Disequilibrium Play in Tennis*

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Abstract

Do the world's best tennis pros play Nash equilibrium mixed strategies? We analyze their choice of serve direction (to the receiver's left, right or body) using data from the Match Charting Project. Using a new approach, we test and reject a key implication of a mixed strategy Nash equilibrium: that the probability of winning the service game is the same for all possible serve strategies. Using dynamic programming, we calculate best-response serve strategies and show that for most elite professional servers, the DP serve strategy significantly increases the probability of winning compared to the mixed strategies they actually use.

Keywords: tennis, games, Nash equilibrium, Minimax theorem, constant sum games, mixed strategies, dynamic directional games, binary Markov games, dynamic programming, structural estimation, muscle memory, magnification effect

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1 Introduction

Walker and Wooders (2001) (WW) analyzed 40 tennis "point games" from Grand Slam tournaments, focusing on the server's choice of first serve direction, left or right. They analyzed first serves to the ad and deuce courts separately, with each treated as repeated *IID* one-shot simultaneous-move games between the server and receiver. They concluded that serve location choices are consistent with mixed strategy Nash equilibrium in their hypothesized static game. In particular, the server's chance of winning a point is the same whether the serve is to the left or the right. Equality of win rates across serve directions has been confirmed in several follow-up studies using additional data. In contrast, our tests typically reject the hypothesis of equal win probabilities across serve locations. We find that most elite pro players such as Roger Federer, Rafael Nadal, and Novak Djokovic could significantly increase their chances of winning if they were to systematically exploit these differences.

Our analysis differs from WW by considering three serve directions (left, right, and *body*) and modeling tennis a *dynamic game*. We allow for body serves because tennis professionals believe they are important, see e.g. Rive and Williams (2011). Dynamics are relevant because the server's strategy and the probability of winning the service game depends on the score state as well as *muscle memory effects* — a serve may be more likely to be successful or a receiver may be more effective in returning a serve hit to the same location as the previous serve. We capture muscle memory via the directions of the two previous first serves and show that it can explain serial correlation in serves even when play is in Nash equilibrium. Previous studies including WW have found serial correlation and interpreted it as evidence against Nash equilibrium.

Accounting for dynamics, a third serve direction, and state-dependent serve direction probabilities leads to more powerful tests of mixed strategy play. Our analysis is based on an online database called the Match Charting Project (MCP), Sackmann (2013), which crowdsources play-by-play data on professional tennis matches and records all three serve directions used in our analysis. Even after restricting to matches played on hard courts,[1] we end up with roughly ten times as many serves per server-receiver pair than WW.

[1] We do this to eliminate a potential source of heterogeneity that could confound our results, since playing characteristics differ across surfaces. We extend our analysis to grass and clay in Section 5.3.

However the main reason why we reject the hypothesis of Nash play is a new methodology that models tennis as a dynamic game in contrast to previous work which treated serves as choices in repeated static games. We model serves as decisions at each *subgame* of the overall *service game* between a server and receiver that ends when one of the players has won at least four points and at least two more points than their opponent. The server chooses the location, speed, and spin of each serve, while the receiver allocates a fixed attention budget to the three serve locations. We test the null hypothesis that observed play in a tennis service game is realization of a Markov Perfect Equilibrium (MPE) in which the server's and receiver's strategies depend only on the muscle memory and score state.[2]

We prove that a MPE exists and is unique in the sense that all subgame perfect equilibria result in the same win probability for the server. We show empirically that serve strategies are *completely mixed,* i.e. at every state of the game the server has a positive probability of choosing any of the three possible serve directions. We define *point outcome probabilities* (POPs) to be the equilibrium probabilities that a serve to a given direction is in, as well as the probability the server wins the rally given the serve is in, both conditional on the current muscle memory and score state. The POPs are endogenous objects since they depend on unobserved choices by the server and receiver. However in a completely mixed MPE the POPs can be treated as fixed and invariant to temporary changes in serve strategy since a receiver would not be able to detect any deviation in serve strategy from a small number of observations when serve directions are chosen randomly at every stage of the service game.

This fact allows us to estimate the POPs from our data as "reduced-form objects" that fully embed the strategic decisions of both players. This converts the dynamic game to a single-agent dynamic programming (DP) problem since the POPs constitute the payoff-relevant beliefs that the server needs to evaluate different serve strategies. According to the *one shot deviation principle* of game theory, a necessary condition for a serve strategy to be a MPE strategy is that there is no deviation at any stage of the dynamic game that strictly increases the server's expected win probability. In most games, this means that any deviation in serve strategy *strictly reduces* the probability of winning. However in a completely mixed MPE a much stronger restriction holds:

[2] While the underlying characteristics of the game do not directly depend on the current score, with muscle memory effects, strategies do generally depend on the score state.

all temporary deviations in serve strategy have the probability of winning. This is an extremely strong implication of a completely mixed MPE that results in infinitely many testable restrictions that we exploit to develop powerful new tests of equilibrium play.

In particular, the probability of winning must be the same for all serve directions in all states of the service game. We test these strong implications of mixed strategy equilibrium by estimating the POPs and the actual serve strategy used in the service game. Since our model has 324 muscle memory/score states and three serve directions, a fully unrestricted estimator of the serve strategy and the POPs would require 4536 parameters for each server-receiver pair — far too many to estimate given the size of our dataset. In Section 4, under a testable assumption that actual serve strategies and POPs are stationary and Markovian (but not necessarily MPE strategies), we estimate flexible reduced-form parametric models of serve strategies and the POPs that include the unrestricted specification as a special case. We use the Akaike Information Criterion (AIC) to select a preferred specification with 44 parameters (12 for the server's strategy and 32 for the POPs) that balances the desire for flexibility against the danger of overfitting.

Rather than separately testing for equal win probabilities across serve locations by aggregating data across individual points (treating first serves as independent, static games as WW did), we introduce a new more powerful *Omnibus Wald test* of the hypothesis of equal win probabilities that must hold *across all possible states of the service game, simultaneously.* We also derive Wald tests of the other key restriction of a completely mixed MPE: win probabilities are the same for all possible deviation strategies. These tests strongly reject equality of win probabilities for all serve directions implied by completely mixed MPE play for the majority of the elite pros we analyzed, including top players such as Federer, Nadal, and Djokovic. The tests are based on recursive calculations of the conditional probability of winning the entire service game in all game states and serve directions for any given serve strategy. Our tests allow for serial correlation in serve directions and the POPs due to muscle memory effects, allowing for state dependence in serve behavior and outcomes that cannot be captured in static approaches to testing for equal win probabilities. Muscle memory not only explains serial correlation in serve directions: accounting for it is the key to the strong rejections of equal win probabilities even using tests similar to the static testing methodology that WW employed that assumes servers maximize the probability of winning each point rather than the overall service game.

To quantify the potential deviation gains from systematically exploiting the unequal win probabilities that our tests reveal, we use DP to calculate best response serve strategies for individual server-receiver pairs using the estimated POPs to provide outcome probabilities for each point given the choice of serve direction. For all the elite pros we analyze, the DP strategy significantly increases win probabilities relative to the mixed serve strategies implied by our reduced-form estimates of their serve behavior. Adopting the DP serve strategy would improve Nadal's probability of winning a service game against Djokovic from 71% (his current win rate) to 91.5%, and Djokovic's chance of winning against Nadal from 83% to 93.7%.[3]

Thus, play of the top tennis pros does not constitute a MPE: our empirical analysis reveals many small advantageous one shot deviations (i.e. changes in serve direction at individual points) and the DP strategy takes maximal advantage of all of them to result in much more significant deviation gains at the level of the entire service game.[4] The reason why we find much larger deviation gains by modeling tennis as a dynamic game rather than as a sequence of repeated static games is an implication of the tennis scoring system we call the *magnification effect*.[5] For example, if we assume each point is an *IID* bernoulli draw with a 50% chance of a win for the server, the server will also win the service game with 50% probability since the rules of tennis imply that points evolve as a random walk with absorbing states of win and loss, respectively. However if a change in serve strategy results in a small increase in winning each point, say 55% (a 10% increase), then tennis scores evolve as a random walk with drift, and the probability of winning the service game increases to 62.3%, a nearly 25% increase.

Though the majority of our analysis focuses on the elite pros playing on hard surfaces, we show that our findings extend to elite women and other less elite professional tennis players as well as to play on clay and grass courts. In general we find that the magnitude of deviation gains from adopting the DP best response serve strategy is a declining function of "relative ability" as proxied by the server's probability of winning the service game against specific opponents. We

[3] Traditional game theory has little to say about "mental ability" since all players are equally rational and intelligent. In the context of our model, these increases in win rates result from a better mental approach to the game. This is because the estimates assume the receiver's strategy and other aspects of the server's play are unchanged under the DP serve strategy, so relative physical ability is held constant.

[4] We find significant improvements from the DP best response serve strategies for all 94 server-receiver-surface combinations for which we have sufficient data to precisely estimate our model. See Section 5.3 for details.

[5] See Appendix D for further discussion of this "magnification effect" that causes our omnibus Wald test of equal win probabilities over the different possible serve directions to have far greater power than the tests WW employed.

do not advise tennis pros to adopt our best-response serve strategies since they are pure strategies that the receiver would eventually learn and adapt to. We calculate "robust" mixed serve strategies that account for estimation error and uncertainty about the POPs and the strategy of the receiver. The robust strategies also significantly increase servers' win probabilities and are more difficult for a receiver to detect and adapt to.

To gain insight into the reasons for suboptimal serve choices, we estimate three dynamic structural models of the directions chosen by the server involving increasing degrees of farsightedness. These models allow for persistent shocks to server performance (muscle memory) as well as *IID* shocks that reflect unobserved transitory factors that affect servers' choices. In the full DP specification, the server uses backward induction to maximize the probability of winning the entire *service game,* which is effectively an infinite horizon problem because service games must be won by at least two points. In the point-myopic model the server solves a two period DP to maximize the probability of winning the current *point* taking into account the option value of a second serve but ignoring the effect of current decisions on the future state of the service game. In the serve-myopic model the server maximizes the probability of winning each *serve,* a completely static problem that ignores even the option value of the second serve.

The serve-myopic model is typically rejected because of signficant differences in serve directions between first and second serves resulting from the option value of the second serve. The full DP model is nearly always rejected because it implies subjective POPs that are too "pessimistic" compared to our unrestricted estimate of the actual objective POPs. In most cases the best fitting model is the point-myopic specification. It implies mixed serve strategies that are close to the ones players actually use, while constituting a nearly optimal response to the "subjective POPs" in the sense that additional increases in win probability from adopting a full DP serve strategy are negligible. The suboptimality in serve behavior we identify is primarily driven by incorrect server beliefs, i.e. a lack of rational expectations of the server's own strengths and weaknesses as well as of the receiver as captured by the POPs, rather than players' inability to optimize.

We address concerns that we only have *estimates* of the POPs rather than the *true* POPs which could result in spurious, upward-biased estimates of the deviation gains using estimated POPs rather than the true POPs. To account for this, we derive an approximate probability distribution for the true POPs based on the observed data. We calculate win probabilities for the fully-dynamic, point-myopic, and serve-myopic serve strategies using a random sample of POPs drawn from the asymptotic distribution centered on the point estimates of the POPs. This robustness exercise confirms our core finding: the fully DP and point-myopic strategies based on "rational POPs" have significantly higher win probabilities (in the sense of 1st order stochastic dominance) relative to those implied by the mixed serve strategies the elite pros actually use.

The paper is organized as follows. In Section 2 we briefly review other previous work on testing for Minimax play in tennis. Section 3 introduces our dynamic models of tennis serve behavior and the relevant implications from game theory for equilibrium play that we test empirically in this paper. In Section 4, we summarize the key findings from our reduced-form empirical analysis of the MCP database, including our key finding: the frequent rejection of the hypothesis of equal win probabilities for all serve directions. In Section 5, we present estimation results for the three structural models of tennis serve behavior discussed above, and we calculate the deviation gains from using unrestricted estimates of the "objective POPs" to compute optimal serve strategies. Section 6 concludes with further discussion/speculation as to why many elite tennis pros appear to fail to adopt optimal serve strategies given the strong incentives to do so.

2 Previous Literature

The first empirical analysis of tennis using statistical methods that we are aware of is by George (1973) who analyzed the decision of whether the serve should be strong (i.e. fast and more difficult to return, but higher probability of faulting) versus weak (i.e. slow and easier to return, but lower probability of faulting). The first analysis of the tennis *service game* using DP that we are aware of is by Norman (1985) who used it to determine "whether to serve fast or slow on either or both serves at each stage in a game, and a simple policy is found" (p. 1985).

We already noted the seminal work of Walker and Wooders (2001) who focused on first serves modeled as independent static games and were unable to reject the hypothesis of equal win probabilities for serving left or right. They also found negative serial correlation in serve directions across individual points in tennis which they interpreted as evidence against equilibrium play. With a larger dataset, Hsu, Huang, and Tang (2007) confirmed WW's conclusions regarding equal win probabilities across observed serve directions, but they did not find serial correlation.

Wiles (2006) showed that serial correlation may not necessarily be evidence of disequilibrium play due to the presence of a "timing variable," which is analagous to our muscle memory effects. In addition, Walker, Wooders, and Amir (2011) showed that if a *monotonicity condition* holds, namely if it is always better to win the current point than lose it, then the strategy that maximizes the probabiity of winning each point also maximizes the probability of winning the service game. This condition could explain why the "point myopic" serve behavior we find is not necessarily suboptimal, as we discuss in Section 3.5. Most recently, Gauriot, Page, and Wooders (2023), using data from 3000 matches and nearly 500,000 serves, confirmed WW's conclusions and noted that "the behavior in the field of more highly ranked (i.e., better) players conforms more closely to theory." But unlike Hsu et al. (2007), they "resoundingly reject the hypothesis that the direction of the serve is serially independent" (p. 1) due to the large size of their dataset and their consideration of non-elite pros.

Other related papers include Klaassen and Magnus (2001) and Klaassen and Magnus (2009). Klaassen and Magnus (2001) tested whether successive points in tennis are independent and identically distributed (*IID*) binary random variables using 481 Wimbledon matches containing nearly 90,000 points. They rejected the *IID* hypothesis, but they found that "Deviations from iid are small, however, and hence the iid hypothesis will still provide a good approximation in many cases." Klaassen and Magnus (2009) abstracted from serve direction and focused on the tradeoff between making a serve hard to return and faulting on the serve, considering both the first and second serves of a point. They rejected the hypothesis that servers optimally solve this tradeoff, but found that "the estimated inefficiencies are not large." In the conclusion we also discuss empirical evidence for disequilibrium play in other sports including soccer, football and baseball. The findings are mixed: strong support for Minimax play in soccer penalty kicks, and strong evidence against equilibrium play in first down decisions in football and pitches in baseball.

3 Modeling Tennis as a Dynamic Game

Tennis is two-player game between a *server* and a *receiver* played in tournaments composed of *matches*. A match consists of a sequence of *sets*.[6] A set, in turn, is a sequence of *service games*

[6] Depending on the tournament, a player must win two of three sets or three of five sets to win the match.

in which one of the two players is the server. The server of the first game is chosen by a flip of a coin, and the identity of the server alternates in each game thereafter. Winning a set typically requires winning six games and leading by two games.[7] Each service game consists of a sequence of sub-games that are called *points*. A point consists of a first serve, plus an option for a second serve after a *faulted*, or missed, first serve.[8] First serves alternate between the right (deuce) and left (ad) side of the court. The service game ends when one of the players wins at least four points in total and at least two more points than their opponent.

3.1 Dynamic Theory of the Service Game

We use the scalar x to track both the cumulative points scored by each player in the current service game and whether or not the server is attempting a first or second serve. Figure 1 is a directed graph of all the transitions for the *point-state* variable x within a service game. The circular nodes indicate first serves, whereas the square nodes indicate second serves. The game starts in state $x = 1$, which corresponds to a first serve at tennis score 0–0. If the server wins the point on the first serve, the point-state transits to $x = 3$, corresponding to a first serve at score 15–0. If the server faults the first serve, the state transits to $x = 2$, which is the second serve, and so forth. There are three possible transitions at every first-serve node, two possible transitions at every second-serve node, and two absorbing states (i.e. terminal nodes whose arrows only point in): the server wins ($x = 37$) or loses ($x = 38$) the game.

The arrows connecting most nodes are uni-directional, leading to higher states x. But state $x = 31$ (Deuce) is connected by a bi-directional arrow to both $x = 33$ and $x = 34$. This follows from the fact that when the players are tied at 40–40 (i.e. Deuce), one of the players must win by two points to win the game. Collectively, we refer to states $31-38$ as the *deuce endgame*.[9]

Given these scoring rules, the probability of winning an individual *point* is generally not the same as the probability of winning the *service game* as illustrated in figure 2. It plots the service

[7] Alternatively, if the score is tied six-all, the set is decided by a *tiebreak*, in which the winner is the first to score seven points and be ahead by at least two.

[8] If a serve touches the net and lands in the field of play (a "let"), then the serve (first or second) is redone. Since our data does not record lets, we do not include them in our model.

[9] Note that states 23 and 24 (29 and 30) are strategically equivalent to 33 and 34 (35 and 36): the transitions to future states in the game including winning or losing are identical.

Figure 1: Score states and transitions in the service game

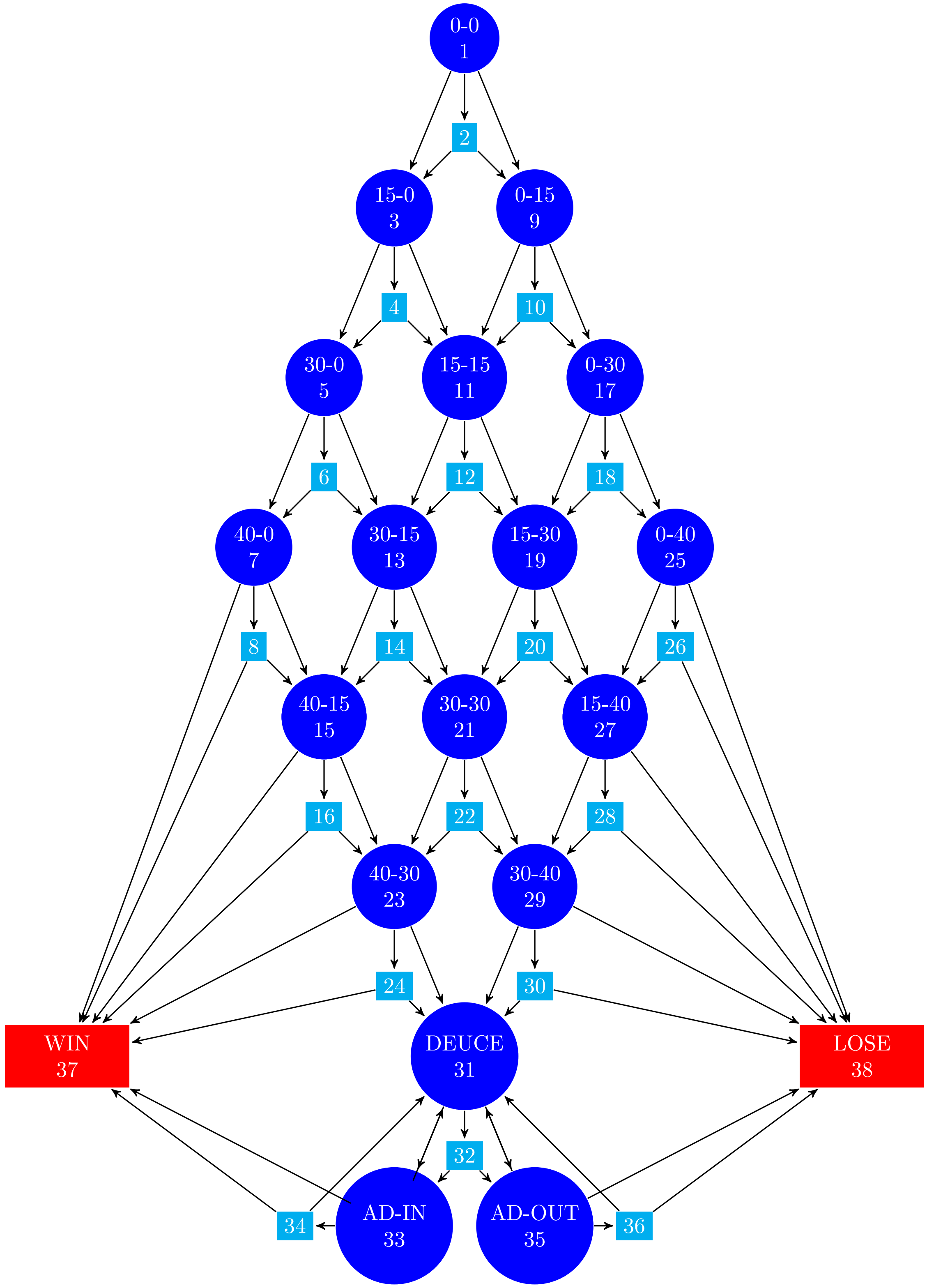
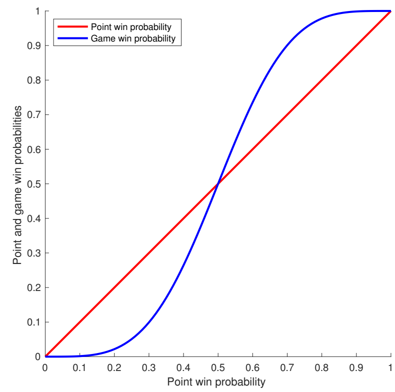


Figure 2: Probabilities of winning a tennis point vs the tennis game



game win probability $g(p)$ as a function of the point win probability p under the assumption that each point of tennis is an *IID* Bernoulli draw with probability p of success. Though we relax the assumption that play at different points are independent draws in our model below, the *IID* Bernoulli assumption implies that the point state in tennis evolves as a random walk with drift, with absorbing states $x = 37$ (win for the server) and $x = 38$ (loss for the server), respectively. The game win probability $g(p)$ equals p at $p = 0.5$ as we noted in the introduction. However any changes in serve strategy that increase the probability of winning each point have a *magnified effect* on the probability of winning the game. Near $p = 0.5$ the slope of the game win probability $g(p)$ is approximately 2.5, so each 10% increase in the point win probability increases the game win probability by 25%. The magnification effect shows how small, hard-to-detect deviation gains at each point of tennis cumulate into much bigger and easier to detect deviation gains in the overall service game, a feature we exploit to derive more powerful tests of Nash play.

At each tennis serve, the server chooses the serve *type* $t = (s,d)$, where $d \in \{l,r,b\}$ indicates the *direction:* to the receiver's left l or right r, or directly into the receiver's body b.[10] Moreover, $s \in \mathcal{S} \subset \mathbb{R}^2$ indicates the *speed* and *spin* of the serve ($\mathcal{S}$ is non-empty, closed, and bounded). The receiver *anticipates* the direction choice of the server. Anticipation includes observable (e.g. where to stand) and unobservable choices. We model anticipation with an attention vector

[10] We follow the literature in assuming that servers "choose" a location, and feel that it is a reasonable fit for the players we analyze. After all, these location categories are broad and our servers are the "best of the best." Tea and Swartz (2022) group serves into our three categories based on "heat maps" of the directions of tens of thousands of men and women's serves at the Grand Slam tournament Roland Garros in 2019 and 2020.

$(a^l, a^r, a^b) \geq 0$, where a^d denotes the attention the receiver devotes to serve location d. We normalize the attention budget $a^l + a^r + a^b = 1$. We assume throughout that the serve direction choice weakly follows the choice of a. This captures the case in which a is a pure location choice, chosen strictly before the server chooses a direction, and the case in which a represents a simultaneous pure mental choice of anticipation.[11]

The probability ℓ that a serve *lands in* (i.e. is not a fault) depends on the court ($c \in \{0,1\}$)[12] and serve type t, while the probability ω that the server wins the subsequent rally (conditional on serving in) depends on the serve type t, court, and attention vector a. We also assume these probabilities can be affected by *muscle memory m* which we encode as the directions previous two first serves, so $m = (d_1, d_2)$ where d_1 is the direction of previous first serve and d_2 is the direction chosen two first serves ago. We track the previous *two* first serves due to the alternation of serves between ad and deuce courts and the possibility that muscle memory may be more affected by the *last serve to the same court rather than by the last serve, which is to a different court.* We initialize muscle memory to null $m = (\emptyset, \emptyset)$ at the start of the service game, and after any first serve we update muscle memory from $m = (d_1, d_2)$ to $m' = f(m,d) = (d, d_1)$ reflecting the direction of the current first serve. We assume muscle memory is only updated at first serve states. This still allows m to capture muscle memory effects of the second serve on the direction of the first, as well as allowing first serve directions to depend on the direction of the previous first serve to the same court.

We assume the probabilities $\ell(m,d,c,s)$ and $\omega(m,d,c,s,a)$ are continuous in (s,a), satisfy $\ell\omega \in [\underline{w}, \overline{w}]$ for some $0 < \underline{w} < \overline{w} < 1$), and are stationary; namely:

Assumption 1 (Stationarity I)**.** *The functions ℓ and ω may vary across server-receiver pairs, but do not vary over time (independent of m and x) or across service games.*

We assume each player's objective is to win the service game, normalizing the winning payoff to 1 and the losing payoff to 0. Since $\ell\omega$ is strictly interior, the game will almost surely end in a finite number of serves. But for completeness' sake, we assume each player earns payoff $1/2$ if the game never ends.

