Chapter 3

Complex Numbers

Contents

3.1 Introduction

3.1.1 Definition and Properties

Definition 3.1.1.1. The *complex numbers* $\mathbb{C}$ are generated by defining $\mathbf{i} = \sqrt{-1}$ and then creating the set of all numbers of the form $a + b\hat{\boldsymbol{\imath}}$ where $a, b \in \mathbb{R}$. The value a is the *real part* and the value b is the *imaginary part*. For a complex number z these are denoted $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.

Standard operations on complex numbers arise obviously from those of real numbers and keeping in mind that $\hat{\boldsymbol{\imath}}^2 = -1$.

Example 3.1. For example $(2+3\hat{\imath})(4-5\hat{\imath}) = 8-10\hat{\imath}+12\hat{\imath}-15\hat{\imath}^2 = 23+2\hat{\imath}$.

Exercise 3.1. Calculate $(4+2\hat{\imath})(8-4\hat{\imath})$.

Definition 3.1.1.2. The *magnitude* (or *absolute value* or *norm*) of a complex number $z = a+b\hat{\imath}$ is denoted $|z|$ and is defined by $|z| = \sqrt{a^2+b^2}$.

Definition 3.1.1.3. A *unit complex number* is a complex number with norm equal to 1.

Definition 3.1.1.4. For a complex number $z = a+b\hat{\imath}$ we define the *complex conjugate* of z, denoted $\bar{z}$, by $\bar{z} = a-b\hat{\imath}$.

Note that for $z = a+b\hat{\imath}$ we have $z\bar{z} = a^2+b^2 = |z|^2$.

Theorem 3.1.1.1. For $z, w \in \mathbb{C}$ we have $\overline{zw} = \bar{z}\bar{w}$.

Proof. Omitted. ∎

Exercise 3.2. Complete the above proof.

The complex conjugate allows us to do division and resolve the result into an obvious complex number. In general for $z_1, z_2 \in \mathbb{C}$ we calculate and simplify

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\bar{z}_2}{\bar{z}_2} = \ldots$$

Example 3.2. To divide $2+3\hat{\imath}$ by $4-5\hat{\imath}$ we do:

$$\frac{2+3\hat{\imath}}{4-5\hat{\imath}} = \frac{2+3\hat{\imath}}{4+5\hat{\imath}}\left(\frac{4-5\hat{\imath}}{4-5\hat{\imath}}\right) = \frac{(2+3\hat{\imath})(4-5\hat{\imath})}{41} = \frac{23+2\hat{\imath}}{41} = \frac{23}{41}+\frac{2}{41}\hat{\imath}$$

Exercise 3.3. Calculate $\frac{3-4\hat{\imath}}{1+\hat{\imath}}$.

As a special case of the above we can always take the *reciprocal* (or multiplicative inverse) $1/z$ of a nonzero complex number.

Exercise 3.4. Find the reciprocal of $5-2\hat{\imath}$.

Both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ can be defined in terms of a complex number and its conjugate.

Theorem 3.1.1.2. For $z \in \mathbb{C}$ we have:

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + \bar{z}) \text{ and } \mathrm{Im}(z) = \tfrac{1}{2i}(z - \bar{z})$$

Proof. Obvious. □

Theorem 3.1.1.3. (Euler's Formula)
If the complex number $a + b\hat{\boldsymbol{\imath}}$ makes an angle of θ with the positive real axis and has norm r then we may rewrite it:

$$a + b\hat{\boldsymbol{\imath}} = re^{\hat{\boldsymbol{\imath}}\theta}$$

Proof. There are many ways to define e^z for $z \in \mathbb{C}$. One classic way is via the Taylor expansion:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

which converges for all $z \in \mathbb{C}$. Under this definition as well as the Taylor expansions for sine and cosine we get the result. □

Corollary 3.1.1.1. As a result of the previous theorem:

(a) For all θ,r we have $re^{i\theta} = r\cos\theta + \hat{\boldsymbol{\imath}}r\sin\theta$

(b) For all θ we have $e^{i\theta} = \cos\theta + \hat{\boldsymbol{\imath}}\sin\theta$

(c) If $z = a + b\hat{\boldsymbol{\imath}} = re^{\hat{\boldsymbol{\imath}}\theta}$ then $\bar{z} = a - b\hat{\boldsymbol{\imath}} = re^{-\hat{\boldsymbol{\imath}}\theta}$.

Example 3.3. The complex number $z = \sqrt{3} + 1\hat{\boldsymbol{\imath}}$ makes an angle of $\theta = \pi/6$ with the positive real axis and has magnitude 2. Therefore:

$$z = \sqrt{3} + 1\hat{\boldsymbol{\imath}} = 2e^{\hat{\boldsymbol{\imath}}(\pi/6)}$$

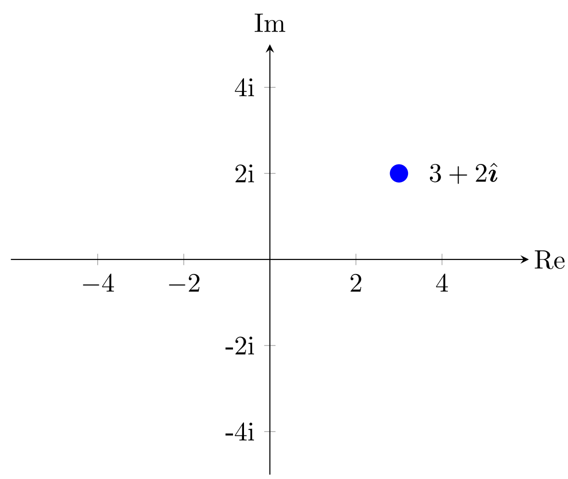
Exercise 3.5. Rewrite $\mathbf{v} = 5 + 5\hat{\boldsymbol{\imath}}$ in exponential form.

