Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

Jean de Becdelievre

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1 Nomenclature

W, C_W	:	glider weight, weight coefficient
L, C_L	:	lift, lift coefficient
$D, Di, D_p, C_D, C_{Di}, C_{D_p}$	:	drag, induced drag, parasite drag, and respective coefficients
b	:	wing span
S	:	wing area
$Æ$	:	wing aspect ratio
y	:	span-wise coordinate, ranging from $-b/2$ to $b/2$
θ	:	remapped span-wise coordinate ranging from 0 to π
N_y	:	Number of span-wise sections
$\Gamma(y)$ or $\Gamma(\theta)$	:	span-wise strength of vortex sheet
A_n	:	nth coefficient of the Fourier expansion of Γ
$\bar{A_n}$	:	A_n/A_1
N_A	:	Number of Fourier coefficients considered for Γ
$c(y)$ or $c(\theta)$	:	span-wise chord distribution
$\left(\frac{c}{b}\right)_k$	:	chord size of section k along the span, divided by the span b
$c_l(y)$ or $c_{l\mathbf{k}}$	:	span-wise lift coefficient distribution, lift coefficient at section k
V	:	airspeed
$\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}$	:	fit of 2D airfoil data
$Re_{\mathbf{lb}}, Re_{\mathbf{ub}}, cl_{\mathbf{lb}}, cl_{\mathbf{ub}}$	:	lower and upper bounds on Re and c_l, region of validity of $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}$
$Re(y)$ or Rek	:	Reynolds number distribution, Reynolds number at spanwise station k
$w_{1:Ny}$	:	positive quadrature weights allowing to write integrals as discrete sums

2 Introduction

This code aims at optimizing the wing planform and airspeed of a glider of a given weight W and fixed airfoil section for maximum L/D. 2D wind tunnel data of the airfoil is available, *i.e.* the function $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l, Re)$ are provided for $c_l \in [c_{l\mathbf{lb}}, c_{l\mathbf{ub}}]$ and $Re \in [Re_{\mathbf{lb}}, Re_{\mathbf{ub}}]$.

Schematically, the optimization problem reads:

$$
\begin{aligned}
\underset{V,b,c(y),c_l(y)}{\text{minimize}} \quad & C_D/C_L \\
\text{subject to} \quad & {C_L}^2 + {C_D}^2 = {C_W}^2 \quad - \quad \textit{gliding flight constraint} \\
& c_{l\mathbf{lb}} \leq c_{l\mathbf{k}} \leq c_{l\mathbf{ub}} \quad \forall k \in [1, N_y] \quad - \quad \textit{domain constraint for } \boldsymbol{{c_{d_p}}^{2D}} \\
& Re_{\mathbf{lb}} \leq Re_k \leq Re_{\mathbf{ub}} \quad \forall k \in [1, N_y] \quad - \quad \textit{domain constraint for } \boldsymbol{{c_{d_p}}^{2D}}
\end{aligned}
$$

First, the optimization problem is described in more details. Second, some insights are obtained from considering simple functions for $\boldsymbol{{c_{d_p}}^{2D}}$. Finally, using a neural network fit of 2d airfoil data, the complete problem is solved and results are analyzed.

3 Fitting Airfoil Data

The model used here for $\boldsymbol{{c_{d_p}}^{2D}}$ comes from a neural network fit of airfoil data from the UIUC database. It is valid for $Re \in [60\,000, 200\,000]$ and $c_l \in [-0.4, 1.25]$. Figure 1 shows a plot of the data, as well as contour lines of the neural network fit.

