Methods 3 - Question Sheet 7

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Question 1. (9 marks for * parts)
Give the general solutions for the following linear partial differential equations and, in each case, find the particular solution satisfying $F(s, 0) = s$.

(a) $F_x - F_y = F$,

(b) * $2F_x + 3F_y = x^2$,

(c) $F_x + 5F_y = xy$,

(d) $xF_x + F_y = 0$,

(e) * $yF_x + xF_y = F$,

(f) $e^y F_x - F_y = xF$.

Answer 1. (a) In this case $F_x - F_y$ is the directional derivative of F in the $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$-direction. We therefore change coordinates to make v into the x'-axis (we can choose the y'-axis arbitrarily) so let's use coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

that is

$$x = x', \qquad y = y' - x'$$

or

$$x' = x, \qquad y' = x + y.$$

By the chain rule we have $\partial/\partial x' = F_x \partial x/\partial x' + F_y \partial y/\partial x' = F_x - F_y$. Therefore in our new coordinates the equation becomes $\partial F/\partial x' = F$, with solution $F(x', y') = C(y')e^{x'}$. Translating back to our original coordinates, $F(x, y) = C(x + y)e^x$.

The particular solution satisfying $F(s, 0) = s$ has $C(s)e^s = s$ so $C(s) = s/e^s$ and the solution is $e^{-y}(x + y)$.

(b) * In this case $2F_x + 3F_y$ is the directional derivative in the $(2, 3)$-direction so we use coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

in which $\partial F/\partial x' = 2F_x + 3F_y$. Thus $x' = x/2$, $y' = y - 3x/2$. Then $F_{x'} = x^2 = 4(x')^2$ so $F = 4(x')^3/3 + C(y') = x^3/6 + C(y - 3x/2)$ is the general solution.

(3 marks)

The particular solution with $F(s,0) = s$ has $s^3/6 + C(-3s/2) = x$ so $C(-3s/2) = s - s^3/6$ or (setting $u = -3s/2$)

$$C(u) = -\frac{2u}{3} + \frac{4u^3}{81}$$

and $F(x,y) = x^3/6 + \frac{4}{81}(y - 3x/2)^3 - \frac{2}{3}(y - 3x/2)$.

(1 marks)

(c) Using coordinates $x = x'$, $y = 5x' + y'$ ($x' = x$, $y' = -5x + y$) we get the solution

$$F(x', y') = y'(x')^2/2 + 5(x')^3/3 + C(y')$$

or

$$F(x,y) = (-5x + y)x^2/2 + 5x^3/3 + C(-5x + y).$$

The particular solution with $F(s,0) = s$ has $C(-5s) = s + 5s^3/6$, or if $w = -5s$

$$C(w) = -w/5 - w^3/150.$$

(d) Now we have varying coefficients: $xF_x + F_y = 0$ and we have to integrate the characteristic vector field $\dot{x} = x$ and $\dot{y} = 1$. This gives characteristic curves $x(t) = ae^t$, $y(t) = t + b$. Setting $b = 0$ we get the coordinate change is $x = ae^t$, $y = t$, which gives $a = xe^{-y}$, $t = y$. The operator $\partial/\partial t$ becomes

$$\frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = ae^t\partial_x + \partial_t = x\partial_x + \partial_y$$

so the equation becomes

$$\partial F/\partial t = 0$$

so the general solution is $F(a,t) = C(a)$ or $F(x,y) = C(xe^{-y})$. The particular solution with $F(s,0) = s$ has $C(se^0) = s$ so $C(s) = s$ and $F(x,y) = xe^{-y}$.

(e) * The characteristic equations are $\dot{x} = y$ and $\dot{y} = x$ so differentiating again with respect to t we get $\ddot{x} = x$ and $\ddot{y} = y$. The first gives $x = Ae^t + Be^{-t}$ and then $y = \dot{x} = Ae^t t - Be^{-t}$.

(2 marks)

Let $A = 1$ so that our coordinates are B and t. In terms of B and t we have $\ln((x + y)/2) = t$ and $(x - y)(x + y)/4 = B = (x^2 - y^2)/4$.

(1 marks)

The equation becomes

$$\partial F/\partial t = F$$

so the general solution is $F = C(B)e^t$.

