Assignment 8

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1 Making a Martingale

We consider the process $X_t = \exp(W_t + g(t))$, where $g : [0, T] \longrightarrow \mathbb{R}$ is a continuous differentiable function. Find all functions g for which the process (X_t) is a martingale.

1.1 Solution

Let $f(x,t) = \exp\{x + g(x)\}$. We have

$$\frac{\delta f}{\delta t}(x,t) = g'(t)e^{x+g(x)} \quad \frac{\delta f}{\delta x}(x,t) = e^{x+g(x)} \quad \frac{\delta 2f}{\delta x^2}(x,t) = e^{x+g(x)}$$

Applying Ito's formula to the given process X_t, we observe the following stochastic differential equation

$$dX_t = g'(t)e^{W_t+g(t)}dt + e^{W_t+g(t)}dW_t + \frac{1}{2}e^{W_t+g(t)}dt$$

Rearranging and writing in integral form, we see

$$X_t = \int_0^t e^{W_s+g(s)}dW_s + \int_0^t e^{W_s+g(s)}(g'(s) + \frac{1}{2})ds$$

For X_t to be a martingale, it must be a stable process. We recognize the stochastic integral is an Ito integral, which is generally stable. Therefore, the following condition must hold.

$$\int_0^t e^{W_s+g(s)}(g'(s) + \frac{1}{2})ds = 0$$

For all $-\infty < W_t + g(t) < \infty$, $e^{W_t+g(t)} > 0$. Therefore $g'(t) + \frac{1}{2} = 0$ must hold. We can then solve for $g(t)$

$$\int_0^t g'(s) + \frac{1}{2}ds = g(t) + \frac{1}{2}t + c \implies g(t) = -\frac{1}{2}t - c$$

This completes the result.

2 Girsanov's theorem

Let (W_t) be a Brownian Motion under some measure P. We consider the process $X_t = W_t + \frac{t^2}{2}$.
(a) Under the measure P, find $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t)$.
(b)Under the measure P, express $P(W_t \leq x)$ and $P(X_t \leq x)$ as functions of Φ, the cummulative distribution function of a standard normal distribution.
(c) Find the appropriate function $\varphi(t)$ for this case. Give an explicit expression for $\frac{dQ}{dP}$ given by Girsanov's theorem.
(d) Under the measure Q, using the expression for $\frac{dQ}{dP}$, calculate explicitly $\mathbb{E}_Q[W_t]$ and $\mathrm{Var}_Q(W_t)$. Is (W_t) a Brownian motion under Q?
(e) Under the measure Q, calculate $\mathbb{E}_Q[X_t]$ and $\mathrm{Var}_Q(X_t)$.
(f) Under the measure Q, calculate $Q(W_t \leq x)$ and $Q(X_t \leq x)$. What can you conclude?
Notice since the cumulative distribution function determines the distribution, the result above proves Girsanov's theorem in the specific case (X_t).

2.1 Solution to Part (a)

We calculate the processes expectation and variance.

$$\mathbb{E}_P[X_t] = \mathbb{E}_P[W_t + \frac{t^2}{2}] = \mathbb{E}_P[W_t] + \frac{t^2}{2} = \frac{t^2}{2}$$

$$\mathrm{Var}_P(X_t) = \mathbb{E}_P[X_t^2] - \mathbb{E}_P[X_t]^2 = \mathbb{E}_P[(W_t + \frac{t^2}{2})^2] - (\frac{t^2}{2})^2$$

$$= \mathbb{E}_P[W_t^2] + t^2\mathbb{E}_P[W_t] + (\frac{t^2}{2})^2 - (\frac{t^2}{2})^2 = \mathbb{E}_P[W_t^2] \implies \mathrm{Var}_P(X_t) = t$$

2.2 Solution to Part (b)

We calculate the probabilities of W_t and X_t relative to some value x by standardizing the variables.

$$P(W_t \leq x) = \Phi(\frac{x - \mathbb{E}_P[W_t]}{\sqrt{\mathrm{Var}_P(W_t)}}) = \Phi(\frac{x}{\sqrt{t}}) = \Phi(Z)$$

$$P(X_t \leq x) = \Phi(\frac{x - \mathbb{E}_P[X_t]}{\sqrt{\mathrm{Var}_P(X_t)}}) = \Phi(\frac{x - \frac{t^2}{2}}{\sqrt{t}}) = \Phi(\frac{x}{\sqrt{t}} - \frac{t^{3/2}}{2}) = \Phi(Z)$$

2.3 Solution to Part (c)

By Girsanov's theorem, we know

$$dQ = ZdP \text{ for } Z = \exp\{-\int_0^T \varphi(s)dW_s - \frac{1}{2}\int_0^T \varphi^2(s)ds\}$$

Letting $\tilde{W}_t = W_t + \frac{t^2}{2} = W_t + \int_0^t \varphi(s)ds$, we see $\varphi(t) = t$. Therefore we can calculate Z to be

$$Z = \exp\{-\int_0^T sdW_s - \frac{1}{2}\int_0^T s^2ds\} = e^{-TW_T - \frac{1}{6}T^3} = \frac{dQ}{dP}$$

