Methods 3 - Question Sheet 4

J. Evans

Question 1. (13 marks for * parts)
Solve the Euler-Lagrange equation for the following variational problems; you may use Beltrami's identity, where appropriate.

(a) * $\int_0^1 \left(x^2 y + \frac{(y')^2}{2}\right) dx$ subject to $y(0) = y(1) = 0$.

(b) * $\int_0^1 \sqrt{y(1+(y')^2)}dx$ (the general solution - leave constants undetermined).

(c) $\int_0^1 \sqrt{(1+y)(1+(y')^2)}dx$ (the general solution - leave constants undetermined).

(d) $\int_0^{\pi/4} (y')^2 \cos^2 x dx$ subject to $y(0) = 0$, $y(\pi/4) = 1$.

(e) $\int_0^{\pi} \left((y')^2 + (\cos^2 x - \sin x)y^2\right) dx$ subject to $y(0) = y(\pi) = 1$ *(Hint: Compute* $\dfrac{d^2(e^{\sin x})}{dx^2}$*.)*

Answer 1. (a) * We must use the full Euler-Lagrange equation

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}$$

(1 marks)

as the integrand depends on all the variables x, y, y'. We have $\dfrac{\partial L}{\partial y'} = y'$ and $\dfrac{\partial L}{\partial y} = x^2$ therefore the Euler-Lagrange equation is

$$y'' = x^2$$

(2 marks)

which integrates up twice to give $y' = x^3/3 + C$ and $y = x^4/12 + Cx + D$.

(2 marks)

The boundary conditions give $D = 0$ and $C + 1/12 = 0$ so $y = (x^4 - x)/12$.

(1 marks)

(b) * Since the integrand $L = \sqrt{y(1+(y')^2)}$ has no explicit x-dependence we can use Beltrami's identity,

$$L - y'\frac{\partial L}{\partial y'} = C$$

(1 marks)

where C is constant. This gives

$$\begin{aligned} C &= \sqrt{y(1+(y')^2)} - y' \frac{\sqrt{y}y'}{\sqrt{1+(y')^2}} \\ &= \sqrt{\frac{y}{1+(y')^2}} \left(1+(y')^2 - (y')^2\right) \\ &= \sqrt{\frac{y}{1+(y')^2}} \end{aligned}$$

(2 marks)

so

$$y' = \sqrt{\frac{y}{C^2} - 1}.$$

Dividing by the RHS and integrating gives

$$\int \frac{C dy}{\sqrt{y - C^2}} = x + D.$$

(2 marks)

You can do the integral by substituting $y = C^2 \sec^2 u$ or by spotting that $d\sqrt{y - C^2} = \frac{dy}{2\sqrt{y-C^2}}$ the result is

$$2C\sqrt{y - C^2} = x + D$$

(2 marks)

or

$$y = \left(\frac{x+D}{2C}\right)^2 + C^2.$$

(c) As usual we can use Beltrami and we get

$$\sqrt{(1+y)(1+(y')^2)} - \sqrt{(1+y)} \frac{(y')^2}{\sqrt{1+(y')^2}} = C$$

or

$$\sqrt{\frac{1+y}{1+(y')^2}} = C$$

which rearranges to give

$$Cy' = \sqrt{y + 1 - C^2}.$$

Dividing by the RHS and integrating gives

$$\int \frac{C dy}{\sqrt{y + 1 - C^2}} = x + D.$$

As in (b), we see that $d\sqrt{y+1-C^2} = dy/2\sqrt{y+1-C^2}$ so the integral is

$$2C\sqrt{y + 1 - C^2} = x + D.$$

This rearranges to give $y = C^2 - 1 + \left(\frac{x+D}{2C}\right)^2$.

(d) The Euler-Lagrange equation for this functional is

$$0 = \frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{d}{dx}(2y' \cos^2 x) = 0$$

so we get $y' \cos^2 x = C$ for some constant C. Therefore $y' = C \sec^2 x$ and $y = C \tan x + D$. The boundary conditions give $D = 0$ and $C = 1$ so the solution is $y = \tan x$.

