# Methods 3 - Question Sheet 1

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**Question 1.** (4 marks)
Let $m, n$ be positive integers. Verify the integral identities:

$$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \delta_{mn}$$

and

$$\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0.$$

**Answer 1.** In the first case we have $\frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \frac{1}{2L}\left(\cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right)\right)$ and since $\int_{-L}^{L}\cos\left(\frac{N\pi x}{L}\right)dx = 0$ unless $N = 0$ the integral of $\frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)$ vanishes unless $n = m$, in which case it becomes $\frac{1}{2L}\int_{-L}^{L}dx = 1$ as required.

(2 marks)

In the second case we have $\frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2L}\left(\sin\left(\frac{(m+n)\pi x}{L}\right) + \sin\left(\frac{(m-n)\pi x}{L}\right)\right)$ and since $\int_{-L}^{L}\sin\left(\frac{N\pi x}{L}\right)dx = 0$ the integral of $\frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$ vanishes as required.

(2 marks)

**Question 2.** (10 marks for * parts)
For each of the following functions, find its half-range sine series on $[0, \pi]$:

(a) * $f(x) = x^3 - \pi^2 x$.

(b) * $f(x) = \cos x + \dfrac{2x}{\pi} - 1$.

(c) $f(x) = e^x - \dfrac{(e^\pi - 1)x}{\pi} - 1$.

(d) $f(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ \pi - x & \text{if } x \in \left[\frac{\pi}{2}, \pi\right] \end{cases}.$

(e) $f(x) = \sin x$.

**Answer 2.** In each case the Fourier coefficient $b_n$ is given by $\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx$.

(a) Integrating by parts we get

$$\begin{aligned}
\frac{2}{\pi}\int_0^\pi (x^3 - \pi^2 x)\sin(nx)dx &= \frac{2}{\pi}\left(\cancelto{0}{\left[-(x^3 - \pi^2 x)\frac{\cos(nx)}{n}\right]_0^\pi} + \frac{1}{n}\int_0^\pi (3x^2 - \pi^2)\cos(nx)dx\right) \\
&= \frac{2}{n\pi}\left(\cancelto{0}{\left[(3x^2 - \pi^2)\frac{\sin(nx)}{n}\right]_0^\pi} - \frac{1}{n}\int_0^\pi 6x\sin(nx)\right) \\
&= -\frac{2}{n^2\pi}\left(\left[-6x\frac{\cos(nx)}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi 6\cos(nx)dx\right) \\
&= \frac{12}{n^3\pi}(\pi(-1)^n) - \cancelto{0}{\frac{12}{n^4\pi}[\sin(nx)]_0^\pi} \\
&= \frac{12(-1)^n}{n^3}.
\end{aligned}$$

(5 marks)

(b) We need to find $\frac{2}{\pi}\int_0^\pi \left(\cos x + \frac{2x}{\pi} - 1\right)\sin(nx)dx$. The cosine integral gives

$$\begin{aligned}
\frac{2}{\pi}\int_0^\pi \cos x \sin(nx)dx &= \frac{1}{\pi}\int_0^\pi (\sin(n+1)x + \sin(n-1)x)\,dx \\
&= -\frac{1}{\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right) \\
&= \frac{2n(1 + (-1)^n)}{\pi(n^2 - 1)}
\end{aligned}$$

(2 marks)

(note that we should treat $n = 1$ separately as we divide by $n - 1$; the integral becomes $(1/\pi)\int_0^\pi(\sin(2x) + \sin(0x))dx = 0$).

