When Does Wealth Inequality Matter for Asset Pricing?

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Caveat: This is very preliminary work

- Work in progress: today more focus on methodology than in economics
- General question: does market incompleteness affect asset prices?
  - Under certain particular circumstances, the answer is no (Krueger and Lusting, JET 2009).
- In general, it is unknown.

The challenge

- HALT: Heterogeneous-Agent Lucas Tree model.
  - Hard to solve, the income-wealth density is an infinite-dimensional stochastic state.
- Proposal: employ the extension of Krusell-Smith to neural networks by Fernández-Villaverde, Hurtado, and Nuno (ECTA, 2023).

Warm-up: representative agent

The Lucas tree

- There is one unit of a tree which generates dividends Y_t each time period.
  - The dividend follows a diffusion process

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dW_t^Y$$

  - The tree trades at price q_t.

Households and equilibrium

- The representative agent has preferences

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(C_t) dt,$$

  - Her budget constraint is

$$q_t \dot{K}_t = Z + Y_t K_t - C_t,$$

    where Z is a constant endowment.

- In equilibrium

$$K_t = 1, \qquad C_t = Z + Y_t.$$

Hamilton-Jacobi-Bellman (HJB) equation

- Let's work with asset holdings K as a state variable (rather than wealth qK).
- The equilibrium asset price will be a function of Y_t,

$$q_t = Q(Y_t)$$

  for a function Q that is to be determined.
  - Given that Y_t follow a diffusion process, so will q_t.
- The representative consumer solves

$$\rho V(K,Y) = \max_C \; u(C) + V_K(K,Y)(Z + YK - C)/Q(Y) + V_Y(K,Y)\mu(Y) + \frac{1}{2} V_{YY}(K,Y)\sigma^2(Y).$$

Solution

$$q_t = Q(Y_t) = \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \frac{u'(Z + Y_s)}{u'(Z + Y_t)} Y_s ds.$$

Asset pricing with heterogeneous agents

Heterogeneous agents

- Assume instead a continuum of individuals with preferences

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_{it}) dt,$$

  - Budget constraint:

$$q_t \dot{a}_{it} = z_{it} + Y_t a_{it} - c_{it}.$$

  - Here z_{it} is idiosyncratic income risk $\rightarrow$ Poisson process $z_{it} \in \{z_1, z_2\}$ with intensities λ_1 and λ_2.
  - No-short-selling constraint

$$a_{it} \geq 0.$$

Kolmogorov Forward (KF) equation

- Let $f_t(a, z)$ be the cross-sectional distribution. It follows the KF equation:

$$\partial_t f = \mathcal{A}^* f \equiv -\partial_a (s_t(a, z) f_t(a, z)) - \lambda_z f_t(a, z) + \lambda_{z'} f_t(a, z'),$$

where

$$s_t(a, z) \equiv \frac{z_{it} + Y_t a_{it} - c_{it}}{q_t}.$$

Asset prices

- The equilibrium asset price will now be given by

$$q_t = Q(Y_t, f_t),$$

that is, f will be a state variable in individual's Bellman equation.

Hamilton-Jacobi-Bellman equation

$$
\begin{aligned}
\rho V(a,z;Y,f) = \max_c \; & u(c) + \partial_a V(a,z;Y,f)(z+Ya-c)/Q(Y,f) + \lambda_z(V(a,z';Y,f) - V(a,z;Y,f)) \\
& + \mu(Y)\partial_Y V(a,z;Y,f) + \frac{1}{2}\sigma^2(Y)\partial_{YY} V(a,z;Y,f) \\
& + \int \frac{\delta V(a,z;Y,f)}{\delta(\tilde{a},\tilde{z})} \mathcal{A}^* f_t(\tilde{a},\tilde{z}) d\tilde{a} d\tilde{z},
\end{aligned}
$$

where $\delta V/\delta f$ is the functional derivative of V with respect to f.

