Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

Jean de Becdelievre

March 26, 2020

1   Nomenclature

W	:	glider weight
L, C_L	:	lift, lift coefficient
$D, Di, D_0, C_D, C_{Di}, C_{D_0}$	:	drag, induced drag, friction drag, and respective coefficients
b	:	wing span
M	:	wing area
$A\!R$	:	wing aspect ratio
y	:	span-wise coordinate, ranging from $-b/2$ to $b/2$
θ	:	remapped span-wise coordinate ranging from 0 to π
N_{theta}	:	Number of span-wise sections
$\Gamma(y)$ or $\Gamma(\theta)$	:	span-wise strength of vortex sheet
A_n	:	nth coefficient of the Fourier expansion of Γ
N_A	:	Number of Fourier coefficients considered for Γ
M	:	matrix of size N_θ, N_A such that: $\forall k, n \quad M_{k,n} = \sin(n\theta_k)$
M'	:	matrix of size N_θ, N_A such that: $\forall k, n \quad M'_{k,n} = n\sin(n\theta_k)/\sin(\theta_k)$
A	:	vector of size N_A containing all the A_n
$\alpha(y)$ or $\alpha(\theta)$	:	span-wise angle of attack distribution
$\alpha_i(y)$ or $\alpha_i(\theta)$	:	span-wise lift-induced angle of attack distribution
$c(y)$ or $c(\theta)$	:	span-wise chord distribution
$c_l(y)$ or $c_l(\theta)$	:	span-wise lift coefficient distribution
$c_d(y)$ or $c_d(\theta)$	:	span-wise drag coefficient distribution
V	:	airspeed
$\boldsymbol{c_l}^{\boldsymbol{2D}}$ and $\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}$	:	fit of 2D airfoil data
Re	:	Reynolds number
$y_k, \theta_k, \alpha_k, \alpha_{i_k}, c_k, c_{l_k}$	:	$y, \theta, \alpha, \alpha_i, c, c_l$ at the kth span section

2 Optimization Problem

This code aims at optimizing the wing planform and airspeed of a glider of a given weight W and fixed airfoil section for maximum L/D. 2D wind tunnel data of the airfoil is available, *i.e.* the functions $\boldsymbol{c_l}^{2D}(\alpha, Re)$ and $\boldsymbol{c_{d_0}}^{2D}(\alpha, Re)$ are provided.

The optimization problem reads:

$$\underset{V,b,c,\alpha,A}{\text{minimize}} \qquad (C_L/(C_{D_i} + C_{D_0}))^{-1}$$

$$\text{subject to} \qquad C_L = \frac{W}{1/2\rho V^2 S}$$

$$c_l(y) = \boldsymbol{c_l}^{2D}\left(\alpha(y) + \alpha_i(y), Re(y)\right)$$

$$c_{d_0}(y) = \boldsymbol{c_{d_0}}^{2D}(\alpha(y) - \alpha_i(y), Re(y))$$

$$C_L = \frac{1}{S} \int_{-b/2}^{b/2} c(y')c_l(y')dy'$$

$$C_{D_0} = \frac{1}{S} \int_{-b/2}^{b/2} c(y')c_{d_0}(y')dy'$$

$$C_{D_i} = \frac{C_L^2}{\pi \mathit{AR} \epsilon}$$

$$\mathit{AR} = \frac{b^2}{S}$$

$$\epsilon = \left(\sum_{n=1}^{N_A} n \left(\frac{A_n}{A_1}\right)^2\right)^{-1}$$

$$\alpha_i(\theta) = -\frac{1}{4b} \sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta)}{\sin(\theta)}$$

$$\forall \theta \in [0, \pi] \qquad \sum_{n=1}^{N_A} A_n \sin(\theta) = \frac{1}{c(\theta)} \boldsymbol{c_l}^{2D}\left(\alpha(\theta) - \alpha_i(\theta), Re(\theta)\right)$$

where θ is a span parametrization such that:

$$\forall y \in [\frac{-b}{2}, \frac{b}{2}], \quad \exists \theta \in [0, \pi], \quad y = \frac{b}{2}\cos\theta$$

