Where is it

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A case study using real world data to formulate optimized dedication, immunization, and other bond portfolios.

Contents

[5]:
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'''
Package Imports
'''
import pandas as pd
import numpy as np
import pulp
from pulp import *
import datetime
import matplotlib.pyplot as plt

import FinOpsCodeDeck as finops

from IPython.display import Markdown as md
```

1 Term Structure

Determine the current term structure of treasury rates (see textbook Section 3.4 or other resources that you can find), and find the present value, duration, and convexity of the stream of liabilities. Please explain the main steps followed in your calculations. Use real world data.

1.1 Deriving Term Structure

In this section, we describe our derivation of the term structure of interest rates. Specifically, we outline our data gathering and transformation techniques and then move to explaining bootstrapping.

1.1.1 Data and Transformations

We begin by pulling current US Treasury issued Bonds and Notes from The Wall Street Journal. We transform this data so we can understand each bonds market. Specifically, we create a bid and ask price for each bond called `'px_bid'` or `'px_ask'`. We also take the maturity of the bond less today's date to get a time to maturity field called `'ttm'`. This time to maturity is a float datatype which represents the years to maturity according to an actual/365 day calendar, the standard calendar of US Treasury Bonds. For sake of simplicity, we use this calendar for the notes as well despite these operating on a 30/360 calendar. Having completed these transformations, we can move to bootstrapping the curve.

1.1.2 Bootstrapping

Bootstrapping is a technique used to find continuous annualized interest rates across all time to maturities. Due to the nature of fixed income securities paying intermediate coupons, bootstrapping is necessary to value a cashflow from one specific point in time to any other. To better understand this, consider the following example.

Example
Let the current market only consist of 2 risk-free bonds that were issued today:
* 1-year zero-coupon bond trading at 99c on the dollar

* 2-year 1.5% annual coupon bond trading at par
To bootstrap the curve, we start with the 1-year zero.

$$99 = 100\exp\{-r\} \implies r = -\log(0.99) \approx 0.01$$

We then use this rate in our calculation with the coupon bond to find the 2-year rate.

$$100 = 1.5\exp\{-0.01(1)\} + 101.5\exp\{-2r\} \implies r \approx 0.0145$$

In this example, we have found the term structure to be given by:

Time to Maturity	Rate
1	1.00%
2	1.45%

So, doing this over all cashflows of all bonds in our data will allow us to derive a term structure across all maturities. This derived term structure will drive our analysis.

NOTE: For sake of simplicity, we round all time to maturity to the nearest hundredth of a year. From a bond trading perspective, this is essentially every 2.5 trading days representing 1 time period. We do this for simplicity in later sections as not all dates marry exactly together. In the event that a particular liability does not have a term structure rate associated with it, we use the closest prior known date. Additionally, in the event there are multiple calculated yields for a particular time to maturity, we take the arithmetic average of them for that time.

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'''
Data Import for Current Term Structure
---
Imports all active treasury bonds data, time indexes them by year
'''
data_prompt = pd.read_excel('Table.xlsx', sheet_name='PromptUse', index_col = 'DateDue')
data_prompt = data_prompt/1000
term_structure_df = pd.read_excel('TableNew.xlsx', sheet_name='d')
term_structure_df['px_ask'] = [i if i>=5 else 100 - i for i in term_structure_df['ASKED'].
 ↪to_list()]
term_structure_df['px_bid'] = [i if i>=5 else 100 - i for i in term_structure_df['BID'].
 ↪to_list()]
term_structure_df['ttm'] = [(i - datetime.datetime.now())/datetime.timedelta(days=365) for i in␣
 ↪term_structure_df['MATURITY']]
```

[7]:
```
'''
Bootstrap yield curve
---
begins with zero-coupon bonds to payout (ttm < 0.5 yrs) & calculates yield
moves to coupon bonds and uses calculated yields to bootstrap further
sorts all bonds into data frame indexed by ttm (by 100th of a year)
NOTE: Averages yields for the same time period
NOTE: assumes yield of period prior if yield for desired period does not exist
'''

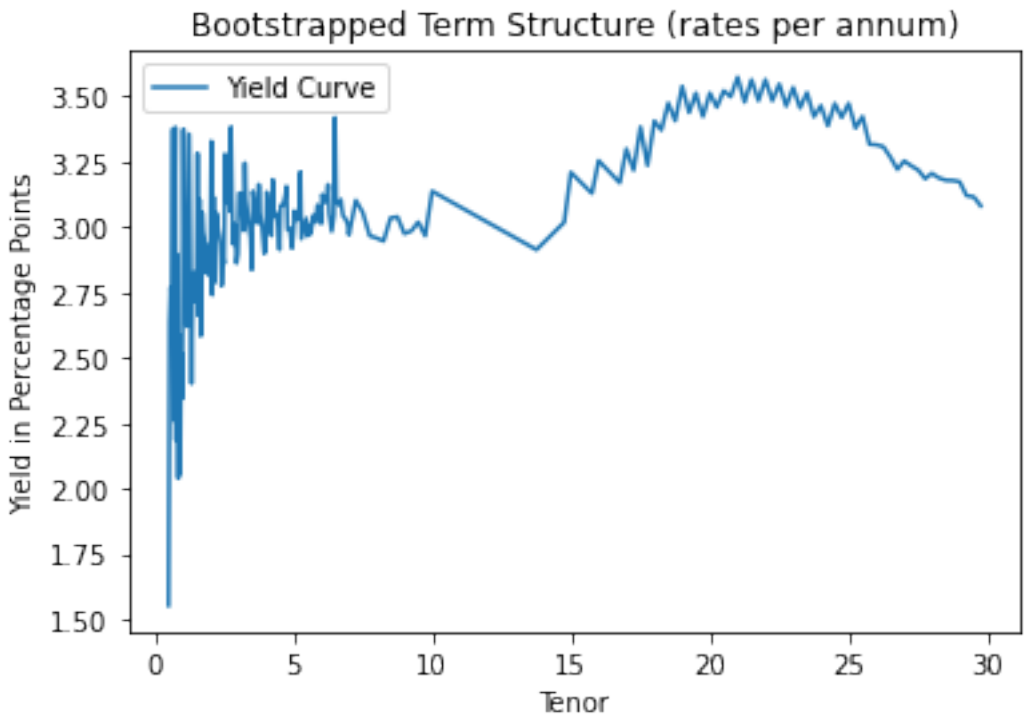
'''short term rates'''
mats = []
```

```
round_to = 2
for bond_tenor in term_structure_df[term_structure_df['ttm'] <= 0.5].index:
    bond = term_structure_df.loc[bond_tenor]
    cpn = bond['COUPON']/2
    ttm = bond['ttm']
    px = bond['px_ask']
    mats.append([np.round(ttm,round_to),np.log((100 + cpn) / bond['px_ask']) / bond['ttm']])
rates = pd.DataFrame(mats, columns=['ttm','rate']).set_index('ttm').groupby('ttm').mean()

'''longer term rates'''
for bond_tenor in term_structure_df[term_structure_df['ttm']>=0.5].index:
    bond = term_structure_df.loc[bond_tenor]
    px = bond['px_ask']
    ttm = bond['ttm']
    cpn = bond['COUPON']/2
    pmts = int(np.ceil(ttm * 2))
    cfs = [cpn if i+1<pmts else 100 + cpn for i in range(pmts)]
    cfs_idx = [np.round(ttm-i*0.5, round_to) for i in reversed(range(pmts))]
    known_rates = [rates[:cfs_idx[i]].iloc[-1,0] for i in range(pmts-1)]
    val = px - sum([cpn * np.exp((-1) * known_rates[i] * cfs_idx[i]) for i in range(pmts-1)])
    yld = (-1) * (np.log(val / (100+cpn)) / cfs_idx[pmts-1])
    add_df = pd.DataFrame([np.round(ttm, round_to), yld], index=['ttm','rate']).transpose().
 ↪set_index('ttm')
    rates = pd.concat([rates,add_df],ignore_index=False)
    rates = rates.groupby('ttm').mean()
```

