Methods 3 - Question Sheet 3

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Question 1. (10 marks for * parts)
Solve Laplace's equation for the following boundary value problems. You may use Fourier series you computed in Sheet 1 and standard formulas for general solutions from lectures provided you state them correctly.

(a) * : $\phi(x,\pi) = x + \pi$, $\phi(0,y) = y$, $\phi(\pi,y) = 2y$, $\phi(x,0) = 0$

(b) * : $\phi(x,\pi) = x^3$, $\phi(0,y) = \sin y$, $\phi(\pi,y) = \pi^3 - \sin y$, $\phi(x,0) = \pi^2 x$

(c) : $\phi(x,\pi) = \pi^2$, $\phi(0,y) = \pi y$, $\phi(\pi,y) = y^2$, $\phi(x,0) = 0$

(d) : $\phi(x,\pi) = -\cos x$, $\phi(0,y) = 1 - 2y/\pi$, $\phi(\pi,y) = 2y/\pi - 1$, $\phi(x,0) = \cos x$

Answer 1. The general solutions referred to in the question are

$\phi(x,\pi) = F(x)$, $\phi(0,y) = 0$, $\phi(\pi,y) = 0$, $\phi(x,0) = 0$

$$\Rightarrow \quad \phi(x,y) = \sum A_n \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh(n\pi y/L)}{\sinh(n\pi)}$$

$\phi(x,\pi) = 0$, $\phi(0,y) = 0$, $\phi(\pi,y) = 0$, $\phi(x,0) = F(x)$

$$\Rightarrow \quad \phi(x,y) = \sum A_n \sin\left(\frac{n\pi x}{L}\right) \frac{\sinh(n\pi(L-y)/L)}{\sinh(n\pi)}$$

$$\begin{array}{ccc} & \phi(x,\pi)=0 & \\ \phi(0,y)=0 & \boxed{} & \phi(\pi,y)=F(y) \\ & \phi(x,0)=0 & \end{array}$$

$$\Rightarrow \quad \phi(x,y) = \sum A_n \sin\left(\tfrac{n\pi y}{L}\right) \frac{\sinh(n\pi x/L)}{\sinh(n\pi)}$$

$$\begin{array}{ccc} & \phi(x,\pi)=0 & \\ \phi(0,y)=F(y) & \boxed{} & \phi(\pi,y)=0 \\ & \phi(x,0)=0 & \end{array}$$

$$\Rightarrow \quad \phi(x,y) = \sum A_n \sin\left(\tfrac{n\pi y}{L}\right) \frac{\sinh(n\pi(L-x)/L)}{\sinh(n\pi)}$$

where A_n is the Fourier coefficient $(2/L)\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$. These formulae may be cited as needed (or derived from the first one by symmetry arguments).

(1 marks)

(a) * We first find a solution

$$\phi_0(x,y) = A + Bx + Cy + Dxy$$

to match the corner values. These are:

$$\boxed{\begin{array}{cc} \pi & 2\pi \\ & \\ 0 & 0 \end{array}}$$

Since $\phi_0(0,0) = 0$ we have $A = 0$. Since $\phi_0(\pi,0) = 0$ we also have $B = 0$. Since $\phi_0(0,\pi) = \pi$ we have $C = 1$ and since $\phi_0(\pi,\pi) = 2\pi$ we have $D = 1/\pi$. Thus $\phi_0(x,y) = ((x/\pi)+1)y$. If we set $\theta = \phi - \phi_0$ then the boundary conditions satisfied by θ vanish identically, so $\theta \equiv 0$. Thus $\phi = \phi_0 = ((x/\pi)+1)y$ is our solution.