(e) We have

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = 2y'', \quad \frac{\partial L}{\partial y} = 2(\cos^2 x - \sin x)y$$

so the Euler-Lagrange equation for this functional is

$$y'' = 2(\cos^2 x - \sin x)y.$$

The hint tells us to compute

$$\frac{d^2(e^{\sin x})}{dx^2} = \frac{d}{dx}\left((\cos x)e^{\sin x}\right) = \cos^2 x e^{\sin x} - \sin x e^{\sin x}$$

so certainly $y = e^{\sin x}$ satisfies the Euler-Lagrange equation. It also satisfies the boundary conditions $y(0) = y(\pi) = e^0 = 1$.

Question 2. (7 marks)

Consider the functional

$$F(y) = \int_a^b \frac{\sqrt{1+(y')^2}}{y} dx$$

for functions y satisfying $y(a) = A$, $y(b) = B$. Find the general solution to the Euler-Lagrange equation for this functional and show that if y is a solution then the graph

$$\{(x, y(x)) \ : \ x \in [a, b]\}$$

is a segment of a circle

$$y^2 + (x - C)^2 = D$$

centred on the x-axis.

Remark 1. This is the equation for a *geodesic*, or shortest path, in an unusual geometry called the *hyperbolic upper-half plane*. The factor of $1/y$ in the integrand means that planar distances count for more towards the boundary of the upper-half plane (i.e. the x-axis) because $1/y$ gets very big when y gets very small. This accounts for the warped shape of the "straight lines" in this geometry (which are now segments of circles centred on the x-axis) and gives rise to extremely pretty pictures like this one due to M. C. Escher.

Answer 2. Since L has no explicit x-dependence, the Euler-Lagrange equation can be integrated up to give the Beltrami identity and we want to solve

$$L - y'\frac{\partial L}{\partial y'} = A,$$

(1 marks)

$$\Rightarrow \quad A = \frac{\sqrt{1+(y')^2}}{y} - y'\frac{y'}{y\sqrt{1+(y')^2}}$$
$$= \frac{1}{y\sqrt{1+(y')^2}}$$

(2 marks)

which gives

$$y' = \sqrt{\frac{1}{y^2A^2} - 1}$$

(2 marks)

Integrating, this gives

$$\begin{aligned}
x - C &= \int \frac{yA dy}{\sqrt{1 - y^2 A^2}} \\
&= -\sqrt{1 - y^2 A^2}/A \\
\Rightarrow &(x - C)^2 + y^2 = 1/A^2
\end{aligned}$$

(2 marks)

which is the equation for a circle centred at $(C, 0)$. Our geodesic is the segment of this circle between $x = a$ and $x = b$.

Question 3.
Show that
$$y(x) = \cos^{-1}(A\cot x) + B$$
is the general solution to the Euler-Lagrange equation for the variational problem associated to the functional
$$\int \sqrt{1 + (y')^2 \sin^2 x} dx.$$
Hint: Use the two substitutions suggested by the solution!

Remark 2. The paths $\gamma(x) = (x, y(x)\cos x, y(x)\sin x)$ are shortest paths ("great circles") on the unit sphere: the functional is just the length functional $\int |\dot\gamma(x)|dx$.

