Chapter 6

Real Projective 2-Space

Contents

6.1 Introduction and Notation

The basic premise here is that $\mathbb{R}^3$ isn't quite good enough. The term "good enough" means that working with certain methods and objects in $\mathbb{R}^3$ is inconvenient and inefficient. Ideally we'd like to represent as much as possible using linear algebra for speed and efficiency.

In an analogous fashion $\mathbb{R}^2$ is not good enough. We saw this in an earlier chapter when we observed that perspective projection cannot be represented by matrix multiplication as it is not linear and likewise for translations as well as many other transformations.

In this chapter we will primarily lay out all the technical details for $\mathbb{R}^2$ and then follow up with the $\mathbb{R}^3$ version with not as much proof. The proofs are essentially the same, so little is missed.

The solution will be to introduce a larger space in which $\mathbb{R}^2$ is essentially a subset. We'll develop methods on this larger set and see that they affect our subset in a useful way.

Extending $\mathbb{R}^3$ to a "larger" space works analogously.

Since we're going to be using lots of vertical vectors I'm going to use the notation $[x; y; z]$ to mean $[x, y, z]^T$. This is basically borrowing notation from Matlab.

6.2 Real Projective 2-Space

6.2.1 Definition

Definition 6.2.1.1. Define *real projective 2-space* denoted $\mathbb{RP}^2$ as the set of nonzero vectors $[X; Y; Z]$ in $\mathbb{R}^3$ with the condition that two vectors are considered to be equivalent if they are nonzero multiples of one another.

The reason for using the "2" exponent is that this newly constructed space is actually two-dimensional due to the degree of freedom lost due to the equivalence.

There are many other ways to define $\mathbb{RP}^2$, some of which will be mentioned later in the chapter, but they are all equivalent. For our purposes this is the most convenient.

Definition 6.2.1.2. We then say that a *projective point* (or just a point when the context is clear) $\mathbf{P}$ in $\mathbb{RP}^2$ is an equivalence class of vectors. Typically we will give just one vector but don't forget that any other equivalent vector is the same point.

Example 6.1. The point $\mathbf{P} = [1; 2; 3]$ represents the set

$$\{\lambda[1; 2; 3] \mid \lambda \neq 0\}$$

Thus $\mathbf{P} = [6; 12; 18]$ and $\mathbf{P} = [-1; -2; -3]$.

Exercise 6.1. List some vectors which are in the equivalence class of $\mathbf{P} = [4; 2; 1]$ and some which are not.

6.2.2 Euclidean Space is Inside

Definition 6.2.2.1. Consider the subset of $\mathbb{RP}^2$ defined by:

$$E^2 = \{[X; Y; Z] \mid Z \neq 0\} \subset \mathbb{RP}^2$$

We call this the *Euclidean patch* for reasons we will see shortly.

Notice that considering equivalence we have:

$$E^2 = \{[X; Y; 1]\}$$

Since all such projective points are distinct when written this way we see that E^2 is essentially a copy of $\mathbb{R}^2$ existing inside $\mathbb{RP}^2$.

Thus we have a mapping from $\mathbb{R}^2$ to $E^2 \subset \mathbb{RP}^2$:

$$\begin{array}{rcl} \mathbb{R}^2 & \rightarrow & E^2 \subset \mathbb{RP}^2 \\ [x; y] & \mapsto & [x; y; 1] = \{[xZ; yZ; Z] \mid Z \neq 0\} \end{array}$$

And we have a mapping from $E^2 \subset \mathbb{RP}^2$ to $\mathbb{R}$:

$$\begin{array}{rcl} E^2 \subset \mathbb{RP}^2 & \rightarrow & \mathbb{R}^2 \\ [X; Y; Z] \equiv [X/Z; Y/Z; 1] & \mapsto & \left[\frac{X}{Z}; \frac{Y}{Z}\right] \end{array}$$

Example 6.2. The Euclidean point $[5; 7] \in \mathbb{R}^2$ correponds to the projective point $[5; 7; 1] \in E^2 \subset \mathbb{RP}^2$ which consists of the set of vectors $\{[5Z; 7Z; Z] \mid Z \neq 0\}$

Example 6.3. The projective point $[3; 5; 2] \equiv [3/2; 5/2; 1] \in E^2 \subset \mathbb{RP}^2$ corresponds to the Euclidean point $\left[\frac{3}{2}; \frac{5}{2}\right] \in \mathbb{R}^2$.

