Glider Optimization with Lifting Line Theory and Airfoil Wind Tunnel Data

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1 Nomenclature

W	:	glider weight
L, C_L	:	lift, lift coefficient
$D, Di, D_0, C_D, C_{Di}, C_{D_0}$	:	drag, induced drag, friction drag, and respective coefficients
b	:	wing span
M	:	wing area
$Æ$	:	wing aspect ratio
y	:	span-wise coordinate, ranging from $-b/2$ to $b/2$
θ	:	remapped span-wise coordinate ranging from 0 to π
N_{theta}	:	Number of span-wise sections
$\Gamma(y)$ or $\Gamma(\theta)$	:	span-wise strength of vortex sheet
A_n	:	nth coefficient of the Fourier expansion of Γ
N_A	:	Number of Fourier coefficients considered for Γ
M	:	matrix of size N_θ, N_A such that: $\forall k, n \quad M_{k,n} = \sin(n\theta_k)$
M'	:	matrix of size N_θ, N_A such that: $\forall k, n \quad M'_{k,n} = n\sin(n\theta_k)/\sin(\theta_k)$
A	:	vector of size N_A containing all the A_n
$\alpha(y)$ or $\alpha(\theta)$	:	span-wise angle of attack distribution
$\alpha_i(y)$ or $\alpha_i(\theta)$	:	span-wise lift-induced angle of attack distribution
$c(y)$ or $c(\theta)$	:	span-wise chord distribution
$c_l(y)$ or $c_l(\theta)$	:	span-wise lift coefficient distribution
$c_d(y)$ or $c_d(\theta)$	:	span-wise drag coefficient distribution
V	:	airspeed
$\boldsymbol{c_l}^{\boldsymbol{2D}}$ and $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}$	:	fit of 2D airfoil data
Re	:	Reynolds number
$y_k, \theta_k, \alpha_k, \alpha_{i_k}, c_k, c_{l_k}$	:	$y, \theta, \alpha, \alpha_i, c, c_l$ at the kth span section

2 Introduction

This code aims at optimizing the wing planform and airspeed of a glider of a given weight W and fixed airfoil section for maximum L/D. 2D wind tunnel data of the airfoil is available, *i.e.* the function $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l, Re)$ are provided for $c_l \in [c_{l\mathbf{lb}}, c_{l\mathbf{ub}}]$ and $Re \in [Re_{\mathbf{lb}}, Re_{\mathbf{ub}}]$.

Schematically, the optimization problem reads:

$$
\begin{aligned}
\underset{V,b,c(y),c_l(y)}{\text{minimize}} \quad & C_D/C_L \\
\text{subject to} \quad & {C_L}^2 + {C_D}^2 = {C_W}^2 \quad - \quad \textit{gliding flight constraint} \\
& c_{l\mathbf{lb}} \leq c_l \leq c_{l\mathbf{ub}} \quad - \quad \textit{domain constraint for } \boldsymbol{c_{d_p}}^{\boldsymbol{2D}} \\
& Re_{\mathbf{lb}} \leq Re \leq Re_{\mathbf{ub}} \quad - \quad \textit{domain constraint for } \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}
\end{aligned}
$$

First, the optimization problem is described in more details. Second, some insights are obtained from considering simple functions for $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}$. Finally, using a neural network fit of 2d airfoil data, the complete problem is solved and results are analysed.

3 Optimization Problem

Loss Function: Using the gliding flight constraint, we have:

$$\frac{{C_L}^2}{{C_D}^2} = \frac{{C_W}^2 - {C_D}^2}{{C_D}^2} = \frac{{C_W}^2}{{C_D}^2} - 1$$

Therefore, minimizing C_D/C_W is equivalent to minimizing C_D/C_L. Moreover, C_D can be decomposed into lift-induced drag and parasite drag:

$$C_D = C_{D_p} + C_{D_i}$$

Remapping the spanwise coordinate: The spanwise coordinate $y \in [\frac{-b}{2}, \frac{b}{2}]$ can be remapped into $\theta \in [\pi, 0]$, such that:

$$y = \frac{b}{2}\cos\theta$$

Parasite Drag : Since we can compute the 2D parasite drag using $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}$, the total C_{D_p} is obtained with integration.

