Scenario Sampling in Large Games

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Peer Effects with Binary Actions

There are $t = 1, \ldots, T$ players, each of whom choose binary action $Y_t \in \{0, 1\}$.

Preferences:

$$\upsilon_t(\mathbf{y}; \mathbf{X}, \mathbf{U}, \theta) = y_t \left(X_t'\beta + \delta s(\mathbf{y}_{-t}) - U_t \right)$$

with

$$s(\mathbf{y}_{-t}) = \sum_{s \text{ friend of } t} y_s.$$

Agents prefer to take action (e.g., ‘smoke’, or buy a consumer good) when more of their peers do so as well ($\delta \geq 0$).

X_t is a vector of observed agent attributes; U_t a random utility term.

Peer Effects with Binary Actions: Equilibria

$\mathbf{Y} = (Y_1, \ldots, Y_T)'$ is a NE in pure strategies if

$$Y_t = \mathbf{1}\left(X_t'\beta + \delta s\left(\mathbf{Y}_{-t}\right) \geq U_t\right)$$

simultaneously for all $t = 1, \ldots, T$.

When $\delta \geq 0$ there exists, for all $\mathbf{U} \in \mathbb{U}^T$, *at least* one NE in pure strategies (Tarski, 1955).

Policy implications of $\delta > 0$ are profound.

A System of Nonlinear Simultaneous Equations

If $U_t | \mathbf{X} \overset{iid}{\sim} \mathcal{N}(0,1)$,then we have a T simultaneous equations 'probit' model (e.g., Heckman, 1978, Maddala, 1983).

Model exhibits both 'simultaneity' and 'completeness' issues.

Simultaneity: Y_t enters the decision rule for player $s \Rightarrow U_t$ and Y_s covary, since Y_s is a component of $s(\mathbf{Y}_{-t})$, $\Rightarrow U_t$ and $s(\mathbf{Y}_{-t})$ will covary as well.

Incompleteness: There may be multiple NE and the model is silent on which one is selected ($\Rightarrow$ distribution of $\mathbf{Y} | \mathbf{X}$ not fully defined).

$\therefore$ a probit fit of Y_t onto X_t and $s(\mathbf{Y}_{-t})$ does not consistently estimate β and/or δ.

This Paper

We show how to make a simulated **ML estimation** consistent with the game theory,

by making an equilibrium assumption

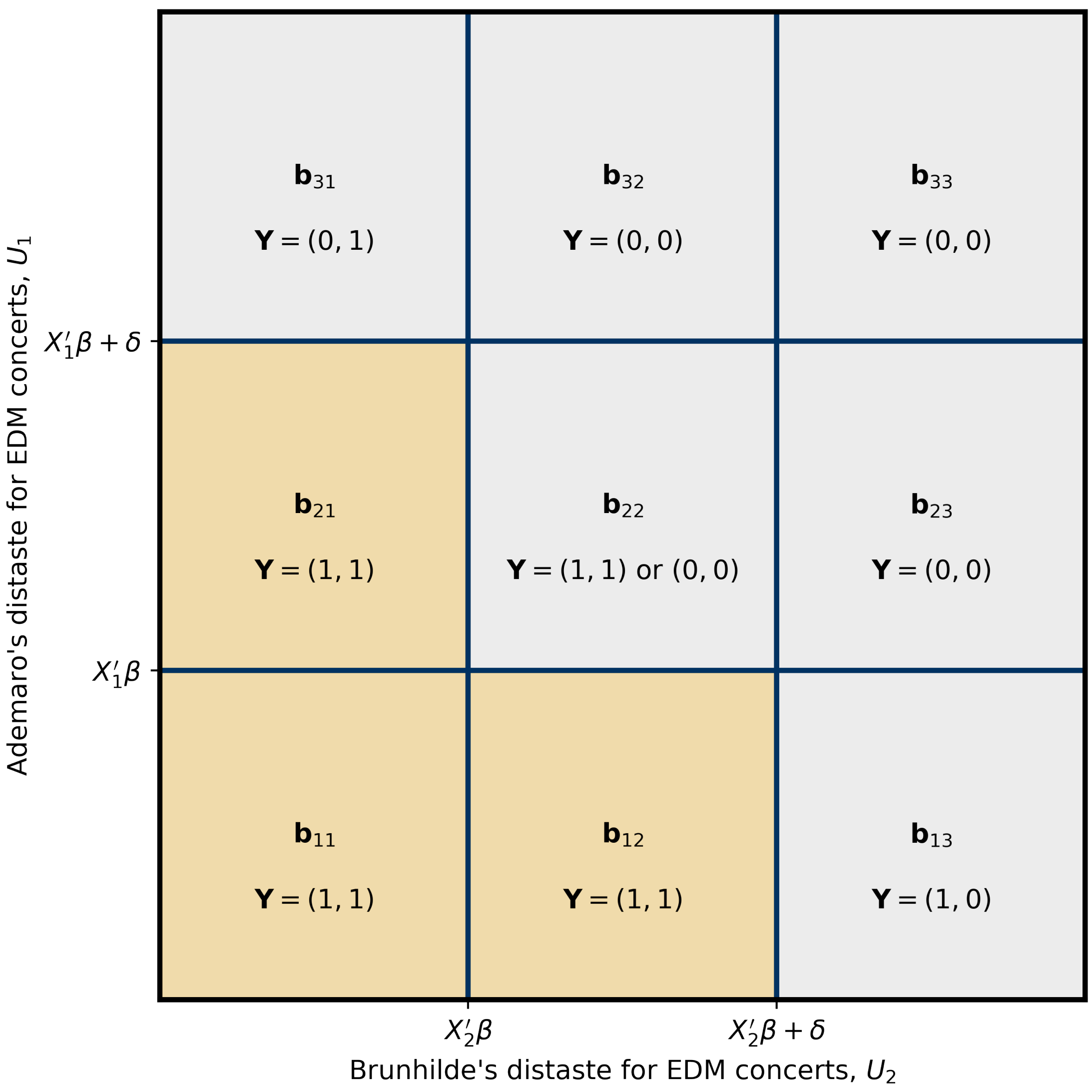
and approximating the log-likelihood function (and its derivatives) by simulation.