[11] All results extend to a model in which the receiver first chooses a subset of the unit triangle and then chooses a specific element of this subset, simultaneous to the server choosing t. This allows for the realistic case in which the physical location of the receiver on the court constrains, but does not fully determine, the attention vector.

[12] Recall that first serves alternate between the deuce ($= 0$) and ad ($= 1$) courts .

Let (σ_S, σ_R) denote the server and receiver's strategies (perhaps mixed and arbitrarily history-dependent) in the service game. And let $\mathcal{W}_S(x,m)$ be the set of probabilities that the server wins the game starting in state (x,m) induced by some pair of (not necessarily Markovian) subgame perfect equilibrium (SPE) strategies (σ_S^*, σ_R^*) for the server and receiver. Appendix A proves:

Theorem 1. *All subgames have a unique value (i.e. $\mathcal{W}_S(x,m)$ is a singleton), and there exists a Markov Perfect Equilibrium (MPE) in which strategies only depend on the current state (x,m).*

Our empirical approach is valid in any MPE in which *all* strategies only depend on (x,m), and Theorem 1 guarantees that there exists an equilibrium with this property. But if there are multiple MPE, then one can construct non-stationary equilibria by making history dependent selections from the set of MPE. In fact, our empirical approach does not rely on Markovian choices of serve direction, it is sufficient for the server's (speed, spin) strategy and the receiver's attention strategy to be Markovian in (x,m). The next assumption guarentees that this is true in any Subgame Perfect Equilibrium (SPE).

Assumption 2. *The win chance ω and chance of not faulting ℓ obey: (i) ω strictly convex in attention a, (ii) $\ell\omega$ strictly concave in (speed, spin) s, and (iii) $\ell(\omega - 1)$ concave in s.*

Theorem 2. *If Assumption 2 holds, then every SPE has the same attention strategy and the same (speed, spin) strategy, and each of these strategies is Markovian in the current state (x,m).*

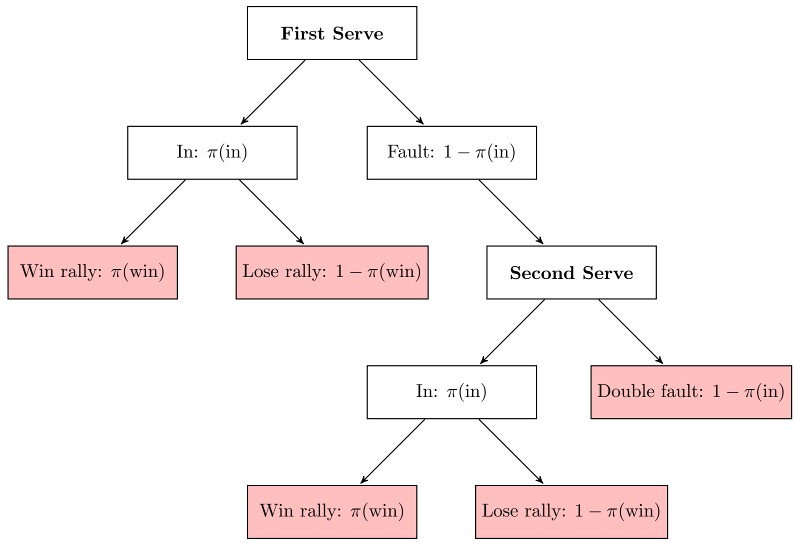
3.2 Optimal Serve Strategies in the Induced Dynamic Program

Our empirical analysis uses Match Charting Project (MCP) data, which does not record speed, spin, or the location of the receiver. To overcome this shortcoming, we use Theorem 1 to project any MPE into the induced DP problem facing a server choosing serve directions to maximize the chances of winning the service game. To do this, let $\rho(s|x,m)$ denote a Markov mixed strategy over the speed and spin vector $s \in \mathcal{S}$ for the server, and let $\alpha(a|x,m)$ denote a Markov mixed strategy over attention for the receiver.

Definition 1. *Given any MPE (ρ^*, α^*), the **Point Outcome Probabilities (POPs)** Π are:*

$$\begin{aligned}
\pi(in|x,m,d) &\equiv \int \ell(m,d,c(x),s)d\rho^*(s|x,m) \\
\pi(win|x,m,d) &\equiv \int\int \omega(m,d,c(x),s,a)d\rho^*(s|x,m)d\alpha^*(a|x,m).
\end{aligned}$$

Figure 3: Detail on the point subgame of tennis



Notice that the mixing probabilities (ρ^*, α^*) will generally depend on the state of the game (x,m), so the POPs will depend on (x,m) even if the underlying conditional probabilities ℓ and ω do not. Given any MPE strategies (ρ^*, α^*), the probabilities π define a single agent "game against nature," a dynamic optimization problem in which the server chooses a serve direction at each node in Figure 1 in order to maximize the probability of winning the service game. Figure 3 illustrates the extensive form of the *point subgame*; namely the subset of the larger directed graph starting at every odd point-state x. In the point subgame, the server chooses a serve direction for the first serve d_1, and in the event of a fault, the direction of a second serve d_2. The point subgame ends with the server winning or losing a point at each pink node.

Building on Norman (1985) we describe the server's DP problem given π. Let $W_S(x,m)$ denote the server's maximal *conditional win probabilities* in state (x,m). Let $W_S(x,m,d)$ be the conditional win probability for the server assuming he serves to direction d on the current serve and behaves optimally on all following serves. Finally, let $x^+(x)$ and $x^-(x)$ denote the successor state in the event that the server wins or loses the point on the current serve, respectively.

The optimal serve strategy can be calculated recursively with the Bellman equation given by:

$$W_S(x,m) = \max_{d \in \{l,b,r\}} W_S(x,m,d). \tag{1}$$

$$\begin{aligned} W_S(x,m,d) &= \pi(\text{in}|x,m,d)\left[\pi(\text{win}|x,m,d)W_S(x^+(x),m') + [1-\pi(\text{win}|x,m,d)]W_S(x^-(x),m')\right] \\ &+ [1-\pi(\text{in}|x,m,d)]W_S(x+1,m') \end{aligned} \tag{2}$$

when x is a first serve state (i.e. x is one of the odd-numbered circular nodes in Figure 1), and

$$\begin{aligned} W_S(x,m,d) &= \pi(\text{in}|x,m,d)\left[\pi(\text{win}|x,m,d)W_S(x^+(x),m) + [1-\pi(\text{win}|x,m,d)]W_S(x^-(x),m)\right] \\ &+ [1-\pi(\text{in}|x,m,d)]W_S(x^-(x),m) \end{aligned} \tag{3}$$

when x is a second serve state (i.e. x is one of the even-numbered square nodes in Figure 1). The optimal serve strategy $D_S^*(x,m)$ is the set of serve directions that maximize the win probability

$$D_S^*(x,m) = \underset{d\in\{l,b,r\}}{argmax}\ W_S(x,m,d). \tag{4}$$

A necessary condition for an MPE serve strategy to be a mixed strategy is that $D_S^*(x,m)$ contains more than one serve direction. A completely mixed MPE serve strategy requires equality of the three win probabilities $\{W_S(x,m,l), W_S(x,m,b), W_S(x,m,r)\}$ in all states (x,m). Since the win probability does not depend on the serve direction in any state (x,m) in a completely mixed MPE, this immediately implies the very strong *strategy independence result* that in equilibrium, *any deviation* in serve strategy implies the same win probability $\mathcal{W}_S(x,m)$ in all states (x,m). We use this strong implication of completely mixed MPE play to construct powerful tests of equilibrium play in Section 4.

Tennis can be viewed as an example of a *directional dynamic game* (DDG) defined by Iskhakov, Rust, and Schjerning (2016), with the exception of the deuce endgame where directionality is not present.[13] While most service games are reasonably short in practice (fewer than 10 points), there is no fixed upper bound on the duration of the *deuce endgame*, the subgame starting at $x = 31$.[14] As a result, tennis must be analzed as an infinite-horizon dynamic game, starting with the deuce endgame which is a fully recursive subgame where win probabilities are determined by solving the Bellman equation simultaenously, as the unique fixed point $W_S = \Gamma(W_S)$. After solving the deuce endgame, we use *state recursion* to solve the rest of the game by backward induction across the remaining directionally ordered states $x < 31$.

[13] Norman (1985) recognized the directionality of tennis and grasped the essence of state recursion when he described how the optimal tennis serve strategy and corresponding win probabilities could be calculated by DP.

[14] The longest deuce endgame that we are aware of was between Anthony Fawcett and Keith Glass in 1975. The score reverted back to deuce 37 times before Glass won the game. Fawcett, however, won the match.

3.3 Calculating Win Probabilities for Stationary Serve Strategies

It is *sufficient* for our empirical analysis, that the unobserved elements of choice (speed, spin, and receiver attention) are Markovian in (x,m), but this is not necessary. Instead we can make the following assumption directly on the induced probabilities.

Assumption 3 (Stationarity II)**.** *The **actual POPs** (those implied even if players are not using MPE strategies) are given by families of conditional probabilities* $\{\pi(in|x,m,d),\pi(win|x,m,d)\}$ *that do not vary over time (independent of* (x,m)*) or across service games.*

Assumption 1 and MPE (serve, speed, and attention) strategies are jointly sufficient, but not necessary, for Assumption 3. While Assumption 3 does not impose equilibrium, it does implicitly assume that the players are unaware if they are failing to play mutual best responses, since otherwise they would have an incentive to alter their strategies, perhaps touching off a learning and adaptation process that would violate stationarity.

When stationarity holds and we have enough data, we can consistently estimate Π and use DP to calculate optimal serve strategies numerically. We then compare optimal win probabilities to win probabilities given actual serve strategies (which can also be consistently estimated given sufficient observations on serve directions). Specifically, let $P(d|x,m)$ be an arbitrary (potentially suboptimal) Markovian serve strategy, i.e. the probability that the server chooses direction d in state (x,m). Let $W_P(x,m)$ be the server's win probability starting in state (x,m) and using strategy P for all future serves, and let $W_P(x,m,d)$ be his win probability when choosing serve direction d in state (x,m) and using strategy P for all future serves. We then have the analog to (1):

$$W_P(x,m) = \sum_{d\in\{l,b,r\}} W_P(x,m,d)P(d|x,m). \tag{5}$$

where $W_P(x,m,d)$ is given by Equations (2) and (3) with W_P in place of W_S. These equations make it clear that W_P is an implicit function of the POPs, Π, and the serve strategy P.

In fact, we can write an expression for W_P as the solution to a system of linear equations, as is well known in the DP literature on *policy evaluation*. Since there are 298 distinct states (x,m):[15]

$$W_P = w_P(P,\Pi) + M_P(P,\Pi)W_P, \tag{6}$$

[15] There is only one possible muscle memory state at the start of the service game $x=1$, three possible muscle memory states for $x=2,3$, and 9, and 9 possible muscle memory states for the remaining 32 score states. Thus, $1\times1+3\times3+32\times9=298$ states (x,m).

where $w_P(P,\Pi)$ is a 298×1 vector providing the probability of directly winning the service game in each state (in most states this is zero), and $M_P(P,\Pi)$ is a 298×298 Markov sub-transition matrix (i.e. not all of its rows sum to 1)[16] representing the probability of transiting between any two states induced by serve strategy P and the POPs Π. Since $M_P(P,\Pi)$ is a Markov sub-transition matrix, the linear system (6) has a unique solution W_P.[17] We can see from (6) that W_P is an implicit function of both (P,Π). We use this result later in the paper to rapidly calculate win probabilities, and via the Implicit Function Theorem, the gradients of the win and conditional win probabilities with respect to model parameters. This enables us to compute standard errors for win probabilities and conduct efficient Wald tests of the hypothesis of equal win probabilities.

With enough observations of service games between a given server and receiver, $W_P(x,m,d)$ can be consistently estimated as the fraction of service games won when the server chose direction d in state (x,m). If the POPs and serve strategies are stationary and Markovian, any win probability W_P must obey identity (6). With 298 states (x,m), non-parametric estimation of W_P involves $298 \times 3 = 894$ individual probabilities $W_P(x,m,d)$. Our analysis also requires an estimate of the actual Markovian serve strategy P and the observed POPs Π. Non-parametric estimation of P requires 298 probabilities and Π a total of $894 \times 2 = 1788$ probabilities. Together, non-parametric estimation of W_P, P, and Π involves a total of $894+298+1788 = 2980$ probabilities, which would require tens of thousands of service games to estimate with any accuracy. In our dataset, we typically have only 100 to 200 service games per server-receiver pair. To overcome this data limitation, we introduce parametric reduced-form models for serve probabilities and the POPs in section 4. We still refer to these as *unrestricted* models of serve behavior because unlike the dynamic structural models we introduce next, we do not require serve direction choices to be best responses to the server's beliefs about the POPs.

3.4 Dynamic Discrete Choice Models of Serve Behavior

To get deeper insight into the behavior of elite servers, we introduce three different structural models of serve behavior that we use in our empirical analysis: 1) a *fully-dynamic model* that assumes the server chooses a strategy that maximizes the probability of winning the entire *service*

[16] The rows of M_P do not all sum to 1, $\ell\omega \in [\underline{w},\overline{w}]$ implies $\pi(\text{in})\pi(\text{win}) \in [\underline{w},\overline{w}]$ for any strategy choices.
[17] We prove this in a corollary of Lemma 1 in Appendix A.

game; 2) a *point-myopic model* that assumes the server chooses serve directions to maximize the probability of winning each *point;* and 3) a *serve-myopic model* that assumes the server chooses serve directions to maximize the probability of winning each *serve*, ignoring the option value of the second serve. For each of these models, we estimate the server's *subjective POPs* that rationalize observed serve behavior as a best response to the server's potentially subjective beliefs about their own performance and the performance of the receiver.

The other important aspect of these dynamic discrete choice models is the introduction of *unobserved shocks* affecting a server's choice of serve direction. These shocks can be interpreted as idiosyncratic factors that affect the server's choice that are not persistent over states of the game, unlike muscle memory. Technically, the introduction of these shocks implies that the server is using a *pure strategy* that only appears to be a mixed strategy due to the effect of the unobserved "serve shocks" though it is tempting to interpret the *conditional choice probabilities* $P(d|x,m)$ implied by these models as mixed strategies.[18]

We assume that these trembles or preference shocks are *IID* across successive serves and are observed only by the server but not by the receiver or the econometrician. Let $\varepsilon(d)$ be the tremble associated with serving to direction d. We further assume that $\{\varepsilon(l),\varepsilon(b),\varepsilon(r)\}$ has a Type 1 extreme value distribution with location parameter normalized so that $E\{\max_d \varepsilon(d)\} = 0$ and scale parameter $\lambda \geq 0$. If λ is large enough, the server's behavior can mimic a mixed serve strategy even when the win probabilities for different serve directions are unequal. However as $\lambda \to 0$, the conditional choice probabilites converge to a mixed strategy only if the subjective POPs satisfy the equal win probability restriction. Thus, dynamic discrete choice models are a natural way to model server behavior while converting the test for equal win probabilities into a simpler test of whether the estimated value of λ equals 0.

Let $\sigma_{FD}(x,m,\varepsilon)$ be the serve strategy under the fully-dynamic structural model as a function

[18] An alternative, game-theoretic interpretation is that these shocks represent *trembles*, or incomplete information on players' preferences that imply a Bayesian Nash equilibrium. McKelvey and Palfrey (1995) and McKelvey and Palfrey (1998) studied games of this type and referred to them as *quantal response equilibria.* However the perspective we take in this paper is to model server direction choice as a single agent dynamic discrete choice problem, taking the receiver's behavior as given and embodied in the POPs. Under this interpretation, following Rust (1987), the ε shocks are unobserved state variables: i.e. idiosyncratic *IID* private information or preference shocks known by the server but not observed by the receiver or the econometrician, but which make the server's behavior appear random even though the actual serve strategy is a pure strategy (i.e. a deterministic function of the server's information).

of the observed state (x,m) and the unobserved trembles $\varepsilon = (\varepsilon(l), \varepsilon(b), \varepsilon(r))$. The fully-dynamic model presumes that for each (x,m,ε), the server chooses the serve direction that maximizes the probability of winning the service game, given by:

$$\sigma_{FD}(x,m,\varepsilon) = \underset{d \in \{l,b,r\}}{argmax}[\lambda \varepsilon(d) + V_\lambda(x,m,d)] \tag{7}$$

where $V_\lambda(x,m,d)$ is a conditional value function, the analog of the conditional win probability $W_S(x,m,d)$ defined in equations (2) and (3) of Section 3.[19] Here, the analog of the function $W_S(x,m)$ given by the Bellman equation (1) is replaced by $V_\lambda(x,m)$ given by:

$$V_\lambda(x,m) = \lambda \log \left(\sum_{d \in \{l,b,r\}} \exp\{V_\lambda(x,m,d)/\lambda\} \right). \tag{8}$$

The serve direction probability implied by the fully-dynamic model is denoted by $P_{FD}(d|x,m)$:

$$P_{FD}(d|x,m) = Pr\{d = \sigma_{FD}(x,m,\varepsilon)|x,m\} = \frac{\exp\{V_\lambda(x,m,d)/\lambda\}}{\sum_{d' \in \{l,b,r\}} \exp\{V_\lambda(x,m,d')/\lambda\}}. \tag{9}$$

$P_{FD}(d|x,m)$ gives the probability of choosing to serve to direction d in observed state (x,m) while accounting for the randomness of the unobserved trembles ε. Since the trembles are *IID* across serves, it would appear that this model should also imply conditional independence of serve directions across successive first and second serves. However, that will actually only be true if there is no muscle memory, i.e. the variable m does not enter $V_\lambda(x,m,d)$ (recall that m is a vector that stores the directions of the two most recent first serves). With muscle memory present, we can still have serial correlation of serves even though the trembles are *IID*.

By Theorem 3 of Iskhakov, Jørgensen, Rust, and Schjerning (2017) we have:

$$W_S(x,m,d) = \lim_{\lambda \downarrow 0} V_\lambda(x,m,d), \tag{10}$$

uniformly for all (x,m,d). This implies that the only way for $P_{FD}(d|x,m)$ to converge to a completely mixed serve strategy as $\lambda \downarrow 0$ is if the limiting conditional win probabilities $W_S(x,m,d)$ obey the equal win probability constraints, $W_S(x,m,l) = W_S(x,m,b) = W_S(x,m,r)$ for all (x,m).

[19] Note that generically the optimal strategy $\sigma_{FD}(x,m,\varepsilon)$ will be a *pure strategy* since the probability of more than one serve direction resulting in the same expected reward (including the serve specific private information shock $\varepsilon(d)$) is zero. Below we characterize a necessary condition for σ_{FD} to converge to a mixed strategy as $\lambda \to 0$: this cannot happen unless the limiting values of V_λ are equal for all choices d. As a reviewer noted, the use of private information shocks to provide an alternative interpretation of what might otherwise appear to be mixed strategies dates back to Harsanyi (1973), and can be used "as a way of justifying the paper's structural model."

The point-myopic and serve-myopic models have the same general structure as the fully-dynamic model, so the serve strategies, value functions, and choice (mixing) probabilities are given by the same equations: (7), (8), and (9). The difference is in the equations defining V_λ. In the serve-myopic model, we have:

$$V_\lambda(x,m,d) = \pi(\text{in}|d,x,m)\pi(\text{win}|d,x,m), \tag{11}$$

i.e. $V_\lambda(x,m,d)$ is the probability of winning the serve. A serve-myopic server maximizes the probability of winning each *serve* while the point-myopic server's objective is to win each *point.* Thus, a point-myopic server performs a two-period backward induction calculation. In any second-serve state the value of the point myopic server $V_\lambda(x,m,d)$ coincides with the serve-myopic formula given in Equation (11) above, but in any non-terminal first serve state V_λ is given by:

$$V_\lambda(x,m,d) = \pi(\text{in}|d,x,m)\pi(\text{win}|d,x,m) + [1-\pi(\text{in}|d,x,m)]V_\lambda(x+1,m'), \tag{12}$$

where $m' = (d^{-1},d)$, and $V_\lambda(x+1,m')$ is the maximum win probability over all second serve directions given in Equation (11).

Note that all three structural models imply probabilistic serve strategies that are entirely determined by the POPs and the scale parameter λ for the trembles. In contrast, the reduced-form model of serve directions does not depend on the POPs since it is estimated separately with a flexible parameterization of serve directions. The structural models can be viewed as restricted special cases of the most flexible specification of the reduced-form model. This enables us to conduct likelihood-ratio specification tests for the three structural models relative to the unrestricted reduced-form specification.[20]

[20] A valid likelihood-ratio specification test would be based on a fully unrestricted version of the reduced-form model with a total of 624 parameters so that it has the flexibility to replicate any conditional probability $P(d|x,m)$. Given the limited number of observations for specific server-receiver pairs, our specification for $P(d|x,m)$ depends on only 12 parameters, though it produces estimates that fit the data well. While our reduced-form specification does not strictly nest the structural models, it has sufficient flexibility to closely approximate the structural serve probabilities We can also do tests using the non-nested specification test of Vuong (1989). However, we prefer the LR tests and also rely on the AIC model selection criterion to select our preferred structural specification, similar to the way we used it to select our preferred specification for the reduced-form model.

3.5 The Monotonicity Condition and Myopic Optimality

Unlike chess, where a player's ability to look ahead and consider the consequences of different moves is critical to success, planning ahead may not be as critical to success in tennis. However the ability to solve at least a two period DP is important, and in section 4.1 we provide clear evidence that the option of a second serve affects the first serve strategy. In particular first serves are signficantly faster but have a higher chance of faulting than second serves. Server's compensate for the slower speed of second serves by hitting a significantly higher fraction of body serves.

However it is less clear whether there is a payoff to solving an infinite horizon DP to determine optimal serve strategies as we did in section 3.2. Indeed, Walker et al. (2011) (WWA) provided a sufficient *Monotonicity Condition* (MC), for "point-myopic play" in tennis to be fully optimal in a version of the game where there are no muscle memory effects or other dynamics that affect play across successive points other than changing the score state x: namely, the probability of winning the service game is always higher after winning any point than losing it. MC is satisfied in tennis if points are *IID* Bernoulli outcomes, since we showed in section 3.1 that this implies that the tennis score state is a random walk with drift, so a strategy that increases the probability of winning any point also increases the probability of winning the service game. WWA proved that MC implies a *decomposition result:* optimal play in the service game decomposes into independent point subgames which can be solved as a two period DP.

empirical thought experiment...

This decomposition result is trivially true in a completely mixed MPE of tennis, since *any* deviation serve strategy is optimal in that case. So in this section we take a disequilibrium play perspective, and assume that the server does face a non-trivial optimization problem. In the presence of dynamic effects such as muscle memory, we could imagine there might be tradeoffs such as serve directions that increase the chance of winning the current serve but which compromise the ability to win subsequent points. For example, serving to the same direction as the previous serve may reduce the chance of faulting due to muscle memory effects, but doing this might improve the receiver's ability to predict serve directions and return them, reducing the server's effectiveness in subsequent states of the game.

The following *generalized monotonicity condition* (GMC) extends WWA's Montonicity Condition, allowing us to prove that point myopic play is optimal in the presence of muscle memory:

Definition 2. ***Generalized Monotonicity Condition (GMC)*** *The probability of winning the game is always higher after winning a point than losing it, and the game win probability for any first serve state is independent of the most recent previous first serve direction d_1:*

$$W_S(x^+(x),m) > W_S(x^-(x),m) \quad \textit{for all } (x,m), \tag{13}$$

and also at any first serve state x, and for any direction d_1 for the previous first serve, if the probability of winning the point is higher for serving to direction d than d', then

$$\begin{aligned} W_S(x^+(x),(d,d_1)) &\geq W_S(x^+(x),(d',d_1)) \\ W_S(x^-(x),(d,d_1)) &\geq W_S(x^-(x),(d',d_1)). \end{aligned} \tag{14}$$

The first inequality in (13) is the same Montonicity Condition that WWA showed is sufficient to establish that an optimal point myopic also maximizes the probability of winning the service game, when there is no "state dependence" in tennis other than via the score state x. When there are dynamic effects such as muscle memory, MC alone will no longer be sufficient to establish this result. The new condition (14) imposes the additional restriction that if a particular serve direction d resuls in a higher probability of winning the current point than some other serve direction d', that the choice of d will not lower the server's probability of winning the service game relative to d' in the subsequent states $x^+(x)$ and $x^-(x)$.

A stronger sufficient condition that implies condition (14) is to require that at any first serve state x the game win probability does not depend on the serve direction of the previous first serve, (though it can depend on the serve direction d_2 two first serves ago when the server was serving to the same court, unless $d_2 = \emptyset$ for the first serves in the game to the ad and deuce courts). When there is muscle memory, choice of first serve direction has future consequences because it affects the evolution of muscle memory, which affects the effectiveness of the server in future states of the game. However if muscle memory only operates across successive serves to the same court, then condition (14) will hold and the server does not have to consider the effects of current serve direction on the probabilities of winning in subsequent game states.

Theorem 3. *If GMC holds, then the optimal point myopic and fully dynamic serve strategies coincide and result in the same game win probabilities for the server.*

The proof of Theorem 3 is in appendix A.4. In section 4 we show that GMC is testable and there are server-receiver pairs for which the GMC fails for our empirically estimated W_S. In these cases, point-myopic strategies are suboptimal, although we show that typically the cost of suboptimality in terms of reduced win probability is small.

