Approximation - On-Policy Prediction

DEPARTMENT OF COMPUTER SCIENCE
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CSCI-531 - Reinforcement Learning

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The Tabular Method Approach

What are Tabular Methods?

- Store **exact value** $q_\pi(s, a)$ for **each** state-action pair

Q-Table

(s_1, a_1)	$q(s_1, a_1)$
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Activity

What issues can arise with this approach for large state spaces?

Problems with Tabular Methods

Scalability Issues

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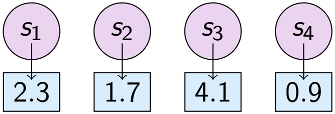
Solution: Function Approximation

The Fundamental Shift: Tables vs Functions

Tabular Method (What We've Done Before)

- **One number per state**: Store $v(s_1), v(s_2), \ldots, v(s_n)$



Tabular: Each state has its own value

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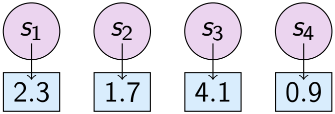

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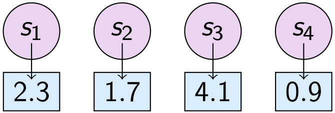

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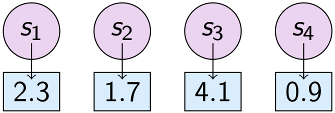


Tabular: Each state has its own value

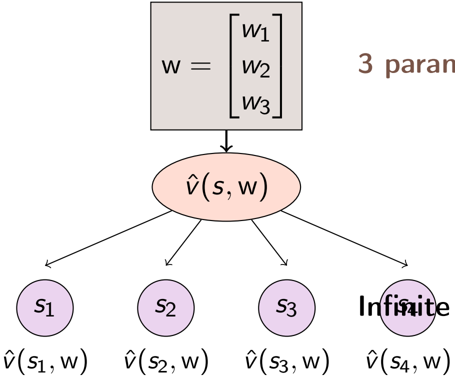
The Problem

What if we have 1 million states? We need 1 million separate values!

Function Approximation: The Core Idea

Key Innovation

Instead of storing individual values, use a **function** with **parameters**



3 parameters

Infinite states possible!

Why Function Approximation Works

The Magic of Shared Parameters

- **Shared parameters**: Same w computes values for ALL states

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Trade-off

We gain efficiency and generalization, but lose the guarantee of exact values

Concrete Example: GridWorld

Scenario: 4×4 GridWorld (16 states)

Tabular: Need 16 separate values

Function Approximation: Maybe just 3 parameters!

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GridWorld States

Function:

$$\hat{v}(x, y, \mathrm{w}) = w_1 \cdot x + w_2 \cdot y + w_3$$

$$\boxed{\mathrm{w} = \begin{bmatrix} 0.5 \\ -0.3 \\ 2.0 \end{bmatrix}}$$

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Key Insight

Changing w_1 affects the value of **all** states. This is both powerful and challenging.

Update Targets in Approximation

General Update Form

$$\textit{NewEstimate} \leftarrow \textit{OldEstimate} + \textit{StepSize}[\textit{Target} - \textit{OldEstimate}]$$

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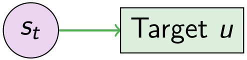

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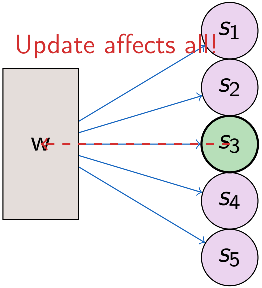
The Generalization Challenge

Key Problem

Updating one state affects the approximation for **all other states**!