Ademaro, Brunhilde and the EDM Concert

Ademaro ($t = 1$) and Brunhilde ($t = 2$) are close friends deciding whether to attend, $y_t \in \{0, 1\}$, a local electronic dance music (EDM) concert.

$$v\left(y_t, y_{-t}; x_t, u_t, \theta\right) = y_t \left(x_t' \beta + \delta y_{-t} - u_t\right). \quad (1)$$

It's more enjoyable to attend the concert with a friend: $\delta > 0$.



$\mathbf{b}_{31}$
$\mathbf{Y} = (0, 1)$
$\mathbf{b}_{32}$
$\mathbf{Y} = (0, 0)$
$\mathbf{b}_{33}$
$\mathbf{Y} = (0, 0)$
$\mathbf{b}_{21}$
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$\mathbf{b}_{22}$
$\mathbf{Y} = (1, 1)$ or $(0, 0)$
$\mathbf{b}_{23}$
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$\mathbf{b}_{11}$
$\mathbf{Y} = (1, 1)$
$\mathbf{b}_{12}$
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$\mathbf{b}_{13}$
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$X_1'\beta + \delta$
$X_1'\beta$
$X_2'\beta$
$X_2'\beta + \delta$
Ademaro's distaste for EDM concerts, U_1
Brunhilde's distaste for EDM concerts, U_2

Scenarios

We can use the utility function and possible peer behaviors to partition the support of U_t in *buckets:*

$$\mathbb{R} = \left(-\infty, X_t'\beta\right] \cup \left(X_t'\beta, X_t'\beta + \delta\right] \cup \left(X_t'\beta + \delta, \infty\right)$$

Bucket boundaries coincide with possible values of the deterministic return to attendance.

Any draw $U_t \sim F_U$ will fall into one, and only one, bucket.

Scenarios

In a similar manner, the support of $\mathbf{U} = (U_1, U_2)'$ can be partitioned into a set of rectangles (e.g., Bresnahan and Reiss, 1991).

$$\mathbb{R}^2 = b^1 \cup b^2 \cup \cdots b^9.$$

We can these rectangles *scenarios* (*szenárien*).

$$\begin{aligned} b^2 &= \left(-\infty, X_1'\beta\right] \times \left(X_2'\beta, X_2'\beta + \delta\right] \\ &= \left(\underline{b}_1^2, \overline{b}_1^2\right] \times \left(\underline{b}_2^2, \overline{b}_2^2\right]. \end{aligned}$$

Equilibrium Selection

For all $\mathbf{U} \in b^2$ Ademaro will go to the EDM concert "no matter what", while Brunhilde is on the fence and only wants to go if Ademaro does.

The NE in this case is $\mathbf{y} = (1, 1)$; they both go.

