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A CORRELATED RANDOM WALK MODEL FOR TWO-DIMENSIONAL DIFFUSION

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Abstract

A two-dimensional lattice random walk is studied in which, at each stage, the direction of the next step is correlated to that of the previous step. An exact expression is obtained from the characteristic function for the dispersion matrix of the n-step transition probabilities, and the limiting diffusion equation is derived.

LATTICE; DISPERSION; CHARACTERISTIC FUNCTION; SPECTRAL DECOMPOSITION

1. Introduction

An argument based on the limiting behaviour of the simple random walk as the length of individual steps decreases whilst the number of steps increases is often used to derive the standard diffusion equation for one-dimensional Brownian motion (see, for example, Feller (1968), pp. 354–359). This equation can also be obtained by a similar argument based on a correlated random walk in which the *direction changes* between steps are independent random variables, rather than the actual *step directions* (Cane (1975), Renshaw and Henderson (1981)). In this paper we extend the correlated random walk approach to two dimensions and obtain the standard two-dimensional diffusion equation

$$A\frac{\partial p}{\partial t}(x, y; t) = \frac{\partial^2 p}{\partial x^2}(x, y; t) + \frac{\partial^2 p}{\partial y^2}(x, y; t) \tag{1.1}$$

where A is an appropriate constant and $p(x, y; t)$ denotes the probability density function at position (x, y) at time $t \geqq 0$.

Although the correlated random walk approach is more complicated than that

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of a simple random walk, it has the distinct advantage of providing a more intuitively plausible description of the motion of particles with non-zero mass. For, owing to its inertia, a particle moving in a certain direction is more likely to continue in that direction in the following time interval than to reverse direction (Klein (1952)). This can be accounted for in a correlated, but not a simple, random walk.

Consider a particle, initially situated at the origin of an infinite square lattice, which moves in unit steps parallel to the X- and Y-axes. The first step is equally likely to be in the positive or negative X- or Y-directions, after which the particle moves at each step one unit either forwards, backwards, to the right or to the left of its previous step direction with probabilities f, b, r and l, respectively, where $f+b+r+l=1$. Here a step to the right implies a 90° clockwise change of direction, whilst a step to the left implies a 90° anti-clockwise change of direction. Thus the direction of each step depends only on the direction of the previous step, the sequence of step directions so forming a Markov chain. Note that if $f=b=r=l=\frac{1}{4}$ then the process reduces to a two-dimensional simple random walk.

Our interest in this process originated from a study of the rooting patterns of the United Kingdom's main forest tree species, Sitka spruce. Roots were found to grow in straight-line segments with random direction *changes* between segments (Henderson and Renshaw (1980)). This type of behaviour could not be modelled by a random walk with independent steps and a form of continuous-space correlated random walk was appropriate, and has been considered by Henderson (1981). Moreover, correlated random walks have many other applications, for example to polymer chemistry (Flory (1962)) and animal diffusion (Skellam (1973)). Further applications are discussed by Henderson (1981) who also gives references to previous investigations.

Several properties of the lattice correlated random walk are studied in this paper before the diffusion equation (1.1) is derived (Section 6). First, an expression is obtained for the characteristic function of the walk (Section 2) and this is used to derive the mean and dispersion matrix of the position of the particle after n steps (Section 3). The recurrence of the walk is then briefly discussed (Section 4) and an asymptotic expression for the expected number of distinct lattice points visited is determined (Section 5). As the algebra involved in deriving some of these results is straightforward, yet tedious, much has been omitted with only the main results being presented.

2. The characteristic function

Consider a single particle performing a correlated random walk, as described above, on the infinite square lattice $\{(s_1, s_2): -\infty < s_1, s_2 < \infty\}$. Denote the characteristic function $\Phi_n(\omega_1, \omega_2)$ for real ω_1, ω_2 by

$$\Phi_n(\omega_1, \omega_2) \equiv \sum_{s_1=-\infty}^{\infty} \sum_{s_2=-\infty}^{\infty} p_n(s_1, s_2) \exp\{is_1\omega_1 + is_2\omega_2\}, \tag{2.1}$$

where $p_n(s_1, s_2)$ is the probability of the particle being at the lattice point (s_1, s_2) after n steps.