[8]:
```
'''
plot yield curve
---
plots yield curve in percentage points
'''
plt.plot(rates[0.5:] * 100)
plt.title('Bootstrapped Term Structure (rates per annum)')
plt.xlabel('Tenor')
plt.ylabel('Yield in Percentage Points')
plt.legend(['Yield Curve'])
```

[8]: <matplotlib.legend.Legend at 0x26114fbff40>



[9]:
```
'''
Liability Stream Analysis
---
Calculates time to maturity (in years) of each obligation
Calculates npv, duration, and convexity of liability stream
Prints stats to markdown for viewing
'''
data_prompt['ttm'] = np.round((data_prompt.index - datetime.datetime.now()) / datetime.
 ↪timedelta(days=365), round_to)
r = [rates[:ttm].iloc[-1,0] for ttm in data_prompt['ttm']]
data_prompt['rates'] = r

npv = sum([data_prompt.iloc[i,0]*np.exp((-1)*data_prompt.iloc[i,1]*data_prompt.iloc[i,2]) for i␣
 ↪in range(len(data_prompt))])
dur = sum([data_prompt.iloc[i,0]*data_prompt.iloc[i,1]*np.exp((-1) * (data_prompt.
 ↪iloc[i,1]+1)*data_prompt.iloc[i,2]) for i in range(len(data_prompt))])
con = sum([data_prompt.iloc[i,0]*data_prompt.iloc[i,1]*(data_prompt.iloc[i,1]+1)*np.
 ↪exp((-1)*(data_prompt.iloc[i,1]+2)*data_prompt.iloc[i,2]) for i in range(len(data_prompt))])
```

[10]:
```
md('''
<center>

The Net Present Value of the Liabilities is $\${:.2f}$ MM

The Macauley Duration of the Liability stream is ${:.2f}$ years

The Convexity of the Liability stream is ${:.2f}$
```

```
</center>
'''.format(npv,dur/npv,con/npv))
```

[10]:

The Net Present Value of the Liabilities is $117.66 MM

The Macauley Duration of the Liability stream is 3.83 years

The Convexity of the Liability stream is 23.31

2 Data

Identify *at least* 30 fixed-income assets that are suitable to construct a dedicated bond portfolio for the municipality liabilities that you have been given. Use assets that are considered risk-free; for example, US government non-callable treasury bonds, treasury bills, or treasury notes. Display in an appropriate table the main characteristics of the bonds you choose. Namely, prices, coupon rates, maturity dates, face value).

[11]:
```
'''
code block
'''
ref_data = ['T ' + '{:.3f}'.format(term_structure_df.iloc[bond,1]) + ' ' +␣
 ↪term_structure_df.iloc[bond,0].strftime('%m/%d/%y') for bond in␣
 ↪term_structure_df.index]
term_structure_df['ref_data'] = ref_data

bonds_clean = term_structure_df[['ref_data', 'px_ask', 'ASKED YIELD']]
bonds_clean.columns = ['Bond', 'Price', 'Yield']
# bonds_clean = bonds_clean.assign(ttm = term_structure_df.ttm.round(2))
bonds_clean = bonds_clean.set_index('Bond')
```

[12]:
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# Hide
list_for_slice = bonds_clean.index.tolist()
bond_dis1 = bonds_clean.loc[:list_for_slice[len(list_for_slice)//4-1],:].
 ↪reset_index()
bond_dis2 = bonds_clean.loc[list_for_slice[len(list_for_slice)//4]:
 ↪list_for_slice[len(list_for_slice)//2-1],:].reset_index()
bond_dis3 = bonds_clean.loc[list_for_slice[len(list_for_slice)//2]:
 ↪list_for_slice[(3*len(list_for_slice))//4-1],:].reset_index()
bond_dis4 = bonds_clean.loc[list_for_slice[(3*len(list_for_slice))//4]:,:].
 ↪reset_index()
multi_col = [(i, j) for i in range(1,5) for j in bond_dis4.columns.to_list()]
side_by_side = pd.DataFrame(index=range(1,92), columns=pd.MultiIndex.
 ↪from_tuples(multi_col))

dis_dict = dict(zip(range(1,5), [bond_dis1, bond_dis2, bond_dis3, bond_dis4]))
for i in range(1,5):
```

