Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

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1 Nomenclature

W	:	glider weight
L, C_L	:	lift, lift coefficient
$D, Di, D_0, C_D, C_{Di}, C_{D_0}$	:	drag, induced drag, friction, and respective coefficients
b	:	span
S	:	wing area
$Æ$	:	wing aspect ratio
y	:	spanwise coordinate, ranging from $-b/2$ to $b/2$
θ	:	remapped spanwise coordinate ranging from 0 to pi
N_{theta}	:	Number of spanwise sections
$\Gamma(y)$ or $\Gamma(\theta)$	:	spanwise strength of vortex sheet
A_n	:	nth coefficient of the Fourrier expansion of Γ
N_A	:	Number of Fourrier coefficients considered for Γ
S	:	matrix of size N_θ, N_A such that: $\forall k, n \quad S_{k,n} = \sin(n\theta_k)$
S'	:	matrix of size N_θ, N_A such that: $\forall k, n \quad {S'}_{k,n} = n\sin(n\theta_k)/\sin(\theta_k)$
A	:	vector of size N_A containing all the A_n
$\alpha(y)$ or $\alpha(\theta)$	:	spanwise angle of attack distribution
$\alpha_i(y)$ or $\alpha_i(\theta)$	:	spanwise lift-induced angle of attack distribution
$c(y)$ or $c(\theta)$	:	spanwise chord distribution
$c_l(y)$ or $c_l(\theta)$	:	spanwise lift distribution
$c_d(y)$ or $c_d(\theta)$	:	spanwise drag distribution
V	:	airspeed
CL_{2D} and $CD0_{2D}$	:	fit of the 2D airfoil data
Re	:	Reynolds number
$y_k, \theta_k, \alpha_k, \alpha_{i_k}, c_k, c_{l_k}$	:	$y, \theta, \alpha, \alpha_i, c, c_l$ at the kth section of the span

2 Optimization Problem

This code aims at optimizing the wing shape and airspeed of a glider of a given weight W and fixed airfoil section. 2D wind tunnel data of the airfoil is available: the functions $C_{L2D}(\alpha, Re)$ and $C_{D02D}(\alpha, Re)$ are provided.

The optimization problem reads:

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & (C_L/(C_{D_i} + C_{D_0}))^{-1} \\
\text{subject to} \quad & c_l(y) = CL_{2D}\left(\alpha(\theta) + \alpha_i(y), Re(y)\right) \\
& c_{d_0} = CD0_{2D}(\alpha - \alpha_i, Re(y)) \\
& C_L = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_l(y')dy' \\
& C_{D_0} = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_{d_0}(y')dy' \\
& C_{D_i} = \frac{C_L}{\pi \mathit{AR} \epsilon} \\
& \mathit{AR} = \frac{b^2}{S} \\
& \epsilon = \left(\sum_{n=1}^{N_A} n\left(\frac{A_n}{A_1}\right)^2\right)^{-1} \\
& \alpha_i(y) = -\frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta)}{\sin(\theta)} \\
\forall\theta \in [0, \pi] \quad & \sum_{n=1}^{N_A} A_n \sin(\theta) = \frac{1}{c(\theta)} CL_{2D}\left(\alpha(\theta) + \alpha_i(\theta), Re(\theta)\right)
\end{aligned}
$$

where θ is another span parametrization such that:

$$\forall\theta \in [0, \pi], \forall y \in [\frac{-b}{2}, \frac{b}{2}], \quad y = \frac{b}{2}\cos\theta$$

We discretize the span into N_{theta} spanwise sections, and perform several simplifications detailed below, such that the final optimizing problem is:

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k + \alpha_{i_k}, Re(c_k, v_k)\right) \\
\text{subject to} \quad & \sum_{n=1}^{N_A} S_{k,n}A_n = c_k CL_{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} S'_{k,n}A_n, Re(c_k, v_k)\right) \quad \text{for } k \in 1,\dots,N_\theta \\
& A_1 = \frac{8W}{\rho V^2 b\pi}
\end{aligned}
$$

3 Lifting Line Theory

From Kutta-Joukovsky theorem:

$$\Gamma(y) = \frac{l(y)}{\rho V} = \frac{1/2\rho V^2 c(y) c_l(y)}{\rho V} = \frac{1}{2} V c(y) c_l(y)$$

The spanwise circulation is defined on a compact $[-b/2, b/2]$, which we remap to $[0, \pi]$ and write a Fourrier series expansion:

$$y = \frac{b}{2}\cos(\theta)$$

$$\Gamma(\theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\theta),\ c_l(\theta) = \frac{1}{c(y)} \sum_{n=1}^{+\infty} A_n \sin(n\theta)$$