Consider the implications for serial independence of serve directions. If there are no muscle memory effects, then WWA's monotonicity condition is equivalent to inequality (13) and implies that any strategy that maximizes the probability of winning each point also maximizes the probability of winning the service game. This leads WW to the wrong conclusion that "In addition to equality of players' winning probabilities, equilibrium play also requires that each player's choices be independent draws from a random process" (p. 1522). Independence in serve directions is *consequence of their assumption* that serve strategies do not depend on previous choices and outcomes. When there is history dependence such as muscle memory, equilibrium strategies will generally depend on both the score state x and the muscle memory m. This history dependence implies that a point myopic serve strategy will generally be suboptimal in terms of maximizing the probability of winning the service game. However even when the stronger form of the Monotonicity Condition — our GMC assumption (14) holds, even though an optimal point myopic serve strategy also maximizes the probability of winning the service game, serial correlation in serve directions will still be a general property of MPE, as we show in Appendix E. Thus, serial independence of serve directions is generally *not* an implication (i.e. necessary condition) of a mixed strategy MPE in the presence of muscle memory, although independence does hold in the absence of dynamic effects such as muscle memory.

We conclude by summarizing the testable implications of the theory we have presented:

1. **Nash equilibrium:** There should not be any other serve strategy that increases the server's probability of winning.

2. **Mixed-strategy equilibrium:** The probability of winning in state (x,m) should be equal for all serve directions chosen with positive probability in state (x,m).

We also test the following behavioral implications of the GMC:

3. **Optimality of myopic serve strategies:** When the GMC holds, it is optimal for the server to adopt a point-myopic strategy that focuses only on the goal of maximizing the probability of winning each point.

4. **Serial independence:** If GMC holds and there are no muscle memory effects, the direction of a first serve should not depend on the direction of any previous first serve.

4 Reduced-Form Analysis of Serve Strategies

In this section, we start with a model-free descriptive analysis of our data. Then we introduce a flexible *reduced form* model of tennis that we use to test several of the key implications of game theory summarized in Section 3, particularly the implication that conditional win probabilities are the same for all serve directions.[21] Most of our analysis focuses on a set of elite professional tennis players, who have all been ranked number one in the world and won multiple Grand Slams. These players are Roger Federer, Rafael Nadal, Novak Djokovic, Andy Murray, Pete Sampras, and Andre Agassi.[22] We focus on these players for two reasons: first, we have the most observations for them, and second, if we can show that they serve suboptimally, that means even the best of the best are susceptible to strategic errors.

4.1 Analysis of Play of Specific Server-Receiver Pairs

We have sufficient observations to analyze serve decisions of specific server-receiver pairs. Table 1 summarizes some of the key statistics for 5 selected elite server-receiver pairs, revealing a great deal of player-specific heterogeneity that would be masked in pooled statistics. The table presents the total number of service games and serves we observe for each pair. A typical service game ends after seven to nine serves. The third column breaks down the total number of serves we observe into first and second serves. We can see that the "crude fault rate" (fraction of total serves that are second serves) differs across servers, ranging from a low of 21% for Nadal serving to Federer to a high of 30% for Sampras serving to Agassi.

The three columns labelled L, B and R list the fraction of first and second serves to the receiver's left, body, and right for each server-receiver pair. We see that in general, servers use mixed strategies, but the mixing probabilities for second serves differ significantly from those for first serves. The last column of the table includes the P-value of a likelihood-ratio test of the null hypothesis that the mixing probabilities for the first and second serves are equal. We see that for all servers, we can decisively reject this hypothesis. In general, the fraction of second serves to the body is about twice as large as for first serves.

[21] Our analysis is not assumption-free, as we maintain Assumption 2 for validity of our statistical tests.
[22] We provide results for women and additional men in Section 5.3.

Table 1: Win probabilities and mixed serve strategies for selected elite server-receiver pairs

Server $\rightarrow$ receiver	Games, serves / Serves/game	1st serves / 2nd serves	Serve directions **L**	**B**	**R**	Win prob (std) / P-value: $P_1 = P_2$
Roger Federer $\rightarrow$	523, 4732	3208	.4402	.1007	.4592	.7686 (.0184)
Rafael Nadal	8.36	1164	.2174	.2698	.5129	5.1×10^{-60}
Rafael Nadal $\rightarrow$	519, 4081	3227	.6616	.2048	.1336	.8092 (.0172)
Roger Federer	7.86	854	.5937	.3208	.0855	6.3×10^{-12}
Roger Federer $\rightarrow$	411, 3501	2524	.4521	.0939	.4540	.8200 (.0190)
Novak Djokovic	8.52	977	.4084	.3408	.2508	6.7×10^{-68}
Novak Djokovic $\rightarrow$	407, 3653	2696	.4640	.1565	.3795	.8010 (.0198)
Roger Federer	8.98	957	.4389	.3365	.2247	1.0×10^{-33}
Rafael Nadal $\rightarrow$	346, 2937	2230	.3964	.2825	.3211	.7197 (.0241)
Novak Djokovic	8.49	707	.4073	.5403	.0523	2.4×10^{-64}
Novak Djokovic $\rightarrow$	356, 2877	2149	.4067	.1619	.4314	.7528 (.0222)
Rafael Nadal	8.08	728	.1484	.2940	.5577	1.2×10^{-40}
Novak Djokovic $\rightarrow$	230, 1958	1447	.4651	.1244	.4105	.7696 (.0278)
Andy Murray	8.51	511	.2192	.4618	.3190	3.0×10^{-53}
Andy Murray $\rightarrow$	230, 2141	1522	.3863	.0841	.5296	.7435 (.0288)
Novak Djokovic	9.31	619	.4233	.4782	.0985	5.8×10^{-122}
Pete Sampras $\rightarrow$	140, 1275	884	.4434	.0724	.4842	.9000 (.0254)
Andre Agassi	9.11	391	.4680	.1765	.3555	5.3×10^{-8}
Andre Agassi $\rightarrow$	135, 1125	825	.5127	.1115	.3758	.8666 (.0293)
Pete Sampras	8.33	300	.5766	.2700	.1533	7.2×10^{-16}

We also see that servers adjust their serve strategy for different receivers. For example, from Table 1, we can see that Nadal uses a different serve strategy when serving to Federer than when serving to Djokovic. The final column of Table 1 shows the empirical service game win probability for the server and its estimated standard error (i.e. the fraction of games the server won). We see quite a bit of variation in service game win probabilities across different server-receiver pairs, ranging from a low of 72% for Nadal serving to Djokovic, to a high of 90% for Sampras serving to Agassi. Even controlling for the same server, we see a fairly big variation in win probabilities depending on the receiver: for example, Nadal has an 81% service game win probability when serving to Federer, as Federer is a weaker receiver than Djokovic. Given the relatively small standard deviations in estimated win probabilities, we can strongly reject the null hypothesis that variation in estimated win probabilities is due to sampling error.

4.2 A Flexible, Agnostic Reduced-Form Probability Model of Tennis

In order to test the key necessary condition for a mixed strategy equilibrium — equality of win probabilities for all serve directions — a deeper econometric analysis is required. As discussed in Section 3, we do not have enough data to estimate a non-parametric model. Instead, we estimate a flexibly parameterized reduced-form specification for serve strategies $P(d|x,m)$ and POPs $(\pi(in|x,m,d),\pi(win|x,m,d))$. Following standard terminology in the dynamic discrete choice literature, we refer to the serve probabilities below as Conditional Choice Probabilities (CCPs). Let $f(x,m,d)$ be a $1\times K_P$ vector of indicators for various subsets of the state/action space. We will describe specific choices for f below. In general, f will partition the state space into subsets where serve direction probabilities are similar. Let θ_P be a conformable $K_P\times 1$ vector of coefficients to be estimated. We use the following flexible logit model for the CCPs:

$$P(d|x,m,\theta_P)=\frac{\exp\{f(x,m,d)'\theta_P\}}{\sum_{\delta\in\{l,b,r\}}\exp\{f(x,m,\delta)'\theta_P\}}. \tag{15}$$

Similarly, let $g_{in}(x,m,d)$ and $g_{win}(x,m,d)$ be $1\times K_{in}$ and $1\times K_{win}$ vectors of indicators used to define the following binary logit models for $\pi(in|x,m,\theta_{in})$ and $\pi(win|x,m,\theta_{win})$ that depend on parameter vectors $(\theta_{in},\theta_{win})$:

$$\pi(in|x,m,d,\theta_{in}) \quad = \quad \frac{\exp\{g_{in}(x,m,d)'\theta_{in}\}}{1+\exp\{g_{in}(x,m,d)'\theta_{in}\}} \tag{16}$$

$$\pi(win|x,m,d,\theta_{win}) \quad = \quad \frac{\exp\{g_{win}(x,m,d)'\theta_{win}\}}{1+\exp\{g_{win}(x,m,d)'\theta_{win}\}} \tag{17}$$

We estimate the parameter vector $\theta=(\theta_P,\theta_{in},\theta_{win})$ by maximum likeihood using the log-likelihood function $L(\theta)$ given by:

$$L(\theta)=\sum_{n=1}^{N}\sum_{s=1}^{S_n}\left[\log\left(P(d_{s,n}|x_{s,n},m_{s,n},\theta_P)\right)+\log\left(h(o_{s,n}|x_{s,n},m_{s,n},d_{s,n},\theta_{in},\theta_{win})\right)\right], \tag{18}$$

where N is the total number of service games observed for a particular server-receiver pair, S_n is the number of serves in game n, and $(d_{s,n},x_{s,n},m_{s,n})$ are the observed serve direction, game state, and muscle memory state at serve s in game n. The variable $o_{s,n}$ is the outcome of serve s of game n and takes one of three possible values: $o_{s,n}=1$ if the serve is in (not faulted) and the server wins the subsequent rally, $o_{s,n}=2$ if the serve is in and the server loses the subsequent rally, or $o_{s,n}=3$ if the serve is faulted. In all non-terminal first serve states (odd values of x), the game state transits to a second serve in the event that $o = 3$, but in any second serve state the server loses the point when $o = 3$ (i.e. the server "double faults"). The conditional probability $h(o|x,m,d,\theta_{in},\theta_{win})$ is defined in terms of the POPs as follows:

$$h(o|x,m,d,\theta_{in},\theta_{win}) = \begin{cases} \pi(in|x,m,d,\theta_{in})\pi(win|x,m,d,\theta_{win}) & \text{if } o = 1 \\ \pi(in|x,m,d,\theta_{in})[1-\pi(win|x,m,d,\theta_{win})] & \text{if } o = 2 \\ 1-\pi(in|x,m,d,\theta_{in}) & \text{if } o = 3. \end{cases} \tag{19}$$

Different specifications correspond to different partitions of the state/action space. The finest partition (where every pair (x,m) is a partition element), yields the full non-parametric specification for (P,Π). Since we do not have sufficient observations to reliably estimate a fully non-parametric model, we face a classic bias/variance tradeoff between estimating a flexible possible model with many parameters, versus a more parsimonius model with sufficiently many observations per parameter estimated to guard against the possibility of overfitting and outlier observations that could distort our estimates of the POPs.

We manage this tradeoff using model selection techniques, particularly the Akaike Information Criterion (AIC) which penalizes model complexity. Specifically, we have $\text{AIC} = 2[K - L(\hat{\theta})]$ where K is the total number of parameters estimated in a given model, $L(\hat{\theta})$ is the maximized value of the log-likelihood function and $\hat{\theta}$ is the maximum likelihood estimate of the parameters of the particular model. We evaluated several different specifications (i.e. choices for f, g_{in} and g_{win} with different numbers of parameters and different partitions of the state space) and chose as our preferred specification (detailed in Appendix B) the model with the lowest AIC.[23]

Our preferred specification still involves a large number of parameters per server-receiver pair. In particular, the serve direction probabilities $P(d|\cdot)$ are governed by 12 parameters, eight for the full set of interactions between direction $d \in \{l,r\}$,[24] court, and serve (first vs. second) dummy variables, as well as four muscle memory parameters interacting the court and serve dummy with a dummy indicating whether the serve direction on the current serve equals the serve direction

[23] We also used the Bayesian Information Criterion $BIC = K * log(n) - 2L(\hat{\theta})$, which has a stronger penalty for model complexity. But we found that the higher complexity penalty caused the BIC to select models with fewer parameters. In cases where one model specification was nested within another encompassing specification, the BIC would choose the more parsimonious restricted specification even though likelihood-ratio tests would lead us to reject the parsimonious restricted specification relative to the less restricted encompassing model.

[24] Since $P(d|x,m,\theta_P)$ sums to 1 across directions, we need only include two dummies for current serve direction.

on the previous first serve to the same court. Each of the POPs $(\theta_{in}, \theta_{win})$ is determined by 16 parameters, which correspond to the same indicators as just described for serve probabilities, except that the current serve direction must include all three directions, since the probabilities $\pi(\text{in})$ and $\pi(\text{win})$ need not sum to 1 across serve directions.[25]

We do not have the space to present all these parameter estimates and the associated standard errors for each of the server-receiver pairs we analyzed, though we provide them for Federer vs. Djokovic in Appendix B, and can provide the rest on request. As we will describe further in the next sections, our preferred specification balances the tradeoff described above: it provides an accurate probability model of the entire service game for individual server-receiver pairs while avoiding the dangers of overfitting. In the remainder of this section, we will use this model to test several of our key assumptions, including the key hypothesis of Nash equilibrium play in tennis.

4.3 Testing for Stationarity Across Matches

We now test Assumption 3, i.e. stationarity of the POPs $(\pi(in|x,m,d,\theta_{in}), \pi(win|x,m,d,\theta_{win}))$ over time and across service games. Suppose the CCPs are also stationary in this same sense. Then the stochastic processes of serves and serve outcomes in any given service game between a given server and receiver on a given type of court are Markovian, and the realizations of these Markov processes are *IID* across successive service games. While the presence of muscle memory and the scoring rules of tennis imply that the sequences of serve directions and serve outcomes will be serially correlated *within a service game,* there will be no dependence across successive games because we assume muscle memory is reset at the start of each service game and there are no other effects that lead to dependence across successive games.

It is easy to think of reasons why Assumption 3 may not hold. For example, if a server injures his shoulder, this can adversely affect the POPs. Or there might be psychological effects, such as confidence or a "hot hand," that could lead to serial correlation across successive service games served by the same player. Finally, if a player is learning and adapting, his strategy may slowly evolve as he learns more about his opponent's weaknesses and adjusts to exploit them.

[25] We also estimated our models with a reduced-form specification that adds a binary partition of the score state capturing how far ahead (or behind) the server is in the current service game to the reduced-form specification of the POPs and CCPs. All of our qualitative results are robust to this alternative specification.

On the other hand, we need to pool across service games to have any hope of efficiently estimating the parameters determining the POPs and the serve strategy. From the previous section, our preferred reduced-form model has a total of 32 parameters (24 if we exclude muscle memory effects). Given that a typical service game lasts for about eight to nine serves, we need at least 100 service games of data to estimate these 32 (or even 24 parameters) with sufficient accuracy. We are particularly concerned with the issue of *overfitting*, along with the possibility that the model's predictions of conditional win probabilities will be incredibly high or low due to the lack of sufficient observations.

The stationarity assumption is testable, and we present results from a simple way of testing for stationarity in Tables 2 and 3 below. For the same set of 10 server-receiver pairs we presented in Table 1, we estimate separate CCPs and POPs for different subsets of service games based on year groupings of our data.[26] For example, for Agassi and Sampras, we divide the data into two subperiods, one from 1995-1999 where we have 67 service games and another from 2000-2002 where we have 60 service games. For Nadal and Federer we have sufficient data to create three subperiods: 2004-2007, 2008-2012, and 2013-2017 with 67, 81, and 91 service games, respectively. We estimate a pooled, or "restricted," model using all games in all years and imposing stationarity. Next, we estimate an "unrestricted" model that allows the CCPs and POPs to be different in each subperiod.

We calculate a likelihood-ratio (LR) test statistic of the stationarity hypothesis by taking twice the difference between the log-likelihood for the unrestricted model (i.e. summing the individual subperiod log-likelihoods) and the log-likelihood for the restricted model. The unrestricted model with two subperiods has a total of $2 \times 32 = 64$ (with muscle memory) or $2 \times 24 = 48$ (without muscle memory) parameters that are estimated separately without placing any equality restrictions across the two sample subsets. Thus, the LR test has 32 degrees of freedom for the specification with muscle memory and 24 degrees of freedom for the specification without muscle memory. For the player pairs where we have enough data to divide the sample into three subperiods, the test has 64 and 48 degrees of freedom, respectively.

From Table 2, we see that we are unable to reject our stationarity Assumption 3 at the 5% level for any of the 10 player pairs we analyzed under the muscle memory specification. For the

[26] Appendix C tests stationarity with an alternative data partition.

Table 2: Tests for stationarity of POPs: $\{\pi(in|x,m,d,\theta_{in}),\pi(win|x,m,d,\theta_{win})\}$

Server → receiver	Muscle Memory			No Muscle Memory		
	Restricted	Unrestricted	LR test (df)	Restricted	Unrestricted	LR test (df)
	LL, AIC	LL, AIC	P-value	LL, AIC	LL, AIC	P-value
Roger Federer →	-1934.3	-1901.9	64.9 (64)	-1940.1	-1916.9	46.5 (48)
Rafael Nadal	**3932.6**	3995.7	.447	**3928.2**	3977.7	.533
Rafael Nadal →	-1880.9	-1843.1	75.7 (64)	-1883.2	-1853.9	58.6 (48)
Roger Federer	**3825.9**	3878.2	.150	**3814.5**	3851.9	.140
Roger Federer →	-2280.7	-2242.0	77.4 (64)	-2284.7	-2256.9	55.8 (48)
Novak Djokovic	**4625.4**	4676.0	.122	**4617.5**	4657.7	.206
Novak Djokovic →	-2403.9	-2363.9	80.1 (64)	-2411.7	-2383.3	56.7 (48)
Roger Federer	**4871.9**	4919.7	.084	**4871.3**	4910.7	.183
Rafael Nadal →	-1414.2	-1402.0	24.3 (32)	-1415.8	-1408.5	14.6 (24)
Novak Djokovic	**2892.4**	2932.1	.832	**2879.6**	2913.0	.932
Novak Djokovic →	-1302.1	-1280.7	42.7 (32)	-1304.5	-1285.9	37.1 (24)
Rafael Nadal	**2668.1**	2689.4	.098	**2656.9**	2667.9	.043*
Novak Djokovic →	-1183.2	-1165.6	35.0 (32)	-1188.7	-1175.4	26.7 (24)
Andy Murray	**2430.3**	2459.3	.326	**2425.5**	2446.8	.317
Andy Murray →	-1280.1	-1258.5	43.0 (32)	-1287.9	-1273.0	30.0 (24)
Novak Djokovic	**2624.1**	2645.1	.092	**2623.9**	2641.9	.186
Pete Sampras →	-1117.9	-1097.4	40.9 (32)	-1124.1	-1107.3	33.7 (24)
Andre Agassi	**2299.7**	2322.9	.135	**2296.2**	2310.5	.091
Andre Agassi →	-1031.1	-1009.1	44.0 (32)	-1032.6	-1012.3	40.6 (24)
Pete Sampras	**2126.2**	2146.2	.077	**2113.2**	2120.6	.019*

specification that excludes muscle memory (which is the preferred one for all 10 under the AIC criterion), we only reject stationarity in the case of Agassi serving to Sampras. Our conclusion is that Assumption 3 is a reasonable approximation to the data, which justifies pooling across service games to get the most reliable possible estimates of serve probabilities and the POPs. Ultimately, we are more concerned about overfitting and the danger of spurious variability in estimated POPs (which could lead to spurious rejection of the hypothesis of equal win probabilities) than we are about the possibility that non-stationarity in server behavior or the POPs across successive games could bias our results.

Table 3 displays the results of LR of stationarity of the CCPs. AIC is lowest for the muscle memory specification for 7 of the 10 player pairs. We reject stationarity of the CCPs for 8 out

Table 3: Tests for stationarity of CCPs: $\{P(d|x,m)\}$

Server $\rightarrow$ receiver	Muscle Memory			No Muscle Memory		
	Restricted	Unrestricted	LR test (df)	Restricted	Unrestricted	LR test (df)
	LL, AIC	LL, AIC	P-value	LL, AIC	LL, AIC	P-value
Roger Federer $\rightarrow$	-1844.8	-1812.0	65.6 (24)	-1874.4	-1844.0	60.6 (16)
Rafael Nadal	3713.5	**3696.0**	.000*	3764.7	**3736.1**	.000*
Rafael Nadal $\rightarrow$	-1688.2	-1636.1	104.2 (24)	-1690.9	-1643.3	95.2 (16)
Roger Federer	3400.3	**3344.1**	.000*	3397.8	**3334.5**	.000*
Roger Federer $\rightarrow$	-2265.1	-2233.0	64.1 (24)	-2293.9	-2264.6	58.4 (16)
Novak Djokovic	4554.1	**4538.1**	.000*	4603.7	**4577.3**	.000*
Novak Djokovic $\rightarrow$	-2423.8	-2392.7	62.2 (24)	-2454.8	-2425.7	58.1 (16)
Roger Federer	4871.6	**4857.4**	.000*	4925.5	**4899.4**	.000*
Rafael Nadal $\rightarrow$	-1432.6	-1419.5	26.2 (12)	-1437.5	-1426.8	21.4 (8)
Novak Djokovic	2889.3	**2887.0**	.010*	2891.0	**2885.6**	.006*
Novak Djokovic $\rightarrow$	-1347.4	-1343.2	8.6 (12)	-1364.2	-1360.3	7.8 (8)
Rafael Nadal	**2718.9**	2734.3	.740	**2744.4**	2752.6	.450
Novak Djokovic $\rightarrow$	-1201.6	-1173.0	57.2 (12)	-1221.0	-1191.7	58.6 (8)
Andy Murray	2427.1	**2393.9**	.000*	2458.0	**2415.4**	.000*
Andy Murray $\rightarrow$	-1250.0	-1182.7	134.6 (12)	-1254.9	-1189.5	130.9 (8)
Novak Djokovic	2524.0	**2413.4**	.000*	2525.9	**2411.0**	.000*
Pete Sampras $\rightarrow$	-1085.4	-1070.3	30.3 (12)	-1096.4	-1083.9	25.0 (8)
Andre Agassi	2194.9	**2188.6**	.003*	2208.8	**2199.8**	.002*
Andre Agassi $\rightarrow$	-931.8	-919.7	24.2 (12)	-945.2	-934.2	22.1 (8)
Pete Sampras	1887.6	**1887.4**	.019*	1906.5	**1900.4**	.005*

of 10 and 9 out of 10 pairs under the muscle memory and no muscle memory specifications, respectively. Under the assumptions in Section 3, if the POPs are stationary, players use MPE strategies, and if the MPE is unique, then the CCPs must be stationary as well. Thus, we conclude that the rejections in Table 3 indicate either a) there are mutiple MPE in tennis and the players "select" different MPEs in different time periods (an explanation we think is unlikely since we conjecture, though have been able to prove formally, that MPE strategies in tennis are generically unique), or b) players are not playing MPE strategies and the variation in CCPs reflects the effect of some sort of learning or experimentation with different serve strategies over time.

4.4 Testing Equality of Win Probabilities Over Directions and Strategies

We now present tests of the key implication of a completely mixed MPE, that point and game win probabilities are independent of serve direction. as well as the stronger implication all deviation serve strategies imply the same win probability. We strongly reject these implications in models that account for muscle memory. As we show below, the data support the presence of muscle memory for almost all player pairs due to strong evidence of serial dependence in serve directions. Accounting for this dependence is key to our ability to detect violations of equal win probabilities.

Table 4 compares the recursively calculated game win probabilities from equation (6) to non-parametric estimates of these probabilities at the first serve of each service game, i.e. the fraction of games won. We restrict attention to the first point of the service game beause it provides the most observations to reliably estimate the game win probability non-parametrically. The final column shows the P-value of a Durbin-Hausman-Wu (DHW) test of our preferred reduced-form specification. Recall that the DHW test compares a consistent but inefficient non-parametric estimator of the game win probability to a relatively efficient estimate of it from equation (6).[27] We see that the calculated win probabilities are close to the non-parametric estimates and are almost always within a standard deviation of each other. The high P-values of the DHW specification tests in the final column of the table show that for all servers except Federer serving to Nadal, we are unable to reject the reduced-form specification and its implied win probability. In the case of Federer serving to Nadal, the RF estimate of the win probability is .796, slightly more than one standard deviation away from the non-parametric estimate of the win probability, .829. The middle columns compare the non-parametric estimates of the conditional win probabilities with the corresponding estimates implied by the reduced-form model, $W_P(1,1,d)$ for $d \in \{l, b.r\}$. The estimates are generally close to each other, though there are some cases where there are large differences due to small numbers of observations resulting in noisy non-parametric estimates.[28]

[27] The DHW specification test compares two estimators of a given quantity or parameter: an inefficient but $\sqrt{N}$-consistent estimator that is consistent under both the null and alternative hypotheses, and an efficient estimator that is also $\sqrt{N}$-consistent for the true parameter under the null hypothesis but may be inconsistent under the alternative hypothesis (N denotes the sample size). In our case, the relevant null hypothesis is that our reduced-form specification for (P,Π) is correct, and the non-parametric estimates of the win probabilities in Table 4 are inefficient but consistent even if the null hypothesis is false (i.e. our reduced-form model is misspecified). Under the null, the DHW test statistic is equal to the square of the two estimates of the win probability divided by the differences in the asymptotic variances, and it converges to a χ^2 random variable with 1 degree of freedom.

