ALGEBRA

Algebra playground is Clifford $Cl_{1,4}(\mathbb{R})$, hence signature is (+ - - - -). The five dimensions are (t, x, y, z, w).
It defines two sets of quaternions with one imaginary unit.
CoQuaternions $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ http://en.wikipedia.org/wiki/Split-quaternion

$$\mathbf{i} = \gamma_t\gamma_w$$
$$\mathbf{j} = \gamma_t\gamma_x\gamma_y\gamma_z$$
$$\mathbf{k} = \gamma_x\gamma_y\gamma_z\gamma_w$$
$$\mathbf{i}^2 = 1$$
$$\mathbf{j}^2 = -1$$
$$\mathbf{k}^2 = 1$$
$$\mathbf{ij} = \gamma_x\gamma_y\gamma_z\gamma_w$$
$$\mathbf{ji} = -\gamma_x\gamma_y\gamma_z\gamma_w$$
$$\mathbf{jk} = \gamma_t\gamma_w$$
$$\mathbf{kj} = -\gamma_t\gamma_w$$
$$\mathbf{ki} = -\gamma_t\gamma_x\gamma_y\gamma_z$$
$$\mathbf{ik} = \gamma_t\gamma_x\gamma_y\gamma_z$$
$$\mathbf{ijk} = 1$$

Quaternions $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ http://en.wikipedia.org/wiki/Quaternion

$$\boldsymbol{i} = \gamma_y\gamma_z$$
$$\boldsymbol{j} = -\gamma_x\gamma_z$$
$$\boldsymbol{k} = \gamma_x\gamma_y$$
$$\boldsymbol{i}^2 = -1$$
$$\boldsymbol{j}^2 = -1$$
$$\boldsymbol{k}^2 = -1$$
$$\boldsymbol{ij} = \gamma_x\gamma_y$$
$$\boldsymbol{ji} = -\gamma_x\gamma_y$$
$$\boldsymbol{jk} = \gamma_y\gamma_z$$
$$\boldsymbol{kj} = -\gamma_y\gamma_z$$
$$\boldsymbol{ki} = -\gamma_x\gamma_z$$
$$\boldsymbol{ik} = \gamma_x\gamma_z$$
$$\boldsymbol{ijk} = -1$$

Imaginary unit i

$$i = \gamma_w$$
$$i^2 = -1$$

Gradient definition

$$\nabla = (\gamma_t \frac{\partial}{\partial t} + \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} + \gamma_w \frac{\partial}{\partial w})$$

http://www.mrao.cam.ac.uk/ clifford/ptIIIcourse/course99/handouts/hout07.ps.gz
Idempotents in $Cl_{1,4}(\mathbb{R})$

$$U_1 : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} + \mathbf{j} - \mathbf{k}) \Rightarrow U_1U_1 - U_1 = 0$$
$$U_2 : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} + \mathbf{j} - \mathbf{k}) \Rightarrow U_2U_2 - U_2 = 0$$
$$U_3 : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} - \mathbf{j} + \mathbf{k}) \Rightarrow U_3U_3 - U_3 = 0$$
$$U_4 : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} - \mathbf{j} + \mathbf{k}) \Rightarrow U_4U_4 - U_4 = 0$$
$$U_5 : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} - \mathbf{j} - \mathbf{k}) \Rightarrow U_5U_5 - U_5 = 0$$
$$U_6 : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} - \mathbf{j} - \mathbf{k}) \Rightarrow U_6U_6 - U_6 = 0$$
$$U_7 : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow U_7U_7 - U_7 = 0$$
$$U_8 : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow U_8U_8 - U_8 = 0$$
$$U_9 : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} - \mathbf{j} + \mathbf{k}) \Rightarrow U_9U_9 - U_9 = 0$$
$$U_{10} : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} - \mathbf{j} + \mathbf{k}) \Rightarrow U_{10}U_{10} - U_{10} = 0$$
$$U_{11} : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} - \mathbf{k}) \Rightarrow U_{11}U_{11} - U_{11} = 0$$
$$U_{12} : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} - \mathbf{k}) \Rightarrow U_{12}U_{12} - U_{12} = 0$$
$$U_{13} : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow U_{13}U_{13} - U_{13} = 0$$
$$U_{14} : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow U_{14}U_{14} - U_{14} = 0$$
$$U_{15} : \frac{1}{2}(1 + \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}) \Rightarrow U_{15}U_{15} - U_{15} = 0$$
$$U_{16} : \frac{1}{2}(1 - \mathbf{j}i)\frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}) \Rightarrow U_{16}U_{16} - U_{16} = 0$$