Example 6.4. The projective point $[-1; 7; 0] \in \mathbb{RP}^2$ doesn't correspond to any point in $\mathbb{R}^2$ because $[-1; 7; 0] \notin E^2$.

Definition 6.2.2.2. We say that points of the form $[X; Y; 0]$ are *points at infinity*. These are points which are not in $E^2 \equiv \mathbb{R}^2$.

Exercise 6.2. Which projective points in $E^2 \subset \mathbb{RP}^2$ correspond to each of the following points in $\mathbb{R}^2$.

(a) $[2; 1]$

(b) $[1; 6]$

(c) $[0.1; 0.7]$

Exercise 6.3. Which points in $\mathbb{R}^2$ correspond to each of the following points in $\mathbb{RP}^2$. One is a trick.

(a) $[2; 5; -2]$

(b) $[10; 5; 7]$

(c) $[7; 2; 0]$

(d) $[6; 0.2; 0.15]$

It's worth noting that there are other ways we could have selected a copy of $\mathbb{R}^2$ inside $\mathbb{RP}^2$, including fixing $X \neq 0$ or $Y \neq 0$.

Note that as we go forward we'll write $E^2 \equiv \mathbb{R}^2$ frequently when the sentence leading up to the equivalence is talking about E^2 and we want to make the point that this is equivalent to $\mathbb{R}^2$ and we'll write $\mathbb{R}^2 \equiv E^2$ frequently when the sentence leading up to the equivalence is talking about $\mathbb{R}^2$ and we want to make the point that this is equivalent to E^2.

For example we might write something like: $[3; 2] \in \mathbb{R}^2 \equiv E^2$ or $[3; 2; 1] \in E^2 \equiv \mathbb{R}^2$.

6.2.3 More Than Euclidean Space

Notice that nothing in $\mathbb{R}^2 \equiv E^2$ matches up with vectors in $\mathbb{RP}^2$ which have the form $[X; Y; 0]$. This means that we've enlarged $E^2 \equiv \mathbb{R}^2$ by adding on these vectors, even given the equivalences.

Look at only these vectors:

- If $Y = 0$ then the set of projective points of the form $[X; 0; 0]$ are all equivalent to the single point $\equiv [1; 0; 0] \in \mathbb{RP}^2$.
- If $Y \neq 0$ then each projective point of the form $[X; Y; 0]$ is equivalent to a projective point of the form $[*; 1; 0]$ which then yields a line of projective points in $\mathbb{RP}^2$.

Therefore:

$$\{[X; Y; 0] \mid X, Y \neq 0\} \equiv [1; 0; 0] \cup \{[X; 1; 0] \mid X \in \mathbb{R}\}$$

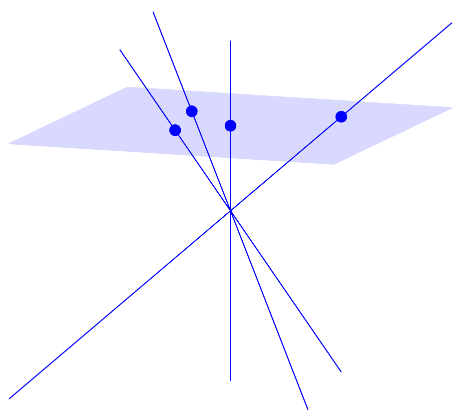
Moreover if we look at vectors of the form $[X; 1; 0]$ as $X \to \infty$ we see that $[X; 1; 0] \equiv [1; 1/X; 0] \to [1; 0; 0]$ and the same happens if $X \to -\infty$ which means that the two ends of the line meet (infinitely far out) at the projective point $[1; 0; 0]$.

Thus the points that we've added form a projective line, which is a circle.

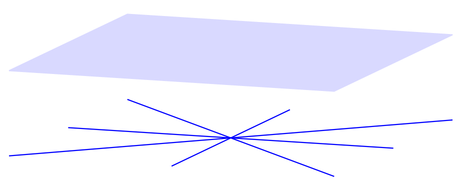
6.2.4 A Graphical Understanding

A useful and graphical way to understand what is going on is to consider that a nonzero vector $[X; Y; Z]$, along with the origin $[0; 0; 0]$, yields a line in $\mathbb{R}^3$. Any nonzero multiple $\lambda[X; Y; Z] = [\lambda X; \lambda Y; \lambda Z]$ will yield exactly the same line:

The lines created using $[X; Y; Z]$ with any $Z \neq 0$ can be thought of as the lines created by $[X; Y; 1]$ can be thought of as the points $[x; y; 1]$, or the plane $z = 1$, which is basically a version of the Euclidean plane.



