ALGEBRA

Algebra playground is Clifford $Cl_{1,4}(\mathbb{R})$, hence signature is $(+----)$. The five dimensions are (t,x,y,z,w). It defines two sets of quaternions with one imaginary unit.
CoQuaternions $(i,\mathbf{i},\mathbf{j},\mathbf{k})$ http://en.wikipedia.org/wiki/Split-quaternion ?

$$\mathbf{i} = \gamma_t \wedge \gamma_w$$
$$\mathbf{j} = \gamma_t \wedge \gamma_x \wedge \gamma_y \wedge \gamma_z \wedge \gamma_w$$
$$\mathbf{k} = \gamma_x \wedge \gamma_y \wedge \gamma_z \wedge \gamma_w$$
$$\mathbf{i}^2 = 1$$
$$\mathbf{j}^2 = 1$$
$$\mathbf{k}^2 = 1$$
$$\mathbf{ij} = -\gamma_x \wedge \gamma_y \wedge \gamma_z$$
$$\mathbf{ji} = -\gamma_x \wedge \gamma_y \wedge \gamma_z$$
$$\mathbf{jk} = \gamma_t$$
$$\mathbf{kj} = \gamma_t$$
$$\mathbf{ki} = -\gamma_t \wedge \gamma_x \wedge \gamma_y \wedge \gamma_z$$
$$\mathbf{ik} = \gamma_t \wedge \gamma_x \wedge \gamma_y \wedge \gamma_z$$
$$\mathbf{ijk} = -\gamma_w$$

Quaternions $(1,\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$ http://en.wikipedia.org/wiki/Quaternion

$$\boldsymbol{i} = \gamma_y \wedge \gamma_z$$
$$\boldsymbol{j} = -\gamma_x \wedge \gamma_z$$
$$\boldsymbol{k} = \gamma_x \wedge \gamma_y$$
$$\boldsymbol{i}^2 = -1$$
$$\boldsymbol{j}^2 = -1$$
$$\boldsymbol{k}^2 = -1$$
$$\boldsymbol{ij} = \gamma_x \wedge \gamma_y$$
$$\boldsymbol{ji} = -\gamma_x \wedge \gamma_y$$
$$\boldsymbol{jk} = \gamma_y \wedge \gamma_z$$
$$\boldsymbol{kj} = -\gamma_y \wedge \gamma_z$$
$$\boldsymbol{ki} = -\gamma_x \wedge \gamma_z$$
$$\boldsymbol{ik} = \gamma_x \wedge \gamma_z$$
$$\boldsymbol{ijk} = -1$$

Imaginary unit i

$$i = \gamma_w$$
$$i^2 = -1$$

Gradient definition

$$\nabla = (\gamma_t \frac{\partial}{\partial t} + \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y} + \gamma_z \frac{\partial}{\partial z} + \gamma_w \frac{\partial}{\partial w})$$

http://www.mrao.cam.ac.uk/ clifford/ptIIIcourse/course99/handouts/hout07.ps.gz
Idempotents in $Cl_{1,4}(\mathbb{R})$

$$U_1 : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1-\mathbf{i}+i-\mathbf{k}) \Rightarrow U_1U_1 - U_1 = 0$$

$$U_2 : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1-\mathbf{i}+i-\mathbf{k}) \Rightarrow U_2U_2 - U_2 = 0$$

$$U_3 : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1-\mathbf{i}-i+\mathbf{k}) \Rightarrow U_3U_3 - U_3 = 0$$

$$U_4 : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1-\mathbf{i}-i+\mathbf{k}) \Rightarrow U_4U_4 - U_4 = 0$$

$$U_5 : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1+\mathbf{i}-i-\mathbf{k}) \Rightarrow U_5U_5 - U_5 = 0$$

$$U_6 : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1+\mathbf{i}-i-\mathbf{k}) \Rightarrow U_6U_6 - U_6 = 0$$

$$U_7 : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1+\mathbf{i}+i+\mathbf{k}) \Rightarrow U_7U_7 - U_7 = 0$$

$$U_8 : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1+\mathbf{i}+i+\mathbf{k}) \Rightarrow U_8U_8 - U_8 = 0$$

$$U_9 : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1+\mathbf{i}-i+\mathbf{k}) \Rightarrow U_9U_9 - U_9 = 0$$

$$U_{10} : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1+\mathbf{i}-i+\mathbf{k}) \Rightarrow U_{10}U_{10} - U_{10} = 0$$

$$U_{11} : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1+\mathbf{i}+i-\mathbf{k}) \Rightarrow U_{11}U_{11} - U_{11} = 0$$