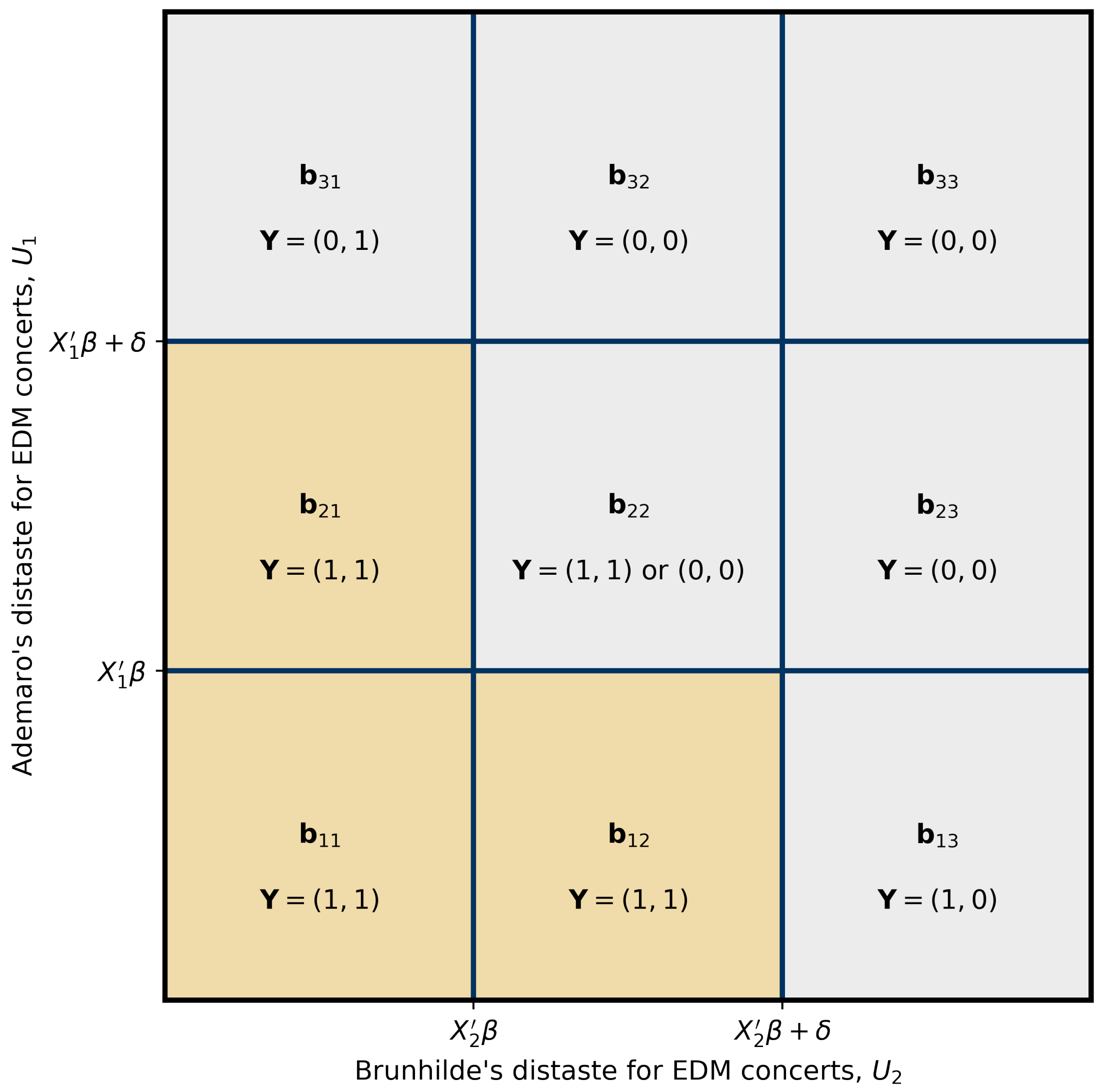
We assume the *minimal* equilibrium is always selected.

Likelihood

With an equilibrium selection assumption in hand, the probability of any game outcome $\mathbf{Y} = \mathbf{y} = (y_1, y_2)'$ corresponds to the probability that $\mathbf{U} = (U_1, U_2)'$ falls into one of the scenarios in which $\mathbf{Y} = \mathbf{y}$ is the (selected) NE.

The probability of observing $\mathbf{Y} = (1,1)'$, for example, corresponds to the ex ante chance that a pair of random utility shocks falls into one of the three cross-hatched scenarios.

For $\mathbf{y} = (1,1)'$ we have $\mathbb{B}_{\mathbf{y}} = \{b_1, b_2, b_4\}$.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Likelihood (continued)

For $\mathbf{y} = (1,1)'$ we integrate $f_{\mathbf{U}}(\mathbf{u}) = f(u_1) f(u_2)$ over the three cross-hatched scenarios.