Exercise 3.6. Rewrite $\mathbf{v} = 4 + 4\sqrt{3}\hat{\boldsymbol{\imath}}$ in exponential form.

Exercise 3.7. Rewrite $5e^{\hat{\boldsymbol{\imath}}(5\pi/6)}$ in standard form.

3.1.2 Graphical Representation

A complex number $z = a + b\hat{\boldsymbol{\imath}}$ can be graphed by plotting the number in the plane using the x-axis as the real axis and the y-axis as the imaginary axis and plotting z at the location (a, b). For example here is the complex number $3 + 2\hat{\boldsymbol{\imath}}$:



It's this graphical representation that allows us to write manipulations of the plane as operations on complex numbers. These operations turn out to be quite simple and convenient.

3.1.3 Translations

So now given a point represented by a complex number it's clear that we can translate the point by adding another complex number. In other words to translate the point represented by $z \in \mathbb{C}$ by a units in the x-direction (real diretion) and b units in the y-direction we simply add:

$$z \mapsto z + (a + b\hat{\imath})$$

3.1.4 Scaling

If we take a complex number and multiply it by a real number then obviously we scale that complex number, moving it away from or towards the origin.

Example 3.4. Multiplying $2 + 3\hat{\imath}$ by 4 results in $4(2 + 3\hat{\imath}) = 8 + 12\hat{\imath}$ which is four times as far from the origin.

3.1.5 Rotations

But what happens graphically if we multiply by a complex number?

Consider the most basic example, multiplication by $\hat{\imath}$. Look at the point $(2, 3)$ represented by $2 + 3\hat{\imath}$. We calculate:

$$\hat{\boldsymbol{\imath}}(2+3\hat{\boldsymbol{\imath}}) = 2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\imath}}^2 = 2\hat{\boldsymbol{\imath}} + 3(-1) = -3 + 2\hat{\boldsymbol{\imath}}$$

Does this remind you of anything? It appears to rotate the point by $\frac{\pi}{2}$ counter-clockwise and in fact it does exactly this for any point:

$$\hat{\boldsymbol{\imath}}(a+b\hat{\boldsymbol{\imath}}) = -b + a\hat{\boldsymbol{\imath}}$$

If we stop and think for a second we might see a connection here. The point $\hat{\boldsymbol{\imath}}$ itself makes an angle of $\pi/2$ with the positive real axis and the result is a rotation by $\pi/2$ radians about the origin.

So how about multiplying by an arbitrary complex number?

Well for starters let's look at an arbitrary unit complex number that makes an angle of θ with the real axis. What happens if we multiply it by the complex number $a+b\hat{\boldsymbol{\imath}}$?

That unit complex number will have the form $q = \cos\theta + \hat{\boldsymbol{\imath}}\sin\theta$ so let's check what happens if we multiply such a unit complex number by $a+b\hat{\boldsymbol{\imath}}$.

$$(\cos\theta + \hat{\boldsymbol{\imath}}\sin\theta)(a+b\hat{\boldsymbol{\imath}}) = (a\cos\theta - b\sin\theta) + \hat{\boldsymbol{\imath}}(a\sin\theta + b\cos\theta)$$

This is really familiar. If we phrase this with points it states:

The point (a,b) goes to the point $(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$.

And if we phrase this with vectors it states:

The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ goes to the vector $\begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$.

So complex numbers can play the role of vectors in $\mathbb{R}^2$ and complex multiplication by unit complex numbers can substitute in for matrix multiplication when it comes to rotation.

Thus we have:

$$\mathrm{Rot}_\theta(a+b\hat{\boldsymbol{\imath}}) = (\cos\theta + \hat{\boldsymbol{\imath}}\sin\theta)(a+b\hat{\boldsymbol{\imath}})$$

Or alternately:

$$\mathrm{Rot}_\theta(z) = e^{\hat{\boldsymbol{\imath}}\theta} z$$

This latter form is really handy and makes it completely clear why the product of two rotations is a rotation.

We summarize all this in a theorem:

Theorem 3.1.5.1. Multiplication by $e^{\hat{\imath}\theta} = \cos\theta + \hat{\imath}\sin\theta$ rotates the complex plane counterclockwise about the origin by θ radians. Alternately put, multiplication by a unit complex number α rotates the complex plane counterclockwise about the origin by the angle that α makes with the positive real axis.

Proof. In the notes above. $\square$

Exercise 3.8. Find the result when $z = 10+7\hat{\imath}$ is rotated clockwise by an angle of $\pi/6$ about the origin.

Exercise 3.9. Find the result when $z = 3e^{\hat{\imath}(\pi/5)}$ is rotated counterclockwise by $5\pi/3$ about the origin.

Exercise 3.10. What happens if we multiply by a complex number which not a unit complex number? Give an example to clarify your answer.

Corollary 3.1.5.1. For any $z \in \mathbb{C}$, multiplication by $\bar{z}/|z|$ rotates the complex plane so that z lands on the positive real axis.

Proof. Multiplication by $\bar{z}/|z|$ rotates the plane clockwise about the origin by the angle which $\bar{z}$ makes with the positive real axis. $\square$

Example 3.5. Multiplication by $(2 - 3\hat{\imath})/\sqrt{13}$ rotates the plane clockwise so that $2 + 3\hat{\imath}$ moves to the positive real axis.

3.1.6 Reflections

How about reflections, specifically those in lines through the origin. Well complex conjugation reflects in the real axis since it negates the imaginary component. Consequently we can reflect in a line through the origin as we did with matrices and vectors, by first rotating the line to the real axis, then reflecting in the real axis, then rotating back.

