Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

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1 Nomenclature

W	:	glider weight
L, C_L	:	lift, lift coefficient
$D, Di, D_0, C_D, C_{Di}, C_{D_0}$	:	drag, induced drag, friction drag, and respective coefficients
b	:	wing span
S	:	wing area
Æ	:	wing aspect ratio
y	:	spanwise coordinate, ranging from $-b/2$ to $b/2$
θ	:	remapped spanwise coordinate ranging from 0 to π
N_{theta}	:	Number of spanwise sections
$\Gamma(y)$ or $\Gamma(\theta)$	:	spanwise strength of vortex sheet
A_n	:	nth coefficient of the Fourrier expansion of Γ
N_A	:	Number of Fourrier coefficients considered for Γ
S	:	matrix of size N_θ, N_A such that: $\forall k, n \quad S_{k,n} = \sin(n\theta_k)$
S'	:	matrix of size N_θ, N_A such that: $\forall k, n \quad S'_{k,n} = n\sin(n\theta_k)/\sin(\theta_k)$
A	:	vector of size N_A containing all the A_n
$\alpha(y)$ or $\alpha(\theta)$	:	spanwise angle of attack distribution
$\alpha_i(y)$ or $\alpha_i(\theta)$	:	spanwise lift-induced angle of attack distribution
$c(y)$ or $c(\theta)$	:	spanwise chord distribution
$c_l(y)$ or $c_l(\theta)$	:	spanwise lift coefficient distribution
$c_d(y)$ or $c_d(\theta)$	:	spanwise drag coefficient distribution
V	:	airspeed
C_L^{2D} and $C_{D_0}^{2D}$	:	fit of 2D airfoil data
Re	:	Reynolds number
$y_k, \theta_k, \alpha_k, \alpha_{i_k}, c_k, c_{l_k}$	:	$y, \theta, \alpha, \alpha_i, c, c_l$ at the kth span section

2 Optimization Problem

This code aims at optimizing the wing planform and airspeed of a glider of a given weight W and fixed airfoil section for maximum L/D. 2D wind tunnel data of the airfoil is available, *i.e.* the functions $C_L^{2D}(\alpha, Re)$ and $C_{D_0}^{2D}(\alpha, Re)$ are provided.

The optimization problem reads:

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \left(C_L/(C_{D_i}+C_{D_0})\right)^{-1} \\
\text{subject to} \quad & C_L = \frac{W}{1/2\rho V^2 S} \\
& c_l(y) = C_L^{2D}\left(\alpha(y)+\alpha_i(y), Re(y)\right) \\
& c_{d_0}(y) = C_{D_0}^{2D}(\alpha(y)-\alpha_i(y), Re(y)) \\
& C_L = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_l(y')dy' \\
& C_{D_0} = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_{d_0}(y')dy' \\
& C_{D_i} = \frac{C_L}{\pi \mathit{AR}\epsilon} \\
& \mathit{AR} = \frac{b^2}{S} \\
& \epsilon = \left(\sum_{n=1}^{N_A} n\left(\frac{A_n}{A_1}\right)^2\right)^{-1} \\
& \alpha_i(\theta) = -\frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta)}{\sin(\theta)} \\
\forall\theta\in[0,\pi] \quad & \sum_{n=1}^{N_A} A_n\sin(\theta) = \frac{1}{c(\theta)}C_L^{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right)
\end{aligned}
$$

where θ is a span parametrization such that:

$$\forall\theta\in[0,\pi], \forall y\in[\frac{-b}{2},\frac{b}{2}], \quad y=\frac{b}{2}\cos\theta$$

We discretize the span into N_{theta} spanwise sections, and perform several simplifications detailed in the rest of this document. The final optimization problem is:

$$\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D}\left(\alpha_k + \alpha_{i_k}, Re(c_k, v_k)\right)\\
\text{subject to} \quad & \sum_{n=1}^{N_A} S_{k,n}A_n = c_k C_L^{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} S'_{k,n}A_n, Re(c_k, v_k)\right) \quad \text{for } k \in 1,\dots,N_\theta\\
& A_1 = \frac{8W}{\rho V^2 b\pi}
\end{aligned}$$