(3 marks)

(b) * The corner values are

$$\boxed{\begin{array}{cc} 0 & \pi^3 \\ & \\ 0 & \pi^3 \end{array}}$$

so $\phi_0(x,y) = \pi^2 x$. The function $\theta = \phi - \phi_0$ satisfies the modified boundary conditions

$$\theta(x,\pi) = x^3 - \pi^2 x$$

$$\theta(0,y) = \sin y \qquad \theta(\pi,y) = -\sin y$$

$$\theta(x,0) = 0$$

(2 marks)

so we consider three problems separately

$$\theta_1(x,\pi) = x^3 - \pi^2 x$$

$$\theta_1(0,y) = 0 \qquad \theta_1(\pi,y) = 0$$

$$\theta_1(x,0) = 0$$

$$\theta_2(x,\pi) = 0$$

$$\theta_2(0,y) = \sin y \qquad \theta_2(\pi,y) = 0$$

$$\theta_2(x,0) = 0$$

$$\theta_4(x,\pi) = 0$$

$$\theta_4(0,y) = 0 \qquad \theta_4(\pi,y) = -\sin y$$

$$\theta_4(x,0) = 0$$

We can solve for θ_4 very easily: the Fourier series of $\sin y$ is just $\sin y$, so the solution is $\theta_4(x,y) = -\sin y \sinh x / \sinh \pi$.

By symmetry we see that $\theta_2(x,y) = -\theta_4(\pi - x, y) = \sin y \sinh(\pi - x)/\sinh(\pi)$.

(2 marks)

Finally, the Fourier series of $x^3 - \pi^2 x$ (Sheet 1, Q. 2(a)) is

$$x^3 - \pi^2 x = \sum_{n=1}^{\infty} \frac{12(-1)^n)}{n^3} \sin(nx)$$

so

(2 marks)

$$\theta_1(x,y) = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \frac{\sinh(ny)}{\sinh(n\pi)} \sin(nx).$$

The final solution is $\phi = \phi_0 + \theta_1 + \theta_2 + \theta_4$.

(c) We first find a solution

$$\phi_0(x,y) = A + Bx + Cy + Dxy$$

to match the corner values. These are:

$$\boxed{\begin{array}{cc} \pi^2 & \pi^2 \\ & \\ 0 & 0 \end{array}}$$

Arguing as in part (a) we get $\phi_0(x, y) = \pi y$. Therefore $\theta = \phi - \phi_0$ satisfies the modified boundary conditions

$$\begin{array}{ccc} & \theta(x,\pi) = 0 & \\ \theta(0,y) = 0 & \boxed{\phantom{\begin{array}{c} xxxx \\ xxxx \\ xxxx \end{array}}} & \theta(\pi,y) = y^2 - \pi y \\ & \theta(x,0) = 0 & \end{array}$$

There is only one nonvanishing boundary condition so we do not need to split into several problems. The Fourier series of $y^2 - \pi y$ is

$$y^2 - y = \sum_{n=1}^{\infty} A_n \sin(n\pi y), \qquad A_n = (2/\pi) \int_0^{\pi} (y^2 - \pi y) \sin(n\pi y) dy.$$

Integrating, we get:

$$A_n = \frac{4}{n^3 \pi}((-1)^n - 1).$$

The solution is therefore

$$\theta(x, y) = \sum_{n=1}^{\infty} A_n \frac{\sinh(nx)}{\sinh(n\pi)} \sin(ny)$$

and the solution to the whole problem is

$$\phi(x, y) = \phi_0(x, y) + \theta(x, y) = \pi y + \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{n^3 \pi} \frac{\sinh(nx)}{\sinh(n\pi)} \sin(ny).$$

(d) The corner values are

$$\boxed{\begin{array}{cc} -1 & 1 \\ & \\ 1 & -1 \end{array}}$$

so $\phi_0(x, y) = (1 - 2x/\pi)(1 - 2y/\pi)$. Then $\theta = \phi - \phi_0$ satisfies

$$\begin{array}{ccc} & \theta(x,\pi) = 1 - \cos x - 2x/\pi & \\ \theta(0,y) = 0 & \boxed{\phantom{\begin{array}{c} xxxx \\ xxxx \\ xxxx \end{array}}} & \theta(\pi,y) = 0 \\ & \theta(x,0) = -1 + \cos x + 2x/\pi & \end{array}$$

