Chapter 9

Bezier Curves

Contents

9.1 Introduction

If we wish to store a curve or surface it's impractical to store every point. The solution is to somehow represent entire curves using small amounts of data. In general there are many ways to do this, for example a circle can be stored using a center and radius, a curve can be stored using a parametrization, and so on.

The approach we will take in this chapter is to define a curve using a number of control points. The simplest example would be defining a straight line using the two end points, and that's where we will start.

9.2 Bezier Curves

9.2.1 Linear Bezier Curves

Definition 9.2.1.1. Given two control points $\mathbf{b}_0$ and $\mathbf{b}_1$ we define the *linear Bezier curve* to be the curve parametrized by:

$$\mathbf{b}(t) = (1-t)\mathbf{b}_0 + t\mathbf{b}_1 \quad \text{for } t \in [0,1]$$

It's clear that this leads to a straight line from $\mathbf{b}_0$ to $\mathbf{b}_1$.

9.2.2 Quadratic Bezier Curves

Definition 9.2.2.1. Given three control points $\mathbf{b}_0$, $\mathbf{b}_1$, and $\mathbf{b}_2$ we define the *quadratic Bezier curve* (*degree 2 Bezier curve*) to be the curve parametrized by:

$$\mathbf{b}(t) = (1-t)^2\mathbf{b}_0 + 2t(1-t)\mathbf{b}_1 + t^2\mathbf{b}_2 \quad \text{for } t \in [0,1]$$

Notice what is happening here. At $t = 0$ the vector $\mathbf{b}_0$ is the only thing accounted for and at $t = 1$ the vector $\mathbf{b}_2$ is the only thing accounted for. For intermediate values of t the function $\mathbf{b}(t)$ assigns a shifting weight to the various control points.

Here are a sampling of the values of the coefficients for each of the three vectors:

t	$(1-t)^2$	$2t(1-t)$	t^2
0	1	0	0
0.1	0.81	0.18	0.01
0.2	0.64	0.32	0.04
0.3	0.49	0.42	0.09
0.4	0.36	0.48	0.16
0.5	0.25	0.50	0.25
0.6	0.16	0.48	0.36
0.7	0.09	0.42	0.36
0.8	0.04	0.32	0.64
0.9	0.01	0.18	0.81
1.0	0	0	1

We see that the coefficients start weighted to that of $\mathbf{b}_0$ and shift at the end to $\mathbf{b}_2$. They never get completely to $\mathbf{b}_1$ although they tend there in the middle. Let's look at a simple example in $\mathbb{R}^2$.

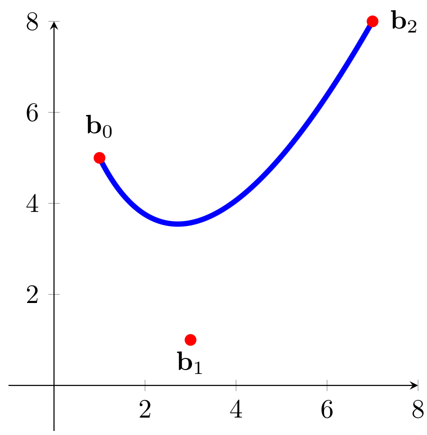
Example 9.1. Given the three control points:

$$\begin{aligned}\mathbf{b}_0 &= [1; 5]\\ \mathbf{b}_1 &= [3; 1]\\ \mathbf{b}_2 &= [7; 8]\end{aligned}$$

The quadratic Bezier curve is given by:

$$\begin{aligned}\mathbf{b}(t) &= (1-t)^2[1; 5] + 2t(1-t)[3; 1] + t^2[7; 8]\\ &= [2t^2 + 4t + 1; 11t^2 - 8t + 5]\end{aligned}$$

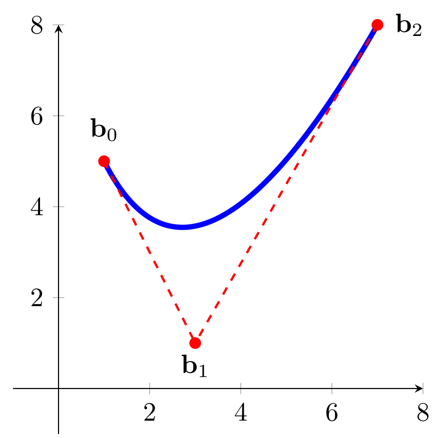
Here is a plot of the curve along with the three control points:



Notice the following:

- The curve passes through the two end control points. It's clear that this will always be the case since $\mathbf{b}(0) = \mathbf{b}_0$ and $\mathbf{b}(1) = \mathbf{b}_2$.
- The curve does not (necessarily) pass through the middle control point.
- The curve exits $\mathbf{b}_0$ in the direction of $\mathbf{b}_1$.
- The curve enters $\mathbf{b}_2$ from the direction of $\mathbf{b}_1$.