As the correlation between step directions prevents us from directly writing down a difference equation for the $p_n(s_1, s_2)$, let us define the particle to be in state E_1, E_2, E_3 or E_4 depending on whether the previous step was in the positive X-, negative X-, positive Y- or negative Y-direction, respectively. These four states are mutually exclusive and exhaustive, so if

$$p_{nj}(s_1, s_2) \equiv \Pr(\text{particle is at position } (s_1, s_2) \text{ and in state } E_j \text{ after } n \text{ steps}),$$

then

$$p_n(s_1, s_2) = \sum_{j=1}^{4} p_{nj}(s_1, s_2). \tag{2.2}$$

By considering the number of lattice points from which the particle may arrive at (s_1, s_2) in one step only we can now write down the following difference equations for $p_{nj}(s_1, s_2)$ $(j = 1, 2, 3, 4)$:

$$\begin{aligned}
p_{n1}(s_1, s_2) &= fp_{n-1,1}(s_1 - 1, s_2) + bp_{n-1,2}(s_1 - 1, s_2) + rp_{n-1,3}(s_1 - 1, s_2) \\
&\quad + lp_{n-1,4}(s_1 - 1, s_2) \\
p_{n2}(s_1, s_2) &= bp_{n-1,1}(s_1 + 1, s_2) + fp_{n-1,2}(s_1 + 1, s_2) + lp_{n-1,3}(s_1 + 1, s_2) \\
&\quad + rp_{n-1,4}(s_1 + 1, s_2) \\
p_{n3}(s_1, s_2) &= lp_{n-1,1}(s_1, s_2 - 1) + rp_{n-1,2}(s_1, s_2 - 1) + fp_{n-1,3}(s_1, s_2 - 1) \\
&\quad + bp_{n-1,4}(s_1, s_2 - 1) \\
p_{n4}(s_1, s_2) &= rp_{n-1,1}(s_1, s_2 + 1) + lp_{n-1,2}(s_1, s_2 + 1) + bp_{n-1,3}(s_1, s_2 + 1) \\
&\quad + fp_{n-1,4}(s_1, s_2 + 1).
\end{aligned} \tag{2.3}$$

If we define the characteristic functions of the improper distributions $\{p_{nj}(s_1, s_2)\}$ for $j = 1, 2, 3, 4$ by

$$\phi_{nj}(\omega_1, \omega_2) \equiv \sum_{s_1=-\infty}^{\infty} \sum_{s_2=-\infty}^{\infty} p_{nj}(s_1, s_2) \exp\{is_1\omega_1 + is_2\omega_2\}, \tag{2.4}$$

then multiplication of each of Equations (2.3) by $\exp\{is_1\omega_1 + is_2\omega_2\}$ and summation over all s_1 and s_2 yields the system of equations

$$\begin{aligned}
\phi_{n1}(\omega_1, \omega_2) = \exp(i\omega_1)[&f\phi_{n-1,1}(\omega_1, \omega_2) + b\phi_{n-1,2}(\omega_1, \omega_2) + r\phi_{n-1,3}(\omega_1, \omega_2) \\
&+ l\phi_{n-1,4}(\omega_1, \omega_2)]
\end{aligned}$$

$$\phi_{n2}(\omega_1, \omega_2) = \exp(-i\omega_1)[b\phi_{n-1,1}(\omega_1, \omega_2) + f\phi_{n-1,2}(\omega_1, \omega_2) + l\phi_{n-1,3}(\omega_1, \omega_2) + r\phi_{n-1,4}(\omega_1, \omega_2)]$$

$$\phi_{n3}(\omega_1, \omega_2) = \exp(i\omega_2)[l\phi_{n-1,1}(\omega_1, \omega_2) + r\phi_{n-1,2}(\omega_1, \omega_2) + f\phi_{n-1,3}(\omega_1, \omega_2) + b\phi_{n-1,4}(\omega_1, \omega_2)]$$

$$\phi_{n4}(\omega_1, \omega_2) = \exp(-i\omega_2)[r\phi_{n-1,1}(\omega_1, \omega_2) + l\phi_{n-1,2}(\omega_1, \omega_2) + b\phi_{n-1,3}(\omega_1, \omega_2) + f\phi_{n-1,4}(\omega_1, \omega_2)]$$

which may be written in the vector-matrix form

$$\boldsymbol{\phi}_n(\omega_1, \omega_2) = \Lambda(\omega_1, \omega_2) Q \boldsymbol{\phi}_{n-1}(\omega_1, \omega_2). \tag{2.5}$$

Here

$$\boldsymbol{\phi}_n(\omega_1, \omega_2) \equiv (\phi_{n1}(\omega_1, \omega_2), \cdots, \phi_{n4}(\omega_1, \omega_2))^{\mathrm{T}},$$

$$\Lambda(\omega_1, \omega_2) = \begin{bmatrix} \exp(i\omega_1) & 0 & 0 & 0 \\ 0 & \exp(-i\omega_1) & 0 & 0 \\ 0 & 0 & \exp(i\omega_2) & 0 \\ 0 & 0 & 0 & \exp(-i\omega_2) \end{bmatrix},$$

and Q is the transition probability matrix for movement between the states E_j $(j = 1, 2, 3, 4)$, namely

$$Q \equiv \begin{bmatrix} f & b & r & l \\ b & f & l & r \\ l & r & f & b \\ r & l & b & f \end{bmatrix}.$$