```
    for j in bond_dis4.columns.to_list():
        side_by_side.loc[:, (i, j)] = dis_dict[i][j].to_list()
almost_dis = side_by_side.set_index((1, 'Bond'),).droplevel(level=0, axis=1)
almost_dis
almost_dis.index.name = 'Bond'
almost_dis
alignment = tuple(['center' for i in range(12)])
```

[14]:
```
# Hide
md('''
{}
'''.format(almost_dis.to_markdown(colalign = alignment))
)
```

[14]:

Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield
T 1.750 05/15/22	100.01	- 1.9931	T 1.625 04/30/23	99.19	2.056	T 2.625 04/15/25	99.164	2.799	T 5.250 11/15/28	113.192	2.935
T 2.125 05/15/22	100.012	- 2.5447	T 2.750 04/30/23	100.206	2.061	T 0.375 04/30/25	93.04	2.815	T 1.500 11/30/28	91.112	2.964
T 0.000 05/17/22	99.352	0.657	T 0.125 05/15/23	98.03	2.066	T 2.875 04/30/25	100.066	2.8	T 1.375 12/31/28	90.194	2.946
T 0.000 05/19/22	99.34	0.669	T 1.750 05/15/23	99.22	2.068	T 2.125 05/15/25	98.016	2.806	T 1.750 01/31/29	92.234	2.951
T 0.000 05/24/22	99.32	0.69	T 0.125 05/31/23	97.306	2.115	T 2.750 05/15/25	99.282	2.791	T 2.625 02/15/29	98.01	2.948
T 0.000 05/26/22	99.322	0.687	T 1.625 05/31/23	99.156	2.12	T 0.250 05/31/25	92.186	2.811	T 5.250 02/15/29	114.03	2.933
T 0.125 05/31/22	99.312	0.6946	T 2.750 05/31/23	100.204	2.124	T 2.875 05/31/25	100.066	2.802	T 1.875 02/28/29	93.152	2.942
T 1.750 05/31/22	100.014	0.608	T 0.250 06/15/23	97.316	2.136	T 0.250 06/30/25	92.116	2.819	T 2.375 03/31/29	96.14	2.951
T 1.875 05/31/22	100.016	0.544	T 0.125 06/30/23	97.23	2.19	T 2.750 06/30/25	99.266	2.805	T 2.875 04/30/29	99.172	2.949
T 0.000 05/31/22	99.307	0.702	T 1.375 06/30/23	99.034	2.181	T 0.250 07/31/25	92.046	2.826	T 2.375 05/15/29	96.134	2.945
T 0.000 06/02/22	99.327	0.682	T 2.625 06/30/23	100.156	2.178	T 2.875 07/31/25	100.06	2.813	T 1.625 08/15/29	91.192	2.918
T 0.000 06/07/22	99.352	0.657	T 0.125 07/15/23	97.19	2.227	T 2.000 08/15/25	97.136	2.833	T 6.125 08/15/29	120.284	2.908
T 0.000 06/09/22	99.347	0.662	T 0.125 07/31/23	97.144	2.271	T 6.875 08/15/25	112.206	2.778	T 1.750 11/15/29	92.076	2.909
T 0.000 06/14/22	99.35	0.659	T 1.250 07/31/23	98.262	2.243	T 0.250 08/31/25	91.29	2.844	T 1.500 02/15/30	90.076	2.915
T 1.750 06/15/22	100.03	0.608	T 2.750 07/31/23	100.19	2.248	T 2.750 08/31/25	99.23	2.839	T 0.625 05/15/30	83.23	2.923
T 0.000 06/16/22	99.372	0.637	T 0.125 08/15/23	97.114	2.277	T 0.250 09/30/25	91.224	2.846	T 6.250 05/15/30	123.232	2.904
T 0.000 06/21/22	99.342	0.667	T 2.500 08/15/23	100.094	2.257	T 3.000 09/30/25	100.166	2.836	T 0.625 08/15/30	83.09	2.92
T 0.000 06/23/22	99.307	0.703	T 6.250 08/15/23	104.296	2.23	T 0.250 10/31/25	91.16	2.85	T 0.875 11/15/30	84.224	2.921
T 0.000 06/28/22	99.275	0.736	T 0.125 08/31/23	97.062	2.344	T 3.000 10/31/25	100.156	2.849	T 1.125 02/15/31	86.076	2.918
T 0.125 06/30/22	99.294	0.755	T 1.375 08/31/23	98.26	2.313	T 2.250 11/15/25	98.004	2.85	T 5.375 02/15/31	118.256	2.923
T 1.750 06/30/22	100.04	0.74	T 2.750 08/31/23	100.176	2.31	T 0.375 11/30/25	91.224	2.855	T 1.625 05/15/31	89.25	2.925
T 2.125 06/30/22	100.052	0.799	T 0.125 09/15/23	97.04	2.33	T 2.875 11/30/25	100.026	2.849	T 1.250 08/15/31	86.162	2.925
T 0.000 06/30/22	99.275	0.736	T 0.250 09/30/23	97.056	2.347	T 0.375 12/31/25	91.166	2.853	T 1.375 11/15/31	87.046	2.935
T 0.000 07/05/22	99.26	0.751	T 1.375 09/30/23	98.222	2.345	T 2.625 12/31/25	99.09	2.835	T 1.875 02/15/32	91.02	2.936

Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield
T 0.000 07/07/22	99.235	0.777	T 2.875 09/30/23	100.23	2.339	T 0.375 01/31/26	91.094	2.864	T 2.875 05/15/32	99.176	2.927
T 0.000 07/12/22	99.237	0.774	T 0.125 10/15/23	96.276	2.389	T 2.625 01/31/26	99.062	2.855	T 4.500 02/15/36	118.09	2.88
T 0.000 07/14/22	99.222	0.789	T 0.375 10/31/23	97.032	2.412	T 1.625 02/15/26	95.194	2.867	T 4.750 02/15/37	121.096	2.957
T 1.750 07/15/22	100.052	0.756	T 1.625 10/31/23	98.286	2.399	T 6.000 02/15/26	111.052	2.84	T 5.000 05/15/37	124.052	2.989
T 0.000 07/21/22	99.21	0.802	T 2.875 10/31/23	100.21	2.413	T 0.500 02/28/26	91.18	2.865	T 4.375 02/15/38	116.29	3.018
T 0.000 07/28/22	99.205	0.807	T 0.250 11/15/23	96.262	2.425	T 2.500 02/28/26	98.232	2.856	T 4.500 05/15/38	118.154	3.033
T 0.125 07/31/22	99.27	0.871	T 2.750 11/15/23	100.156	2.413	T 0.750 03/31/26	92.092	2.867	T 3.500 02/15/39	105.026	3.108
T 1.875 07/31/22	100.064	0.902	T 0.500 11/30/23	97.01	2.475	T 2.250 03/31/26	97.24	2.867	T 4.250 05/15/39	115.006	3.106
T 2.000 07/31/22	100.072	0.915	T 2.125 11/30/23	99.172	2.431	T 0.750 04/30/26	92.042	2.868	T 4.500 08/15/39	118.124	3.114
T 0.000 08/04/22	99.085	0.93	T 2.875 11/30/23	100.202	2.454	T 2.375 04/30/26	98.062	2.861	T 4.375 11/15/39	116.14	3.146