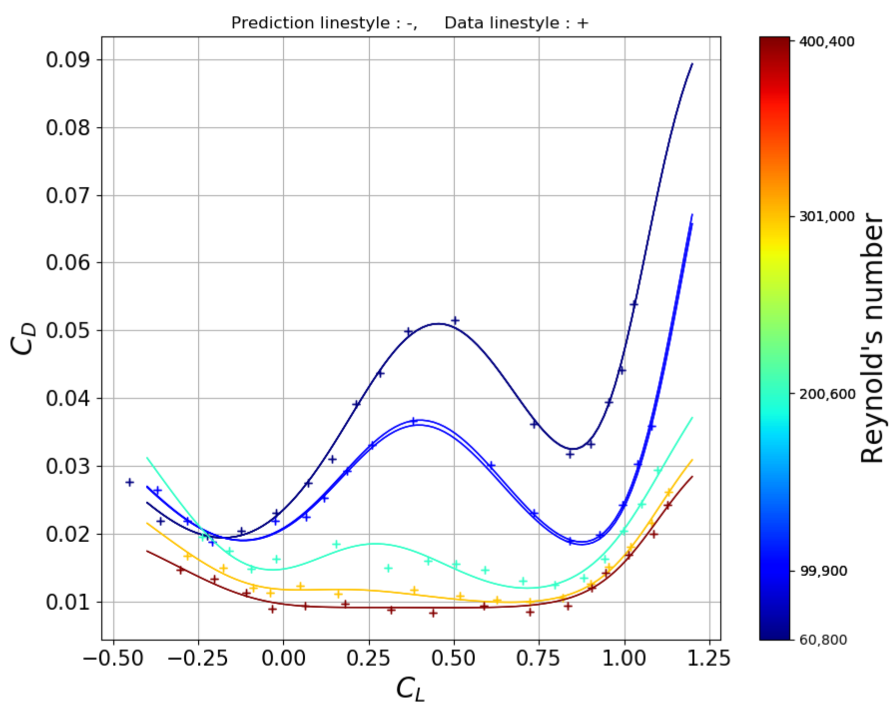

Figure 1: Neural Network Model of the Airfoil Data

4 Optimization Problem

Loss Function: Using the gliding flight constraint, we have:

$$\frac{{C_L}^2}{{C_D}^2} = \frac{{C_W}^2 - {C_D}^2}{{C_D}^2} = \frac{{C_W}^2}{{C_D}^2} - 1$$

Therefore, minimizing C_D/C_W is equivalent to minimizing C_D/C_L. Moreover, C_D can be decomposed into lift-induced drag and parasite drag:

$$C_D = C_{D_p} + C_{D_i}$$

Parasite Drag : Since we can compute the 2D parasite drag using ${\boldsymbol{c_{d_p}}}^{\boldsymbol{2D}}$, the total C_{D_p} is obtained with integration.

$$C_{D_p} = \frac{1}{S}\int_{-b/2}^{b/2} c(y){\boldsymbol{c_{d_p}}}^{\boldsymbol{2D}}(c_l(y), Re(y))dy = A\!\!\!R \sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k {\boldsymbol{c_{d_p}}}^{\boldsymbol{2D}}(c_{l\mathbf{k}}, Re_k)$$

where:

- N_y is the number of spanwise panels
- $w_{1,\dots,N_y}$ a set of quadrature weights that depend on the spacing and location of the panels, but not on their size or on the total span
- $\left(\frac{c}{b}\right)_k = c_k/b$ is the chord of the kth spanwise panel, non-dimensionalized by the total span ratio b.

Induced Drag: We use a far field estimate of the induced drag. Assuming a planar wake, we use a sine basis expansion of the vortex sheet strength:

$$\Gamma(\theta) = 2bV \sum_{n=1}^{N_a} A_n sin(n\theta)$$

N_a is the order of this expansion. Using lifting line theory, this give us:

$$C_{D_i} = \frac{C_L^2}{\pi}\left(1 + \sum_{n=2}^{N_a} n{\bar{A}_n}^2\right)$$

where we defined $\bar{A}_n = A_n/A_1$.

Approximate Handling of the Equality Constraint: We simplify the problem by assuming that the glide angle γ is small. In such case,

$$C_W = \frac{C_L}{\cos\gamma} \approx C_L \tag{1}$$

The validity of the small angle assumption is discussed in appendix A. Instead of solving the original nonlinear constrained problem, we insert the value of A_1 into the objective function. $\pi Æ A_1^2$ appears in the expression of C_{D_i}, so we get:

$$C_{D_i} = \frac{C_W^2}{\pi Æ}\left(1 + \sum_{n=2}^{N_a} n\bar{A}_n^{\;2}\right)$$

Objective Function: Putting together all of the above, we obtain the following objective function:

$$\frac{C_D}{C_W} = \frac{C_W}{\pi Æ}\left(1 + \sum_{n=2}^{N_a} n\bar{A}_n^{\;2}\right) + \frac{Æ}{C_W}\sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_{l\mathbf{k}}, Re_k)$$

Replacing C_W by its value, we actually optimize:

$$\frac{4W}{\pi\rho^2(Vb)^2}\left(1 + \sum_{n=2}^{N_a} n\bar{A}_n^{\;2}\right) + (Vb)^2\sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_{l\mathbf{k}}, Re_k)$$

Reynolds and Lift Coefficient Bounds: We can write the Reynolds number Re_k and the lift coefficient $c_{l\mathbf{k}}$ at the local section k as a function of other variables used previously:

$$Re_k = \frac{\rho}{\nu}\left(\frac{c}{b}\right)_k (Vb)$$

$$c_{l\mathbf{k}} = \frac{8W}{\pi\rho(Vb)^2\left(\frac{c}{b}\right)_k}\left(\sin(\theta_k) + \sum_{n=2}^{N_a}\bar{A}_n \sin(n\theta_k)\right)$$

The collected optimization problem is written and analyzed in the next section.

