Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

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1 Optimization Problem

2 Lifting Line Theory

$$\Gamma(y) = \frac{l(y)}{\rho V} = \frac{1/2 \rho V^2 c(y) c_l(y)}{\rho V} = \frac{1}{2} V c(y) c_l(y)$$

$$\theta = \frac{b}{2}\cos(\theta)$$

$$\Gamma(\theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\theta),\ c_l(\theta) = \frac{1}{c(y)} \sum_{n=1}^{+\infty} A_n \sin(n\theta)$$

$$\begin{aligned}
\alpha_i(y) &= \frac{-1}{4\pi V} \int_{-b/2}^{b/2} \frac{\frac{d\Gamma(t)}{dt}}{y - t} dt \\
\alpha_i(\theta) &= \frac{-1}{4\pi V} \int_{\pi}^{0} \frac{\left(\frac{d\Gamma(\Theta)}{d\Theta}\frac{d\Theta}{dt}\right)}{b/2\cos(\theta) - b/2\cos(\Theta)} \frac{dt}{d\Theta} d\Theta \\
&= -\frac{-1}{2b\pi V} \int_0^{\pi} \frac{\frac{d\Gamma(\Theta)}{d\Theta}}{\cos(\theta) - \cos(\Theta)} d\Theta
\end{aligned}$$

$$\frac{d\Gamma(\Theta)}{d\Theta} = \frac{d}{d\Theta}\frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\Theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta)$$

$$\begin{aligned}
\alpha_i(\theta) &= -\frac{-1}{2b\pi} \int_0^{\pi} \frac{\frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta \\
&= -\frac{-1}{4b\pi} \sum_{n=1}^{+\infty} n A_n \int_0^{\pi} \frac{\cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta
\end{aligned}$$

$$\alpha_i(\theta) = \frac{-1}{4b}\sum_{n=1}^{+\infty} nA_n \frac{\sin(n\theta)}{\sin(\theta)}$$

The general equation that needs to be solved to find the $A_{1:+\infty}$ is:

$$\begin{aligned} c_l(\theta) &= CL_{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right) \\ \sum_{n=1}^{+\infty} A_n\sin(n\theta) &= c(\theta)CL_{2D}\left(\alpha(\theta) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta)\right) \end{aligned}$$

If we only compute A_n for $n \in [0, N_A]$, we can choose a discrete set of θ_k for $k \in [0, N_\theta]$:

$$SA = c(\theta)CL_{2D}\left(\alpha(\theta) - \frac{1}{4b}S'A_n, Re(\theta)\right) \tag{1}$$

3 L over D

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{Di}+C_{D0}}$$

From lifting line theory:

$$\begin{aligned} C_L &= \frac{1}{S}\int_{b/2}^{b/2} c(y)c_l(y)dy \\ &= \frac{b}{2S}\int_0^{\pi}\sum_{n=1}^{+\infty} A_n\sin(n\theta)\sin(\theta)d\theta \\ &= \frac{b\pi A_1}{4S} \end{aligned} \tag{2}$$

$$\begin{aligned} c_{d_i}(\theta) &= -c_l(\theta)\alpha_i(\theta) \\ &= \frac{1}{4bc(\theta)}\sum_{n_a=0}^{+\infty} n_aA_{n_a}\frac{\sin(n_a\theta)}{\sin(\theta)}\sum_{n_b=0}^{+\infty} A_{n_b}\sin(n_b\theta) \end{aligned}$$

$$\begin{aligned} C_{Di} &= \frac{b}{2S}\int_0^{\pi} c_{d_i}(\theta)c(\theta)d\theta \\ C_{Di} &= \frac{b}{2S}\int_0^{\pi}\frac{1}{4b}\sum_{n_a=0}^{+\infty} n_aA_{n_a}\frac{\sin(n_a\theta)}{\sin(\theta)}\sum_{n_b=0}^{+\infty} A_{n_b}\sin(n_b\theta)\sin(\theta)d\theta \\ C_{Di} &= \frac{1}{8S}\int_0^{\pi}\sum_{n_a=0}^{+\infty}\sum_{n_b=0}^{+\infty} n_aA_{n_a}A_{n_b}\sin(n_a\theta)\sin(n_b\theta)d\theta \end{aligned} \tag{3}$$