As the first step is equally likely to be any one of the four directions possible we have

$$\boldsymbol{\phi}_0(\omega_1, \omega_2) = (1/4)(1, 1, 1, 1)^{\mathrm{T}} \equiv (1/4)\boldsymbol{h} \qquad \text{(say)}.$$

Thus solving (2.5) recursively gives

$$\boldsymbol{\phi}_n(\omega_1, \omega_2) = (1/4)(\Lambda(\omega_1, \omega_2)Q)^n \boldsymbol{h},$$

and as from (2.1), (2.2) and (2.4)

$$\Phi_n(\omega_1, \omega_2) = \boldsymbol{h}^{\mathrm{T}} \boldsymbol{\phi}_n(\omega_1, \omega_2)$$

we see that

$$\Phi_n(\omega_1, \omega_2) = (1/4)\boldsymbol{h}^{\mathrm{T}}(\Lambda(\omega_1, \omega_2)Q)^n \boldsymbol{h}. \tag{2.6}$$

This is the expression for the characteristic function which we shall use in the following section to obtain the mean and dispersion matrix of the position of the particle after n steps.

3. The mean and dispersion matrix

Denote the mean position of the particle after n steps by the vector $\boldsymbol{\mu}_n = (\mu_{n1}, \mu_{n2})^{\mathrm{T}}$, and the associated 2×2 dispersion matrix by $\Sigma_n = (\sigma^2_{njk})$ where $j, k = 1, 2$. From the definition of $\Phi_n(\omega_1, \omega_2)$ we have

$$\mu_{nj} = -i \frac{\partial}{\partial \omega_j} \Phi_n(\omega_1, \omega_2) \Big|_{\omega_1 = \omega_2 = 0} \qquad (j = 1, 2) \tag{3.1}$$

and

$$\sigma^2_{njk} = \frac{-\partial^2}{\partial \omega_j \partial \omega_k} \Phi_n(\omega_1, \omega_2) \Big|_{\omega_1 = \omega_2 = 0} - \mu_{nj}\mu_{nk} \qquad (j, k = 1, 2), \tag{3.2}$$

and so these moments can be obtained by differentiating (2.6). Many terms involved in the resulting derivatives can be simplified at $\omega_1 = \omega_2 = 0$ because $\Lambda(0, 0) = I_4$, the 4×4 identity matrix, and Q is doubly stochastic. This means that terms such as $(\Lambda(\omega_1, \omega_2)Q)^m \boldsymbol{h}$ and $\boldsymbol{h}^{\mathrm{T}}(\Lambda(\omega_1, \omega_2)Q)^m$ reduce to $\boldsymbol{h}$ and $\boldsymbol{h}^{\mathrm{T}}$, respectively, at $\omega_1 = \omega_2 = 0$ for any positive integer m.

For brevity we write $\partial \Phi_n(0, 0)/\partial \omega_j$ and $\partial^2 \Phi_n(0, 0)/\partial \omega_j \partial \omega_k$ for the first and second derivatives of $\Phi_n(\omega_1, \omega_2)$ evaluated at $\omega_1 = \omega_2 = 0$, respectively. Similarly, we write $\partial \Lambda(0, 0)/\partial \omega_j$ for the matrix of derivatives of $\Lambda(\omega_1, \omega_2)$ evaluated at $\omega_1 = \omega_2 = 0$. After performing the differentiations we finally have

$$\frac{\partial}{\partial \omega_j} \Phi_n(0, 0) = (n/4)\boldsymbol{h}^{\mathrm{T}} \left\{ \frac{\partial}{\partial \omega_j} \Lambda(0, 0) \right\} \boldsymbol{h} \qquad (j = 1, 2) \tag{3.3}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial \omega_j \partial \omega_k} \Phi_n(0, 0) = &-(n/2)\delta_{jk} + (1/4)\boldsymbol{h}^{\mathrm{T}} \left[\frac{\partial}{\partial \omega_j} \Lambda(0, 0) \left\{ \sum_{m=1}^{n-1} (n - m)Q^m \right\} \right. \\ &\left. \times \frac{\partial}{\partial \omega_k} \Lambda(0, 0) \right] \boldsymbol{h} \\ &+ (1/4)\boldsymbol{h}^{\mathrm{T}} \left[\frac{\partial}{\partial \omega_k} \Lambda(0, 0) \left\{ \sum_{m=1}^{n-1} (n - m)Q^m \right\} \frac{\partial}{\partial \omega_j} \Lambda(0, 0) \right] \boldsymbol{h}, \end{aligned} \tag{3.4}$$

where δ_{jk} is the Kronecker delta function.