$$
\begin{aligned}
C_{D_p} &= \frac{1}{S}\int_{-b/2}^{b/2} c(y)\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l(y), Re(y))dy \\
&= \frac{b}{2S}\int_0^{\pi} c(\theta)\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l(\theta), Re(\theta))\sin(\theta)d\theta \\
&= \frac{b}{2S}\sum_{k=0}^{N_y} w_k c_k c_{d_p\mathbf{k}}
\end{aligned}
$$

where N_y is the number of spanwise panels, $\theta_{1,\dots,N_y}$ their location, and $w_{1,\dots,N_y}$ a set of quadrature weights that depend on the spacing and location of the panels. We also wrote:

$$c(\theta_k) = c_k \quad \text{and} \quad \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l(\theta_k), Re(\theta)k)) = c_{d_p\mathbf{k}}$$

In the rest of this document, we almost always describe the chord distribution with the non-dimensional elements c_k/b, that we write $\left(\frac{c}{b}\right)_k$. In the expression of C_{D_p}, this gives:

$$C_{D_p} = \frac{AR}{2} \sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k c_{d_p\mathbf{k}}$$

Induced Drag: We use a far field estimate of the induced drag. Assuming a planar wake, we use a sine basis expansion of the vortex sheet strength:

$$\Gamma(\theta) = 2bV \sum_{n=1}^{N_a} A_n sin(n\theta)$$

N_a is the order of this expansion. This gives us:

$$C_{D_i} = \pi AR \sum_{n=1}^{N_a} nA_n^2$$

Weight Coefficient: Factorizing by AR, we write the weight coefficient as:

$$C_W = \frac{W}{1/2\rho V^2 S} = \frac{W}{1/2\rho} AR (Vb)^{-2}$$

In this study, ρ is assumed to be constant, and will be dropped from the objective function below.

Objective Function: Putting together all of the above, we obtain the following objective function:

$$\frac{C_D}{C_W} = (Vb)^2 \left(\pi \sum_{n=1}^{N_a} nA_n^2 + \frac{1}{2} \sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k c_{d_p\mathbf{k}}\right)$$

Approximate Handling of the Equality Constraint: We simplify the problem by assuming that the glide angle γ is small. In such case,

$$C_W = \frac{C_L}{\cos\gamma} \approx C_L = \pi AR A_1 \tag{1}$$

The validity of the small angle assumption is discussed in appendix A. Instead of solving the original nonlinear constrained problem, we insert the value of A_1 into the objective function.

$\pi \mathcal{R} A_1^2$ appears in the expression of C_{D_i}, so we get:

$$\begin{aligned} C_{D_i} &= \pi \mathcal{R} A_1^2 + \pi \mathcal{R} \sum_{n=2}^{N_a} n A_n^2 \\ &= \frac{\mathcal{R}}{\pi} \left(\frac{C_W}{\mathcal{R}} \right)^2 + \pi \mathcal{R} \sum_{n=2}^{N_a} n A_n^2 \\ &= \frac{4W^2}{\rho^2 \pi} \mathcal{R} (Vb)^{-4} + \pi \mathcal{R} \sum_{n=2}^{N_a} n A_n^2 \end{aligned}$$

Inequality Constraints: We can write the Reynold's number and the lift coefficient $c_{l\mathbf{k}}$ at the local section k as a function of other variables used previously:

$$c_{l\mathbf{k}} = 4 \left(\frac{c}{b} \right)_k^{-1} \sum_{n=1}^{N_a} A_n \sin(n\theta_k) = 4 \left(\frac{c}{b} \right)_k^{-1} \left(A_1 \sin(\theta_k) + \sum_{n=2}^{N_a} A_n \sin(n\theta_k) \right)$$

Therefore:

$$c_{l\mathbf{k}} = 4 \left(\frac{c}{b} \right)_k^{-1} \left(\frac{W}{1/2\rho\pi} (Vb)^{-2} \sin(\theta_k) + \sum_{n=2}^{N_a} A_n \sin(n\theta_k) \right)$$

Also:

$$Re_k = \frac{\rho}{\nu} \left(\frac{c}{b} \right)_k \sqrt{(Vb)^2}$$

The collected optimization problem is written and analyzed in the next section.

