Finite Dependence and Unobserved Heterogeneity

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DSE at Lausanne

August 2023

Introduction

Two outstanding questions

- In the previous lecture I said:
  1. Let's separate inference from counterfactuals but remain consistent to the data generating process.
  2. assume conditional independence.
- Then we can:
  1. derive necessary and sufficient conditions for identification.
  2. finesse computing the equilibrium fixed point within estimation.
- But what happens to CCP estimators when:
  1. choice probabilities are poorly estimated by their sample counterparts (be they relative frequencies or nonparametrically estimated conditional expectation functions characterizing the CCPs)?
  2. the conditional independence function fails because some of the state variables in the model are unobserved in the data?

- With respect to imprecisely estimated CCPs arising from data sparsity:
  - The CCPs (and transition probability estimates) are sufficient statistics.
  - Only the precision of the structural parameter estimates count.
  - Improving the precision of the structural estimates comes at the cost of making functional form assumptions.
  - There are multiple paths to construct the dynamic correction terms used in computing the conditional value functions.
  - Nevertheless the estimation of dynamic discrete choice models is complicated by the calculation of payoffs that occur only very infrequently and/or in the distant future.
- If transition matrices exhibit a property called *finite dependence* this drastically reduces the number of CCPs used to estimate the structural parameters.
- Finite dependence is the topic of the first half of this lecture.

- With respect to the problem of unobserved heterogeneity:
  - Identification can only be achieved by imposing further functional form assumptions.
  - Ignoring unobserved heterogeneity is fundamentally a misspecification problem and therefore applies to maximum likelihood too.
  - Maximum likelihood estimates correcting for unobserved heterogeneity typically require multiple integration over all future unobserved states.
- Cheap CCP estimators can be constructed and implemented using the EM algorithm, the topic of the second half of this lecture.
  - I provide an example of how it works in practice
- The lecture draws heavily from joint work with Peter Arcidiacono:
  - "Conditional choice probability estimation of dynamic discrete choice models with unobserved heterogeneity," *Econometrica*, 2011 pp. 1823 - 1867.
  - "Nonstationary dynamic models with finite dependence," *Quantitative Economics*, 2019 pp. 853 - 890.

Finite Dependence
Short panels, when S < T

- Suppose the sampling period, S, falls short of the time horizon T.
- Rather than express $u_{jt}(x)$ as a sum to T, we express u_{jt} as a sum to S and then use the value function at $S+1$:

$$
\begin{aligned}
u_{jt}(x) &= u_{1t}(x) + \psi_{1t}(x) - \psi_{jt}(x) \qquad (1) \\
&+ \sum_{\tau=t+1}^{S} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \left\{ \begin{array}{l} [u_{1\tau}(x_\tau) + \psi_{1t}(x_\tau)] \times \\ [\kappa_{\tau-1}(x_\tau|x,1) - \kappa_{\tau-1}(x_\tau|x,j)] \end{array} \right\} \\
&+ \sum_{x_{S+1}=1}^{X} \beta^{S-t} V_{S+1}(x_{S+1})] \left[\kappa(x_{S+1}|x,1) - \kappa(x_{S+1}|x,j)\right]
\end{aligned}
$$

- Since the CCPs and state transitions are identified up to S, the utility flows would be exactly identified if $V_{S+1}(x)$ was known.
- However $V_{S+1}(x)$ is endogenous and depends on CCPs that occur after the sample ends.

Introduction

Conditional value function representation

- For all $\rho \leq T - t$ and $\tau = t+1, \ldots, t+\rho$, define any weights $\omega_{k\tau}(t, x_\tau, j)$ satisfying:

$$\left|\omega_{k\tau}(t, x_\tau, j)\right| < \infty \text{ and } \sum_{k=1}^{J} \omega_{k\tau}(t, x_\tau, j) = 1$$

.