It follows from the definition of $\Lambda(\omega_1, \omega_2)$ that

$$\boldsymbol{h}^{\mathrm{T}} \left\{ \frac{\partial}{\partial \omega_j} \Lambda(0, 0) \right\} \boldsymbol{h} = 0 \qquad (j = 1, 2),$$

whence from (3.1) and (3.3) we have

$$\mu_{nj} = 0 \qquad (j = 1, 2).$$

Thus the mean position of the particle after any number of steps is at the origin, as indeed one would anticipate from the symmetry of the initial conditions.

To derive the dispersion matrix we use the spectral decomposition of the matrix Q. This matrix has eigenvalues $\{\lambda_i\}$ given by $\lambda_1 = 1$, $\lambda_2 = f + b - r - l$, $\lambda_3 = f - b + i(r - l)$ and $\lambda_4 = f - b - i(r - l)$, and the spectral decomposition

$$Q = \sum_{d=1}^{4} \lambda_d A_d$$

where

$$A_1 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \frac{1}{4}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix},$$

$$A_3 = \frac{1}{4}\begin{bmatrix} 1 & -1 & -i & i \\ -1 & 1 & i & -i \\ i & -i & 1 & -1 \\ -i & i & -1 & 1 \end{bmatrix} \text{ and } A_4 = \frac{1}{4}\begin{bmatrix} 1 & -1 & i & -i \\ -1 & 1 & -i & i \\ -i & i & 1 & -1 \\ i & -i & -1 & 1 \end{bmatrix}.$$

If we use the property that

$$Q^m = \sum_{d=1}^{4} \lambda_d^m A_d$$

for any positive integer m, then (3.4) may be rearranged as

$$\frac{\partial^2}{\partial\omega_j \partial\omega_k}\Phi_n(0,0) = -(n/2)\delta_{jk} + (1/4)\sum_{d=1}^{4}\left[\left\{\sum_{m=1}^{n-1}\lambda_d^m(n-m)\right\}\left\{\boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_j}\Lambda(0,0)A_d\frac{\partial}{\partial\omega_k}(0,0)\boldsymbol{h} + \boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_k}\Lambda(0,0)A_d\frac{\partial}{\partial\omega_j}\Lambda(0,0)\boldsymbol{h}\right\}\right] \tag{3.5}$$

for $j, k = 1, 2$. Now by inspection we see that

$$\boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_j}\Lambda(0,0)A_d\frac{\partial}{\partial\omega_k}\Lambda(0,0)\boldsymbol{h} = 0 \qquad (j,k,d = 1,2),$$

$$\boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_j}\Lambda(0,0)A_d\frac{\partial}{\partial\omega_j}\Lambda(0,0)\boldsymbol{h} = -1 \qquad (j = 1,2;\ d = 3,4),$$

$$\boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_1}\Lambda(0,0)A_3\frac{\partial}{\partial\omega_2}\Lambda(0,0)\boldsymbol{h} = -\boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_2}\Lambda(0,0)A_3\frac{\partial}{\partial\omega_1}\Lambda(0,0)\boldsymbol{h} = i$$

and

$$\boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_1}\Lambda(0,0)A_4\frac{\partial}{\partial\omega_2}\Lambda(0,0)\boldsymbol{h} = -\boldsymbol{h}^{\mathrm{T}}\frac{\partial}{\partial\omega_2}\Lambda(0,0)A_4\frac{\partial}{\partial\omega_1}\Lambda(0,0)\boldsymbol{h} = -i.$$

Thus on using these results in (3.5) we obtain

$$\frac{\partial^2}{\partial \omega_j^2}\Phi_n(0,0) = -(n/2)-(1/2)\sum_{m=1}^{n-1}(n-m)(\lambda_3^m+\lambda_4^m) \qquad (j=1,2)$$

and

$$\frac{\partial^2}{\partial \omega_j \partial \omega_k}\Phi_n(0,0) = 0 \qquad (j \neq k).$$

On using (3.2) we therefore have

$$\begin{aligned}\Sigma_n = (1/2)[&n\{1+\lambda_3(1-\lambda_3)^{-1}+\lambda_4(1-\lambda_4)^{-1}\}-\lambda_3(1-\lambda_3^n)(1-\lambda_3)^{-2}\\ &-\lambda_4(1-\lambda_4^n)(1-\lambda_4)^{-2}]I_2,\end{aligned} \tag{3.6}$$

where I_2 is the 2×2 identity matrix. Thus the two coordinates of the particle's position are uncorrelated, and each has variance asymptotically equal to the number of steps (n) times $(1/2)\{1+\lambda_3(1-\lambda_3)^{-1}+\lambda_4(1-\lambda_4)^{-1}\}$. This constant of proportionality may be rearranged as

$$\frac{(1/2)\{1-(f-b)^2-(r-l)^2\}}{\{1-2(f-b)+(f-b)^2+(r-l)^2\}}$$

provided that $f \neq 1$.