The lines created using nonzero $[X; Y; 0]$ cannot be created using $[X; Y; 1]$ and hence cannot be thought of this way. However can be thought of as lines through the origin which are horizontal, not meeting $z = 1$.



6.2.5 Compatibility with Euclidean Space

It's important to note that in $\mathbb{RP}^2$ objects must behave as espected when we focus on the Euclidean patch. For example a line in $\mathbb{RP}^2$ must look like a line when we restrict our view to the Euclidean patch, otherwise we've just built something completely new and possibly useless. We'll see how this works as we move forward.

6.3 Standard Transformations of Points

Now that we've gained some familiarity with projective space let's return to our original issue of managing transformations.

6.3.1 Rotations

We saw in Chapter 2 that in $\mathbb{R}^2$ we can rotate around the origin counterclockwise by θ radians by left-multiplying the vector $[x; y]$ by the matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Consider now the matrix:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that this acts on $\mathbb{RP}^2$ as follows:

$$\begin{aligned} R_Z(\theta)[X; Y; Z] &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \\ &= \begin{bmatrix} X\cos\theta + Y\sin\theta \\ Y\cos\theta - X\sin\theta \\ Z \end{bmatrix} \end{aligned}$$

Specifically note that this maps E^2 to E^2 in such a manner as to rotate about the origin clockwise by θ radians. It fixes Z which is why we've used the notation R_Z.

This rotation should not really be described as rotation around an axis as far as $\mathbb{RP}^2$ is concerned but rather as transformation of $\mathbb{RP}^2$ which fixes Z and for each fixed Z_0 acts on $[X; Y; Z_0]$ as a rotation with regards to X and Y. With regards to $E^2 \equiv \mathbb{R}^2$ it acts equivalently to a counterclockwise rotation about the origin.

Example 6.5. To rotate $[2;1] \in \mathbb{R}^2 \equiv E^2$ about the origin by 0.2 radians we write $[2;1;1] \in E^2 \equiv \mathbb{R}^2$ and calculate:

$$R_Z(0.2)[2;1;1] = \begin{bmatrix} \cos(0.2) & -\sin(0.2) & 1 \\ \sin(0.2) & \cos(0.2) & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 2.76 \\ 2.38 \\ 1 \end{bmatrix}$$

So the result in $\mathbb{R}^2$ is approximately $[2.76; 2.38]$.

Exercise 6.4. What is the point that results when we rotate $[2;3]$ about the origin counterclockwise by $\frac{\pi}{6}$ radians?

6.3.2 Translations

This is where things get interesting and useful. Recall that translation of $\mathbb{R}^2$ cannot be achieved by matrix multiplication because it's not linear.

However consider this matrix:

$$T_Z(a,b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that this acts on $\mathbb{RP}^2$ as follows:

$$\begin{aligned} T_Z(a,b)[X;Y;Z] &= \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \\ &= \begin{bmatrix} X + aZ \\ Y + bZ \\ Z \end{bmatrix} \end{aligned}$$

Specifically note that this maps E^2 to E^2 in such a manner as to translate $[X;Y;1]$ to $[X+a; Y+b; 1]$. It fixes Z which is why we've used the notation T_Z.

Thus effectively this matrix multiplication translates $\mathbb{R}^2 \equiv E^2$ by $[a;b]$.

Example 6.6. Observe that:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} X + 3 \\ Y + 2 \\ 1 \end{bmatrix}$$

Exercise 6.5. Write down the matrix for translation by $[-5;3]$ and show how it affects the point $[40;31]$.

6.3.3 Reflections

In an earlier chapter we built our reflections based upon the reflection in the y-axis. We can essentially do that here via:

$$F_Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We're using the notation F_Y because this flip negates the Y-component in $\mathbb{RP}^2$. It keeps both X and Z constant so there would ambiguity in writing F_X instead.

As far as $E^2 \equiv \mathbb{R}^2$ is concerned this is indeed a flip over the $X \equiv x$-axis.