2.4 Solution to Part (d)

We calculate the expectation of W_t under Q to be

$$\mathbb{E}_Q[W_t] = \mathbb{E}_P[W_t Z_t] = \mathbb{E}_P[W_t e^{-TW_T - \frac{1}{6}T^3}] = e^{-\frac{t^3}{6}}\mathbb{E}_P[W_t e^{-tW_t}] = e^{-\frac{t^3}{6}}\mathbb{E}_P[\sqrt{t}Xe^{-t\sqrt{t}X}]$$

We let say $W_t = \sqrt{t}X$ where $X \sim \mathcal{N}(0,1)$. We also note

$$\mathbb{E}[Xe^{\lambda X}] = \frac{d}{d\lambda}\mathbb{E}[e^{\lambda X}] = \frac{d}{d\lambda}e^{\frac{\lambda^2}{2}} = \lambda e^{\frac{\lambda^2}{2}}$$

where the second term is the MGF of a normal distribution. Letting $\lambda = -t^{\frac{3}{2}}$, we observe

$$e^{-\frac{t^3}{6}}\mathbb{E}_P[\sqrt{t}Xe^{-t\sqrt{t}X}] = e^{-\frac{t^3}{6}}\sqrt{t}\frac{d}{d\lambda}\mathbb{E}_P[e^{\lambda X}] = e^{-\frac{t^3}{6}}\sqrt{t}\lambda e^{\frac{\lambda^2}{2}} = e^{-\frac{t^3}{6}}\sqrt{t}(t^{\frac{3}{2}})e^{\frac{(-t^{\frac{3}{2}})^2}{2}}$$

$$= -e^{-\frac{t^3}{6}}t^2 e^{\frac{t^3}{2}} = -t^2 e^{\frac{1}{3}t^3} = \mathbb{E}_Q[W_t]$$

2.5 Solution to Part (e)

2.6 Solution to Part (f)

3 Feynman-Kac Application

We consider a process $(X_t)_{t\geq 0}$ given by $X_t = e^{W_t}$, where (W_t) is a Brownian Motion.

Part A

(a) Determine the Stochastic Differential Equation satisfied by the process (X_t).

(b) We want to determine $g(t, X_t) = \mathbb{E}[X_T^2|\mathcal{F}_t]$. Use Feynman-Kac formula and the result of (a) to write the PDE (with its terminal condition) that the function g has to solve.

(c) We guess that g has to be a polynomial of degree 2 in x. Determine g by solving the PDE.

(d) We can directly calculate the result using properties of conditional expectation. Verify that you obtain the same solution.

Part B

(e) We consider the function

$$h(x) = \mathbb{1}_{(1,\infty)}(x) = \begin{cases} 1 \ x > 1 \\ 0 \ x \leq 1 \end{cases}$$

We would like to determine $\mathbb{E}[h(X_T)|\mathcal{F}_t] = g(t, X_t)$. Use Feynman-Kac formula to write the PDE (with its terminal condition) that g has to solve.

(f) We would like to solve the PDE obtained in (b). Let $f(t, x)$ be a function such that

$$g(t, x) = f(T - t, \ln(x)) \Longleftrightarrow f(t, x) = g(T - t, e^x)$$

Use the PDE in (e) to determine the PDE (with *initial* condition) that the function f has to solve.

(g) We know that the solution to the heat equation $\frac{\delta u}{\delta t} - \frac{1}{2}\frac{\delta^2 y}{\delta x^2} = 0$ with intial condition u_0 is given by

$$u(t, x) = \int_{y\in\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(x-y)^2}{2t}\} u_0(y) dy$$

Use this result to determine the function f and, then, the function g.

(h) Notice that the result gives us an estimate of the probability $P(X_T > 1|\mathcal{F}_t)$, i.e. the best approximation of $P(X_T > 1)$ given the observation until time t. Determine $P(X_T > 1|\mathcal{F}_0)$ from (g) and verify that it corresponds to the result obtained by the opposite direction.

3.1 Part A

3.1.1 Solution to Part (a)

We use Ito's formula to write the processes SDE. we observe this to be

$$\begin{cases} dX_t = X_t dW_t + \frac{1}{2} X_t dt \\ X_0 = 1 \end{cases}$$

3.1.2 Solution to Part (b)

We have $g(t, X_t) = \mathbb{E}[X_T^2|\mathcal{F}_t] = \mathbb{E}[h(X_T)|\mathcal{F}_t]$ implies that $h(x) = x^2$. We observe from (a) that $\mu = \frac{x}{2}$ and $\sigma = x$. Therefore by Feynman-Kac

$$\begin{cases} \frac{\delta g}{\delta t} + \frac{x}{2}\frac{\delta g}{\delta x} + \frac{1}{2}x^2\frac{\delta^2 g}{\delta x^2} = 0 \\ g(T, x) = x^2 \end{cases}$$

