Methods 3 - Question Sheet 8

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Question 1. (10 marks)
Solve the wave equation $\partial_t^2\phi = \partial_x^2\phi$ with the initial conditions

(a) * $\phi(x,0) = x \qquad \partial_t\phi(x,0) = x^2$.

(b) $\phi(x,0) = x^2 \qquad \partial_t\phi(x,0) = x$.

(c) $\phi(x,0) = 0 \qquad \partial_t\phi(x,0) = k\sin kx$.

(d) * $\phi(x,0) = \begin{cases} 0 & \text{if } |x| > 1 \\ 1 & \text{if } |x| \leq 1, \end{cases} \qquad \partial_t\phi(x,0) = \begin{cases} 0 & \text{if } |x| > 1 \\ -1 & \text{if } |x| \leq 1. \end{cases}$

In this last case illustrate your answer with a spacetime diagram.

Answer 1. We know that the general D'Alembert solution to the wave equation is a function $\phi(x,t)$ of the form

$$\phi(x,t) = F(x+t) + G(x-t).$$

We use this in what follows.

(1 marks)

(a) * The initial conditions give us

$$\phi(x,0) = F(x) + G(x) = x, \qquad \partial_t\phi(x,0) = F'(x) - G'(x) = x^2.$$

Integrating the second of these gives

$$F(x) - G(x) = x^3/3 + C$$

(we will see soon that the constant can be ignored) so solving for $F(x)$ gives us

$$F(x) = G(x) + C + x^3/3 = -G(x) + x$$

which implies

$$G(x) = \frac{1}{2}(x - x^3/3 - C)$$

which in turn implies

$$F(x) = x - \frac{1}{2}(x - x^3/3 - C) = \frac{1}{2}(x + x^3/3 + C)$$

so $F(x+t) + G(x-t) = \frac{1}{2}(x + t + (x+t)^3/3) + \frac{1}{2}(x - t - (x-t)^3/3)$ (the constants cancel out as mentioned earlier).

(3 marks)

(b) The initial conditions give us

$$\phi(x,0) = F(x) + G(x) = x^2, \qquad \partial_t\phi(x,0) = F'(x) - G'(x) = x.$$

Integrating the second of these gives

$$F(x) - G(x) = x^2/2 + C$$

(we will see soon that the constant can be ignored) so solving for $F(x)$ gives us

$$F(x) = G(x) + C + x^2/2 = -G(x) + x^2$$

which implies

$$G(x) = \frac{1}{2}(x^2/2 - C)$$

which in turn implies

$$F(x) = x^2 - \frac{1}{2}(x^2/2 - C) = \frac{1}{2}(3x^2/2 + C)$$

so $F(x+t) + G(x-t) = \frac{3}{4}(x+t)^2 + \frac{1}{4}(x-t)^2$ (the constants cancel out as mentioned earlier).

(c) The initial conditions give us

$$\phi(x,0) = F(x) + G(x) = 0, \qquad \partial_t\phi(x,0) = F'(x) - G'(x) = k\sin kx.$$

Integrating the second of these gives

$$F(x) - G(x) = -\cos kx + C$$

(we will see soon that the constant can be ignored). Since the first equation gives $F(x) = -G(x)$, the second becomes

$$2F(x) = -\cos kx + C = -2G(x)$$

so $F(x+t) + G(x-t) = \frac{1}{2}\cos(k(x-t)) - \frac{1}{2}\cos(k(x+t))$ (the constants cancel out as mentioned earlier).

(d) * The initial conditions give us

$$\phi(x,0) = F(x) + G(x) = \begin{cases} 0 & \text{if } x < -1, x > 1 \\ 1 & \text{if } -1 \leq x \leq 1 \end{cases},$$

$$\partial_t\phi(x,0) = F'(x) - G'(x) = \begin{cases} 0 & \text{if } x < -1, x > 1 \\ -1 & \text{if } -1 \leq x \leq 1 \end{cases}.$$

Integrating the second of these gives

$$F(x) - G(x) = \begin{cases} 0 & \text{if } x < -1 \\ -1 - x & \text{if } -1 \leq x \leq 1 \\ -2 & \text{if } x > 1. \end{cases}$$

(ignoring constants) so solving for $F(x)$ and $G(x)$ gives us

$$2F(x) = \begin{cases} 0 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x \leq 1 \\ -2 & \text{if } x > 1. \end{cases},$$

$$2G(x) = \begin{cases} 0 & \text{if } x < -1 \\ 2 + x & \text{if } -1 \leq x \leq 1 \\ 2 & \text{if } x > 1. \end{cases}$$

i.e.