T 0.000 08/11/22	99.055	0.96	T 0.125 12/15/23	96.126	2.46	T 1.625 05/15/26	95.094	2.879	T 4.625 02/15/40	120.014	3.143
T 1.500 08/15/22	100.04	0.999	T 0.750 12/31/23	97.07	2.508	T 0.750 05/31/26	91.292	2.884	T 1.125 05/15/40	70.18	3.309
T 1.625 08/15/22	100.05	0.999	T 2.250 12/31/23	99.202	2.481	T 2.125 05/31/26	97.05	2.875	T 4.375 05/15/40	116.1	3.178
T 7.250 08/15/22	101.184	0.942	T 2.625 12/31/23	100.08	2.466	T 0.875 06/30/26	92.074	2.886	T 1.125 08/15/40	70.03	3.324
T 0.000 08/18/22	99.01	1.006	T 0.125 01/15/24	96.046	2.5	T 1.875 06/30/26	96.052	2.868	T 3.875 08/15/40	108.282	3.227
T 0.000 08/25/22	99.01	1.007	T 0.875 01/31/24	97.072	2.542	T 0.625 07/31/26	91.024	2.891	T 1.375 11/15/40	73.066	3.326
T 0.125 08/31/22	99.234	1.042	T 2.500 01/31/24	99.302	2.532	T 1.875 07/31/26	96.024	2.87	T 4.250 11/15/40	113.294	3.243
T 1.625 08/31/22	100.052	1.057	T 0.125 02/15/24	95.286	2.534	T 1.500 08/15/26	94.144	2.896	T 1.875 02/15/41	79.282	3.325
T 1.875 08/31/22	100.076	1.037	T 2.750 02/15/24	100.112	2.542	T 6.750 08/15/26	115.122	2.878	T 4.750 02/15/41	121.106	3.225
T 0.000 09/01/22	98.957	1.06	T 1.500 02/29/24	98.056	2.546	T 0.750 08/31/26	91.134	2.89	T 2.250 05/15/41	84.262	3.335
T 0.000 09/08/22	98.912	1.106	T 2.125 02/29/24	99.092	2.533	T 1.375 08/31/26	93.302	2.885	T 4.375 05/15/41	115.164	3.272
T 1.500 09/15/22	100.03	1.214	T 2.375 02/29/24	99.232	2.531	T 0.875 09/30/26	91.246	2.89	T 1.750 08/15/41	77.146	3.348
T 0.000 09/15/22	98.9	1.12	T 0.250 03/15/24	95.286	2.556	T 1.625 09/30/26	94.262	2.894	T 3.750 08/15/41	106.144	3.294
T 0.000 09/22/22	98.885	1.135	T 2.125 03/31/24	99.07	2.554	T 1.125 10/31/26	92.21	2.893	T 3.125 11/15/41	97.11	3.311
T 0.000 09/29/22	98.887	1.133	T 2.250 03/31/24	99.13	2.576	T 1.625 10/31/26	94.234	2.893	T 2.000 11/30/41	80.292	3.34
T 0.125 09/30/22	99.184	1.258	T 0.375 04/15/24	95.286	2.584	T 2.000 11/15/26	96.07	2.903	T 2.375 02/15/42	86.136	3.316
T 1.750 09/30/22	100.056	1.266	T 2.000 04/30/24	98.286	2.581	T 6.500 11/15/26	115.034	2.893	T 3.125 02/15/42	97.094	3.312
T 1.875 09/30/22	100.074	1.244	T 2.250 04/30/24	99.12	2.579	T 1.250 11/30/26	93.032	2.882	T 3.000 05/15/42	95.104	3.322
T 0.000 10/06/22	98.845	1.176	T 2.500 04/30/24	99.27	2.582	T 1.625 11/30/26	94.222	2.879	T 2.750 08/15/42	91.074	3.349
T 0.000 10/13/22	98.757	1.266	T 0.250 05/15/24	95.144	2.601	T 1.250 12/31/26	92.304	2.889	T 2.750 11/15/42	91.022	3.356
T 1.375 10/15/22	100.004	1.336	T 2.500 05/15/24	99.25	2.613	T 1.750 12/31/26	95.034	2.887	T 3.125 02/15/43	96.232	3.345
T 0.000 10/20/22	98.732	1.292	T 2.000 05/31/24	98.264	2.593	T 1.500 01/31/27	93.292	2.891	T 2.875 05/15/43	92.264	3.354
T 0.000 10/27/22	98.697	1.329	T 0.250 06/15/24	95.066	2.628	T 2.250 02/15/27	97.042	2.9	T 3.625 08/15/43	104.124	3.335

Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield
T 0.125 10/31/22	99.14	1.365	T 1.750 06/30/24	98.06	2.632	T 6.625 02/15/27	116.114	2.912	T 3.750 11/15/43	106.112	3.334
T 1.875 10/31/22	100.072	1.375	T 2.000 06/30/24	98.214	2.647	T 1.125 02/28/27	92.052	2.888	T 3.625 02/15/44	104.09	3.346
T 2.000 10/31/22	100.09	1.379	T 0.375 07/15/24	95.064	2.669	T 1.875 02/28/27	95.172	2.878	T 3.375 05/15/44	100.094	3.356
T 0.000 11/03/22	98.675	1.352	T 1.750 07/31/24	98.014	2.665	T 0.625 03/31/27	89.244	2.891	T 3.125 08/15/44	96.11	3.359
T 0.000 11/10/22	98.605	1.424	T 2.125 07/31/24	98.264	2.674	T 2.500 03/31/27	98.09	2.88	T 3.000 11/15/44	94.084	3.366
T 1.625 11/15/22	100.022	1.483	T 0.375 08/15/24	94.31	2.693	T 0.500 04/30/27	89.002	2.897	T 2.500 02/15/45	86.084	3.369
T 7.625 11/15/22	103.032	1.346	T 2.375 08/15/24	99.092	2.702	T 2.750 04/30/27	99.126	2.881	T 3.000 05/15/45	94.094	3.358
T 0.125 11/30/22	99.08	1.522	T 1.250 08/31/24	96.25	2.708	T 2.375 05/15/27	97.164	2.913	T 2.875 08/15/45	92.116	3.35
T 2.000 11/30/22	100.084	1.505	T 1.875 08/31/24	98.042	2.721	T 0.500 05/31/27	88.24	2.916	T 3.000 11/15/45	94.174	3.337
T 0.000 12/01/22	98.64	1.389	T 0.375 09/15/24	94.224	2.735	T 0.500 06/30/27	88.184	2.916	T 2.500 02/15/46	86.076	3.344
T 1.625 12/15/22	100.012	1.557	T 1.500 09/30/24	97.07	2.717	T 0.375 07/31/27	87.274	2.903	T 2.500 05/15/46	86.074	3.338
T 0.000 12/29/22	98.515	1.52	T 2.125 09/30/24	98.21	2.713	T 2.250 08/15/27	96.24	2.922	T 2.250 08/15/46	82.016	3.335
T 0.137 12/31/22	99.016	1.653	T 0.625 10/15/24	95.034	2.723	T 6.375 08/15/27	116.224	2.92	T 2.875 11/15/46	92.234	3.31
T 2.125 12/31/22	100.09	1.669	T 1.500 10/31/24	97.02	2.745	T 0.500 08/31/27	88.06	2.926	T 3.000 02/15/47	94.266	3.307