The final two notes can be clarified by drawing the lines tangent to the curve at the start and end control points:



This can be proven easily:

Theorem 9.2.2.1. For control points $\mathbf{b}_0$, $\mathbf{b}_1$ and $\mathbf{b}_2$ the lines tangent to the quadratic Bezier curve at $t = 0$ and $t = 1$ both intersect $\mathbf{b}_1$.

Proof. Observe that for $t = 0$ we have:

$$\begin{aligned}
\mathbf{b}(t) &= (1-t)^2\mathbf{b}_0 + 2t(1-t)\mathbf{b}_1 + t^2\mathbf{b}_2 \\
\mathbf{b}'(t) &= (2t-2)\mathbf{b}_0 + (2-4t)\mathbf{b}_1 + 2t\mathbf{b}_2 \\
\mathbf{b}'(0) &= -2\mathbf{b}_0 + 2\mathbf{b}_1 \\
\mathbf{b}'(0) &= 2(\mathbf{b}_1 - \mathbf{b}_0)
\end{aligned}$$

Since this is a multiple of $\mathbf{b}_1 - \mathbf{b}_0$ it follows that the vector tangent to the curve at $\mathbf{b}_0$ points from $\mathbf{b}_0$ towards $\mathbf{b}_1$. The proof for the other endpoint is similar. $\square$

Exercise 9.1. Prove the result for the other endpoint.

Exercise 9.2. Find the parametrization for the quadratic bezier curve with the three control points:

$$\begin{aligned}
\mathbf{b}_0 &= [1; 1] \\
\mathbf{b}_1 &= [10; 1] \\
\mathbf{b}_2 &= [10; 10]
\end{aligned}$$

9.2.3 Cubic Bezier Curves

Definition 9.2.3.1. Given four control points $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ we define the *cubic Bezier curve* (*degree 3 Bezier curve*) to be the curve parametrized by:

$$\mathbf{b}(t) = (1-t)^3\mathbf{b}_0 + 3t(1-t)^2\mathbf{b}_1 + 3t^2(1-t)\mathbf{b}_2 + t^3\mathbf{b}_3 \quad \text{for } t \in [0,1]$$

Let's look at a simple example in $\mathbb{R}^2$.

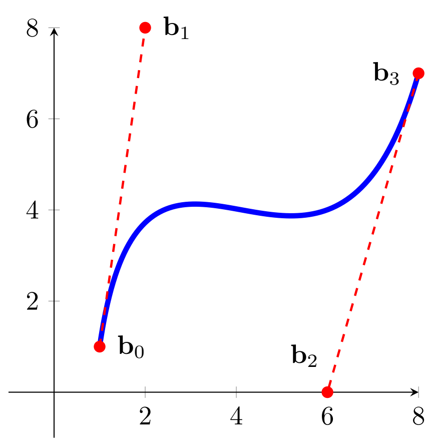
Example 9.2. Given the four control points:

$$\begin{aligned} \mathbf{b}_0 &= [1;1] \\ \mathbf{b}_1 &= [2;8] \\ \mathbf{b}_2 &= [6;0] \\ \mathbf{b}_3 &= [8;7] \end{aligned}$$

The cubic Bezier curve is given by:

$$\begin{aligned} \mathbf{b}(t) &= (1-t^3)[1;1] + 3t(1-t)^2[2;8] + 3t^2(1-t)[6;0] + t^3[8;7] \\ &= [-5t^3 + 9t^2 + 3t + 1; 30t^3 - 45t^2 + 21t + 1] \end{aligned}$$

Here is a plot of the curve along with the four control points. In addition we've added the tangent lines at the start and end points:



We see similar behavior as the quadratic in that the tangent lines at the start and end points intersubsection the point directly after and before them respectively.

