, w \in V$ implies $v + w \in V$.
- $v \in V$ and $\lambda \in \mathbb{R}$ implies $\lambda v \in V$.



Definition

A subset $V \subset \mathbb{R}^n$ is an *affine subspace* if there exists a vector w and a linear subspace V' such that

$$V = \{v \in \mathbb{R}^n \ : \ v = w + v', \ v' \in V'\}.$$



Subspace	dimension	codimension

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Line		

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Line	1	

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Plane		

Subspace	dimension	codimension
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Subspace	dimension	codimension
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$\vdots$	$\vdots$	

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Plane	2	
$\vdots$	$\vdots$	
Hyperplane		

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Hyperplane	$n-1$	

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Hyperplane	$n-1$	
Ambient space $\mathbb{R}^n$	n	

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$\vdots$	$\vdots$	
Hyperplane	$n-1$	1
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Hyperplane	$n-1$	1
Ambient space $\mathbb{R}^n$	n	0

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$r^{\perp}$ is the *orthogonal complement* to $r^T = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$.

Intersections of hyperplanes

An m-by-n matrix A defines m linear hyperplanes:

$$\begin{aligned} A_{11}x_1 + \cdots + A_{1n}x_n &= 0 \\ \vdots \qquad \vdots \qquad &\vdots \\ A_{m1}x_1 + \cdots + A_{mn}x_n &= 0. \end{aligned}$$

i.e. $Ax = 0$. A solution x to this system of equations represents a point in the intersection of these hyperplanes.

Definition

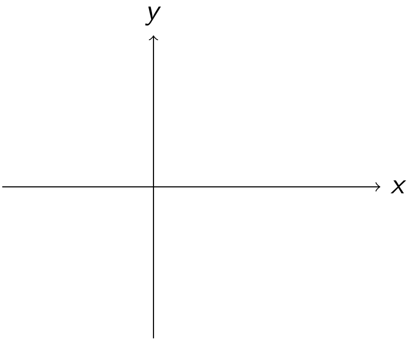
The kernel $\ker(A)$ is defined to be this intersection, i.e.

$$\ker(A) = \{x \in \mathbb{R}^n \ : \ Ax = 0\}.$$

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$Ax = b$ defines three *lines* (hyperplanes in $\mathbb{R}^2$):



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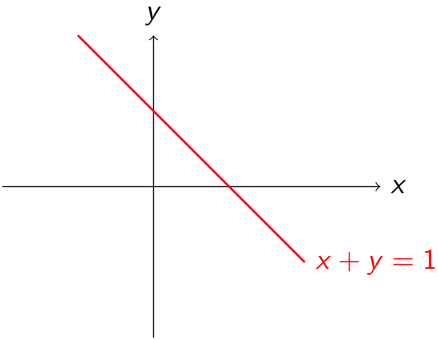

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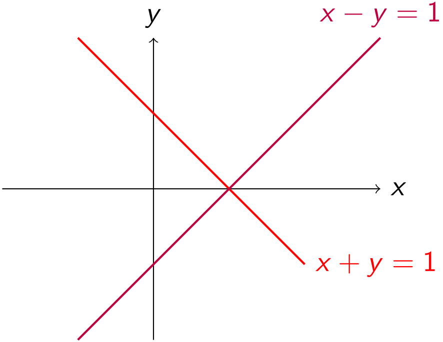

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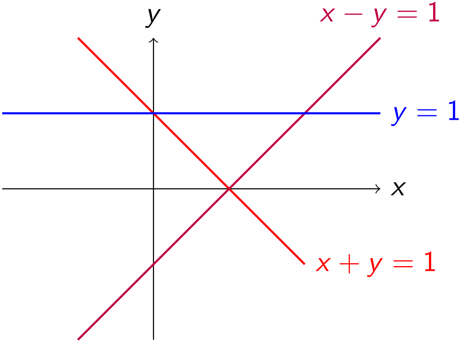

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

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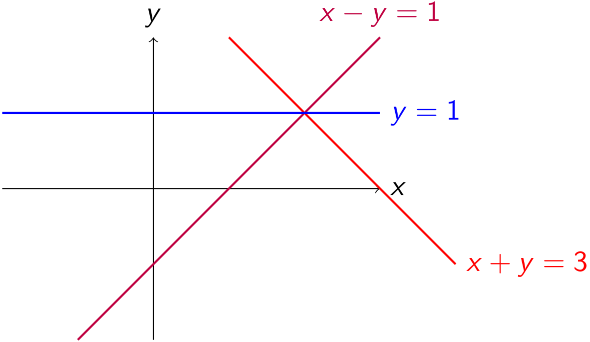

Definition

Nullity of $A = \dim \ker(A)$.