A line through the origin in the complex plane can be represented by a single nonzero complex number just like a vector. So to reflect in the line represented by $a + b\hat{\imath} \neq 0$ we first find the corresponding angle θ and then for any z we can reflect it by first rotating the complex plane by $-\theta$, then conjugating, then rotating back.

$$\begin{aligned}
\text{Refl}_\theta(z) &= (\cos\theta + i\sin\theta)\overline{(\cos(-\theta) + \hat{\boldsymbol{\imath}}\sin(-\theta))z} \\
&= (\cos\theta + i\sin\theta)\overline{(\cos(\theta) - \hat{\boldsymbol{\imath}}\sin(\theta))z} \\
&= (\cos\theta + i\sin\theta)(\cos\theta + \hat{\boldsymbol{\imath}}\cos\theta)\bar{z} \\
&= \ldots \\
&= (\cos(2\theta) + \hat{\boldsymbol{\imath}}\sin(2\theta))\bar{z}
\end{aligned}$$

Which can also be written as:

$$\text{Refl}_\theta(x + y\hat{\boldsymbol{\imath}}) = (\cos(2\theta) + \hat{\boldsymbol{\imath}}\sin(2\theta))(x - y\hat{\boldsymbol{\imath}})$$

or as:

$$\text{Refl}_\theta(z) = e^{\hat{\boldsymbol{\imath}}\theta}\overline{e^{\hat{\boldsymbol{\imath}}(-\theta)}z} = e^{\hat{\boldsymbol{\imath}}\theta}e^{\hat{\boldsymbol{\imath}}\theta}\bar{z} = e^{2\hat{\boldsymbol{\imath}}\theta}\bar{z}$$

This is quite interesting because it asserts that reflecting in the line with angle θ is equivalent to taking the conjugate and rotating it by an angle of 2θ. This should not be at all surprising as we saw the same behavior with matrices in an earlier chapter.

Example 3.6. To find the result when $z = 2 + 5\hat{\boldsymbol{\imath}}$ is reflected in the line represented by $3 + 1\hat{\boldsymbol{\imath}}$ notice that although θ is not obvious or nice we do know that $\sin\theta = 1/\sqrt{10}$ and $\cos\theta = 3/\sqrt{10}$. Consequently:

$$\begin{aligned}
\sin(2\theta) &= 2\sin\theta\cos\theta = 2\left(1/\sqrt{10}\right)\left(3/\sqrt{10}\right) = 3/5 \\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta = \left(3/\sqrt{10}\right)^2 - \left(1/\sqrt{10}\right)^2 = 4/5
\end{aligned}$$

Thus reflection is given by the product:

$$\text{Refl}_\theta(x + y\hat{\boldsymbol{\imath}}) = \left(\frac{4}{5} + \frac{3}{5}\hat{\boldsymbol{\imath}}\right)(x - y\hat{\boldsymbol{\imath}})$$

and so

$$\text{Refl}_\theta(2 + 5\hat{\boldsymbol{\imath}}) = \left(\frac{4}{5} + \frac{3}{5}\hat{\boldsymbol{\imath}}\right)(2 - 5\hat{\boldsymbol{\imath}}) = \frac{23}{5} - \frac{14}{5}\hat{\boldsymbol{\imath}}$$

Exercise 3.11. Find the result when $z = 10 + 7\hat{\boldsymbol{\imath}}$ is reflected in the line which makes an angle of $\theta = \pi/3$ with the positive real axis.

Exercise 3.12. Find the result when $z = 3 - 1\hat{\boldsymbol{\imath}}$ is reflected in the line through the origin and through $5 + 8\hat{\boldsymbol{\imath}}$.

Exercise 3.13. As with matrices and vectors composing two reflections results in a rotation. Show how this works with complex numbers.

Exercise 3.14. Show algebraically that if $z \in \mathbb{C}$ is on the line represented by $z_0 \in \mathbb{C}$ that reflecting z in that line just returns z.

Exercise 3.15. Assuming $a + b\hat{\imath}$ makes an angle of θ with the positive real-axis rewrite the formula:

$$\mathrm{Refl}_\theta(x + y\hat{\imath}) = (\cos(2\theta) + \hat{\imath}\sin(2\theta))(x - y\hat{\imath})$$

without any sines or cosines.

Exercise 3.16. Consider the assertion:

$$\mathrm{Refl}_\theta(z) = e^{-i\left(\frac{\pi}{2}-\theta\right)}\left[-\overline{e^{i\left(\frac{\pi}{2}-\theta\right)}z}\right]$$

(a) Show algebraically that this assertion is true.

(b) Provide a geometric explanation for the assertion.
Hint: The original development used complex conjugation as a key point to reflect over the real axis. How would we go about reflecting over the complex axis and using this instead?

3.1.7 Combinations

We may of course combine these operations, for example to rotate about a point other than the origin we translate, rotate, and translate back.

Example 3.7. To rotate by $\pi/4$ about $1 + 2\hat{\imath}$ we first subtract $1 + 2\hat{\imath}$, multiply by $e^{\hat{\imath}(\pi/4)}$, and then add $1 + 2\hat{\imath}$. That is:

$$z \mapsto e^{\hat{\imath}(\pi/4)}(z - (1 + 2\hat{\imath})) + (1 + 2\hat{\imath})$$

Alternately we can rewrite $a + b\hat{\imath}$ in place of z, and $\cos(\pi/4) + \hat{\imath}\sin(\pi/4)$ in place of $e^{\hat{\imath}(\pi/4)}$ and then distribute and simplify.

Exercise 3.17. For the previous example do the rewrites and distribute and simplify.

