Questions for enthusiasts

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Question 1. (Special case of the maximum principle)
Suppose that $\phi(x, y)$ solves Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

A nondegenerate critical point is a critical point where the Hessian

$$\begin{pmatrix} \dfrac{\partial^2\phi}{\partial x^2} & \dfrac{\partial^2\phi}{\partial x\partial y} \\ \dfrac{\partial^2\phi}{\partial x\partial y} & \dfrac{\partial^2\phi}{\partial y^2} \end{pmatrix}$$

is an invertible matrix (in particular it cannot have zero eigenvalues). Show that ϕ has no nondegenerate maxima or minima.

Hint: Take the trace of the Hessian (the sum of its diagonal values) and recall that the trace does not change under conjugation.

Question 2. (Uniqueness of solutions to Laplace's equation)
Let $U \subset \mathbf{R}^2$ be an open set with boundary curve ∂U. Recall Green's theorem $\int_U \nabla \cdot v dx dy = \int_{\partial U} v \cdot \hat{n} ds$ where $\hat{n}$ is the unit outward normal to ∂U and ds is the length element on ∂U. Let Δ denote the Laplacian operator $\Delta\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}$.

(a) Suppose that $\phi\colon U \to \mathbf{R}$ is a function with $\phi(x) = 0$ for $x \in \partial U$. By considering $v = \phi\nabla\phi$, show that $\int_U \phi\Delta\phi dx dy = -\int_U |\nabla\phi|^2 dx dy$.

(b) Deduce that if $\Delta\phi = 0$ then $\phi(x) = 0$ for all $x \in U$.

(c) Deduce that if ϕ_1 and ϕ_2 are solutions to Laplace's equation and $\phi_1(x) = \phi_2(x)$ for all $x \in \partial U$ then $\phi_1(x) = \phi_2(x)$ for all $x \in U$.

Question 3. (Hurwitz's solution to the isoperimetric problem)
Suppose that $\gamma(t) = (x(t), y(t))$ is a path in the plane such that $\gamma(t + 2\pi) = \gamma(t)$. Suppose that $\gamma([0, 2\pi])$ has length K and that γ is parametrised by arc-length, so that $\sqrt{\dot{x}^2 + \dot{y}^2} = K/2\pi$. Note that because it is parametrised by arc-length, we have

$$K^2 = 2\pi \int_0^{2\pi} \left(\dot{x}^2 + \dot{y}^2\right) dt.$$

If

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(kt) + b_k \sin(kt)\right)$$
$$y(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(kt) + d_k \sin(kt)\right)$$

are Fourier series for the functions $x(t)$ and $y(t)$, write expressions for K^2 and the area

$$A = \int_0^{2\pi} x\dot{y}dt$$

bounded by γ in terms of the Fourier coefficients. Use these formulae to deduce the *isoperimetric inequality*:

$$K^2 - 4\pi A \geq 0.$$

When does equality hold?

Question 4. (Gram-Schmidt orthogonalisation)
Let V be the vector space of (square-integrable) functions on the interval $[-1, 1]$ and let

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

denote the L^2-inner product on V.

The Gram-Schmidt process in linear algebra starts with a sequence of linearly independent vectors $v_1, v_2, v_3, \ldots$ and produces an orthonormal basis $u_1, u_2, u_3, \ldots$ (normalise the first vector v_1 to get u_1, then set u_2 to be the normalisation of $v_2 - (v_2 \cdot u_1)u_1$, etc.). One can apply Gram-Schmidt to the sequence of polynomials $1, x, x^2, x^3, \ldots$ in V to produce an basis orthonormal with respect to the L^2-inner product. Compute the first few orthonormal polynomials.

Question 5. (The Γ-function)
Define $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$ for $r > 0$ a positive real number.

(a) Prove that $\Gamma(r+1) = r\Gamma(r)$. We define $\Gamma(r)$ for $r \in (-1,0)$ by setting $\Gamma(r) := \Gamma(r+1)/r$ (and inductively we can extend to all negative non-integers). In particular $\Gamma(-1/2) = 2\Gamma(1/2)$.

(b) Compute $\Gamma(1)$ and prove that $\Gamma(n+1) = n!$ if n is a positive integer.

(c) Show that $\Gamma(1/2) = \sqrt{\pi}$.

(d) Compute $\int_0^\infty \sqrt{y}e^{-y^3}dy$ and $\int_0^1 \frac{dx}{\sqrt{-\ln(x)}}$.

Question 6. (Sturm-Liouville systems) Let $p(x), q(x)$ be functions and define the differential operator

$$Dy = \frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy.$$

An eigenfunction of D with eigenvalue λ is a (nonzero, possible complex-valued) function y such that $Dy = \lambda y$.

(a) Show that $\langle f, Dg\rangle = \langle Df, g\rangle$.

(b) Prove that if λ is an eigenvalue of D then $\lambda \in \mathbf{R}$.

(c) Prove that if y_i is a λ_i eigenfunction, for $i = 1, 2$, and $\lambda_1 \neq \lambda_2$ then y_1 and y_2 are orthogonal.

Question 7. (The minimal surface equation)
Show that the Euler-Lagrange equation for the functional

$$\int_0^1\int_0^1 \sqrt{1 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2}\,dxdy$$

is the *minimal surface equation*

$$\frac{\partial^2\phi}{\partial x^2}\left(1 + \left(\frac{\partial\phi}{\partial y}\right)^2\right) + \frac{\partial^2\phi}{\partial y^2}\left(1 + \left(\frac{\partial\phi}{\partial x}\right)^2\right) = 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}.$$

Question 8.
Consider the Schrödinger equation with potential $V(x)$

$$-i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi, \quad V(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \in [0,1] \\ 1 & \text{if } x > 1 \end{cases}$$

for a complex-valued function ψ defined on the whole real line. Recall that the separated solutions $X(x)T(t)$ have $T(t) = e^{iEt/\hbar}$.

(a) Assuming that $0 < E < 1$, show that the general solution for X is

$$X(x) = \begin{cases} A_1 e^{\mu x} + B_1 e^{-\mu x} & x \in (-\infty, 0] \\ A_2 \sin\lambda x + B_2 \cos\lambda x & x \in [0,1] \\ A_3 e^{\mu x} + B_3 e^{-\mu x} & x \in [1,\infty) \end{cases}$$

where $\mu = \sqrt{2m(1-E)/\hbar^2}$ and $\lambda = \sqrt{2mE/\hbar^2}$.

(b) We will assume the "boundary condition at infinity" that X is bounded as $x \to \infty$. Show that this means $B_1 = A_3 = 0$.

(c) By further requiring that X and dX/dx are continuous at 0 and at 1, find a system of equations relating A_1, A_2, B_2 and B_3.

(d) Show that these equations imply

$$\cot\left(\frac{2mE}{\hbar^2}\right) = \frac{E - \frac{1}{2}}{\sqrt{E(1-E)}}.$$

(e) Sketch the graph of the solution X you have found (assuming the coefficients are real).

This potential is called the square-well potential. The physical interpretation of ψ is that

$$\int_a^b |\psi(x,t)|^2 dx$$

is the probability of finding a particle in the interval $[a,b]$ at time t. Since ψ is nonzero in the region outside $[0,1]$ we see that there is a chance the particle can escape the potential well. Classically, a particle could never escape the square-well potential if its energy was less than the "potential energy barrier" (in this case $E < 1$). The fact that, quantum mechanically, a particle can escape is a phenomenon called quantum tunnelling. For example, this is the mechanism which allows alpha particles to escape from atomic nuclei in radioactive decay.