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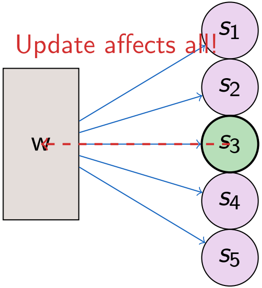

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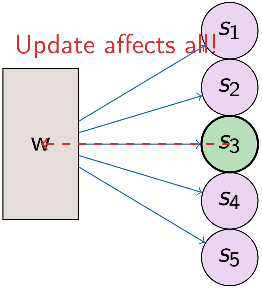

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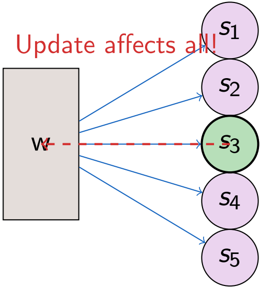

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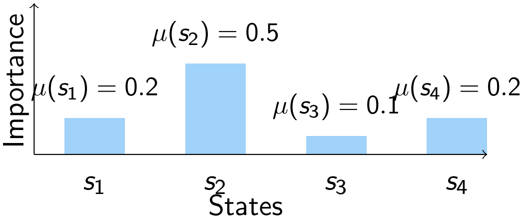
- Need to prioritize which states matter most

Defining State Importance

State Distribution $\mu(s)$

- $\mu(s) \geq 0$ for all states s



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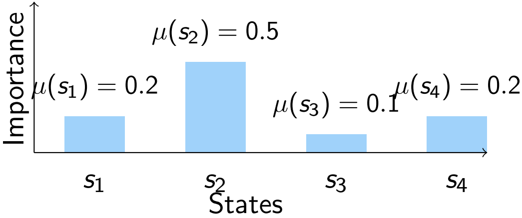

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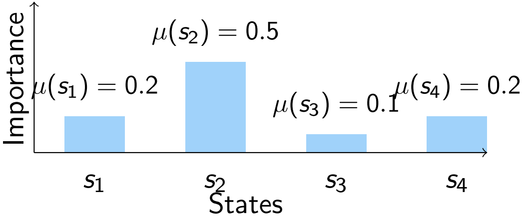

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- $\mu(s)$ = fraction of time spent in state s under policy π
- Focus learning on **frequently visited** states

The Objective Function

Mean Squared Value Error (MSVE)

$$\overline{\mathrm{VE}}(\mathbf{w}) = \sum_{s \in S} \mu(s) \left[v_\pi(s) - \hat{v}(s, \mathbf{w}) \right]^2 \tag{1}$$

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- $\mu(s)$: **State importance** weighting

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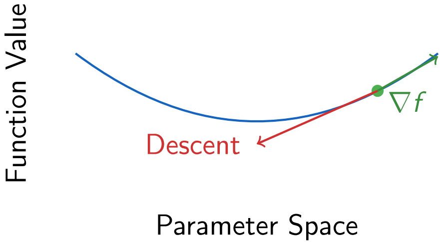
- Quality depends on function approximation choice

Gradient Descent Intuition

What is a Gradient?

- **Slope** of a function at a specific point



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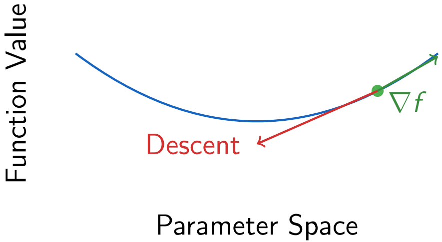

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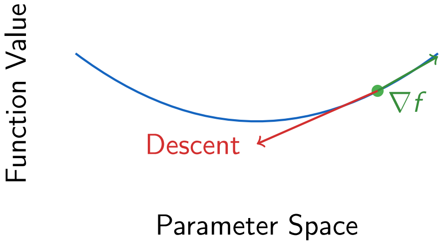

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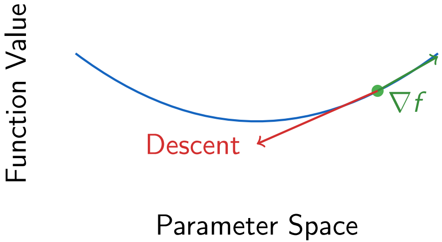