We split this into two problems

$$\theta_1(x,\pi) = 0 \qquad\qquad \theta_3(x,\pi) = 1 - \cos x - 2x/\pi$$

$$\theta_1(0,y) = 0 \qquad \theta_1(\pi,y) = 0 \qquad\qquad \theta_3(0,y) = 0 \qquad \theta_3(\pi,y) = 0$$

$$\theta_1(x,0) = -1 + \cos x + 2x/\pi \qquad\qquad \theta_3(x,0) = 0$$

It is clear from symmetry considerations that $\theta_1(x,y) = -\theta_3(x,1-y)$.

We solve for θ_2. The Fourier series of $1 - 2x/\pi - \cos x$ is (Sheet 1, Q.2(b))

$$1 - 2x/\pi - \cos x = -2\sum_{n=2}^{\infty} \frac{(-1)^n + 1}{(n^2-1)n\pi}$$

so the solution is

$$\theta_3(x,y) = -2\sum_{n=2}^{\infty} \frac{((-1)^n + 1)}{(n^2-1)n\pi} \frac{\sinh(ny)}{\sinh(n\pi)} \sin(nx)$$

The full solution is $\phi = \phi_0 + \theta_1 + \theta_3$.

Question 2. (5 marks for * part)
Solve the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}$$

on $x \in [0, 1]$ (so $L = 1$) with boundary conditions $\phi(0, t) = \phi(1, t) = 0$ for each of the initial conditions.

(a) * $\phi(x, 0) = x - x^2$, $\dfrac{\partial \phi}{\partial t}(x, 0) = 0$.

(b) $\phi(x, 0) = x - x^3$, $\dfrac{\partial \phi}{\partial t}(x, 0) = 0$.

(c) $\phi(x, 0) = \cos 2\pi x - 1$, $\dfrac{\partial \phi}{\partial t}(x, 0) = 0$.

Answer 2. In each case, the solution will be given as a linear combination of separated solutions

$$\sum_{n=1}^{\infty} \sin(n\pi x)(A_n \cos(n\pi t) + B_n \sin(n\pi t))$$

(where we have used the Dirichlet boundary conditions $\phi(0, t) = \phi(1, t) = 0$ to rule out terms involving $\cos(n\pi x)$). The initial condition $\dfrac{\partial \phi}{\partial t}(x, 0) = 0$ in each case gives

$$\sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x) = 0$$

so $B_n = 0$. The other initial condition $\phi(x, 0) = F(x)$ gives

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x) = F(x)$$

so A_n is the nth half-range sine Fourier coefficient of F.

(3 marks)

(a) * In this case the Fourier coefficient is

$$\begin{aligned}
2\int_0^1 (x - x^2) \sin(n\pi x) dx &= 2\left(\cancelto{0}{\left[-(x - x^2)\frac{\cos(n\pi x)}{n\pi} \right]_0^1} + \frac{1}{n\pi}\int_0^1 (1 - 2x)\cos(n\pi x) dx \right) \\
&= \frac{2}{n\pi}\left(\cancelto{0}{\left[(1 - 2x)\frac{\sin(n\pi x)}{n\pi} \right]_0^1} - \frac{1}{n\pi}\int_0^1 (-2)\sin(n\pi x) dx \right) \\
&= \frac{4}{n^2\pi^2}\left[-\frac{\cos(n\pi x)}{n\pi} \right]_0^1 \\
&= \frac{4(1 + (-1)^{n+1})}{n^3\pi^3}
\end{aligned}$$

(2 marks)

so the solution is

$$\phi(x,t) = \sum_{n=1}^{\infty} \frac{4(1+(-1)^{n+1})}{n^3\pi^3} \sin(n\pi x)\cos(n\pi t).$$

(b) $\phi(x,t) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3} \sin(n\pi x)\cos(n\pi t)$.