$$
\begin{aligned}
\Pr(\mathbf{Y} = \mathbf{y} | \, \mathbf{X}; \theta) &= \sum_{b \in \mathbb{B}_{\mathbf{y}}} \int_{\mathbf{u} \in b} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} \\
&= \int_{\mathbf{u} \in b^1} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} + \int_{\mathbf{u} \in b^2} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} + \int_{\mathbf{u} \in b^4} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} \\
&= \sum_{j=1,2,4} \left[F\left(\overline{b}_1^j\right) - F\left(\underline{b}_1^j\right) \right] \left[F\left(\overline{b}_2^j\right) - F\left(\underline{b}_2^j\right) \right] \\
&= F\left(X_1'\beta\right) F\left(X_2'\beta\right) + F\left(X_1'\beta\right) \left[F\left(X_2'\beta + \delta\right) - F\left(X_2'\beta\right) \right] \\
&\quad + \left[F\left(X_1'\beta + \delta\right) - F\left(X_1'\beta\right) \right] F\left(X_2'\beta\right). \qquad (2)
\end{aligned}
$$

Simulated Likelihood (continued)

An accept/reject Monte Carlo ("dartboard") simulation estimate is

$$\widehat{\Pr}\left(\mathbf{Y}=\mathbf{y}|\,\mathbf{X};\theta\right)=\frac{1}{S}\sum_{s=1}^{S}\mathbf{1}\left(B^{(s)}\in\mathbb{B}_{\mathbf{y}}\right). \tag{3}$$

with $B^{(s)}$ now a random draw from $\mathbb{B}$ with distribution $\zeta\left(b;\theta\right)$.

$\widehat{\Pr}\left(\mathbf{Y}=\mathbf{y}|\,\mathbf{X};\theta\right)=\frac{1}{S}\sum_{s=1}^{S}\mathbf{1}\left(\mathbf{y}\text{ is } \textit{the} \text{ NE at } \mathbf{U}^{(s)}\right)$.

Unfortunately in large games we will have $\mathbf{1}\left(B^{(s)}\in\mathbb{B}_{\mathbf{y}}\right)=0$ with very high probability.

Importance Sampling Scenarios

Let $\lambda_{\mathbf{y}}\left(b;\theta\right)$ be a function which assigns probabilities to the elements of $\mathbb{B}_{\mathbf{y}}$.

We require that

1. $\lambda_{\mathbf{y}}\left(b;\theta\right)$ be strictly greater than zero for any $b \in \mathbb{B}_{\mathbf{y}}$ and zero otherwise (i.e., $b \in \mathbb{B} \setminus \mathbb{B}_{\mathbf{y}}$);

2. satisfy the adding up condition $\sum_{b \in \mathbb{B}_{\mathbf{y}}} \lambda_{\mathbf{y}}\left(b;\theta\right) = 1$.

Importance Sampling Scenarios (continued)

Rewrite the likelihood function as an *average* over those scenarios in the set $\mathbb{B}_{\mathbf{y}}$.

Let $\theta^{(0)}$ be some (fixed) value for the parameter; we have that

$$\begin{aligned}
\Pr\left(\mathbf{Y}=\mathbf{y}\middle|\,\mathbf{X};\theta\right) &= \sum_{b\in\mathbb{B}_{\mathbf{y}}} \zeta\left(b;\theta\right) \\
&= \sum_{b\in\mathbb{B}_{\mathbf{y}}} \frac{\zeta\left(b;\theta\right)}{\lambda_{\mathbf{y}}\left(b;\theta^{(0)}\right)} \lambda_{\mathbf{y}}\left(b;\theta^{(0)}\right) \\
&= \mathbb{E}_{\tilde{B}}\left[\frac{\zeta\left(\tilde{B};\mathbf{X},\theta\right)}{\lambda_{\mathbf{y}}\left(\tilde{B};\theta^{(0)}\right)}\right], \qquad (4)
\end{aligned}$$

where $\tilde{B}$ denotes a random draw from $\lambda_{\mathbf{y}}\left(b;\theta^{(0)}\right)$.

Importance Sampling Scenarios (continued)

Let $\tilde{B}^{(s)}$ be $s = 1, \ldots, S$ independent draws from $\lambda_{\mathbf{y}}\left(b; \theta^{(0)}\right)$.

An importance sampling Monte Carlo estimate of the likelihood function is:

$$\widehat{\Pr}\left(\mathbf{Y} = \mathbf{y} \middle| \mathbf{X}; \theta\right) = \frac{1}{S} \sum_{s=1}^{S} \frac{\zeta\left(\tilde{B}^{(s)}; \theta\right)}{\lambda_{\mathbf{y}}\left(\tilde{B}^{(s)}; \theta^{(0)}\right)}. \tag{5}$$

This estimate, because the cardinality of $\mathbb{B}_{\mathbf{y}}$ is finite, is consistent as $S \to \infty$.

All summands in (5) are non-zero.

This Paper

Develops an algorithm for sampling scenarios from $\mathbb{B}_{\mathbf{y}}$.