4 Optimization Problem Insights and Discussion

Putting together all of the last section we have:

$$\begin{aligned} \underset{(Vb)^2, \left(\frac{c}{b}\right)_{1:N_y}, A_{2:N_a}}{\text{minimize}} \quad & \frac{4W^2}{\rho^2 \pi} (Vb)^{-2} + (Vb)^2 \left(\pi \sum_{n=2}^{N_a} n A_n^2 + \frac{1}{2} \sum_{k=0}^{N_y} w_k \left(\frac{c}{b} \right)_k \boldsymbol{c_{d_p}}^{\boldsymbol{2D}} (c_{l\mathbf{k}}, Re_k) \right) \\ \text{subject to} \quad & c_{l\mathbf{lb}} \leq c_l \leq c_{l\mathbf{ub}} \\ & Re_{\mathbf{lb}} \leq Re \leq Re_{\mathbf{ub}} \end{aligned}$$

with:

$$\begin{aligned} c_{l\mathbf{k}} &= 4 \left(\frac{c}{b} \right)_k^{-1} \left(\frac{W}{1/2\rho\pi} (Vb)^{-2} \sin(\theta_k) + \sum_{n=2}^{N_a} A_n \sin(n\theta_k) \right) \\ Re_k &= \frac{\rho}{\nu} \left(\frac{c}{b} \right)_k \sqrt{(Vb)^2} \end{aligned}$$

Design Variables: The first interesting note is that the optimization of the planform and airspeed only requires:

- $(Vb)^2$ which is the only place where the airspeed appears. Note that we could equivalently use $C_W/\text{Æ}$, which makes it obvious that this design variable is tightly connected to the nominal C_L.
- $\left(\frac{c}{b}\right)_{1:N_y}$ is simply related to the planform shape
- $A_{1:N_a}$ describes the lift distribution. Note that A_1 is not a design variable, it is directly set by the value of $C_W/\text{Æ}$ (see equation 1).

Once the optimization has converged, choosing either a span value b, an airspeed V, or the chord c_k of any of the chord sections allows to recover the full dimensional planform shape.

Origin of each term: In the objective function, we can connect all of the terms with their physical origin:

$$\underbrace{\frac{4W^2}{\rho^2\pi}(Vb)^{-2}}_{(1)} + \underbrace{(Vb)^2}_{\text{division by } C_W} \left(\underbrace{\pi \sum_{n=2}^{N_a} n A_n^2}_{(2)} + \underbrace{\frac{1}{2} \sum_{k=0}^{N_y} w_k \left(\frac{c}{b}\right)_k \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_{l\mathbf{k}}, Re_k)}_{\text{parasite drag}} \right)$$

(1) and (2) are the two parts of the induced drag, divided by C_W:

- (1) comes from the induced drag of an elliptically loaded wing. In essence, it is proportional to $C_L/(\pi \text{Æ})$.
- (2) is the sum of all additional terms due to the non elliptical loading.

The rest of this section develops some insights for this problem.