PHYSICS
The following symbols are defined :
Energy $E \in \mathbb{R}$
Mass $m \in \mathbb{R}$
Momentum $\mathbf{p}$ is defined with $p_x, p_y, p_z \in \mathbb{R}$

$$\mathbf{p} = p_x \boldsymbol{i} + p_y \boldsymbol{j} + p_z \boldsymbol{k}$$

$$\mathbf{p} = p_z \gamma_x \gamma_y - p_y \gamma_x \gamma_z + p_x \gamma_y \gamma_z$$

Electric field $\mathbf{E}$ is defined with $E_x, E_y, E_z \in \mathbb{R}$

$$\mathbf{E} = E_x \boldsymbol{i} + E_y \boldsymbol{j} + E_z \boldsymbol{k}$$

$$\mathbf{E} = E_z \gamma_x \gamma_y - E_y \gamma_x \gamma_z + E_x \gamma_y \gamma_z$$

Magnetic field $\mathbf{B}$ is defined with $B_x, B_y, B_z \in \mathbb{R}$

$$\mathbf{B} = B_x \boldsymbol{i} + B_y \boldsymbol{j} + B_z \boldsymbol{k}$$

$$\mathbf{B} = B_z \gamma_x \gamma_y - B_y \gamma_x \gamma_z + B_x \gamma_y \gamma_z$$

Wave : K is a constant and f is a function of (t, x, y, z, w)

$$\psi = K e^f$$

First derivative

$$\nabla \psi = \nabla (K e^f)$$

$$\nabla \psi = (\nabla K) e^f + K \nabla (e^f)$$

$$\nabla K = 0$$

$$\nabla \psi = K (\nabla f) e^f$$

f is the exponential function

$$(E_z z + E_y y + E_x x)\, \gamma_t - B_y z \gamma_x - B_z x \gamma_y - B_x y \gamma_z + (p_y y + p_x x + p_z z - E t)\, \gamma_w$$

DIRAC

$$http://en.wikipedia.org/wiki/Dirac\_equation$$

$$0=(\gamma_0\frac{\partial}{\partial t}+\gamma_1\frac{\partial}{\partial x}+\gamma_2\frac{\partial}{\partial y}+\gamma_3\frac{\partial}{\partial z}+im)\psi$$

Substitution in Dirac equation

$$-im\psi=(-\mathbf{i}\frac{\partial}{\partial t}-\boldsymbol{i}\mathbf{k}\frac{\partial}{\partial x}-\boldsymbol{j}\mathbf{k}\frac{\partial}{\partial y}-\boldsymbol{k}\mathbf{k}\frac{\partial}{\partial z})\psi$$

With the above gradients, identify the Dirac algebra aka gamma matrices

$$\gamma_0=-\gamma_t\gamma_w$$

$$-\mathbf{i}=-\gamma_t\gamma_w$$

$$\gamma_0^2=1$$

$$\gamma_1=\gamma_x\gamma_w$$

$$-\boldsymbol{i}\mathbf{k}=\gamma_x\gamma_w$$

$$\gamma_1^2=-1$$

$$\gamma_2=\gamma_y\gamma_w$$

$$-\boldsymbol{j}\mathbf{k}=\gamma_y\gamma_w$$

$$\gamma_2^2=-1$$

$$\gamma_3=\gamma_z\gamma_w$$

$$-\boldsymbol{k}\mathbf{k}=\gamma_z\gamma_w$$

$$\gamma_3^2=-1$$

$$\gamma_5=-\gamma_t\gamma_x\gamma_y\gamma_z\gamma_w$$

$$-i\mathbf{j}=-\gamma_t\gamma_x\gamma_y\gamma_z\gamma_w$$

$$\gamma_5^2=1$$

Exponential function with energy and momentum only, electric and magnetic fields are null f

$$f=(p_y y+p_x x+p_z z-Et)\,\gamma_w$$

Gradient for f

$$\nabla f=-E\gamma_t\gamma_w-p_x\gamma_x\gamma_w-p_y\gamma_y\gamma_w-p_z\gamma_z\gamma_w$$