[28] For example, in the case of Sampras serving to Agassi, due to the low probability that Sampras serves to the

Table 4: Estimated 1st serve win and conditional probabilities. selected elite servers

Server → receiver	Est.	Win prob 1st serve	Conditional win probability, 1st serve **L**	**B**	**R**	DHW test P-value
Roger Federer →	NP	.796 (.026)	.816 (.025)	.650 (.030)	.803 (.025)	.004
Rafael Nadal	RF	.829 (.023)	.828 (.024)	.819 (.027)	.833 (.022)	
Rafael Nadal →	NP	.786 (.026)	.748 (.028)	.896 (.020)	.762 (.028)	.107
Roger Federer	RF	.807 (.023)	.808 (.023)	.807 (.025)	.803 (.025)	
Roger Federer →	NP	.810 (.024)	.844 (.022)	.867 (.021)	.767 (.025)	.504
Novak Djokovic	RF	.818 (.020)	.826 (.020)	.812 (.023)	.813 (.021)	
Novak Djokovic →	NP	.782 (.025)	.769 (.026)	.710 (.028)	.815 (.024)	.910
Roger Federer	RF	.781 (.022)	.792 (.022)	.769 (.026)	.774 (.024)	
Rafael Nadal →	NP	.712 (.035)	.685 (.036)	.726 (.035)	.750 (.034)	.992
Novak Djokovic	RF	.712 (.034)	.712 (.034)	.701 (.035)	.718 (.034)	
Novak Djokovic →	NP	.829 (.029)	.868 (.026)	.735 (.034)	.833 (.029)	.278
Rafael Nadal	RF	.848 (.023)	.854 (.023)	.830 (.027)	.849 (.023)	
Novak Djokovic →	NP	.794 (.034)	.759 (.036)	.750 (.036)	.841 (.031)	.871
Andy Murray	RF	.791 (.029)	.796 (.031)	.758 (.034)	.799 (.029)	
Andy Murray →	NP	.721 (.038)	.816 (.033)	.500 (.042)	.701 (.038)	.675
Novak Djokovic	RF	.717 (.036)	.735 (.036)	.712 (.039)	.703 (.037)	
Pete Sampras →	NP	.885 (.028)	.894 (.027)	1.00 (.000)	.859 (.030)	.150
Andre Agassi	RF	.866 (.024)	.866 (.025)	.839 (.029)	.872 (.024)	
Andre Agassi →	NP	.874 (.029)	.907 (.026)	.867 (.030)	.852 (.032)	.362
Pete Sampras	RF	.859 (.024)	.861 (.026)	.853 (.026)	.859 (.024)	

In contrast, the middle columns of table 4 reveal big differences in game win probabilities for different serve directions. The largest gap is a 31 percentage point difference between the win probability of serving left (81.6%) vs body (50%) for Murray serving to Djokovic, roughly 10 times higher than their estimated standard errors. The average value of the maximum deviation in game win probabilities over all states and serve directions in Table 4 is 19 percentage points, nearly 4 times as large as the estimated standard errors of these maximum deviations.

Though table 4 reassures us that our recursive calculation of game win probabilities results in accurate and efficient estimates, some readers may be skeptical that the evidence against equal

body (approximately 7%, see Table 1) and the relatively low number of games in which we observe him serving (140), the non-parametric estimate of the conditional win probability of serving to the body equals 1. Of course, this non-parametric estimate is probably not a reasonable estimate: instead it is likely to be a lucky outcome for Sampras who happened to win every one of the 8 games where he served to Agassi's body on the very first serve of the game.

game win probabilities is as convincing as tests of equal *point* win probabilities that WW and most of the subsequent literature have focused on testing. In table 5 we present Omnibus Wald tests of equality of the *point win probabilities* at all states (x,m) of tennis simultaneously. Recall that under the point myopic theory of play, the server does not consider the future consequences of different serve directions and instead focuses on maximizing the probability of winning each point which is a 2 period DP problem. Starting at the second serve, the restriction that point win probabilities are the same for all serve directions holds holds if the serve win probability $V(x,m,d)$ given by

$$V(x,m,d) = \pi(\text{in}|x,m,d)\pi(\text{win}|x,m,d), \tag{20}$$

is the same for all 3 serve directions in all second serve states (x,m). In any first serve state, the point win probability $V(x,m,d)$ given by

$$\begin{aligned} V(x,m,d) &= \pi(\text{in}|x,m,d)\pi(\text{win}|x,m,d) + \\ &\quad [1-\pi(\text{in}|x,m,d)][\sum_{d'\in\{l,b,r\}} P(d'|x,m')\pi(\text{in}|x+1,m',d')\pi(\text{win}|x+1,m',d')], \end{aligned} \tag{21}$$

and it should also be the same for all d where $m' = f(m,d')$ is the new muscle memory state implied by serve direction d' which is updated only in first serve states.

Table 5 provides the Wald test statistics, their P values, and degrees of freedom for the Omnibus test of equality of conditional win probabilities for all serve directions, i.e. the restrictions that $V(x,m,l) = V(x,m,b) = V(x,m,r)$ for all 298 states (x,m), where $V(x,m,d)$ is given in equations (20) and (22) above. We see that there are strong rejections of the hypothesis of equal win probabilities for all player pairs except for Federer serving to Djokovic. Overall, we also see big differences in point win probabilities across different serve directions: the average maximum deviation over all 10 player pairs is .275 with a standard deviation of .083.

Why are we able to reject the hypothesis of equal point win probabilities so strongly when previous studies were unable to do so? We believe that accounting for muscle memory is a large part of the story. If we repeat the Wald tests in table 5 under the no muscle memory specification, we find smaller maximum deviations in win probabilities over serve directions over the reduced state space and we only reject the equal win probability hypothesis for 2 of the 10 player pairs above.[29] Previous tests such as those by WW focused only on 1st serves and pooled all 1st serve

[29] These pairs are Djokovic serving to Murray and Djokovic serving to Nadal (Wald statistics are 18.4 and 23 with

Table 5: Wald tests of equal point win probabilities, selected elite servers, MM specification

Server → Receiver	Wald statistic	Degrees of freedom	P-value
Roger Federer → Rafael Nadal	405.4	29	5.9×10^{-68}
Rafael Nadal → Roger Federer	243.2	30	2.9×10^{-35}
Roger Federer → Novak Djokovic	23.6	30	.75
Novak Djokovic → Roger Federer	274.5	27	8.9×10^{-43}
Rafael Nadal → Novak Djokovic	83.5	29	3.5×10^{-7}
Novak Djokovic → Rafael Nadal	69.6	28	2.1×10^{-5}
Novak Djokovic → Andy Murray	52.3	30	0.007
Andy Murray → Novak Djokovic	212.0	30	2.7×10^{-29}
Pete Sampras → Andre Agassi	146.4	30	2.9×10^{-17}
Andre Agassi → Pete Sampras	198.6	30	8.9×10^{-27}

observations into just two groups: to the ad and deuce courts, respectively. Pooling the data in this way masks big differences in win probabilities for different serve directions that appear once we control for serial correlation in serves by conditioning on previous serve history via the muscle memory state. The importance of controlling for muscle memory is confirmed in Appendix D where we present the results of Wald tests of equal win probabilities under the no muscle memory specification, essentially replicating WW's approach to testing but using our data and including second serves. Similar to WW, these tests usually fail to reject the hypothesis that win probabilities are the same for all serve directions.

We now turn to testing for *equal game win probabilities* using the fully dynamic version of the model with recursively calculated game win probabilities W_P using equation (5), and equations (2), (3), and (5) to calculate the direction-specific win probabilities $W_P(x,m,d)$ entering the recursive formula for $W_P(x,m)$. The Omnibus test of the hypothesis of equal win probabilities for all serve directions involves testing the equality restrictions in all 298 states (x,m)

$$W_P(x,m,l) = W_P(x,m,b) = W_P(x,m,r). \tag{22}$$

In our preferred specification with muscle memory, the Omnibus Wald test amounts to a test of 596 equality restrictions of the form given in (22).[30]

8 degrees of freedom and P values of .018 and .003, respectively). The average value of the maximum difference in point win probabilities over all serve directions and states for these 10 pairs is .20 with a standard deviation of .08.

[30] The specification with muscle memory is our preferred specification, since the no muscle memory specification is strongly rejected for all but one of the player-pairs, see table 7 in section 4.5 below.

Table 6: Wald tests of equal game win probabilities, selected elite servers, MM specification

Server → Receiver	4 fixed serve strategies at 3 states, 9 df	RF serve strategy at 4 states, 12 df
Roger Federer → Rafael Nadal	1.4×10^{-11}	.605
Rafael Nadal → Roger Federer	.873	.018
Roger Federer → Novak Djokovic	6.5×10^{-30}	1.6×10^{-68}
Novak Djokovic → Roger Federer	.0009	.220
Rafael Nadal → Novak Djokovic	2.2×10^{-254}	.526
Novak Djokovic → Rafael Nadal	4.0×10^{-91}	4.5×10^{-44}
Novak Djokovic → Andy Murray	1.4×10^{-69}	.018
Andy Murray → Novak Djokovic	9.1×10^{-91}	.00001
Pete Sampras → Andre Agassi	.787	.003
Andre Agassi → Pete Sampras	.764	.667

Since the conditional win probabilities are implicit functions of (P,Π) and (P,Π) are functions of the 44-dimensional parameter vector $\hat{\theta} = (\hat{\theta}_P, \hat{\theta}_{in}, \hat{\theta}_{win})$ we use the delta method to construct the Omnibus Wald test statistic. This is a quadratic form in the 596×1 vector of differences in conditional win probabilities between serve directions over all states, using the Moore-Penrose inverse of the 596×596 covariance matrix of win probability differences, expressed as a sandwich formula in terms of the 44×44 variance covariance matrix for the reduced-form parameter vector $\hat{\theta}$. We need to use the Moore-Penrose inverse rather than the standard matrix inverse because the rank of the covariance matrix (which equals the degrees of freedom of the χ^2 distribution of the Omnibus test statistic under the null hypothesis) is at most 44.[31]

We find that the Omnibus Wald test results in even stronger rejections of the hypothesis of equal game win probabilities than we obtained when testing for equality of point win probabilities in table 5, with P-values nearly 0 for all player pairs in Table 6. However there are reasons to distrust such strong rejections due to small sample numerical issues with the Moore-Penrose inverse, which is not a continuous function of its matrix argument. The discontinuity can invalidate the standard Chi-squared asymptotic distribution of the Wald test statistic under the null hypothesis. Andrews (1987) provides a sufficient condition for of "generalized Wald tests" that rely on the Moore-Penrose inverse to have the usual asymptotic Chi-square distribution: namely,

[31] The rank of the covariance matrix is generally even lower than 44 because the rank of 596×44 gradient matrix for the win probability differences is often less than the number of parameters, 44.

the rank of the finite sample covariance matrix of the restrictions converges with probability 1 to the rank of the limiting covariance matrix. Matrix rank is not a continuous function either, but it is semi-continuous so the rank condition of Andrews (1987) should hold generically.[32] Nevertheless, we have observed a tendency for Wald test statistics to grow rapidly with the total number of retrictions being tested, so we have opted to adopt a more conservative approach to testing for equal win probabilities using *small subsets of the total number of restrictions.* Since matrix inversion is continuous, our conservative approach reduces the problem of spurious rejections, though it does lead to power/size tradeoffs in the choice of how many resrictions to test and requires additional choices over which subset of restrictions we choose to test.

The last column of Table 6 presents the P-values for our more conservative test of equal win probabilities at a subset of 6 points in the state space: 1) 0-0, 2) 15-0, 3) 0-15, 4) 40-15, 5) 15-50, and 6) deuce. This is a test of 12 restrictions and since the covariance matrix for these restrictions is invertible, this test has 12 degrees of freedom. This test rejects the hypothesis of equal win probabilities (at the 5% level) in 6 of the 10 player pairs in the table.[33]

The second column of Table 6 reports P-values for a Wald test of the invariance of win probabilities with respect to strategy deviations which must hold when MPE serve strategies are completely mixed. We computed the win probabilities of four different fixed strategies at three points in the state space, resulting in a test with 9 restrictions and degrees of freedom (since the covariance matrix for this reduced set of restrictions is invertible). The four fixed serve strategies are: 1) always serve left, 2) always serve to the body, 3) always serve right, and 4) a uniform disribution over serve directions, i.e. serving to each direction with probability 1/3. The three particular score states used in these tests are 1) 40-15, 2) 15-40, and 3) deuce, 40-40. This test strongly rejects the hypothesis of equal win probabilities for 7 out of the 10 player pairs.

In general, our ability to reject the hypothesis of equal win probabilities for all serve directions and the hypothesis that win probabilities are invariant with respect to changes in serve strategy increases with the number of restrictions we choose to test. We conclude that our new approach

[32] A sufficient condition for the rank condition in Andrews (1987) to hold is that the limit covariance matrix of the restrictions is *regular* (i.e. the rank is the same for all covariance matrices in a neighborhood of the limiting value), and Proposition 4 of A. D. Lewis (2009) establishes that the set of regular matrices is an open and dense set of the space of all matrices.

[33] A test of equality of point win probabilities restricted to the same 6 score states rejects for 3 of the 10 pairs at the 5% level.

to testing the strong restrictions implied by mixed strategy Nash equilibrium play, combined with allowing for serial correlation in serve directions via muscle memory effects, as well as the incorporation of body serves and the greater number of observations we have compared to WW's original analysis, explains why we are able to reject the hypothesis of equal win probabilities in the majority of elite player pairs we have analyzed.

4.5 Testing for "Muscle Memory" Effects

We conclude this section by presenting evidence of serial dependence in serve directions and, to a lesser extent, the POPs. We have already shown in Section 4.1 that there are significant differences between the mixture probabilities that servers use for first and second serves, so it should not be surprising that we also find significant serial dependence between first and second serves. However, this serial dependence is not necessarily inconsistent with equilibrium play, since the server considers the option value of the second serve when choosing the speed and direction of the first serve.

The more important question is whether there is serial correlation *across successive first serves.* Our preferred specification for our reduced-form model of serve directions allows for serial dependence in successive first serves. First, similar to WW, our specification conditions on court, so serve strategy can alternate between the ad and deuce courts. This effect induces serial dependence in serve strategy but not serial correlation in serve directions. We capture serial correlation in serve directions via muscle memory. The muscle memory specification also induces serial correlation between the first and second serves, since the serve direction probabilities for second serves depend on the location of the faulted first serve. The specification without muscle memory allows for the first effect, serial dependence as play alternates between courts, but implies zero serial correlation in the directions of successive first serves.

To test for serial dependence in serve directions, we use likelihood-ratio tests of a restricted version of our reduced-form model that excludes the muscle memory variable m. As we showed in Section 3, the directions of serves become serially independent under this specification. Table 7 presents the results of LR tests of the hypothesis of "no muscle memory effects." The last column of the table shows that except for the case of Nadal serving to Federer, we can reject the

Table 7: Tests for muscle memory effects in (P,Π), selected elite servers

Server $\rightarrow$ receiver	Model	No muscle memory		Muscle memory		LR test
		AIC	**LL**	**AIC**	**LL**	P-value
Roger Federer $\rightarrow$	Serves	3764.7	-1874.4	3713.5*	-1844.8	4.3×10^{-12}
Rafael Nadal	POPs	3928.3*	-1940.1	3932.6	-1934.3	.170
Rafael Nadal $\rightarrow$	Serves	3397.8*	-1690.9	3400.3	-1688.2	.249
Roger Federer	POPs	3814.5*	-1883.3	3824.9	-1880.9	.779
Roger Federer $\rightarrow$	Serves	4603.7	-2293.9	4554.1*	-2265.1	9.3×10^{-12}
Novak Djokovic	POPs	4617.4*	-2284.8	4625.4	-2280.7	.414
Novak Djokovic $\rightarrow$	Serves	4925.5	-2454.8	4871.6*	-2423.8	1.1×10^{-12}
Roger Federer	POPs	4871.3*	-2411.7	4871.9	-2403.9	.048
Rafael Nadal $\rightarrow$	Serves	2891.0	-1437.5	2889.3*	-1432.6	.044
Novak Djokovic	POPs	2879.6*	-1415.8	2892.4	-1414.2	.921
Novak Djokovic $\rightarrow$	Serves	2744.8	-1364.2	2718.9*	-1347.4	9.0×10^{-7}
Rafael Nadal	POPs	2656.9*	-1304.5	2668.1	-1302.1	.779
Novak Djokovic $\rightarrow$	Serves	2458.0	-1221.0	2427.1*	-1201.6	7.7×10^{-8}
Andy Murray	POPs	2425.5*	-1188.7	2430.3	-1183.2	.202
Andy Murray $\rightarrow$	Serves	2525.9	-1254.9	2524.0*	-1250.0	.044
Novak Djokovic	POPs	2623.9	-1287.9	2524.1*	-1280.1	.049
Pete Sampras $\rightarrow$	Serves	2208.8	-1096.4	2194.9*	-1085.4	2×10^{-4}
Andre Agassi	POPs	2296.2*	-1124.1	2299.7	-1117.9	.134
Andre Agassi $\rightarrow$	Serves	1906.5	-945.3	1887.6*	-931.8	2×10^{-5}
Pete Sampras	POPs	2113.2*	-1032.6	2126.2	-1031.1	.934

hypothesis of no muscle memory at the 5% significance level. However, when it comes to the POPs, we have far weaker evidence of serial correlation. For most of the server-receiver pairs in Table 7, we are unable to reject the hypothesis of no muscle memory effects.

Why would that be the case? We think it may have to do with the receiver's behavior. Specifically, if muscle memory effects are real, and the receiver shifts his position accordingly, then the receiver could effectively cancel out any effect that muscle memory would impart on the POPs. As a result, we would observe serial correlation in the server's directional choices but not in the POPs.[34] This can be consistent with Nash equilibrium play as we demonstrate in Appendix E.

[34] We solved for the MPE in a two-direction version of our model and observe that muscle memory effects induce much larger changes in server's and receiver's equilibrium mixed strategies than in the POPs.

5 Dynamic Structural Analysis of Serve Strategies

In the previous section, we estimated an unrestricted reduced-form model of serve directions and POPs and showed that this flexible, agnostic model of tennis rejects the key implication of a mixed strategy Nash equilibrium: namely that the POPs satisfy the restriction that the probability of winning is the same for all serve directions in every state of the game (and thus, all possible serve strategies have equal win probabilities). These tests did not require us to make any assumptions about server behavior beyond the Stationarity Assumption 3. This section provides more insight into server behavior by presenting estimation results for the three structural models of serve behavior we introduced in section 3.4. We estimate their parameters by maximum likelihood using the full panel likelihood function (18) on data from hard courts for the 10 elite server-receiver pairs listed in Table 8.[35] Based on our findings in section 4.5, which provide strong evidence of serial correlation in serve directions across successive points, we focus on the specification with muscle memory. For comparability we use the same specification of the POPs as in our reduced-form results presented in Section 4, so our structural models involve a total of 33 parameters: the 32×1 vector of POP parameters $(\theta_{in}, \theta_{win})$, plus the extreme-value scaling parameter λ.

The structural estimates of the POPs can be regarded as estimates of the server's *subjective beliefs* that may or may not correspond to rational *objective beliefs* about the true POPs, which we estimate via our unrestricted estimates of the POPs. As we discussed in section 3.4, the structural model implies mixed strategy Nash equilibrium play if two key restrictions are satisfied: 1) $\lambda = 0$ (i.e. players use mixed strategies, which can only hold if the POPs obey the equal win probability resrictions), and 2) subjective POPs equal objective POPs.

Unlike the reduced-form specification, the assumption of optimal play implicit in the structural models imposes "cross equation restrictions" on the serve probabilities: they are implicit functions of the POP parameters as well as the scale parameter λ for the extreme value distributed trembles. This implies that the likelihood function is no longer block-diagonal between the POP parameters $(\theta_{in}, \theta_{win})$ and λ, unlike the unrestricted reduced form model where we do have block-diagonality between the 12×1 parameter θ_P determining serve probabilities and the

[35] We extend our analysis to grass and clay courts and a much larger set of server-receiver pairs in Section 5.3.

Table 8: Summary of structural estimation results for selected elite pro server-receiver pairs

Player pair	**Reduced-Form**	**Serve-Myopic**	**Point-Myopic**	**Fully-Dynamic**
Server→ receiver	LL, N AIC	LL, $\hat{\lambda}$ AIC, LR P-value	LL, $\hat{\lambda}$ AIC, LR P-value	LL, $\hat{\lambda}$ AIC, LR P-value
Roger Federer → Rafael Nadal	-3779.1, 2011 7646.1	-3788.2, 6.1×10^{-4} 7642.7, .074	-3783.8, 5.5×10^{-3} **7633.7, .571**	-3817.3, 2.9×10^{-5} 7700.7, 7.1×10^{-12}
Rafael Nadal → Roger Federer	-3569.1, 1882 7226.2	-3571.3, 6.1×10^{-3} 7208.6, .957	-3570.6, 2.7×10^{-3} **7207.3, .990**	-3632.1, 8.4×10^{-5} 7330.3, 1.1×10^{-21}
Roger Federer → Novak Djokovic	-4545.8, 2333 9179.5	-4551.2, .010 **9168.4, .457**	-4552.2, 4.4×10^{-3} 9170.4, .300	-4576.0, 9.0×10^{-4} 9128.0, 7.5×10^{-9}
Novak Djokovic → Roger Federer	-4827.7, 2372 **9743.5**	-4840.0, .011 9746.0, .010	-4842.0, 1.8×10^{-3} 9750.0, 2.7×10^{-3}	-4844.8, 2.4×10^{-4} 9755.6, 3.5×10^{-4}
Rafael Nadal → Novak Djokovic	-2846.8, 1405 5781.7	-2853.8, 5.8×10^{-6} 5773.7, .232	-2853.2, 6.5×10^{-6} **5772.4, .310**	-2864.5, 6.4×10^{-7} 5795.0, 2.2×10^{-4}
Novak Djokovic → Rafael Nadal	-2649.5, 1344 5387.0	-2659.9, .070 5385.9, .035	-2656.1,.097 5378.2, .285	-2654.7, 5.8×10^{-6} **5375.3, .505**
Novak Djokovic → Andy Murray	-2384.7, 1201 4857.5	-2396.2, 9.6×10^{-3} **4858.3, .018**	-2396.9, .044 4859.8, .011	-2413.0, 2.2×10^{-4} 4892.0, 4.1×10^{-8}
Andy Murray → Novak Djokovic	-2530.1, 1328 5148.1	-2536.4, .014 **5138.9, .310**	-2539.8, 6.8×10^{-3} 5145.7, .052	-2556.3, $1.1\times1-^{-5}$ 5178.6, 2.2×10^{-7}
Pete Sampras → Andre Agassi	-2203.3, 1181 **4494.6**	-2219.6, .031 4505.3, 5.9×10^{-4}	-2217.7, .037 4501.4 2.5×10^{-3}	-2240.2, 3.1×10^{-4} 4546.3, 2.4×10^{-11}
Andre Agassi → Pete Sampras	-1962.9, 1050 4013.8	-1973.0, 6.7×10^{-4} 4011.9, .043	-1970.9, 3.2×10^{-6} **4007.8, .140**	-2005.8, 5.1×10^{-6} 4077.6, 1.1×10^{-13}

POP parameters $(\theta_{in}, \theta_{win})$. Thus, in the structural model there is a tension between maximizing the likelihood for the POPs versus the likelihood for serve directions. As we will see, maximum likelihood resolves this tension by distorting the estimates of the POPs while also driving the estimate of λ close to zero. As we noted in equation (10) of section 3.4, the only way the model can explain mixed strategy play as $\lambda \downarrow 0$ is to force the POPs to obey the equal win probability restrictions. The benefit from distorting the POPs so they satisfy these restrictions is that it improves the ability of the models to match observed serve direction probabilities.

Table 8 summarizes the structural estimation results for the same 10 elite server-receiver pairs that we analyzed in Section 4.[36] For comparison, we show the optimized log-likelihood function

[36] Due to limited space, we do not provide the 32 parameter estimates of $(\theta_{in}, \theta_{win})$ and their standard errors for all 10 servers for all three structural models. We are happy to provide these results to interested readers on request.

for the reduced-form model and the number of serve observations used to estimate the parameters, along with the point estimates of λ for each of the structural models. The second row of numbers for each server-receiver pair reports the AIC value along with the P-value of a "likelihood-ratio test" of each structural model relative to the reduced-form model. As per our discussion above, these models are not strictly nested within each other, though the reduced-form model is the more flexible specification with a total of 44 parameters.

In view of this, we follow the approach in Section 4 and select our preferred model as the one with the smallest value of the AIC, labelled in bold font. Notice that the best-fitting model selected by the AIC is also the model that has the highest P-value for a quasi-likelihood-ratio test of each the structural models relative to the reduced-form model. Thus, the model with the lowest AIC is generally also the model for which there is the least evidence against it relative to the reduced-form model using the likelihood-ratio test. In two cases, Djokovic serving to Federer and Sampras serving to Agassi, the AIC selects the reduced-form model, and the likelihood-ratio test strongly rejects all three structural models.