Or, in an even more compact form by plugging in the last constraint for A_1 in both other equations:

$$\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{W}{\rho b} + \frac{V^2\pi}{8}\sum_{n=2}^{N_A} nA_n^2 + V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D}\left(\alpha_k - \frac{1}{4b}S'_{k,i}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=1}^{N_A} S'_{k,n}A_n, Re(c_k, v_k)\right)\\
\text{subject to} \quad & S_{k,1}\frac{8W}{\rho V^2 b\pi} + \sum_{n=2}^{N_A} S_{k,n}A_n =\\
& c_k C_L^{2D}\left(\alpha_k - \frac{1}{4b}S'_{k,i}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=1}^{N_A} S'_{k,n}A_n, Re(c_k, v_k)\right) \quad \text{for } k \in 1,\dots,N_\theta
\end{aligned}$$

3 Lifting Line Theory

From Kutta-Joukovsky theorem:

$$\Gamma(y) = \frac{l(y)}{\rho V} = \frac{1/2\rho V^2 c(y) c_l(y)}{\rho V} = \frac{1}{2}V c(y) c_l(y)$$

The strength of the vortex sheet Γ is defined on a closed interval $[-b/2, b/2]$, which we remap to $[0, \pi]$. On this interval, a Fourrier series expansion can be written:

$$y = \frac{b}{2}\cos(\theta)$$

$$\Gamma(\theta) = \frac{1}{2}V\sum_{n=1}^{+\infty} A_n \sin(n\theta),\ c_l(\theta) = \frac{1}{c(y)}\sum_{n=1}^{+\infty} A_n \sin(n\theta)$$

Using Biot and Savart law, the induced angle of attack on at each spanwise coordinate can be written as:

$$\alpha_i(y) = \frac{-1}{4\pi V}\int_{-b/2}^{b/2} \frac{\frac{d\Gamma(t)}{dt}}{y-t}dt$$

$$\begin{aligned}\alpha_i(\theta) &= \frac{-1}{4\pi V}\int_{\pi}^{0} \frac{\left(\frac{d\Gamma(\Theta)}{d\Theta}\frac{d\Theta}{dt}\right)}{b/2\cos(\theta) - b/2\cos(\Theta)}\frac{dt}{d\Theta}d\Theta \\ &= -\frac{-1}{2b\pi V}\int_{0}^{\pi} \frac{\frac{d\Gamma(\Theta)}{d\Theta}}{\cos(\theta)-\cos(\Theta)}d\Theta\end{aligned}$$

$$\frac{d\Gamma(\Theta)}{d\Theta} = \frac{d}{d\Theta}\frac{1}{2}V\sum_{n=1}^{+\infty} A_n\sin(n\Theta) = \frac{1}{2}V\sum_{n=1}^{+\infty} nA_n\cos(n\Theta)$$

$$\begin{aligned}\alpha_i(\theta) &= -\frac{-1}{2b\pi}\int_{0}^{\pi} \frac{\frac{1}{2}V\sum_{n=1}^{+\infty} nA_n\cos(n\Theta)}{\cos(\theta)-\cos(\Theta)}d\Theta \\ &= -\frac{-1}{4b\pi}\sum_{n=1}^{+\infty} nA_n\int_{0}^{\pi} \frac{\cos(n\Theta)}{\cos(\theta)-\cos(\Theta)}d\Theta\end{aligned}$$

Directly plugging in the solution of the Glauert integral:

$$\alpha_i(\theta) = \frac{-1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta)}{\sin(\theta)}$$

The value of the lift coefficient on each section is given by the fit of the 2D airfoil data:

$$c_l(\theta) = C_L^{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right)$$

Plugging in the value of c_l in terms of the Fourrier coefficients of Γ, we obtain the general equation that needs to be solved to find the $A_{1:+\infty}$ is:

$$\sum_{n=1}^{+\infty} A_n\sin(n\theta) = c(\theta)C_L^{2D}\left(\alpha(\theta) - \frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta)\right)$$

If we only compute A_n for $n \in [0, N_A]$, we can choose a discrete set of θ_k for $k \in [0, N_\theta]$:

$$SA = c(\theta)C_L^{2D}\left(\alpha(\theta) - \frac{1}{4b}S'A, Re(\theta)\right) \tag{1}$$

with:

S a matrix of size N_θ, N_A such that: $\forall m,n \quad S_{m,n} = \sin(n\theta_m)$