Exercise 9.3. Find the parametrization for the cubic bezier curve with the four control points:

$$\begin{aligned}\mathbf{b}_0 &= [0; 0]\\ \mathbf{b}_1 &= [10; 0]\\ \mathbf{b}_2 &= [0; 10]\\ \mathbf{b}_3 &= [10; 10]\end{aligned}$$

Exercise 9.4. Find the parametrization for the cubic bezier curve with the four control points:

$$\begin{aligned}\mathbf{b}_0 &= [5; 5]\\ \mathbf{b}_1 &= [10; 5]\\ \mathbf{b}_2 &= [10; 10]\\ \mathbf{b}_3 &= [15; 10]\end{aligned}$$

The order of the points is of course important.

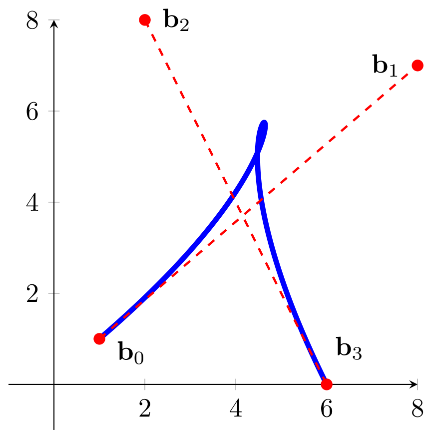
Example 9.3. Given the four control points from our first example but rearranged:

$$\begin{aligned}\mathbf{b}_0 &= [1; 1]\\ \mathbf{b}_1 &= [8; 7]\\ \mathbf{b}_2 &= [2; 8]\\ \mathbf{b}_3 &= [6; 0]\end{aligned}$$

The cubic Bezier curve is given by:

$$\begin{aligned}\mathbf{b}(t) &= (1 - t^3)[1; 1] + 3t(1 - t)^2[8; 7] + 3t^2(1 - t)[2; 8] + t^3[6; 0]\\ &= [23t^3 - 39t^2 + 21t + 1; -4t^3 - 15t^2 + 18t + 1]\end{aligned}$$

Here is a plot of the curve along with the four control points. In addition we've added the tangent lines at the start and end points:



We see similar behavior as the quadratic in that the tangent lines at the start and end points intersubsection the point directly after and before them respectively.

9.2.4 General Bezier Curves

Definition 9.2.4.1. Given $n+1$ control points $\mathbf{b}_0$, $\mathbf{b}_2$, ... ,$\mathbf{b}_n$ we define the *degree* n *Bezier curve* to be the curve parametrized by:

$$\mathbf{b}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \mathbf{b}_i \quad \text{for } t \in [0,1]$$

9.2.5 Properties of Bezier Curves

Bezier curves have several properties, some of which we've seen. We present them here as a series of theorems.

Theorem 9.2.5.1. The coefficients of $\mathbf{b}_0, ..., \mathbf{b}_n$ add to 1.

Proof. By the Binomial Theorem we have:

$$\sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} = (t + (1-t))^n = 1^n = 1$$

□

Theorem 9.2.5.2. Given n control points $\mathbf{b}_0$ through $\mathbf{b}_n$, the line tangent to $\mathbf{b}(t)$ at $\mathbf{b}_0$ passes through $\mathbf{b}_1$ and the line tangent to $\mathbf{b}(t)$ at $\mathbf{b}_n$ passes through $\mathbf{b}_{n-1}$.

Proof. We have:

$$\begin{aligned}
\mathbf{b}(t) &= (1-t)^n\mathbf{b}_0 + nt(1-t)^{n-1}\mathbf{b}_1 \\
&+ \left[\sum_{i=2}^{n-1}\binom{n}{i} t^i(1-t)^{n-i}\mathbf{b}_i\right] \\
&+ t^n\mathbf{b}_n \\
\mathbf{b}'(t) &= -n(1-t)^{n-1}\mathbf{b}_0 + n\left((1-t)^{n-1} - t(n-1)(1-t)^{n-2}\right)\mathbf{b}_1 \\
&+ \left[\sum_{i=2}^{n-1}\binom{n}{i}\left(it^{i-1}(1-t)^{n-i} - t^i(n-i)(1-t)^{n-i-1}\right)\mathbf{b}_i\right] \\
&+ nt^{n-1}\mathbf{b}_n \\
\mathbf{b}'(0) &= -n(1-0)^{n-1}\mathbf{b}_0 + n\left((1-0)^{n-1} - 0(n-1)(1-0)^{n-2}\right)\mathbf{b}_1 \\
&+ \left[\sum_{i=2}^{n-1}\binom{n}{i}\left(i0^{i-1}(1-0)^{n-i} - 0^i(n-i)(1-0)^{n-i-1}\right)\mathbf{b}_i\right] \\
&+ n0^{n-1}\mathbf{b}_n \\
\mathbf{b}'(0) &= -n\mathbf{b}_0 + n\mathbf{b}_1 \\
&= n(\mathbf{b}_1 - \mathbf{b}_0)
\end{aligned}$$