For the other eight servers, we see that the AIC selects the fully-dynamic model in only one case, Djokovic serving to Nadal. The AIC selects the point-myopic model as the best model for four other servers, and it selects the serve-myopic model for three of the servers. We would expect the serve-myopic model to be resoundingly rejected because it does not allow the server to look ahead even, even just one serve ahead to take advantage of the option of a second serve on a first serve. However, the serve myopic model does implicitly reflect adjustments in serve strategy via the POPs that may reflect a server's ability to look ahead. For example, the estimated POPs for the second serve in the serve-myopic sepecification show a lower proability of faulting (presumably because the server reduces the speed of the second serve), but a lower probability of winning the rally given that the second serve is in (presumably due to the receiver's improved ability to return a slower serve). So the serve-myopic model is able to reflect state-dependence in tennis serves via their effect on the POPs and this is why it is not so surprising that the serve-myopic specification performs as well as it does.

Notice that the estimated scale parameters $\hat{\lambda}$ for all specifications are uniformly small, so we find limited role for "trembles" to explain the observed mixed serve strategies of these players. Instead, the maximum-likelihood estimates of the POPs $(\hat{\theta}_{in}, \hat{\theta}_{win})$ are distorted in a manner that

Table 9: Win probabilities and Hausman tests for selected elite pro server-receiver pairs

Player pair	**Nonparametric win probability**	**Reduced-Form**	**Serve-Myopic**	**Point-Myopic**	**Fully-Dynamic**
Server$\rightarrow$ receiver	$\hat{W}(1,1)$	$W(1,1)$ P-value	$W(1,1)$ P-value	$W(1,1)$ P-value	$W(1,1)$ P-value
Roger Federer $\rightarrow$ Rafael Nadal	.796 (.026)	.829 (.023) .004	.825 (.021) .050	.830 (.021) .025	.749 (.021) 1.3×10^{-3}
Rafael Nadal $\rightarrow$ Roger Federer	.786 (.026)	.807 (.023) .107	.806 (.022) .147	.807 (.022) .132	.641 (.022) 1.0×10^{-22}
Roger Federer $\rightarrow$ Novak Djokovic	.810 (.024)	.818 (.020) .504	.818 (.020) .506	.818 (.020) .509	.759 (.020) 1.0×10^{-4}
Novak Djokovic $\rightarrow$ Roger Federer	.781 (.025)	.781 (.023) .910	.779 (.022) .838	.778 (.022) .756	.746 (.021) .011
Rafael Nadal $\rightarrow$ Novak Djokovic	.712 (.035)	.712 (.034) .992	.703 (.033) .485	.706 (.033) .675	.650 (.032) 4.1×10^{-5}
Novak Djokovic $\rightarrow$ Rafael Nadal	.829 (.029)	.848 (.023) .278	.846 (.023) .318	.850 (.023) .231	.797 (.024) .052
Novak Djokovic $\rightarrow$ Andy Murray	.794 (.034)	.792 (.029) .871	.791 (.030) .840	.791 (.030) .844	.750 (.029) .012
Andy Murray $\rightarrow$ Novak Djokovic	.721 (.038)	.717 (.036) .675	.717 (.034) .792	.718 (.035) .825	.584 (.032) 1.6×10^{-11}
Pete Sampras $\rightarrow$ Andre Agassi	.885 (.028)	.866 (.024) .150	.863 (.024) .130	.866 (.024) .187	.757 (.028) 0
Andre Agassi $\rightarrow$ Pete Sampras	.874 (.029)	.859 (.024) .362	.854 (.026) .183	.853 (.026) .150	.715 (.028) 1.4×10^{-58}

results in conditional win probabilities much closer to equality than the ones implied by the reduced-form estimates of the POPs. Note that the λ estimates decline for the structural models that require increasingly "far sighted" calculations by the server. When λ is sufficiently small, the conditional value functions $V_\lambda(x,m,d)$ are extremely close to the conditional win probabilities as per the limiting result in equation (10). But when λ is larger, the trembles play a more important role in the mixed serve strategies, allowing more freedom for the conditional value functions (and the conditional win probabilities) to differ across serve directions.

Table 9 provides the estimated win probabilities and the P-values of Hausman-Wu-Durbin specification tests of the different model specifications. Recall this test is based on a comparison of the implied win probabilities calculated via equation (6) to the non-parametric estimate of the win probability, where the latter is simply the fraction of the games between a given server-receiver pair that the server won. The first column of Table 9 presents the non-parametric estimate of the win probability and its standard error, and the remaining columns present the estimated win probabilities implied by equation (6) with standard errors calculated via the delta method.[37]

We see that the specification tests strongly reject the fully-dynamic model, with the exception of Djokovic serving to Nadal. Recall from Table 8 that the AIC criterion selects the fully-dynamic model as the preferred specification for Djokovic serving to Nadal, so it is reassuring to know that it is not rejected by the specification tests. But for the the other servers, we note that the fully-dynamic model typically significantly underestimates the true win probability. This is caused by the need to distort the POPs to rationalize serve behavior as a best response to the estimated POPs in the fully-dynamic model. As we will show in the next subsection, the serve strategy for the fully-dynamic model is close to the "true" serve strategy captured by the reduced-form model, but the estimated POPs from the fully-dynamic model imply far less favorable performance for the server than the POPs estimated from the reduced-form model. Indeed, the fully-dynamic POPs generally imply both a higher probability of faults and a lower probability of winning the rally given a serve is in compared to the reduced-form POPs. In contrast, the specification tests are generally unable to reject the point-myopic and serve-myopic models. This is consistent with the results we reported in Table 8, where we showed that these models were the ones most frequently selected as having the lowest AIC values.

Note that when λ is sufficiently small, the structural models predict that the effect of trembles are negligible, and servers will choose to serve to the direction with the highest win probability. In this situation, in order to fit the observed mixed serve strategies, the model is forced to equate conditional win probabilities. We see this most clearly in the inability of the Omnibus Wald test to reject the hypothesis of equal conditional win probabilities for the fully-dynamic model (not reported due to space considerations). For the point-myopic and serve-myopic models, we showed that the estimated λ values were larger, so trembles play a greater role in explaining serves. This greater freedom allows these models to rationalize the observed mixed strategies without having to equate conditional win probabilities, and this is reflected by the greater frequency of rejection

[37] Note that the model estimates are relatively efficient estimates of the win probability (as reflected by their smaller standard errors), but they are consistent only if the model specification is correct. The less efficient non-parametric estimator of the win probability is consistent regardless.

of the equal win probability hypothesis for these specifications, especially for the serve-myopic model. The reduced-form model places no constraint on the estimation of the POPs since it estimates separate parameters and likelihoods for serves and POPs. This flexibility results in nearly unbiased estimates of the POPs and implied win probabilities.

We also observe significant *dynamic attenuation* in the restricted structural estimates of the POPs. That is, as we noted in the previous section, the reduced-form estimation results reveal much stronger evidence of serial correlation in serve behavior compared to the POPs. In the fully-dynamic model, the degree of serial correlation in both serves and the POPs is attenuated (i.e. closer to zero; and thus, more likely to be statistically insignificant). In fact, for most servers, the fully-dynamic model does not exhibit any statistically detectable serial correlation in the structural estimates of the POPs, though it does predict serial correlation in serves. What explains this paradox? The explanation is that when λ is close to zero, *serve strategies are very sensitive to small changes in the POPs*, since trembles play a negligible role and the server chooses to serve to the direction with the highest win probability. Thus, it is possible to produce significant muscle memory effects in serve strategies (i.e. dependence of the current serve direction to the direction of the previous serve to the same court) via *very tiny, oscillations in the POPs which are hard to detect statistically*.

Now we return to the key question of this paper: do these distorted/attenuated estimates of the POPs enable the structural models to rationalize observed serve behavior as mixed strategies consistent with Nash equilibrium? We have shown that at best, the structural models are able to rationalize observed serve behavior as a best response, but only relative to the server's *subjective perception* of their environment and receiver, as captured by the structural estimates of the POPs. These subjective beliefs are distorted estimates of the true POPs, which are consistently estimated by the unrestricted reduced-form model. A Nash equilibrium entails a key assumption of *rationality* i.e. the players' subjective beliefs about each other coincide with the truth. In the next section, we will directly calculate best response strategies to our estimates of the true POPs using DP and compare how well these strategies perform relative to the mixed serve strategies the players actually use.

5.1 Calculating Best-Response Serve Strategies

We now provide a more powerful direct test of Nash equilibrium play in tennis: we construct alternative *deviation* serve strategies that significantly increase a server's chance of winning the service game compared to the mixed strategy they are currently using. If the hypothesis of Nash equilibrium is correct, it should be impossible to construct any such deviation strategies. *We construct optimal deviation strategies via DP, setting* $\lambda = 0$ *and using the reduced-form estimates of the POPs.* The DP solution results in pure serve strategies that exploit the unequal win probabilities reflected in the reduced-form estimates of the POPs. At each stage of the game, the DP serve strategy chooses the serve direction that has the maximum conditional win probability (see equation (4) of Section 3), where the optimal conditional win probability $W_S(x,m,d)$ is calculated via the Bellman equations given in equations (1), (2) and (3) of Section 3.

Table 10 presents the optimal DP game win probability as well as game win probabilities implied by three other potentially suboptimal serve strategies. For convenience we repeat the first three columns of table 9 which show the 10 player-pairs, the non-parametric win probability and the reduced-form estimate of the win probability calculated from the estimated POPs and mixed serve strategies of the reduced form model using equation 5 in section 3.3. As we noted, the reduced-form estimates generally closely match the non-parametric estimates, and thus constitute our best estimates of each server's win probability implied by the mixed serve strategy that they actually use. The next three columns of table 10 present counterfactual game win probabilities for the serve-myopic, point-myopic and full DP serve strategies, also using equation 5. In all three cases, we calculated game win probabilities using the reduced-form estimates of the POPs, not the structural estimates of the POPs shown in table 9 that are based on the distorted structural estimates of the POPs. We also fixed $\lambda = 0$, so we do not allow for any "trembles" in our calculated serve strategies.

By construction, the full DP serve strategy maximizes the game win probability and we see that in table 10. However if GMC holds, the optimal point-myopic serve strategy coincides with the full DP serve strategy and implies the same game win probability. Therefore failures in GMC are revealed by cases where the DP game win probability is strictly higher than the win probability implied by the optimal point-myopic serve strategy. We do observe some violations of GMC in table 10, but in all cases the incremental gain from using DP to compute a fully optimal dynamic serve strategy is small.

The last column of table 10 presents the P-value of a Wald test for Nash equilibrium. The test is constructed by appealing to the *one shot deviation principle*, which states that there is *no deviation at any stage of a dynamic game that can increase the server's chance of winning, given the strategy of the receiver, and the service game continuation values*. We find that there are profitable one-shot deviations at many stages of the service game. While each such deviation yields a modest improvement in the win probability, the cumulative effect of all profitable deviations is often a large improvement in the overall game win probability. Of course, if a server were to switch to the DP best response, the receiver will eventually detect the change and adjust his own strategy, which would change the POPs, offsetting some of the gains we predict.

Table 10: Improvements in game win probabilities for selected elite pro server-receiver pairs

Player pair	**Non Parametric**	**Reduced Form**	**Serve Myopic**	**Point Myopic**	**Full DP**	**Wald test P value**
Roger Federer → Rafael Nadal	.796 (.026)	.829 (.023)	.856	.894	.894	.0016
Rafael Nadal → Roger Federer	.785 (.026)	.807 (.023)	.840	.884	.884	.014
Roger Federer → Novak Djokovic	.810 (.024)	.818 (.020)	.823	.870	.877	.024
Novak Djokovic → Roger Federer	.782 (.025)	.781 (.023)	.850	.863	.869	.002
Rafael Nadal → Novak Djokovic	.712 (.035)	.712 (.034)	.855	.916	.916	.0004
Novak Djokovic → Rafael Nadal	.829 (.029)	.848 (.023)	.937	.927	.937	.00013
Novak Djokovic → Andy Murray	.794 (.034)	.792 (.029)	.901	.905	.905	.00019
Andy Murray → Novak Djokovic	.721 (.038)	.717 (.036)	.845	.860	.869	.001
Pete Sampras → Andre Agassi	.885 (.028)	.866 (.024)	.942	.949	.949	.003
Andre Agassi → Pete Sampras	.874 (.029)	.859 (.024)	.912	.935	.936	.040

Recall that σ_S was used in Section 3 to denote the optimal serve strategy, which is an implicit function of the POPs, Π, that we now make explicit by writing $\sigma_S(\Pi)$. Let Π^* and P^* denote the true equilibrium POPs and mixed serve strategy, respectively, in a MPE in tennis. By assumption, the players have common knowledge of these POPs. While we do not directly observe Π^* and P^*, we can consistently estimate them with sufficient data. In particular, the hypothesis of Nash equilibrium implies that for any alternative serve strategy σ, we have:

$$W_S(P^*,\Pi^*) \geq W_S(\sigma,\Pi^*). \tag{23}$$

Let $\sigma_S(\Pi^*)$ be the optimal dynamic serve strategy (generally a pure strategy) calculated by DP for the true Nash equilibrium POPs Π^*. Then by definition of optimality, we have:

$$W_S(\sigma_S(\Pi^*),\Pi^*) \geq W_S(P^*,\Pi^*) \geq W_S(\sigma,\Pi^*) \tag{24}$$

for all stationary Markovian serve strategies σ. Together, inequalities (23) and (24) imply the key equality:

$$W_S(P^*,\Pi^*) = W_S(\sigma_S(\Pi^*),\Pi^*), \tag{25}$$

which serves as the basis for our direct test of a mixed-strategy Nash equilibrium in tennis: the optimal DP serve strategy should not result in a higher win probability compared to the mixed serve strategy P^* that the server actually used.

Using consistent estimators of the game win probabilities on the left and right hand sides of equation (25) we can construct a test statistic based on the squared standardized difference in these win probabilities, which has a Chi-squared distribution with 1 degree of freedom if the null hypothesis is true. The last column of table 10 presents the P-values for this test, and it shows that we strongly reject the best response property for a mixed strategy equilibrium in equation (25) for all 10 player pairs.

5.2 Evaluating the Robustness of Deviation Gains

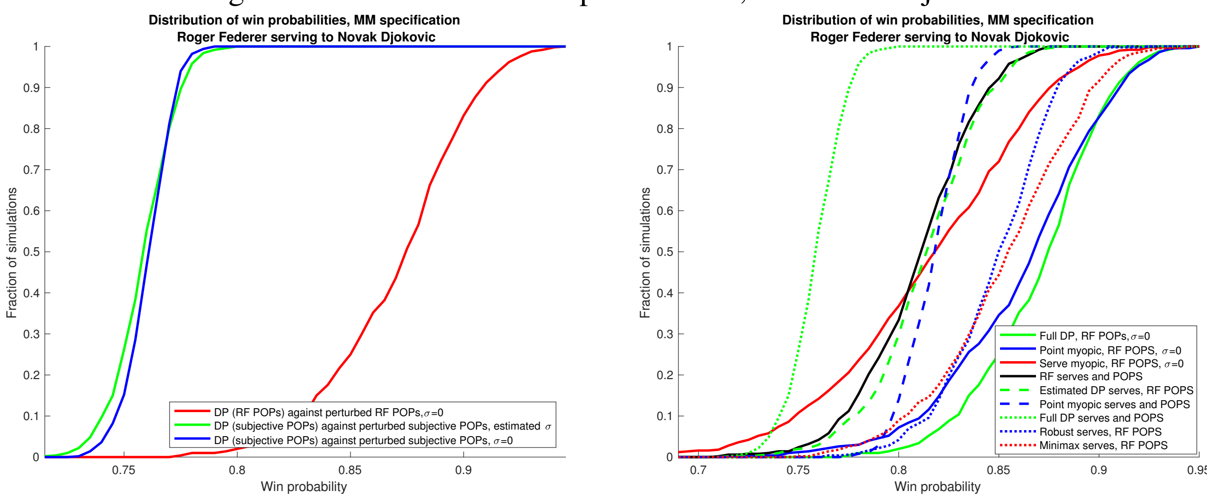
A shortcoming of our approach to testing the hypothesis of Nash equilibrium is that our tests are based on *estimates* of the POPs rather than the *true* POPs. In small samples, estimation error in the POPs could result in spurious, upward-biased estimates of the win probability using noisy estimate of the POPs to calculate a best response strategy via DP instead of the true POPs which we cannot observe. To deal with this, we investigate the robustness of our conclusions via stochastic simulations to compare win probabilities of the estimated mixed serve strategy and our calculated DP best response strategy over a large number randomly drawn POPs.

We draw the random POPs from the asymptotic distribution of the maximum likelihood estimator centered on the point estimates of the reduced-form POP parameters $(\hat{\theta}_{in}, \hat{\theta}_{win})$. We then calculate POPs implied by these simulated parameter values to generate a set of POPs that are randomly distributed about the true POPs. For each realization of the POPs we calculated the win probability fixing the mixed serve strategy of the server at its estimated value, $\hat{P}$, and fixing our estimated DP best response serve strategy at the value calculated using the reduced-form point estimate of the POPs, $\sigma(\hat{\Pi})$. This results in a distribution over simulated win probabilities for the two fixed serve strategies, allowing us to determine if the DP serve strategy outperforms the estimated mixed serve strategy in a range of environments near the true POPs, eliminating the advantage the DP strategy obtains from assuming the estimated POPs are the same as the true POPs. Thus, we force the DP strategy to confront POPs it was not "expecting."

We also calculated similar distributions of win probabilities, but using simulated draws from the *structural estimates of the POPs.* We refer to these random draws as the *perturbed POPs* and the left panel of figure 4 shows the CDFs of simulated game win probabilities for three different cases: 1) red line: the performance of the DP serve strategy $\sigma(\hat{\Pi})$ against 500 random perturbations of the reduced-form point estimates of the POPs, $\hat{\Pi}$, 2) green line: the performance of the point estimate of the fully dynamic structural serve strategy against 500 random perturbations of the structural point estimate of the POPs, and 3) blue line: the performance of the DP serve strategy (calculated with $\lambda = 0$) as the best response to the structural point estimate of the POPs against 500 random perturbations about this point estimate. Note that the serve strategy used to construct the green CDF is a *mixed strategy* whereas the blue and red CDFs are *pure strategies.*

We see that the blue and green CDFs are nearly identical, which is an illustration of the "no deviation gains" condition in equation (25) when the equal win probability condition holds. Even though equal win probabilities do not hold for the 500 perturbations of the point estimate of the structural POPs (which does satisfy the equal win probability restriction), they are sufficiently close to holding so that the full DP serve strategy is unable to systematically outperform the mixed serve strategy. The red CDF shows that DP serve strategy based on reduced-form estimates of the POPs results in a distribution of win probabilities that stochastically dominates the other two CDFs, illustrating the large increase in win probabilities resulting from a serve strategy based on an unbiased estimate of the POPs. As we already noted, the structural estimates of the POPs are distorted to rationalize the observed mixed serve strategy as a best response to the POPs. Even though the reduced-form estimates of the POPs may reflect some small sample noise, they indicate sufficiently large departures from the equal win probability restriction to result the large deviation gains illustrated by the red CDF in figure 4. Thus, even though the structural models can fit serve strategies, they must do so by imputing that players have irrational subjective beliefs.

Figure 4: Distribution of win probabilities, Federer → Djokovic



The right hand panel of figure 4 plots distributions of CDFs for other serve strategies. The solid black line is the CDF for game win probabilities implied by the estimated mixed serve strategy from the reduced-form model. The dashed green line is the CDF of win probabilities computed by the estimated serve strategy from the fully dynamic structural model. Both CDFs are calculated using random perturbations of the reduced form estimates of the POPs. We see that the green dashed CDF lies nearly on top of the black CDF, indicating that the the estimated serve strategy from the dynamic structural model is virtually the same as the mixed serve strategy estimated by the reduced-form model. The dashed blue line is the CDF of win probabilities implied by the structural estimate of the point-myopic serve strategy against perturbations of the reduced-form estimate of the POPs. This CDF does not match the black CDF as well, and reflects the fact that the estimated value of λ for the point-myopic structural model is about 10 times larger than for the fully dynamic structural model. Thus, the point-myopic strategy reflects the effect of more "noise" (random trembles) that make it a worse approximation the reduced-form estimate of the mixed serve strategy.

The green dotted line, the left-most CDF in the figure, is the CDF of win probabilities implied by the structural estimate of the serve strategy but using perturbations of the structural estimates of the POPs. It is the same as the solid green CDF in the left panel of the figure. This is another illustration of how the distorted estimates of the structural POPs are too pessimistic, and result in significantly smaller estimates of win probabilities, a feature that we used to reject the fully dynamic structural model of tennis using the Durbin-Hausman-Wu test in table 9.

The rightmost solid green CDF is the distribution of win probabilities computed by DP (with $\lambda = 0$) using the reduced-form estimates of the POPs, for 500 random perturbations of the reduced-form POPs. It is the same as the solid red CDF in the left panel. The solid blue CDF in the right panel of figure 4 plots the CDF of game win probabilities implied by the optimal point-myopic serve strategy (also calculated using the reduced-form POPs with $\lambda = 0$) against perturbations of the reduced-form POPs. We see it performs almost as well as the full DP serve strategy, but the gap between the blue and green CDFs is an illustration of the failure of GMC: the full DP serve strategy results in higher game win probabilities compared to the point myopic serve strategy as we also see for Federer serving to Djokovic in table 10.

We also see a big gap between the dashed blue and solid blue CDFs: the former is the distribution of win probabilities implied by the point myopic serve strategy that was computed from structural estimates of the POPs whereas the latter is the distribution of win probabilities of a point-myopic serve strategy computed using the reduced-form estimate of the POPs. This shows that there are significant deviation gains from using better estimates of the POPs (i.e. the reduced-form estimates) whether Federer was to use an optimal point myopic serve strategy or a fully optimal DP serve strategy. However the smaller gap between the solid blue and green CDFs in the right hand panel of figure 4 illustrates the much smaller incremental gains from "planning ahead."

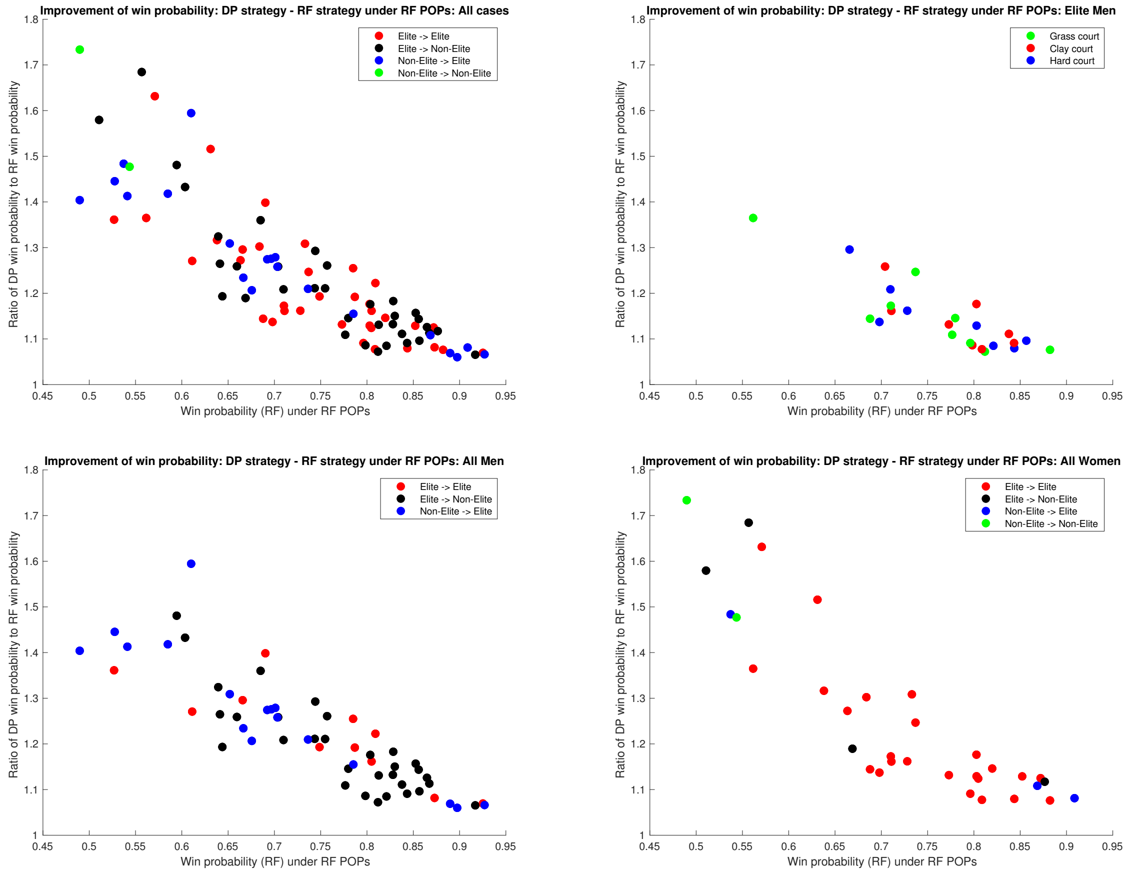
We also calculated serve strategies using an informal "robust control" approach that calculates a serve strategy by averaging the optimal serve strategies that are individually calculated by DP for each randomly drawn POP. This robust serve strategy is a mixed strategy, which is a desirable property if receivers have more difficulty finding best responses to mixed than pure serve strategies. We also calculated a minimax serve strategy by using the DP server strategy calculated for the worst case draw of the POPs, i.e. the DP best response serve strategy for the POPs that resulted in the smallest win probability over the set of randomly drawn POPs. The CDFs of game win probabilties implied by these strategies (computed using 500 random perturbations of the reduced-form POPs) are illustrated by the dotted red and blue lines in the right hand panel of figure 4. Both of these CDF stochastically dominate the black CDF, which captures the performance of Federer's actual mixed serve strategy. However surprisingly, neither of the robust serve strategies does as well as the full DP serve strategy computed from the reduced-form point estimates of the POPs (solid green CDF).