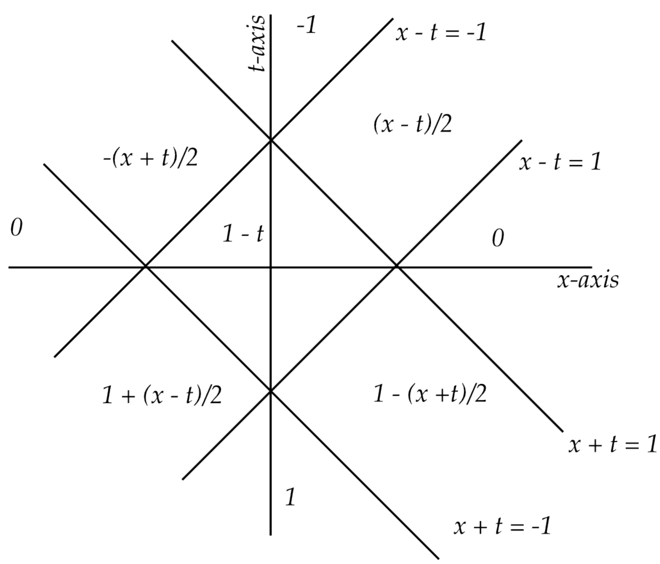
$$F(x+t) = \begin{cases} 0 & \text{if } x + t < -1 \\ -(x+t)/2 & \text{if } x + t \in [-1, 1] \\ -1 & \text{if } x + t > 1 \end{cases}$$

$$G(x-t) = \begin{cases} 0 & \text{if } x - t < -1 \\ 1 + (x-t)/2 & \text{if } x - t \in [-1, 1] \\ 1 & \text{if } x - t > 1 \end{cases}$$

(4 marks)

The resulting function is best illustrated by a plot:

(2 marks)



Question 2. (10 marks for * parts)
For each of the following equations:

(i) * Find coordinates u, v such that the left-hand side of the equation becomes $\partial_u \partial_v F$.

(ii) * Find the general solution of the equation.

(iii) * Find the particular solution for the given initial conditions.

(a) * $F_{xx} + 5F_{xy} + 6F_{yy} = 0,\ F(x, 0) = \sin(x),\ \partial_y F(x, 0) = \cos(x)$.

(b) * $2F_{xx} + 3F_{xy} + F_{yy} = x,\ F(x, 0) = x,\ \partial_y F(x, 0) = 0$.

(c) $F_{xy} + F_{yy} = \sin x,\ F(x, 0) = x,\ \partial_y F(x, 0) = x$.

Answer 2. In each case we use the proposition in the notes which says that if α and β are the roots of $AT^2 + BT + C = 0$ then under the change of coordinates $x = s + t$ and $y = -\beta s - \alpha t$ the operator $A\partial_x^2 + B\partial_x\partial_y + C\partial_y^2$ becomes $A\partial_s\partial_t$.

(1 marks)

(a) * In this case we have $A = 1$, $B = 5$ and $C = 6$. The roots are $\alpha = -2$ and $\beta = -3$ so the coordinates we desire are s, t where $x = s + t$ and $y = 2s + 3t$, so $t = y - 2x$ and $s = 3x - y$. The equation is now $F_{st} = 0$ with general solution $M(t) + N(s) = M(y - 2x) + N(3x - y)$ for arbitrary functions M and N.

(2 marks)

The initial conditions now give

$$M(-2x) + N(3x) = \sin x, \qquad M'(-2x) - N'(3x) = \cos x$$

so integrating the second gives $-M(2x)/2 - N(3x)/3 = \sin x$ therefore solving these simultaneous equations for M and N we get $M(-2x) = -8\sin(x)$ and $N(3x) = 9\sin x$, so $M(u) = 8\sin(u/2)$ and $N(u) = 9\sin(u/3)$ which gives

(3 marks)

$$F(x, y) = 8\sin((y - 2x)/2) + 9\sin((3x - y)/3)$$

It is always a good idea to check that the final F does indeed satisfy the initial conditions (this one does).

(b) * In this case we have $A = 2$, $B = 3$ and $C = 1$. The roots are $\beta = -1$ and $\alpha = -1/2$ so the coordinates we desire are s, t where $x = s+t$ and $y = s+t/2$, so $t = 2x-2y$ and $s = 2y - x$. The equation is now $2F_{st} = x = s + t$ with general solution (integrating with respect to s and then t)

$$(s^2t + st^2)/4 + M(s) + N(t) = \frac{3x^2y - 2xy^2 - x^3}{2} + M(2y - x) + N(2x - 2y)$$

for arbitrary functions M and N.