T 1.500 01/15/23	99.274	1.712	T 2.250 10/31/24	98.274	2.733	T 0.500 10/31/27	87.262	2.932	T 3.000 05/15/47	94.282	3.302
T 0.000 01/26/23	98.452	1.586	T 0.750 11/15/24	95.054	2.752	T 2.250 11/15/27	96.186	2.927	T 2.750 08/15/47	90.276	3.285
T 0.125 01/31/23	98.27	1.771	T 2.250 11/15/24	98.244	2.765	T 6.125 11/15/27	116.04	2.928	T 2.750 11/15/47	90.302	3.277
T 1.750 01/31/23	100	1.749	T 7.500 11/15/24	111.176	2.685	T 0.625 11/30/27	88.082	2.937	T 3.000 02/15/48	95.21	3.25
T 2.375 01/31/23	100.142	1.739	T 1.500 11/30/24	96.292	2.766	T 0.625 12/31/27	88.02	2.944	T 3.125 05/15/48	98.04	3.232
T 1.375 02/15/23	99.22	1.795	T 2.125 11/30/24	98.142	2.763	T 0.750 01/31/28	88.166	2.949	T 3.000 08/15/48	95.242	3.241
T 2.000 02/15/23	100.05	1.788	T 1.000 12/15/24	95.204	2.758	T 2.750 02/15/28	99.004	2.937	T 3.375 11/15/48	103.004	3.205
T 7.125 02/15/23	104.006	1.708	T 1.750 12/31/24	97.134	2.775	T 1.125 02/29/28	90.136	2.934	T 3.000 02/15/49	96.142	3.199
T 0.000 02/23/23	98.332	1.713	T 2.250 12/31/24	98.22	2.771	T 1.250 03/31/28	90.284	2.95	T 2.875 05/15/49	94.084	3.194
T 0.125 02/28/23	98.206	1.855	T 1.125 01/15/25	95.25	2.777	T 1.250 04/30/28	90.24	2.955	T 2.250 08/15/49	82.304	3.191
T 1.500 02/28/23	99.232	1.849	T 1.375 01/31/25	96.114	2.779	T 2.875 05/15/28	99.192	2.948	T 2.375 11/15/49	85.126	3.175
T 2.625 02/28/23	100.192	1.853	T 2.500 01/31/25	99.09	2.776	T 1.250 05/31/28	90.2	2.956	T 2.000 02/15/50	78.066	3.189
T 0.500 03/15/23	98.276	1.879	T 1.500 02/15/25	96.184	2.798	T 1.250 06/30/28	90.146	2.965	T 1.250 05/15/50	64.106	3.184
T 0.000 03/23/23	98.282	1.772	T 2.000 02/15/25	97.296	2.786	T 1.000 07/31/28	88.296	2.965	T 1.375 08/15/50	66.176	3.177
T 0.125 03/31/23	98.134	1.954	T 7.625 02/15/25	112.286	2.725	T 2.875 08/15/28	99.162	2.961	T 1.625 11/15/50	71.07	3.166
T 1.500 03/31/23	99.19	1.97	T 1.125 02/28/25	95.186	2.779	T 5.500 08/15/28	114.156	2.944	T 1.875 02/15/51	75.31	3.152
T 2.500 03/31/23	100.146	1.964	T 2.750 02/28/25	99.286	2.787	T 1.125 08/31/28	89.164	2.964	T 2.375 05/15/51	85.156	3.141
T 0.250 04/15/23	98.13	2.016	T 1.750 03/15/25	97.052	2.798	T 1.250 09/30/28	90.026	2.968	T 2.000 08/15/51	78.106	3.137
T 0.000 04/20/23	98.127	1.933	T 0.500 03/31/25	93.222	2.798	T 1.375 10/31/28	90.226	2.966	T 1.875 11/15/51	76.014	3.124

Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield	Bond	Price	Yield
T 0.125 04/30/23	98.056	2.057	T 2.625 03/31/25	99.176	2.787	T 3.125 11/15/28	100.312	2.959	T 2.250 02/15/52	83.134	3.108

3 Dedication Portfolio

3.1 Prompt

Formulate a linear programming model to find the lowest cost bond dedicated portfolio that covers the stream of liabilities. To eliminate the possibility of any interest risk, assume that a 0% reinvestment rate on cash balances carried out from one date to the next. Assume no short selling of bonds is allowed. What is the cost of your portfolio? How does this cost compares with the NPV of the liabilities? What is the composition of the portfolio?

$$\min \quad z_0 + \sum_{i=1}^{N} P_i x_i$$

$$\text{s.t.} \quad \sum_{i=1,\ldots,\ n:\ M_i > t-1} C_i x_i + \sum_{i=1,\ldots,\ n:\ M_i = t} 100 x_i + (1 + r_f) z_{t-1} - z_t = L_t \quad \text{for t} \in \{1, \ldots, 16\}$$

where,

P_i : Price of the bond $i = 1, \ldots, N$

x_i : the amount purchased of bond $i = 1, \ldots, N$

z_t : Excess cashflows after Liability as been paid for period $t = 1, \ldots, 16$

C_i : The coupon payment from bond $i = 1, \ldots, N$

L_t : Liability in period $t = 1, \ldots, 16$

M_i : The maturity year of bond$i = 1, \ldots, N$

r_f : the risk free rate that will be used as the carrying interest for our last period surplus

In a dedication protfolio we are "cash matching". By this we mean to match the sum total of Cash Flows from our pruchased fixed income assets for every and all period Liabilties. We must state that we are not allowed to take a short position in any stock therefore, we are trying minimize the cost of the purchases portfolio to meet our liabilities. We have a surplus term that will account for our excess cash flows after liabiltiies paid for a certain period. These funds may be used for a period in which the inflow from the bonds is not enough to cover liabilties.

NOTE: The objective of this problem does have the risk-free rate as 0% on assumption. Thus the interest earned term is just $(1 + r_f) z_{t-1} = z_{t-1}$

3.2 Code

[15]:
```
'''
Data Manipulation
'''
term_by_maturity = term_structure_df.set_index('MATURITY')
possibilities = term_by_maturity.drop(
    index=[i for i in term_by_maturity.index.to_list() if i > data_prompt.index.to_list()[-1]],
```

