

# Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

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## 1 Nomenclature

$W$	:	glider weight
$L, C_L$	:	lift, lift coefficient
$D, D_i, D_0, C_D, C_{D_i}, C_{D_0}$	:	drag, induced drag, friction drag, and respective coefficients
$b$	:	wing span
$M$	:	wing area
$\mathcal{R}$	:	wing aspect ratio
$y$	:	span-wise coordinate, ranging from $-b/2$ to $b/2$
$\theta$	:	remapped span-wise coordinate ranging from 0 to $\pi$
$N_{theta}$	:	Number of span-wise sections
$\Gamma(y)$ or $\Gamma(\theta)$	:	span-wise strength of vortex sheet
$A_n$	:	$n$ th coefficient of the Fourier expansion of $\Gamma$
$N_A$	:	Number of Fourier coefficients considered for $\Gamma$
$M$	:	matrix of size $N_\theta, N_A$ such that: $\forall k, n \quad M_{k,n} = \sin(n\theta_k)$
$M'$	:	matrix of size $N_\theta, N_A$ such that: $\forall k, n \quad M'_{k,n} = n \sin(n\theta_k) / \sin(\theta_k)$
$A$	:	vector of size $N_A$ containing all the $A_n$
$\alpha(y)$ or $\alpha(\theta)$	:	span-wise angle of attack distribution
$\alpha_i(y)$ or $\alpha_i(\theta)$	:	span-wise lift-induced angle of attack distribution
$c(y)$ or $c(\theta)$	:	span-wise chord distribution
$c_l(y)$ or $c_l(\theta)$	:	span-wise lift coefficient distribution
$c_d(y)$ or $c_d(\theta)$	:	span-wise drag coefficient distribution
$V$	:	airspeed
$C_L^{2D}$ and $C_{D_0}^{2D}$	:	fit of 2D airfoil data
$Re$	:	Reynolds number
$y_k, \theta_k, \alpha_k, \alpha_{i_k}, c_k, c_{l_k}$	:	$y, \theta, \alpha, \alpha_i, c, c_l$ at the $k$ th span section

## 2 Optimization Problem

This code aims at optimizing the wing planform and airspeed of a glider of a given weight  $W$  and fixed airfoil section for maximum  $L/D$ . 2D wind tunnel data of the airfoil is available, *i.e.* the functions  $C_L^{2D}(\alpha, Re)$  and  $C_{D_0}^{2D}(\alpha, Re)$  are provided.

The optimization problem reads:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && (C_L / (C_{D_i} + C_{D_0}))^{-1} \\
& \text{subject to} && C_L = \frac{W}{1/2 \rho V^2 S} \\
& && c_l(y) = C_L^{2D}(\alpha(y) + \alpha_i(y), Re(y)) \\
& && c_{d_0}(y) = C_{D_0}^{2D}(\alpha(y) - \alpha_i(y), Re(y)) \\
& && C_L = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_l(y') dy' \\
& && C_{D_0} = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_{d_0}(y') dy' \\
& && C_{D_i} = \frac{C_L^2}{\pi \mathcal{R} \epsilon} \\
& && \mathcal{R} = \frac{b^2}{S} \\
& && \epsilon = \left( \sum_{n=1}^{N_A} n \left( \frac{A_n}{A_1} \right)^2 \right)^{-1} \\
& && \alpha_i(\theta) = -\frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta)}{\sin(\theta)} \\
& \forall \theta \in [0, \pi] && \sum_{n=1}^{N_A} A_n \sin(\theta) = \frac{1}{c(\theta)} C_L^{2D}(\alpha(\theta) - \alpha_i(\theta), Re(\theta))
\end{aligned}$$

where  $\theta$  is a span parametrization such that:

$$\forall y \in \left[ \frac{-b}{2}, \frac{b}{2} \right], \quad \exists \theta \in [0, \pi], \quad y = \frac{b}{2} \cos \theta$$