- We showed $v_{jt}(x_t) =$

$$\begin{aligned} & u_{jt}(x_t) + \sum_{x=1}^{X} \beta^{\rho+1} V_{t+\rho+1}(x) \kappa_{t+\rho+1}(x | t, x_t, j) \\ & + \sum_{\tau=t+1}^{t+\rho} \sum_{k=1}^{J} \sum_{x_\tau=1}^{X} \beta^{\tau-t} \left[\begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right] \omega_{k\tau}(t, x_\tau, j) \kappa_\tau(x_\tau | t, x_t, j) \end{aligned} \tag{2}$$

where $\kappa_{t+1}(x_{t+1} | t, x_t, j) \equiv f_{jt}(x_{t+1} | x_t)$ and:

$$\kappa_{\tau+1}(x_{\tau+1} | t, x_t, j) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}\left(t, x_\tau, j\right) f_{k\tau}(x_{\tau+1} | x_\tau) \kappa_\tau(x_\tau | t, x_t, j)$$

- The pair of choices $\{i, j\}$ exhibits *ρ-period dependence* at (t, x_t) if there exist a pair of sequences of decision weights:

$$\{\omega_{k\tau}(t, x_\tau, i)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)} \text{ and } \{\omega_{k\tau}(t, x_\tau, j)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$$

  such that for all $x_{t+\rho+1} \in \{1, \ldots, X\}$:

$$\kappa_{t+\rho+1}(x_{t+\rho+1} | t, x_t, i) = \kappa_{t+\rho+1}(x_{t+\rho+1} | t, x_t, j)$$

- Finite dependence:
  1. trivially holds for $\rho = T - t$ when $T < \infty$, but only merits attention when $\rho < T - t$.
  2. extends to games by conditioning on the player as well.
  3. can be tested without specifying utilities.
  4. might hold for some choice pairs but not others, and for certain states but not others.
  5. could be defined for mixed choices to start the sequence, not just deterministic moves.

Finite Dependence

Representing utility

- Intuitively, ρ period dependence holds when two sequences of weighted choices leading off from different initial choices generate the same distribution of state variables $\rho + 1$ periods later.
- Most empirical applications have the finite dependence property.
- If there is finite dependence for (t, x_t, i, j), then:

$$
\begin{aligned}
& u_{jt}(x_t) + \psi_j[p_t(x_t)] - u_{it}(x_t) - \psi_i[p_t(x_t)] = \\
& \sum_{(k,\tau,x_\tau)=(1,t+1,1)}^{(J,t+\rho,X)} \beta^{\tau-t} \left\{ \begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right\} \left[\begin{array}{l} \omega_{k\tau}(t, x_\tau, i)\kappa_\tau(x_\tau | t, x_t, i) \\ -\omega_{k\tau}(t, x_\tau, j)\kappa_\tau(x_\tau | t, x_t, j) \end{array} \right]
\end{aligned} \tag{3}
$$

- To derive this equation:
  - replace $v_{jt}(x)$ with $V_t(x) - \psi_j[p_t(x_t)]$ in (2)
  - form an analogous equation for i
  - difference the two resulting equations and note the $V_t(x)$ terms cancel.

- To apply finite dependence in estimation:
  - estimate the transition terms $f_{jt}(x_{t+1}|x)$
  - form $\kappa_\tau(x_\tau|t, x_t, j)$ terms with transitions and weights $\omega_{k\tau}(t, x_\tau, j)$
  - estimate the CCPs $p_t(x_t)$
  - plug CCPs into $\psi_j[p_t(x_t)]$ terms
  - estimate utility $u_{jt}(x_t)$ terms using (3) using Minimum Distance.
- As before, if $u_{k\tau}(x_\tau)$ is linear in x_τ for all (τ, x_τ, k), then this is a linear estimation problem with a quadratic objective function and a closed form solution.

Simple Examples of Finite Dependence

Terminal choices

- Terminal choices and renewal choices are widely assumed in econometric applications of dynamic optimization problems and games.
- A *terminal choice* ends the evolution of the state variable with an *absorbing state* that is independent of the current state.
- If the first choice denotes a terminal choice, then:

$$f_{1t}(x_{t+1}|x) \equiv f_{1t}(x_{t+1})$$

for all $(t, x) \in \mathbb{T} \times \mathbb{X}$ and hence:

$$\sum\nolimits_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) = f_{1,t+1}(x_{t+2})$$

- Setting $\omega_{k\tau}(t, x, i) = 0$ for all (x, i) and $k \neq 1$, Equation (3) implies:

$$\begin{aligned} & u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\ = \ & \sum\nolimits_{x_{t+1}=1}^{X} \beta \left\{ u_{1,t+1}(x_{t+1}) + \psi_1\left[p_{t+1}(x_{t+1})\right]\right\} f_{jt}(x_{t+1}|x_t) \end{aligned}$$