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Gradient Descent Update

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla f(\mathbf{w}_t)$$

where α is the **learning rate**

Example: Gradient Calculation - Setup

Function: $f(x, y) = 0.5x^2 + y^2$

Find the gradient at point $(x = 10, y = 10)$

Step 1: Partial Derivatives

$$\frac{\partial f}{\partial x} = x$$
$$\frac{\partial f}{\partial y} = 2y$$

Example: Gradient Calculation - Vector Form

Step 2: Gradient Vector

$$\nabla f(x, y) = \begin{bmatrix} x \\ 2y \end{bmatrix}$$

Step 3: Evaluate at $(10, 10)$

$$\nabla f(10, 10) = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

Activity: 2D Gradient Descent Practice

Your Turn

Execute two steps of gradient descent starting from $(x_0, y_0) = (2, 3)$ with $\alpha = 0.1$ for:

$$f(x, y) = x^2 + 2y^2 - 4x - 6y + 10$$

Steps to follow:

1. Find the gradient $\nabla f(x, y)$
2. Calculate (x_1, y_1) after first step
3. Calculate (x_2, y_2) after second step

Solution: Step 1 - Gradient Calculation

Gradient of $f(x, y) = x^2 + 2y^2 - 4x - 6y + 10$

Taking partial derivatives:

$$\frac{\partial f}{\partial x} = 2x - 4 \tag{4}$$

$$\frac{\partial f}{\partial y} = 4y - 6 \tag{5}$$

Gradient Vector

$$\nabla f(x, y) = \begin{bmatrix} 2x - 4 \\ 4y - 6 \end{bmatrix}$$

Solution: Step 2 - First Iteration

Starting Point: $(x_0, y_0) = (2, 3)$

Evaluate gradient at $(2, 3)$:

$$\nabla f(2,3) = \begin{bmatrix} 2(2) - 4 \\ 4(3) - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Update Rule: $\theta_{new} = \theta_{old} - \alpha \nabla f(\theta_{old})$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 0.1 \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 2.4 \end{bmatrix}$$

Solution: Step 3 - Second Iteration

Starting Point: $(x_1, y_1) = (2.0, 2.4)$

Evaluate gradient at $(2.0, 2.4)$:

$$\nabla f(2.0, 2.4) = \begin{bmatrix} 2(2.0) - 4 \\ 4(2.4) - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3.6 \end{bmatrix}$$

Second Update

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 2.4 \end{bmatrix} - 0.1 \begin{bmatrix} 0 \\ 3.6 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 2.04 \end{bmatrix}$$

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In RL, we want to learn value functions $v_\pi(s)$ or $q_\pi(s, a)$ but:

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Goal

Find weights w that make $\hat{v}(s, \mathrm{w}) \approx v_\pi(s)$ for all states

From Supervised Learning to RL

Supervised Learning Setting

- Have training data: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
- Know true labels y_i
- Minimize loss: $\sum_i L(f(x_i), y_i)$

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The Problem

We don't know the true values $v_\pi(s_i)$! That's what we're trying to learn!

SGD for Value Function Approximation

Assumptions

- Approximate value function $\hat{v}(s, \mathrm{w})$ is differentiable

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Gradient of Squared Error

$$\nabla \left[v_\pi(s_t) - \hat{v}(s_t, \mathrm{w})\right]^2 = -2\left[v_\pi(s_t) - \hat{v}(s_t, \mathrm{w})\right] \nabla \hat{v}(s_t, \mathrm{w})$$

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- States appear according to distribution μ

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Solution: Unbiased Target Estimates

Replace true value $v_\pi(s_t)$ with unbiased estimate U_t

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Key Insight

The target U_t must be an unbiased estimate of $v_\pi(s_t)$ for convergence guarantees

Target Approximations: Detailed Explanation

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For convergence, we need:

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Key Takeaway

There's a bias-variance tradeoff in choosing targets for SGD

Gradient Monte Carlo Method

Algorithm Gradient Monte Carlo for Value Function Approximation

Input: Policy π, number of episodes N, step size $\alpha \in (0, 1]$
Initialize: $\mathrm{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathrm{w} = 0$)
for N episodes **do**
    Generate episode using π: $S_0, A_0, R_1, S_1, \ldots, S_{T-1}, A_{T-1}, R_T$
    for each step $t = 0, 1, \ldots, T-1$ **do**
        $\mathrm{w} \leftarrow \mathrm{w} + \alpha[G_t - \hat{v}(S_t, \mathrm{w})]\nabla \hat{v}(S_t, \mathrm{w})$
    end for
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    for each step t = 0, 1, ..., T − 1 do
        w ← w + α[G_t − v̂(S_t, w)]∇v̂(S_t, w)
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Algorithm Semi-Gradient TD(0)

Input: Policy π, step size $\alpha \in (0, 1]$
Initialize: $\mathrm{w} \in \mathbb{R}^d$ arbitrarily
repeat
    Initialize S
    repeat
        $A \leftarrow \pi(\cdot|S)$
        Take action A, observe R, S'
        $\mathrm{w} \leftarrow \mathrm{w} + \alpha[R + \gamma \hat{v}(S', \mathrm{w}) - \hat{v}(S, \mathrm{w})]\nabla \hat{v}(S, \mathrm{w})$
        $S \leftarrow S'$
    until S is terminal
until convergence

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- Still converges but to different solution than true gradient

From State Values to Action Values

Extending to Control

- Need action-value function: $\hat{q}(s, a, \mathbf{w})$ instead of $\hat{v}(s, \mathbf{w})$

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General Update for Action Values

$$\mathrm{w}_{t+1} = \mathrm{w}_t + \alpha \left[U_t - \hat{q}(S_t, A_t, \mathrm{w}_t) \right] \nabla \hat{q}(S_t, A_t, \mathrm{w}_t)$$

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For one-step SARSA: $U_t = R_{t+1} + \gamma \hat{q}(S_{t+1}, A_{t+1}, \mathrm{w}_t)$

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- Use ϵ-greedy: $a^* = \arg\max_a \hat{q}(s, a, \mathbf{w})$

Semi-Gradient SARSA Algorithm

Input: $\hat{q} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^d \rightarrow \mathbb{R}$, step size α, $\epsilon > 0$
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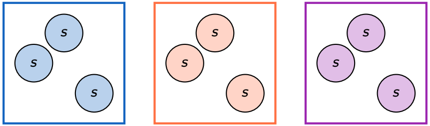
repeat
    Initialize S
    $A \leftarrow \epsilon$-greedy$(\hat{q}(S, \cdot, \mathrm{w}))$
    repeat
        Take action A, observe R, S'
        if S' is terminal **then**
            $\mathrm{w} \leftarrow \mathrm{w} + \alpha[R - \hat{q}(S, A, \mathrm{w})]\nabla\hat{q}(S, A, \mathrm{w})$
            break
        end if
        $A' \leftarrow \epsilon$-greedy$(\hat{q}(S', \cdot, \mathrm{w}))$
        $\mathrm{w} \leftarrow \mathrm{w} + \alpha[R + \gamma\hat{q}(S', A', \mathrm{w}) - \hat{q}(S, A, \mathrm{w})]\nabla\hat{q}(S, A, \mathrm{w})$
        $S \leftarrow S'$, $A \leftarrow A'$

State Aggregation

Basic Idea

- Group **similar states** together

State Groups



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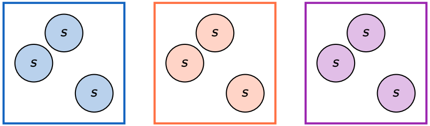

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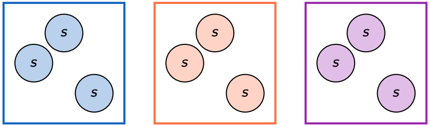

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Linear Methods

Linear Approximation

$$\hat{v}(s, \mathbf{w}) = \mathbf{w}^T \mathbf{x}(s) = \sum_{i=1}^{d} w_i x_i(s)$$

where $\mathbf{x}(s) = (x_1(s), x_2(s), \ldots, x_d(s))^T$ is the feature vector

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Higher-Order Polynomial

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