It appears that the optimal DP serve strategy, which is a pure strategy, performs surprisingly well in environments it was not "expecting." This may be due to the fact that it is a pure strategy and pure strategies may be fairly robust to perturbations in the POPs because they are frequently "corner solutions" that will not change in response to sufficiently small changes in the POPs. In any event, we leave further exploration of this topic, and a deeper assessment of the value of more sophisticated verions of robust control, to future work. We do find that the optimal pure serve strategies that we calculate by DP to be intuitive and relatively simple to describe verbally. For example, in the case of Djokovic serving to Nadal, the fully-dynamic serve strategy generally entails serving to Nadal's right (i.e. backhand since Nadal is a lefty) on first serves, whereas on second serves, the optimal direction depends on the whether Djokovic is serving to the deuce or ad court. To the deuce court, he should serve to Nadal's backhand, whereas to the ad court, he should serve to Nadal's forehand. That is, Djokovic should hit his second serve wide.

5.3 Results for Additional Server-Receiver Pairs and Surfaces

We analyzed additional server-receiver-surface combinations where we had sufficient data to precisely estimate our model. In total, we analyzed 76 distinct server-receiver pairs and 94 distinct server-receiver-surface combinations. We decisively reject the hypothesis that the estimated mixed serve strategies are consistent with equilibrium play for all 94 combinations using the

Figure 5: Relation between player ability and deviation gains for different player groups



Wald test of absence of deviation gains, i.e. tests that equation (25) holds.

Figure 5 summarizes the deviation gains we found. The figure has four panels that plot the gain in win probability for different groups of players. In each panel, the vertical axis is the ratio of the mean win probabilities from the fully-dynamic best-response serve strategy to the win probability implied by our reduced form estimates of each server's actual serve strategy. And the horizontal axis is the mean win probability under the actual serve strategy. Thus, the ratio of mean win probabilities shows the relative improvement in the win probability from adopting the fully-dynamic serve strategy.

We calculated the probabilities in Figure 5 using the same procedure as in section 5.2, namely, we calculated win probabilities that are robust to estimation error in the POPs. Specifically, we estimated our structural model for each server-receiver pair. We then calculated win probabilities for the observed server strategies and the fully dynamic best response for 500 different POPs that are *IID* draws from the estimated asymptotic distribution about the point estimates of the POPs for each server in each server-receiver pair.

The top-left panel shows a scatter plot of the improvements for all 76 top-ranked professional server-receiver pairs for which we had sufficient data from the Match Charting Project to reliably estimate the POPs. The points are color-coded so that red dots are our calculated mean deviation gains for “elite” servers playing other “elite” receivers (where we classify a player as “elite” if they were ranked first or second worldwide at some point in their career), the black points are elite servers playing non-elite receivers, the blue dots are non-elite servers playing elite receivers, and the green points are non-elite servers playing non-elite receivers. The most striking finding in this graph is the obvious downward sloping pattern in the scatter plot: we predict that servers with lower win probabilities experience the biggest relative deviation gains from switching to the DP serve strategy. Of course, since win probabilities cannot be higher than 1, the relative gain is constrained to decline as the win probability under the server’s existing serve strategy approaches 1. Nevertheless, the results indicate a clear correlation in “ability” as measured by the server’s existing win probability and the extent of their suboptimality: we predict that the lowest ability servers have the most to gain from using DP to optimize their serve strategies.

The top-right panel of Figure 5 plots the deviation gains across surfaces (green for grass, red for clay, and blue for hard courts) for the 10 “most elite” server-receiver pairs that we focused our analysis on in this paper (e.g. the 10 pairs in Table 10). We see that these players have higher service game win probabilities and also lower deviation gains compared to the set of all players plotted in the left-hand panel. On average for all players, the mean win probability under the existing serve strategy is 73%, and they can expect a 24% increase in win probability on average from adopting the fully-dynamic serve strategy. However, for the most elite players, the average win probability when serving is 76%, and they can expect a 15% increase in win probability on average from deviating to the fully-dynamic strategy. There does not appear to be any clear relation between court type, server ability, and the deviation gain for these most elite players.

Finally, the bottom two panels of Figure 5 plot the results for men (left panel) and women (right panel). The same negative correlation between deviation gains and ability as measured by win probability under their existing serve strategy is apparent for both men and women servers.

The relationship between average win probability and gain to switching to the fully-dynamic serve strategy is also robust across the two sexes. In particular, the male servers in our analysis have a higher average probability of winning under their current serve strategy (74% for men vs 70% for women), and a lower average deviation gain from switching to the fully-dynamic serve strategy (22% for men vs 28% for women).

6 Conclusion

There is substantial evidence against Nash equilibrium and minimax play in laboratory experiments: see, e.g. Brown and Rosenthal (1990) and Camerer (2003). However, a standard critique is that laboratory subjects are not sufficiently trained and incentivized to behave sufficiently closely to the predictions of game theory. The influential study by Walker and Wooders (2001) concludes that "the theory has performed far better in explaining the play of top professional tennis players in our data set" (p. 1535). Similar results have been found in to other sports such as soccer (see, e.g. Chiappori, Levitt, and Groseclose (2002)) who study the direction of penalty kicks. The general conclusion is encapsulated in the title of the study by Palacios-Huerta (2003), "Professionals Play Minimax" (see also Palacios-Huerta (2014)).

In contrast, we show that the serve strategies of elite tennis pros are inconsistent with the minimax prediction. Though they use mixed strategies, the probability of winning is not the same for all serve locations — the key restriction of the Nash equilibrium/minimax solution. There has also been considerable work on testing for serial independence in serve directions as an additional implication of mixed strategy equilibrium. We argue that serial dependence, which has been found in many previous studies including Walker and Wooders (2001), is not necessarily inconsistent with equilibrium play when we account for *muscle memory effects* that reflect natural improvements from repeating recently-performed actions, and show that such muscle memory effects can induce both positive and negative serial correlation in serve directions. Our empirical analysis confirms that muscle memory effects are important to explain observed serve behavior.

Our empirical analysis exploits a new source of data, the Match Charting Project, that allows us to analyze a large number of professional tennis matches at the level of individual server-receiver pairs. Unlike previous analyses, we have also used body serves — a feature of the MCP data — in addition to the left and right serves that have been the focus of the previous literature. Tennis players and coaches consider body serves to be an important component of an optimal server strategy, a view supported by our analysis, since they are used frequently in the data and in the calculated optimal serve strategies.

However, the inclusion of body serves and access to more observations are not the main reason for the rejection of the hypothesis of Nash equilibrium play. Our main innovation is to provide new, more powerful tests of Nash equilibrium. We have introduced an omnibus Wald test for equal win probabilities for all serve directions that for rejects the hypothesis of equal win probabilities for the majority of the 10 elite professional server-receiver pairs we analyzed as well as the majority of an additional 66 male and female top ranked professional pairs. We also introduced an alternative direct test of the key implication of Nash equilibrium: namely, that there is no deviation strategy that can strictly improve the payoff of the players. Using numerical dynamic programming and our econometric estimates of the point outcome probabilities (POPs) that capture the probabilistic outcomes of serves to each possible direction, we reject the hypothesis that the observed mixed strategies of these elite pro servers are best responses.

Previous approaches to testing for equal win probabilities across serve directions have focused on testing how serve directions affect the *probability of winning individual points*, whereas we recursively calculate how the choice of serve direction affects the *probability of winning the entire service game*. Tests based on the former have low power to detect evidence of disequilibrium play because the deviation gains at the level of individual points are smaller and statistically more difficult to detect without a large number of serve observations as we show in Appendix D. By focusing on the conditional win probabilities for the entire service game, we developed much more powerful tests of the key implications of Nash equilibrium play that exploit the *magnification effect* — the smaller deviation gains at each individual point of the service game cumulate into much more substantial and easier to detect deviation gains in the service game as a whole.

We have used dynamic programming to construct best-response serve strategies, and we showed that they significantly increase the probability of winning the overall service game. We used stochastic simulations to show that our calculated deviation gains are *robust* in the sense that they result in significantly higher win probabilities even if the true POPs differ from the estimated POPs that these strategies were "expecting."

Similar to WW, our conclusion is based on a key *stationarity assumption* that all learning and strategy experimentation has already taken place, and that strategies do not change across games. However, our stationarity assumption is substantially weaker than WW's: like them, we assume stationarity in play across different service games, but unlike WW, we relax the assumption of stationary play *over different states within a service game.* We have shown that serve strategies and win probabilities vary significantly across states within an individual service game in tennis. We show that the stationarity assumption is testable, and that we cannot reject stationarity of the POPs, though we do reject stationarity of the serve strategies (CCPs). We interpret the latter rejection as further evidence against Minimax play, since if the POPs are stationary and serve strategies correspond to a unique MPE of the service game, then serve strategies should be stationary as well.

A reviewer of this article raised the critique that our structural model fails to account for persistent private information of the players, such as relating to their health or stamina during a match that could affect serve strategies in a way our muscle memory specification cannot account for, such as a declining "stamina" during a match or game. As the reviewer observed, "A tired player may be significantly less likely to win serves that lead to prolonged rallies, and it seems likely that certain serve directions are more likely to lead to such rallies than others." Additionally, the reviewer points out that our models do not account for private information about the POPs and this can result "in the econometrician observing POPs that are very different from the POPs observed by the player. Therefore, the econometrician will be using the wrong statistics to test the 'equal win probabilities' hypothesis, rendering the test invalid."

We acknowledge these thoughtful critiques and agree that our structural models, which account for private information shocks in an *IID* manner and persistent shocks via muscle memory, do not adequately capture the effects of persistent health shocks and declining stamina which likely affect play including serve strategy. Extending our model to allow for persistent private information would be a major undertaking, and would require a different notion of equilibrium (e.g. Perfect Bayesian Equilibrium) that may not even entail mixed strategy play as a necessary condition for equilibrium, similar to the way the addition of additive *IID* private information shocks results in a pure rather than mixed strategy equilibrium that only appears to be mixed due to unobserved shocks only the players observe. We also acknowledged from the start that there are many unobserved actions players take as well, such as the speed and spin of a serve, the receiver's "focus" in returning a serve, and the many decisions that players make during a rally.

However while it is true that there are many states and decisions in tennus that we cannot observe, there are key decisions and outcomes we do observe: the server's choice of direction and outcomes of individual points and games. our perspective has been to treat our models as *projections* of the higher dimensional strategies and POPs that depend on both states and actions we do not observes onto the information we can observe. Let $W(\xi)$ denote the expected win probability at the start of a tennis service game where the private information of the players at the start of the game is denoted by the random vector ξ. While we do not observe ξ, using our data we can estimate $W = E\{W(\xi)\}$, which can be treated as the "projection" of the random variable $W(\xi)$ onto the information we do observe. In this paper we have shown there exist serve strategies that depend on the limited public information we do observe: the players, the court and (x,m) (the score state and muscle memory state at each point in the service game) that significantly increase the expected win probability W. That is, we construct alternative, feasible serve strategies P' that imply a win probability W' satisfying $W' > W = E\{W(\xi)\}$. If we have convinced the reader that these counterfactual serve strategies constitute "one shot deviations" that can increase the expected win probability not only in our simplified, but admittedly misspecified model of tennis, but can also increase win probabilities if implemented in practice, then this suggests that our failure to observe and model all relevant states and decisions in the game of tennis does not affect our basis conclusion of disequilibrium play in tennis.

Our conclusion that many elite tennis pros fail to play serve strategies that are best responses to their opponents may also seem surprising given the stakes involved in top level tennis matches, and is clearly contrary to the consensus in the literature noted above. We believe that we have convincing evidence of suboptimal serve strategies, but the ultimate test would be to run field experiments to verify whether our DP serve strategies really do deliver the increased win probabilities that we predict. Our predicted gains may dissipate rapidly in the field as the receiver recognizes and adapts to a change in the server's strategy. Ultimately, the issues raised by the possibility of learning and adaptation to changes in strategy are beyond the scope of this analysis.

More generally, while we are convinced that many of the elite pro tennis players are not playing best responses, we are not entirely sure why. Indeed, the monetary rewards to increasing the service game win chances by the magnitudes we estimate are very high. The usual presumption in economics and much of the previous literature on tennis is that when there are high rewards, we can expect to see behavior that is consistent with Nash equilibrium. Or at least we should not see *large* gains left unexploited. An alternative hypothesis that is consistent with our findings is the principle of *satisficing* of Simon (1956): "Both from these scanty data and from an examination of the postulates of the economic models it appears probable that however adaptive the behavior of organisms in learning and choice situations, this adaptiveness falls far short of the ideal of 'maximizing' postulated in economic theory. Evidently, organisms adapt well enough to 'satisfice'; they do not, in general, 'optimize.' "

We estimated dynamic structural models of serve choices to gain insight into why these elite pro servers fail to use Nash equilibrium strategies. These models rationalize serve strategies as best responses, but instead of to the true POPs, they are best responses to distorted, subjective POPs that deviate significantly from the true POPs. We have no explanation for this seeming failure of rational expectations on the part of elite pro servers, but it constitutes our best explanation of the behavior we observe. We also introduced a generalized version of the monotonicity condition of Walker et al. (2011) and confirmed empirically that it typically holds for the server-receiver pairs we analyze, even in the presence of "muscle memory." When the generalized monotoncity condition holds, the tennis server can maximize the chance of winning the overall game by solving a much simpler two period DP problem that maximizes the probability of winning each point rather than having to solve a full (infinite horizon) DP problem that maximizes the probability of winning the overall game. Consequently, identifying profitable deviation strategies does not require sophisticated dynamic analysis. This suggests that disequilibrium play in tennis is unlikely to be driven by the inability of top tennis players to do the relevant mental calculations; for instance, it does not take complex calculations for Djokovic to hit more first serves to Nadal's backhand or hit more second serves wide.

Thus, our findings lead us to conclude that even the elite pro tennis players may have inadequate statistical knowledge or an inadequate mental model of the POPs, the point outcome probabilities that implicitly embody their own strengths and weaknesses as tennis players as well as those of their opponents. The rising industry of *sports analytics* may lead to new awareness and changes in behavior that motivates tennis players to change their strategies to gain advantages over their opponents. The steady state outcome of such learning and experimentation could well be something that more closely approximates Nash equilibrium play.

We note that we are not the first study to have provided evidence that suggests highly compensated and motivated sports professionals may not be behaving optimally. There is the famous book *Moneyball* by M. Lewis (2003) that showed how sports analytics could improve the performance of entire baseball teams. Focusing on individual baseball players, Bhattacharya and Howard (2022) showed that while pitchers use mixed strategies over pitch types (fastball, curveball, etc), “payoffs differ significantly across pitch types” (p. 350). In football, Romer (2006) used dynamic programming to demonstrate that most teams make suboptimal decisions regarding when to go for it on fourth down, punt, or kick a field goal. We feel that tennis may be another sport where econometrics, dynamic programming, and analytics could affect thinking, change behavior, and help guide players to play in a way that more closely corresponds to the predictions of Nash equilibrium.

Appendices

A Existence and Uniqueness

A.1 A Value Contraction for Markov Chains with Terminal Rewards

Let Z be a finite set of states. Let $q(z',z)$ be the transition chance from state z to z', and let $q^n(z',z)$ be the chance that the stochastic process is in state z' after exactly n steps when starting in state z. Define $q(Z',z) \equiv \sum_{z' \in Z'} q(z',z)$ and $q^n(Z',z) \equiv \sum_{z' \in Z'} q^n(z',z)$ for all $Z' \subseteq Z$. States $Z_A \subseteq Z$ are terminating states, i.e. $q(z,z) = 1$ for all $z \in Z_A$. Assume terminal reward function $g : Z_A \to [0,1]$, i.e. $g(z)$ is the reward when the process first enters terminal state z. Let $\mathcal{W}$ be the space of functions mapping from Z to $[0,1]$, with $W(z) = g(z)$ for all $z \in Z_A$ and define the value mapping:

$$TW(z) = \sum_{z'} q(z',z)W(z') \quad \forall z \in Z \setminus Z_A \text{ and } W \in \mathcal{W} \tag{26}$$

The iterated map is defined recursively: $T^2W = T(TW)$ and $T^nW = T(T^{n-1}W)$.

Lemma 1. *Assume there exists an integer $k < \infty$ and a constant $q_0 > 0$, s.t. $q^k(Z_A,z) \geq q_0$ for all $z \in Z \setminus Z_A$, then T^k is a contraction of modulus $1-q_0$ in the sup-norm.*

Proof. First, we show via induction on k that:

$$T^kW(z) = \sum_{z' \in Z_A} q^k(z',z)g(z') + \sum_{z' \in Z \setminus Z_A} q^k(z',z)W(z') \quad \forall z \in Z \setminus Z_A \text{ and } W \in \mathcal{W} \tag{27}$$

To verify (27) for $k = 1$, simply rewrite (26), as follows:

$$TW(z) = \sum_{z' \in Z_A} q(z',z)g(z') + \sum_{z' \in Z \setminus Z_A} q(z',z)W(z') \quad \forall z \in Z \setminus Z_A \text{ and } W \in \mathcal{W}$$

For the inductive step assume (27) is satisfied for $k-1$, then for $z \in Z \setminus Z_A$ and $W \in \mathcal{W}$:

$$\begin{aligned}
T^kW(z) &= \sum_{z' \in Z_A} q(z',z)g(z') + \sum_{z' \in Z \setminus Z_A} q(z',z)T^{k-1}W(z') \\
&= \sum_{z' \in Z_A} q(z',z)g(z') + \sum_{z' \in Z \setminus Z_A} q(z',z)\left[\sum_{z'' \in Z_A} q^{k-1}(z'',z')g(z'') + \sum_{z'' \in Z \setminus Z_A} q^{k-1}(z'',z')W(z'')\right] \\
&= \sum_{z' \in Z_A} q(z',z)g(z') + \sum_{z'' \in Z \setminus Z_A} q^k(z'',z)g(z'') + \sum_{z'' \in Z \setminus Z_A} q^k(z'',z)W(z'') \\
&= \sum_{z' \in Z_A} q^k(z',z)g(z') + \sum_{z' \in Z \setminus Z_A} q^k(z',z)W(z')
\end{aligned}$$

Having verified (27), the fact that T^k is a contraction of modulus $1-q_0$ whenever $q^k(Z_A,z) > q_0$ for all $z \in Z \setminus Z_A$ follows easily from standard reasoning. $\square$

A.2 Value and MPE Existence: Proof of Theorem 1

Proof. STEP 1: A UNIQUE EQUILIBRIUM VALUE. Let $Z = \{1,2,\ldots,38\} \times \{l,r,b\}^2$ be the set of states, and $\Gamma(z)$ be the subgame starting in state z. For any pure strategies (d,s,a), define the transition function $q(z',z|d,s,a)$. Easily, q inherits continuity in (s,a) from ℓ and ω. For any function $v : \{1,2,\ldots,38\} \times \{l,r,b\}^2 \mapsto [0,1]$ with $v(38,m) = 0$ and $v(37,m) = 1$, define the static zero-sum game with server payoff:

$$u(z,d,s,a|v) \equiv \sum_{z' \in Z} q(z',z|d,s,a)v(z')$$

Let $\mathcal{B}$ be the set of probability distributions over $\{l,r,b\} \times \mathcal{S}$ and let $\mathcal{A}$ be the set of probability distributions over receiver attention vectors. Since q is continuous in (s,a), u is continuous in (s,a) for any fixed v; the Minimax theorem in Ville (1938) applies:

$$\min_{\alpha \in \mathcal{A}} \max_{\beta \in \mathcal{B}} \int u(z,d,s,a|v)d\beta(d,s)d\alpha(a) = \max_{\beta \in \mathcal{B}} \min_{\alpha \in \mathcal{A}} \int u(z,d,s,a|v)d\beta(d,s)d\alpha(a)$$

Altogether, the recursive sub-game $\Gamma(z)$ meets the premise of Theorem 6 in Everett (1957); and thus, there exists a unique value $v^* : \{1,2,\ldots,38\} \times \{l,r,b\}^2 \mapsto [0,1]$ in each $\Gamma(z)$.

STEP 2: PAYOFFS ARE WELL-DEFINED FOR ALL STATIONARY STRATEGIES. Let $Z_A = \{37,38\} \times \{l,r,b\}^2$, $g(37,m) = 1$, and $g(38,m) = 0$ for all muscle memory states m. Stationary pure strategies are functions $B : Z \setminus Z_A \mapsto \{l,r,b\} \times \mathcal{S}$ and $A : Z \setminus Z_A \mapsto \mathcal{A}$. Any pair of pure strategies induces transition chance $q(z',z|B(z),A(z))$. Since $\omega\ell \geq \underline{w} > 0$, there is an integer $k < \infty$ and a uniform lower bound $q_0 > 0$ on the chance that the server wins ($x = 37$) within k serves starting from any non-terminal state. Thus, the transition function $q^n(z',z|B,A)$ obeys the premise of Lemma 1. Consequently, the chance that the server wins the service game starting in any state z given strategies B,A is the unique fixed point of the contraction mapping (27), which we write as:

$$T^k W(z|B,A) = \sum_{z' \in Z_A} q^k(z',z|B,A)g(z') + \sum_{z' \in Z \setminus Z_A} q^k(z',z|B,A)W(z'|B,A) \tag{28}$$

to emphasize the dependence on strategies (B,A). Let $U(z|B,A)$ be this unique fixed point.

Step 3: MPE Existence. We now consider variation in strategies; and thus, reinterpret (28) as a mapping on the space of functions that map from $Z \setminus Z_A \times \{l,b,r\} \times S \times A$ to $[0,1]$ and are continuous in (s,a). Since, $q(\cdot|d,s,a)$ is continuous in (s,a) by Step 1 and T^k is a contraction, the fixed point $U(z|B,A)$ is continuous in (B,A). Furthermore, the strategy sets S and A are both compact. Thus, by Glicksberg (1952) there exists a Nash Equilibrium $(\sigma_R^*(z), \sigma_S^*(z))$ of the static game with payoffs $U(z|B,A)$. By construction this is an MPE of the dynamic game. □

A.3 Unique Attention, Speed, and Spin: Proof of Theorem 2

Step 0: Winning a Serve is Strictly Preferred to Losing. Easily, if we fix the serve direction, the server strictly prefers: winning on the current serve to faulting, faulting a first serve to losing the point on the first serve, and winning a point to losing i.e.:

$$\begin{aligned} W(x^+(x),m) &> W(x+1,m) \quad \text{and} \quad W(x+1,m) > W(x^-(x),m) \quad \forall x \text{ odd} \\ W(x^+(x),m) &> W(x^-(x),m) \quad \forall x \end{aligned}$$

This follows easily from our assumption that $\ell\omega$ is bounded away from zero and 1 for all strategy pairs. In particular, for any arbitrarily (perhaps history contingent) equilibrium strategy for the receiver, the server could always adopt the (perhaps history contingent) strategy following a point win (or faulted first serve) that he would have adopted in the alternate universe that he lost the point. Easily, whatever this strategy is, the chance of eventually winning the game is strictly higher following a point win (or fault) than following a lost point.

Step 1: The Set of MPE is Convex. Since the equilibrium value is unique, we can write the static zero sum payoff for all first serves (x odd) as:

$$\begin{aligned} u(x,m,d,s,a) \;\equiv\; & W(x+1,m') + \ell(m,d,c(x),s)\omega(m,d,c(x),s,a)\left[W(x^+(x),m') - W(x+1,m')\right] \\ & + \; \ell(m,d,c(x),s)(\omega(m,d,c(x),s,a)-1)\left[W(x+1,m') - W(x^-(x),m')\right] \end{aligned} \tag{29}$$

where $m' = (d, m_1)$, and for all second serves (x even):

$$u(x,m,d,s,a) \equiv W(x^-(x),m) + \ell(m,d,c(x),s)\omega(m,d,c(x),s,a)\left[W(x^+(x),m) - W(x^-(x),m)\right]$$

Note that W is not a function of a or s, and all terms in square brackets are strictly positive by Step 0; and thus, by Assumption 2, u is strictly convex in a and strictly concave in s for both odd and even x. Furthermore, the server has a finite choice of serve directions d. Altogether, by Proposition 3.1(i) in Hwang and Rey-Bellet (2020) the set of MPE is convex.

STEP 2: UNIQUE a AND s FOR ALL (x,m). We prove that there is a unique NE attention in each state (x,m). The proof of uniqueness of the server's NE (speed, spin) choice in each state follows parallel steps. First, given any arbitrary mixed server strategy $\beta \in \mathcal{B}$ define the payoff:

$$\nu(x,m,\beta,a) \equiv \int u(x,m,d,s,a)d\beta(d,s)$$

Trivially, ν is linear in β, and inherits strict convexity in a from u. Since the receiver's best response attention minimizes ν, which is strictly convex in a, the receiver uses a pure strategy at every (x,m) in any NE.

