Methods 3 - Question Sheet 6

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In this sheet ϕ_x denotes $\dfrac{\partial \phi}{\partial x}$.

Question 1. (10 marks for * parts)
Find the Euler-Lagrange equation for the following functionals (we have only written the Lagrangian in each case - if you need to, assume that the domain of integration is $(x, y) \in [0, 1]^2$)

(a) * $\frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + \frac{1}{2}K\phi^2$ (where K is a constant),

(b) * $\phi^2\phi_x^2 + \phi_y^2$,

(c) * $\phi_x\phi_y$ (in this case, also find the general solution to the Euler-Lagrange equation),

(d) $\frac{1}{2}\left(\phi_x^2 + \phi_y^2\right)$ subject to the constraint $\int \phi^2 dxdy = K$.

(e) $\frac{1}{\phi_x} + \frac{1}{\phi_y}$.

Answer 1. (a) The Euler-Lagrange equation

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial \phi_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial \phi_y} = \frac{\partial L}{\partial \phi}$$

becomes

$$\frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} = K\phi$$

that is the Helmholtz equation $\Delta\phi = K\phi$.

(3 marks)

(b) The Euler-Lagrange equation

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial \phi_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial \phi_y} = \frac{\partial L}{\partial \phi}$$

becomes

$$\frac{\partial}{\partial x}\left(2\phi^2\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(2\frac{\partial \phi}{\partial y}\right) = 2\phi\left(\frac{\partial \phi}{\partial x}\right)^2$$

or

$$4\phi\left(\frac{\partial \phi}{\partial x}\right)^2 + 2\phi^2\frac{\partial^2 \phi}{\partial x^2} + 2\frac{\partial^2 \phi}{\partial y^2} = 2\phi\left(\frac{\partial \phi}{\partial x}\right)^2.$$

Simplifying gives

(3 marks)

$$0 = \phi^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \phi \left(\frac{\partial \phi}{\partial x} \right)^2 .$$

(c) The Euler-Lagrange equation

$$\frac{\partial}{\partial x} \frac{\partial L}{\partial \phi_x} + \frac{\partial}{\partial y} \frac{\partial L}{\partial \phi_y} = \frac{\partial L}{\partial \phi}$$

becomes

$$\frac{\partial}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial}{\partial y} \frac{\partial \phi}{\partial x} = 0$$

or

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0.$$

The general solution is $F(x) + G(y)$ for arbitrary functions F and G.

(4 marks)

(d) Modify the functional to

$$\frac{1}{2} \left(\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right) - \lambda(\phi^2 - K)$$

the Euler-Lagrange equation

$$\frac{\partial}{\partial x} \frac{\partial L}{\partial \phi_x} + \frac{\partial}{\partial y} \frac{\partial L}{\partial \phi_y} = \frac{\partial L}{\partial \phi}$$

becomes

$$\frac{\partial}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y} \frac{\partial \phi}{\partial y} = -2\lambda\phi$$

or $\Delta\phi = -2\lambda\phi$ (again this is Helmholtz's equation).

(e) The Euler-Lagrange equation

$$\frac{\partial}{\partial x} \frac{\partial L}{\partial \phi_x} + \frac{\partial}{\partial y} \frac{\partial L}{\partial \phi_y} = \frac{\partial L}{\partial \phi}$$

becomes

$$\frac{\partial}{\partial x} \left(-\frac{1}{\phi_x^2} \right) + \frac{\partial}{\partial y} \left(-\frac{1}{\phi_y^2} \right) = 0$$

or

$$\frac{2\phi_{xx}}{\phi_x^3} + \frac{2\phi_{yy}}{\phi_y^3} = 0.$$

Equivalently

$$\phi_y^3 \phi_{xx} + \phi_x^3 \phi_{yy} = 0.$$

Question 2.
A string has its endpoints fixed at $(0,0)$ and $(L,0)$. If its height at x and time t is $\phi(x,t)$ then (to a good approximation) its total kinetic energy is $E(\phi) = \frac{\rho}{2}\int_0^L \phi_t^2 dx$ and its total potential energy (coming from stretching tension) is $T(\phi) = \frac{\tau}{2}\int_0^1 \phi_x^2 dx$. The string moves to minimise the integral

$$\int_0^1 (E(\phi) - T(\phi))dt.$$

Show that the string obeys the *wave equation*

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$$

where $c = \sqrt{\tau/\rho}$.

Answer 2. The Euler-Lagrange equation for $\int_0^1\int_0^L (\frac{\rho}{2}(\partial_t\phi)^2 - \frac{\tau}{2}(\partial_x\phi)^2)dxdt$ is

$$\frac{\rho}{2}\frac{\partial}{\partial t}\partial_t\phi - \frac{\tau}{2}\frac{\partial}{\partial x}\partial_x\phi = 0$$

or

$$\frac{\rho}{\tau}\partial_t^2\phi = \partial_x^2\phi$$

which is the wave equation.

Question 3. (10 marks)

(a) Find the half-range sine series of the function

$$F(x) = \begin{cases} x^2 & \text{if } x \in [0, 1/2] \\ (x-1)^2 & \text{if } x \in [1/2, 1] \end{cases}$$

(b) Derive the Euler-Lagrange equation for the following functional

$$\int_0^1 \int_0^1 \left(\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right) dx dy$$

(c) Solve this Euler-Lagrange equation given the boundary values

$$\phi(x,1) = 0$$

$$\phi(0,y) = F(y) \qquad \phi(1,y) = 0$$

$$\phi(x,0) = 0$$

where F is defined in part (a) of the question.