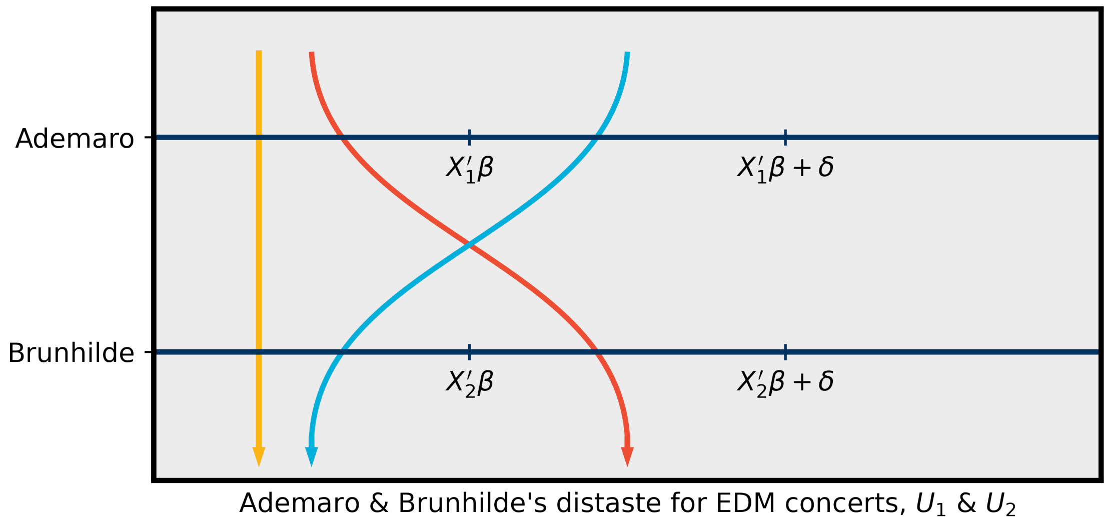
Allows for SML estimation of a class of supermodular games.

The analyst observes $N \geq 1$ games.

The space of action profiles $\mathbb{Y}$ for each game has cardinality 2^T.

Can easily handle examples with T in the tens of thousands.

Key Idea



Ademaro
Brunhilde
$X_1'\beta$
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Ademaro & Brunhilde's distaste for EDM concerts, U_1 & U_2

Key Idea

We proceed by drawing $\mathbf{U}$ such that $\mathbf{U} \in \tilde{B}$, $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ with probability one.

If we draw the elements of $\mathbf{U} = (U_1, \ldots, U_T)'$ *independently*, then $\mathbf{U} \in B$, but $B \in \mathbb{B}_{\mathbf{y}}$ with negligible probability.

Instead we draw $U_1, U_2, \ldots$ *sequentially*.

The support of U_t will depend on the realizations of U_s for $s < t$. We vary the support such that, in the end, $\mathbf{U} \in \tilde{B}$, $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ with probability one.

The logic of NE (and supermodularity) allows us to find the correct support for each draw.

Simulation Algorithm

1. $\mathbf{y}$ is target NE. We want $\mathbf{U} \in \tilde{B}$ with $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$.

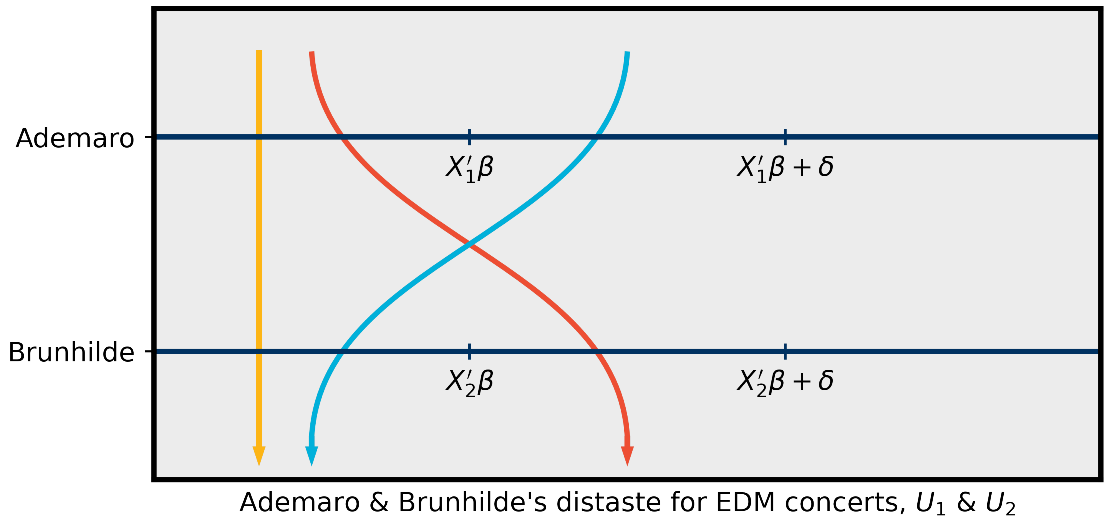
2. Start with $y_t = 0$ cases: draw $U_t \in \left(X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta, \infty\right)$.