(1 marks)

In other words

$$F(x,y) = C(x^2 - y^2)(x+y)/2$$

(note that the factor of 1/4 in B can be absorbed into the undetermined function C). Given that $F(x,0) = x$ we have $C(x^2)x/2 = x$ so $C(x^2) = 2$ or $C(w) = 2$. So the particular solution is $(x+y)$.

(1 marks)

Note that although our new coordinate system was only valid in the region $x + y > 0$, the solution we have found turns out to be defined everywhere.

(f) The characteristic equations are $\dot{x} = e^y$ and $\dot{y} = -1$ so the solutions are $y = -t + b$ and $x = -e^{-t} + a$. Thus (setting $b = 0$) $a = x + e^y$. The equation becomes

$$\partial F/\partial t = (a - e^{-t})F$$

with solution

$$\ln F = at + e^{-t} + C(a)$$

Thus $F = \exp(C(x + e^y) + e^y - (x + e^y)y)$. The particular solution with $F(s, 0) = s$ has

$$C(s + 1) + 1 = \ln s$$

so (setting $w = s + 1$)

$$C(w) = \ln(w - 1) - 1.$$

Question 2. (9 marks for * parts)
For each of the following initial value problems:

(i) try to find the best collection of adjectives to describe the equation at hand (e.g. inhomogeneous linear, quasilinear, linear with constant coefficients,...);

(ii) * write down the characteristic vector field in $\mathbf{R}^3$ and find the characteristic curves passing through the initial condition;

(iii) * give the solution to this initial value problem implicitly as a solution surface $(s,t) \mapsto (x(s,t), y(s,t), z(s,t))$;

(iv) * find and sketch the caustic of the solution surface;

(v) using a computer, plot (1) the solution surface; (2) some of the projections to the xy-plane of the characteristic curves;

(vi) * find a function whose graph is equal to the solution surface (remember to specify the domain on which this function is defined).

(a) $FF_x - F_y = y$, $F(s,0) = s^2$.

(b) $F_x - F_y = 1$, $F(s,s) = s$.

(c) * $(x+F)F_x + F_y = F$, $F(s,0) = s$.

(d) $xyF_x - F_y + F^2 = 0$, $F(s,0) = s$.

(e) * $xF_x - FF_y = 1$, $F(s,0) = 0$.

(f) $x^2F - F_x - xF_y = 0$, $F(s,s) = s$.

Answer 2. (a) This equation is quasilinear. The characteristic vector field is

$$\dot{x} = z, \;\; \dot{y} = -1, \;\; \dot{z} = y.$$

The characteristic curves therefore have $y = -t + b$ and so $z = -t^2/2 + bt + a$ and $x = -t^3/6 + bt^2/2 + at + c$. The condition $F(s,0) = s^2$ means that at $t = 0$, along $y = 0$, if $x = s$ then $z = s^2$, which translates into

$$b = 0, \;\; c = s, \;\; a = s^2$$

Therefore the solution surface can be parametrised as

$$(s,t) \mapsto (-t^3/6 + s^2 t + s, -t, -t^2/2 + s^2)$$

We can find the caustic by calculating the determinant

$$\det \begin{pmatrix} \partial_s x & \partial_t x \\ \partial_s y & \partial_t y \end{pmatrix} = \det \begin{pmatrix} 2st+1 & -t^2/2 + s^2 \\ 0 & -1 \end{pmatrix}$$

and setting it equal to zero, which gives

$$2st = -1.$$

Substituting back in, the caustic has $x = -t^3/6 - 1/4t$ and $y = -t$ as a parametric curve, or

$$x = y^3/6 + 1/4y.$$

Now $x = -s^2y + s + y^3/6$ so $s = \frac{1\pm\sqrt{1+2y^4/3-4xy}}{-2y}$ and $z = s^2 - t^2/2$ so

$$z = \left(\frac{1 \pm \sqrt{1+2y^4/3-4xy}}{2y}\right)^2 - y^2/2$$

as a function of x and y (defined on $1 + 2y^4/3 - 4xy > 0$, $y \neq 0$).