S' a matrix of size N_θ, N_A such that: $\forall m,n \quad S'_{m,n} = n\sin(n\theta_m)/\sin(\theta_m)$

A a vector of size N_A containing all the A_n

4 L over D

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{Di} + C_{D0}}$$

Let us start with C_L:

$$\begin{aligned} C_L &= \frac{1}{S}\int_{b/2}^{b/2} c(y)c_l(y)dy \\ &= \frac{b}{2S}\int_0^{\pi} \sum_{n=1}^{+N_A} A_n \sin(n\theta)\sin(\theta)d\theta \\ &= \frac{b}{2S}\sum_{n=1}^{+N_A} A_n \int_0^{\pi} \sin(n\theta)\sin(\theta)d\theta \end{aligned} \tag{2}$$

Since the Fourrier basis is orthogonal, we have for any k, n:

$$\int_0^{\pi} \sin(n\theta)\sin(k\theta)d\theta = \begin{cases} 0 & k \neq n \\ \frac{\pi}{2} & k = n \end{cases}$$

Therefore:

$$C_L = \frac{b\pi A_1}{4S} \tag{3}$$

Now, to get C_{Di}, we look at it's spanwise distribution c_{d_i}:

$$\begin{aligned} c_{d_i}(\theta) &= -c_l(\theta)\alpha_i(\theta) \\ &= \frac{1}{4bc(\theta)}\sum_{n_a=0}^{+\infty} n_a A_{n_a}\frac{\sin(n_a\theta)}{\sin(\theta)}\sum_{n_b=0}^{+\infty} A_{n_b}\sin(n_b\theta) \end{aligned}$$

Integrating to get C_{Di}:

$$\begin{aligned} C_{Di} &= \frac{b}{2S}\int_0^{\pi} c_{d_i}(\theta)c(\theta)d\theta \\ C_{Di} &= \frac{b}{2S}\int_0^{\pi} \frac{1}{4b}\sum_{n_a=0}^{N_A} n_a A_{n_a}\frac{\sin(n_a\theta)}{\sin(\theta)}\sum_{n_b=0}^{N_A} A_{n_b}\sin(n_b\theta)\sin(\theta)d\theta \\ C_{Di} &= \frac{1}{8S}\int_0^{\pi}\sum_{n_a=0}^{N_A}\sum_{n_b=0}^{N_A} n_a A_{n_a}A_{n_b}\sin(n_a\theta)\sin(n_b\theta)d\theta \end{aligned} \tag{4}$$

Using again that the Fourrier basis functions are orthogonal, we obtain:

$$C_{Di} = \frac{\pi}{16S}\sum_{n=1}^{N_A} nA_n^2 \tag{5}$$

Finally, C_{D_0} is also obtained by integration:

$$\begin{aligned}
c_{d_0}(\theta) &= C_{D_0}^{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right) \\
C_{D_0}(\theta) &= \frac{b}{2S}\int_0^{\pi} c(\theta)C_{D_0}^{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right)\sin(\theta)d\theta \\
C_{D_0}(\theta) &= \frac{b}{2S}\int_0^{\pi} c(\theta)C_{D_0}^{2D}\left(\alpha(\theta)-\frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta)}{\sin(\theta)}, Re(\theta)\right)\sin(\theta)d\theta
\end{aligned}$$

If $C_{D_0}^{2D}$ has an analytic expression, this can be simplified further. In the our case however, it is a general function fitted from wind tunnel data, therefore the integration is performed numerically with Clenshaw–Curtis quadrature.

Calling $w_\theta^{(k)}$ and θ_k respectively the weights and the points of this quadrature, we get:

$$C_{D_0}(\theta) = \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k)C_{D_0}^{2D}\left(\alpha(\theta_k)-\frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) \tag{6}$$

Finally:

$$\begin{aligned}
\frac{L}{D} &= \frac{C_L}{C_{Di}+C_{D0}} \\
\frac{L}{D} &= \frac{\frac{b\pi A_1}{4S}}{\frac{\pi}{16S}\sum_{n=1}^{+\infty} nA_n^2 + \frac{b}{2S}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k)C_{D_0}^{2D}\left(\alpha(\theta_k)-\frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)} \\
\frac{L}{D} &= \left(\frac{1}{4bA_1}\sum_{n=1}^{+\infty} nA_n^2 + \frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c(\theta_k)C_{D_0}^{2D}\left(\alpha(\theta_k)-\frac{1}{4b}\sum_{n=1}^{+\infty} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right)\right)^{-1}
\end{aligned} \tag{7}$$