- Similarly a *renewal choice* yields a probability distribution of the state variable next period that does not depend on the current state.
- If the first choice is a renewal choice, then for all $j \in \{1, \dots, J\}$:

$$\begin{aligned}\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) f_{jt}(x_{t+1}|x_t) &= \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2}) \sum_{x_{t+1}=1}^{X} f_{jt}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2}) \qquad (4)\end{aligned}$$

- In this case Equation (3) implies:

$$\begin{aligned}& u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\ = \quad & \sum_{x=1}^{X} \beta \{u_{1,t+1}(x) + \psi_1 [p_{t+1}(x)]\} [f_{jt}(x|x_t) - f_{1t}(x|x_t)]\end{aligned}$$

- Consider a simple search model in which all jobs are temporary, lasting only one period.
- Each period $t \in \{1, \dots, T\}$ an individual may:
  - stay home by setting $d_{1t} = 1$
  - or apply for temporary employment setting $d_{2t} = 1$.
- Job applicants are successful with probability λ_t, time varying job offer arrival rates.
- Experience $x \in \{1, \dots, X\}$ increases by one unit with each period of work, up to X, and does not depreciate.
- Current utility $u_{jt}(x_t)$ depends on choices, time and experience.

- For all (t, x_t) with $x_t < X$ set:
  - $d_{1t} = 1$ (stay home) and then "apply for employment" with weight:

$$\begin{aligned} \lambda_t/\lambda_{t+1} &= \omega_{k=2,t+1}(t, x_t, i=1) \\ &= 1 - \omega_{k=1,t+1}(t, x_t, i=1) \end{aligned}$$

  - $d_{2t} = 1$ (seek work) and then stay home:

$$\omega_{k=1,t+1}\left(t, x_t, j=2\right) = \omega_{k=1,t+1}\left(t, x_t+1, j=2\right) = 1$$

  - to attain one-period dependence since:

$$\kappa_3(x_{t+3}|t, x_t, 1) = \kappa_3(x_{t+3}|t, x_t, 2) = \begin{cases} 1-\lambda_t \text{ for } x_{t+3} = x_t \\ \lambda_t \text{ for } x_{t+3} = x_t + 1 \end{cases}$$

- Note that if $\lambda_t > \lambda_{t+1}$ then $\omega_{2,t+1}(t, x_t, 1) > 1$ and $\omega_{1,t+1}(t, x_t, 1) = 1 - \lambda_t/\lambda_{t+1} < 0$.

Distributional Assumptions about the Unobserved Variables

Motivating example: Rust's (1987) bus engine revisited

- What if we want to relax assumption that the distribution of unobserved variables is known?
- Then we must place identifying assumptions on the way systematic payoffs are parameterized.
- Recall Mr. Zurcher decides whether to replace the existing engine ($d_{1t} = 1$), or keep it for at least one more period ($d_{2t} = 1$).
- Bus mileage advances 1 unit ($x_{t+1} = x_t + 1$) if Zurcher keeps the engine ($d_{2t} = 1$) and is set to zero otherwise($x_{t+1} = 0$ if $d_{1t} = 1$).
- Transitory iid choice-specific shocks, ϵ_{jt} are Type 1 Extreme value.
- Zurcher sequentially maximizes expected discounted sum of payoffs:

$$E\left\{\sum_{t=1}^{\infty}\beta^{t-1}\left[d_{2t}(\theta_1 x_t + \theta_2 s + \epsilon_{2t}) + d_{1t}\epsilon_{1t}\right]\right\}$$

Motivating Example

ML Estimation when CCP's are known (infeasible)

- To show how the EM algorithm helps, consider the infeasible case where $s \in \{1, \ldots, S\}$ is unobserved but $p(x, s)$ is known.
- Let π_s denote population probability of being in unobserved state s.
- Supposing β is known the ML estimator for this "easier" problem is:

$$\{\hat{\theta}, \hat{\pi}\} = \arg\max_{\theta, \pi} \sum_{n=1}^{N} \ln \left[\sum_{s=1}^{S} \pi_s \prod_{t=1}^{T} l(d_{nt} | x_{nt}, s, p, \theta) \right]$$

where $p \equiv p(x, s)$ is a string of probabilities assigned/estimated for each (x, s) and $l(d_{nt} | x_{nt}, s_n, p, \theta)$ is derived from our representation of the conditional valuation functions and takes the form:

$$\frac{d_{1nt} + d_{2nt} \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln [p(0, s)] - \beta \ln [p(x_{nt} + 1, s)]}{1 + \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln [p(0, s)] - \beta \ln [p(x_{nt} + 1, s)])}$$

- Maximizing over the sum of a log of summed products is computationally burdensome.