The $2x/\pi$ term gives

$$\begin{aligned}
\frac{4}{\pi^2}\int_0^\pi x\sin(nx)dx &= \frac{4}{\pi^2}\left(\left[-x\frac{\cos(nx)}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)dx\right) \\
&= -\frac{2}{\pi}\left(\frac{2(-1)^n}{n}\right) + \cancelto{0}{\frac{4}{n^2\pi^2}[\sin(nx)]_0^\pi}
\end{aligned}$$

(2 marks)

and the constant term gives

$$\frac{2}{\pi}\int_0^\pi (-1)\sin(nx)dx = \frac{2((-1)^n - 1)}{n\pi}$$

(1 marks)

so in total we have

$$\begin{aligned}
b_n &= \frac{2}{\pi}\left(\frac{n(1+(-1)^n)}{n^2-1} - \frac{2(-1)^n}{n} + \frac{(-1)^n - 1}{n}\right) \\
&= \frac{2}{\pi}((-1)^n + 1)\left(\frac{n}{n^2-1} - \frac{1}{n}\right) \\
&= \frac{2((-1)^n + 1)}{\pi n(n^2-1)}
\end{aligned}$$

(and $b_1 = 0$).

(c) $b_n = \frac{2(e^\pi(-1)^n - 1)}{n\pi(n^2+1)}$.

(d) $b_n = \frac{4\sin(n\pi/2)}{n^2\pi}$.

(e) $b_1 = 1$, $b_n = 0$ for $n > 1$.

**Question 3.** (6 marks)

(a) Suppose that $F(x)$ is an odd function on $[-\pi, \pi]$. Prove that $\int_{-\pi}^{\pi} F(x)^2 dx = 2\int_0^{\pi} F(x)^2 dx$.

Consider the odd function $F(x)$ on $[-\pi, \pi]$ equal to $x(\pi - x)$ on $[0, \pi]$. The Fourier series of $F$ was found in lectures:

$$F(x) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx)}{n^3}.$$

(b) Use this to show that

$$T := \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots = \frac{\pi^6}{960}.$$

(c) Deduce that if

$$S := \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$$

then $S = T + \frac{S}{64}$ and hence calculate $S$.

**Answer 3.** (a) $\int_{-\pi}^{\pi} F(x)^2 dx = \left(\int_{-\pi}^{0} + \int_0^{\pi}\right) F(x)^2 dx$. Changing variables $x \to u = -x$ in the first integral gives $\int_{\pi}^{0} F(-u)^2(-du) = \int_0^{\pi} (-F(u))^2 du = \int_0^{\pi} F(x)^2 dx$ hence $\int_{-\pi}^{\pi} F(x)^2 dx = 2\int_0^{\pi} F(x)^2 dx$.

(1 marks)

(b) Parseval's theorem gives

$$\frac{64}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^6} = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x)^2 dx = \frac{1}{\pi} \int_0^{\pi} x^2(\pi - x)^2 dx$$

by part (a). This integral is

$$\begin{aligned} \frac{2}{\pi} \int_0^{\pi} (x^2\pi^2 - 2x^3\pi + x^4) dx &= \frac{2}{\pi} \left[ \frac{x^3\pi^2}{3} - 2\frac{x^4\pi}{4} + \frac{x^5}{5} \right]_{x=0}^{\pi} \\ &= 2\pi^4 \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) \\ &= \frac{\pi^4}{15}. \end{aligned}$$

(2 marks)

Thus

$$\sum_{n \text{ odd}} \frac{1}{n^6} = \frac{\pi^6}{15 \times 64} = \frac{\pi^6}{960}$$

as required.

(1 marks)

(c) We have $S = \sum_{n \text{ odd}} \frac{1}{n^6} + \sum_{n \text{ even}} \frac{1}{n^6} = T + \sum_n \frac{1}{(2n)^6} = T + \frac{S}{64}$. Therefore $S(1 - 1/64) = 63S/64 = T = \pi^6/(15 \times 64)$ so $S = \pi^6/(15 \times 63)$ or

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

(2 marks)

**Question 4.**
Find the half-range cosine series of $f(x) = x^4 - 2\pi^2 x^2$ on $[0, \pi]$. By considering $f(0)$, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}$$

**Answer 4.** $a_0 = -\frac{14\pi^4}{15}$, $a_n = \frac{48(-1)^{n+1}}{n^4}$ so

$$f(x) = -\frac{7\pi^4}{15} + 48\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}\cos(nx).$$