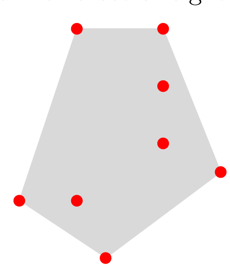
The result follows. The proof for $\mathbf{b}_n$ is similar. □

Exercise 9.5. Prove the result for $\mathbf{b}_n$.

Definition 9.2.5.1. Given a set of $n+1$ points $\mathbf{b}_0$, ..., $\mathbf{b}_n$ the *convex hull* of the points is defined as:

$$CH(\mathbf{b}_0, ..., \mathbf{b}_n) = \left\{ b_0\mathbf{b}_0 + .. + b_n\mathbf{b}_n \,|\, b_0, ..., b_n \in [0,1] \wedge \sum_{i=1}^{n} b_i = 1 \right\}$$

Visually speaking the convex hull can be pictured by stretching an elastic band so that it is as small as possible and still contains all the control points. For example here is the convex hull for a set of eight points:



Theorem 9.2.5.3. For all t, $\mathbf{b}(t)$ lies within the convex hull of the control

points.

Proof. Since $\mathbf{b}(t)$ is a subset of CH the result follows. $\square$

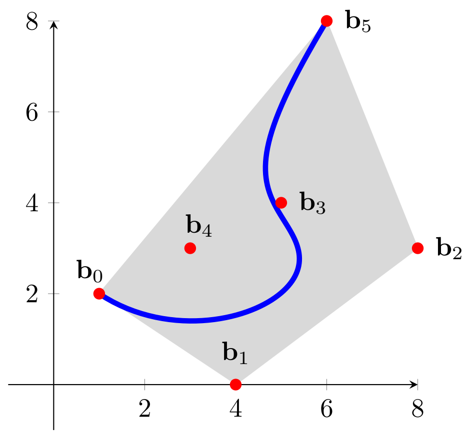
Example 9.4. Consider the six control points:

$$\begin{aligned}
\mathbf{b}_0 &= [1; 2] \\
\mathbf{b}_1 &= [4; 0] \\
\mathbf{b}_2 &= [8; 3] \\
\mathbf{b}_3 &= [5; 4] \\
\mathbf{b}_4 &= [3; 3] \\
\mathbf{b}_5 &= [6; 8]
\end{aligned}$$

The degree 5 Bezier curve is:

$$\mathbf{b}(t) = [-20t^5 + 80t^4 - 80t^3 + 10t^2 + 15t + 1; t^5 + 35t^4 - 70t^3 + 50t^2 - 10t + 2]$$

Here is a plot of the curve along with the six control points and the convex hull.



Theorem 9.2.5.4. For any linear transformation T the Bezier curve constructed using the $T(\mathbf{b}_i)$ is equivalent to the Bezier curve constructed using the $\mathbf{b}_i$ and then transformed using T.

Proof. This follows from the fact that the Bezier curve is constructed as a linear combination of the $\mathbf{b}_i$ and the fact that the transformation is linear. $\square$

9.2.6 Visual Construction

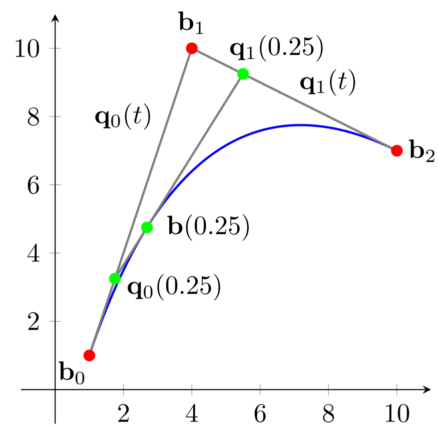
Visually speaking how is a Bezier curve being constructed? We'll describe this as we move through the degrees.