Now, toward a contradiction assume that $(\hat{\beta},\hat{a})$ and (β',a') are both Nash Equilibria in state (x,m) with $\hat{a} \neq a'$. Define $(\beta'',a'') \equiv \lambda(\hat{\beta},\hat{a}) + (1-\lambda)(\beta',a')$ for some $\lambda \in (0,1)$. Then use $\nu^* = \nu(x,m,\hat{\beta},\hat{a}) = \nu(x,m,\beta',a')$, followed by the fact that $\hat{\beta}(\beta')$ maximizes ν given $\hat{a}$ (a'), to get:

$$\begin{aligned}
\nu^* &= \nu(x,m,\hat{\beta},\hat{a}) &\geq \nu(x,m,\beta'',\hat{a}) \\
\nu^* &= \nu(x,m,\beta',a') &\geq \nu(x,m,\beta'',a')
\end{aligned}$$

Now take a $(\lambda, 1-\lambda)$ weighted average of the above two inequalities, and then use ν strictly convex in a and $\hat{a} \neq a'$ to discover:

$$\nu^* \geq \lambda\nu(x,m,\beta'',\hat{a}) + (1-\lambda)\nu(x,m,\beta'',a') > \nu(x,m,\beta'',a'')$$

And thus, (β'',a'') cannot be a NE at (x,m), since all NE must have the same value ν^*. But this contradicts Step 1, i.e. the set of NE convex for all (x,m). □

A.4 Optimality of Point-Myopic Play: Proof of Theorem 3

Suppose GMC holds, see equations (13) and (14) of Assumption 2 in section 3.5. We will now show that a strategy that maximizes the probability of winning any point of the service game also maximizes the probability of winning the service game overall. Consider the second serve of any point. Equation (11) of section 3.4 gives the choice-specific probability of winning the second serve. Let $V_2(x,m,d)$ denote this probability. It follows that an optimal point-myopic second serve strategy is the serve direction d that maximizes $V_2(x,m,d)$ given in equation (11) (if there are multiple direction hat maximize this win probability, we can interp this as a case of a mixed strategy where the server mixes over the set of optimal serve directions).

Now consider a fully dynamic optimal serve strategy at the same second serve state. In this case the optimal serve direction must maximize the *game win probability* $W_S(x,m,d)$ given in equation (3) of section 3.2. Subtracting the optimal game win probability $W_S((x^-(x),m)$ corresponding to losing the serve at second serve state x from both sides of equation (3) does not change the serve direction that maximizes the left hand side of this equation. Rearrange terms after doing this subtraction to get

$$\begin{aligned} W_S(x,m,d) - W_S(x^-(x),m) &= \pi(\text{in}|x,m,d)\pi(\text{win}|x,m,d)[W_S(x^+(x),m) - W(x^-(x),m)], \\ &= V_2(x,m,d)[W_S(x^+(x),m) - W(x^-(x),m)], \end{aligned} \tag{30}$$

where we have used our assumption that muscle memory m is not updated at 2nd serve states, and hence m is not affected by the choice of serve direction d. By inequality (13) of the GMC the term $[W_S(x^+(x),m) - W(x^-(x),m)]$ is strictly positive and independent of d. Thus, a serve direction d is an element of an optimal point myopic strategy in a second serve state (i.e. a maximizer of $V_2(x,m,d)$) if and only if it is an element of an optimal fully dynamic strategy, i.e. a maximizer of of probability of winning the entire game, $W_s(x,m,d)$. Thus, GMC implies that for all second serves the point-myopic serve strategy and the fully dynamic serve strategies coincide.

Now consider any first serve state, x, which via our encoding of tennis score states must be an odd integer. A point myopic serve strategy at the first serve maximizes the probability of winning the point, taking the option of the second serve into account. The value of making the first serve to direction d under a point myopic serve strategy is given by equation (12) of section 3.4, where the option value of a second serve is captured by the value function $V_2(x+1,m')$ given by

$$V_2(x+1,m') = V_2(x+1,(d,d_1)) = \max_{d' \in \{l,b,r\}} \pi(\text{in}|d',x+1,(d,d_1))\pi(\text{win}|d',x+1,(d,d_1)). \tag{31}$$

Note how the presence of muscle memory causes the server to account for how their choice of first serve direction d will affect their muscle memory state $m' = (d,d_1)$ at the second serve, and thus the value of the second serve $V_2(x+1,(d,d_1))$. Thus, the probability of winning the point for a point-myopic server who serves to direction d is given by

$$\begin{aligned}
V_1(d,x,m) &= V_1(d,x,(d_1,d_2)) \qquad (32)\\
&= \pi(\text{in}|d,x,(d_1,d_2))\pi(\text{win}|d,x,(d_1,d_2)) + [1-\pi(\text{in}|d,x,(d_1,d_2))]V_2(x+1,(d,d_1))\\
&= \pi(\text{in}|d,x,(d_1,d_2))\pi(\text{win}|d,x,(d_1,d_2)) + \\
&\quad [1-\pi(\text{in}|d,x,(d_1,d_2))]\left[\max_{d'\in\{l,b,r\}} \pi(\text{in}|d',x+1,(d,d_1))\pi(\text{win}|d',x+1,(d,d_1))\right].
\end{aligned}$$

Now consider the probability of winning the service game at the same first serve state (x,m) when serving to direction d. This probability is $W_s(x,m,d)$, the choice-specific value function is given by equation (2) of section 3.2. In all first serve states the muscle memory state is updated from its current value $m=(d_1,d_2)$ to $m'=(d,d_1)$ to reflect the current choice of serve direction (where recall d_1 is the serve direction of the most previous first serve and d_2 is the serve direction of the first serve two first serves ago, unless $x=1$ in which case $m=(\emptyset,\emptyset)$ or $x\in\{3,9\}$ in which case $m=(d_1,\emptyset)$). Once again we can do the trick of subtracting $W_s(x^-(x),m')$ from both sides of the equation for $W_s(x,m,d)$ in equation (2). Doing some algebra, we get

$$\begin{aligned}
W_S(x,m,d) &= \pi(\text{in}|x,m,d)\pi(\text{win}|x,m,d)[W_S(x^+(x),(d,d_1)) - W_S(x^-(x),(d,d_1))] + \qquad (33)\\
&\quad [1-\pi(\text{in}|x,m,d)][W_S(x+1,(d,d_1)) - W_S(x^-(x),(d,d_1))]\\
&= V_1(d,x,m)\left[W_S(x^+(x),(d,d_1)) - W_S(x^-(x),(d,d_1))\right] - W_S(x^-(x),(d,d_1)),
\end{aligned}$$

where $V_1(d,x,m)$ is the point win probability at the first serve to direction d given in equation (33). But by the GMC assumption in equation (13) of Assumption 2 in section 3.5, the game win probabilities on the right hand size of the last equation (34) are independent of d. It follows that whichever serve direction maximizes the probability of winning the point at any first serve state, $V_1(d,x,m)$ also maximizes the the probability of winning the entire service game, $W_S(x,m,d)$. This implies that in all first serve states the optimal point myopic serve strategy coincides with the optimal dynamic strategy that maximizes the probability of winning the entire service game. It is also evident that if there are equal point win probabilities across different serve directions d at any state (x,m), then the corresponding game win probabilities are equal as well. It follows that if GMC holds, we have equal point win probabilities across any subset of serve directions in any state of the game if and only if there are equal game win probabilities for the same set of serve directions in the same corresponding states. □

B Preferred Specification Estimates: Federer and Djokovic

Tables 11 and 12 present the maximum-likelihood estimates of the parameters $\hat{\theta} = (\hat{\theta}_P, \hat{\theta}_{in}, \hat{\theta}_{win})$ of our preferred specification of the reduced-form model of serves and POPs, where we restrict the sample to hard courts only. Our preferred specification has the smallest AIC over the four alternative specifications we consider, including a restricted specification that rules out muscle memory effects by excluding the variable m from the set of indicator functions $\{f(x,m,d), g_{in}(x,m,d), g_{win}(x,m,d)\}$ defining each specification. Our preferred specification has 12 parameters for $P(d|x,m,\theta_P)$ and 32 parameters for $(\pi(in|x,m,d), \pi(win|x,m,d))$ (16 parameters each). We do not have the space to present the estimates for all the players we analyze in this paper. However, we do show the parameters for two specific server-receiver pairs, Djokovic serving to Federer and Federer serving to Djokovic.

In Table 11, we present the parameter estimates $\hat{\theta}_P$ determining the reduced-form serve probabilities $P(d|x,m,\hat{\theta}_P)$. Recall that we code muscle memory as $m = (d_{-2}, d_{-1})$, where d_{-2} is the direction of the serve two first serves ago (i.e. the direction of the previous first serve to the same court) and d_{-1} is the direction of the previous first serve. Parameters 3 and 9 are significantly positive for both Djokovic and Federer, which indicates that their first serves exhibit *positive* serial correlation: there is an increased likelihood of serving to direction $d = d_{-2}$, where the latter is the direction of the most recent first serve to the same court. However, the coefficients on Parameters 6 and 12 are not significantly different from zero, which is consistent with a lack of significant serial correlation between the directions of the faulted first and subsequent second serves.

The other parameter estimates in Table 11 determine the directions of the first and second serves to the ad and deuce courts separately. Our specification normalizes $f(x,m,d) = 0$ when $d = b$ (i.e. body serves), so the coefficient estimates for the other serve directions $d \in \{l, r\}$ are enough to determine all three probabilities $P(d|x,m)$ for any given (x,m) value. The implicit restriction in our specification is that serve directions only depend on whether the current serve is a first serve or second serve, and whether it is to the ad or deuce court. The large positive values of Parameters 1, 2, 7, and 8 indicate that on first serves, Djokovic and Federer typically serve more often to the left or right than to the body (approximately 18% to the body for Djokovic, which is still significantly greater than 11% for Federer). However, the negative estimates of

Table 11: ML parameter estimates, reduced-form model of serve directions, $P(d|x,m,\theta_P)$

Parameter Name		**Djokovic** $\rightarrow$ Federer		**Federer** $\rightarrow$ Djokovic	
		Estimate	Standard Error	Estimate	Standard Error
1	1st serve, ad court, $d = l$	.861	(.095)	1.367	(.133)
2	1st serve, ad court, $d = r$	.601	(.100)	1.319	(.132)
3	1st serve, ad court, $d = d_{-2}$	.294	(.079)	.396	(.086)
4	2nd serve, ad court, $d = l$	.462	(.146)	.551	(.152)
5	2nd serve, ad court, $d = r$	$-.386$	(.183)	$-.215$	(.171)
6	2nd serve, ad court, $d = d_{-1}$	$-.051$	(.136)	$-.143$	(.142)
7	1st serve, deuce court, $d = l$	.836	(.102)	1.234	(.145)
8	1st serve, deuce court, $d = r$	.795	(.106)	1.402	(.143)
9	1st serve, deuce court, $d = d_{-2}$	.566	(.077)	.483	(.103)
10	2nd serve, deuce court, $d = l$	$-.208$	(.145)	$-.293$	(.132)
11	2nd serve, deuce court, $d = r$	$-.650$	(.151)	$-.531$	(.159)
12	2nd serve, deuce court, $d = d_{-1}$	$-.002$	(.168)	.137	(.144)
Observations, log-likelihood		2372, -2324.8		2333, -2265.06	
AIC, BIC		4871.6, 4940.8		4554.1, 4623.2	

Parameters 10 and 11 indicate that on second serves to the deuce court, both players increase their probability of hitting a body serve significantly (to about 43% for both servers).

Table 12 shows the parameter estimates of our preferred specification for the POPs $(\hat{\theta}_{in}, \hat{\theta}_{win})$. None of the parameters for indicators that depend on the muscle memory state variable m are significant for $\pi(in|x,m,\hat{\theta}_{in})$ (see Parameters 4, 8, 12 and 16), which indicates that serving to the same direction as the previous serve does not reduce their probability of faulting. However, Parameter 4 of $\hat{\theta}_{win}$ is positive and significant for Djokovic serving to Federer, so here, serving to the same direction as the previous first serve to the ad court increases Djokovic's probability of winning given that the serve is in.

Comparing the parameters (1,2,3,9,10,11) of the vector $\hat{\theta}_{win}$, we see that they are uniformly (i.e. across all three directions) significantly larger for Federer than for Djokovic. This implies that conditional on a first serve going in, Federer has a higher chance of winning the subsequent rally when serving to Djokovic than Djokovic does when serving to Federer. However, when we do a similar comparison of the corresponding coefficient estimates of $\hat{\theta}_{in}$, the inequality is almost uniformly reversed, indicating that Djokovic has a higher probability of making his first

Table 12: ML parameter estimates of POPs $(\pi(in|x,m,d,\theta_{in}),\pi(win|x,m,d,\theta_{win}))$

Parameter		**Djokovic** → Federer		**Federer** → Djokovic	
θ_{in}		Estimate	Standard Error	Estimate	Standard Error
1	1st serve, ad court, $d = l$	.465	(.147)	.486	(.140)
2	1st serve, ad court, $d = b$	.945	(.213)	.864	(.230)
3	1st serve, ad court, $d = r$	.744	(.134)	.614	(.115)
4	1st serve, ad court, $d = d_{-2}$	−.093	(.156)	−.113	(.145)
5	2nd serve, ad court, $d = l$	3.468	(.572)	2.277	(.403)
6	2nd serve, ad court, $d = b$	2.137	(.288)	3.249	(.459)
7	2nd serve, ad court, $d = r$	2.150	(.445)	1.898	(.360)
8	2nd serve, ad court, $d = d_{-2}$	.277	(.513)	.118	(.440)
9	1st serve, deuce court, $d = l$	.604	(.159)	.122	(.148)
10	1st serve, deuce court, $d = b$	.928	(.173)	.530	(.232)
11	1st serve, deuce court, $d = r$	.813	(.142)	.422	(.128)
12	1st serve, deuce court, $d = d_{-2}$	−.138	(.165)	.164	(.154)
13	2nd serve, deuce court, $d = l$	2.214	(.406)	2.652	(.362)
14	2nd serve, deuce court, $d = b$	1.917	(.352)	3.820	(.719)
15	2nd serve, deuce court, $d = r$	1.430	(.374)	2.033	(.387)
16	2nd serve, deuce court, $d = d_{-2}$	.294	(.452)	.329	(.499)
θ_{win}		Estimate	Standard Error	Estimate	Standard Error
1	1st serve, ad court, $d = l$	.641	(.182)	1.092	(.208)
2	1st serve, ad court, $d = b$	.470	(.196)	.760	(.250)
3	1st serve, ad court, $d = r$	.439	(.143)	.795	(.154)
4	1st serve, ad court, $d = d_{-2}$	.456	(.188)	.314	(.210)
5	2nd serve, ad court, $d = l$	.975	(.235)	.148	(.247)
6	2nd serve, ad court, $d = b$	.593	(.195)	.073	(.182)
7	2nd serve, ad court, $d = r$	.650	(.290)	.063	(.285)
8	2nd serve, ad court, $d = d_{-2}$	−.537	(.273)	.038	(.270)
9	1st serve, deuce court, $d = l$	.878	(.185)	1.397	(.232)
10	1st serve, deuce court, $d = b$	.614	(.223)	1.055	(.305)
11	1st serve, deuce court, $d = r$	.728	(.171)	.933	(.167)
12	1st serve, deuce court, $d = d_{-2}$	−.182	(.194)	−.392	(.216)
13	2nd serve, deuce court, $d = l$	−.090	(.221)	.292	(.221)
14	2nd serve, deuce court, $d = b$	−.282	(.222)	−.022	(.222)
15	2nd serve, deuce court, $d = r$	.530	(.295)	.142	(.263)
16	2nd serve, deuce court, $d = d_{-2}$	.489	(.281)	.206	(.259)
Observations, log-likelihood		2333, -2403.9		2372, -2324.8	
AIC, BIC		4871.9, 5056.6		4625.4, 4809.6	

serve than Federer.[38] Thus, our estimates reflect an intuitive trade-off: a faster serve or one aimed closer to the lines has a higher chance of missing, but conditional on it going in, the receiver has a lower chance of returning it successfully or winning the subsequent rally.

C Alternative Subsample Stationarity Tests

We revisit the stationarity tests in Section 4.3, except that here we divide all serves on hard courts for each of our ten elite server-receiver pairs into serves from the first set in the match (no matter when that match was played) and serves from the later sets of a match. We then estimate POPs and CCPs separately for these two subsamples and perform stationarity tests as in Section 4.3.

Table 13 presents results for the POPs, which are strikingly similar to those for the calendar year groupings in Section 4.3. We get one rejection at the 5% level for the muscle memory specification, and no rejections (even at the 10%) level for the no muscle memory specification. As in Section 4.3, stationarity is rejected more frequently for the CCPs (Table 14).

[38] Though for second serves to the deuce court, Federer has a uniformly lower chance of double faulting.

Table 13: Within-Match Tests for stationarity of POPs: $\{\pi(in|x,m,d,\theta_{in}),\pi(win|x,m,d,\theta_{win})\}$

Server → receiver	Muscle Memory			No Muscle Memory		
	Restricted	Unrestricted	LR test (df)	Restricted	Unrestricted	LR test (df)
	LL, AIC	LL, AIC	P-value	LL, AIC	LL, AIC	P-value
Roger Federer →	-1934.3	-1919.8	29.0 (32)	-1940.1	-1928.0	24.3 (24)
Rafael Nadal	**3932.6**	3967.6	.620	**3928.2**	3952.0	.445
Rafael Nadal →	-1880.9	-1860.3	41.3 (32)	-1883.2	-1867.5	31.4 (24)
Roger Federer	**3825.9**	3848.6	.125	**3814.5**	3831.1	.142
Roger Federer →	-2280.7	-2265.7	30.0 (32)	-2284.7	-2272.6	24.2 (24)
Novak Djokovic	**4625.4**	4659.4	.570	**4617.5**	4641.3	.448
Novak Djokovic →	-2403.9	-2389.0	29.9 (32)	-2411.7	-2399.0	25.2 (24)
Roger Federer	**4871.9**	4906.0	.572	**4871.3**	4894.1	.393
Rafael Nadal →	-1414.2	-1397.0	34.4 (32)	-1415.8	-1404.7	22.2 (24)
Novak Djokovic	**2892.4**	2922.0	.355	**2879.6**	2905.4	.565
Novak Djokovic →	-1302.1	-1289.7	24.7 (32)	-1304.5	-1294.3	20.3 (24)
Rafael Nadal	**2668.1**	2707.5	.819	**2656.9**	2684.6	.681
Novak Djokovic →	-1183.2	-1166.0	34.4 (32)	-1188.7	-1179.3	19.0 (24)
Andy Murray	**2430.3**	2459.9	.355	**2425.5**	2454.5	.753
Andy Murray →	-1280.1	-1263.7	32.7 (32)	-1287.9	-1277.1	21.6 (24)
Novak Djokovic	**2624.1**	2655.4	.431	**2623.9**	2650.2	.601
Pete Sampras →	-1117.9	-1103.5	28.6 (32)	-1124.1	-1111.9	24.4 (24)
Andre Agassi	**2299.7**	2335.1	.638	**2296.2**	2319.7	.437
Andre Agassi →	-1031.1	-1005.3	51.6 (32)	-1032.6	-1017.4	30.3 (24)
Pete Sampras	**2126.2**	2183.6	.015*	**2113.2**	2130.9	.174

Table 14: Within-match tests for stationarity of CCPs: $\{P(d|x,m)\}$

Server → receiver	Muscle Memory			No Muscle Memory		
	Restricted	Unrestricted	LR test (df)	Restricted	Unrestricted	LR test (df)
	LL, AIC	LL, AIC	P-value	LL, AIC	LL, AIC	P-value
Roger Federer →	-1844.8	-1837.9	13.7 (12)	-1874.4	-1870.6	7.5 (8)
Rafael Nadal	**3713.5**	3723.8	.321	**3764.7**	3773.2	.479
Rafael Nadal →	-1688.2	-1680.3	15.7 (12)	-1690.9	-1687.9	5.9 (8)
Roger Federer	**3400.3**	3408.6	.205	**3397.8**	3407.9	.659
Roger Federer →	-2265.1	-2256.5	17.1 (12)	-2293.9	-2290.4	6.9 (8)
Novak Djokovic	**4554.1**	4561.0	.145	**4603.7**	4612.8	.545
Novak Djokovic →	-2423.8	-2411.7	24.2 (12)	-2454.8	-2441.2	27.2 (8)
Roger Federer	4871.6	**4871.3**	.019*	4925.5	**4914.3**	.001*
Rafael Nadal →	-1432.6	-1420.2	24.8 (12)	-1437.5	-1431.9	11.1 (8)
Novak Djokovic	2889.3	**2888.5**	.016*	**2891.0**	2895.9	.198
Novak Djokovic →	-1347.4	-1340.5	13.8 (12)	-1364.2	-1360.7	7.0 (8)
Rafael Nadal	**2718.9**	2729.1	.315	**2744.4**	2753.4	.539
Novak Djokovic →	-1201.6	-1191.7	19.8 (12)	-1221.0	-1217.4	7.2 (8)
Andy Murray	**2427.1**	2431.3	.071	**2458.0**	2466.8	.514
Andy Murray →	-1250.0	-1240.7	18.5 (12)	-1254.9	-1248.2	13.4 (8)
Novak Djokovic	**2524.0**	2529.5	.100	**2525.9**	2528.5	.098
Pete Sampras →	-1085.4	-1077.0	16.9 (12)	-1096.4	-1092.5	7.7 (8)
Andre Agassi	**2194.9**	2202.0	.154	**2208.8**	2217.1	.466
Andre Agassi →	-931.8	-923.6	16.4 (12)	-945.2	-939.5	11.5 (8)
Pete Sampras	**1887.6**	1895.2	.176	**1906.5**	1911.0	.176

Online Appendices

D Replicating Walker and Wooders' (2001) Analysis

We now replicate the static approach to testing for equal win probabilities used by Walker and Wooders (2001) (WW), but with our much larger dataset from the Match Charting Project. As we noted in Section 3, the WW approach is valid if 1) the Generalized Monotonicity Condition (GMC) holds (i.e. the Markov Perfect Equilbrium (MPE) for the overall service game decomposes into independent MPEs for the individual "point subgames" of the service game), and 2) their stationarity assumption holds (i.e. serves to the ad or deuce courts by a particular server-receiver pair on a given court surface can be treated as repeated static games for which serve directions and the "stage game" outcomes are *IID* random variables across points within a single service game and across service games on the same court surface for any server-receiver pair).

Our approach to testing the hypothesis that win probabilities are the same for all serve directions via the "omnibus Wald test" introduced in Section 4 is robust to relaxations of both of these assumptions. In addition, our test is significantly more powerful due to the"magnification effect;" namely, that the variation in service game win chances are typically much larger than the variation in the probability of winning any particular point in the service game. Thus, the variability in win probabilities of individual points across serve directions is far harder to detect compared to the variability in conditional win probabilities for the entire service game.

However, it is useful for comparison purposes to see if WW's findings can be replicated using our larger data set using their approach to testing for equal win probabilities. It is also useful to see if their findings are robust to several other changes to how we model the "point subgame" that they analyzed, as well as changes to the statistical method used to test the null hypothesis of equal win probabilities. We follow their assumptions and approach, except that in our "replication" below we:

1. Incorporate a third serve direction, body serves, not just serves to the left and right,
2. Include second serves after a faulted first serve, not just first serves,
3. Use both Wald and likelihood-ratio tests of the null hypothesis of equal win probabilities.

We continue to impose WW's stationarity assumption. That is, all serve directions and point-subgame outcomes to the ad and deuce courts, respectively, are *IID* random variables reflecting repeated play of the same point-subgame MPE at all states of the service game. However, unlike WW (who only considered first serves), we allow for the option of a second serve by treating the point subgame as a two-person, two-stage, constant-sum game between the server and receiver, as illustrated in Figure 3 of Section 3.2. As we noted in Table 1, there is clear evidence that servers use different mixing probabilities for first vs. second serves, which we account below in our tests of equal win probabilities for all serve directions for first and second serves.

Our tests for equal win probabilities are based on a conditional likelihood function for four possible outcomes of the point subgame conditional on the serve directions d_1 and d_2, which are, respectively, the first and second serves of the subgame. Since second serves are only attempted in response to a faulted first serve, we let $d_2 = \emptyset$ denote a null outcome when the first serve was not faulted, thereby obviating the need for a second serve. Let o denote the outcome of a generic point subgame. It has four possible values $o \in \{1,2,3,4\}$ defined as follows:

$o = 1$ First serve is in, and the server wins the subsequent rally,

$o = 2$ First serve is in, and the server loses the subsequent rally,

$o = 3$ First serve is out (faulted first serve), and the server wins with the second serve,

$o = 4$ First serve is out (faulted first serve), and the server loses with the second serve.

Let $\pi(\text{in}_1|d_1)$ denote the probability that the first serve is in (i.e. not faulted) conditional on its direction being $d_1 \in \{l,b,r\}$. Let $\pi(\text{win}_1|d_1)$ denote the probability that the server wins the rally on the first serve conditional on the first serve being in and its direction being d_1. Similarly, let $\pi(\text{lose}_1|d_1)$ be the probability that the server loses the rally following the first serve, conditional on the first serve being in and its direction being d_1. Then we have:

$$1 = \pi(\text{in}_1|d_1)\pi(\text{win}_1|d_1) + \pi(\text{in}_1|d_1)\pi(\text{lose}_1|d_1) + [1 - \pi(\text{in}_1|d_1)] \tag{34}$$

reflecting the three possible outcomes of the first serve: 1) first serve is in and the server wins the rally, 2) first serve is in and the server loses the rally, or 3) the server faults the first serve.