Exercise 3.18. Find the resulting point when $z = 6 + 2\hat{\imath}$ is rotated by $11\pi/6$ clockwise around $5 + 3\hat{\imath}$. Write this in the form $a + b\hat{\imath}$.

Exercise 3.19. Find the resulting point when $z = -2 + 1\hat{\imath}$ is reflected in the line given in Euclidean coordinates by $y = 3x + 2$. Write this in the form $a + b\hat{\imath}$.

3.2 Representing Lines in the Complex Plane

There are several ways to represent linex in $\mathbb{C}$ and we'll investigate several of them here.

3.2.1 Lines Through the Origin

A line through the origin may be represented by either the angle θ_0 that it makes with the positive real axis or by a nonzero complex number which lies on the line.

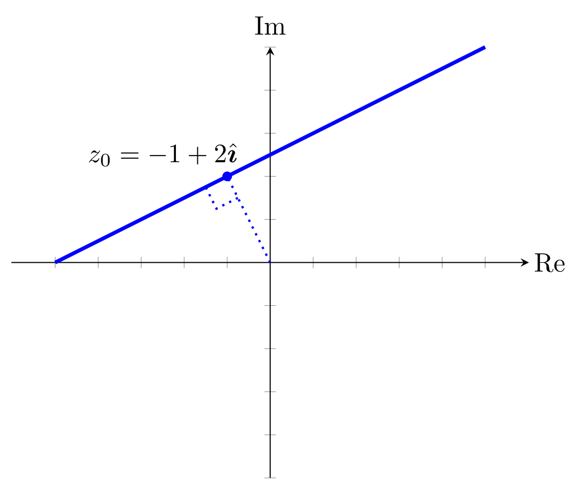
Notice that this is not unique unless we put restrictions on θ, like demanding $\theta \in [0, \pi)$ or $\theta \in (-\pi/2, pi/2]$.

3.2.2 Closest Point Representation

Lines not through the origin may (as in $\mathbb{R}^2$) be represented by the point on the line z_0 closest to the origin.

One great aspect of this approach is that the representation is unique, meaning each point gives only one line and each line only has one closest point.

For example the point $z_0 = -1 + 2\hat{\boldsymbol{\imath}}$ represents the line shown:



3.2.3 Transformations of Lines

It turns out that rotations and reflections are easy with both of these representations, and that translations are harder.

Clearly rotating a line through the origin about the origin results in another line through the origin and is an easy calculation.

Example 3.8. Rotating the line represented by $\theta_0 = \pi/6$ by an angle of $\theta = \pi/3$ results in a line represented by $\theta_0 + \theta = \pi/6 + \pi/3 = \pi/2$.

Reflecting a line through the origin about the origin can be calculated by creating a complex number from the angle and reflecting that. In other words an angle θ_0 can be represented by $e^{i\theta_0}$ and reflecting this in a line with angle θ yields:

$$e^{i2\theta}\overline{e^{i\theta_0}} = e^{i(2\theta-\theta_0)}$$

and so the mapping is:

$$\theta_0 \mapsto 2\theta - \theta_0$$

As with lines through the origin rotations work well with lines represented by their closest point. In fact it's straightfoward, we simply rotate the point that represents the line and we get the point that represents the rotated line.

Example 3.9. Consider the line represented by $z_0 = 5 + 2\hat{\boldsymbol{\imath}}$. If we rotate this by $\theta_0 = \pi/3$ about the origin the resulting line will be represented by:

$$(\cos(\pi/3) + \hat{\boldsymbol{\imath}}\sin(\pi/3))(5+2\hat{\boldsymbol{\imath}}) = \left(\frac{1}{2} + \hat{\boldsymbol{\imath}}\frac{\sqrt{3}}{2}\right)(5+2\hat{\boldsymbol{\imath}}) = \frac{5-2\sqrt{3}}{2} + \hat{\boldsymbol{\imath}}\frac{2+5\sqrt{3}}{2}$$

Similarly to reflect a line represented by z_0 we simply reflect z_0. That is:

$$z_0 \mapsto e^{i2\theta}\bar{z}_0$$

As with lines through the origin translations may or may not be straightforward.

There are four cases to deal with when it comes to translations:

(a) A line through the origin could translate to another line through the origin.

(b) A line not through the origin could translate to a line through the origin.

(c) A line not through the origin could translate to a line not through the origin.

(d) A line through the origin could translate to a line not through the origin.

Exercise 3.20. Under which conditions could each of the cases arise? Give a specific example of each.

We'll tackle these cases one-by-one.

(a) Suppose $\mathcal{L}$ passes through the origin and we translate by $\alpha \in \mathbb{C}$ such that $T(\mathcal{L})$ also passes through the origin.

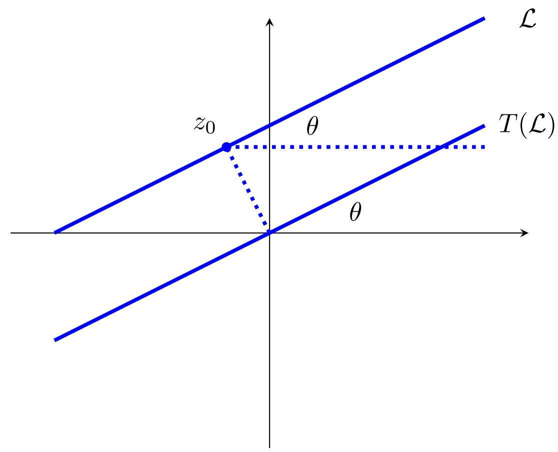
In this case $\mathcal{L}$ is represented by θ and α must translate along the same line in order for the line to remain on the origin. Therefore $T(\mathcal{L})$ is also represented by θ because it's the same line.