```
    columns=['BID', 'ASKED', 'ASKED YIELD']
    )

'''List of bond maturities less than liability maturity'''
date_lists_to_change_to_periods = [
    [i for i in possibilities.index.to_list() if i <= t]
    for t in data_prompt.index.tolist()
    ]
'''Removing the duplicates from each one'''
for i in reversed(range(1,len(date_lists_to_change_to_periods))):
    for j in range(0,len(date_lists_to_change_to_periods[i-1])):
        date_lists_to_change_to_periods[i].remove(date_lists_to_change_to_periods[i-1][j])

for i in range(0,len(date_lists_to_change_to_periods)):
    possibilities.loc[date_lists_to_change_to_periods[i],'period'] = i+1

possibilities['face'] = 100
possibilities['bond#'] = range(1,len(possibilities)+1)
possibilities = possibilities.set_index('bond#')

'''for labeling later'''
dec_var_names = possibilities['ref_data']
```

[16]:
```
'''Getting data ready for the solver'''

'''Exmpty Array'''
cfs = np.zeros((len(possibilities),len(date_lists_to_change_to_periods)))

'''CF Matrix'''
'''Will make function later'''
for i in range(0, len(cfs)):
    for j in range(1, len(cfs[0])+1):
        if possibilities.loc[i+1,'period'] == j and possibilities.loc[i+1,'COUPON'] == 0:
            cfs[i][j-1] = possibilities.loc[i+1,'face']
        elif possibilities.loc[i+1,'period'] == j and possibilities.loc[i+1,'COUPON'] != 0:
            cfs[i][0:j-1] = possibilities.loc[i+1,'COUPON']/2
            cfs[i][j-1] = possibilities.loc[i+1,'face'] + possibilities.loc[i+1,'COUPON']/2

cf_matrix = cfs.tolist()
prices = possibilities['px_ask'].values.tolist()
liabilities = data_prompt['Amount'].values.tolist()
```

[17]:
```
'''Solving for the dedicated portfolio'''

# Making variable list of strings
periods = [i for i in range(0,len(cf_matrix[0])+1)]

# Dictionary of period constraints
period_dict = {}
for i in range(0,len(cf_matrix[0])):
    period_dict['Period {}'.format(i+1)] = dict(zip(dec_var_names,[cf_matrix[j][i] for j in␣
 ↪range(0,len(cf_matrix))]))

objective = dict(zip(dec_var_names, prices))

# Decision Vars
quantity = LpVariable.dict('', dec_var_names, lowBound=0)
excess = LpVariable.dict('ExcessCF', periods, lowBound=0)

# Intializing the Problem
dedication_1 = LpProblem('Dedicated', LpMinimize)

# Objective function
dedication_1 += excess[0]+lpSum([objective[i]*quantity[i] for i in dec_var_names])

# Constraints
```

```
for i in range(0,len(cf_matrix[0])):
    dedication_1 += lpSum([period_dict['Period {}'.format(i+1)][j]*quantity[j] for j in dec_var_names]) +
 ↪excess[i]- excess[i+1] == liabilities[i]

dedication_1.solve()
```

[17]: 1

3.3 Cost and Composition

[18]:
```
'''
Print Solution to dedication_1 portfolio
'''
'''
Print Solution to dedication_1 portfolio with dedication_1ing Limit
'''
ded_df = pd.DataFrame(
    [v.varValue for v in dedication_1.variables() if v.varValue != 0],
    index=[str(v.name[1:9].replace('_', ' ') + v.name[9:].replace('_', '/'))
        if v.name[0] != 'E'
        else str(r'$\text{' + v.name[:-2] + '}' + v.name[-2:] + '$') for v in
 ↪[g for g in dedication_1.variables() if g.varValue != 0]],
    columns=['Quantity']
    )
mature = ['20'+ i[-2:] + '-'+i[-8:-6] + '-'+i[-5:-3]
    if i[0] != 'E'
    else '2100-01-01'
    for i in ded_df.index.tolist()]
bond_s_dis = ded_df.reset_index()
bond_s_dis['m'] = mature
bond_s_dis.set_index('m', inplace=True)
bond_s_dis.sort_index(inplace=True)
bond_s_dis.set_index('index', inplace=True)
bond_s_dis.index.name = ''

md('''