- The EM algorithm is a computationally attractive alternative to directly maximizing the likelihood.
- Denote by $d_n \equiv (d_{n1}, \ldots, d_{nT})$ and $x_n \equiv (x_{n1}, \ldots, x_{nT})$ the full sequence of choices and mileages observed in the data for bus n.
- At the m^{th} iteration:

$$
\begin{aligned}
q_{ns}^{(m+1)} &= \Pr\left\{ s \,\middle|\, d_n, x_{n,} \theta^{(m)}, \pi_s^{(m)}, p \right\} \\
&= \frac{\pi_s^{(m)} \prod_{t=1}^{T} l(d_{nt}|x_{nt}, s, p, \theta^{(m)})}{\sum_{s'=1}^{S} \pi_{s'}^{(m)} \prod_{t=1}^{T} l(d_{nt}|x_{nt}, s', p, \theta^{(m)})}
\end{aligned}
$$

$$
\pi_s^{(m+1)} = N^{-1} \sum_{n=1}^{N} q_{ns}^{(m+1)}
$$

$$
\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln[l(d_{nt}|x_{nt}, s, p, \theta)]
$$

Our algorithm begins by setting initial values for $\theta^{(1)}$, $\pi^{(1)}$, and $p^{(1)}(\cdot)$:

Step 1 Compute $q_{ns}^{(m+1)}$ as:

$$q_{ns}^{(m+1)} = \frac{\pi_s^{(m)} \prod_{t=1}^{T} l\left[d_{nt}|x_{nt}, s, p^{(m)}, \theta^{(m)}\right]}{\sum_{s'=1}^{S} \pi_s^{(m)} \prod_{t=1}^{T} l\left(d_{nt}|x_{nt}, s', p^{(m)}, \theta^{(m)}\right)}$$

Step 2 Compute $\pi_s^{(m+1)}$ according to:

$$\pi_s^{(m+1)} = \frac{\sum_{n=1}^{N} q_{ns}^{(m+1)}}{N}$$

Step 3 Update $p^{(m+1)}(x, s)$ using one of two rules below

Step 4 Obtain $\theta^{(m+1)}$ from:

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln\left[l\left(d_{nt}|x_{nt}, s_n, p^{(m+1)}, \theta\right)\right]$$

- Take a weighted average of decisions to replace engine, conditional on x, where weights are the conditional probabilities of being in unobserved state s.

Step 3A Update CCP's with:

$$p^{(m+1)}(x, s) = \frac{\sum_{n=1}^{N} \sum_{t=1}^{T} d_{1nt} q_{ns}^{(m+1)} I(x_{nt} = x)}{\sum_{n=1}^{N} \sum_{t=1}^{T} q_{ns}^{(m+1)} I(x_{nt} = x)}$$

- Or in a stationary infinite horizon model use identity from model that likelihood returns CCP of replacing the engine:

Step 3B Update CCP's with:

$$p^{(m+1)}(x_{nt}, s_n) = l(d_{nt1} = 1 | x_{nt}, s_n, p^{(m)}, \theta^{(m)})$$

First Monte Carlo

Finite horizon renewal problem

- Suppose $s \in \{0, 1\}$ equally weighted.
- There are two observed state variables
  1. total accumulated mileage:

$$x_{1t+1} = \begin{cases} \Delta_t \text{ if } d_{1t} = 1 \\ x_{1t} + \Delta_t \text{ if } d_{2t} = 1 \end{cases}$$

  2. permanent route characteristic for the bus, x_2, that systematically affects miles added each period.
- We assume $\Delta_t \in \{0, 0.125, \ldots, 24.875, 25\}$ is drawn from:

$$f(\Delta_t | x_2) = \exp[-x_2(\Delta_t - 25)] - \exp[-x_2(\Delta_t - 24.875)]$$

and x_2 is a multiple 0.01 drawn from a discrete equi-probability distribution between 0.25 and 1.25.