We discretize the span into N_{theta} span-wise sections, and perform several simplifications detailed in the rest of this document. The final optimization problem is:

$$\underset{V,b,c,\alpha,A}{\text{minimize}} \qquad \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k \boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha_k + \alpha_{i_k}, Re(c_k, v_k)\right)$$

$$\text{subject to} \qquad \sum_{n=1}^{N_A} M_{k,n} A_n = c_k \boldsymbol{c_l}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} M'_{k,n} A_n, Re(c_k, v_k)\right) \qquad \text{for } k \in 1,\dots,N_\theta$$

$$A_1 = \frac{8W}{\rho V^2 b\pi}$$

Or, in an even more compact form by plugging in the last constraint for A_1 in both other equations:

$$\underset{V,b,c,\alpha,A}{\text{minimize}} \quad \frac{W}{\rho b} + \frac{V^2\pi}{8}\sum_{n=2}^{N_A} nA_n^2 + V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k \boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}M'_{k,1}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=2}^{N_A} M'_{k,n}A_n, Re(c_k, v_k)\right)$$

$$\text{subject to} \qquad M_{k,1}\frac{8W}{\rho V^2 b\pi} + \sum_{n=2}^{N_A} M_{k,n} A_n =$$

$$c_k \boldsymbol{c_l}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}M'_{k,1}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=2}^{N_A} M'_{k,n}A_n, Re(c_k, v_k)\right) \qquad \text{for } k \in 1,\dots,N_\theta$$

3   Lifting Line Theory

From Kutta-Joukovsky theorem:

$$\Gamma(y) = \frac{l(y)}{\rho V} = \frac{1/2\rho V^2 c(y) c_l(y)}{\rho V} = \frac{1}{2}V c(y) c_l(y)$$

The strength of the vortex sheet Γ is defined on a closed interval $[-b/2, b/2]$, which we remap to $[\pi, 0]$. On this interval, a Fourier series expansion can be written:

$$y = \frac{b}{2}\cos(\theta)$$

$$\Gamma(\theta) = \frac{1}{2}V\sum_{n=1}^{+\infty} A_n \sin(n\theta), \; c_l(\theta) = \frac{1}{c(y)}\sum_{n=1}^{+\infty} A_n \sin(n\theta)$$

Using Biot and Savart law, the induced angle of attack on at each span-wise coordinate

can be written as:

$$\alpha_i(y) = \frac{-1}{4\pi V} \int_{-b/2}^{b/2} \frac{\frac{d\Gamma(t)}{dt}}{y-t} dt$$

$$\alpha_i(\theta) = \frac{-1}{4\pi V} \int_{\pi}^{0} \frac{\left(\frac{d\Gamma(\Theta)}{d\Theta} \frac{d\Theta}{dt}\right)}{b/2\cos(\theta) - b/2\cos(\Theta)} \frac{dt}{d\Theta} d\Theta$$

$$= \frac{1}{2b\pi V} \int_{0}^{\pi} \frac{\frac{d\Gamma(\Theta)}{d\Theta}}{\cos(\theta) - \cos(\Theta)} d\Theta$$

$$\frac{d\Gamma(\Theta)}{d\Theta} = \frac{d}{d\Theta} \frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\Theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta)$$

$$\alpha_i(\theta) = \frac{1}{2b\pi} \int_{0}^{\pi} \frac{\frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta$$

$$= \frac{1}{4b\pi} \sum_{n=1}^{+\infty} n A_n \int_{0}^{\pi} \frac{\cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta$$

Directly plugging in the solution of the Glauert integral:

$$\alpha_i(\theta) = \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)}$$

The value of the lift coefficient on each section is given by the fit of the 2D airfoil data:

$$c_l(\theta) = \boldsymbol{c_l}^{\boldsymbol{2D}} \left(\alpha(\theta) - \alpha_i(\theta), Re(\theta) \right)$$

Plugging in the value of c_l in terms of the Fourier coefficients of Γ, we obtain the general equation that needs to be solved to find the $A_{1:+\infty}$ is:

$$\sum_{n=1}^{+\infty} A_n \sin(n\theta) = c(\theta) \boldsymbol{c_l}^{\boldsymbol{2D}} \left(\alpha(\theta) - \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta) \right)$$

