

Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

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1 Nomenclature

W	:	glider weight
L, C_L	:	lift, lift coefficient
$D, D_i, D_0, C_D, C_{D_i}, C_{D_0}$:	drag, induced drag, friction drag, and respective coefficients
b	:	wing span
S	:	wing area
\mathcal{R}	:	wing aspect ratio
y	:	span-wise coordinate, ranging from $-b/2$ to $b/2$
θ	:	remapped span-wise coordinate ranging from 0 to π
N_{theta}	:	Number of span-wise sections
$\Gamma(y)$ or $\Gamma(\theta)$:	span-wise strength of vortex sheet
A_n	:	n th coefficient of the Fourier expansion of Γ
N_A	:	Number of Fourier coefficients considered for Γ
S	:	matrix of size N_θ, N_A such that: $\forall k, n \quad S_{k,n} = \sin(n\theta_k)$
S'	:	matrix of size N_θ, N_A such that: $\forall k, n \quad S'_{k,n} = n \sin(n\theta_k) / \sin(\theta_k)$
A	:	vector of size N_A containing all the A_n
$\alpha(y)$ or $\alpha(\theta)$:	span-wise angle of attack distribution
$\alpha_i(y)$ or $\alpha_i(\theta)$:	span-wise lift-induced angle of attack distribution
$c(y)$ or $c(\theta)$:	span-wise chord distribution
$c_l(y)$ or $c_l(\theta)$:	span-wise lift coefficient distribution
$c_d(y)$ or $c_d(\theta)$:	span-wise drag coefficient distribution
V	:	airspeed
C_L^{2D} and $C_{D_0}^{2D}$:	fit of 2D airfoil data
Re	:	Reynolds number
$y_k, \theta_k, \alpha_k, \alpha_{i_k}, c_k, c_{l_k}$:	$y, \theta, \alpha, \alpha_i, c, c_l$ at the k th span section

2 Optimization Problem

This code aims at optimizing the wing planform and airspeed of a glider of a given weight W and fixed airfoil section for maximum L/D . 2D wind tunnel data of the airfoil is available, *i.e.* the functions $C_L^{2D}(\alpha, Re)$ and $C_{D_0}^{2D}(\alpha, Re)$ are provided.

The optimization problem reads:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && (C_L / (C_{D_i} + C_{D_0}))^{-1} \\
& \text{subject to} && C_L = \frac{W}{1/2 \rho V^2 S} \\
& && c_l(y) = C_L^{2D}(\alpha(y) + \alpha_i(y), Re(y)) \\
& && c_{d_0}(y) = C_{D_0}^{2D}(\alpha(y) - \alpha_i(y), Re(y)) \\
& && C_L = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_l(y') dy' \\
& && C_{D_0} = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_{d_0}(y') dy' \\
& && C_{D_i} = \frac{C_L}{\pi R \epsilon} \\
& && R = \frac{b^2}{S} \\
& && \epsilon = \left(\sum_{n=1}^{N_A} n \left(\frac{A_n}{A_1} \right)^2 \right)^{-1} \\
& && \alpha_i(\theta) = -\frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta)}{\sin(\theta)} \\
& \forall \theta \in [0, \pi] && \sum_{n=1}^{N_A} A_n \sin(\theta) = \frac{1}{c(\theta)} C_L^{2D}(\alpha(\theta) - \alpha_i(\theta), Re(\theta))
\end{aligned}$$

where θ is a span parametrization such that:

$$\forall \theta \in [0, \pi], \forall y \in \left[-\frac{b}{2}, \frac{b}{2}\right], \quad y = \frac{b}{2} \cos \theta$$

We discretize the span into N_{theta} span-wise sections, and perform several simplifications detailed in the rest of this document. The final optimization problem is:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && \frac{V^2 \pi}{8} \sum_{n=1}^{N_A} n A_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} (\alpha_k + \alpha_{i_k}, \text{Re}(c_k, v_k)) \\
& \text{subject to} && \sum_{n=1}^{N_A} S_{k,n} A_n = c_k C_L^{2D} \left(\alpha_k - \frac{1}{4b} \sum_{n=1}^{N_A} S'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \quad \text{for } k \in 1, \dots, N_\theta \\
& && A_1 = \frac{8W}{\rho V^2 b \pi}
\end{aligned}$$