Answer 3. The Euler-Lagrange equation is
$$0 = \frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{d}{dx}\left(\frac{y'\sin^2 x}{\sqrt{1+(y')^2\sin^2 x}}\right)$$
so
$$\frac{y'\sin^2 x}{\sqrt{1+(y')^2\sin^2 x}} = C$$
for some constant C. This rearranges to give
$$(y')^2(\sin^2 x - C^2)\sin^2 x = C^2$$
or
$$y' = \frac{C}{\sin x\sqrt{\sin^2 x - C^2}}.$$
Integrating gives
$$y = C\int \frac{dx}{\sin x\sqrt{\sin^2 x - C^2}}$$
and this can be done by first substituting $u = \cot x$ ($du = -(1+u^2)dx$, $1+u^2 = 1/\sin^2 x$) to get
$$y = -C\int \frac{du}{\sqrt{1 - C^2 - C^2u^2}} = -\int \frac{du}{\sqrt{(1-C^2)/C^2 - u^2}}$$
and then substituting $u = \left(\sqrt{(1-C^2)/C^2}\right)\cos v$ ($du = -\left(\sqrt{(1-C^2)/C^2}\right)\sin v dv$) which gives
$$y = \int dv = v + B$$
and if $A = 1/\sqrt{(1-C^2)/C^2}$ then $v = \cos^{-1}(Au) = \cos^{-1}(A\cot x)$, as required.

Question 4.
Consider a function $x(t)$ describing the position at time t of a particle of mass m sitting in the force field F whose strength is the gradient of a potential $-V(x)$. Find the Euler-Lagrange equation for the functional

$$\int_0^1 \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) dt$$

and show that solutions obey Newton's law of motion $F = ma$. Interpret Beltrami's identity physically in this situation.

Answer 4. The Euler-Lagrange equation is

$$\frac{\partial L}{\partial x} = \frac{d}{dx}\frac{\partial L}{\partial \dot{x}} \;\Rightarrow\; -\frac{dV}{dx} = m\ddot{x}$$

which is precisely Newton's law given that $F = -\dfrac{dV}{dx}$ and $a = \ddot{x}$. Beltrami's identity becomes

$$C = L - \dot{x}\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m\dot{x}^2 - V(x) - m\dot{x}^2 = -(m\dot{x}^2 + V).$$

The RHS is the total energy (kinetic plus potential) so Beltrami's identity tells us that energy is conserved (i.e. constant in time).

Question 5.
Let $\gamma(t) = (t, t^2\cos(\theta(t)), t^2\sin(\theta(t)))$ be a parametric curve in $\mathbf{R}^3$.

(a) Check that $\gamma(t)$ lies on the surface $S = \{y^2 + z^2 = x^4\}$.

The length of this curve is defined to be the integral $\int |\dot{\gamma}(t)| dt$ where $\dot{\gamma}$ denotes the vector whose components are the t-derivatives of the components of γ.

(b) Write out this integral explicitly as a function of t and $\dot{\theta}(t)$.

(c) Show that if θ solves the corresponding Euler-Lagrange equation (i.e. if γ minimises length amongst paths on S) then

$$\theta(t) = \int \frac{C}{t^2}\sqrt{\frac{1+4t^2}{t^4 - C^2}} dt$$

for some constant C.

Answer 5. (a) We have $y^2 + z^2 = t^4(\cos^2\theta(t) + \sin^2\theta(t)) = t^4 = x^4$.

(b) The vector $\dot{\gamma}(t)$ is

$$(1, 2t\cos\theta - t^2\dot{\theta}\sin\theta, 2t\sin\theta + t^2\dot{\theta}\cos\theta)$$

which has length

$$\sqrt{1 + 4t^2 + t^4\dot{\theta}^2}$$

so the curve has length

$$\int_0^1 |\dot{\gamma}| dt = \int_0^1 \sqrt{1 + 4t^2 + t^4\dot{\theta}^2} dt.$$

(c) We compute the Euler-Lagrange equation for this integral, considered as a functional in θ:

$$0 = \frac{d}{dt}\frac{t^4\dot{\theta}}{\sqrt{1 + 4t^2 + t^4\dot{\theta}^2}}$$

so

$$t^4\dot{\theta} = C\sqrt{1 + 4t^2 + t^4\dot{\theta}^2}$$

for some constant C. This rearranges to give

$$\dot{\theta} = \frac{C}{t^2}\sqrt{\frac{1+4t^2}{t^4 - C^2}}$$

as required.