If we only compute A_n for $n \in [0, N_A]$, we can choose a discrete set of θ_k for $k \in [0, N_\theta]$:

$$MA = c\boldsymbol{c_l}^{\boldsymbol{2D}} \left(\alpha - \frac{1}{4b} M'A, Re(c, V) \right) \tag{1}$$

with:

c and α two vectors of size N_θ containing the chord and angle of attack at each θ_k

M and a M' two matrices of size N_θ, N_A such that:

$\forall k \in \{0, \ldots, N_\theta\}, \forall n \in \{0, \ldots, N_A\} \quad M_{k,n} = \sin(n\theta_k) \quad M'_{k,n} = n\sin(n\theta_k)/\sin(\theta_k)$

A a vector of size N_A containing all the A_n

4

4 Lift To Drag Ratio

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{Di} + C_{D0}}$$

Let us start with C_L:

$$
\begin{aligned}
C_L &= \frac{1}{S} \int_{b/2}^{b/2} c(y) c_l(y) dy \\
&= \frac{b}{2S} \int_0^\pi \sum_{n=1}^{+N_A} A_n \sin(n\theta) \sin(\theta) d\theta \\
&= \frac{b}{2S} \sum_{n=1}^{+N_A} A_n \int_0^\pi \sin(n\theta) \sin(\theta) d\theta
\end{aligned}
$$

$$(2)$$

Since the Fourier basis is orthogonal, we have for any k, n:

$$\int_0^\pi \sin(n\theta) \sin(k\theta) d\theta = \begin{cases} 0 & k \neq n \\ \frac{\pi}{2} & k = n \end{cases}$$

Therefore:

$$C_L = \frac{b\pi A_1}{4S} \tag{3}$$

Now, to get C_{Di}, we look at it's span-wise distribution c_{d_i}:

$$
\begin{aligned}
c_{d_i}(\theta) &= c_l(\theta) \alpha_i(\theta) \\
&= \frac{1}{4bc(\theta)} \sum_{n_a=0}^{+\infty} n_a A_{n_a} \frac{\sin(n_a\theta)}{\sin(\theta)} \sum_{n_b=0}^{+\infty} A_{n_b} \sin(n_b\theta)
\end{aligned}
$$

Integrating to get C_{Di}:

$$
\begin{aligned}
C_{Di} &= \frac{b}{2S} \int_0^\pi c_{d_i}(\theta) c(\theta) d\theta \\
C_{Di} &= \frac{b}{2S} \int_0^\pi \frac{1}{4b} \sum_{n_a=0}^{N_A} n_a A_{n_a} \frac{\sin(n_a\theta)}{\sin(\theta)} \sum_{n_b=0}^{N_A} A_{n_b} \sin(n_b\theta) \sin(\theta) d\theta \\
C_{Di} &= \frac{1}{8S} \int_0^\pi \sum_{n_a=0}^{N_A} \sum_{n_b=0}^{N_A} n_a A_{n_a} A_{n_b} \sin(n_a\theta) \sin(n_b\theta) d\theta
\end{aligned}
\tag{4}
$$

Using again that the Fourier basis functions are orthogonal, we obtain:

$$C_{Di} = \frac{\pi}{16S} \sum_{n=1}^{N_A} n A_n^2 \tag{5}$$

Finally, C_{D_0} is also obtained by integration:

$$c_{d_0}(\theta) = \boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha(\theta) - \alpha_i(\theta), Re(\theta)\right)$$

$$C_{D_0}(\theta) = \frac{b}{2S}\int_0^\pi c(\theta)\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha(\theta) - \alpha_i(\theta), Re(\theta)\right)\sin(\theta)d\theta$$

$$C_{D_0}(\theta) = \frac{b}{2S}\int_0^\pi c(\theta)\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha(\theta) - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta)\right)\sin(\theta)d\theta$$

If $\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}$ has an analytic expression, this can be simplified further. In the our case however, it is a general function fitted from wind tunnel data, therefore the integration is performed numerically with Clenshaw–Curtis quadrature.