Or, in an even more compact form by plugging in the last constraint for A_1 in both other equations:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && \frac{W}{\rho b} + \frac{V^2 \pi}{8} \sum_{n=2}^{N_A} n A_n^2 + V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left(\alpha_k - \frac{1}{4b} S'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=1}^{N_A} S'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \\
& \text{subject to} && S_{k,1} \frac{8W}{\rho V^2 b \pi} + \sum_{n=2}^{N_A} S_{k,n} A_n = \\
& && c_k C_L^{2D} \left(\alpha_k - \frac{1}{4b} S'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=2}^{N_A} S'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \quad \text{for } k \in 1, \dots, N_\theta
\end{aligned}$$

3 Lifting Line Theory

From Kutta-Joukowski theorem:

$$\Gamma(y) = \frac{l(y)}{\rho V} = \frac{1/2 \rho V^2 c(y) c_l(y)}{\rho V} = \frac{1}{2} V c(y) c_l(y)$$

The strength of the vortex sheet Γ is defined on a closed interval $[-b/2, b/2]$, which we remap to $[0, \pi]$. On this interval, a Fourier series expansion can be written:

$$y = \frac{b}{2} \cos(\theta)$$

$$\Gamma(\theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\theta), \quad c_l(\theta) = \frac{1}{c(y)} \sum_{n=1}^{+\infty} A_n \sin(n\theta)$$

Using Biot and Savart law, the induced angle of attack on at each span-wise coordinate

can be written as:

$$\begin{aligned}
\alpha_i(y) &= \frac{-1}{4\pi V} \int_{-b/2}^{b/2} \frac{\frac{d\Gamma(t)}{dt}}{y-t} dt \\
\alpha_i(\theta) &= \frac{-1}{4\pi V} \int_{\pi}^0 \frac{\left(\frac{d\Gamma(\Theta)}{d\Theta} \frac{d\Theta}{dt} \right)}{b/2 \cos(\theta) - b/2 \cos(\Theta)} \frac{dt}{d\Theta} d\Theta \\
&= \frac{1}{2b\pi V} \int_0^{\pi} \frac{\frac{d\Gamma(\Theta)}{d\Theta}}{\cos(\theta) - \cos(\Theta)} d\Theta \\
\frac{d\Gamma(\Theta)}{d\Theta} &= \frac{d}{d\Theta} \frac{1}{2} V \sum_{n=1}^{+\infty} A_n \sin(n\Theta) = \frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta) \\
\alpha_i(\theta) &= \frac{1}{2b\pi} \int_0^{\pi} \frac{\frac{1}{2} V \sum_{n=1}^{+\infty} n A_n \cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta \\
&= \frac{1}{4b\pi} \sum_{n=1}^{+\infty} n A_n \int_0^{\pi} \frac{\cos(n\Theta)}{\cos(\theta) - \cos(\Theta)} d\Theta
\end{aligned}$$

Directly plugging in the solution of the Glauert integral:

$$\alpha_i(\theta) = \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)}$$

The value of the lift coefficient on each section is given by the fit of the 2D airfoil data:

$$c_l(\theta) = C_L^{2D} (\alpha(\theta) - \alpha_i(\theta), Re(\theta))$$

Plugging in the value of c_l in terms of the Fourier coefficients of Γ , we obtain the general equation that needs to be solved to find the $A_{1:+\infty}$ is:

$$\sum_{n=1}^{+\infty} A_n \sin(n\theta) = c(\theta) C_L^{2D} \left(\alpha(\theta) - \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta) \right)$$

If we only compute A_n for $n \in [0, N_A]$, we can choose a discrete set of θ_k for $k \in [0, N_\theta]$:

$$SA = c C_L^{2D} \left(\alpha - \frac{1}{4b} S' A, Re(c, V) \right) \quad (1)$$

with:

c and α two vectors of size N_θ containing the chord and angle of attack at each θ_k