Thus the variance is determined by the differences $(f-b)$ and $(r-l)$ rather than the actual values of the parameters. In particular, if $f=b$ and $r=l$ the variance is constant whatever the choice of f. This is a very interesting result for which we can suggest no intuitive explanation. On the other hand when $f=b$ but $r \neq l$ the variance decreases with $|r-l|$. This is because if $|r-l|$ is large there is high probability that the particle's path loops around the origin, in a clockwise direction if $r>l$ and an anticlockwise direction if $l>r$, with the particle remaining close to its original position throughout the walk. If $r=l$ but $f \neq b$ the variance increases with f, because the probability of moving away from the origin without any direction change increases. Similarly, if $r=l$ and $f \neq b$ the variance decreases as b increases, because the probability of continually reversing direction and so remaining near the origin increases. The minimum variance occurs when $b=1$ and the distribution is then concentrated on $(0,0)$, $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$. The maximum variance occurs when $f=1$ and each of the sites $(n,0)$, $(-n,0)$, $(0,n)$ and $(0,-n)$ is occupied with probability $1/4$.

Note that the dispersion matrix Σ_n is diagonal only because of our choice that the first step of the particle is equally likely to be in any of the four permissible directions, and for other initial conditions the two coordinates of the particle's position are not necessarily uncorrelated. Derivation of the dispersion matrix under other initial conditions will be very difficult: the method described above

is no longer applicable because the 4×1 vector $\boldsymbol{h}$ in expression (2.6) must be replaced by some other 4×1 vector reflecting the initial probability of each of the four states E_1, E_2, E_3 and E_4, and it is no longer possible to exploit the reduction of terms such as $P^m\boldsymbol{h}$ to $\boldsymbol{h}$.

4. Recurrence

Recurrence of the walk in the special case of $r = l$ is proved by Henderson, Renshaw and Ford (1983). In the more general case of $r \neq l$ a theorem of Holgate (1966) implies that the walk is also recurrent provided that $f \neq 1$. Holgate showed that two-dimensional walks whose steps form a Markov chain with integer-valued components are recurrent provided that a certain symmetry condition is satisfied (his expression (6)). The direction of the $(n+1)$th step of the correlated random walk depends only upon the direction of the nth, and the sequence of steps therefore forms a Markov chain with integer-valued components. So as Holgate's symmetry condition can easily be shown to be satisfied, the walk is recurrent.

Given that a random walk is recurrent the next step is to determine the mean recurrence time, i.e. the mean number of steps until the first return to the origin. Henderson (1981) shows that just as for the simple random walk, the mean recurrence time for the correlated random walk is infinite provided that none of the parameters f, b, r or l is unity. This implies that no matter how large the value of b, the reversal probability, and therefore how high the probability of immediate return to the origin after only two steps, provided that $b \neq 1$ and return is not certain after two steps the mean recurrence time is infinite.

5. Number of distinct lattice points visited

As the number of steps increases the particle visits more and more lattice points for the first time. The expected number of distinct lattice points visited during the walk is of interest and an asymptotic expression for this number can be obtained by following a method used by Montroll (1964) for simple random walks. The method is based on the generating function

$$U(z, s_1, s_2) \equiv \sum_{n=0}^{\infty} z^n p_n(s_1, s_2) \qquad (0 \leqq z < 1) \tag{5.1}$$

and in particular requires evaluation of $U(z,0,0)$ as $z \to 1$.

An expression for $U(z,0,0)$ may be derived from the matrix formulation (2.6) for the characteristic function $\Phi_n(\omega_1, \omega_2)$. For

$$p_n(0,0) = (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Phi_n(\omega_1, \omega_2) d\omega_1 d\omega_2,$$

and so we may use (2.6) together with (5.1) to give

$$U(z,0,0) = (4\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[\sum_{n=0}^{\infty} \{z^n \boldsymbol{h}^{\mathrm{T}} (\Lambda(\omega_1, \omega_2) Q)^n \boldsymbol{h}\} \right] d\omega_1 d\omega_2$$

which may be rewritten as

$$U(z,0,0) = (4\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \boldsymbol{h}^{\mathrm{T}} (I_4 - z\Lambda(\omega_1, \omega_2) Q)^{-1} \boldsymbol{h} d\omega_1 d\omega_2$$

where I_4 is the 4×4 identity matrix.