Using Biot and Savart law, the induced angle of attack on at each spanwise coordinate can be written as:

$$
\begin{aligned}
\alpha_i(y) &= \frac{-1}{4\pi V}\int_{-b/2}^{b/2} \frac{\frac{d\Gamma(t)}{dt}}{y-t} dt \\
\alpha_i(\theta) &= \frac{-1}{4\pi V}\int_{\pi}^{0} \frac{\left(\frac{d\Gamma(\Theta)}{d\Theta}\frac{d\Theta}{dt}\right)}{b/2\cos(\theta) - b/2\cos(\Theta)} \frac{dt}{d\Theta} d\Theta \\
&= -\frac{-1}{2b\pi V}\int_0^{\pi} \frac{\frac{d\Gamma(\Theta)}{d\Theta}}{\cos(\theta) - \cos(\Theta)} d\Theta
\end{aligned}
$$

$$\frac{d\Gamma(\Theta)}{d\Theta} = \frac{d}{d\Theta}\frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\Theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} nA_n \cos(n\Theta)$$

$$
\begin{aligned}
\alpha_i(\theta) &= -\frac{-1}{2b\pi}\int_0^{\pi} \frac{\frac{1}{2} V \sum_{n=1}^{+\infty} nA_n \cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta \\
&= -\frac{-1}{4b\pi}\sum_{n=1}^{+\infty} nA_n \int_0^{\pi} \frac{\cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta
\end{aligned}
$$

Directly plugging in the solution of the Glauert integral:

$$\alpha_i(\theta) = \frac{-1}{4b}\sum_{n=1}^{+\infty} nA_n \frac{\sin(n\theta)}{\sin(\theta)}$$

The general equation that needs to be solved to find the $A_{1:+\infty}$ is:

$$c_l(\theta) = CL_{2D}\left(\alpha(\theta) + \alpha_i(\theta), Re(\theta)\right)$$
$$\sum_{n=1}^{+\infty} A_n \sin(n\theta) = c(\theta) CL_{2D}\left(\alpha(\theta) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n \frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta)\right)$$

If we only compute A_n for $n \in [0, N_A]$, we can choose a discrete set of θ_k for $k \in [0, N_\theta]$:

$$SA = c(\theta) CL_{2D}\left(\alpha(\theta) - \frac{1}{4b} S' A, Re(\theta)\right) \tag{1}$$

with:

S a matrix of size N_θ, N_A such that: $\forall m, n \quad S_{m,n} = \sin(n\theta_m)$

S' a matrix of size N_θ, N_A such that: $\forall m, n \quad {S'}_{m,n} = n\sin(n\theta_m)/\sin(\theta_m)$

A a vector of size N_A containing all the A_n

4 L over D

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{Di} + C_{D0}}$$

Let us start with C_L:

$$\begin{aligned} C_L &= \frac{1}{S}\int_{b/2}^{b/2} c(y) c_l(y) dy \\ &= \frac{b}{2S}\int_0^\pi \sum_{n=1}^{+N_A} A_n \sin(n\theta)\sin(\theta) d\theta \\ &= \frac{b}{2S}\sum_{n=1}^{+N_A} A_n \int_0^\pi \sin(n\theta)\sin(\theta) d\theta \end{aligned} \tag{2}$$

Since the Fourrier basis is orthogonal, we have for any k, n:

$$\int_0^\pi \sin(n\theta)\sin(k\theta) d\theta = \begin{cases} 0 & k \neq n \\ \frac{\pi}{2} & k = n \end{cases}$$

Therefore:

$$C_L = \frac{b\pi A_1}{4S}$$

Now, to get C_{Di}, we look at it's spanwise distribution c_{d_i}:

$$\begin{aligned}
c_{d_i}(\theta) &= -c_l(\theta)\alpha_i(\theta) \\
&= \frac{1}{4bc(\theta)} \sum_{n_a=0}^{+\infty} n_a A_{n_a} \frac{\sin(n_a\theta)}{\sin(\theta)} \sum_{n_b=0}^{+\infty} A_{n_b} \sin(n_b\theta)
\end{aligned}$$