Answer 3. (a) We need to calculate the nth Fourier coefficient $F_n = 2\int_0^1 F(x)\sin(n\pi x)dx$. We split the integral as

$$2\int_0^{1/2} x^2 \sin(n\pi x)dx + 2\int_{1/2}^1 (x-1)^2 \sin(n\pi x)dx$$

The first gives

$$\begin{aligned} \int_0^{1/2} x^2 \sin(n\pi x)dx &= \left[-x^2 \frac{\cos(n\pi x)}{n\pi} \right]_0^{1/2} + 2\int_0^{1/2} x \frac{\cos(n\pi x)}{n\pi} dx \\ &= \frac{-\cos(n\pi/2)}{4n\pi} + \frac{2}{n\pi}\left[x\frac{\sin(n\pi x)}{n\pi} \right]_0^{1/2} - \frac{2}{n\pi}\int_0^{1/2} \frac{\sin(n\pi x)}{n\pi} dx \\ &= \frac{-\cos(n\pi/2)}{4n\pi} + \frac{\sin(n\pi/2)}{n^2\pi^2} + \frac{2}{n^3\pi^3}\left(\cos(n\pi/2) - 1\right) \end{aligned}$$

the second gives

$$\begin{aligned} \int_{1/2}^1 (x-1)^2 \sin(n\pi x)dx &= \left[-(x-1)^2 \frac{\cos(n\pi x)}{n\pi} \right]_{1/2}^1 + 2\int_{1/2}^1 (x-1) \frac{\cos(n\pi x)}{n\pi} dx \\ &= \frac{\cos(n\pi/2)}{4n\pi} + \frac{2}{n\pi}\left[(x-1)\frac{\sin(n\pi x)}{n\pi} \right]_{1/2}^1 - \frac{2}{n\pi}\int_{1/2}^1 \frac{\sin(n\pi x)}{n\pi} dx \\ &= \frac{\cos(n\pi/2)}{4n\pi} + \frac{\sin(n\pi/2)}{n^2\pi^2} + \frac{2}{n^3\pi^3}\left((-1)^n - \cos(n\pi/2)\right) \end{aligned}$$

Adding these two integrals (and reinserting the factor of 2) gives

(6 marks)

$$F_n = \frac{4}{n^2\pi^2}\left(\sin(n\pi/2) + \frac{(-1)^n - 1}{n\pi}\right)$$

so

$$F(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\sin(n\pi/2) + \frac{(-1)^n - 1}{n\pi}\right)\sin(n\pi x).$$

(b) The Euler-Lagrange equation

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial \partial_x \phi} + \frac{\partial}{\partial y}\frac{\partial L}{\partial \partial_y \phi} = 0$$

is

$$\frac{\partial}{\partial x}2\phi_x + \frac{\partial}{\partial y}2\phi_y = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

This is just Laplace's equation.

(1 marks)

(c) So we need to solve Laplace's equation with the given boundary conditions. Since all the corner values vanish we can make the Ansatz

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi y)\sinh(n\pi(1-x))$$

and deduce from $\phi(0,y) = F(y)$ that

$$\sum_{n=1}^{\infty} A_n \sin(n\pi y)\sinh(n\pi) = \sum_{n=1}^{\infty} F_n sin(n\pi y)$$

and so A_n is $F_n \sinh(n\pi)$. Therefore the solution is

(3 marks)

$$\sum_{n=1}^{\infty} \frac{4}{n^2\pi^2\sinh(n\pi)}\left(\sin(n\pi/2) + \frac{(-1)^n - 1}{n\pi}\right)\sin(n\pi y)\sinh(n\pi(1-x))$$

Question 4.
Show that the Euler-Lagrange equation for the functional

$$\int_0^1 \int_0^1 \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dxdy$$

is the *minimal surface equation*

$$\frac{\partial^2 \phi}{\partial x^2}\left(1 + \left(\frac{\partial \phi}{\partial y}\right)^2\right) + \frac{\partial^2 \phi}{\partial y^2}\left(1 + \left(\frac{\partial \phi}{\partial x}\right)^2\right) = 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y}.$$

Answer 4. We will write ϕ_x for $\partial_x \phi$ and $j = \sqrt{1 + \phi_x^2 + \phi_y^2}$ for brevity. The Euler-Lagrange equation is

$$\frac{\partial}{\partial x}\frac{\phi_x}{j} + \frac{\partial}{\partial y}\frac{\phi_y}{j} = 0$$

We have $\frac{\partial}{\partial x}\frac{\phi_x}{j} = \frac{\phi_{xx}}{j} - \frac{\phi_x}{j^2}\frac{\partial j}{\partial x}$ and $\frac{\partial j}{\partial x} = \frac{\phi_x \phi_{xx} + \phi_y \phi_{yx}}{j}$ (with a similar equation for the second term). Therefore the Euler-Lagrange equation implies

$$0 = \frac{\phi_{xx} + \phi_{yy}}{j} - \frac{1}{j^3}\left(\phi_x^2 \phi_{xx} + 2\phi_x \phi_y \phi_{xy} + \phi_y^2 \phi_{yy}\right)$$

or

$$0 = \phi_{xx}(1 + \phi_y^2) + \phi_{yy}(1 + \phi_x^2) - 2\phi_x \phi_y \phi_{xy}.$$

Question 5. Check that the function $\phi(x, y) = \sqrt{1 - x^2 - y^2}$ satisfies the *constant mean curvature equation*

$$-\lambda = \frac{\partial}{\partial x}\left(\frac{\phi_x}{\sqrt{1 + \phi_x^2 + \phi_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\phi_y}{\sqrt{1 + \phi_x^2 + \phi_y^2}}\right)$$

for a suitable value of λ. This proves that hemispherical soap bubbles can exist!