Exercise 3.21. Suppose a line represented by $\theta = \pi/4$ is translated by $\alpha = -3 - 3\hat{\boldsymbol{\imath}}$. Which angle represents the translated line?

(b) Suppose $\mathcal{L}$ does not pass through the origin and we translate by α such that $T(\mathcal{L})$ passes through the origin.

Suppose $\mathcal{L}$ is represented by closest point z_0.

Consider the following picture:



In this case the angle θ that $T(\mathcal{L})$ makes with the positive real axis is the same angle that $\mathcal{L}$ makes with the horizontal. This angle equals the angle that z_0 makes minus $\pi/2$:

$$\theta = \arg(z_0) - \frac{\pi}{2} = \text{atan2}\left(\text{Im}(z_0), \text{Re}(z_0)\right) - \frac{\pi}{2}$$

Note: The atan2(y, x) function returns the angle from the positive x-axis to the ray from the origin to (x, y). It's more succinct to use arg but often atan2 is found in programming languages.

Note also that this may not give a result in the desired range for θ. Depending on our desired range we might need other manipulations.

Exercise 3.22. Suppose the line represented by $2 - 2\hat{\imath}$ is translated by $\alpha = 3$. Which angle represents the translated line?

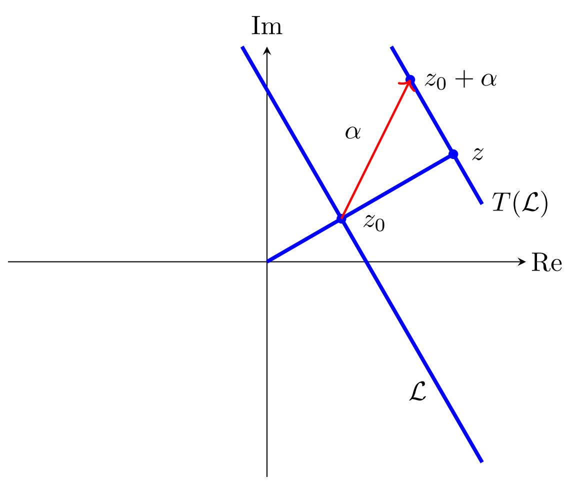
(c) Suppose $\mathcal{L}$ does not pass through the origin and we translate by α such that $T(\mathcal{L})$ does not pass through the origin.

Theorem 3.2.3.1. Suppose L does not pass through the origin and has closest point z_0. Let T denote translation by $\alpha \in \mathbb{C}$. Assuming $T(\mathcal{L})$ does not contain the origin show that the closest point representing $T(\mathcal{L})$ is:

$$z_0 + \frac{z_0}{|z_0|^2}\mathrm{Re}(\bar{z}_0\alpha)$$

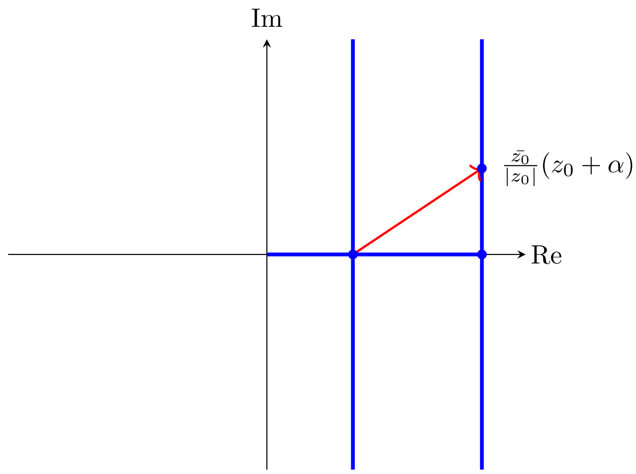
Proof. Consider the following picture where the line $\mathcal{L}$ has been translated by α to the line $T(\mathcal{L})$.

The goal is to find the point z.



We proceed by first rotating the picture so that the line joining 0 to z_0 to z lies along the positive real axis. We can do this by applying the clockwise rotation resulting from multiplication by $\bar{z}_0/|z_0|$.

We have marked the most important point.



The point z is obtained by taking our most important point, taking the real part (dropping to the real axis), and then rotating back. The result is:

$$\begin{aligned} z &= \frac{z_0}{|z_0|}\mathrm{Re}\left(\frac{\bar{z_0}}{|z_0|}(z_0+\alpha)\right) \\ &= \frac{z_0}{|z_0|^2}\mathrm{Re}\left(\bar{z_0}(z_0+\alpha)\right) \\ &= \frac{z_0}{|z_0|^2}\mathrm{Re}\left(|z_0|^2+\bar{z_0}\alpha\right) \\ &= \frac{z_0}{|z_0|^2}\left[|z_0|^2+\mathrm{Re}\left(\bar{z_0}\alpha\right)\right] \\ &= z_0+\frac{z_0}{|z_0|^2}\mathrm{Re}(\bar{z_0}\alpha) \end{aligned}$$

□

Exercise 3.23. Suppose the line represented by closest point $z_0 = -1-3\hat{\boldsymbol{\imath}}$ is translated by $\alpha = 2+\hat{\boldsymbol{\imath}}$. Which point represents the translated line?

Exercise 3.24. Suppose the line represented by closest point $z_0 = 2e^{\hat{\boldsymbol{\imath}}\pi/5}$ is translated by $\alpha = 3e^{\hat{\boldsymbol{\imath}}\pi/3}$. Which point represents the translated line?

(d) Supose $\mathcal{L}$ passes through the origin and we translate by α such that $T(\mathcal{L})$ does not pass through the origin.