Calling $w_\theta^{(k)}$ and θ_k respectively the weights and the points of this quadrature, we get:

$$C_{D_0}(\theta) = \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k)\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) \qquad (6)$$

Finally:

$$\frac{L}{D} = \frac{C_L}{C_{Di} + C_{D0}}$$

$$\frac{L}{D} = \frac{\frac{b\pi A_1}{4S}}{\frac{\pi}{16S}\sum_{n=1}^{+\infty} nA_n^2 + \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k)\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)}$$

$$\frac{L}{D} = \left(\frac{1}{4bA_1}\sum_{n=1}^{+\infty} nA_n^2 + \frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k)\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)\right)^{-1}$$
$$(7)$$

5   Optimization Problem

We start with:

$$\underset{V,b,c,\alpha,A}{\text{minimize}} \qquad (C_L/(C_{D_i} + C_{D_0}))^{-1}$$

$$\text{subject to} \qquad C_L = \frac{W}{1/2\rho V^2 S}$$

$$c_l(y) = \boldsymbol{c_l^{2D}}\left(\alpha(y) + \alpha_i(y), Re(y)\right)$$

$$c_{d_0}(y) = \boldsymbol{c_{d_0}}^{\boldsymbol{2D}}(\alpha(y) - \alpha_i(y), Re(y))$$

$$C_L = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_l(y')dy'$$

$$C_{D_0} = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_{d_0}(y')dy'$$

$$C_{D_i} = \frac{C_L^2}{\pi \mathcal{R}\epsilon}$$

$$\mathcal{R} = \frac{b^2}{S}$$

$$\epsilon = \left(\sum_{n=1}^{N_A} n\left(\frac{A_n}{A_1}\right)^2\right)^{-1}$$

$$\alpha_i(\theta) = -\frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta)}{\sin(\theta)}$$

$$\forall \theta \in [0,\pi] \qquad \sum_{n=1}^{N_A} A_n \sin(\theta) = \frac{1}{c(\theta)}\boldsymbol{c_l^{2D}}\left(\alpha(\theta) - \alpha_i(\theta), Re(\theta)\right)$$

Discretizing, and using equations 3, 5, 6, 7, we get:

$$\underset{V,b,c,\alpha,A}{\text{minimize}} \qquad \frac{1}{4bA_1}\sum_{n=1}^{N_A} nA_n^2 +$$

$$\frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k \boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) \qquad (8)$$

$$\text{subject to} \qquad MA = c\boldsymbol{c_l^{2D}}\left(\alpha - \frac{1}{4b}M'A, Re(c,V)\right) \qquad (9)$$

$$A_1 = \frac{8W}{\rho V^2 b\pi} \qquad (10)$$

Using 10 to simplify 8 further, we obtain:

$$
\underset{V,b,c,\alpha,A}{\text{minimize}} \qquad \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2+
$$

$$
V^2b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k \boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) \qquad (11)
$$

$$
\text{subject to} \qquad MA = c\boldsymbol{c_l}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}M'A_n, Re(\theta_k)\right) \qquad (12)
$$

$$
A_1 = \frac{8W}{\rho V^2 b\pi} \qquad (13)
$$

because:

$$
\frac{1}{4bA_1} = \frac{\rho V^2 \pi}{8W}
$$

$$
\frac{\pi}{2A_1} = \frac{\rho V^2 b}{W}
$$

Finally, we incorporate 13 into the objective function and the constraint.

$$
\underset{V,b,c,\alpha,A}{\text{minimize}} \qquad \frac{W}{\rho b} + \frac{V^2\pi}{8}\sum_{n=2}^{N_A} nA_n^2 + V^2b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k\boldsymbol{c_{d_0}}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}M'_{k,1}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=1}^{N_A} M'_{k,n}A_n, Re(c_k, v_k)\right)
$$

$$
\text{subject to} \qquad M_{k,1}\frac{8W}{\rho V^2 b\pi} + \sum_{n=2}^{N_A} M_{k,n}A_n =
$$

$$
c_k\boldsymbol{c_l}^{\boldsymbol{2D}}\left(\alpha_k - \frac{1}{4b}M'_{k,1}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=2}^{N_A} M'_{k,n}A_n, Re(c_k, v_k)\right) \qquad \text{for } k \in 1,\ldots,N_\theta
$$