(2 marks)

In terms of x and y this is

$$F(x,y) = \frac{3x^2y - 2xy^2 - x^3}{2} + M(2y-x) + N(2x-2y).$$

The initial conditions now become

$$F(x,0) = -x^3/2 + M(-x) + N(2x) = x, \qquad \partial_y F(x,0) = 3x^2/2 + 2M'(-x) - 2N'(2x) = 0.$$

Differentiating the first gives

$$-3x^2/2 - M'(-x) + 2N'(2x) = 1$$

and adding this to the second equation we get

$$M'(-x) = 1$$

which we integrate:

$$\int M'(-x)dx = -M(-x) = x$$

to get $M(z) = z$. Then

$$N(2x) = x - M(-x) + x^3/2$$

we get $N(z) = z/2 + z/2 + (z/2)^3/2 = z + z^3/16$. So the solution is

$$\frac{3x^2y - 2xy^2 - x^3}{2} + (2y - x) + (2x - 2y + (2x-2y)^3/16)$$

or

$$\frac{3x^2y - 2xy^2 - x^3}{2} + x + (x-y)^3/2.$$

(3 marks)

(c) In this example $A = 0$ so we need to swap the roles of x and y. At the outset we will relabel all ys to xs and vice versa, reversing this at the end of the problem. So we need to solve

$$F_{xx} + F_{xy} = \sin y, \qquad F(0,y) = y,\ \partial_x F(0,y) = y.$$

Now we have $A = B = 1$ so the polynomial $AT^2 + BT + C = 0 = T^2 + T$ has roots $\beta = -1$ and $\alpha = 0$. In the new coordinates $x = s + t$, $y = s$, (or $s = y$, $t = x - y$) the operator $\partial_x\partial_x + \partial_x\partial_y$ becomes $\partial_s\partial_t$ and the equation is

$$F_{st} = \sin y = \sin s.$$

Integrating this gives

$$F = -t\cos s + M(s) + N(t) = -(x-y)\cos y + M(y) + N(x-y).$$

Now the initial conditions give

$$F(0,y) = y = y\cos y + M(y) + N(-y), \qquad \partial_x F(0,y) = -\cos y + N'(-y) = y$$

so $N'(-y) = y + \cos y$ and $\int N'(-y)dy = -N(-y) = \int (y + \cos y)dy = y^2/2 + \sin y$ so

$$N(z) = -z^2/2 + \sin z.$$

This gives

$$M(z) = z - z\cos z + z^2/2 + \sin z$$

so

$$\begin{aligned} F(x,y) &= (y-x)\cos y + y - y\cos y + y^2/2 + \sin y + \sin(x-y) - (x-y)^2/2 \\ &= y - x\cos y + \sin y + \sin(x-y) - x^2/2 + xy. \end{aligned}$$

Finally we must remember to reverse x and y:

$$F(x,y) = x - y\cos x + \sin x + \sin(y-x) - y^2/2 + xy.$$

Question 3.
Consider the parabolic differential equation $F_{xx} + 2F_{xy} + F_{yy} = 0$. Let

$$u(x, y) = Px + Qy, \qquad v(x, y) = Rx + Sy, \qquad PS - QR \neq 0$$

be a linear change of coordinates. Find a choice[1] of P, Q, R, S so that

$$F_{uu} = F_{xx} + 2F_{xy} + F_{yy} = 0$$

and hence find the general solution of the given parabolic equation in terms of x and y.

Answer 3. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denote the inverse of $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then, by the chain rule, we have

$$\partial_u = \frac{\partial x}{\partial u}\partial_x + \frac{\partial y}{\partial u}\partial_y = a\partial_x + c\partial_y$$

and

$$\partial_v = \frac{\partial x}{\partial v}\partial_x + \frac{\partial y}{\partial v}\partial_y = b\partial_x + d\partial_y$$

so

$$\partial_u \partial_u = a^2\partial_x^2 + 2ac\partial_x\partial_y + c^2\partial_y^2.$$

If this is supposed to be $\partial_x^2 + 2\partial_x\partial_y + \partial_y^2$ then we may as well pick $a = c = 1$ and b and d can be anything provided the matrix is invertible. If we take $b = 0$, $d = 1$ then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

which gives[2] $u = x$, $v = y - x$. The differential equation now becomes

$$\partial_u^2 F = 0$$

whose general solution is

$$F(u, v) = A(v)u + B(v)$$

for arbitrary functions A and B, that is

$$F(x, y) = A(y - x)x + B(y - x).$$

[1] There are many choices that will work! Make sure that your choice satisfies $PS - QR \neq 0$ otherwise it's not a valid (invertible) change of coordinates.

[2] It is a confusing fact of life that $u = x$ but $\partial_u = \partial_x + \partial_y$ (instead of being ∂_x). This you can compute easily from the chain rule. The point is that the unit vector in the x-direction points one unit in the u-direction but also one unit in the v-direction, and the partial derivative ∂_x is the directional derivative along this vector (which is now clearly different from the vector in the u-direction).