(c) $\phi(x,t) = \sum_{\substack{n=1 \\ n\neq 2}}^{\infty} \frac{8(1+(-1)^{n+1})}{n\pi(n^2-4)} \sin(n\pi x)\cos(n\pi t)$. Note that the case $n = 2$ must be dealt with separately as one ends up dividing by $n-2$ at some point in the solution (when integrating $\sin((n-2)\pi)$).

Question 3. (5 marks)
Suppose that ϕ satisfies the wave equation $\partial_t^2\phi = c^2\partial_x^2\phi$ where c depends discontinuously on x:

$$c(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x > 0. \end{cases}$$

We impose the boundary conditions $\phi(-L,t) = \phi(L,t) = 0$ and the conditions that ϕ and $\partial_x\phi$ are continuous at $x = 0$. Let $X(x)T(t)$ be a separated solution; the boundary conditions imply $X(-L) = X(L) = 0$. If $T''/T = -m^2$, prove that m satisfies

$$\tan(mL) = -2\tan(mL/2).$$

Answer 3. Separated solutions to the wave equation with $T''/T = -m^2$ are

$$X = A\sin(mx/c) + B\cos(mx/c), \quad T = C\sin(mt) + D\cos(mt).$$

If $X(x)T(t)$ is the separated solution for $x < 0$ then (using the formula for c):

$$X(x) = \begin{cases} A_1\sin(mx) + B_1\cos(mx) & x < 0 \\ A_2\sin(mx/2) + B_2\cos(mx/2) & x < 0 \end{cases}$$

The continuity and differentiability of X at $x = 0$ imply

$$B_1 = B_2, \quad mA_1 = mA_2/2$$

so $A_2 = 2A_1$ and the solution is

$$X(x) = \begin{cases} A_1\sin(mx) + B_1\cos(mx) & x < 0 \\ 2A_1\sin(mx/2) + B_1\cos(mx/2) & x < 0 \end{cases}$$

(2 marks)

Now we fix the boundary conditions $\phi(-L,t) = \phi(L,t) = 0$.

The conditions $\phi(\pm L,t) = 0$ imply that $X(-L) = 0$ and $X(L) = 0$, so

$$\begin{aligned} A_1\sin(-mL) + B_1\cos(-mL) &= 0 \\ 2A_1\sin(mL/2) + B_1\cos(mL/2) &= 0 \end{aligned}$$

(1 marks)

If $A_1 = 0$ then the only way to get a nontrivial solution is if $\cos(mL) = \cos(mL/2) = 0$. This implies $m = 2n\pi/L$ for some integer n (and the equation $\tan(mL) = -2\tan(mL/2)$ is trivially satisfied because both sides vanish).

If $A_1 \neq 0$ then we get $B_1/A_1 = \tan(mL) = -2\tan(mL/2)$ as required.

(2 marks)

There is a discrete set of m for which $\tan(mL) = -2\tan(mL/2)$, though it looks like a transcendental problem to give a general expression for such an m.

Question 4.

Consider the wave equation $\dfrac{\partial^2\phi}{\partial t^2} = \dfrac{\partial^2\phi}{\partial x^2}$ on the interval $[0,1]$.

(a) Find the separated solutions satisfying $\phi(0,t) = 0$, $\dfrac{\partial\phi}{\partial x}(1,t) = 0$.

(b) Find the full solution satisfying the boundary conditions in (a) and the initial conditions

$$\phi(x,0) = \cos(x\pi/2), \quad \frac{\partial\phi}{\partial t}(x,0) = 0.$$

Answer 4. The factors of separated solutions $X(x)T(t)$ obey the ODEs $X'' = -\lambda X$, $T'' = -\lambda T$. Thus

$$X = \begin{cases} A\sin(px) + B\cos(px) & \text{if } \lambda = p^2 > 0 \\ Ax + B & \text{if } \lambda = 0 \\ A\sinh(px) + B\cosh(px) & \text{if } \lambda = -p^2 < 0. \end{cases}$$

Case 1: $\lambda = p^2$. Setting $X(0) = 0$ gives $B = 0$. Setting $X'(1) = 0$ gives $p = \left(n + \frac{1}{2}\right)\pi$.