6.3.4 Combinations

Since we may now use matrix multiplication to do rotations, translations and reflections we may use them to do combinations. Not only that but they're now all represented by matrices.

For example now we can rotate around points other than the origin.

Example 6.7. To rotate by 0.2 radians counterclockwise about the point [3; 8] we use the matrix product:

$$\begin{aligned} &T_Z(3,8)R_Z(0.2)T_Z(-3,-8) \\ &\qquad = \begin{bmatrix} \cos(0.2) & -\sin(0.2) & 8\sin(0.2) - 3\cos(0.2) + 3 \\ \sin(0.2) & \cos(0.2) & 8 - 3\sin(0.2) - 8\cos(0.2) \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

So then to do this to the point [30; 70] we would do:

$$\begin{aligned} T_Z(3,8)R_Z(0.2)T_Z(-3,-8)\begin{bmatrix} 30 \\ 70 \\ 1 \end{bmatrix} &= \begin{bmatrix} 27\cos(0.2) - 62\sin(0.2) + 3 \\ 62\cos(0.2) - 27\sin(0.2) + 8 \\ 1 \end{bmatrix} \\ &\approx \begin{bmatrix} 17.14 \\ 74.13 \\ 1 \end{bmatrix} \end{aligned}$$

Exercise 6.6. Write down the (approximate) matrix which rotates counterclockwise by 4.12 radians about the point [5; 2] and show how it rotates the point [10; 10].

Exercise 6.7. Write down the (approximate) matrix which reflects in the line $y = 2x + 1$ and show how it reflects the point [6; 1].

6.3.5 Managing Sets of Points

Since objects are made up of points it's entirely likely that we will need to apply our transformations to set of points rather than to individual points.

Keeping in mind that the definition of matrix multiplication is:

$$M\left[\mathbf{x}_1\,\mathbf{x}_2\,...\,\mathbf{x}_n\right] = \left[M\mathbf{x}_1\,M\mathbf{x}_2\,...\,M\mathbf{x}_n\right]$$

It follows that when working with several points we can simply store them all together in a single matrix P, each as a column, and then calculate MP in order to apply the transformation M to all of them simultaneously.

Example 6.8. To rotate the three points $[0;0]$, $[2;0]$, and $[1;2]$ by 6.3 radians about the point $[4;5]$ we do:

$$T_Z(4,5)R_Z(6.2)TZ(-4,-5)\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} -0.40 & 1.59 & 0.76 \\ 0.35 & 0.18 & 2.26 \\ 1.00 & 1.00 & 1.00 \end{bmatrix}$$

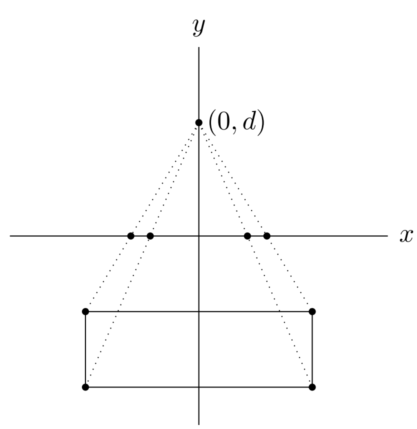
So the resulting points are $[-0.40;0.35]$, $[1.59;0.18]$, and $[0.76;2.26]$.

6.4 Perspective Projection

6.4.1 Method

Most of this is a repeat of the calculation in an earlier chapter but it's worth revisiting it here.

Imagine an object in the xy-plane below the x-axis. If you position your eye at the point $(0;d)$ and imagine that the viewing plane is the x-axis then you will see the object as if it is projected with perspective to the x-axis:



A simple calculation with similar triangles tells us that the point $(x;y)$ maps to the point $(x';0)$ via:

$$\frac{x'}{d} = \frac{x}{d-y}$$
$$x' = \frac{dx}{d-y}$$

This is not a linear transformation, meaning there is no 2×2 matrix P such that:

$$P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

This is easy to see - if such a mapping were a linear transformation then it would take the basis vectors $[1;0]$ and $[0;1]$ to $[d(1)/(d-0);0] = [1;0]$ and $[d(0)/(d-1);0] = [0;0]$ respectively, meaning it would be the matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

However clearly this matrix does not do what we're asking of it:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \neq \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

However consider the following matrix multiplication in $\mathbb{RP}^2$, specifically how it affects $E^2 \equiv \mathbb{R}^2$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1/d & 1 \end{bmatrix}}_{P(d)}\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ 1-Y/d \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ (d-Y)/d \end{bmatrix} \equiv \begin{bmatrix} dX/(d-Y) \\ 0 \\ 1 \end{bmatrix}$$

What this shows us is that multiplication by the matrix $P(d)$ takes $\mathbb{RP}^2$ to itself in such a manner that the Euclidean patch is projected exactly as we want it to be.

