Methods 3 - Question Sheet 2

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Question 1. (7 marks)

(a) For each of the following PDEs for $\phi(x, y)$, separate variables ($\phi(x, y) = X(x)Y(y)$) and find the ODEs satisfied by X and Y.

(i) $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial\phi}{\partial y} + \phi.$
(Telegraph equation, governing lossy wave transmission)

(ii) $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = K\phi$ where K is a constant.
(Helmholtz equation, governing static temperature distributions in 2D)

(b) For equations (i) and (ii) above, solve the ODEs you found.
Be careful to distinguish the values of the separation constant λ where the behaviour of X changes and those where the behaviour of Y changes.

Answer 1. (a) (i) We have $X''Y = XY''+XY'+XY$ so $X''/X = (Y''+Y'+Y)/Y = -\lambda$ is constant. Therefore

$$X'' = -\lambda X, \quad Y'' + Y' + (1 + \lambda)Y = 0.$$

(ii) $X''Y + XY'' = KXY$ so $X''/X = -Y''/Y + K = -\lambda$ is constant. Therefore

$$X'' = -\lambda X, \quad Y'' = (K + \lambda)Y.$$

(2 marks)

(b) In both cases the equation $X'' = -\lambda X$ has the solutions

$$X = \begin{cases} A\cos(px) + B\sin(px) & \text{if } \lambda = p^2 > 0 \\ Ax + B & \text{if } \lambda = 0 \\ A\cosh(px) + B\sinh(px) & \text{if } \lambda = -p^2 < 0. \end{cases}$$

(1 marks)

Let us analyse the Y equations:

(i) The auxiliary quadratic is

$$t^2 + t + (1 + \lambda) = 0$$

with roots $-\dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{-(3+4\lambda)}$ so the equation has solutions

$$Y = \begin{cases} e^{-y/2}(A\cos(qy) + B\sin(qy)) & \text{if } \frac{-3-4\lambda}{4} = -q^2 < 0 \\ e^{-y/2}(Ay + B) & \text{if } -3-4\lambda = 0 \\ e^{-y/2}(A\cosh(qy) + B\sinh(qy)) & \text{if } \frac{-3-4\lambda}{4} = q^2 > 0. \end{cases}$$

(2 marks)

(ii) The solution is

$$Y = \begin{cases} A\cos(qy) + B\sin(qy) & \text{if } K + \lambda = -q^2 < 0 \\ Ay + B & \text{if } K + \lambda = 0 \\ A\cosh(qy) + B\sinh(qy) & \text{if } K + \lambda = q^2 > 0. \end{cases}$$

(2 marks)

Question 2. (6 marks for * parts)

In each case, solve the heat equation $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$ for a temperature distribution $\phi(x,t)$ on the rod $x \in [0, \pi]$ with the given initial and boundary conditions. You may quote any Fourier series you need from Question Sheet 1:

(a) * $\phi(x,0) = x^3$, $\phi(0,t) = 0$, $\phi(\pi,t) = \pi^3$ (Dirichlet).

(b) * $\phi(x,0) = \cos x$, $\phi(0,t) = 1$, $\phi(\pi,t) = -1$ (Dirichlet).

(c) $\phi(x,0) = x^4 - 2\pi^2 x^2$, $\frac{\partial \phi}{\partial x}(0,t) = 0$, $\frac{\partial \phi}{\partial x}(\pi,t) = 0$ (Neumann).

(d) $\phi(x,0) = \cos x$, $\frac{\partial \phi}{\partial x}(0,t) = 0$, $\frac{\partial \phi}{\partial x}(\pi,t) = 0$ (Neumann).

(e) $\phi(x,0) = \begin{cases} x & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ \pi - x & \text{if } x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$, $\phi(0,t) = 0$, $\phi(\pi,t) = 0$ (Dirichlet).

(f) $\phi(x,0) = e^x$, $\phi(0,t) = 1$, $\phi(\pi,t) = e^\pi$ (Dirichlet).