We discretize the span into  $N_{theta}$  span-wise sections, and perform several simplifications detailed in the rest of this document. The final optimization problem is:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && \frac{V^2 \pi}{8} \sum_{n=1}^{N_A} n A_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} (\alpha_k + \alpha_{i_k}, \text{Re}(c_k, v_k)) \\
& \text{subject to} && \sum_{n=1}^{N_A} M_{k,n} A_n = c_k C_L^{2D} \left( \alpha_k - \frac{1}{4b} \sum_{n=1}^{N_A} M'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \quad \text{for } k \in 1, \dots, N_\theta \\
& && A_1 = \frac{8W}{\rho V^2 b \pi}
\end{aligned}$$

Or, in an even more compact form by plugging in the last constraint for  $A_1$  in both other equations:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && \frac{W}{\rho b} + \frac{V^2 \pi}{8} \sum_{n=2}^{N_A} n A_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left( \alpha_k - \frac{1}{4b} M'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=1}^{N_A} M'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \\
& \text{subject to} && M_{k,1} \frac{8W}{\rho V^2 b \pi} + \sum_{n=2}^{N_A} M_{k,n} A_n = \\
& && c_k C_L^{2D} \left( \alpha_k - \frac{1}{4b} M'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=2}^{N_A} M'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \quad \text{for } k \in 1, \dots, N_\theta
\end{aligned}$$

### 3 Lifting Line Theory

From Kutta-Joukowski theorem:

$$\Gamma(y) = \frac{l(y)}{\rho V} = \frac{1/2 \rho V^2 c(y) c_l(y)}{\rho V} = \frac{1}{2} V c(y) c_l(y)$$

The strength of the vortex sheet  $\Gamma$  is defined on a closed interval  $[-b/2, b/2]$ , which we remap to  $[\pi, 0]$ . On this interval, a Fourier series expansion can be written:

$$y = \frac{b}{2} \cos(\theta)$$

$$\Gamma(\theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\theta), \quad c_l(\theta) = \frac{1}{c(y)} \sum_{n=1}^{+\infty} A_n \sin(n\theta)$$

Using Biot and Savart law, the induced angle of attack on at each span-wise coordinate

can be written as:

$$\begin{aligned}
\alpha_i(y) &= \frac{-1}{4\pi V} \int_{-b/2}^{b/2} \frac{\frac{d\Gamma(t)}{dt}}{y-t} dt \\
\alpha_i(\theta) &= \frac{-1}{4\pi V} \int_{\pi}^0 \frac{\left( \frac{d\Gamma(\Theta)}{d\Theta} \frac{d\Theta}{dt} \right)}{b/2 \cos(\theta) - b/2 \cos(\Theta)} \frac{dt}{d\Theta} d\Theta \\
&= \frac{1}{2b\pi V} \int_0^{\pi} \frac{\frac{d\Gamma(\Theta)}{d\Theta}}{\cos(\theta) - \cos(\Theta)} d\Theta \\
\frac{d\Gamma(\Theta)}{d\Theta} &= \frac{d}{d\Theta} \frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\Theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta) \\
\alpha_i(\theta) &= \frac{1}{2b\pi} \int_0^{\pi} \frac{\frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta \\
&= \frac{1}{4b\pi} \sum_{n=1}^{+\infty} n A_n \int_0^{\pi} \frac{\cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta
\end{aligned}$$

Directly plugging in the solution of the Glauert integral:

$$\alpha_i(\theta) = \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)}$$

The value of the lift coefficient on each section is given by the fit of the 2D airfoil data:

$$c_l(\theta) = C_L^{2D} (\alpha(\theta) - \alpha_i(\theta), Re(\theta))$$

Plugging in the value of  $c_l$  in terms of the Fourier coefficients of  $\Gamma$ , we obtain the general equation that needs to be solved to find the  $A_{1:+\infty}$  is:

$$\sum_{n=1}^{+\infty} A_n \sin(n\theta) = c(\theta) C_L^{2D} \left( \alpha(\theta) - \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta) \right)$$

If we only compute  $A_n$  for  $n \in [0, N_A]$ , we can choose a discrete set of  $\theta_k$  for  $k \in [0, N_\theta]$ :

$$MA = c C_L^{2D} \left( \alpha - \frac{1}{4b} M' A, Re(c, V) \right) \quad (1)$$

with:

$c$  and  $\alpha$  two vectors of size  $N_\theta$  containing the chord and angle of attack at each  $\theta_k$

$M$  and a  $M'$  two matrices of size  $N_\theta, N_A$  such that:

$$\forall k \in \{0, \dots, N_\theta\}, \forall n \in \{0, \dots, N_A\} \quad M_{k,n} = \sin(n\theta_k) \quad M'_{k,n} = n \sin(n\theta_k) / \sin(\theta_k)$$

$A$  a vector of size  $N_A$  containing all the  $A_n$

## 4 Lift To Drag Ratio

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{Di} + C_{D0}}$$

Let us start with  $C_L$ :

$$\begin{aligned} C_L &= \frac{1}{S} \int_{b/2}^{b/2} c(y) c_l(y) dy \\ &= \frac{b}{2S} \int_0^\pi \sum_{n=1}^{+N_A} A_n \sin(n\theta) \sin(\theta) d\theta \\ &= \frac{b}{2S} \sum_{n=1}^{+N_A} A_n \int_0^\pi \sin(n\theta) \sin(\theta) d\theta \end{aligned} \tag{2}$$

Since the Fourier basis is orthogonal, we have for any  $k, n$ :

$$\int_0^\pi \sin(n\theta) \sin(k\theta) d\theta = \begin{cases} 0 & k \neq n \\ \frac{\pi}{2} & k = n \end{cases}$$

Therefore:

$$C_L = \frac{b\pi A_1}{4S} \tag{3}$$

Now, to get  $C_{Di}$ , we look at it's span-wise distribution  $c_{di}$ :

$$\begin{aligned} c_{di}(\theta) &= c_l(\theta) \alpha_i(\theta) \\ &= \frac{1}{4bc(\theta)} \sum_{n_a=0}^{+\infty} n_a A_{n_a} \frac{\sin(n_a \theta)}{\sin(\theta)} \sum_{n_b=0}^{+\infty} A_{n_b} \sin(n_b \theta) \end{aligned}$$

Integrating to get  $C_{Di}$ :

$$\begin{aligned} C_{Di} &= \frac{b}{2S} \int_0^\pi c_{di}(\theta) c(\theta) d\theta \\ C_{Di} &= \frac{b}{2S} \int_0^\pi \frac{1}{4b} \sum_{n_a=0}^{N_A} n_a A_{n_a} \frac{\sin(n_a \theta)}{\sin(\theta)} \sum_{n_b=0}^{N_A} A_{n_b} \sin(n_b \theta) \sin(\theta) d\theta \\ C_{Di} &= \frac{1}{8S} \int_0^\pi \sum_{n_a=0}^{N_A} \sum_{n_b=0}^{N_A} n_a A_{n_a} A_{n_b} \sin(n_a \theta) \sin(n_b \theta) d\theta \end{aligned} \tag{4}$$

Using again that the Fourier basis functions are orthogonal, we obtain:

$$C_{Di} = \frac{\pi}{16S} \sum_{n=1}^{N_A} n A_n^2 \tag{5}$$

Finally,  $C_{D_0}$  is also obtained by integration:

$$\begin{aligned}
c_{d_0}(\theta) &= C_{D_0}^{2D}(\alpha(\theta) - \alpha_i(\theta), Re(\theta)) \\
C_{D_0}(\theta) &= \frac{b}{2S} \int_0^\pi c(\theta) C_{D_0}^{2D}(\alpha(\theta) - \alpha_i(\theta), Re(\theta)) \sin(\theta) d\theta \\
C_{D_0}(\theta) &= \frac{b}{2S} \int_0^\pi c(\theta) C_{D_0}^{2D} \left( \alpha(\theta) - \frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta) \right) \sin(\theta) d\theta
\end{aligned}$$

If  $C_{D_0}^{2D}$  has an analytic expression, this can be simplified further. In the our case however, it is a general function fitted from wind tunnel data, therefore the integration is performed numerically with Clenshaw–Curtis quadrature.