Theorem

Let A be an m-by-n matrix and $b \in \mathbb{R}^m$ be a vector. The space of solutions to $Ax = b$ is either empty or a translate of $\ker(A)$.

Corollary

$\dim \ker(A) =$ *number of free indices of reduced echelon form of A.*

Subspaces, II

Definition

A *linear combination* of vectors $v_1, \dots, v_k \in \mathbb{R}^n$ is any vector of the form

$$\lambda_1 v_1 + \cdots + \lambda_k v_k, \qquad \lambda_i \in \mathbb{R}.$$

Definition

The *subspace spanned by* $v_1, \dots, v_k$ is the set of all their linear combinations:

$$\mathrm{span}(v_1, \dots, v_k) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \ : \ \lambda_i \in \mathbb{R}\}.$$

Lemma

$\mathrm{span}(v_1, \dots, v_k)$ *is a linear subspace of* $\mathbb{R}^n$*.*

Definition

If $V = \mathrm{span}(v_1, \ldots, v_k)$, we call $\{v_1, \ldots, v_k\}$ a *spanning set* for V. A minimal spanning set is called a *basis*.

Theorem

Two bases for the same subspace V have the same size (this size is called the dimension *of V).*

Image

Definition

The image $\mathrm{im}(A)$ of an m-by-n matrix A is the subspace of $\mathbb{R}^m$ consisting of all $b \in \mathbb{R}^m$ such that $Ax = b$ has a solution $x \in \mathbb{R}^n$.

Lemma

$\mathrm{im}(A)$ *is a subspace of* $\mathbb{R}^n$*. It is spanned by the columns of* A*.*

Definition

The *rank* of A is the dimension of $\mathrm{im}(A)$.

Theorem

The rank of A is the number of leading indices of the reduced echelon form of A.

Corollary (Rank-nullity theorem)

If A is m-by-n then $\mathrm{rank(A)} + \mathrm{null}(A) = n$.

Summary

Theorem

Let A be an m-by-n matrix and $b \in \mathbb{R}^m$ be a vector.

- *$Ax = b$ has a solution if and only if $b \in \mathrm{im}(A)$,*
- *the space of solutions is a translate of $\ker(A)$.*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\quad A \quad} & \mathbb{R}^m \\ \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} \\ \ker(A) & & \mathrm{im}(A) \end{array}$$

Linear maps

Vector space

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for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$.

Vector space

Definition

A ($\mathbb{C}$-)vector space is a set v together with:

- a map $V \times V \to V$, written $(v, w) \mapsto v + w$,
- a map $\mathbb{C} \times V \to V$, written $(\lambda, v) \mapsto \lambda v$,
- an element $0 \in V$,

such that:

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for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{C}$.

Vector space

Definition

A (k-)vector space is a set v together with:

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- a map $k \times V \to V$, written $(\lambda, v) \mapsto \lambda v$,
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such that:

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\end{aligned}
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for all $u, v, w \in V$ and $\lambda, \mu \in k$.
k can be any *field*: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}/p$, $\mathbb{Q}_p$,...

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Linear maps

Definition

A map $T\colon V \to W$ is called *linear* if

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad \text{and} \quad T(\lambda v) = \lambda T(v)$$

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If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear then there exists an m-by-n matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$. Conversely, a matrix defines a linear map $v \mapsto Av$.

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Example

Differentiation defines a linear map $\frac{d}{dx} \colon \mathcal{C}^1(\mathbb{R}) \to \mathcal{C}^0(\mathbb{R})$.