Answer 2. (a) * $\phi(x,0) = x^3$, $\phi(0,t) = 0$, $\phi(\pi,t) = \pi^3$ (Dirichlet).

First we subtract a steady solution of the form $\phi_0(x,t) = Ax + B$ to make the boundary conditions vanish. We need $\phi_0(x,t) = \pi^2 x$. Defining $\Theta(x,t) = \phi(x,t) - \phi_0(x,t)$, we see that Θ is still a solution to the heat equation by linearity and that it now satisfies the Dirichlet conditions $\Theta(0,t) = \Theta(\pi,t) = 0$, $\Theta(x,0) = x^3 - \pi^2 x$.

(2 marks)

Our Ansatz for Θ will be

$$\sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}$$

and the coefficients b_n are determined by the initial condition

$$\Theta(x,0) = x^3 - \pi^2 x = \sum_{n=1}^{\infty} b_n \sin(nx).$$

(1 marks)

In other words ,b_n should be the Fourier coefficients of $x^3 - \pi^2 x$ which we calculated on Sheet 1 Q. 2(a) to be

$$b_n = \frac{12(-1)^n}{n^3}.$$

Therefore the final solution is

$$\phi(x,t) = \phi_0(x,t) + \Theta(x,t) = \pi^2 x + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin(nx) e^{-n^2 t}.$$

(b) * $\phi(x,0) = \cos x$, $\phi(0,t) = 1$, $\phi(\pi,t) = -1$ (Dirichlet).

The structure of the argument is the same as (a). We get $\phi_0(x,t) = 1 - 2x/\pi$ and hence

(2 marks)

$$\Theta(x,0) = \cos x + \frac{2x}{\pi} - 1 = \sum_{n=1}^{\infty} \frac{2((-1)^n + 1)}{\pi n(n^2 - 1)} \sin(nx)$$

by Sheet 1, Q. 2(b). Therefore

(1 marks)

$$\phi(x,t) = 1 - \frac{2x}{\pi} + \sum_{n=1}^{\infty} \frac{2((-1)^n + 1)}{\pi n(n^2 - 1)} \sin(nx) e^{-n^2 t}.$$

(c) $\phi(x,0) = x^4 - 2\pi^2 x^2$, $\dfrac{\partial \phi}{\partial x}(0,t) = 0$, $\dfrac{\partial \phi}{\partial x}(\pi,t) = 0$ (Neumann).

This time there is no need to modify the boundary conditions as they are already zero. If we take the half-range cosine series $a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx)$ of $x^4 - 2\pi^2 x^2$ then

$$\phi(x,t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-n^2 t}$$

will be a solution of the heat equation satisfying the correct initial condition and the Neumann boundary conditions (because each term $\cos(nx)$ has vanishing derivative at 0 and π). In Sheet 1, Q. 4, we calculated this cosine series:

$$x^4 - 2\pi^2 x^2 = -\frac{14\pi^4}{15} + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos(nx)$$

so the required solution is

$$\phi(x,t) = -\frac{14\pi^4}{15} + 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos(nx) e^{-n^2 t}.$$

(d) $\phi(x,0) = \cos x$, $\dfrac{\partial \phi}{\partial x}(0,t) = 0$, $\dfrac{\partial \phi}{\partial x}(\pi,t) = 0$ (Neumann).

Again there is no need to modify the boundary conditions and this time the cosine series is very easy to find! It is just $\cos x$, so the required solution is $\phi(x,t) = (\cos x)e^{-t}$.

(e) $\phi(x,0) = \begin{cases} x & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ \pi - x & \text{if } x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$, $\phi(0,t) = 0$, $\phi(\pi,t) = 0$ (Dirichlet).