Bounded Rationality

- The problem above is not only impossible to compute but also crazy:
  - it assumes that individuals keep track of the entire cross-sectional distribution. So it seems natural to instead assume some bounded rationality.
- A natural assumption is that individuals form beliefs about prices directly:

$$dq_t = \mu^q(q_t, Y_t)dt + \sigma^q(q_t, Y_t)dW_t$$

for some unknown functions μ^q and σ^q that can potentially be very complicated and non-linear.

HJB under bounded rationality

- Under this assumption the HJB simplifies to

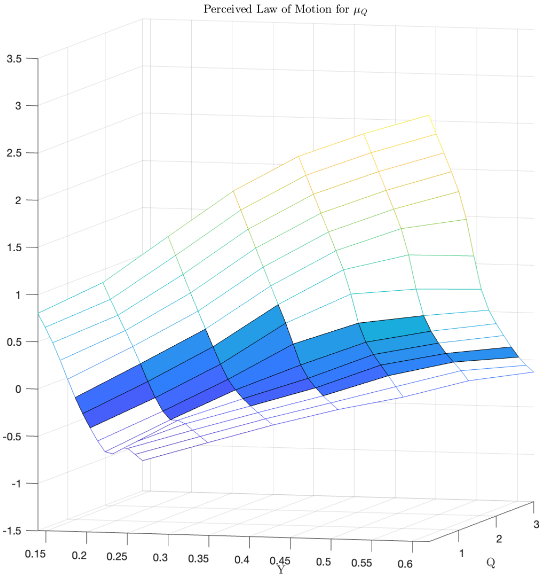
$$
\begin{aligned}
\rho V(a,z;q,Y) = \max_c \; & u(c) + \partial_a V(a,z;q,Y)(z + Ya - c)/q + \lambda_z(V(a,z';q,Y) - V(a,z;q,Y)) \\
& + \mu(Y)\partial_Y V(a,z;q,Y) + \frac{1}{2}\sigma^2(Y)\partial_{YY} V(a,z;q,Y) \\
& + \mu^q(q,Y)\,\partial_q V(a,z;q,Y) + \frac{1}{2}\sigma^q(q,Y)^2\,\partial_{qq} V(a,z;q,Y) \\
& + \sigma(Y)\sigma^q(q,Y)\partial_{qY} V(a,z;q,Y).
\end{aligned}
$$