- A linear Bezier curve we understand, $\mathbf{b}(t)$ is simply the point t of the way along the straight line from $\mathbf{b}_0$ to $\mathbf{b}_1$.
- For a quadratic Bezier curve first construct the linear Bezier curve $\mathbf{q}_0(t)$ using $\mathbf{b}_0$ and $\mathbf{b}_1$ and construct the linear Bezier curve $\mathbf{q}_1(t)$ using $\mathbf{b}_1$ and $\mathbf{b}_2$. Then we find $\mathbf{b}(t)$ by interpolating t of the way between these two.

Example 9.5. Given the three control points:

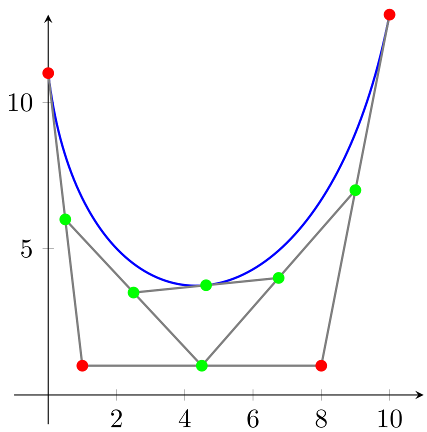
$$\begin{aligned}\mathbf{b}_0 &= [1; 1]\\ \mathbf{b}_1 &= [4; 10]\\ \mathbf{b}_2 &= [10; 7]\end{aligned}$$

To find $\mathbf{b}(0.25)$ we construct linear Bezier curves $\mathbf{q}_0(t)$ from $[1; 1]$ to $[4; 10]$ and $\mathbf{q}_1(t)$ from $[4; 10]$ to $[10; 7]$. We then go 0.25 of the way along the line from $\mathbf{q}_0(0.25)$ to $\mathbf{q}_1(0.25)$



- For a cubic Bezier curve we need to go recursively further. We first construct the linear Bezier curves $\mathbf{q}_0(t)$ using $\mathbf{b}_0$ and $\mathbf{b}_1$, $\mathbf{q}_1(t)$ using $\mathbf{b}_1$ and $\mathbf{b}_2$, and $\mathbf{q}_2(t)$ using $\mathbf{b}_2$ and $\mathbf{b}_3$. We next construct the linear Bezier surves $\mathbf{r}_0(t)$ using $\mathbf{q}_0(t)$ and $\mathbf{q}_1(t)$ and $\mathbf{r}_1(t)$ using $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$. Then we find $\mathbf{b}(t)$ by interpolating t of the way between these two.

Example 9.6. Here is an example without too much cluttering detail for the points $[0; 11]$, $[1; 1]$, $[8; 1]$, and $[10; 13]$ at $t = 0.5$.



- The pattern continues.

9.3 The de Casteljau Algorithm

Even though the formula for a Bezier curve is explicit and closed it is fairly computationally intensive. For any given t-value we must calculate a sum and each term in the sum involves an $\binom{n}{m}$ function which is itself fairly intensive.

Given a value of t the de Casteljau Algorithm can locate a specific point fairly efficiently. The algorithm is recursive as follows:

Theorem 9.3.0.1. Suppose $\mathbf{b}(t)$ is the degree n Bezier curve for control points $\{\mathbf{b}_0, ..., \mathbf{b}_n\}$. To find $\mathbf{b}(t)$ proceed as follows: Define:

$$\begin{aligned}
\mathbf{b}_{0,i} &= \mathbf{b}_i && \text{For } i = 0, ..., n \\
\mathbf{b}_{1,i} &= (1-t)\mathbf{b}0, i + t\mathbf{b}_{0,i+1} && \text{For } i = 0, ..., n-1 \\
\mathbf{b}_{2,i} &= (1-t)\mathbf{b}_{1,i} + t\mathbf{b}_{1,i+1} && \text{For } i = 0, ..., n-2 \\
&\vdots \\
\mathbf{b}_{j,i} &= (1-t)\mathbf{b}_{j-1,i} + t\mathbf{b}_{j-1,i+1} && \text{For } i = 0, ..., n-j \\
&\vdots \\
\mathbf{b}_{n,i} &= (1-t)\mathbf{b}_{n-1,i} + t\mathbf{b}_{n-1,i+1} && \text{For } i = 0
\end{aligned}$$