The boundary conditions are already vanishing so we do not need to modify them. We just need to find the Fourier series of $\phi(x,0)$ which was given in Sheet 1, Q. 2(d), as

$$\sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{n^2 \pi} \sin(nx)$$

so the required solution is

$$\phi(x,t) = \sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{n^2 \pi} \sin(nx) e^{-n^2 t}.$$

(f) $\phi(x,0) = e^x$, $\phi(0,t) = 1$, $\phi(\pi,t) = e^{\pi}$ (Dirichlet).

This time we proceed (as in (a)) by modifying the boundary condition. We need to subtract $\phi_0(x,t) = Ax + B$ where $\phi_0(0,t) = 1$, $\phi_0(\pi,t) = e^{\pi}$ so we take $\phi_0(x,t) = (e^{\pi} - 1)\frac{x}{\pi} + 1$. Define $\Theta(x,t) = \phi(x,t) - \phi_0(x,t)$ which now has vanishing boundary conditions and initial condition $\Theta(x,0) = e^x - \frac{e^{\pi}-1}{\pi}x - 1$. The Fourier series of this initial condition is $\sum_{n=1}^{\infty} \frac{2(e^{\pi}(-1)^n - 1)}{n\pi(n^2+1)} \sin(nx)$ by Sheet 1, Q. 2(c) so the required solution is

$$\phi(x,t) = (e^{\pi} - 1)\frac{x}{\pi} + 1 + \sum_{n=1}^{\infty} \frac{2(e^{\pi}(-1)^n - 1)}{n\pi(n^2+1)} \sin(nx) e^{-n^2 t}.$$

Question 3. (7 marks)
Suppose $\phi(x,y) = X(x)Y(y)$ is a nontrivial separated solution of the Helmholtz equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = K\phi$ satisfying the boundary conditions $X(0) = X(1) = 0$, $Y(0) = Y(1) = 0$.

(a) Show that $\phi(x,y) = C\sin(px)\sin(qy)$ for some $p, q \in \pi\mathbf{Z}$, $C \in \mathbf{R}$.

(b) Deduce that if there is a nontrivial separated solution then $K \in \pi^2\mathbf{Z}$.

(c) If $K = -2\pi^2$, deduce that the only solutions have the form $C\sin(\pi x)\sin(\pi y)$, $C \in \mathbf{R}$, and sketch the graph of such a solution.

(d) How many solutions can you find when $K = -5\pi^2$? When $K = -50\pi^2$?

(e) Why is the space of solutions to the Helmholtz equation (for fixed K) a vector space? What is its dimension?

Answer 3. (a) By Q. 1(a.ii) we know that (for some separation constant λ)

$$X = \begin{cases} A\cos(px) + B\sin(px) & \text{if } \lambda = p^2 > 0 \\ Ax + B & \text{if } \lambda = 0 \\ A\cosh(px) + B\sinh(px) & \text{if } \lambda = -p^2 < 0. \end{cases}$$

$$Y = \begin{cases} C\cos(qy) + D\sin(qy) & \text{if } K + \lambda = -q^2 < 0 \\ Cy + D & \text{if } K + \lambda = 0 \\ C\cosh(qy) + D\sinh(qy) & \text{if } K + \lambda = q^2 > 0. \end{cases}$$

If we impose the boundary conditions $X(0) = X(1) = 0$ then we get:

- $\lambda > 0$: $A = 0$, $p \in \pi\mathbf{Z}$ (so $X = B\sin(px)$).
- $\lambda = 0$: $A = B = 0$.
- $\lambda < 0$: $A = B = 0$.

If we impose the boundary conditions $Y(0) = Y(1) = 0$ then we get

- $K + \lambda < 0$: $C = 0$, $q \in \pi\mathbf{Z}$ (so $Y = D\sin(qy)$)
- $K + \lambda = 0$: $C = D = 0$.
- $K + \lambda > 0$: $C = D = 0$.

so to get a nontrivial solution we need $E\sin(px)\sin(qy)$ with $p, q \in \pi\mathbf{Z}$, $E = BD \in \mathbf{R}$.