$$U_{12} : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1+\mathbf{i}+i-\mathbf{k}) \Rightarrow U_{12}U_{12} - U_{12} = 0$$

$$U_{13} : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1-\mathbf{i}+i+\mathbf{k}) \Rightarrow U_{13}U_{13} - U_{13} = 0$$

$$U_{14} : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1-\mathbf{i}+i+\mathbf{k}) \Rightarrow U_{14}U_{14} - U_{14} = 0$$

$$U_{15} : \frac{1}{2}(1+\mathbf{j})\frac{1}{2}(1-\mathbf{i}-i-\mathbf{k}) \Rightarrow U_{15}U_{15} - U_{15} = 0$$

$$U_{16} : \frac{1}{2}(1-\mathbf{j})\frac{1}{2}(1-\mathbf{i}-i-\mathbf{k}) \Rightarrow U_{16}U_{16} - U_{16} = 0$$

PHYSICS
The following symbols are defined :
Energy $E \in \mathbb{R}$
Mass $m \in \mathbb{R}$
Momentum $\mathbf{p}$ is defined with $p_x, p_y, p_z \in \mathbb{R}$

$$\mathbf{p} = p_x \boldsymbol{i} + p_y \boldsymbol{j} + p_z \boldsymbol{k}$$

$$\mathbf{p} = p_z \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} - p_y \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} + p_x \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$

Electric field $\mathbf{E}$ is defined with $E_x, E_y, E_z \in \mathbb{R}$

$$\mathbf{E} = E_x \boldsymbol{i} + E_y \boldsymbol{j} + E_z \boldsymbol{k}$$

$$\mathbf{E} = E_z \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} - E_y \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} + E_x \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$

Magnetic field $\mathbf{B}$ is defined with $B_x, B_y, B_z \in \mathbb{R}$

$$\mathbf{B} = B_x \boldsymbol{i} + B_y \boldsymbol{j} + B_z \boldsymbol{k}$$

$$\mathbf{B} = B_z \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} - B_y \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} + B_x \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$

Wave : K is a constant and f is a function of (t, x, y, z, w)

$$\psi = K e^f$$

First derivative

$$\nabla \psi = \nabla (K e^f)$$

$$\nabla \psi = (\nabla K) e^f + K \nabla (e^f)$$

$$\nabla K = 0$$

$$\nabla \psi_L = (\nabla f) K e^f$$

$$\nabla \psi_R = K e^f (\nabla f)$$

$$\nabla \psi_R = K (\nabla f) e^f$$

Find solutions that fullfills left and right multiplication, (start with right multiplication)
f is the exponential function

$$f = (E_x x + E_y y + E_z z)\, \boldsymbol{\gamma_t} - B_y z \boldsymbol{\gamma_x} - B_z x \boldsymbol{\gamma_y} - B_x y \boldsymbol{\gamma_z} + (-Et + p_x x + p_y y + p_z z)\, \boldsymbol{\gamma_w}$$

DIRAC
http://en.wikipedia.org/wiki/Dirac_equation

$$0 = (\gamma_0 \frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} + im)\psi$$

Define the Dirac algebra (aka gamma matrices) with the two sets of quaternions

$$\gamma_0 = -\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w}$$
$$-\mathbf{i} = -\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w}$$
$$\gamma_0 \wedge 2 = 1$$
$$\gamma_1 = \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w}$$
$$-\boldsymbol{i}\mathbf{k} = \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w}$$
$$\gamma_1 \wedge 2 = -1$$
$$\gamma_2 = \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w}$$
$$-\boldsymbol{j}\mathbf{k} = \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w}$$
$$\gamma_2 \wedge 2 = -1$$
$$\gamma_3 = \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$
$$-\boldsymbol{k}\mathbf{k} = \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$
$$\gamma_3 \wedge 2 = -1$$
$$\gamma_5 = -\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$
$$-\mathbf{j} = -\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$
$$\gamma_5 \wedge 2 = 1$$

Substitution in Dirac equation

$$-i\frac{\partial}{\partial w}\psi = (-\mathbf{i}\frac{\partial}{\partial t} - \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial x} - \boldsymbol{j}\mathbf{k}\frac{\partial}{\partial y} - \boldsymbol{k}\mathbf{k}\frac{\partial}{\partial z})\psi$$

Exponential function with energy and momentum only, electric and magnetic fields are null f

$$f = -mw + (-Et + p_x x + p_y y + p_z z)\,\boldsymbol{\gamma_w}$$

Gradient for f

$$\nabla f = m\boldsymbol{\gamma_w} - E\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} - p_x\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w} - p_y\boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w} - p_z\boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$