The practical upshot of this is that if we treat $\mathbb{R}^2$ as the Euclidean patch inside $\mathbb{RP}^2$ then we can perform linear (matrix) transformations to take care of perspective projection.

Example 6.9. If your eye is at $[0;5]$ the point $[7;-2]$ gets mapped to the x-axis as follows, with the calculation done in $\mathbb{RP}^2$:

$$\begin{aligned} P(5)\begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1/5 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 0 \\ 1+2/5 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 0 \\ 7/5 \end{bmatrix} \\ &\equiv \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Thus to the point $[5;0]$.

Exercise 6.8. If your eye is at $[0;10]$ calculate where the point $[8;-5]$ gets mapped on the x-axis by this perspective projection.

Note that the calculation doesn't actually demand that the point be below the x-axis.

Exercise 6.9. If your eye is at $[0;10]$ calculate where the point $[2;3]$ gets mapped on the x-axis by this perspective projection. Draw a plot of these three ponts so you can see what happened.

Things go a bit crazy if the point is above the eye though.

Exercise 6.10. If your eye is at $[0;10]$ calculate where the point $[-2;12]$ gets mapped on the x-axis by this perspective projection. Draw a plot of these three ponts so you can see what happened.

6.4.2 Computation Note

From a computational perspective each vector must be scaled so that its third coordinate equals 1. This is not a linear operation, unfortunately, in the sense that it cannot be represented by a matrix product. In addition if we apply perspective projections to multiple points then we must normalize them all independently.

However this only needs to be done once, after all the transformations are applied and before the points are rendered, so the computational overhead is minimum.

Example 6.10. Given the square with corners $[1;-4]$, $[1;-2]$, $[3;-2]$ and $[3;-4]$ if we wish to rotate this about the origin by -0.3 radians, then translate down by 2, and then project with $d = 7$ we proceed as follows:

$$P(7)T_Z(0,-2)R_Z(-0.3)\begin{bmatrix} 1 & 1 & 3 & 3 \\ -4 & -2 & -2 & -4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.2267443375 & 0.3642960758 & 2.2749690541 & 1.6839286407 \\ 0 & 0 & 0 & 0 \\ 1.8738380233 & 1.6008847407 & 1.6853190855 & 1.9582723681 \end{bmatrix}$$

However then we need to scale each column independently by dividing each column by its third entry. The result is, approximately:

$$\approx \begin{bmatrix} -0.12 & 0.28 & 1.35 & 0.86 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus the projected points are approximately $[-0.12;0]$, $[0.28;0]$, $[1.35;0]$ and $[0.86;0]$.

6.5 Points, Lines, and Duality

6.5.1 Points

As we've seen, a point in $\mathbb{RP}^2$ is defined by the equivalence class of a triple $[X;Y;Z]$. For simplicity we'll just say "the point" $[X;Y;Z]$ when we mean the entire equivalence class.

6.5.2 Lines

In order to define a line in $\mathbb{RP}^2$ we need to establish exactly what a line is. There are many ways to do this but one common-sense one (which is perfect for us) would be to take an equation which defines a one-dimensional set of points in $\mathbb{RP}^2$ and which, when restricted to $E^2 \equiv \mathbb{R}^2$, yields a line in the normal $\mathbb{R}^2$ sense.

Definition 6.5.2.1. A *line in projective space* $\mathcal{L}$ in $\mathbb{RP}^2$ is defined by giving a nonzero vector $\mathbf{L} = [a;b;c] \in \mathbb{RP}^2$ and looking at all those $[X;Y;Z]$ satisfying:

$$\mathcal{L} = \{[X; Y; Z] \,|\, aX + bY + cZ = 0\}$$

Equivalently $[a, b, c][X; Y; Z] = 0$ or $[a; b; c] \cdot [X; Y; Z] = 0$.