Then:

$$\mathbf{b}(t) = \mathbf{b}_{n,0}$$

Proof. Omitted. However the general idea can be seen in the visual construction from earlier. We contruct a succession of sets of intermediate points each as a linear combination of the previous set. □

While this may look particular confusing the algorithm can be displayed easily as a set of rows of numbers:

$$\begin{array}{lllllll}
\mathbf{b}_{0,0}=\mathbf{b}_0 & \mathbf{b}_{0,1}=\mathbf{b}_1 & \mathbf{b}_{0,2}=\mathbf{b}_2 & \dots & \mathbf{b}_{0,n-2}=\mathbf{b}_{n-2} & \mathbf{b}_{0,n-1}=\mathbf{b}_{n-1} & \mathbf{b}_{0,n}=\mathbf{b}_n \\
\mathbf{b}_{1,0} & \mathbf{b}_{1,1} & \mathbf{b}_{1,2} & \dots & \mathbf{b}_{1,n-2} & \mathbf{b}_{1,n-1} & \\
\mathbf{b}_{2,0} & \mathbf{b}_{2,1} & \mathbf{b}_{2,2} & \dots & \mathbf{b}_{2,n-2} & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
\mathbf{b}_{n-1,0} & \mathbf{b}_{n-1,1} & & & & & \\
\mathbf{b}_{n,0} & & & & & &
\end{array}$$

In the above, the first row is simply $\mathbf{b}_0$ through $\mathbf{b}_n$. Each entry in succesive rows is a linear combination of the entry directly above it and the entry above and to the right, using the weights $1-t$ and t respectively.

Example 9.7. Given the four control points:

$$\begin{aligned}
\mathbf{b}_0 &= [1;1] \\
\mathbf{b}_1 &= [2;8] \\
\mathbf{b}_2 &= [6;0] \\
\mathbf{b}_3 &= [8;7]
\end{aligned}$$

For reference the cubic Bezier curve is parametrized by:

$$\begin{aligned}
\mathbf{b}(t) &= (1-t^3)[1;1] + 3t(1-t)^2[2;8] + 3t^2(1-t)[6;0] + t^3[8;7] \\
&= [-5t^3+9t^2+3t+1; 30t^3-45t^2+21t+1]
\end{aligned}$$

Suppose we only needed to find $\mathbf{b}(0.1)$. Using the algorithm we proceed as follows:

$$\begin{array}{llll}
[1;1] & [2;8] & [6;0] & [8;7] \\
[1.1;1.7] & [2.4;7.2] & [6.2;0.7] & \\
[1.23;2.25] & [2.78;6.55] & & \\
[1.385;2.68] & & &
\end{array}$$

In this table, for example, the $b_{1,0}$ entry is $[1.1, 1.7] = (1-0.1)[1;1] + 0.1[2;8]$.

Thus $\mathbf{b}(0.1) = [1.385; 2.68]$. This checks out in the parametrization above.

Moreover the algorithm gives us a method of subdividing the curve. To give an example of how this would be useful, imagine an object in $\mathbb{R}^2$ described by a set of curves where each curve is the Bezier curve for a given set of control points. Suppose now that for just one of these curves we need to display part of the curve. Perhaps the rest is hidden. Suppose we only wish to display the part from $t=0$ to $t=0.1$. How could we isolate just that part?

Theorem 9.3.0.2. Suppose $\mathbf{b}(t)$ is the parametrization of the degree n Bezier curve with control points $\mathbf{b}_0$, ..., $\mathbf{b}_n$. Suppose $t_0 \in [0,1]$ is chosen. The curves parametrized by $\mathbf{b}(t)$ on $[0,t_0]$ and $\mathbf{b}(t)$ on $[t_0,1]$ are also degree n Bezier curves and moreover we may obtain control points for these two curves according to the following:

(a) For $\mathbf{b}(t)$ on $[0,t_0]$ use the left column of the table going downwards. In other words use:

$$\mathbf{b}_{0,0}, \mathbf{b}_{1,0}, \mathbf{b}_{2,0}, ..., \mathbf{b}_{n,0}$$

(b) For $\mathbf{b}(t)$ on $[t_0,1]$ use the diagonal edge of the table going up and to the right. In other words use:

$$\mathbf{b}_{n,0}, \mathbf{b}_{n-1,1}, \mathbf{b}_{n-2,2}, ..., \mathbf{b}_{1,n-1}, \mathbf{b}_{0,n}$$