<center> Portfolio Value of ${:.2f} MM  </center> <br>

<center>

{}

'''.format(value(dedication_1.objective), bond_s_dis.to_markdown(colalign =
 ↪("right", "center")))
)
```

[18]: Portfolio Value of $117.77 MM

	Quantity
T 7.250 08/15/22	0.0836671
T 7.125 02/15/23	0.0667
T 6.250 08/15/23	0.0590762
T 2.000 05/31/24	0.0509223
T 1.500 11/30/24	0.0714316
T 7.625 02/15/25	0.0819673
T 6.875 08/15/25	0.0750923
T 0.750 05/31/26	0.107674
T 6.750 08/15/26	0.0780774
T 6.625 02/15/27	0.0507125
T 6.375 08/15/27	0.0423924
T 1.250 05/31/28	0.0637436
T 5.500 08/15/28	0.084142
T 2.875 04/30/29	0.0664559
T 6.125 08/15/29	0.0774112
T 0.625 05/15/30	0.0697819

From our results we can see the prtfolio cost is $ 0.1% $ more than the NPV of the Liabilties. Thus, we will be paying a premium to hold onto the treasuries to meet our liabilties. This premium is $ $ .11MM $ as the NPV is $ 117.66. This is to be expected as we are not actively trading this portfolio and holding all bonds till their maturities.

4 Sensitivity Analysis

Use the linear programming sensitivity analysis information to determine the term structure of interest rates implied by the optimal bond portfolio you found in the previous question. Use a plot to compare these rates with the current term structure of treasury rates you found in the first question.

4.1 Shadow Prices

[19]:
```
'''
Pull sensitivity analysis
---
https://s3.amazonaws.com/assets.datacamp.com/production/course_8835/slides/
 ↪chapter4.pdf
'''
o = [{'name':name, 'shadow price':c.pi} for name, c in dedication_1.constraints.
 ↪items()]
shadow_px = pd.DataFrame(o).set_index(data_prompt.index).drop('name',axis=1)
clean_shadow_px = pd.DataFrame(o).set_index(data_prompt.index.strftime('%m/%d/
 ↪%y')).drop('name',axis=1)
md('''

<center>
```

```
{}

'''.format(clean_shadow_px.to_markdown(colalign = ("right", 'center')))
)
```

[19]:

DateDue	shadow price
12/15/22	0.976444
06/15/23	0.970693
12/15/23	0.952351
06/15/24	0.944203
12/15/24	0.927139
06/15/25	0.906415
12/15/25	0.896101
06/15/26	0.884951
12/15/26	0.870135
06/15/27	0.856877
12/15/27	0.8426
06/15/28	0.834112
12/15/28	0.820296
06/15/29	0.812113
12/15/29	0.795826
06/15/30	0.788304

4.2 Term Structures (Actual vs Implied)

[20]:

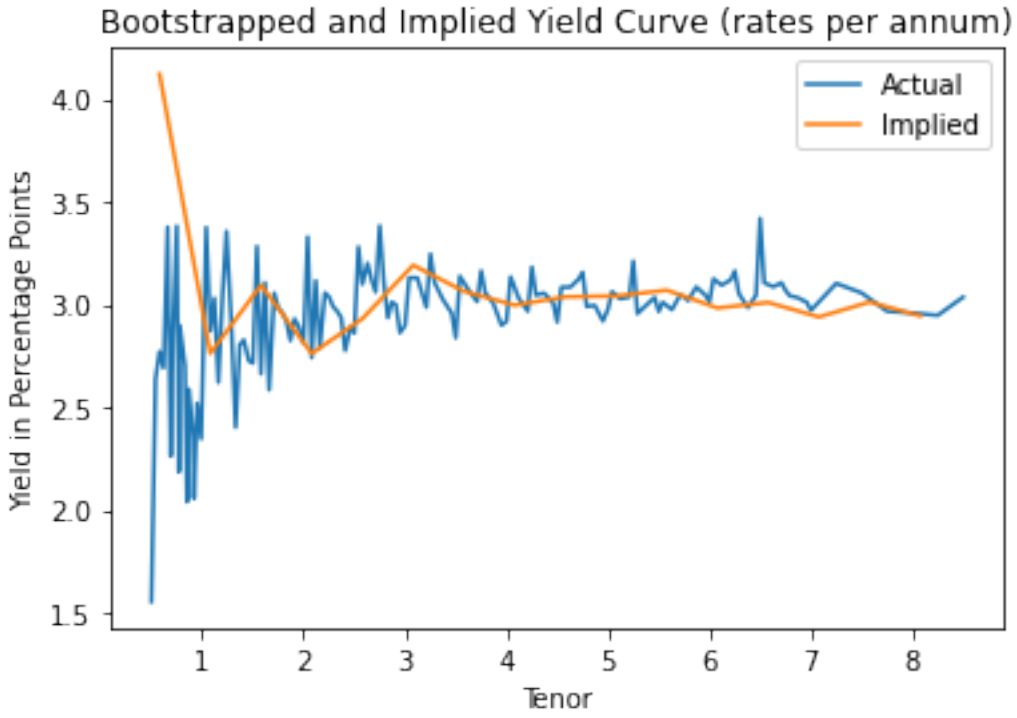
```
'''
Presents implied and actual yield curve as a plot
'''
shadow_ttm = ((shadow_px.index - datetime.datetime.now()) / datetime.
 ↪timedelta(days=365)).to_list()
shadow_factors = shadow_px['shadow price'].to_list()
implied_rates = [-np.log(shadow_factors[i]) / shadow_ttm[i] for i in␣
 ↪range(len(shadow_ttm))]

implied_rates_df = pd.DataFrame(
                        data = [shadow_ttm, implied_rates],
                        index=['ttm','implied_rate']
                    )

implied_rates_df = (implied_rates_df
                        .transpose()
                        .round({'ttm':round_to})
                        .set_index('ttm')
                    )
```

```
plt.plot(rates[0.5:8.5] * 100)
plt.plot(implied_rates_df * 100)
plt.title('Bootstrapped and Implied Yield Curve (rates per annum)')
plt.xlabel('Tenor')
plt.ylabel('Yield in Percentage Points')
plt.legend(['Actual', 'Implied'])
```

[20]: <matplotlib.legend.Legend at 0x261187ddf10>



We can see how the dedication will match the term structure of interest as each can be a function of their cahsflows and if we are essentially investing in the bonds that make up the term structure, will be very close to it but not exactly. We will find the cheapest "path" through the term strauctre, based on our liabilties.

4.3 Implied Rates

[21]:
```
md('''