Square of the gradient is a Lorentz invariant

$$\nabla f^2 = E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2$$

Particular solutions K

$$K_1 = (-\mathbf{i}E + im + \mathbf{kp})$$
$$K_2 = -(-\mathbf{i}E + im - \mathbf{kp})\mathbf{k}$$
$$K_3 = -(-\mathbf{i}E + im + \mathbf{kp})\mathbf{j}$$
$$K_4 = (-\mathbf{i}E - im - \mathbf{kp})\mathbf{i}$$
$$K_5 = (-\mathbf{i}E - im + \mathbf{kp})\mathbf{j}i$$
$$K_6 = -(-\mathbf{i}E - im + \mathbf{kp})i$$
$$K_7 = (-\mathbf{i}E + im - \mathbf{kp})\mathbf{i}i$$
$$K_8 = -(-\mathbf{i}E - im - \mathbf{kp})\mathbf{k}i$$

Check first derivative is null $K\nabla f$

$$K_1\nabla f = E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2$$
$$K_2\nabla f = \left(E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\right) \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$
$$K_3\nabla f = \left(E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\right) \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$
$$K_4\nabla f = \left(E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\right) \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w}$$
$$K_5\nabla f = \left(E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\right) \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$
$$K_6\nabla f = \left(E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\right) \boldsymbol{\gamma_w}$$
$$K_7\nabla f = \left(E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\right) \boldsymbol{\gamma_t}$$
$$K_8\nabla f = \left(E^2 - m^2 - (p_x)^2 - (p_y)^2 - (p_z)^2\right) \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$

Right multiplication, left multiplication and combinations of the eight particular solutions

$$U_1 = \frac{1}{4} - \frac{1}{4}\boldsymbol{\gamma_t} + \frac{1}{4}\boldsymbol{\gamma_w} - \frac{1}{4}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} + \frac{1}{4}\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} - \frac{1}{4}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} - \frac{1}{4}\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w} + \frac{1}{4}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$

$$\begin{aligned}
U_1K_1\nabla f =& \frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2 \\
&+ \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} + \left(\frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_w} \\
&+ \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} \\
&+ \left(\frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \\
&+ \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} + \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w} \\
&+ \left(\frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}
\end{aligned}$$

$$\begin{aligned}
\nabla f K_1U_1 =& \frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2 \\
&+ \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} + \left(\frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_w} \\
&+ \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} \\
&+ \left(\frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \\
&+ \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} + \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w} \\
&+ \left(\frac{E^2}{4} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}
\end{aligned}$$

$$\psi_L = -e^{-f}K_1U_1$$
$$\psi_R = U_1K_1e^f$$
$$\psi = \psi_L\psi_R$$
$$\psi : -e^{-f}K_1U_1U_1K_1e^f = -e^{-f}K_1U_1K_1e^f$$

this is the definition of a rotation :

$$e^{-f}e^{f} = 1$$

IBOZOO UU ? $-K_1U_1K_1 = \Psi$
for the same phase f, we can write 8 K combined with 16 U : 128 solutions ?

$$\begin{aligned}
\Psi = & -\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2 \\
& + \left(-\frac{E^2}{4} + \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} - \frac{Ep_x}{2}\boldsymbol{\gamma_x} - \frac{Ep_y}{2}\boldsymbol{\gamma_y} - \frac{Ep_z}{2}\boldsymbol{\gamma_z} + \left(\frac{E^2}{4} - \frac{Em}{2} + \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_w} \\
& + \frac{mp_x}{2}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} + \frac{mp_y}{2}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_y} + \frac{mp_z}{2}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_z} + \left(\frac{E^2}{4} - \frac{Em}{2} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} + \left(\frac{p_x}{2}(E-m)\right)\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w} + \left(\frac{p_y}{2}(E-m)\right)\boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w} + \left(\frac{p_z}{2}(E-m)\right)\boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w} \\
& + \left(\frac{p_z}{2}(-E+m)\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} + \left(\frac{p_y}{2}(E-m)\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} + \left(\frac{p_x}{2}(-E+m)\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} + \left(-\frac{E^2}{4} + \frac{Em}{2} - \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} + \frac{mp_z}{2}\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w} - \frac{mp_y}{2}\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w} + \frac{mp_x}{2} \\
& + \left(-\frac{E^2}{4} + \frac{Em}{2} - \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} - \frac{Ep_z}{2}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w} + \frac{Ep_y}{2}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w} - \frac{Ep_x}{2}\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w} + \left(-\frac{E^2}{4} + \frac{m^2}{4} - \frac{1}{4}(p_x)^2 - \frac{1}{4}(p_y)^2 - \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \\
& + \left(-\frac{E^2}{4} + \frac{m^2}{4} + \frac{1}{4}(p_x)^2 + \frac{1}{4}(p_y)^2 + \frac{1}{4}(p_z)^2\right)\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}
\end{aligned}$$