Here's a warm up:

Exercise 3.25. Suppose $\mathcal{L}$ is represented by $1+1\hat{\boldsymbol{\imath}}$. If T is translation by $1+0\hat{\boldsymbol{\imath}}$, which closest point represents $T(\mathcal{L})$? This can be done by inspection so draw a picture.

Exercise 3.26. Suppose $\mathcal{L}$ is represented by $1 + 2\hat{\boldsymbol{\imath}}$. If T is translation by $1 - 7\hat{\boldsymbol{\imath}}$, which closest point represents $T(\mathcal{L})$? This probably cannot be done by inspection but a picture can help you figure it out.

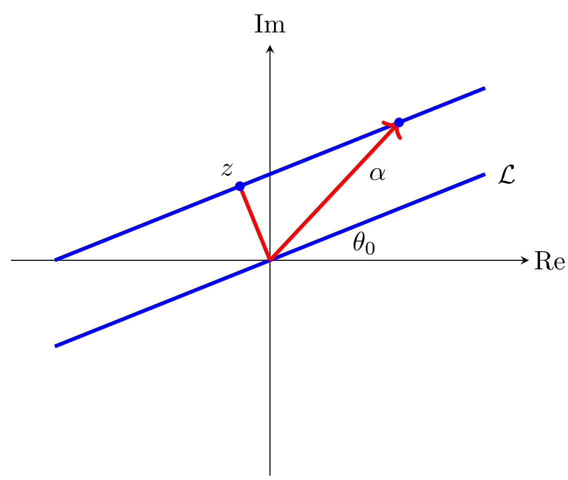
Theorem 3.2.3.2. Suppose $\mathcal{L}$ lies through the origin and makes an angle of θ with the positive real axis. Let T denote translation by $\alpha \in \mathbb{C}$. Assuming $T(\mathcal{L})$ does not contain the origin show that the closest point representing $T(\mathcal{L})$ is:

$$z = \hat{\boldsymbol{\imath}} e^{\hat{\boldsymbol{\imath}}\theta} \mathrm{Im}\left(\alpha e^{-\hat{\boldsymbol{\imath}}\theta}\right)$$

Note that if θ is not given but instead $\mathcal{L}$ is given by a point on the line then this formula can be rewritten using sines and cosines to avoid having to find θ explicitly.

Proof. Omit. □

Exercise 3.27. Complete the above proof.
Hint: This is not hard and is based on the following picture. The goal in the picture is to find z given α and θ_0. Think about rotating $\mathcal{L}$ to the positive real axis first. If you're totally confused you might look at (d) first since the proof is given. Note that (c) was in fact a harder proof and not quite the same but it might give you some ideas.



Exercise 3.28. In what way does the previous theorem/proof/exercise fail to work when $T(\mathcal{L})$ contains the origin?

Exercise 3.29. Suppose a line $\mathcal{L}$ represented by $\theta_0 = 2\pi/3$ is translated by $\alpha = 4 - 3\hat{\boldsymbol{\imath}}$. Which point represents the translated line?

Exercise 3.30. Suppose a line $\mathcal{L}$ passes through the origin and through

$2 - 1\hat{\imath}$. If this line is translated by $\alpha = 5 + 2\hat{\imath}$ which point represents the translated line?

Summary:

In summary we have:

Through Origin to Through Origin	$\theta_0 \mapsto \theta = \theta_0$
Not Through Origin to Through Origin	$z_0 \mapsto \theta = \arg(z_0) - \frac{\pi}{2}$
Not Through Origin to Not Through Origin	$z_0 \mapsto z = z_0 + \frac{z_0}{\lvert z_0\rvert^2}\mathrm{Re}(\bar{z_0}\alpha)$
Through Origin to Not Through Origin	$\theta_0 \mapsto z = \hat{\imath}e^{\hat{\imath}\theta_0}\mathrm{Im}\left(\alpha e^{-\hat{\imath}\theta_0}\right)$

3.2.4 More on Transformation of Lines

Of course this can get even more contorted if, for example, We want to rotate a line about a point which is not the origin. Naturally we know how to do it, we translate so the point is at the origin, then rotate, then translate back. However we need to understand that either of these two translations could result in any of the four complications which arose in the previous section. Of course it's worth noting that most lines (in some statistical sense) don't intersect the origin.

Exercise 3.31. Find the resulting line (either represented by an angle or point, whichever is appropriate) when the line represented by $3 + 2\hat{\imath}$ is rotated by $\pi/6$ about the point $1 + 0\hat{\imath}$.

Exercise 3.32. Find the resulting line (either represented by an angle or point, whichever is appropriate) when the line represented by $3 + 2\hat{\imath}$ is reflected in the line with Euclidean equation $y = 2x + 1$.

Exercise 3.33. Let $\mathcal{L}$ be the line represented by $1 + 2\hat{\imath}$. If we rotate $\mathcal{L}$ about $1 + 1\hat{\imath}$ by θ radians, what must θ be to ensure that the rotated line meets the origin?

3.2.5 Locating Points on Lines

Using the angle representation of a line through the origin makes it fairly easy to see if a point is on that line, we simply check if the point makes the same angle as the line.

However for the closest point representation of a line not through the origin it's not so obvious. It's not hard, it's just not obvious.

Example 3.10. As an example if $-1 + 2\hat{\imath}$ represents a line, how can we tell if $10 + 18\hat{\imath}$ is on the line? One way would be to subtract them and see if the result is perpendicular to $-1+2\hat{\imath}$ treating complex numbers as vectors. To frame this differently, we could find the slope between the points and see if it equals the negative reciprocal of the slope joining $-1+2\hat{\imath}$ to the origin.

The downside to this approach is that we move out of the complex numbers to do the work. Another approach would be to rotate the point of interest by $\pi/2$ about the representing point and see if the result is a scalar multiple of the representing point.

