Methods 3 - Question Sheet 5

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Question 1. (10 marks for * parts)
Solve the Euler-Lagrange equation for the following constrained functionals:

(a) * $F(y) = \int_0^{\pi/2}((y')^2 + 2xyy')dx$ subject to $\int_0^{\pi/2} ydx = K$ and the boundary conditions $y(0) = 0 = y(\pi/2)$.

(b) * $F(y) = \int_0^1((y')^2 + y^2)dx$ subject to $\int_0^1 xy = \frac{1}{6} + \frac{1}{e}$ and the boundary conditions $y(0) = 0$, $y(1) = \sinh(1) + 1/2$.

(c) $F(y) = \int_0^1 \sqrt{1 + (y')^2}dx$ subject to $\int_0^1 \sqrt{y}dx = K$ and the boundary conditions $y(0) = A$, $y(1) = B$ (you need not compute any constants in the solution and may leave the solution in implicit form).

(d) $F(y) = \int_0^{\pi/2}((y')^2 - y^2)dx$ subject to $\int_0^{\pi/2} ydx = 6 - \pi$ and the boundary conditions $y(0) = 1 = y(\pi/2)$.

(e) $F(y) = \int_a^b(y')^2dx$ subject to $\int_a^b \frac{dx}{y} = K$ (give your solution in implicit form, specifying x in terms of y).

Answer 1. (a) * The Lagrangian is

$$(y')^2 + 2xyy' - \lambda(y - 2K/\pi)$$

which gives the Euler-Lagrange equation

$$2y'' + 2xy' + 2y = -\lambda + 2xy'$$

or

(2 marks)

$$y'' + y = -\lambda/2.$$

The general solution is $\alpha \sin x + \beta \cos x - \lambda/2$ and the boundary conditions give $y(0) = y(\pi/2) = 0$ so $\alpha = \lambda/2 = \beta$ and the constraint gives $\alpha + \beta - \lambda\pi/4 = K$ so $\lambda = K/(1 - \pi/4)$.

(3 marks)

(b) * The Lagrangian is

$$(y')^2 + y^2 - \lambda(xy - 2/\pi)$$

which gives the Euler-Lagrange equation

(2 marks)

$$2y'' = 2y - \lambda x$$

which has general solution

$$y = \alpha \sinh x + \beta \cosh x + \lambda x/2.$$

The boundary conditions give

$$y(0) = \beta = 0, \qquad y(1) = \alpha \sinh(1) + \lambda/2 = \sinh(1) + 1/2.$$

We have

$$\int (\alpha x \sinh x + \lambda x^2/2) dx = \frac{\lambda}{6} + \frac{\alpha}{e}$$

so the solution to these two simultaneous equations for α and λ is $\alpha = \lambda = 1$. So the final answer is

(3 marks)

$$y(x) = \sinh(x) + x/2.$$

(c) The Beltrami identity holds so we get

$$\frac{1}{\sqrt{1+(y')^2}} - \lambda(\sqrt{y} - K) = C$$

for some C. This gives

$$y' = \sqrt{\frac{1}{(C + \lambda\sqrt{y} - \lambda K)^2} - 1}$$

so we have to integrate

$$\int dx = \int \frac{dy}{\sqrt{\frac{1}{(a+b\sqrt{y})^2} - 1}} = \int \frac{(a + b\sqrt{y})dy}{\sqrt{1 - (a+b\sqrt{y})^2}}$$

where for convenience, $a = C - \lambda K$, $b = \lambda$. We do this integral by substituting $a + b\sqrt{y} = \sin\theta$, so

$$dy = \frac{2}{b}\sqrt{y}\cos\theta d\theta = \frac{2}{b^2}\cos\theta(\sin\theta - a)d\theta$$

and the integral becomes

$$\frac{2}{b^2}\int (\sin^2\theta - a\sin\theta)d\theta$$

which gives

$$\frac{2}{b^2}(a\cos\theta + (\theta - \sin\theta\cos\theta)/2$$

or, substituting back, we get

$$x = \frac{2}{b^2}\left(a\sqrt{1-(a+b\sqrt{y})^2} + \sin^{-1}(a+b\sqrt{y}) - \frac{1}{2}(a+b\sqrt{y})\sqrt{1-(a+b\sqrt{y})^2}\right) + \text{const.}$$

Thankfully we can leave it in this form and not worry about constants!

(d) This one turns out to be easier without Beltrami. The Euler-Lagrange equation is

$$\frac{d}{dx}(2y') = -2y - \lambda$$

or

$$y'' + y = -\lambda/2.$$

This has general solution $\alpha \sin x + \beta \cos x - \lambda/2$. The boundary conditions and constraint imply:

$$y(0) = \beta - \lambda/2 = 1, \qquad y(\pi/2) = \alpha - \lambda/2 = 1$$

and

$$\int_0^{\pi/2} (\alpha \sin x + \beta \cos x - \lambda/2) dx = \alpha + \beta - \frac{\lambda\pi}{4} = 6 - \pi$$

so we have

$$\lambda = 4, \qquad \alpha = \beta = 3.$$

(e) Consider the modified functional

$$\int_a^b ((y')^2 - \lambda \left(\frac{1}{y} - \frac{K}{b-a} \right) dx.$$

The Euler-Lagrange equation reduces to Beltrami's identity

$$(y')^2 - \lambda \left(\frac{1}{y} - \frac{K}{b-a} \right) - 2(y')^2 = C$$

for some constant C. This gives

$$y' = \sqrt{\frac{K\lambda}{b-a} - C - \frac{\lambda}{y}}.$$

Let us write $A = K\lambda/(b-a) - C$, so that

$$\int \frac{\sqrt{y}dy}{\sqrt{Ay - \lambda}} = \int dx.$$

This integral is very similar to the one which occurred in the brachistochrone problem. It gives

$$\frac{\lambda}{A^{3/2}} \log \left(\sqrt{A}\sqrt{Ay - \lambda} + A\sqrt{y} \right) + \frac{\sqrt{y}\sqrt{Ay - \lambda}}{A} + B = x.$$