S and a S' two matrices of size N_θ, N_A such that:

$$\forall k \in \{0, \dots, N_\theta\}, \forall n \in \{0, \dots, N_A\} \quad S_{k,n} = \sin(n\theta_k) \quad S'_{k,n} = n \sin(n\theta_k) / \sin(\theta_k)$$

A a vector of size N_A containing all the A_n

4 Lift To Drag Ratio

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{Di} + C_{D0}}$$

Let us start with C_L :

$$\begin{aligned} C_L &= \frac{1}{S} \int_{b/2}^{b/2} c(y) c_l(y) dy \\ &= \frac{b}{2S} \int_0^\pi \sum_{n=1}^{+N_A} A_n \sin(n\theta) \sin(\theta) d\theta \\ &= \frac{b}{2S} \sum_{n=1}^{+N_A} A_n \int_0^\pi \sin(n\theta) \sin(\theta) d\theta \end{aligned} \tag{2}$$

Since the Fourier basis is orthogonal, we have for any k, n :

$$\int_0^\pi \sin(n\theta) \sin(k\theta) d\theta = \begin{cases} 0 & k \neq n \\ \frac{\pi}{2} & k = n \end{cases}$$

Therefore:

$$C_L = \frac{b\pi A_1}{4S} \tag{3}$$

Now, to get C_{Di} , we look at it's span-wise distribution c_{di} :

$$\begin{aligned} c_{di}(\theta) &= c_l(\theta) \alpha_i(\theta) \\ &= \frac{1}{4bc(\theta)} \sum_{n_a=0}^{+\infty} n_a A_{n_a} \frac{\sin(n_a \theta)}{\sin(\theta)} \sum_{n_b=0}^{+\infty} A_{n_b} \sin(n_b \theta) \end{aligned}$$

Integrating to get C_{Di} :

$$\begin{aligned} C_{Di} &= \frac{b}{2S} \int_0^\pi c_{di}(\theta) c(\theta) d\theta \\ C_{Di} &= \frac{b}{2S} \int_0^\pi \frac{1}{4b} \sum_{n_a=0}^{N_A} n_a A_{n_a} \frac{\sin(n_a \theta)}{\sin(\theta)} \sum_{n_b=0}^{N_A} A_{n_b} \sin(n_b \theta) \sin(\theta) d\theta \\ C_{Di} &= \frac{1}{8S} \int_0^\pi \sum_{n_a=0}^{N_A} \sum_{n_b=0}^{N_A} n_a A_{n_a} A_{n_b} \sin(n_a \theta) \sin(n_b \theta) d\theta \end{aligned} \tag{4}$$

Using again that the Fourier basis functions are orthogonal, we obtain:

$$C_{Di} = \frac{\pi}{16S} \sum_{n=1}^{N_A} n A_n^2 \tag{5}$$

Finally, C_{D_0} is also obtained by integration:

$$\begin{aligned}
c_{d_0}(\theta) &= C_{D_0}^{2D}(\alpha(\theta) - \alpha_i(\theta), Re(\theta)) \\
C_{D_0}(\theta) &= \frac{b}{2S} \int_0^\pi c(\theta) C_{D_0}^{2D}(\alpha(\theta) - \alpha_i(\theta), Re(\theta)) \sin(\theta) d\theta \\
C_{D_0}(\theta) &= \frac{b}{2S} \int_0^\pi c(\theta) C_{D_0}^{2D} \left(\alpha(\theta) - \frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta) \right) \sin(\theta) d\theta
\end{aligned}$$

If $C_{D_0}^{2D}$ has an analytic expression, this can be simplified further. In the our case however, it is a general function fitted from wind tunnel data, therefore the integration is performed numerically with Clenshaw–Curtis quadrature.