Question 4.
Find complex coordinates $(u(x,y), v(x,y))$ so that the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ becomes $\phi_{uv} = 0$. Deduce that a solution of Laplace's equation can be written $\phi(x,y) = F(u) + G(v)$ for some complex functions F, G. Express the solutions $\phi(x,y) = xy$ and $\phi(x,y) = \sin nx \sinh ny$ in this form.

Answer 4. As in the previous question, let

$$u(x,y) = Px + Qy, \qquad v(x,y) = Rx + Sy, \qquad PS - QR \neq 0$$

and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denote the inverse of $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then, by the chain rule, we have

$$\partial_u = \frac{\partial x}{\partial u}\partial_x + \frac{\partial y}{\partial u}\partial_y = a\partial_x + c\partial_y$$

and

$$\partial_v = \frac{\partial x}{\partial v}\partial_x + \frac{\partial y}{\partial v}\partial_y = b\partial_x + d\partial_y$$

so

$$\partial_u\partial_v = ab\partial_x^2 + (ad+bc)\partial_x\partial_y + cd\partial_y^2.$$

This is supposed to be $\partial_x^2 + \partial_y^2$. If we pick $a = b = 1$ then we need $c + d = 0$ and $cd = 1$ so $-c^2 = 1$. Therefore we take $c = -d = i$ so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

which gives

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

so $u = \frac{x-iy}{2}$ and $v = \frac{x+iy}{2}$.

Therefore the general solution of Laplace's equation has the form[3]

$$F(x-iy) + G(x+iy)$$

for example

$$xy = \frac{i}{4}(x-iy)^2 - \frac{i}{4}(x+iy)^2$$

and

$$\sin nx \sinh ny = \frac{1}{2i}\cos(n(x-iy)) - \frac{1}{2i}\cos(n(x+iy)).$$

[3] I have absorbed the factors of $1/2$ into the unknown functions F and G.

Question 5. *This question is a little different: in each case we study a function $y(x,t)$ where $x \geq 0$ as well as $t \geq 0$ and so we impose a boundary condition along $x = 0$ as well as the usual boundary condition along $t = 0$.*

A prisoner is attached to an infinitely long chain parametrised by a coordinate $x \in [0, \infty)$. At time $t = 0$ the prisoner starts jumping up and down on the spot $x = 0$ so that his height at time $t \geq 0$ is $y(0,t) = 1 - \cos(t)$. The chain starts off with

$$y(x,0) = 0, \ x \geq 0 \qquad \partial_t y(x,0) = 0, \ x \geq 0,$$

and obeys the wave equation

$$y_{tt} = y_{xx}.$$

By substituting D'Alembert's solution $y(x,t) = F(x+t) + G(x-t)$ into the initial conditions show that for some constant k, $F(z) = k$ and $G(z) = -k$ for all $z \geq 0$. Using the condition $y(0,t) = 1 - \cos t$ for $t \geq 0$, deduce that $G(z) = 1 - k - \cos(z)$ for $z < 0$. Find $y(x,t)$ for all $x \geq 0$, $t \geq 0$ (be careful to separate the cases $x - t \leq 0$ and $x - t \geq 0$) and sketch $y(x, 3\pi)$.

A prison guard is sitting at $x = 8$, watching the chain. At what point does he notice that the prisoner is jumping up and down?

Answer 5. We set $y(x,t) = F(x+t) + G(x-t)$ and substitute it into

$$y(x,0) = 0, \ x \geq 0, \qquad \partial_t y(x,0) = 0, \ x \geq 0.$$

This gives

$$F(x) + G(x) = 0, \ x \geq 0, \qquad F'(x) - G'(x) = 0, \ x \geq 0$$

so

$$F(x) = -G(x), \ F(x) = G(x) + 2k, \qquad x \geq 0$$

for some constant k, which gives $F(x) = k$, $G(x) = -k$ on $x \geq 0$.

The remaining boundary condition is

$$y(0,t) = 1 - \cos t, \ t \geq 0$$

or

$$F(t) + G(-t) = 1 - \cos t, \ t \geq 0$$

so

$$G(-t) = 1 - k - \cos t, \ t \geq 0$$

or $G(z) = 1 - k - \cos z$ for $z \leq 0$.

Now $y(x,t) = F(x+t) + G(x-t)$. Since $x + t$ is always positive for $x, t \geq 0$ we have $F(x,t) = k$. However it is possible for $x - t$ to be positive or negative. This gives

$$y(x,t) = \begin{cases} 1 - \cos(x-t) & \text{if } x - t \leq 0 \\ 0 & \text{if } x - t \geq 0. \end{cases}$$

If $t = 3\pi$ then the chain looks like the graph of $1 + \cos x$ on $x \leq 3\pi$ and zero beyond that. At time $t = 8$ the range $x - t \leq 0$ will include the guard's position at $x = 8$ and the chain will start to move at that point.