3. Go through $y_t = 1$ cases one at a time and

    (a) check how many "defections" would occur if t – contrary to fact – doesn't take action ($\Rightarrow$ new NE with $\tilde{\mathbf{y}} \leq \mathbf{y}$);

    (b) get threshold $\bar{h}_t \in \left(X_t'\beta, X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta\right]$ such that if $U_t \leq \bar{h}_t$ our sequence "stays on track."

Illustration



Random Utility Draws for $y_t = 1$ Cases

Finding the appropriate range restriction on U_t for the $y_t = 1$ cases is key.

1. Since $s(\mathbf{y}_{-t})'\delta \geq 0$, if $U_t \in (-\infty, X_t'\beta]$ the action will be taken (strictly dominant strategy).

2. Also possible that a draw of $U_t \in \left(X_t'\beta, X_t'\beta + s(\mathbf{y}_{-t})'\delta\right]$ is sufficiently low such that agent t would still choose to take the action.

3. If $U_t \in \left(X_t'\beta + s(\mathbf{y}_{-t})'\delta, \infty\right)$ agent t will not take the action (no matter what other agents do).

Random Utility Draws for $y_t = 1$ Cases (continued)

We can conclude that there exists an agent-by-action-specific *threshold* $\bar{h}_t \in \left(X_t'\beta, X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta \right]$, such that

- if $U_t \leq \bar{h}_t$, then it is possible to construct subsequent draws such that, in the end, $\mathbf{U} \in \tilde{B}$ with $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ (as needed),

- whereas if $U_t > \bar{h}_t$, it will not be possible.

Algorithm 1: Scenario sampler

Inputs: $\mathbf{z} = (\mathbf{X}, \mathbf{y})$, θ (i.e., a target pure strategy combination and a utility/payoff function)

1. Initialize $\mathbf{U} = (U_1, \ldots, U_T)' = \underline{0}_T$.

2. For $t = 1, \ldots, T$

   (a) If $y_t = 0$, then sample $U_t \in \left[X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta, \infty\right)$ from the conditional density $\frac{f(u)}{1-F\left(X_t'\beta+s(\mathbf{y}_{-t})'\delta\right)} \stackrel{def}{\equiv} \omega_t f\left(u\right)$.

3. For $t = 1, \ldots, T$

(a) If $y_t = 1$, then

i. determine $\bar{h}_t$ using Threshold($\mathbf{z}$, θ, $\mathbf{U}$, t);

ii. sample $U_t \in \left(-\infty, \bar{h}_t\right]$ from the conditional density $\frac{f(u)}{F(\bar{h}_t)} \overset{def}{\equiv} \omega_t f\left(u\right).$

4. Find $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ such that $\mathbf{U} \in \tilde{B}$.

Outputs: The $T \times 1$ weight vector $\underline{\omega} = (\omega_1, \ldots, \omega_T)'$, the vector of utility shifters $\mathbf{U}$ and a (random) scenario $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$.

Algorithm 2: Threshold finder

Inputs: $\mathbf{z} = (\mathbf{X}, \mathbf{y})$, θ, $\mathbf{U}$, t

1. For $t' = 1, \ldots, T$

   (a) if $y_{t'} = 0$, then set $\tilde{U}_{t'} = U_{t'}$;

   (b) if $y_{t'} = 1$, then

      i. if $t' < t$, then set $\tilde{U}_{t'} = U_{t'}$ ($\bar{h}_{t'}$ already found)

      ii. if $t' > t$, then set $\tilde{U}_{t'} = X_t'\beta - 1$ ($\bar{h}_{t'}$ not already found; force $\tilde{Y}_{t'} = 1$)

2. Set $\tilde{U}_t = X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta + 1$ (ensures that player t *will not* want to choose $\tilde{Y}_t = 1$ in Step 3 below)

3. Find the minimal NE, $\tilde{\mathbf{Y}}$, associated with $\tilde{\mathrm{U}}$. Set $\bar{h}_t = X_t'\beta + s\left(\tilde{\mathbf{Y}}_{-t}\right)'\delta$

Output: The threshold, $\bar{h}_t$.

Threshold finder (intuition)

By forcing player t to not take the action (Step 2), some players – for whom we have already simulated utility shocks ($t' < t$) – will choose to also now not take action (even thought $y_{t'} = 1$). This induces a new NE (step 3) with $\tilde{\mathbf{Y}} \leq \mathbf{y}$ (supermodularity).