Since for any k, n:

$$\int_0^{\pi} \sin(n\theta)\sin(k\theta)d\theta = \begin{cases} 0 & k \neq n \\ \frac{\pi}{2} & k = n \end{cases}$$

we have:

$$C_{Di} = \frac{\pi}{16S}\sum_{n=1}^{+\infty} nA_n^2 \tag{4}$$

Finally, C_{D_0} must be computed by integration:

$$\begin{aligned}
c_{d_0}(\theta) &= CD0_{2D}\left(\alpha(\theta) + \alpha_i(\theta), Re(\theta)\right) \\
C_{D_0}(\theta) &= \frac{b}{2S}\int_0^{\pi} c(\theta) CD0_{2D}\left(\alpha(\theta) + \alpha_i(\theta), Re(\theta)\right)\sin(\theta)d\theta \\
C_{D_0}(\theta) &= \frac{b}{2S}\int_0^{\pi} c(\theta) CD0_{2D}\left(\alpha(\theta) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta)\right)\sin(\theta)d\theta
\end{aligned}$$

If $CD0_{2D}$ has an analytic expression, this can be simplified further. In this case however, it is a general function fitted from wind tunnel data, therefore the integration is performed numerically with Clenshaw–Curtis quadrature.

Calling $w_\theta^{(k)}$ and θ_k respectively the weights and the points of this quadrature, we get:

$$C_{D_0}(\theta) = \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) CD0_{2D}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) \tag{5}$$

Finally:

$$\begin{aligned}
\frac{L}{D} &= \frac{C_L}{C_{Di} + C_{D0}} \\
\frac{L}{D} &= \frac{\frac{b\pi A_1}{4S}}{\frac{\pi}{16S}\sum_{n=1}^{+\infty} nA_n^2 + \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) CD0_{2D}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)} \\
\frac{L}{D} &= \left(\frac{1}{4bA_1}\sum_{n=1}^{+\infty} nA_n^2 + \frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) CD0_{2D}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)\right)^{-1}
\end{aligned} \tag{6}$$

4 Optimization Problem

$$\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & (L/D)^{-1} \\
\text{subject to} \quad & c_l(\theta_k) = CL_{2D}\left(\alpha(\theta_k) + \alpha_i(\theta_k), Re(\theta_k)\right) && (7) \\
& C_L = \frac{W}{1/2\rho V^2 S} && (8)
\end{aligned}$$

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{1}{4bA_1}\sum_{n=1}^{N_A} nA_n^2 + \\
& \frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) && (9)\\
\text{subject to} \quad & SA = CL_{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (10)\\
& A_1 = \frac{2W}{\rho V^2 b\pi} && (11)
\end{aligned}
$$

where it was included in 7 that $C_L = \frac{b\pi A_1}{4S}$. Using 8 to simplify 15 further, we obtain:

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + \\
& V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) && (12)\\
\text{subject to} \quad & SA = CL_{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (13)\\
& A_1 = \frac{8W}{\rho V^2 b\pi} && (14)
\end{aligned}
$$

because:

$$
\begin{aligned}
\frac{1}{4bA_1} &= \frac{\rho V^2 \pi}{8W}\\
\frac{\pi}{2A_1} &= \frac{\rho V^2 b}{W}
\end{aligned}
$$

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (15)\\
\text{subject to} \quad & SA = CL_{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (16)\\
& A_1 = \frac{8W}{\rho V^2 b\pi} && (17)
\end{aligned}
$$

$$
\begin{aligned}
\underset{V,b,c,\alpha,\alpha_i A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k + \alpha_{i,k}, Re(\theta_k)\right) && (18)\\
\text{subject to} \quad & SA = CL_{2D}\left(\alpha_k + \alpha_{i,k}, Re(\theta_k)\right) && (19)\\
& A_1 \geq \frac{8W}{\rho V^2 b\pi} && (20)\\
& \alpha_{i,k} \geq -\frac{1}{4b}S' A_n && (21)
\end{aligned}
$$