Example 3.11. To check if $10+18\hat{\imath}$ is on the line represented by $-1+2\hat{\imath}$ we calculate:

$$i(10+18\hat{\imath}-(-1+2\hat{\imath}))+(-1+2\hat{\imath})=-17+13\hat{\imath}$$

and observe that this is not a scalar multiple of $-1+2\hat{\imath}$ and hence $10+18\hat{\imath}$ is not on this line.

Example 3.12. To check if $5+5\hat{\imath}$ is on the line represented by $-1+2\hat{\imath}$ we calculate:

$$i(5+5\hat{\imath}-(-1+2\hat{\imath}))+(-1+2\hat{\imath})=-4+8\hat{\imath}$$

and observe that this is a scalar multiple of $-1+2\hat{\imath}$ and hence $5+5\hat{\imath}$ is on this line.

Exercise 3.34. Use this approach to check if the following points are on the line represented by $3-1\hat{\imath}$:

(a) $6+18\hat{\imath}$

(b) $-4-8\hat{\imath}$

(c) $-2-6\hat{\imath}$

Exercise 3.35. Use this approach to find a generic criteria under which $x+y\hat{\imath}$ is on the line represented by closest point $a+b\hat{\imath}\neq 0$. Your criteria should read something like:

$x+y\hat{\imath}$ is on the line represented by $a+b\hat{\imath}$
iff
???? is a scalar multiple of ???

Note that checking whether $\mathbf{v}$ and $\mathbf{w}$ are scalar multiples is an existence check but it can be rephrased as an equality check since $\mathbf{v}$ and $\mathbf{w}$ are scalar multiples if when normalized they are either the same or opposites.

3.2.6 Parametrization

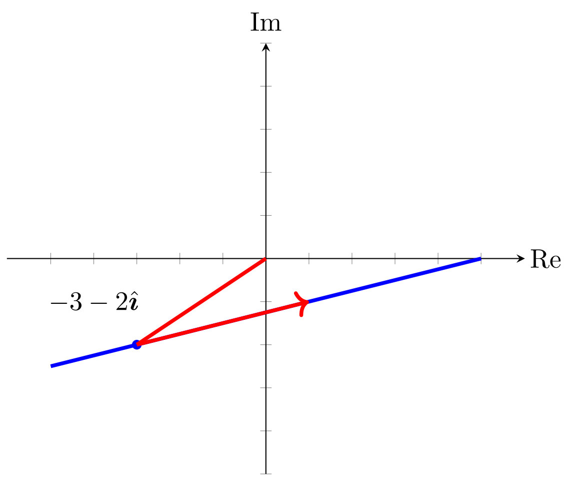
As a final comment we can represent a line by a point $z_0 \in \mathbb{C}$ and an additional $\alpha \in \mathbb{C}$ which indicates a direction. This pair (z_0, α) then represents a line in that all points $z = z_0 + \alpha t$ with $t \in \mathbb{R}$.

The disadvantage to this is the introduction of another variable in the way we're describing the line in the sense that calculations involving the line almost always require this other variable. For example checking if a point is on the line involves solving for t. We're not implying this is good or bad, just that it's an added factor.

Example 3.13. The line represented by the pair:

$$(z_0, \alpha) = (-3 - 2\hat{\boldsymbol{\imath}}, 4 + 1\hat{\boldsymbol{\imath}})$$

is shown here:



Interestingly this representation is strongly tied into the closest point parametrization in that if z_0 is the closest point on $\mathcal{L}$ to the origin then the line can be written as the set of points

$$z = z_0 + \hat{\boldsymbol{\imath}} z_0 t$$

This is fairly clear. If z_0 is the closest point then the line through the origin and z_0 is perpendicular to $\mathcal{L}$. This perpendicular line consists of points $z_0 t$ and when we do $\hat{\boldsymbol{\imath}} z_0 t$ we rotate by $\pi/2$ and get points through the origin perpendicular to the line. The z_0+ simply translates those points to lie on the point z_0.

This representation is easily understood because it's similar to the traditional parametrization of lines used in courses like Calculus 3:

$$\mathbf{r}(t) = (x_0 + at)\hat{\boldsymbol{\imath}} + (y_0 + bt)\hat{\boldsymbol{\jmath}} + (z_0 + ct)\hat{\boldsymbol{k}}$$

The major downside to this representation is the fact that it involves another variable t. Consequently finding points on the line and checking if points are on the line invariably results in solving for t, which can be tricky.

On the other hand it's fairly easy to apply transformatons to parametrized lines because the parametrization $z = z_0 + \alpha t$ explicitly returns complex numbers which can then be operated on individually.

Example 3.14. Consider the line parametrized by $z = (2 + 3\hat{\boldsymbol{\imath}}) + (4 - 5\hat{\boldsymbol{\imath}})t$. If we rotate this about the origin by $\pi/4$ radians the result can be written a number of was. For example here's one way:

$$\begin{aligned}(\cos(\pi/4) + \hat{\boldsymbol{\imath}}\sin(\pi/4))z &= (\cos(\pi/4) + \hat{\boldsymbol{\imath}}\sin(\pi/4))\left((2 + 3\hat{\boldsymbol{\imath}}) + (4 - 5\hat{\boldsymbol{\imath}})t\right)\\ &= (\sqrt{2}/2 + \hat{\boldsymbol{\imath}}\sqrt{2}/2)\left((2 + 3\hat{\boldsymbol{\imath}}) + (4 - 5\hat{\boldsymbol{\imath}})t\right)\\ &= (-\sqrt{2}/2 + 5\hat{\boldsymbol{\imath}}\sqrt{2}/2) + (9\sqrt{2}/2 - \hat{\boldsymbol{\imath}}\sqrt{2}/2)t\end{aligned}$$

The rewrite is done so that it's clear from the result what the anchor point and the direction complex number are. In other words the result has been rewritten in the standard form.