- Let θ_{0t} be an aggregate shock (denoting cost fluctuations say).
- The difference in current payoff from retaining versus replacing the engine is:

$$u_{2t}(x_{1t}, s) - u_{1t}(x_{1t}, s) \equiv \theta_{0t} + \theta_1 \min\{x_{1t}, 25\} + \theta_2 s$$

- Denoting the observed state variables by $x_t \equiv (x_{1t}, x_2)$, this translates to:

$$\begin{aligned} v_{2t}(x_t, s) - v_{1t}(x_t, s) \quad = \quad & \theta_{0t} + \theta_1 \min\{x_{1t}, 25\} + \theta_2 s \\ & + \beta \sum_{\Delta_t \in \Lambda} \left\{ \ln\left[\frac{p_{1t}(0, s)}{p_{1t}(x_{1t} + \Delta_t, s)} \right] \right\} f(\Delta_t | x_2) \end{aligned}$$

First Monte Carlo

Table 1 of Arcidiacono and Miller (2011, page 1854)

MONTE CARLO FOR THE OPTIMAL STOPPING PROBLEM[a]

		s Observed			s Unobserved		Time Effects	
				Ignoring s			s Observed	s Unobserved
	DGP	FIML	CCP	CCP	FIML	CCP	CCP	CCP
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
θ_0 (intercept)	2	2.0100	1.9911	2.4330	2.0186	2.0280		
		(0.0405)	(0.0399)	(0.0363)	(0.1185)	(0.1374)		
θ_1 (mileage)	−0.15	−0.1488	−0.1441	−0.1339	−0.1504	−0.1484	−0.1440	−0.1514
		(0.0074)	(0.0098)	(0.0102)	(0.0091)	(0.0111)	(0.0121)	(0.0136)
θ_2 (unobs. state)	1	0.9945	0.9726		1.0073	0.9953	0.9683	1.0067
		(0.0611)	(0.0668)		(0.0919)	(0.0985)	(0.0636)	(0.1417)
β (discount factor)	0.9	0.9102	0.9099	0.9115	0.9004	0.8979	0.9172	0.8870
		(0.0411)	(0.0554)	(0.0591)	(0.0473)	(0.0585)	(0.0639)	(0.0752)
Time (minutes)		130.29	0.078	0.033	275.01	6.59	0.079	11.31
		(19.73)	(0.0041)	(0.0020)	(15.23)	(2.52)	(0.0047)	(5.71)

[a]Mean and standard deviations for 50 simulations. For columns 1–6, the observed data consist of 1000 buses for 20 periods. For columns 7 and 8, the intercept (θ_0) is allowed to vary over time and the data consist of 2000 buses for 10 periods. See the text and the Supplemental Material for additional details.

- CCP estimators also apply to dynamic games of incomplete information, where players integrate over the actions of their rivals.
- Intuitively rivals are treated like nature in the state transitions.
- This final example illustrate with a dynamic game of entry and exit.
- Entrants (but not incumbents) pay startup cost to compete.
- Paying the startup cost transforms entrant into incumbent.
- Declining to compete in any given period is tantamount to exit.
- When a firm exits another firm potentially enters next period.
- There are two sources of dynamics in this model:
  1. An entrant depreciates startup cost over its anticipated lifetime.
  2. Becoming an incumbent reduces the probability of other firms entering the market, and hence increases expected profits.

Second Monte Carlo

Two observed state variables

- Each market has a permanent market characteristic, denoted by x_1, common to each player within the market and constant over time, but differing independently across markets, with equal probabilities on support $\{1, \ldots, 10\}$.
- The number of firm exits in the previous period is also common knowledge to the market, and this variable is indicated by:

$$x_{2t} \equiv \sum_{h=1}^{I} d_{1,t-1}^{(h)}$$

- This variable is a useful predictor for the number of firms that will compete in the current period.
- Intuitively, the more players paying entry costs, the lower the expected number of competitors.