Integrating to get C_{Di}:

$$\begin{aligned}
C_{Di} &= \frac{b}{2S}\int_0^\pi c_{d_i}(\theta)c(\theta)d\theta \\
C_{Di} &= \frac{b}{2S}\int_0^\pi \frac{1}{4b} \sum_{n_a=0}^{N_A} n_a A_{n_a} \frac{\sin(n_a\theta)}{\sin(\theta)} \sum_{n_b=0}^{N_A} A_{n_b} \sin(n_b\theta)\sin(\theta)d\theta \\
C_{Di} &= \frac{1}{8S}\int_0^\pi \sum_{n_a=0}^{N_A}\sum_{n_b=0}^{N_A} n_a A_{n_a} A_{n_b} \sin(n_a\theta)\sin(n_b\theta)d\theta \qquad (3)
\end{aligned}$$

Using again that the Fourrier basis functions are orthogonal, we obtain:

$$C_{Di} = \frac{\pi}{16S}\sum_{n=1}^{N_A} nA_n^2 \qquad (4)$$

Finally, C_{D_0} is also obtained by integration:

$$\begin{aligned}
c_{d_0}(\theta) &= CD0_{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right) \\
C_{D_0}(\theta) &= \frac{b}{2S}\int_0^\pi c(\theta)CD0_{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right)\sin(\theta)d\theta \\
C_{D_0}(\theta) &= \frac{b}{2S}\int_0^\pi c(\theta)CD0_{2D}\left(\alpha(\theta) - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta)\right)\sin(\theta)d\theta
\end{aligned}$$

If $CD0_{2D}$ has an analytic expression, this can be simplified further. In the our case however, it is a general function fitted from wind tunnel data, therefore the integration is performed numerically with Clenshaw–Curtis quadrature.

Calling $w_\theta^{(k)}$ and θ_k respectively the weights and the points of this quadrature, we get:

$$C_{D_0}(\theta) = \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k)CD0_{2D}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) \qquad (5)$$

Finally:

$$\frac{L}{D} = \frac{C_L}{C_{Di} + C_{D0}}$$

$$\frac{L}{D} = \frac{\frac{b\pi A_1}{4S}}{\frac{\pi}{16S}\sum_{n=1}^{+\infty} nA_n^2 + \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) CD0_{2D}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)}$$

$$\frac{L}{D} = \left(\frac{1}{4bA_1}\sum_{n=1}^{+\infty} nA_n^2 + \frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) CD0_{2D}\left(\alpha(\theta_k) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)\right)^{-1} \tag{6}$$

5 Optimization Problem

$$\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & (L/D)^{-1} \\
\text{subject to} \quad & c_l(\theta_k) = c(\theta_k) CL_{2D}\left(\alpha(\theta_k) + \alpha_i(\theta_k), Re(\theta_k)\right) && (7) \\
& C_L = \frac{W}{1/2\rho V^2 S} && (8)
\end{aligned}$$

$$\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{1}{4bA_1}\sum_{n=1}^{N_A} nA_n^2 + \\
& \frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) && (9) \\
\text{subject to} \quad & SA = c(\theta_k) CL_{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (10) \\
& A_1 = \frac{2W}{\rho V^2 b\pi} && (11)
\end{aligned}$$

where it was included in 7 that $C_L = \frac{b\pi A_1}{4S}$. Using 8 to simplify 15 further, we obtain:

$$\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + \\
& V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) && (12) \\
\text{subject to} \quad & SA = c(\theta_k) CL_{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (13) \\
& A_1 = \frac{8W}{\rho V^2 b\pi} && (14)
\end{aligned}$$

because:

$$\frac{1}{4bA_1} = \frac{\rho V^2 \pi}{8W}$$
$$\frac{\pi}{2A_1} = \frac{\rho V^2 b}{W}$$

$$\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k - \frac{1}{4b}S' A_n, Re(\theta_k)\right) && (15)\\
\text{subject to} \quad & SA = c(\theta_k) CL_{2D}\left(\alpha_k - \frac{1}{4b}S' A_n, Re(\theta_k)\right) && (16)\\
& A_1 = \frac{8W}{\rho V^2 b\pi} && (17)
\end{aligned}$$

$$\begin{aligned}
\underset{V,b,c,\alpha,\alpha_i A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k CD0_{2D}\left(\alpha_k + \alpha_{i_k}, Re(\theta_k)\right) && (18)\\
\text{subject to} \quad & SA = c(\theta_k) CL_{2D}\left(\alpha_k + \alpha_{i_k}, Re(\theta_k)\right) && (19)\\
& A_1 \geq \frac{8W}{\rho V^2 b\pi} && (20)\\
& \alpha_{i_k} \geq -\frac{1}{4b}S' A_n && (21)
\end{aligned}$$