Exercise 3.36. Find the result when the line $z = (5+2\hat{\boldsymbol{\imath}})+(4-3\hat{\boldsymbol{\imath}})t$ is translated by $3 + 7\hat{\boldsymbol{\imath}}$.

Exercise 3.37. Find the result when the line $z = (5+2\hat{\boldsymbol{\imath}})+(4-3\hat{\boldsymbol{\imath}})t$ is reflected in the line represented by $\bar{v} = 4 + 1\hat{\boldsymbol{\imath}}$.
Note: If you choose your approach carefully this isn't particularly bad and the result simplifies.

3.3 Complex Affine Transformations

As we've seen, rotations are performed by multiplication by unit complex numbers, scaling by multiplication by real numbers, and translation by addition of complex numbers.

It follows that this comprehensively covers all transformations which satisy the following definition:

Definition 3.3.0.1. A *complex affine transformation* is a transformation $\phi : \mathbb{C} \to \mathbb{C}$ which may be written in the from $\phi(z) = az + b$ where $a, b \in \mathbb{C}$.

Theorem 3.3.0.1. The composition of two affine transformations is an affine transformation.

Proof. Omitted. □

Exercise 3.38. Show that the above theorem is true.

From this we get the following:

Theorem 3.3.0.2. Every combination of rotations, scalings and translations is affine and every affine transformation is a combination of those.

Proof. The first part follows from the previous theorem. To see the second part we simply take an affine transformation and rewrite it:

$$z \mapsto az + b = |a|\frac{a}{|a|}z + b$$

From here we can see that z undergoes first a rotation, then a scaling, then a translation. □

Not all transformations $\mathbb{C} \to \mathbb{C}$ are affine, however. In fact reflections are not, and even complex conjugation is not.

Exercise 3.39. Show that complex conjugation is not affine. One way to do this is to assume it is and find a problem. If it were, this would mean that there are $a, b \in \mathbb{C}$ such that $\phi(z) = az + b$ satisfies $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(\hat{\boldsymbol{\imath}}) = -\hat{\boldsymbol{\imath}}$. Show why this would be a problem.

Exercise 3.40. Show that reflections in general are not affine.

Exercise 3.41. Show that the following transformations are not affine:

(a) The mapping $x + y\hat{\boldsymbol{\imath}} \mapsto x - y\hat{\boldsymbol{\imath}}$.

(b) The mapping $x + y\hat{\boldsymbol{\imath}} \mapsto x + 2y\hat{\boldsymbol{\imath}}$.

(c) The mapping $x + y\hat{\boldsymbol{\imath}} \mapsto (x + y) + y\hat{\boldsymbol{\imath}}$.

It's worth noting that in $\mathbb{R}^2$ all of the above are affine (we'll say $\mathbb{R}^2$ affine) in the sense that they may be written as:

$$\phi(\mathbf{v}) = A\mathbf{v} + \mathbf{b} \text{ with } A \text{ a } 2 \times 2 \text{ matrix and } \mathbf{b} \in \mathbb{R}^2$$

For example the second from the exercises above is easy to see:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$

It follows that the notion of being affine depends upon how the space is being managed.

Exercise 3.42. Show that the three problems from the last exercise are $\mathbb{R}^2$ affine.

Exercise 3.43. Show that the mapping $x + y\hat{\imath} \mapsto x^2 + y\hat{\imath}$ is not complex affine nor $\mathbb{R}^2$ affine.

Which really goes to show that the complex numbers are in some ways at a bit of a disadvantage. There are parallels between $\mathbb{R}^2$ and $\mathbb{C}$ but they're certainly not equivalent in terms of the ease of computations that we need to do.

3.4 Matrix Representation of Complex Numbers

We've seen in the chapter some parallels of how matrices and complex numbers may achieve the same goals although phrased somewhat differently.

Interestingly complex numbers themselves may be represented as matrices. If we set up the following representation:

$$a + b\hat{\imath} \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Then we have:

Theorem 3.4.0.1. Under the correspondance given above we have:

(a) Addition of complex numbers is represented by addition of matrices.

(b) Multiplication of complex numbers is represented by multiplication of matrices.

(c) The inverses of a complex number is represented by the inverse of a matrix.

(d) Division of complex numbers is taken care of by inverses.

(e) Complex conjugation is represented by the matrix transpose.

(f) The magnitude of a complex number is represented by the determinant of a matrix.

Proof. Here's a proof of (b): Observe that:

$$(a + b\hat{\imath})(c + d\hat{\imath}) = (ac - bd) + (ad + bc)\hat{\imath}$$

and that:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{bmatrix}$$

We immediately see that the representation of the product matches the product.

□

For example rotation of $a + b\hat{\boldsymbol{\imath}}$ by θ degrees was given by:

$$\text{Rot}_\theta(a + b\hat{\boldsymbol{\imath}}) = (\cos\theta + \hat{\boldsymbol{\imath}}\sin\theta)(a + b\hat{\boldsymbol{\imath}})$$

In the language of matrices this becomes:

$$\text{Rot}_\theta\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Not only is this useful because it reduces much of the handling of complex numbers to basic matrix calculations but this property will extend to quaternions, as we will see later.

Exercise 3.44. Prove (a),(c),(d),(e),(f) of the theorem above. All of them are fairly straighforward.

3.5 Taking this to 3D

Given that complex numbers can at most represent two dimensions via the real and imaginary parts it's not clear how this could extend to three dimensions. For that we need quaternions, which extend complex numbers.

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