Square of the gradient is a Lorentz invariant

$$\nabla f^2 = -{p_z}^2 + E^2 - {p_y}^2 - {p_x}^2$$

Particular solutions K

$$K_1 = (-\mathbf{i}E + im + \mathbf{k}\mathbf{p})$$
$$K_2 = -(-\mathbf{i}E + im - \mathbf{k}\mathbf{p})\mathbf{k}$$
$$K_3 = -(-\mathbf{i}E - im + \mathbf{k}\mathbf{p})\mathbf{j}$$
$$K_4 = (-\mathbf{i}E - im - \mathbf{k}\mathbf{p})\mathbf{i}$$
$$K_5 = (-\mathbf{i}E + im + \mathbf{k}\mathbf{p})\mathbf{j}i$$
$$K_6 = -(-\mathbf{i}E - im + \mathbf{k}\mathbf{p})i$$
$$K_7 = (-\mathbf{i}E + im - \mathbf{k}\mathbf{p})\mathbf{i}i$$
$$K_8 = -(-\mathbf{i}E - im - \mathbf{k}\mathbf{p})\mathbf{k}i$$

Check first derivative is null $K(\nabla f + im)$

$$K_1(\nabla f + im) = -m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2$$
$$K_2(\nabla f + im) = \left(-m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2\right)\gamma_x\gamma_y\gamma_z\gamma_w$$
$$K_3(\nabla f + im) = \left(-m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2\right)\gamma_t\gamma_x\gamma_y\gamma_z$$
$$K_4(\nabla f + im) = \left(-m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2\right)\gamma_t\gamma_w$$
$$K_5(\nabla f + im) = \left(-m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2\right)\gamma_t\gamma_x\gamma_y\gamma_z\gamma_w$$
$$K_6(\nabla f + im) = \left(-m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2\right)\gamma_w$$
$$K_7(\nabla f + im) = \left(-m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2\right)\gamma_t$$
$$K_8(\nabla f + im) = \left(-m^2 - {p_z}^2 + E^2 - {p_y}^2 - {p_x}^2\right)\gamma_x\gamma_y\gamma_z$$

MAXWELL

Exponential function with electric and magnetic fields only f

$$f = (E_z z + E_y y + E_x x)\,\gamma_t - B_y z \gamma_x - B_z x \gamma_y - B_x y \gamma_z$$

Gradient for f

$$\nabla f = E_x \gamma_t \gamma_x + E_y \gamma_t \gamma_y + B_z \gamma_x \gamma_y + E_z \gamma_t \gamma_z - B_y \gamma_x \gamma_z + B_x \gamma_y \gamma_z$$

Square of the gradient is a Lorentz invariant

$$\nabla f^2 = {E_z}^2 - {B_y}^2 + {E_x}^2 + {E_y}^2 - {B_z}^2 - {B_x}^2 + (2B_y E_y + 2B_z E_z + 2B_x E_x)\,\gamma_t \gamma_x \gamma_y \gamma_z$$

DIRAC MAXWELL

∇f is the electromagnetic field F with a Dirac component

$$
\begin{aligned}
F_{DiracMaxwell} &= E_x\gamma_t\gamma_x \\
&+ E_y\gamma_t\gamma_y \\
&+ B_z\gamma_x\gamma_y \\
&+ E_z\gamma_t\gamma_z \\
&- B_y\gamma_x\gamma_z \\
&+ B_x\gamma_y\gamma_z \\
&- E\gamma_t\gamma_w \\
&- p_x\gamma_x\gamma_w \\
&- p_y\gamma_y\gamma_w \\
&- p_z\gamma_z\gamma_w
\end{aligned}
$$

$$F_{Maxwell} = \mathbf{B} - \mathbf{jE}$$

$$
\begin{aligned}
F_{Maxwell} &= E_x\gamma_t\gamma_x \\
&+ E_y\gamma_t\gamma_y \\
&+ B_z\gamma_x\gamma_y \\
&+ E_z\gamma_t\gamma_z \\
&- B_y\gamma_x\gamma_z \\
&+ B_x\gamma_y\gamma_z
\end{aligned}
$$

$$F_{Dirac} = (\mathbf{kp} - \mathbf{i}E)$$

$$
\begin{aligned}
F_{Dirac} &= -E\gamma_t\gamma_w \\
&- p_x\gamma_x\gamma_w \\
&- p_y\gamma_y\gamma_w \\
&- p_z\gamma_z\gamma_w
\end{aligned}
$$