(b) This problem is inhomogeneous linear with constant coefficients. The characteristic vector field is

$$\dot{x} = 1,\ \dot{y} = -1,\ \dot{z} = 1$$

so the characteristic curves are $x = t + c$, $y = -t + a$ and $z = t + b$. $F(s, s) = s$ at $t = 0$ gives $c = s$, $b = s$ and $a = s$. Therefore the solution surface is

$$(s, t) \mapsto (t + s, s - t, t + s)$$

The caustic is defined by the vanishing of

$$\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2$$

as a function of s, t, but $-2 \neq 0$ so the caustic is empty. We can write $z = t + s = x$ so $F(x, y) = x$ is the solution we seek (defined globally).

(c) * This equation is quasilinear. The characteristic vector field is

$$\dot{x} = x + z,\ \dot{y} = 1,\ \dot{z} = z.$$

This has integral curves $y = t + c$, $z = be^t$ and $x = (a + bt)e^t$. The initial condition $F(s, 0) = s$ gives $c = 0$, $b = s$ and $a = s$ along $t = 0$. The parametrised solution surface is therefore

$$(s, t) \mapsto (s(1 + t)e^t, t, se^t)$$

The caustic is the locus where

$$\det \begin{pmatrix} (1+t)e^t & s(2+t)e^t \\ 0 & 1 \end{pmatrix} = 0$$

that is

$$(1 + t)e^t = 0$$

i.e.

$$t = -1$$

The caustic is therefore the point $y = -1$, $x = 0$ (just substituting $t = -1$ into the parametric surface). Note that $z = se^t = se^y$ and $s = xe^{-y}/(1 + y)$ so

$$F(x, y) = \frac{x}{1 + y}$$

is the solution (defined away from $y = -1$).

(d) This equation is quasilinear (not linear, because of the F^2 term). The characteristic vector field is

$$\dot{x} = xy,\ \dot{y} = -1,\ \dot{z} = -z^2$$

and integrating gives $y = -t + c$, $z = \frac{1}{t+b}$ and $x = ae^{(-t+c)^2/2}$. The initial condition $F(s, 0) = s$ gives $c = 0$, $s = \frac{1}{b}$ and $s = a$ along $t = 0$. Therefore the solution surface is parametrised by

$$(s, t) \mapsto \left(se^{-t^2/2}, -t, \frac{1}{t + 1/s}\right).$$

This has caustic

$$0 = \det \begin{pmatrix} e^{-t^2/2} & -ste^{-t^2/2} \\ 0 & -1 \end{pmatrix} = -e^{-t^2/2}$$

Since this function is nowhere vanishing, the caustic is empty. We have $s = xe^{t^2/2} = xe^{y^2/2}$ and

$$F(x, y) = z = \frac{1}{t + 1/s} = \frac{1}{-y + x^{-1}e^{-y^2/2}} = \frac{x}{e^{-y^2/2} - xy}$$

(defined on $xy \neq e^{-y^2/2}$). Note that this function is not globally defined even though there is no caustic - the solution surface has no vertical tangency: it goes to infinity too fast along $xy = e^{-y^2/2}$.

(e) * This equation is quasilinear. The characteristic vector field is

$$\dot{x} = x,\ \dot{y} = -z,\ \dot{z} = 1$$

which has integral curves $z = t + c$, $x = ae^t$, $y = -t^2/2 - ct + b$.

(1 marks)

The initial condition $F(s, 0) = 0$ gives $a = s$, $c = 0$ and $b = 0$ along $t = 0$ so the solution surface is parametrised by

$$(s, t) \mapsto (se^t, -t^2/2, t).$$

(1 marks)

This has caustic

$$0 = \det \begin{pmatrix} e^t & se^t \\ 0 & -t \end{pmatrix} = -te^t$$

that is $t = 0$.

(1 marks)

This corresponds to the line $y = 0$. The solution satisfies

$$F(x, y) = z = \pm\sqrt{-2y}.$$

(1 marks)

It is quite easy to visualise what is going on here: the solution is double-valued so there are two branches of the solution over $y < 0$. These branches meet along the preimage of the caustic $y = 0$, with a vertical tangency. The solution is defined on the set $\{y < 0\}$.