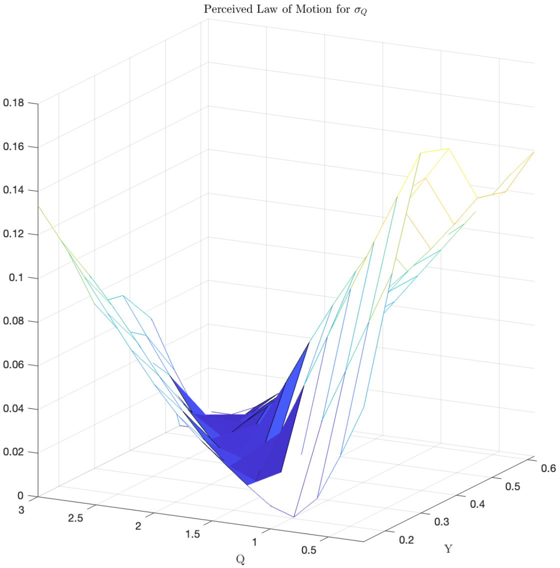
Enter neural networks

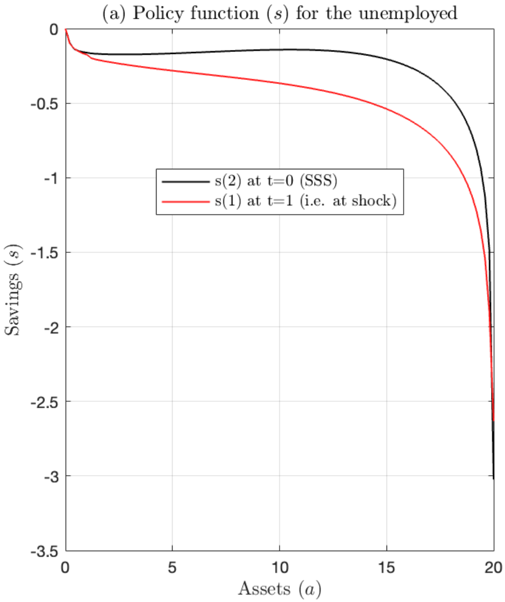
- We can approximate the unknown functions $\mu^q(q, Y)$ and $\sigma^q(q, Y)$ using a neural network.
- This is a low-dimensional problem, so other function approximation schemes could be used as well.
- However, neural networks are convenient for two reasons:
  - Good extrapolation properties (in contrast to Chebyshev, for instance)
  - Richness in software tools

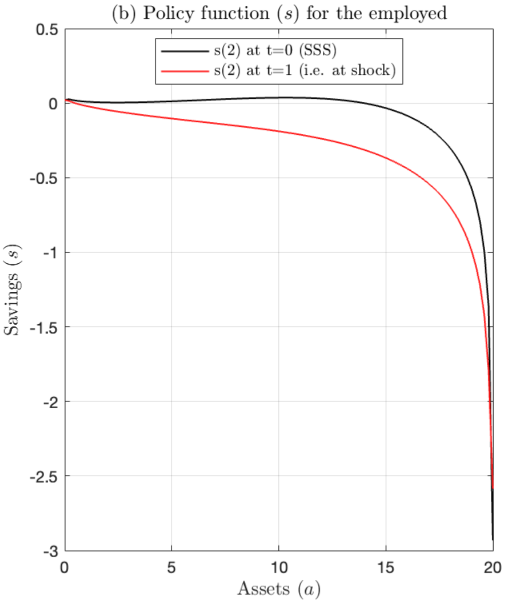
Algorithm

1. Make an initial guess of the perceived law of motion $\mu^q\left(q, Y; \boldsymbol{\theta^0}\right)$ and $\sigma^q\left(q, Y; \phi^{\mathbf{0}}\right)$, $n = 0$.
2. Then, for $n := n + 1$:
   1. Solve the HJBusing $\mu^q\left(q, Y; \boldsymbol{\theta^{n-1}}\right)$ and $\sigma^q\left(q, Y; \phi^{\mathbf{n-1}}\right)$
   2. Simulate samples of Y_t and q_t
   3. Train two neural networks, one on $\mu^q\left(q, Y; \boldsymbol{\theta^n}\right)$ and the other on $\sigma^q\left(q, Y; \phi^{\mathbf{n}}\right)$
   4. Check the norms $\left\|\mu^q\left(q, Y; \boldsymbol{\theta^n}\right) - \mu^q\left(q, Y; \boldsymbol{\theta^{n-1}}\right)\right\|$ and $\left\|\sigma^q\left(q, Y; \boldsymbol{\theta^n}\right) - \sigma^q\left(q, Y; \boldsymbol{\theta^{n-1}}\right)\right\|$


Perceived Law of Motion for μ_Q
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
Y
1
2
3
Q


Perceived Law of Motion for σ_Q
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
3
2.5
2
1.5
1
0.5
Q
0.2
0.3
0.4
0.5
0.6
Y


(a) Policy function (s) for the unemployed
s(2) at t=0 (SSS)
s(1) at t=1 (i.e. at shock)
Savings (s)
Assets (a)
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
0
5
10
15
20


(b) Policy function (s) for the employed
s(2) at t=0 (SSS)
s(2) at t=1 (i.e. at shock)
Savings (s)
Assets (a)
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
5
10
15
20

Next steps

- Compare HA and RA models.
- Extend the number of assets to risky bonds
- Consider realistic asset allocation rules
- Stay tuned!