<center>

{}

'''.format((implied_rates_df*100).to_markdown(colalign = ("right",)))
)
```

[21]:

ttm	implied_rate
0.58	4.12025
1.08	2.76134
1.58	3.0928
2.08	2.76039
2.58	2.93078
3.08	3.19027
3.58	3.06319
4.08	2.99571
4.58	3.0364
5.08	3.04062
5.58	3.06851
6.08	2.98203
6.58	3.00864
7.08	2.93839
7.58	3.01126
8.08	2.94298

5 Immunization Portfolio

Formulate a linear programming model to find the lowest cost bond immunized portfolio that matches the present value, duration, and convexity of a stream of liabilities. Assume that no short rates are allowed. What is the cost of your portfolio? How much would you save by using this immunization strategy instead of the dedication one? Is your portfolio immunized against non-parallel shifts in the term structure? Explain why or why not.

Now, we will approach this problem from an Immunization perspective. This means that we are going to find a new portfolio of bonds that match the Net Present Value, Duration, and Convexity of the stream of Liabilities required, while still minimizing the cost of the actual portfolio. For the purposes of this portfolio, we will assume that short rates aren't allowed.

To achieve this, we first calculated the NPV, Duration, and Convexity of each of the bonds cash flows'. Then, to be able to match them to the NPV, Duration, and Convexity of the Liabilities (previously calculated), we multiply them by the amount of bonds to be bought, and sum them. This is explicitly stated in the Mathematical Formulation.

5.1 Mathematical Formulation

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} P_i x_i \\
\text{s.t.} \quad & \sum_{i=1,\ldots,n} NPV_i * x_i = NPV_{Liabilities} \\
& \sum_{i=1,\ldots,n} Duration_i * x_i = Duration_{Liabilities} \\
& \sum_{i=1,\ldots,n} Convexity_i * x_i = Convexity_{Liabilities} \\
& x_i \geq 0
\end{aligned}
$$

where,

x_i: quantity purchased of bond i;

P_i: ask price of bond i.

5.2 Code

[22]:
```
'''
Aggregates cashflow matrix and ref data for immunization
---
Puts cashflow matrix into a dataframe for merging
merges possible bond ref data with cashflow matrix
cleans resulting dataframe

NOTE: MATH NEEDS WORK HERE BUT WE CAN FIGURE OUT
from here: use ttm and col_num against calculated curve to find appropriate␣
 ↪measure
       pv_factor = exp{-rt} = exp{- () * (ttm)}
'''
cf_df = pd.DataFrame(cf_matrix, index=dec_var_names)

cf_df = pd.merge(                                                    #␣
 ↪Combines possible bonds with cashflow matrix
    left = possibilities,                                            #␣
 ↪possible bonds - SAME DF AS DEDICATION
    right = cf_df,                                                   #␣
 ↪Cashflow matrix - NP ARRAY FROM DEDICATION AS DF FOR MERGING
    how='inner',                                                     #␣
 ↪Catches any missed bonds on merge
    left_on='ref_data',
    right_index=True                                                 #␣
 ↪Casflow df indexed by bond name
)

cf_df = (cf_df
```

```
            .drop(['COUPON','period','face'],axis=1)                    #␣
 ↪Drops unnecessary ref data
            .set_index('ref_data')                                      #␣
 ↪Sets index to bond name
            .round({'ttm':round_to})                                    #␣
 ↪rounds time to maturity to 2 decimal places -- allows use of derived term␣
 ↪structure (indexed by hundredths)
       )
```

[23]:
```
'''
Create Present Value, Duration, and Convexity factors for all possible time␣
 ↪index based on derived rates curve
'''
t = rates.index
r = rates['rate']
npv_factor = np.exp(-r*t)
dur_factor = t*np.exp(-r*(t+1))
con_factor = t*(t+1)*np.exp(-r*(t+2))
```

[24]:
```
'''
Calculates npv, duration, and convexity terms for all bonds consiuder in problem
'''
npvs=[]
durs=[]
cons=[]
for bond in cf_df.index:
    bond_df = cf_df.loc[bond]
    bond_ttm = bond_df.loc['ttm']
    bond_cf_stream = bond_df.loc[0:]
    eo_cfs = bond_cf_stream.idxmax()
    cpn_ttm = [(bond_ttm - 0.5*i).round(round_to) for i in range(eo_cfs+1)]
    bond_cf_ttm = pd.Series(data=bond_df.loc[0:eo_cfs].to_list(),␣
 ↪index=reversed(cpn_ttm))

    bond_npv = sum([bond_cf_ttm.loc[i] * npv_factor.loc[:i].iloc[-1] for i in␣
 ↪bond_cf_ttm.index])
    bond_dur = sum([bond_cf_ttm.loc[i] * dur_factor.loc[:i].iloc[-1] for i in␣
 ↪bond_cf_ttm.index])
    bond_con = sum([bond_cf_ttm.loc[i] * con_factor.loc[:i].iloc[-1] for i in␣
 ↪bond_cf_ttm.index])

    npvs.append(bond_npv)
    durs.append(bond_dur)
    cons.append(bond_con)
```

```
immunization_df = pd.DataFrame([npvs, durs, cons], columns=cf_df.index,␣
 ↪index=['npv','duration','convexity']).transpose()
```

[25]:
```
'''
Solves immunization portfolio
'''
bond_count = LpVariable.dicts('Bonds',dec_var_names,lowBound=0)

immunization = LpProblem('immunization', LpMinimize)

immunization += lpSum([cf_df['px_ask'].loc[i] * bond_count[i] for i in␣
 ↪dec_var_names])
immunization += lpSum([immunization_df['npv'].loc[i] * bond_count[i] for i in␣
 ↪dec_var_names]) == npv
immunization += lpSum([immunization_df['duration'].loc[i] * bond_count[i] for i␣
 ↪in dec_var_names]) == dur
immunization += lpSum([immunization_df['convexity'].loc[i] * bond_count[i] for␣
 ↪i in dec_var_names]) == con

immunization.solve()
```

[25]: 1