(f) The equation is homogeneous linear but with varying coefficients. The characteristic vector field is

$$\dot{x} = -1,\ \dot{y} = -x,\ \dot{z} = -x^2 z$$

which has solutions

$$x = -t + a,\ y = t^2/2 - at + b,\ z = ce^{-t^3/3 - a^2 t + at^2}.$$

The initial condition $F(s, s) = s$ along $t = 0$ gives

$$a = s,\ b = s,\ c = s$$

so the solution surface is

$$(s, t) \mapsto (-t + s, t^2/2 - st + s, se^{-t^3/3 - s^2 t + st^2}).$$

This has caustic

$$0 = \det \begin{pmatrix} 1 & -1 \\ 1 - t & t - s \end{pmatrix} = 1 - s$$

so the caustic is $s = 1$. This means $x = 1 - t$, $y = 1 - t + t^2/2$ so the caustic is the curve

$$y = x + (x - 1)^2/2 = x^2/2 + 1/2$$

We also have $x = s - t$ and $y = s - st + t^2/2 = s - st + t^2/2 + s^2/2 - s^2/2 = x^2/2 + s - s^2/2$. Therefore

$$s^2 - 2s + 2y - x^2 = 0$$

and

$$s = \frac{2 \pm 2\sqrt{1 - 2y + x^2}}{2}$$

so

$$t = s - x = 1 - x \pm \sqrt{1 - 2y + x^2}.$$

Substituting this into $se^{-t^3/3 - s^2 t + st^2}$ gives F as a function of x and y (will be a mess). This solution is defined on the set $1 - 2y + x^2 > 0$.

Question 3. (2 marks)
In this question, we will prove that the Burgers equation

$$\partial_t u + u\partial_x u = 0$$

is satisfied by the velocity field of a non-viscous fluid in one dimension.

Suppose that the real line is filled with a fluid whose particles at point x are moving with velocity $u(t, x)$ at time t. Suppose that the particles don't interact with one another or experience any external force (so by Newton's law, they have zero acceleration). Let $\gamma(t)$ be the path of one of the fluid particles so that $\dot{\gamma}(t) = u(t, \gamma(t))$. Given that γ has no acceleration, deduce that u satisfies the Burgers equation.

Answer 3. If we differentiate $\dot{\gamma}(t) = u(t, \gamma(t))$ using the chain rule then we get

$$\ddot{\gamma}(t) = \frac{\partial u}{\partial t} + \dot{\gamma}\frac{\partial u}{\partial x}$$

so $\ddot{\gamma} = 0$ gives the Burgers equation.

(2 marks)

Question 4.
For some function $G(x, y)$, consider the linear PDE

$$-y\partial_x F + x\partial_y F = G(x, y)$$

with the initial condition $F(x, 0) = 0$ for $x > 0$. Show that this initial-value problem has a single-valued solution on $\mathbf{R}^2 \setminus \{0\}$ if and only if $\int_0^{2\pi} G(A\cos\theta, A\sin\theta)d\theta = 0$ for all A. For $G(x, y) = x$ find this solution explicitly.

Answer 4. The characteristics are the integral curves $(x(t), y(t))$ of the vector field

$$\dot{x} = -y, \qquad \dot{y} = x.$$

Differentiating with respect to t again gives $\ddot{x} = -x$ so $x = A\cos t + B\sin t$ and $y = -\dot{x} = A\sin t - B\cos t$. So the integral curves are circles centred at the origin. Imposing $F(x, 0) = 0$ at $t = 0$ we see that $x(0) = A$, $y(0) = -B$ so $B = 0$ and $(x, y) = (A\cos t, A\sin t)$. If $u(t) = F(A\cos t, A\sin t)$ then $\dot{u} = G(A\cos t, A\sin t)$ is the ODE to which the PDE reduces along the characteristics. In particular, since the half-line $\{(x, 0) \ : x \in (0, \infty)\}$ is given in the coordinates (A, t) by $t = 0$, the solution is

$$u(T) = \int_0^T G(A\cos t, A\sin t)dt$$

This is single-valued if and only if $\int_0^{2\pi} G(A\cos t, A\sin t)dt = 0$ for all A. In particular this condition is satisfied by $G(x, y) = x$. The solution is then

$$u(T) = \int_0^T A\cos t dt = A\sin T$$

or, in terms of (x, y), $F(x, y) = y$. Indeed one can verify that

$$\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} = 1$$

so

$$-y\partial_x F + x\partial_y F = x$$

as desired.