Second Monte Carlo

Unobserved (Markov chain state) variables, and price equation

- The unobserved state variable $s_t \in \{1, \ldots, 5\}$ follows a first order Markov chain.
- We assume that the probability of the unobserved variable remaining unchanged in successive periods is fixed at some $\pi \in (0, 1)$, and that if the state does change, any other state is equally likely to occur with probability $(1 - \pi)/4$.
- We generated also price data on each market, denoted by w_t, with the equation:

$$w_t = \alpha_0 + \alpha_1 x + \alpha_2 s_t + \alpha_3 \sum_{h=1}^{I} d_{1t}^{(h)} + \eta_t$$

where η_t is distributed as a standard normal disturbance independently across markets and periods, revealed to each market after the entry and exit decisions are made.

- The flow payoff of an active firm i in period t, net of private information $\epsilon_{2t}^{(i)}$ is modeled as:

$$U_2\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) = \theta_0 + \theta_1 x + \theta_2 s_t + \theta_3 \sum_{h=1}^{I} d_{1t}^{(h)} + \theta_4 d_{1,t-1}^{(i)}$$

- We normalize exit utility as $U_1\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) = 0$
- We assume $\epsilon_{jt}^{(i)}$ is distributed as Type 1 Extreme Value.
- The number of firms in each market in our experiment is 6.
- We simulated data for $3,000$ markets, and set $\beta = 0.9$.
- Starting at an initial date with 6 entrants in the market, we ran the simulations forward for twenty periods.

Second Monte Carlo

Table 2 of Arcidiacono and Miller (2011, page 1862)

MONTE CARLO FOR THE ENTRY/EXIT GAME[a]

	DGP (1)	s_t Observed (2)	Ignore s_t (3)	CCP Model (4)	CCP Data (5)	Two-Stage (6)	No Prices (7)
Profit parameters							
θ_0 (intercept)	0	0.0207 (0.0779)	−0.8627 (0.0511)	0.0073 (0.0812)	0.0126 (0.0997)	−0.0251 (0.1013)	−0.0086 (0.1083)
θ_1 (obs. state)	0.05	−0.0505 (0.0028)	−0.0118 (0.0014)	−0.0500 (0.0029)	−0.0502 (0.0041)	−0.0487 (0.0039)	−0.0495 (0.0038)
θ_2 (unobs. state)	0.25	0.2529 (0.0080)		0.2502 (0.0123)	0.2503 (0.0148)	0.2456 (0.0148)	0.2477 (0.0158)
θ_3 (no. of competitors)	−0.2	−0.2061 (0.0207)	0.1081 (0.0115)	−0.2019 (0.0218)	−0.2029 (0.0278)	−0.1926 (0.0270)	−0.1971 (0.0294)
θ_4 (entry cost)	−1.5	−1.4992 (0.0131)	−1.5715 (0.0133)	−1.5014 (0.0116)	−1.4992 (0.0133)	−1.4995 (0.0133)	−1.5007 (0.0139)
Price parameters							
α_0 (intercept)	7	6.9973 (0.0296)	6.6571 (0.0281)	6.9991 (0.0369)	6.9952 (0.0333)	6.9946 (0.0335)	
α_1 (obs. state)	−0.1	−0.0998 (0.0023)	−0.0754 (0.0025)	−0.0995 (0.0028)	−0.0996 (0.0028)	−0.0996 (0.0028)	
α_2 (unobs. state)	0.3	0.2996 (0.0045)		0.2982 (0.0119)	0.2993 (0.0117)	0.2987 (0.0116)	
α_3 (no. of competitors)	−0.4	−0.3995 (0.0061)	−0.2211 (0.0051)	−0.3994 (0.0087)	−0.3989 (0.0088)	−0.3984 (0.0089)	
π (persistence of unobs. state)	0.7			0.7002 (0.0122)	0.7030 (0.0146)	0.7032 (0.0146)	0.7007 (0.0184)
Time (minutes)		0.1354 (0.0047)	0.1078 (0.0010)	21.54 (1.5278)	27.30 (1.9160)	15.37 (0.8003)	16.92 (1.6467)

[a]Mean and standard deviations for 100 simulations. Observed data consist of 3000 markets for 10 periods with 6 firms in each market. In column 7, the CCP's are updated with the model. See the text and the Supplemental Material for additional details.