5 Insights For Individual Design Variable

5.1 $(Vb)^2$

Fixing all other design variables, the optimization problem from $(Vb)^2$ looks like the following:

$$\begin{aligned} \underset{(Vb)^2}{\text{minimize}} \quad & \frac{k_1}{(Vb)^2} + (Vb)^2 k_2 \\ \text{subject to} \quad & lb_1 \leq 1/(Vb)^2 + k_3 \leq ub_1 \\ & lb_2 \leq \sqrt{(Vb)^2} \leq ub_2 \end{aligned}$$

where k_1, k_2 and k_3 are constant positive numbers. The landscape for this function is shown on figure 1

Figure 1: Optimization landscape for $(Vb)^2$

5.2 $\left(\frac{c}{b}\right)_{1:N_y}$

Fixing all other design variables, the optimization problem from $\left(\frac{c}{b}\right)_{1:N_y}$ looks like the following:

$$\begin{array}{ll} \underset{\left(\frac{c}{b}\right)_{1:N_y}}{\text{minimize}} & \displaystyle\sum_{k=1}^{N_y} k_i \left(\frac{c}{b}\right)_k \boldsymbol{c_{d_p}}^{\boldsymbol{2D}} \left(\frac{c}{b}\right)_k \\ \text{subject to} & lb \leq \left(\frac{c}{b}\right)_k \leq ub \end{array}$$

where $k_{0:N_y}$ are constant positive numbers, and $(lb,\, ub)$ are bounds derived from the c_l and Re bounds.

Very clearly, unless $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}$ varies with $\left(\frac{c}{b}\right)_k^{-1}$ to a power bigger than one, the optimizer will always set each $\left(\frac{c}{b}\right)_k$ to its lowest bound (from the Re or from c_l, whichever is tightest). Interestingly, if the airfoil is a flat plate, then:

$$\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(Re) \approx 2\frac{0.074}{Re^{0.2}}$$

With only a power 0.2 on $\left(\frac{c}{b}\right)_k^{-1}$, this is not a big enough penalty on small chords to prevent the maximum L/D to happen at the lowest possible chord.

5.3 $A_{1:N_a}$

Fixing all other design variables, the optimization problem from $A_{2:N_a}$ looks like the following:

$$\begin{array}{ll} \underset{A_{2:N_a}}{\text{minimize}} & \displaystyle\sum_{n=2}^{N_a} nA_n^2 + \sum_{k=0}^{N_y} k_i \boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(\sum_{n=2}^{N_a} A_n sin(n\theta_k)) \\ \text{subject to} & \displaystyle lb_k \leq \sum_{n=2}^{N_a} A_n sin(n\theta_k) \leq ub_k \end{array}$$

where $(lb_k \leq 0)$ and $(ub_k \geq 0)$ are bounds derived from the c_l bounds. Pretty clearly, the main incentive is to set all of the

6 Confirming the Insight on Simplified 2D Sections

6.1 Constant $\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}$

Let's assume:

$$\boldsymbol{c_{d_p}}^{\boldsymbol{2D}}(c_l, Re) = c_{d_p} \quad - \quad \textit{(constant)}$$

Then the optimization problem becomes:

$$
\begin{aligned}
\underset{(Vb)^2, \left(\frac{c}{b}\right)_{1:N_y}, A_{2:N_a}}{\text{minimize}} \quad & \frac{4W^2}{\rho^2\pi}(Vb)^{-2}+(Vb)^2\left(\pi\sum_{n=2}^{N_a} nA_n^2 + \frac{\text{Æ}}{2}\right) \\
\text{subject to} \quad & c_{l\mathbf{lb}} \le c_{l\le} c_{l\mathbf{ub}} \\
& Re_{\mathbf{lb}} \le Re \le Re_{\mathbf{ub}}
\end{aligned}
$$

with:

$$
\begin{aligned}
c_{l\mathbf{k}} &= 4\left(\frac{c}{b}\right)_k^{-1}\sum_{n=1}^{N_y} A_n\sin(n\theta_k) \\
&= 4\left(\frac{c}{b}\right)_k^{-1}\left(\sum_{n=1}^{N_y} A_n\sin(n\theta_k)\sum_{n=2}^{N_y} A_n\sin(n\theta_k)\right) \\
Re_k &= \frac{\rho}{\nu}\left(\frac{c}{b}\right)_k\sqrt{(Vb)^2}
\end{aligned}
$$

6.2 Flat Plate Model

6.3 Flat Plate with Exponential C_L Penalty

A Validity of the small angle assumptions

TBD