Case 2: $\lambda = 0$. Setting $X(0) = 0$ gives $B = 0$. Setting $X'(1) = 0$ gives $A = 0$.

Case 3: $\lambda = -p^2$. Setting $X(0) = 0$ gives $B = 0$. Setting $X'(1) = 0$ gives $A = 0$.

Thus the nontrivial solutions are $\cos\left(\left(n + \frac{1}{2}\right)\pi x\right)$.

We make an Ansatz for the solution of the form

$$\phi(x,t) = \sum_{n=0}^{\infty} \cos\left(\left(n + \frac{1}{2}\right)\pi x\right)\left(A_n \sin\left(\left(n + \frac{1}{2}\right)\pi t\right) + B_n \cos\left(\left(n + \frac{1}{2}\right)\pi t\right)\right).$$

Setting $t = 0$ gives

$$\phi(x,0) = \sum B_n \cos\left(\left(n + \frac{1}{2}\right)\pi x\right)$$

which equals $\cos(x\pi/2)$ by the initial condition, so the only nonvanishing term is $B_0 = 1$. Differentiating the Ansatz with respect to t and setting $t = 0$ gives

$$\frac{\partial\phi}{\partial t}(x,0) = \sum A_n \left(n + \frac{1}{2}\right)\pi\cos\left(\left(n + \frac{1}{2}\right)\pi x\right) = 0$$

which tells us that $A_n = 0$ for all n. Therefore the solution is

$$\phi(x,t) = \cos(\pi x/2)\cos(\pi t/2).$$

Question 5.
Consider the 2-dimensional heat equation for a temperature distribution $\phi(x, y, t)$

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}.$$

(a) By separating variables $\phi(x, y, t) = X(x)Y(y)T(t)$, show that

$$T = e^{Kt}, \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = K\phi$$

(remember this second equation is the Helmholtz equation from Sheet 2).

For a separated solution, the Dirichlet conditions

$$\phi(x, 0, t) = \phi(x, 1, t) = \phi(0, y, t) = \phi(1, y, t) = 0$$

mean that we require $X(0) = X(1) = Y(0) = Y(1) = 0$. The possible separated solutions to the Helmholtz equation with these boundary conditions were $\sin(px)\sin(qx)$ with $p, q \in \pi\mathbf{Z}$, $K = -(p^2 + q^2)$.

(b) What is the solution to the 2-dimensional heat equation with the initial condition $\phi(x, y, 0) = x(1 - x)\sin(\pi y)$?

Answer 5. (a) Separating variables we get

$$XYT' = X''YT + XY''T$$

so

$$T'/T = (X''Y + XY'')/XY = K$$

for some constant K. The T equation gives $T = e^{Kt}$, the other equation is $X''Y + XY'' = KXY$ or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = K\phi.$$

(b) Our Ansatz for a solution is a linear combination of separated solutions

$$\phi(x, y, t) = \sum_{m,n} B_{m,n} \sin(n\pi x)\sin(m\pi y)e^{-(n^2+m^2)\pi^2 t}$$

satisfying

$$\phi(x, y, 0) = \sum_{m,n} B_{m,n} \sin(n\pi x)\sin(m\pi y) = x(1 - x)\sin(\pi y).$$

Multiply by $\sin(k\pi y)$ and integrate from -1 to 1 - the Fourier integral identities tell you that

$$\sum_n B_{k,n} \sin(n\pi x) = x(1 - x)\delta_{1,k}$$

so take $B_{1,n}$ to be the nth Fourier coefficient of $x(1 - x)$ and all other B to be zero. Therefore the solution is

$$\sum_{n=1}^{\infty} \frac{2((-1)^{n+1} + 1)}{n^3\pi^3} \sin(n\pi x)\sin(\pi y)e^{-(n^2+1)\pi^2 t}.$$