Although the algebra involved in the inversion of the integrand matrix and the calculation of the integrals is very tedious, only standard integrals are involved and $U(z,0,0)$ can be expressed in terms of elliptic integrals of the first, second and third kinds. The value as z is increased to unity is of most interest and in this case further simplification is possible. Details are given by Henderson (1981) and the result is that for z sufficiently close to unity

$$U(z,0,0) \sim (2\pi)^{-1} A(z) \log[16\{1 - B(z)\}^{-1}].$$

Here $A(z)$ and $B(z)$ are rational polynomials in f, b, r, l and z, with $0 \leqq B(z) < 1$ for $0 \leqq z < 1$, and where

$$A(1) = \frac{2\{1 - 2(f-b) + (f-b)^2 + (r-l)^2\}}{\{1 - (r-l)^2 - (f-b)^2\}}$$

and

$$B(1) = 1.$$

By following Montroll's method and using the asymptotic expression for $U(z,0,0)$, an asymptotic expression for the expected number of distinct lattice points visited can be determined. Again details are given by Henderson (1981), and the result is that, as the number of steps (n) taken by the particle increases, the expected number of distinct lattice points visited is asymptotically equal to

$$2\pi n / \{A(1) \log n\}.$$

In the special case $f = b = r = l = 1/4$ the correlated random walk reduces to the simple random walk, and the above expression reduces to

$$\pi n / \log n.$$

This agrees with the simple random walk result first obtained by Dvoretsky and Erdös (1950).

Note that $1/A(1)$ is the same multiplying factor as occurs in the asymptotic expression for the dispersion matrix, namely $(1/A(1))nI_2$. Thus the expected number of distinct lattice points visited is directly related to the variance of the particle's position and both are influenced by the differences $(f-b)$ and $(r-l)$, not the actual values of f, b, r or l.

The relationship between the number of lattice points visited during a random walk and the variance has not, to our knowledge, been considered in the literature. There is no obvious connection between either the definitions of the two terms or the derivation of the two expressions, and no apparent reason for each expression to have the same multiplying factor. This relationship *may* apply only to the correlated random walk considered here but a further investigation into whether other, more general, random walks have similar properties would be useful.

6. The diffusion approximation

We now proceed to consider the behaviour of the walk as the step length decreases whilst the number of steps increases. Let us therefore assume that the distance between lattice points is now δx in the X-direction and δy in the Y-direction and write $x_1 = s_1\delta x$ and $y_2 = s_2\delta y$. Furthermore, let the time interval between steps be δt and write $t = n\delta t$. In addition, write $q_j(x, y; t)$ for $p_{nj}(s_1, s_2)$ $(j = 1,2,3,4)$ and $v_1(x, y; t)$ for $p_n(s_1, s_2)$. Then our purpose is to study the behaviour of $v_1(x, y; t)$ as δx, δy and δt tend to 0.

If $\boldsymbol{q}(x, y; t)$ denotes the 4×1 vector with elements $q_j(x, y; t)$, then expansion of the four difference equations (2.3) by Taylor's series yields the vector-matrix equation

$$\begin{aligned}\boldsymbol{q} = Q\left\{\boldsymbol{q} - \delta t\frac{\partial \boldsymbol{q}}{\partial t}\right\} &+ \begin{bmatrix} -f & -b & -r & -l \\ b & f & l & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \delta x \frac{\partial \boldsymbol{q}}{\partial x} \\ &+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -l & -r & -f & -b \\ r & l & b & f \end{bmatrix} \delta y \frac{\partial \boldsymbol{q}}{\partial y} \\ &+ (1/2)\begin{bmatrix} f & b & r & l \\ b & f & l & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (\delta x)^2 \frac{\partial^2 \boldsymbol{q}}{\partial x^2} \\ &+ (1/2)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ l & r & f & b \\ r & l & b & f \end{bmatrix} (\delta y)^2 \frac{\partial^2 \boldsymbol{q}}{\partial y^2} \\ &+ O(\delta t^2, \delta x^3, \delta y^3).\end{aligned} \tag{6.1}$$

We are interested in

$$v_1 = q_1 + q_2 + q_3 + q_4,$$

but in order to rearrange (6.1) to involve v_1 we also require the following terms:

$$v_2 = q_1 + q_2 - q_3 - q_4$$

$$v_3 = q_1 - q_2 - iq_3 + iq_4$$

and

$$v_4 = q_1 - q_2 + iq_3 - iq_4.$$

By multiplying each side of the equation (6.1) by a matrix of left-hand eigenvalues of Q, namely