Note that this is in fact a one-dimensional set of points $[X; Y; Z]$ because we lose a dimension due to projetive equivalence and we lose a dimension due to the equation.

Notice that any nonzero multiple of $\mathbf{L} = [a; b; c]$ defines the same line. In this way lines behave a bit like points. This is more true than we might realize right now.

Theorem 6.5.2.1. When at least one of a and b is nonzero this definition of a line matches our intuition in the sense that a line in projective space is a Euclidean line when restricted to $E^2 \equiv \mathbb{R}^2$ and every Euclidean line in $\mathbb{R}^2 \equiv E^2$ arises from this definition. When $a = b = 0$ we get all the points at infinity.

Proof. Given $[a; b; c]$ consider the set

$$\mathcal{L} = \{[X; Y; Z] \,|\, aX + bY + cZ = 0\}$$

If $a = b = 0$ then $c \neq 0$ and:

$$\mathcal{L} = \{[X; Y; Z] \,|\, (0)X + (0)Y + cZ = 0\} = \{[X; Y; 0] \,|\, X, Y \in \mathbb{R}\} - \{[0; 0; 0]\}$$

which is precisely the set of points at infinity.

If one of a and b is nonzero then if we isolate our view to the Euclidean patch. The point $[x; y] \in \mathbb{R}^2$ corresponds to $[x; y; 1] \in E^2 \subset \mathbb{RP}^2$ so for this point to be on the line we must have

$$a(x) + b(y) + c(1) = 0$$

This is a line in $\mathbb{R}^2$.

Likewise every line in $\mathbb{R}^2 \equiv E^2$ may be written in the form $ax + by + c = 0$ with one of a, b nonzero and hence in $\mathbb{RP}^2$ may be represented by $[a; b; c]$. $\square$

Theorem 6.5.2.2. A line in $\mathbb{RP}^2$ represented by $[a; b; c]$ with at least one of a and b nonzero contains a single point at infinity.

Proof. To identify this point, note that it must have the form $[X; Y; 0]$ and must satisfy $aX + bY + c(0) = 0$.

Observe that for such a point:

- If $a \neq 0$ then $Y \neq 0$ (otherwise $AX = 0$ so $X = 0$) and so:

$$[X; Y; 0] \equiv [aX; aY; 0] \equiv [-bY; aY; 0] \equiv [-b; a; 0] \equiv [b; -a; 0]$$

- If $b \neq 0$ then $X \neq 0$ (otherwise $BY = 0$ so $Y = 0$) and so:

$$[X; Y; 0] \equiv [bX; bY; 0] \equiv [bX; -aX; 0] \equiv [-b; a; 0] \equiv [b; -a; 0]$$

So that the point satisfying this is $[-b; a; 0] \equiv [b; -a; 0]$.

□

Example 6.11. The vector $[1; 2; 3]$ defines the line consisting of the set of points $[X; Y; Z]$ satisfying:

$$\begin{aligned} [1, 2, 3][X; Y; Z] &= 0 \\ X + 2Y + 3Z &= 0 \end{aligned}$$

In $\mathbb{R}^2 \equiv E^2$ the points which lie on this projective line satisfy the equation:

$$x + 2y + 3 = 0$$

The point at infinity on this line is $[-b; a; 0] = [-2; 1; 0]$.

Exercise 6.11. For the line defined by $[5; 2; 6]$ which Euclidean equation does this correspond to in $E^2 \equiv \mathbb{R}^2$ and which point at infinity is on this line?

Theorem 6.5.2.3. For a line in $\mathbb{R}^2 \equiv E^2$ the point at infinity which is picked up in E^2 is the natural limit point at the ends of the line, meaning if we algebraically follow the line to infinity the result will be the asssociated point at infinity.

Proof. Suppose we start with a line of the form $y = mx + b$. This is equivalent to $mx - 1y + b = 0$ and hence is represented by the vector $[m; -1; b]$. The point at infinity we pick up is then $[1; m; 0]$ and observe that for points on the line we have $[x; y; 1] = [x; mx + b; 1] \equiv [1; m + b/x; 1/x]$ and so as $x \to \pm\infty$ we have $[x; y; 1] \to [1; m; 0]$.