3.1.3 Solution to Part (c)

We say g takes the form $g(t, x) = a(t) + b(t)x + c(t)x^2$. Rewriting the first equation of the PDE using g of this form, we see

$$(c'(t)x^2 + b'(t)x + a'(t)) + \frac{x}{2}(2c(t)x + b(t)) + \frac{1}{2}x^2(2c(t)) = 0$$

Rewriting this to isolate the a's, b's, and c's, we see

$$a'(t) = 0 \implies a(t) = A$$

$$b'(t)x + \frac{1}{2}b(t)x = 0 \implies \frac{1}{2}b(t) = -b'(t) \implies b(t) = Be^{-2t}$$

$$c'(t)x^2 + 2c(t)x^2 = 0 \implies 2c(t) = -c'(t) \implies c(t) = Ce^{-\frac{1}{2}t}$$

We can then use this to determine that

$$g(t, x) = Ce^{-\frac{1}{2}t}x^2 + Be^{-2t}x + A$$

Evaluating at time T we observe

$$g(T, x) = x^2 = Ce^{-\frac{1}{2}T}x^2 + Be^{-2T}x + A$$

We can clearly see that $A = B = 0$ at the terminal condition. We then evaluate C.

$$x^2 = Ce^{-\frac{1}{2}T}x^2$$

$$\implies C = e^{\frac{1}{2}T}$$

Therefore, we have

$$g(t, x) = e^{\frac{1}{2}(T-t)}x^2$$

3.1.4 Solution to Part (d)

We have from (c) $g(T, x) = x^2$ implies $g(t, W_t) = e^{2W_t}$. We calculate this using conditional expectation below to validate.

$$\mathbb{E}[X_T^2|\mathcal{F}_t] = \mathbb{E}[e^{2W_T}|\mathcal{F}_t] = \mathbb{E}[e^{2(W_T-W_t+W_t-W_0}|\mathcal{F}_t] = e^{2W_t}\mathbb{E}[e^{2(W_T-W_t)}] = e^{2W_t}e^{\frac{T-t}{2}} = g(t, W_t)$$

This validates the result.

3.2 Part B

3.2.1 Solution to Part (e)

We have $g(T, X_t) = \mathbb{1}_{(1,\infty)}(x)$. We observe from (a) that $\mu = \frac{x}{2}$ and $\sigma = x$. Therefore by Feynman-Kac

$$\begin{cases} \frac{\delta g}{\delta t} + \frac{x}{2}\frac{\delta g}{\delta x} + \frac{1}{2}x^2\frac{\delta^2 g}{\delta x^2} = 0 \\ g(T, x) = \mathbb{1}_{(1,\infty)}(x) \end{cases}$$

3.2.2 Solution to Part (f)

We say $g(t, x) = f(T - t, \ln(x))$. We observe

$$g'(x) = \frac{1}{x}f'(\ln(x)) \implies xg'(x) = f'(\ln(X))$$

$$g''(x) = \frac{1}{x^2}f''(\ln(X)) - \frac{1}{x^2}f'(\ln(X)) \implies x^2g''(x) = f''(\ln(X)) - f'(\ln(X))$$

$$g'(t) = -f'(t)$$

Replacing these terms into the differential equation we see

$$0 = (-f'(t)) + \frac{1}{2}(f'(\ln(x))) + \frac{1}{2}((f''(\ln(x))) - (f'(\ln(x)))) = \frac{1}{2}f''(x) - f'(t)$$

We then evaluate the initial/terminal condition. We have

$$g(T, x) = \mathbb{1}_{(1,\infty)}(x) = f(0, \ln(x)) \implies f(0, x) = \mathbb{1}_{(1,\infty)}(e^x) = \mathbb{1}_{(0,\infty)}(x)$$

We can then replace these terms into the PDE and observe

$$\begin{cases} \frac{\delta f}{\delta t} - \frac{1}{2}\frac{\delta^2 f}{\delta x^2} = 0 \\ f(0, x) = \mathbb{1}_{(0,\infty)}(x) \end{cases}$$

3.2.3 Solution to Part (g)

We use the provided solution to the heat equation to observe

$$f(t, x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}\exp\{-\frac{(x-y)^2}{2t}\}\mathbb{1}_{(0,\infty)}(y)dy$$

Using the indicator function we observe

$$f(t, x) = \int_0^\infty \frac{1}{\sqrt{2\pi t}}\exp\{-\frac{(x-y)^2}{2t}\}dy = \frac{1}{2}$$

We recognize this to be the CDF of a normal random variable with mean 0, implying the area under the curve is a half. Since f represents the risk neutral process, it makes sense for this to be a constant. To derive g, we observe

$$g(t, x) = f(T - t, \ln(x)) = \int_0^\infty \frac{1}{\sqrt{2\pi(T-t)}}\exp\{-\frac{(\ln(x) - y)^2}{2(T-t)}\}dy$$

3.2.4 Solution to Part (h)

We see from the above result that the function is half the CDF of a normal distribution implying the probability of the process ending in the money is one half.