5 Optimization Problem Insights and Discussion

$$\begin{aligned}
\underset{Vb,\left(\frac{c}{b}\right)_{1:N_y},\bar{A}_{2:N_a}}{\text{minimize}} \quad & \frac{4W}{\pi\rho^2(Vb)^2}\left(1 + \sum_{n=2}^{N_a} n\bar{A}_n^{\;2}\right) + (Vb)^2\sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_{l\mathbf{k}}, Re_k) \\
\text{subject to} \quad & c_{l\mathbf{lb}} \leq c_{l\mathbf{k}} \leq c_{l\mathbf{ub}} \quad \forall k \in [1, N_y] \\
& Re_{\mathbf{lb}} \leq Re_k \leq Re_{\mathbf{ub}} \quad \forall k \in [1, N_y]
\end{aligned}$$

with:

$$Re_k = \frac{\rho}{\nu}\left(\frac{c}{b}\right)_k (Vb)$$

$$c_{l\mathbf{k}} = \frac{8W}{\pi\rho(Vb)^2\left(\frac{c}{b}\right)_k}\left(\sin(\theta_k) + \sum_{n=2}^{N_a}\bar{A}_n \sin(n\theta_k)\right)$$

Design Variables: The first interesting note is that the optimization of the planform and airspeed only requires:

- $(Vb)^2$ which is the only place where the airspeed appears. Note that we could equivalently use $C_W/\text{Æ}$, which makes it obvious that this design variable is tightly connected to the nominal C_L.
- $\left(\frac{c}{b}\right)_{1:N_y}$ is simply related to the planform shape
- $\bar{A}_{2:N_a}$ describes the lift distribution. Note that A_1 is not a design variable, it is directly set by the value of $C_W/(\pi \text{Æ})$.

This set of variables is not sufficient to recover the wing dimensions, or even the airspeed. Once the optimization has converged, choosing either a span value b, an airspeed V, or the chord c_k of any of the chord sections allows to recover the full dimensional planform shape.

Feasibility Range: This problem is feasible for any weight W, air density ρ or viscosity ν. As W varies, it becomes more complicated to make the c_l bound. But, even though there is a bound on $\left(\frac{c}{b}\right)_k (Vb)$ (from the bound on Re), there is no bound on $\left(\frac{c}{b}\right)_k (Vb)^2$ in the current format. Therefore we can always find a set of $\left(\frac{c}{b}\right)_k (Vb)^2$ such that the c_l bound is respected.

6 Insights from Simplified 2D Section Models

6.1 Constant ${c_{d_p}}^{2D}$

Let us assume:

$$\boldsymbol{{c_{d_p}}^{2D}}(c_l, Re) = c_{d_p} \,\textit{(constant)}$$

Then the optimization problem becomes:

$$\begin{aligned}
\underset{Vb,\left(\frac{c}{b}\right)_{1:N_y},\bar{A}_{2:N_a}}{\text{minimize}} \quad & \frac{4W}{\pi\rho^2(Vb)^2}\left(1+\sum_{n=2}^{N_a} n\bar{A}_n^{\,2}\right)+(Vb)^2\sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k c_{d_p} \\
\text{subject to} \quad & c_{l\mathbf{lb}} \le c_{l\mathbf{k}} \le c_{l\mathbf{ub}} \quad \forall k \in [1, N_y] \\
& Re_{\mathbf{lb}} \le Re_k \le Re_{\mathbf{ub}} \quad \forall k \in [1, N_y]
\end{aligned}$$

We can see that:

- $\bar{A}_{2:N_a} \to 0$ to minimize induced drag
- The chord elements $\left(\frac{c}{b}\right)_{1:N_y}$ will all tend to 0, or to their minimal value allowed by the Re bound.
- Vb will reach a finite, non-zero optimal value (to the extent that this makes sense if $\left(\frac{c}{b}\right)_{1:N_y} = 0$)

Without a proper 2D section model, this problem has limited interest: the optimal L/D is always obtained at the minimal allowed chord.