Suppose on the other hand we start with a line of the form $x = a$. This is equivalent to $1x + 0y - a = 0$ and hence is represented by the vector $[1; 0; -a]$. The point at infinity we pick up is then $[0; 1; 0]$ and observe that for points on the line we have $[x; y; 1] = [a; y; 1] \equiv [a/y; 1; 1/y]$ and so as $y \to \pm\infty$ we have $[x; y; 1] \to [0; 1; 0]$. □

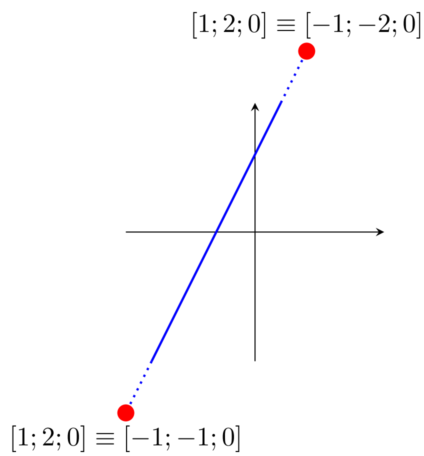
Just to clarify what this is saying, here are two examples:

Example 6.12. The line $y = 2x + 3$ can be rewritten as $2x - y - 3 = 0$ and therefore picks up the point at infinity $[-b; a; 0] = [1; 2; 0]$. Moreover as we go to infinity at the ends of the line, meaning $x \to \pm\infty$, we see:

$$[x; y; 1] = [x; 2x + 3; 1] \equiv [1; 2 + 3/x; 1/x] \to [1; 2; 0]$$

Thus as we go to infinity at the ends of the line we encounter the point at infinity at both ends. What's really happening is that at infinity the ends of the line are meeting at that point.

Visually we have the following:

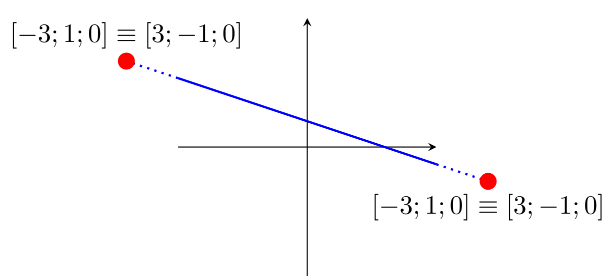


Example 6.13. The line $y = -\frac{1}{3}x + 1$ can be rewritten as $x + 3y - 3 = 0$ and therefore picks up the point at infinity $[-b; a; 0] = [-3; 1; 0]$. Moreover as we go to infinity at the ends of the line, meaning $x \to \pm\infty$, we see:

$$[x; y; 1] = \left[x; -\frac{1}{3}x + 1; 1\right] \equiv \left[1; -\frac{1}{3} + \frac{1}{x}; 1/x\right] \to [1; -\frac{1}{3}; 0] \equiv [-3; 1; 0]$$

Thus as we go to infinity at the ends of the line we encounter the point at infinity at both ends. What's really happening is that at infinity the ends of the line are meeting at that point.

Visually we have the following:



Summary: With the exception of the line composed of all points at infinity, a line in $\mathbb{RP}^2$ is made up of a familiar line in $E^2 \equiv \mathbb{R}^2$ as well as a single point at infinity. If the familar line has the equation $ax + by + c = 0$ then the single point at infinity is $[-b; a; 0]$. As we head "to infinity" at the ends of the line we "reach" the point at infinity. Essentially the two ends of the line "meet up" at that point at infinity.

Exercise 6.12. Which point at infinity is picked up when the line $y = 5x - 7$ is moved into $\mathbb{RP}^2$?

Exercise 6.13. Which point at infinity is picked up when the line $x = 17$ is moved into $\mathbb{RP}^2$?

Exercise 6.14. Which point at infinity is picked up when the line $y = 5$ is moved into $\mathbb{RP}^2$?

6.5.3 Intersections of Lines

Suppose we have two lines and we wish to find the point where they intersect.