Let $\pi(\text{win}_2|d_2)$ denote the conditional probability of winning with the second serve given that the first serve is faulted (thus resulting in a second serve) and the second serve direction is d_2. Since there are only two possible outcomes for the second serve (i.e. the server wins or loses with it), it follows that $\pi(\text{lose}_2|d_2) = 1 - \pi(\text{win}_2|d_2)$. And similarly for first serves, $\pi(\text{lose}_1|d_1) = 1 - \pi(\text{win}_1|d_1)$.

Let $f(o|d_1,d_2)$ denote the probability of the outcome o of a point subgame conditional on the serve directions for the first and second serves (d_1,d_2), where $d_2 = \emptyset$ in cases where the first serve is not faulted. We can express this conditional probability in terms of the other conditional probabilities π defined above as:

$$\begin{aligned}
f(1|d_1,d_2) &= \pi(\text{in}_1|d_1)\pi(\text{win}_1|d_1) \\
f(2|d_1,d_2) &= \pi(\text{in}_1|d_1)\pi(\text{lose}_1|d_1) \\
f(3|d_1,d_2) &= [1-\pi(in_1|d_1)]\pi(\text{win}_2|d_2) \\
f(4|d_1,d_2) &= [1-\pi(in_1|d_1)][1-\pi(\text{win}_2|d_2)]. \qquad (35)
\end{aligned}$$

It is easy to see that $\sum_{o=1}^{4} f(o|d_1,d_2) = 1$, so $f(o|d_1,d_2)$ is a valid conditional probability. We use f as the basis for our estimation and tests of equality of win probabilities for all serve directions. Note that this distribution is fully described by nine parameters:

1. $\pi(\text{in}_1|d_1)$, $d_1 \in \{l,b,r\}$
2. $\pi(\text{win}_1|d_1)$, $d_1 \in \{l,b,r\}$
3. $\pi(\text{win}_2|d_2)$, $d_2 \in \{l,b,r\}$.

This nine-parameter model for $f(o|d_1,d_2)$ forms the basis for what we refer to as the *unrestricted log-likelihood* that does not impose any constraint that win probabilities for the first or second serve are equal across all serve directions. Note that we can define the *ex ante* win probability for the overall point subgame, accounting for the option of a second serve, as follows:

$$\text{Prob}\{\text{win point subgame}|d_1,d_2) = \pi(\text{in}_1|d_1)\pi(\text{win}_1|d_1) + [1-\pi(\text{in}_1|d_1)]\pi(\text{win}_2|d_2). \qquad (36)$$

Thus, the *hypothesis of equal win probabilities for all serve directions* (for both first and second serves) amounts to the following four restrictions on the probabilities (i.e. parameters) of the unrestricted model:

$$\begin{aligned}
\pi(\text{win}_2|l) &= \pi(\text{win}_2|r) \equiv \Pi(\text{win}_2) \qquad (37) \\
\pi(\text{win}_2|b) &= \pi(\text{win}_2|r) \equiv \Pi(\text{win}_2) \\
\pi(\text{in}_1|l)\pi(\text{win}_1|l) + [1-\pi(\text{in}_1|l)]\Pi(\text{win}_2) &= \pi(\text{in}_1|r)\pi(\text{win}_1|r) + [1-\pi(\text{in}_1|r)]\Pi(\text{win}_2) \\
\pi(\text{in}_1|b)\pi(\text{win}_1|b) + [1-\pi(\text{in}_1|b)]\Pi(\text{win}_2) &= \pi(\text{iin}_1|r)\pi(\text{win}_1|r) + [1-\pi(\text{in}_1|r)]\Pi(\text{win}_2).
\end{aligned}$$

We use both Wald and likelihood-ratio (LR) tests to test the hypothesis of equal win probabilities above. Under the Wald test, we estimate the nine parameters of the unrestricted model by maximum likelihood and then use a standard Wald test to see if the four nonlinear restrictions on the parameters given in Equation (37) hold. For the LR test, we estimate a restricted likelihood that imposes the equal win probability restrictions (37). Call the log-likelihood of the restricted model L_r and the log-likelihood of the unrestricted model L_u. Then the LR test statistic is $\text{LR} = 2(L_u - L_r)$, and it is asymptotically distributed as a χ^2 random variable with 4 degrees of freedom if the null hypothesis of equal win probabilities for all serve directions is true. Of course, the Wald test statistic is also asymptotically distributed as a χ^2 random variable with 4 degrees of freedom under the null hypothesis.

Tables 15 and 16 present the results of these tests for the subsets of first serves to the ad and deuce courts, respectively. We also present maximum-likelihood estimates of the conditional win probabilities for the first and second serves, as defined above, and we show the total number of observations of first and second serves on which these estimates are based. The numbers in parentheses below the estimated conditional win probabilities are the estimated standard errors of the parameters. These are computed from the inverse of the Hessian matrix of the unrestricted likelihood L_u at the maximum-likelihood estimates. The final column of each table shows the P-values of the LR and Wald tests.

In Table 15, we reject the null hypothesis of equal win probabilities at the 5% significance level for only one pair, Djokovic serving to Nadal. In Table 16, we reject the null hypothesis for 3 of the 10 pairs shown: 1) Djokovic serving to Murray, 2) Murray serving to Djokovic, and 3) Sampras serving to Agassi. In general, both the Wald and LR tests result in similar test statistics and P-values, and both tests agree on the server pairs for which we reject equal win probabilities.

We can get more insight into why these rejections occur by looking at the estimated con-

Table 15: Tests of equal point win probabilities, Ad court, selected elite server-receiver pairs

Server $\rightarrow$	Serves	1st serve win probs			2nd serve win probs			P-values
receiver	1st, 2nd	**L**	**B**	**R**	**L**	**B**	**R**	Wald, LR
Roger Federer $\rightarrow$	692	.650	.456	.665	.493	.609	.508	.275
Rafael Nadal	274	(.025)	(.066)	(.028)	(.057)	(.059)	(.044)	.271
Rafael Nadal $\rightarrow$	698	.647	.636	.741	.593	.527	.400	.351
Roger Federer	188	(.022)	(.043)	(.041)	(.044)	(.067)	(.155)	.331
Roger Federer $\rightarrow$	805	.649	.622	.636	.563	.490	.493	.581
Novak Djokovic	329	(.024)	(.053)	(.026)	(.039)	(.051)	(.058)	.580
Novak Djokovic $\rightarrow$	848	.598	.614	.596	.496	.393	.533	.463
Roger Federer	288	(.025)	(.039)	(.029)	(.042)	(.052)	(.064)	.474
Rafael Nadal $\rightarrow$	512	.575	.567	.579	.463	.467	.750	.783
Novak Djokovic	148	(.034)	(.041)	(.039)	(.068)	(.053)	(.217)	.849
Novak Djokovic $\rightarrow$	476	.641	.587	.716	.710	.606	.453	.0006
Rafael Nadal	161	.035)	(.050)	(.033)	(.082)	(.060)	(.062)	.0005
Novak Djokovic $\rightarrow$	414	.641	.492	.657	.548	.591	.531	.134
Andy Murray	157	(.033)	(.062)	(.040)	(.077)	(.060)	(.071)	.122
Andy Murray $\rightarrow$	448	.543	.534	.623	.469	.427	.440	.662
Novak Djokovic	203	(.040)	(.065)	(.031)	(.051)	(.055)	(.100)	.661
Pete Sampras $\rightarrow$	392	.642	.650	.666	.475	.600	.436	.717
Andre Agassi	176	(.038)	(.107)	(.033)	(.050)	(.110)	(.068)	.724
Andre Agassi $\rightarrow$	369	.674	.680	.666	.640	.555	.312	.123
Pete Sampras	128	(.030)	(.093)	(.046)	(.047)	(.165)	(.116)	.164

ditional win probabilties. For Djokovic serving to Nadal in the ad court, Table 15 shows big differences in conditional win probabilities across different serve directions, ranging from a low of 45.2% for second serves to the right to a high of 71.0% for second serves to the left. This difference of 25.8% is more than two standard errors in magnitude (when using the standard error for either win probability). The spread in conditional win probabilities for first serves is also more than two standard errors (though the standard errors for first serves are about half as large as those for second serves due to 466 first serve vs. 157 second serve observations).

Even in the cases where we are unable to reject equal win probabilities, we see pretty big differences in win probabilities across serve directions. However, since these differences are typically within two standard errors of each other, the Wald and LR tests do not reject. However, this suggests to us that the failure of these tests to reject in so many of the player pairs may

Table 16: Tests of equal point win probabilities, Deuce court, selected elite server-receiver pairs

Server → receiver	Serves 1st, 2nd	1st serve win probs **L**	**B**	**R**	2nd serve win probs **L**	**B**	**R**	P-values Wald, LR
Roger Federer →	769	.637	.645	.680	.467	.459	.529	.772
Rafael Nadal	276	(.027)	(.061)	(.023)	(.074)	(.064)	(.038)	.767
Rafael Nadal →	758	.613	.635	.578	.492	.616	.607	.316
Roger Federer	238	(.022)	(.036)	(.047)	(.045)	(.052)	(.092)	.320
Roger Federer →	875	.652	.624	.645	.495	.500	.458	.672
Novak Djokovic	324	(.025)	(.050)	(.023)	(.049)	(.043)	(.055)	.672
Novak Djokovic →	914	.661	.612	.637	.652	.558	.556	.270
Roger Federer	322	(.024)	(.039)	(.024)	(.045)	(.042)	(.059)	.282
Rafael Nadal →	551	.588	.578	.641	.574	.504	.615	.187
Novak Djokovic	194	(.035)	(.040)	(.034)	(.060)	(.047)	(.135)	.205
Novak Djokovic →	527	.665	.610	.680	.591	.511	.602	.493
Rafael Nadal	180	(.032)	(.056)	(.030)	(.105)	(.075)	(.036)	.473
Novak Djokovic →	465	.626	.515	.679	.667	.424	.560	.017
Andy Murray	165	(.035)	(.060)	(.032)	(.086)	(.054)	(.070)	.019
Andy Murray →	488	.639	.600	.567	.461	.436	.632	.040
Novak Djokovic	189	(.034)	(.077)	(.031)	(.057)	(.051)	(.111)	.047
Pete Sampras →	427	.675	.590	.730	.422	.628	.583	.021
Andre Agassi	186	(.033)	(.078)	(.032)	(.059)	(.074)	(.058)	.018
Andre Agassi →	405	.658	.625	.683	.481	.594	.704	.356
Pete Sampras	148	(.038)	(.061)	(.034)	(.069)	(.059)	(.088)	.384

be due to their limited power when there are relatively few observations. The lack of sufficient observations results in high estimated standard errors in conditional win probabilities, which makes it difficult to determine if win probabilities really are different across serve directions, or if these differences are purely a reflection of random sampling error.

To try to get a handle on the question of the power of these tests, we present a summary of a limited Monte Carlo study in Tables 17 and 18, which display results for serves to the ad and deuce courts, respectively. Each table summarizes the results of a Monte Carlo experiment where we simulate the play of 46 elite server-receiver pairs (which include the 10 pairs analyzed above and in the paper) across 2000 tennis points. We use our point estimates from the Match Charting Project data of the CCPs and POPs for each pair as the true data generating process. Since the point estimates generally entail unequal win probabilties across serve directions, we expect that a

Table 17: Summary of equal point win probability tests, Ad court, 46 elite player pairs

Item	**Data**	**Unrestricted Simulation**	**Restricted Simulation**
Average number of points	227	2000	2000
Rejections at 5% Wald test	1 out of 46, or 2.1% of cases	34 out of 46, or 73.9% of cases	1 out of 46, or 2.1% of cases
Average P-value Wald rejections	.0006	.008	.0005
Rejections at 5% LR test	1 out of 46, or 2.1% of cases	34 out of 46, or 73.9% of cases	3 out of 46, or 6.5% of cases
Average P-value LR rejections	.0005	.008	.0005
Average number of points for rejected pairs	476	2000	2000

sufficiently powerful test will reject the null hypothesis in the majority of the 46 cases considered.

This is indeed the case, as we can see in the "Unrestricted Simulation" column. Here, the Wald test rejects equal win probabilities at the 5% significance level for 39 of the 46 pairs when serving to the ad court, and for 34 of the 46 pairs when serving to the deuce court. In contrast, when we perform the tests on the actual Match Charting Project data, the Wald test rejects the null hypothesis for only three and one of the 46 player pairs, respectively. We conclude that power may be a concern, and having a dataset that is approximately ten times larger than the Match Charting Project (which in turn is about ten times larger than WW's dataset) is necessary to have sufficient power to detect even fairly substantial violations of the equal win probabilities for all serve directions.

There is, of course, a related concern about the possibility of "over-rejecting" the null hypothesis in large sample sizes. This danger is measured by the Type 1 error rate, which is controlled to be 5% in our example. The last column of Tables 17 and 18 titled "Restricted Simulation" presents the results of Wald and likelihood-ratio tests for another set of simulated samples of size 2000 for the ad and deuce courts, repectively. But in this case, we use as the true data-generating process maximum-likelihood estimates of the conditional win probabilities under the constraint of equal win probabilities for all three serve directions. Although we perform only a single Monte Carlo simulation per server-receiver pair, when considering all 46 pairs, we see that we have re-

Table 18: Summary of equal point win probability tests, Deuce court, 46 elite player pairs

Item	**Data**	**Unrestricted Simulation**	**Restricted Simulation**
Average number of points	224	2000	2000
Rejections at 5% Wald test	3 out of 46, or 6.5% of cases	40 out of 46, or 87.0% of cases	2 out of 46, or 4.3% of cases
Average P-value Wald rejections	.026	.009	.028
Rejections at 5% LR test	3 out of 46, or 6.5% of cases	40 out of 46, or 87.0% of cases	2 out of 46, or 4.3% of cases
Average P-value LR rejections	.028	.008	.028
Average number of points for rejected pairs	460	2000	2000

jections of equal win probabilities ranging from 2.2% to 8.7% of the pairs. This is the range of rejections we would expect to see from a test of size 5% when the null hypothesis is really true.

Thus, we have attempted to replicate WW's main findings using a testing approach similar to the one they used using our larger Match Charting Project dataset. We reach a different conclusion than they do: we do not conclude that the inability to reject the null hypothesis of equal win probabilities justifies a conclusion that elite tennis pros are playing minimax serve strategies. Instead we interpret the tests of equal win probabilities as frequently failing to reject as a result of low power due to limited number of observations of first and second serves. Our Monte Carlo experiments suggest that if we had approximately 10 times as many point game observations as we currently have (even in our larger data set, which already has about 10 times more point game observations than the Wimbledon tennis match data that WW analyzed), there would be sufficient power to reject the null hypothesis of equal win probabilities for the most of the server/receiver pairs we analyzed.

E Muscle Memory and Serial Correlation in Serve Locations

We now explore serial correlation in serve location choices in MPE. We do this in a simple version of the model, but the core insights remain valid in the general model.

E.1 Three Sources of Serial Correlation in Serve Locations

Assume that attention to serve location d, only directly impacts win rates if the server chooses location d, i.e. $\omega(m,d,s,a) = \omega(m,d,a^d)$, and that $\omega(m,d,a^d)$ is differentiable, strictly decreasing, and weakly convex in a^d for all m. Let speed and spin choices in some MPE be given by $s^*(x,m)$ and define the induced conditional probabilities:

$$\ell^d(x,m) \equiv \ell(m,d,s^*(x,m)) \quad \text{and} \quad \omega^d(x,m,a^d) \equiv \omega(m,d,s^*(x,m),a^d),$$

Recall that $W_S(x,m)$ is the server's chance of wining the service game in state (x,m) and let $w^d(x,m,a^d)$ be the conditional chance that the server wins the service game when choosing location d given that the receiver chooses attention a^d at location d. Then for first serves (x odd):

$$\begin{aligned} W^d(x,m,a^d) \;&=\; \ell^d(x,m)\Big(\omega^d(x,m,a^d)W_S(x^+(x),(m_2,d)) + (1-\omega^d(x,m,a^d))W_S(x^-(x),(m_2,d))\Big) \\ &\quad + (1-\ell^d(x,m))W_S(x+1,(m_2,d)) \end{aligned}$$

To further simplify, assume an MPE in which the server strictly mixes over l,r with respective chances $\sigma_S(x,m), 1-\sigma_S(x,m)$ on first serves.[39] Since the receiver has no direct effect on the muscle memory state, the receiver will best respond by setting $a^b = 0$; and thus, we have $a^r = 1 - a^l$. Altogether, the chance that the server wins the service game in state (x,m) for first serves given receiver attention a^l is:

$$W(x,m,a^l) \equiv \sigma_S(x,m)W^l(x,m,a^l) + (1-\sigma_S(x,m))W^r(x,m,a^r)$$

Further assuming $a^l \in (0,1)$,[40] it must be the case that the receiver cannot lower this probability by adjusting a^l up *or* down; and thus, the receiver's MPE attention $a^l(x,m)$ must obey $W_{a^l}(x,m,a^l(x,m)) = 0$, i.e.:

$$\frac{\sigma_S(x,m)}{1-\sigma_S(x,m)} = \frac{W^r_{a^r}(x,m,1-a^l(x,m))}{W^l_{a^l}(x,m,a^l(x,m))} = \frac{\ell^r(x,m)\omega^r_{a^r}(x,m,1-a^l(x,m))\Delta^r(x,m_2)}{\ell^l(x,m)\omega^l_{a^l}(x,m,a^l(x,m))\Delta^l(x,m_2)} \tag{38}$$

[39] A sufficient condition for an MPE with no body first serves is that serving left or right on first serves gives the server a better chance of winning the current point *and* enhances his chances of winning future points: $\ell^b(x,m)\omega^b(x,m,a^b) < \ell^d(x,m)\omega^d(x,m,a^d)$ for all $m,a,d \in \{l,r\}$, and x odd *and* $\ell^d(x,(d'',b))\omega^d(x,(d'',b),a^d) \leq \ell^d(x,(d'',d'))\omega^d(x,(d'',d'),a^d)$ for all x,d'',a^d and $d,d' \in \{l,r\}$.

[40] An assumption that *implies* $a^l \in (0,1)$ is that the ratio $\omega^l_{a^l}(x,m,a^l)/\omega^r_{a^r}(x,m,1-a^l)$ converges to ∞ as $a^l \to 0$ and converging to 0 as $a^l \to 1$.

where $\Delta^d(x, m_2) = W_S(x^+(x), (m_2, d)) - W_S(x^-(x), (m_2, d))$ is the increase in the service game win chance from winning vs. losing the current point on the first serve.

Equation (38) affords a way to formalize equilibrium serial correlation in first serve strategies. Specifically, a sufficient condition for negatively serial correlation is that the server is less likely to serve left following a left serve, i.e. when $\sigma(x, (d, l)) < \sigma(x, (d, r))$, for all odd x and first serve locations d chosen two first serves prior, which by (38) is equivalent to:

$$\frac{W^r_{a^r}(x, (d, l), 1 - a^l(x, (d, l)))}{W^l_{a^l}(x, (d, l), a^l(x, (d, l)))} < \frac{W^r_{a^r}(x, (d, r), 1 - a^l(x, (d, r)))}{W^l_{a^l}(x, (d, r), a^l(x, (d, r)))} \tag{39}$$

Since $W^d_{a^d} = \ell^d \omega^d_{a^d} \Delta$, inequality (39) compares the product of three separate ratios, and thus, there are three logically separate ways to generate negative serial correlation with muscle memory. One is when muscle memory affects the server's chance of landing a serve in, as follows:

$$\frac{\ell^r(x, (d, l))}{\ell^l(x, (d, l))} < \frac{\ell^r(x, (d, r))}{\ell^l(x, (d, r))} \tag{40}$$

This comparison of likelihood ratios states that the server's relative chance of landing a right serve in is higher following a right serve. While this makes intuitive sense, it is an empirical question whether such short term muscle memory effects exist for elite pro serves.

Muscle memory can also generate (39) if the following inequality holds:

$$\frac{\omega^r_{a^r}(x, (d, l), 1 - a^l(x, (d, l)))}{\omega^l_{a^l}(x, (d, l), a^l(x, (d, l)))} < \frac{\omega^r_{a^r}(x, (d, r), 1 - a^l(x, (d, r)))}{\omega^l_{a^l}(x, (d, r), a^l(x, (d, r)))} \tag{41}$$

For an interpretation of this condition, notice that $\omega^r_{a^r}/\omega^l_{a^l}$ is the marginal rate of technical substitution (MRTS) between attention at location r and attention at location l. Inequality (41) states that this MRTS is larger following a serve to the right than it is following a serve to the left. This could be the result of a *direct effect* of muscle memory, the MRTS larger following a serve to the right holding the receiver's attention constant, or an *indirect effect*, the receiver's attention changes following a serve to the right inducing a larger MRTS.

Notice that inequalities (40) and (41) are about the impact of *past* serve locations on the *current* point game ratios ℓ^r/ℓ^d and $\omega^r_{a^r}/\omega^l_{a^l}$. These effects may be present even if the server and receiver behave *myopically*, maximizing their chances of winning the current point and ignoring the future. When the players are forward looking, there is third potential source of negative serial correlation; namely:

$$\frac{\Delta^r(x, l)}{\Delta^l(x, l)} < \frac{\Delta^r(x, r)}{\Delta^l(x, r)}$$

which after substituting in for all Δ^d becomes:

$$\frac{W_S(x^+(x),(l,r))-W_S(x^-(x),(l,r))}{W_S(x^+(x),(l,l))-W_S(x^-(x),(l,l))}<\frac{W_S(x^+(x),(r,r))-W_S(x^-(x),(r,r))}{W_S(x^+(x),(r,l))-W_S(x^-(x),(r,l))} \tag{42}$$

For an interpretation, recall that muscle memory $m=(m_1,m_2)$, where m_1 is the location of the previous first serve and m_2 is the location of the second to last first serve. Now, arbitrarily order serve locations $l>r$ (or vice versa), then inequality (42) states that the increase in the probability of winning the service game from winning vs. losing the *current* point $W_S(x^+(x),m)-W_S(x^-(x),m)$ is strictly log-supermodular in (m_1,m_2). This necessarily requires muscle memory to depend on the two previous serve locations.

E.2 Example: Serial Correlation in A Linear Model

We now simplify the model further in order to sign the equilibrium serial correlation in serve locations and show that this serial correlation can be strictly negative (or strictly positive), even if the POPs in Definition 1 are independent of muscle memory.

The *two location linear model* removes spin, speed and body serves as choice variables and assumes that ℓ and ω only depend on the prior first serve location, m_1, and that the conditional win chance ω is linear in attention, i.e. $\omega^d(m_1)=\bar{\omega}-\eta^d(m_1)a^d$ with $\bar{\omega}\in(0,1)$ and $\eta^d(m_1)\in(0,\bar{\omega})$. Thus, this model is fully determined by the nine scalars: $\bar{\omega}$ and $\ell^d(m_1),\eta^d(m_1)$ for $(d,m_1)\in\{l,r\}^2$. The two location linear model is *log-supermodular* when $\ell^r(l)\ell^l(r)\eta^r(l)\eta^l(r)<\ell^r(r)\ell^l(l)\eta^r(r)\eta^l(l)$ and *log-submodular* when the opposite inequality obtains. The *symmetric two location linear model* further restricts: $\ell^d(m_1)=\bar{\ell}$, $\eta^r(l)=\eta^l(r)=\eta$, and $\eta^l(l)=\eta^r(r)=\hat{\eta}$.

Theorem 4. *Serve locations are negatively (positively) serially correlated in the log-supermodular (log-submodular) two location linear model in any MPE in which the server strictly mixes over first serve locations. The POPs are independent of muscle memory in the symmetric two location linear model.*

STEP 1: SERIAL CORRELATION IN SERVE LOCATIONS. Direct substitution establishes that inequality (39) is equivalent to $\ell^r(l)\ell^l(r)\eta^r(l)\eta^l(r)<\ell^r(r)\ell^l(l)\eta^r(r)\eta^l(l)$ (i.e. log-supermodularity) in the two location linear model; and thus, negative serial correlation obtains. Similarly, under log-submodularity, inequality (39) flips, implying positive serial correlation.

STEP 2: SERIALLY INDEPENDENT POPS. The chance of serving in ℓ is a constant by assumption. Routine algebra establishes that the following strategies constitute a MPE:

$$\frac{a^l(r)}{1-a^l(r)} = \frac{1-a^l(l)}{a^l(l)} = \frac{\sigma_S(r)}{1-\sigma_S(r)} = \frac{1-\sigma_S(l)}{\sigma_S(l)} = \frac{\hat{\eta}}{\eta} \tag{43}$$

Given these strategies, the win chance is $\omega^d(m_1) = \bar{\omega} - \frac{\eta\hat{\eta}}{\eta+\hat{\eta}}$, independent of muscle memory. This implies the continuation value function W_S is independent of muscle memory; and thus, the generalized monotonicity condition 2 holds. Altogether, the service game can be decomposed into a sequence of identical static games. It is straightforward to verify that strategies (43) constitute the unique equilibrium in these static games. □

The symmetric model is log-supermodular when $\eta < \hat{\eta}$ and log-submodular when $\eta > \hat{\eta}$; and thus, serve locations are generically either negatively serially correlated or positively serially correlated in the symmetric two location linear model, despite the fact that the POPs are independent of muscle memory.

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