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \\ 1 & -1 & i & -i \end{bmatrix},$$

the vector-matrix equation can be rearranged as four partial differential equations in v_1, v_2, v_3 and v_4. These are

$$\begin{aligned} v_1 = {} & \lambda_1 v_1 - \lambda_1 \delta t \frac{\partial v_1}{\partial t} - (1/2)\delta x \left(\lambda_3 \frac{\partial v_3}{\partial x} + \lambda_4 \frac{\partial v_4}{\partial x}\right) \\ & - (i/2)\delta y \left(\lambda_3 \frac{\partial v_3}{\partial y} - \lambda_4 \frac{\partial v_4}{\partial y}\right) + (1/4)\delta x^2 \left(\lambda_1 \frac{\partial^2 v_1}{\partial x^2} + \lambda_2 \frac{\partial^2 v_2}{\partial x^2}\right) \\ & + (1/4)\delta y^2 \left(\lambda_1 \frac{\partial^2 v_1}{\partial y^2} - \lambda_2 \frac{\partial^2 v_2}{\partial y^2}\right) + O(\delta t^2, \delta x^3, \delta y^3), \end{aligned} \tag{6.2}$$

$$\begin{aligned} v_2 = {} & \lambda_2 v_2 - \lambda_2 \delta t \frac{\partial v_2}{\partial t} - (1/2)\delta x \left(\lambda_3 \frac{\partial v_3}{\partial x} + \lambda_4 \frac{\partial v_4}{\partial x}\right) \\ & + (i/2)\delta y \left(\lambda_3 \frac{\partial v_3}{\partial y} - \lambda_4 \frac{\partial v_4}{\partial y}\right) + (1/4)\delta x^2 \left(\lambda_1 \frac{\partial^2 v_1}{\partial x^2} + \lambda_2 \frac{\partial^2 v_2}{\partial x^2}\right) \\ & - (1/4)\delta y^2 \left(\lambda_1 \frac{\partial^2 v_1}{\partial y^2} - \lambda_2 \frac{\partial^2 v_2}{\partial y^2}\right) + O(\delta t^2, \delta x^3, \delta y^3), \end{aligned} \tag{6.3}$$

$$\begin{aligned} v_3 = {} & \lambda_3 v_3 - \lambda_3 \delta t \frac{\partial v_3}{\partial t} - (1/2)\delta x \left(\lambda_1 \frac{\partial v_1}{\partial x} + \lambda_2 \frac{\partial v_2}{\partial x}\right) \\ & + (i/2)\delta y \left(\lambda_1 \frac{\partial v_1}{\partial y} - \lambda_2 \frac{\partial v_2}{\partial y}\right) + (1/4)\delta x^2 \left(\lambda_3 \frac{\partial^2 v_3}{\partial x^2} + \lambda_4 \frac{\partial^2 v_4}{\partial x^2}\right) \\ & + (1/4)\delta y^2 \left(\lambda_3 \frac{\partial^2 v_3}{\partial y^2} - \lambda_4 \frac{\partial^2 v_4}{\partial y^2}\right) + O(\delta t^2, \delta x^3, \delta y^3) \end{aligned} \tag{6.4}$$

and

$$v_4 = \lambda_4 v_4 - \lambda_4 \delta t \frac{\partial v_4}{\partial t} - (1/2)\delta x \left(\lambda_1 \frac{\partial v_1}{\partial x} + \lambda_2 \frac{\partial v_2}{\partial x}\right)$$

$$- (i/2)\delta y \left(\lambda_1 \frac{\partial v_1}{\partial y} - \lambda_2 \frac{\partial v_2}{\partial y}\right) + (1/4)\delta x^2 \left(\lambda_3 \frac{\partial^2 v_3}{\partial x^2} + \lambda_4 \frac{\partial^2 v_4}{\partial x^2}\right) \tag{6.5}$$

$$- (1/4)\delta y^2 \left(\lambda_3 \frac{\partial^2 v_3}{\partial y^2} - \lambda_4 \frac{\partial^2 v_4}{\partial y^2}\right) + O(\delta t^2, \delta x^3, \delta y^3).$$

To derive the diffusion equation we now consider the process as the length of individual steps becomes small but the steps are separated by such short time intervals that the motion of the particle appears continuous. First, we set $\delta y = \delta x$ and as usual ensure that the variance of the particle's position after a large number of steps remains finite and positive by choosing $\delta t = \delta x^2/\sigma^2$ for some positive parameter σ^2. Our objective is to allow δx to tend to 0 and hence obtain a limiting form, independent of δx, for the density $v_1(x, y, t)$.