(2 marks)

(b) In particular this means $\lambda = p^2 \in \pi^2\mathbf{Z}$ and $-K - \lambda = q^2 \in \pi^2\mathbf{Z}$, hence $-K = q^2 + p^2 \in \pi^2\mathbf{Z}$.

(1 marks)

(c) Suppose that $p = m\pi$ and $q = n\pi$, $m, n \in \mathbf{Z}$. If $K = -2\pi^2$ then, since $K = -(p^2+q^2) = -\pi^2(m^2+n^2)$ we have $m^2 + n^2 = 2$. But the only way to write 2 as a sum of squares is as $1+1$, hence $m = n = 1$. Thus $\phi(x, y) = E\sin(\pi x)\sin(\pi y)$.

(1 marks)

(d) 5 can be written as $1 + 4$ or $4 + 1$ so any linear combination of $\sin(x)\sin(2y)$ and $\sin(2x)\sin(y)$ gives a solution. 50 can be written as $1 + 49$, $25 + 25$, $49 + 1$ so any linear combination of $\sin(\pi x)\sin(7\pi x)$, $\sin(5\pi x)\sin(5\pi y)$ and $\sin(7\pi x)\sin(\pi y)$ gives a solution.

(1 marks)

(e) The space of solutions is a vector space because the equation is linear. In general, the dimension of the space of solutions for $K = -N\pi^2$ is the number of ways of writing N as a sum of two positive square numbers m^2+n^2 (where the order of m and n matters, so $m = 1$, $n = 2$ is not the same as $m = 2$, $n = 1$).

(2 marks)

Here's a little more explanation about the final part of the question, going into a lot more detail than I needed for the two marks available... (thanks to the person who asked me the questions which made me decide to write this out!). The space of all functions is a vector space: we can add functions, we can multiply them by any real number and we have a function 0. Let's restrict ourselves to functions F which vanish at 0 and L and for which $\int_0^L F^2 dx < \infty$ (this is still a vector space! It's called $L^2([0, L])$). When doing Fourier theory, we discussed the fact that $\{\sin\left(\frac{n\pi x}{L}\right)\}_{n=1}^{\infty}$ is something like a basis for this vector space: any function can be expanded as $F(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$. For a basis we should really only take finite linear combinations of the basis vectors $\sin\left(\frac{n\pi x}{L}\right)$, so this is really a "Hilbert space basis" (the infinite sum *converges* to F). You can think of the Fourier coefficients A_n as the components of the "vector" F in terms of the basis $\sin\left(\frac{n\pi x}{L}\right)$.

Now there's another vector space, $L^2([0, L]\times[0, L])$ consisting of functions $F(x, y)$ with $\int_0^L \int_0^L F^2 dxdy < \infty$. If you fix y you can Fourier expand $F(x, y) = \sum_{n=1}^{\infty} A_n(y)\sin\left(\frac{n\pi x}{L}\right)$. You can then Fourier expand the coefficient $A_n(y) = \sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)$. Overall you get

$$F(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right).$$

Let $L = 1$. Inside the vector space $L^2([0, 1] \times [0, 1])$ there is a subspace of solutions to $\Delta\phi = K\phi$. To see it's a vector subspace, note that, by linearity of the equation, you can add two solutions or multiply them by a constant and the result is a solution. Our goal is to prove that the set $\sin(px)\sin(qy)$ where $p = n\pi$, $q = m\pi$, $m, n \in \mathbf{Z}$ and $K = -(n^2 + m^2)\pi^2$ forms a basis for this subspace. Note that there are only finitely many possibilities for m and n, so the space of solutions will be finite-dimensional.