$\bar{h}_t$ is the maximal value of U_t such that the "defections" in $\tilde{\mathbf{Y}}$ don't occur,

If $U_t \in \left(-\infty, \bar{h}_t\right]$, then player t will take the action as desired, and those players $t' < t$ which "defected" in $\tilde{\mathbf{Y}}$ will also take the action.

OTH, if $U_t > \bar{h}_t$, then player t not taking the action, and some subset of players $t' < t$ also not taking action, yields a minimal NE ($\tilde{\mathbf{Y}}$) below the target.

Monte Carlo Experiments, peer effects

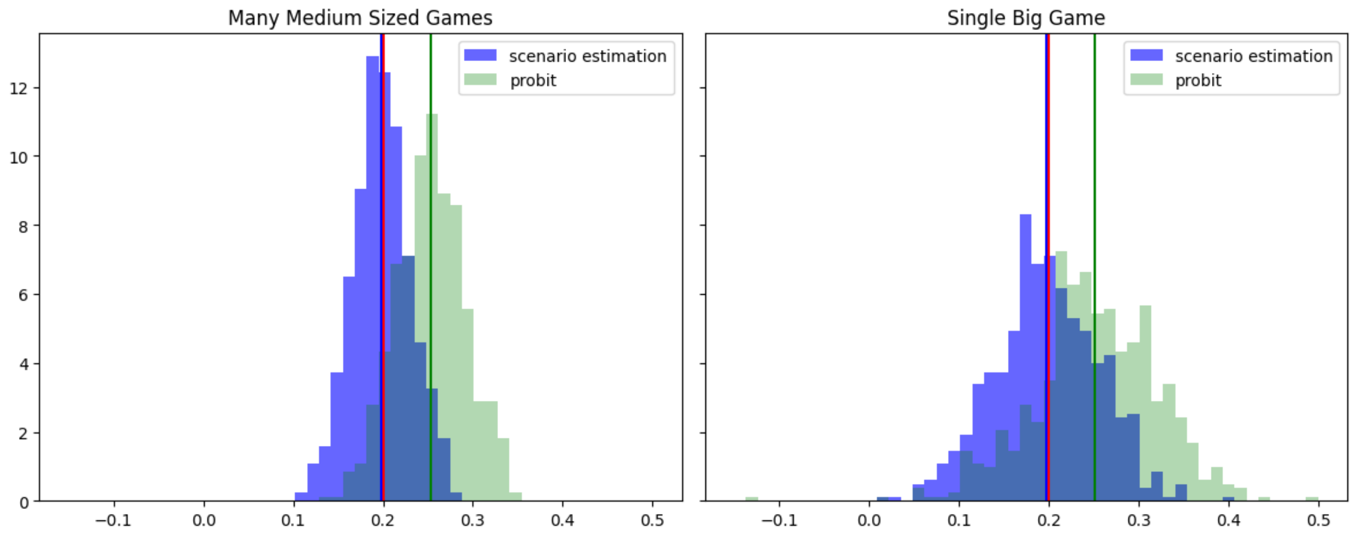
$$\upsilon_t(\mathbf{y};\mathbf{x},\mathbf{u},\theta) \stackrel{def}{\equiv} y_t\left(x_t'\beta + \delta\left(\sum_{s \text{ is friend of } t} y_s\right) - u_t\right).$$

Friendships generated by a random geometric network. Four covariates, two discrete, two continuous.