Now the eigenvalues λ_2, λ_3 and λ_4 are not unity for genuinely two-dimensional walks. Examination of Equations (6.3), (6.4) and (6.5) shows that the functions v_2, v_3 and v_4 cannot therefore be independent of δx as otherwise the equations are inconsistent as $\delta x \to 0$. The equations can only remain consistent as $\delta x \to 0$ if v_2, v_3 and v_4 are of the form

$$v_2(x, y; t) \equiv \delta x^2 v_2'(x, y; t),$$

$$v_3(x, y; t) \equiv \delta x v_3'(x, y; t),$$

and

$$v_4(x, y; t) \equiv \delta x v_4'(x, y; t)$$

for some functions $v_2'(x, y; t)$, $v_3'(x, y; t)$ and $v_4'(x, y; t)$ which are independent of δx. Hence on dividing each side of (6.2) and (6.3) by δx^2, and each side of (6.4) and (6.5) by δx, and then allowing $\delta x \to 0$, we obtain

$$\sigma^{-2} \frac{\partial v_1}{\partial t} = -(1/2)\left(\lambda_3 \frac{\partial v_3'}{\partial x} + \lambda_4 \frac{\partial v_4'}{\partial x}\right) - (i/2)\left(\lambda_3 \frac{\partial v_3'}{\partial y} - \lambda_4 \frac{\partial v_4'}{\partial y}\right)$$

$$+ (1/4)\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right), \tag{6.6}$$

$$(1 - \lambda_2) v_2' = -(1/2)\left(\lambda_3 \frac{\partial v_3'}{\partial x} + \lambda_4 \frac{\partial v_4'}{\partial x}\right) + (i/2)\left(\lambda_3 \frac{\partial v_3'}{\partial y} - \lambda_4 \frac{\partial v_4'}{\partial y}\right)$$

$$+ (1/4)\left(\frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial y^2}\right), \tag{6.7}$$

$$(1 - \lambda_3) v_3' = -(1/2)\left(\frac{\partial v_1}{\partial x} - i \frac{\partial v_1}{\partial y}\right), \tag{6.8}$$

and

$$(1-\lambda_4)v_4' = -(1/2)\left(\frac{\partial v_1}{\partial x} + i\frac{\partial v_1}{\partial y}\right). \tag{6.9}$$

The function $v_2'(x, y; t)$ is of no direct interest and does not appear in Equations (6.6), (6.8) or (6.9). Thus we may ignore (6.7) and use (6.8) and (6.9) to substitute for v_3' and v_4' in (6.6). This gives

$$4\sigma^{-2}\frac{\partial v_1}{\partial t} = \{1 + \lambda_3(1-\lambda_3)^{-1} + \lambda_4(1-\lambda_4)^{-1}\}\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right) \tag{6.10}$$

which is of the same form as the standard diffusion equation (1.1). Hence the two-dimensional diffusion equation can be obtained by taking suitable limits of a correlated random walk. The solution to (6.10) can be shown to be

$$v_1(x, y; t) = \frac{\exp[(\sigma^2 t)^{-1}\{1 + \lambda_3(1-\lambda_3)^{-1} + \lambda_4(1-\lambda_4)^{-1}\}^{-1}(x^2 + y^2)]}{\pi t\sigma^2\{1 + \lambda_3(1-\lambda_3)^{-1} + \lambda_4(1-\lambda_4)^{-1}\}} \tag{6.11}$$

(see, for example, Pielou (1977), p. 170). Thus in the limit as $\delta x \to 0$ the distribution of the particle's position is normal with zero mean and dispersion matrix

$$(\sigma^2 t/2)\{1 + \lambda_3(1-\lambda_3)^{-1} + \lambda_4(1-\lambda_4)^{-1}\}I_2.$$

When $f = b = r = l = 1/4$ the correlated random walk is equivalent to a simple random walk on a two-dimensional square lattice. In this case $\lambda_3 = \lambda_4 = 0$ and the limiting distribution has dispersion matrix $(\sigma^2 t/2)I_2$. Thus the effect of correlation between step directions is simply to introduce a scaling factor $\{1 + \lambda_3(1-\lambda_3)^{-1} + \lambda_4(1-\lambda_4)^{-1}\}$. This can be rearranged as

$$\frac{1-(f-b)^2-(r-l)^2}{1-2(f-b)+(f-b)^2+(r-l)^2},$$

and the comments at the end of Section 3 concerning the properties of the dispersion matrix for different parameter values also apply to the dispersion of the limiting distribution.

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