6.2 Flat Plate Model

We now assume:

$$\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l, Re) = c_{d_p} Re^{-\eta} = c_{d_p} \left(\frac{\rho}{\nu} \left(\frac{c}{b}\right)_k (Vb) \right)^{-\eta}$$

Typically, for a turbulent flat plate, we would have $\eta = 0.2$. Then we obtain:

$$\begin{aligned} \underset{Vb, \left(\frac{c}{b}\right)_{1:N_y}, \bar{A}_{2:N_a}}{\text{minimize}} \quad & \frac{4W}{\pi\rho^2(Vb)^2}\left(1 + \sum_{n=2}^{N_a} n\bar{A}_n{}^2\right) + (Vb)^{2-\eta}\sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k^{1-\eta}\left(\frac{\rho}{\nu}\right)^{-\eta} c_{d_p} \\ \text{subject to} \quad & c_{l\mathbf{lb}} \leq c_{l\mathbf{k}} \leq c_{l\mathbf{ub}} \quad \forall k \in [1, N_y] \\ & Re_{\mathbf{lb}} \leq Re_k \leq Re_{\mathbf{ub}} \quad \forall k \in [1, N_y] \end{aligned}$$

In such case:

- There still is no incentive to do anything else than an elliptic lift distribution: $\bar{A}_{2:N_a} \to 0$ to minimize induced drag.
- The chord elements $\left(\frac{c}{b}\right)_{1:N_y}$ will all tend to 0 for any value of η smaller than 1.
- Again, Vb will reach a finite, non-zero optimal value unless η becomes bigger than 2.

Here we can see that this problem starts to make sense if the C_{D_p} penalty for decreasing the chord is more than linear in $1/\left(\frac{c}{b}\right)_k$, i.e. $\eta \geq 1$. This is not likely to come from a variation of C_{D_p} with the Reynolds number, but could come from a variation with c_l.

6.3 Flat Plate with Quadratic C_L Dependency

We now assume:

$$\begin{aligned} \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l, Re) &= c_{d_p\mathbf{0}} Re^{-\eta} + c_{d_p\mathbf{1}} {c_l}^2 \\ &= {c_{d_p}}'_{\mathbf{0}} \left(\frac{c}{b}\right)_k^{-\eta} (Vb)^{-\eta} + {c_{d_p}}'_{\mathbf{1}} \frac{W^2}{(Vb)^4 \left(\frac{c}{b}\right)_k^2} \left(\sin(\theta_k) + \sum_{n=2}^{N_a} \bar{A}_n \sin(n\theta_k)\right)^2 \end{aligned} \tag{2}$$

where several constant terms are regrouped under ${c_{d_p}}'_{\mathbf{0}}$ and ${c_{d_p}}'_{\mathbf{1}}$. The terms of the objective function can be split in the following way:

$$\frac{4W}{\pi\rho^2(Vb)^2}\left(1 + \sum_{n=2}^{N_a} n\bar{A}_n{}^2\right) \qquad \text{span-dependent induced drag} \tag{3a}$$

$$+ {c_{d_p}}'_{\mathbf{0}}(Vb)^{2-\eta}\sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k^{1-\eta} \qquad Re\text{-dependent parasite drag} \tag{3b}$$

$$+ W^2 {c_{d_p}}'_{\mathbf{1}} \sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k^{-1} (Vb)^{-2} \left(\sin(\theta_k) + \sum_{n=2}^{N_a} \bar{A}_n \sin(n\theta_k)\right)^2 \qquad c_l\text{-dependent parasite drag} \tag{3c}$$

The following behavior is observed:

- The chord elements $\left(\frac{c}{b}\right)_{1:N_y}$ do not tend to 0 anymore. The penalty for increasing c_l 3c and reducing Re 3b act together to prevent it. We note that the c_l penalty is is far stronger.

- Again, Vb will reach a finite, non-zero optimal value unless η becomes bigger than 2. The optimal of Vb is much higher what is estimated in the previous case because of the incentive from 3c to reduce c_l.

- The effect of $\bar{A_{2:N_a}}$ from 3c dominates the induced drag penalty 3a for a non-elliptical wing distribution. $\bar{A_{2:N_a}}$ is mostly chosen in order to minimize the average c_l^2.

As the airplane weight varies, we observe:

- For small W, 3b has main biggest role, which will push for shorter chord, and higher Vb

- For slightly higher W, the penalty on lift per unit span 3a kicks in, and higher spans are encouraged.

- At high W, the dominant term is 3c. Bigger chords, lower Vb and non-elliptic lift distribution are promoted.

6.4 Results With Polynomial Fit

A fit of the form show in equation 2 of the 2D data shown on Figure 1 is realized.

The fit captures both essential facts that c_{d_p} decreases with Re, and that there is a sharp increase in c_l after approximately $c_l = 0.8$.

6.5 Gathered insights

This section illustrates the dramatic changes in the optimal solution that can occur when changing the 2D section parasite drag model. Especially, not taking into account c_l dependance leads to unrealistic solutions, where the optimal L/D only happens at the minimum chord.

A physical solution only happens when a penalty on 2d section c_l is included, and the c_l dependance has a major impact on the final solution. This motivates the use of an accurate 2d section model.

A Validity of the small angle assumptions

TBD