Question 2. (5 marks)
Suppose that $(x(t), y(t))$ is a vector-valued function of t such that $(x(a), y(a)) = (x(b), y(b)) = (0, 0)$. If $L(t, x, y, \dot{x}, \dot{y})$ is a Lagrangian and $F(x, y) = \int_a^b L(t, x, y, \dot{x}, \dot{y})dt$ is the corresponding functional, show that $(x(t), y(t))$ is a critical point of F if and only if the equations

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}, \quad \frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$$

both hold.

Answer 2. If we make a variation $x(t) \mapsto x(t) + \delta(t)$, $y(t) \mapsto y(t) + \epsilon(t)$ with $\delta(a) = \delta(b) = \epsilon(a) = \epsilon(b) = 0$ then, to first order, the variation in L is

$$L(t, x + \delta, y + \epsilon, \dot{x} + \dot{\delta}, \dot{y} + \dot{\epsilon}) - L(t, x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial x}\delta + \frac{\partial L}{\partial y}\epsilon + \frac{\partial L}{\partial \dot{x}}\dot{\delta} + \frac{\partial L}{\partial \dot{y}}\dot{\epsilon}$$

The corresponding variation in $F(x, y)$ is

(2 marks)

$$d_{(x,y)}F(\delta, \epsilon) = \int_a^b \left(\frac{\partial L}{\partial x}\delta + \frac{\partial L}{\partial y}\epsilon + \frac{\partial L}{\partial \dot{x}}\dot{\delta} + \frac{\partial L}{\partial \dot{y}}\dot{\epsilon}\right) dt$$

Integrating by parts this gives

(1 marks)

$$\int_a^b \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x}\right) \delta dt + \int_a^b \left(\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y}\right) \epsilon dt + \left[\frac{\partial L}{\partial \dot{x}}\delta + \frac{\partial L}{\partial \dot{y}}\epsilon\right]_a^b$$

and the boundary terms vanish because δ and ϵ both vanish at a and b. Setting $\epsilon = 0$ we see that $\int_a^b \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) \delta dt = 0$ for all δ (so the integrand must vanish) and setting $\delta = 0$ we see, likewise, that $\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$.

(2 marks)

Question 3. (5 marks)
Let P be a polynomial of even degree 2 or more and consider the functional

$$F(y) = \int_0^1 P(y')dx$$

for functions y satisfying $y(0) = 0$, $y(1) = 1$. Find the Euler-Lagrange equation and show that if y is a solution then y' is constant. Deduce that $y(x) = x$.

Hint: When computing the Euler-Lagrange equation, use the chain rule. Your Euler-Lagrange equation should involve the derivative P' of P.

Answer 3. The Euler-Lagrange equation is $\frac{d}{dx}(P'(y')) = 0$ so $P'(y') = C$ for some constant C.

(2 marks)

Therefore y' is a zero of the polynomial $P' - C$.

(1 marks)

Since polynomials have a finite number of zeros, $y' \equiv A$ for one of the zeros $P'(A) - C = 0$, so $y(x) = Ax + B$ (note that since the polynomial has even degree, $P' - C$ has at least one zero). The boundary conditions now imply that $A = 1$, $B = 0$ (hence $C = P'(1)$) and so $y(x) = x$ is the only solution.

(2 marks)

Question 4.
A smooth probability distribution on $\mathbf{R}$ with second moment σ^2 is a smooth function $\rho\colon \mathbf{R} \to [0, \infty)$ satisfying

$$(\star) \quad \int_{\mathbf{R}} \rho(x) = 1, \qquad (\star\star) \quad \int_{\mathbf{R}} x^2 \rho(x) dx = \sigma^2.$$

Show that if ρ is a smooth probability distribution maximising the *entropy functional*

$$S(\rho) = -\int_{\mathbf{R}} \rho(x) \ln(\rho(x)) dx$$

amongst all smooth probability distributions with second moment σ^2 then

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Hint: Introduce two Lagrange multipliers: one for $(\star)$ and one for $(\star\star)$. When imposing the constraints $(\star)$ and $(\star\star)$ it may help to remember that $\int_{\mathbf{R}} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ and $\int_{\mathbf{R}} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$.

Answer 4. We need to introduce the modified functional with two Lagrange multipliers

$$\int_{\mathbf{R}} \big(-\rho \ln \rho - \lambda_1(\rho - \kappa) - \lambda_2(x^2\rho - \kappa)\big)\, dx$$

where κ is an arbitrary function with $\int_{\mathbf{R}} \kappa dx = 1$ (over a finite interval of length L we could just use $\kappa = 1/L$, but this doesn't work for the whole real line!). The Euler-Lagrange equation is

$$-\ln \rho - 1 - \lambda_1 - \lambda_2 x^2 = 0$$

because there is no ρ'-dependence. This implies

$$\rho = \exp(-1 - \lambda_1) \exp(-\lambda_2 x^2)$$

so this is a normal distribution with mean zero. Let's call $\exp(-1 - \lambda_1) = A$. The integral of this distribution over $\mathbf{R}$ is $A\sqrt{\pi/\lambda_2}$ and the second moment is $\frac{1}{2}A\sqrt{\pi/\lambda_2^3}$. This gives $\lambda_2 = 1/2\sigma^2$ and $A = 1/\sigma\sqrt{2\pi}$.