Dirac observables handout 11 chapter 3.1

MAXWELL

Exponential function with electric and magnetic fields only f

$$f = (E_x x + E_y y + E_z z)\,\boldsymbol{\gamma_t} - B_y z \boldsymbol{\gamma_x} - B_z x \boldsymbol{\gamma_y} - B_x y \boldsymbol{\gamma_z}$$

Gradient for f

$$\nabla f = E_x \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} + E_y \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_y} + E_z \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_z} + B_z \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} - B_y \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} + B_x \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$

Square of the gradient is a Lorentz invariant

$$\nabla f^2 = -(B_x)^2 - (B_y)^2 - (B_z)^2 + (E_x)^2 + (E_y)^2 + (E_z)^2 + (2B_x E_x + 2B_y E_y + 2B_z E_z)\,\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$

DIRAC MAXWELL

∇f is the electromagnetic field F with a Dirac component

$$f = (E_x x + E_y y + E_z z)\,\boldsymbol{\gamma_t} - B_y z \boldsymbol{\gamma_x} - B_z x \boldsymbol{\gamma_y} - B_x y \boldsymbol{\gamma_z} + (-Et + p_x x + p_y y + p_z z)\,\boldsymbol{\gamma_w}$$

$$F_{DiracMaxwell} : \nabla f = E_x \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} + E_y \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_y} + E_z \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_z} - E \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} + B_z \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} - B_y \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} - p_x \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w} + B_x \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z} - p_y \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w} - p_z \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$

$$F_{Maxwell} = -i(i\mathbf{B} - \mathbf{jE})$$

$$F_{Maxwell} = E_x \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x} + E_y \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_y} + E_z \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_z} + B_z \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y} - B_y \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z} + B_x \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}$$

$$F_{Dirac} = (\mathbf{kp} - \mathbf{i}E)$$

$$F_{Dirac} = -E \boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} - p_x \boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w} - p_y \boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w} - p_z \boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$

DIRAC COUPLING

$$-(im + \mathbf{i}q\phi + \mathbf{k}q\mathbf{A})\psi = (-\mathbf{i}\frac{\partial}{\partial t} - \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial x} - \boldsymbol{j}\mathbf{k}\frac{\partial}{\partial y} - \boldsymbol{k}\mathbf{k}\frac{\partial}{\partial z})\psi$$

Exponential function with energy and momentum only, and a potential that depends on w
Electric and magnetic field are null

$$f = \phi qw\boldsymbol{\gamma_t} + A_x qw\boldsymbol{\gamma_x} + A_y qw\boldsymbol{\gamma_y} + A_z qw\boldsymbol{\gamma_z} + (-Et + p_x x + p_y y + p_z z)\,\boldsymbol{\gamma_w}$$

Gradient for f

$$\nabla f = (-E + \phi q)\,\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w} + (A_x q - p_x)\,\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w} + (A_y q - p_y)\,\boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w} + (A_z q - p_z)\,\boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}$$

Full exponential function

$$f = (E_x x + E_y y + E_z z + \phi qw)\,\boldsymbol{\gamma_t} + (A_x qw - B_y z)\,\boldsymbol{\gamma_x} + (A_y qw - B_z x)\,\boldsymbol{\gamma_y} + (A_z qw - B_x y)\,\boldsymbol{\gamma_z} + (-Et + p_x x + p_y y + p_z z)\,\boldsymbol{\gamma_w}$$

Gradient for f

$$\begin{aligned}\nabla f =& E_x\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_x}\\ &+ E_y\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_y}\\ &+ E_z\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_z}\\ &+ (-E + \phi q)\,\boldsymbol{\gamma_t} \wedge \boldsymbol{\gamma_w}\\ &+ B_z\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_y}\\ &- B_y\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_z}\\ &+ (A_x q - p_x)\,\boldsymbol{\gamma_x} \wedge \boldsymbol{\gamma_w}\\ &+ B_x\boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_z}\\ &+ (A_y q - p_y)\,\boldsymbol{\gamma_y} \wedge \boldsymbol{\gamma_w}\\ &+ (A_z q - p_z)\,\boldsymbol{\gamma_z} \wedge \boldsymbol{\gamma_w}\end{aligned}$$