Calling  $w_\theta^{(k)}$  and  $\theta_k$  respectively the weights and the points of this quadrature, we get:

$$C_{D_0}(\theta) = \frac{b}{2S} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) C_{D_0}^{2D} \left( \alpha(\theta_k) - \frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right) \quad (6)$$

Finally:

$$\begin{aligned}
\frac{L}{D} &= \frac{C_L}{C_{D_i} + C_{D_0}} \\
\frac{L}{D} &= \frac{\frac{b\pi A_1}{4S}}{\frac{\pi}{16S} \sum_{n=1}^{+\infty} n A_n^2 + \frac{b}{2S} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) C_{D_0}^{2D} \left( \alpha(\theta_k) - \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right)} \\
\frac{L}{D} &= \left( \frac{1}{4bA_1} \sum_{n=1}^{+\infty} n A_n^2 + \frac{2}{\pi A_1} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) C_{D_0}^{2D} \left( \alpha(\theta_k) - \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right) \right)^{-1} \quad (7)
\end{aligned}$$

## 5 Optimization Problem

We start with:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && (C_L / (C_{D_i} + C_{D_0}))^{-1} \\
& \text{subject to} && C_L = \frac{W}{1/2\rho V^2 S} \\
& && c_l(y) = C_L^{2D} (\alpha(y) + \alpha_i(y), Re(y)) \\
& && c_{d_0}(y) = C_{D_0}^{2D} (\alpha(y) - \alpha_i(y), Re(y)) \\
& && C_L = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_l(y') dy' \\
& && C_{D_0} = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_{d_0}(y') dy' \\
& && C_{D_i} = \frac{C_L^2}{\pi R \epsilon} \\
& && R = \frac{b^2}{S} \\
& && \epsilon = \left( \sum_{n=1}^{N_A} n \left( \frac{A_n}{A_1} \right)^2 \right)^{-1} \\
& && \alpha_i(\theta) = -\frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta)}{\sin(\theta)} \\
& \forall \theta \in [0, \pi] && \sum_{n=1}^{N_A} A_n \sin(\theta) = \frac{1}{c(\theta)} C_L^{2D} (\alpha(\theta) - \alpha_i(\theta), Re(\theta))
\end{aligned}$$

Discretizing, and using equations 3, 5, 6, 7, we get:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && \frac{1}{4bA_1} \sum_{n=1}^{N_A} n A_n^2 + \\
& && \frac{2}{\pi A_1} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left( \alpha_k - \frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right) \quad (8)
\end{aligned}$$

$$\begin{aligned}
& \text{subject to} && M A = c C_L^{2D} \left( \alpha - \frac{1}{4b} M' A, Re(c, V) \right) \quad (9)
\end{aligned}$$

$$A_1 = \frac{8W}{\rho V^2 b \pi} \quad (10)$$

Using 10 to simplify 8 further, we obtain:

$$\begin{aligned} \underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8} \sum_{n=1}^{N_A} nA_n^2 + \\ & V^2b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left( \alpha_k - \frac{1}{4b} \sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, \text{Re}(\theta_k) \right) \end{aligned} \quad (11)$$

$$\text{subject to} \quad MA = cC_L^{2D} \left( \alpha_k - \frac{1}{4b} M' A_n, \text{Re}(\theta_k) \right) \quad (12)$$

$$A_1 = \frac{8W}{\rho V^2 b \pi} \quad (13)$$

because:

$$\begin{aligned} \frac{1}{4bA_1} &= \frac{\rho V^2 \pi}{8W} \\ \frac{\pi}{2A_1} &= \frac{\rho V^2 b}{W} \end{aligned}$$

Finally, we incorporate 13 into the objective function and the constraint.

$$\begin{aligned} \underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{W}{\rho b} + \frac{V^2\pi}{8} \sum_{n=2}^{N_A} nA_n^2 + V^2b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left( \alpha_k - \frac{1}{4b} M'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=1}^{N_A} M'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \\ \text{subject to} \quad & M_{k,1} \frac{8W}{\rho V^2 b \pi} + \sum_{n=2}^{N_A} M_{k,n} A_n = \\ & c_k C_L^{2D} \left( \alpha_k - \frac{1}{4b} M'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=2}^{N_A} M'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \quad \text{for } k \in 1, \dots, N_\theta \end{aligned}$$