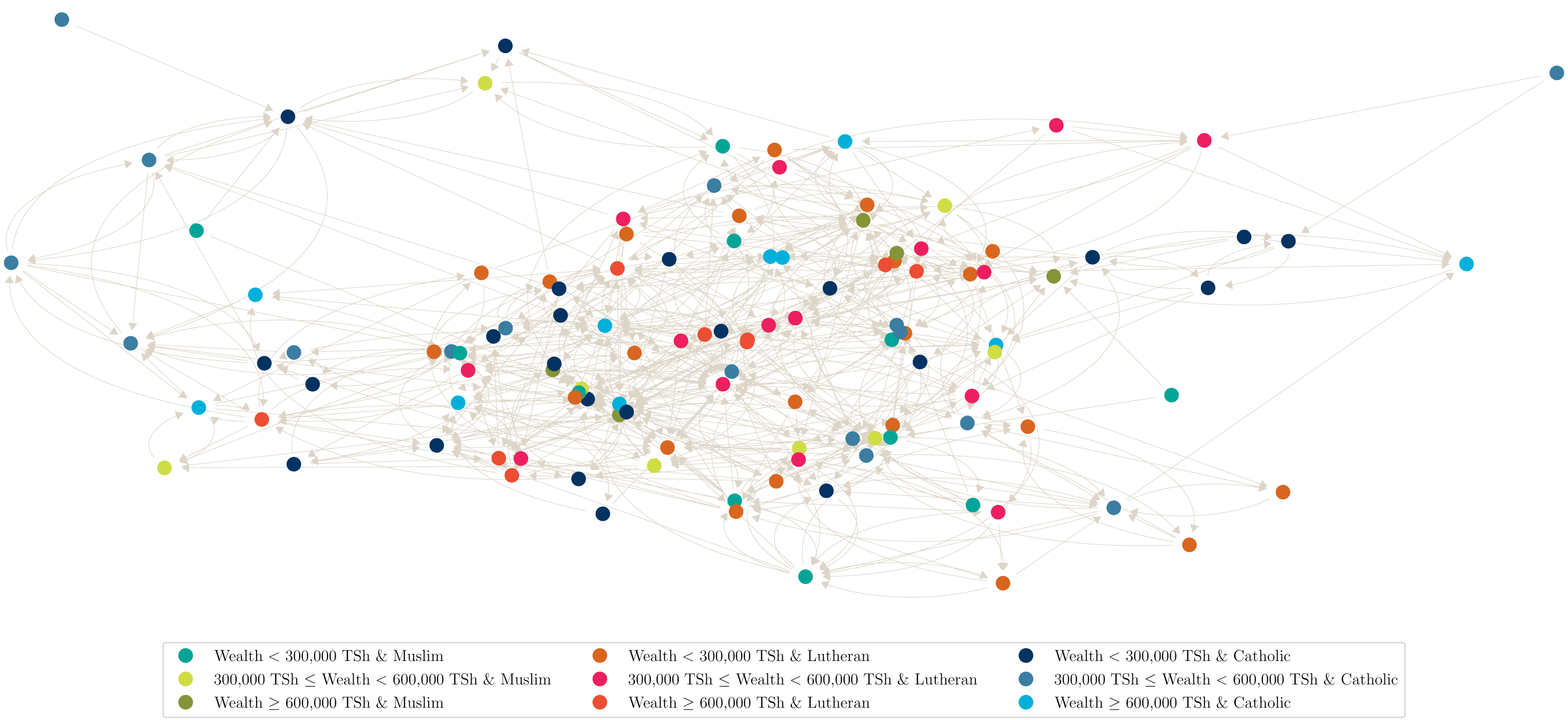
Two cases: 1. 2000 agents in 100 distinct friendship networks; 2. 500 agents in a single friendship network.

$\delta_0 = 0.20$	mean	0.198	0.198
	std. dev.	0.032	0.058
	coverage	0.954	0.885

Monte Carlo Experiments (continued)



Application: Nyakatoke



“Regressor"	Probit	SMLE ($S=1$)	SMLE ($S=10$)	SMLE ($S=100$)
Support	0.183	0.166	0.127	0.146
($\sum_{r=1}^{T} y_{rt}y_{rs}$)	(0.031)	(0.015)	(0.014)	(0.031)
Parents, children	1.485	1.511	1.509	1.510)
and siblings	(0.116)	(0.113)	(0.113)	(0.117)
Nephews, nieces, uncles, aunts,	0.919	0.897	0.921	0.929)
cousins, grandparents, grandchildren	(0.128)	(0.127)	(0.127)	(0.128)
Other blood relative	0.697	0.691	0.714	0.702
	(0.102)	(0.100)	(0.100)	(0.101)
Distance (km)	-1.375	-1.396	-1.420	-1.394
	(0.100)	(0.099)	(0.099)	(0.101)
Same religion (Catholic,	0.169	0.156	0.168	0.172
lutheran or Muslim)	(0.049)	(0.048)	(0.048)	(0.048)
Same clan	0.008	0.018	0.006	0.011
	(0.079)	(0.078)	(0.078)	(0.079)
Both t and s household heads	-0.097	-0.082	-0.100	-0.097
have completed primary school	(0.156)	(0.155)	(0.156)	(0.156)
Activity overlap (0 to 1)	-0.012	-0.013	-0.011	-0.011
	(0.015)	(0.014)	(0.014)	(0.015)
Absolute household head age	-0.082	-0.080	-0.084	-0.081
difference (decades)	(0.021)	(0.021)	(0.021)	(0.021)
Absolute wealth difference	-0.025	-0.024	-0.026	-0.025
(000,000s of Tanzanian Shillings)	(0.008)	(0.008)	(0.008)	(0.008)

Recap

1. Our approach sampled **ML estimation** feasible in supermodular games with many agents (T) and/or many actions (M).

2. Because we can also construct score estimates, we can fit high dimensional models (i.e., don't need to rely on grid searches).

3. Opens up a wide variety of large games to formal/structural empirical analysis.