Calling $w_\theta^{(k)}$ and θ_k respectively the weights and the points of this quadrature, we get:

$$C_{D_0}(\theta) = \frac{b}{2S} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) C_{D_0}^{2D} \left(\alpha(\theta_k) - \frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right) \quad (6)$$

Finally:

$$\begin{aligned}
\frac{L}{D} &= \frac{C_L}{C_{D_i} + C_{D_0}} \\
\frac{L}{D} &= \frac{\frac{b\pi A_1}{4S}}{\frac{\pi}{16S} \sum_{n=1}^{+\infty} n A_n^2 + \frac{b}{2S} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) C_{D_0}^{2D} \left(\alpha(\theta_k) - \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right)} \\
\frac{L}{D} &= \left(\frac{1}{4bA_1} \sum_{n=1}^{+\infty} n A_n^2 + \frac{2}{\pi A_1} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k) C_{D_0}^{2D} \left(\alpha(\theta_k) - \frac{1}{4b} \sum_{n=1}^{+\infty} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right) \right)^{-1} \quad (7)
\end{aligned}$$

5 Optimization Problem

We start with:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && (C_L / (C_{D_i} + C_{D_0}))^{-1} \\
& \text{subject to} && C_L = \frac{W}{1/2\rho V^2 S} \\
& && c_l(y) = C_L^{2D} (\alpha(y) + \alpha_i(y), Re(y)) \\
& && c_{d_0}(y) = C_{D_0}^{2D} (\alpha(y) - \alpha_i(y), Re(y)) \\
& && C_L = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_l(y') dy' \\
& && C_{D_0} = \frac{1}{S} \int_{-b/2}^{b/2} c(y') c_{d_0}(y') dy' \\
& && C_{D_i} = \frac{C_L}{\pi R \epsilon} \\
& && R = \frac{b^2}{S} \\
& && \epsilon = \left(\sum_{n=1}^{N_A} n \left(\frac{A_n}{A_1} \right)^2 \right)^{-1} \\
& && \alpha_i(\theta) = -\frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta)}{\sin(\theta)} \\
& \forall \theta \in [0, \pi] && \sum_{n=1}^{N_A} A_n \sin(\theta) = \frac{1}{c(\theta)} C_L^{2D} (\alpha(\theta) - \alpha_i(\theta), Re(\theta))
\end{aligned}$$

Discretizing, and using equations 3, 5, 6, 7, we get:

$$\begin{aligned}
& \underset{V, b, c, \alpha, A}{\text{minimize}} && \frac{1}{4bA_1} \sum_{n=1}^{N_A} n A_n^2 + \\
& && \frac{2}{\pi A_1} \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left(\alpha_k - \frac{1}{4b} \sum_{n=1}^{N_A} n A_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k) \right) \quad (8)
\end{aligned}$$

$$\begin{aligned}
& \text{subject to} && S A = c C_L^{2D} \left(\alpha - \frac{1}{4b} S' A, Re(c, V) \right) \quad (9)
\end{aligned}$$

$$A_1 = \frac{2W}{\rho V^2 b \pi} \quad (10)$$

Using 10 to simplify 8 further, we obtain:

$$\begin{aligned} \underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8} \sum_{n=1}^{N_A} nA_n^2 + \\ & V^2b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left(\alpha_k - \frac{1}{4b} \sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, \text{Re}(\theta_k) \right) \end{aligned} \quad (11)$$

$$\text{subject to} \quad SA = C_L^{2D} \left(\alpha_k - \frac{1}{4b} S' A_n, \text{Re}(\theta_k) \right) \quad (12)$$

$$A_1 = \frac{8W}{\rho V^2 b \pi} \quad (13)$$

because:

$$\begin{aligned} \frac{1}{4bA_1} &= \frac{\rho V^2 \pi}{8W} \\ \frac{\pi}{2A_1} &= \frac{\rho V^2 b}{W} \end{aligned}$$

Finally, we incorporate 13 into the objective function and the constraint.

$$\begin{aligned} \underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{W}{\rho b} + \frac{V^2\pi}{8} \sum_{n=2}^{N_A} nA_n^2 + V^2b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D} \left(\alpha_k - \frac{1}{4b} S'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=1}^{N_A} S'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \\ \text{subject to} \quad & S_{k,1} \frac{8W}{\rho V^2 b \pi} + \sum_{n=2}^{N_A} S_{k,n} A_n = \\ & c_k C_L^{2D} \left(\alpha_k - \frac{1}{4b} S'_{k,1} \frac{8W}{\rho V^2 b \pi} - \frac{1}{4b} \sum_{n=2}^{N_A} S'_{k,n} A_n, \text{Re}(c_k, v_k) \right) \quad \text{for } k \in 1, \dots, N_\theta \end{aligned}$$