Suppose that $F(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$ is a solution to $\Delta F = KF$. Then

$$\begin{aligned}\Delta F &= \sum_{m,n=1}^{\infty} A_{mn}(-n^2 - m^2)\pi^2 \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right) \\ &= KF \\ &= \sum_{m,n=1}^{\infty} KA_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)\end{aligned}$$

so comparing the coefficients of the basis element $\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$ we see that $A_{mn}K = -A_{mn}\pi^2(n^2 + m^2)$. So $A_{mn}(K + \pi^2(n^2 + m^2)) = 0$ and either $A_{mn} = 0$ or $K = -\pi^2(n^2 + m^2)$. Therefore the solution is

$$F(x,y) = \sum A_{mn} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$$

where the sum extends over just those m and n which satisfy $(n^2 + m^2)\pi^2 = -K$. This is what we wanted to show!

Question 4.

The Schrödinger equation for the complex probability amplitude $\psi(x,t)$ of a free particle is

$$-i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

where $\hbar$, m are constants and $i = \sqrt{-1}$.

(a) Separate variables $\psi(x,t) = X(x)T(t)$ and show that $T(t) = e^{iEt/\hbar}$ for some constant E (called the *energy* - in our language it arises as a separation constant).

(b) Assume that $X(0) = X(L)$ (i.e. we are considering a free particle living in the interval $[0, L]$). Show that the energy of a separated solution is *quantised*: it can only take on values $\dfrac{n^2\hbar^2\pi^2}{2mL^2}$.

Answer 4. (a) Separating variables we get

$$-\frac{\hbar^2}{2m}\frac{X''}{X} = E = -i\hbar\frac{T'}{T}$$

so $T' = iET/\hbar$. This has solution $T = e^{iEt/\hbar}$.

(b) The other equation gives $X'' = -\frac{2mE}{\hbar^2}X$ so

$$X = \begin{cases} A\cos(px) + B\sin(px) & \text{if } \frac{2mE}{\hbar^2} = p^2 > 0 \\ Ax + B & \text{if } \frac{2mE}{\hbar^2} = 0 \\ A\cosh(px) + B\sinh(px) & \text{if } \frac{2mE}{\hbar^2} = -p^2 < 0. \end{cases}$$

The boundary conditions $X(0) = X(L) = 0$ give

- $\frac{2mE}{\hbar^2} > 0$: $A = 0$, $p \in (\pi/L)\mathbf{Z}$ (so $X = B\sin(px)$).
- $\frac{2mE}{\hbar^2} = 0$: $A = B = 0$.
- $\frac{2mE}{\hbar^2} < 0$: $A = B = 0$.

so $p = n\pi/L$ and hence $\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2}$. This implies $E = \frac{n^2\pi^2\hbar^2}{2mL^2}$ for some $n \in \mathbf{Z}$.

Question 5.
Let $X(x)T(t)$ be a separated solution to the heat equation and suppose that it satisfies Neumann boundary conditions $X'(0) = X'(L) = 0$. Show that $X = A\cos\left(\frac{n\pi x}{L}\right)$ for some $n \in \mathbf{Z}$ (*Note: $n = 0$ is allowed*).

Answer 5. Separation of variables implies $X'' = -\lambda X$ for some λ so

$$X(x) = \begin{cases} A\cos px + B\sin px & \text{if } \lambda = p^2 > 0 \\ A + Bx & \text{if } \lambda = 0 \\ A\cosh px + B\sinh px & \text{if } \lambda = -p^2 < 0. \end{cases}$$

If $\lambda < 0$ then $X'(0) = Bp\cosh(0) = 0$ so $B = 0$ and $X'(L) = Ap\sinh(L) = 0$ so $A = 0$.

If $\lambda = 0$ then $X'(0) = X'(L) = B = 0$. It is still possible that $A \neq 0$ in which case $X = A = A\cos\left(\frac{0\pi x}{L}\right)$.

If $\lambda > 0$ then $X'(0) = Bp\cos(0) = 0$ so $B = 0$ and $X'(L) = -Ap\sin(pL) = 0$ so $\sin(pL) = 0$ so $pL = n\pi$ for some $n \in \mathbf{Z}$.