5 Optimization Problem

We start with:

$$
\begin{aligned}
&\underset{V,b,c,\alpha,A}{\text{minimize}} && (C_L/(C_{D_i}+C_{D_0}))^{-1} \\
&\text{subject to} && C_L = \frac{W}{1/2\rho V^2 S} \\
& && c_l(y) = C_L^{2D}\left(\alpha(y)+\alpha_i(y), Re(y)\right) \\
& && c_{d_0}(y) = C_{D_0}^{2D}(\alpha(y)-\alpha_i(y), Re(y)) \\
& && C_L = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_l(y')dy' \\
& && C_{D_0} = \frac{1}{S}\int_{-b/2}^{b/2} c(y')c_{d_0}(y')dy' \\
& && C_{D_i} = \frac{C_L}{\pi Æ\epsilon} \\
& && Æ = \frac{b^2}{S} \\
& && \epsilon = \left(\sum_{n=1}^{N_A} n\left(\frac{A_n}{A_1}\right)^2\right)^{-1} \\
& && \alpha_i(\theta) = -\frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta)}{\sin(\theta)} \\
&\forall\theta\in[0,\pi] && \sum_{n=1}^{N_A} A_n\sin(\theta) = \frac{1}{c(\theta)}C_L^{2D}\left(\alpha(\theta)+\alpha_i(\theta), Re(\theta)\right)
\end{aligned}
$$

Discretizing, and using equations 3, 5, 6, 7, we get:

$$
\begin{aligned}
&\underset{V,b,c,\alpha,A}{\text{minimize}} && \frac{1}{4bA_1}\sum_{n=1}^{N_A} nA_n^2 + \\
& && \frac{2}{\pi A_1}\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n\frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) && (8) \\
&\text{subject to} && SA = C_L^{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (9) \\
& && A_1 = \frac{2W}{\rho V^2 b\pi} && (10)
\end{aligned}
$$

Using 10 to simplify 8 further, we obtain:

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{V^2\pi}{8}\sum_{n=1}^{N_A} nA_n^2 + \\
& V^2 b \sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D}\left(\alpha_k - \frac{1}{4b}\sum_{n=1}^{N_A} nA_n \frac{\sin(n\theta_k)}{\sin(\theta_k)}, Re(\theta_k)\right) && (11)\\
\text{subject to} \quad & SA = C_L^{2D}\left(\alpha_k - \frac{1}{4b}S'A_n, Re(\theta_k)\right) && (12)\\
& A_1 = \frac{8W}{\rho V^2 b\pi} && (13)
\end{aligned}
$$

because:

$$
\begin{aligned}
\frac{1}{4bA_1} &= \frac{\rho V^2 \pi}{8W}\\
\frac{\pi}{2A_1} &= \frac{\rho V^2 b}{W}
\end{aligned}
$$

Finally, we incorporate 13 into the objective function and the constraint.

$$
\begin{aligned}
\underset{V,b,c,\alpha,A}{\text{minimize}} \quad & \frac{W}{\rho b} + \frac{V^2\pi}{8}\sum_{n=2}^{N_A} nA_n^2 + V^2 b\sum_{k=1}^{N_\theta} w_\theta^{(k)} c_k C_{D_0}^{2D}\left(\alpha_k - \frac{1}{4b}S'_{k,i}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=1}^{N_A} S'_{k,n}A_n, Re(c_k, v_k)\right)\\
\text{subject to} \quad & S_{k,1}\frac{8W}{\rho V^2 b\pi} + \sum_{n=2}^{N_A} S_{k,n}A_n = \\
& c_k C_L^{2D}\left(\alpha_k - \frac{1}{4b}S'_{k,i}\frac{8W}{\rho V^2 b\pi} - \frac{1}{4b}\sum_{n=1}^{N_A} S'_{k,n}A_n, Re(c_k, v_k)\right) \quad \text{for } k \in 1, \ldots, N_\theta
\end{aligned}
$$