Proof. Omit. □

Example 9.8. Using the above cubic if we split the Bezier curve at $t = 0.1$ then the curve parametrized by $\mathbf{b}(t)$ on $[0,0.1]$ can be contructed using control points:

$$[1;1], [1.1;1.7], [1.23;2.25], [1.385;2.68]$$

and the curve parametrized by $\mathbf{b}(t)$ on $[0.1,1]$ can be contructed using control points:

$$[1.385;2.68], [2.78;6.55], [6.2;0.7], [8;7]$$

9.4 TrueType Font Design

9.4.1 Font Specification

TrueType fonts are created using Bezier curives in that each letter is created by a set of linear and quadratic Bezier curves.

Example 9.9. For example suppose we wished to create a version of the Greek letter π. We might use a horizontal bar for the top, a vertical bar for the left lower part. and a vertical bar which curves to the right for the right lower part.

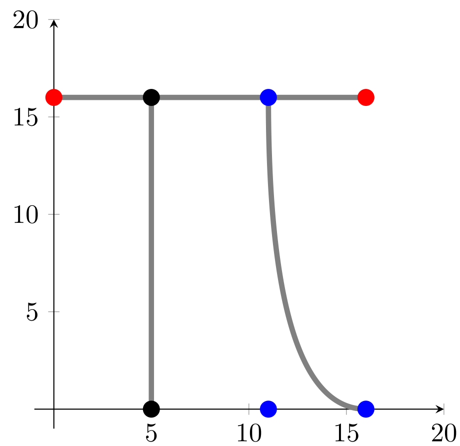
We might create each of these as follows in a 16×16 grid:

Horizontal bar: $[0\,;\,16]$ and $[16\,;\,16]$
Lower left: $[5\,;\,0]$ and $[5\,;\,16]$
Lower right: $[11\,;\,16]$ and $[11\,;\,0]$ and $(16\,;\,0]$.

If we create the Bezier curves for each of these we get the following.

Horizontal bar: $[16t\,;\,16]$
Lower left: $[5\,;\,16t]$
Lower right: $[5t^2+11\,;\,16t^2-32t+16]$

Here is the result. The control points have been drawn in red, black and blue for each of the horizontal, lower left and lower right respectively.



9.4.2 Transformations

Because Bezier curves may be tranformed by tranforming the control points, this allows us to translate, rotate, reflect, and scale by doing these things to the control points.

Example 9.10. Suppose we wish to rotate our letter π by 1.2 radians about the point $[2;3]$. To do this we simply rotate the points and recreate the curves. To rotate the points we multiply by the transformation $T(2,3)R(1.2)T(-2,-3)$.

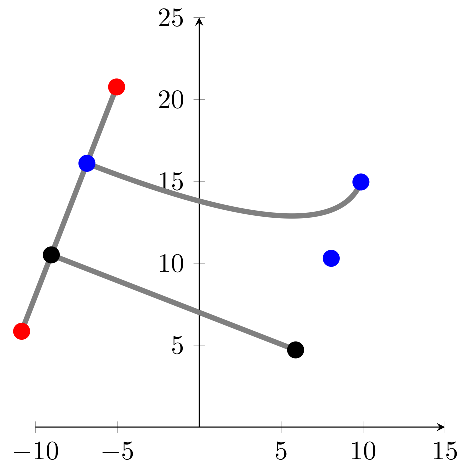
The resulting rotated points are:

Horizontal bar: $[-10.84\,;\,5.847]$ and $[-5.043\,;\,20.76]$
Lower left: $[5.883\,;\,4.709]$ and $[-9.029\,;\,10.51]$
Lower right: $[-6.855\,;\,16.1]$ and $[8.057\,;\,10.3]$ and $[9.869\,;\,14.96]$

If we create the Bezier curves for each of these we get the following.

Horizontal bar: $[5.798t-10.84\,;\,14.91t+5.847]$
Lower left: $[5.883-14.91t\,;\,5.798t+4.709]$
Lower right: $[-13.1t^2+29.83t-6.855\,;\,10.46t^2-11.6t+16.1]$

Here is the picture:


25
20
15
10
5
−10
−5
5
10
15

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