Setting $x = 0$ gives $f(0) = 0$ and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{48 \times 15} = \frac{7\pi^4}{720}.$$

**Question 5.**
Imagine there were a function $\delta(x)$ with the property that

$$\int_{-1}^{1} \delta(x)g(x)dx = g(0)$$

for any function $g$. What would the Fourier series of $\delta$ be? What might the graph of the function $\delta$ look like? Let $\delta_N$ denote the approximation to $\delta$ obtained by summing the first $N$ terms of its Fourier series. Show that $\delta_N(x) = \frac{\sin((N+1/2)\pi x)}{2\sin(\pi x/2)}$. Use a computer to plot $\delta_N$ for some small values of $N$. As $N$ increases, does the plot start to look like the graph you imagined?

*$\delta$ is called the Dirac delta function; it is not really a function, but fits into the theory of "distributions". The sequence of truncations is called the Dirichlet kernel; analysis of the integral $\int \delta_N(x)F(x+y)dx$ is involved in proving that the Fourier series of $F$ converges in the $L^2$-sense to $F$.*

**Answer 5.** We would have

$$\int_{-1}^{1} \delta(x)\sin(n\pi x)dx = \sin(n\pi 0) = 0$$

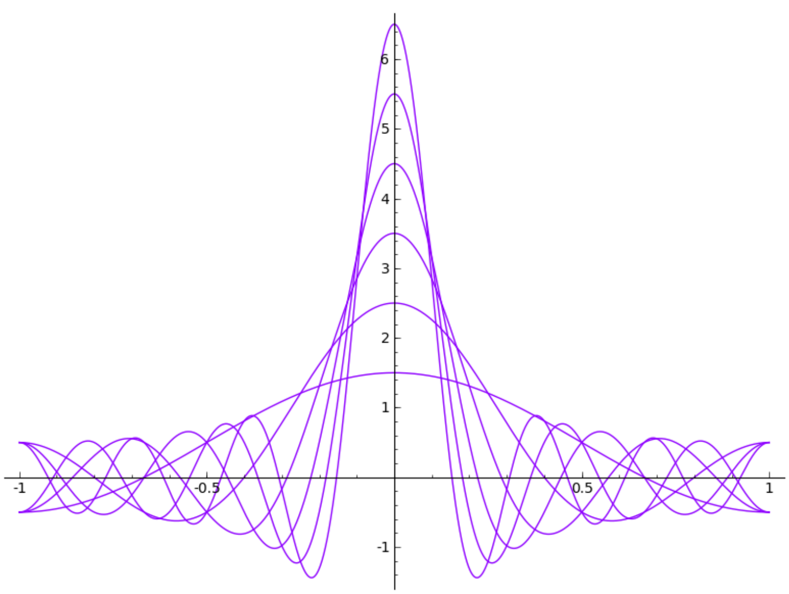
and

$$\int_{-1}^{1} \delta(x)\cos(n\pi x)dx = \cos(n\pi 0) = 1$$

so

$$\delta(x) = \frac{1}{2} + \cos(\pi x) + \cos(2\pi x) + \cos(3\pi x) + \cdots$$

Here are plots of the first six truncations, gradually tending to an infinitely tall spike concentrated near 0.




To see that $\delta_N(x) = \frac{\sin((N+1/2)\pi x)}{2\sin(\pi x/2)}$, note that

$$\sin(\pi x/2)\cos(n\pi x) = \frac{1}{2}(\sin((n+1/2)\pi x) - \sin((n-1/2)\pi x))$$

so there are many cancelling terms in the sum

$$\sin(\pi x/2)(\cos(\pi x) + \cos(2\pi x) + \cdots + \cos(N\pi x))$$

which simplifies to

$$\frac{1}{2}(\sin((N+1/2)\pi x) - \sin(\pi x/2))$$

Thus

$$\cos(\pi x) + \cos(2\pi x) + \cdots + \cos(N\pi x) = \frac{\sin((N+1/2)\pi x)}{2\sin(\pi x/2)} - \frac{1}{2}$$

as required.