Theorem 6.5.3.1. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbb{R}^2$ are defined by $\mathbf{L_1}$ and $\mathbf{L_2}$ respectively. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ meet at the point $\mathbf{L}_1 \times \mathbf{L}_2$

Proof. If we put $\mathbf{L_1} = [a_1; b_1; c_1]$ and $\mathbf{L_2} = [a_2; b_2; c_2]$ Then we are looking for X, Y, Z satisfying the system:

$$a_1X + b_1Y + c_1Z = 0$$
$$a_2X + b_2Y + c_2Z = 0$$

Of course this does not have a unique solution in $\mathbb{R}^3$ since there are three variables and two equations but in $\mathbb{RP}^2$ it will have a unique solution.

To see this, first multiply the first equation by b_2 and the second by b_1 and then subtract, canceling the Y terms:

$$\begin{aligned} a_1b_2X + b_1b_2Y + b_2c_1Z &= 0 \\ a_2b_1X + b_1b_2Y + b_1c_2Z &= 0 \\ (a_1b_2 - a_2b_1)X + (b_2c_1 - b_1c_2)Z &= 0 \\ (a_1b_2 - a_2b_1)X &= (b_1c_2 - b_2c_1)Z \end{aligned}$$

A similar calculation canceling the X terms yields:

$$(a_1b_2 - a_2b_1)Y = (a_2c_1 - a_1c_2)Z$$

This is satisfied only by multiples of the vector:

$$[b_1c_2 - b_2c_1; a_2c_1 - a_1c_2; a_1b_2 - a_2b_1] = \mathbf{L_1} \times \mathbf{L_2}$$

□

This is extremely computationally useful.

Example 6.14. Consider the two lines defined by $\mathbf{L_1} = [2; 1; -3]$ and $\mathbf{L_2} = [0; 4; 7]$.

Observe that these points meet at:

$$\mathbf{L_1} \times \mathbf{L_2} = [21; -14; 8]$$

Notice that in the Euclidean patch the two lines have equations $2x + y - 3 = 0$ and $0x + 4y + 7 = 0$ and the meeting point is $[21; -14; 8] \equiv [21/8; -7/4; 1]$ which is $[21/8; -7/4] \in \mathbb{R}^2$.

Exercise 6.15. Use the above method to calculate the point at which the lines $y = 5x + 1$ and $y = -2x - 7$ meet.

Exercise 6.16. Use the above method to calculate the point at which the lines $x = 0$ and $y = 0$ meet.

At this point we might be surprised to notice that the above formula works even when the lines are parallel in the Euclidean patch! Let's see how this works out in a specific example:

Example 6.15. Consider the two lines defined by $\mathbf{L_1} = [4; 3; 5]$ and $\mathbf{L_2} = [4; 3; 9]$. Observe that in the Euclidean patch these correspond to the parallel lines with equations $4x + 3y + 5 = 0$ and $4x + 3y + 9 = 0$ respectively, and don't meet.

However the above calculation yields:

$$\mathbf{L_1} \times \mathbf{L_2} = [12; -16; 0] \equiv [3; -4; 0]$$

So these two lines do meet in $\mathbb{RP}^2$ at a point at infinity. Notice that this point at infinty is not unexpected since it's the point at infinity for each of the lines individually.

Exercise 6.17. Use the above method to calculate the point at which the lines $y = 3x + 1$ and $y = 3x + 7$ meet.

The fact that this works for parallel lines makes sense when coupled with the term "projective". Imagine two parallel lines going to the horizon. Those lines don't meet in reality but if we include the horizon as part of our space then the lines intuitively do meet there. Projective geometry is formalizing this concept.

6.5.4 Line Joining Two Points

Suppose we have two points and we wish to find the line joining them.

Theorem 6.5.4.1. Suppose $\mathbf{P}_1$ and $\mathbf{P}_2$ are two points in $\mathbb{RP}^2$. Then the vector defining the line containing both is $\mathbf{P}_1 \times \mathbf{P}_2$

Proof. Suppose $\mathbf{P_1} = [X_1; Y_1; Z_1]$ and $\mathbf{P_2} = [X_2; Y_2; Z_2]$. Let $[a; b; c]$ represent the line containing both. Then we have:

$$aX_1 + bY_1 + cZ_1 = 0$$
$$aX_2 + bY_2 + cZ_2 = 0$$

At this point we have exactly the same system as in the previous theorem except we're solving for a, b, c instead of X, Y, Z and consequently the equivalent computation applies, resulting in the solution being all multiples of the vector:

$$[Y_1Z_2 - Y_2Z_1; X_2Z_1 - X_1Z_2; X_1Y_2 - X_2Y_1] = \mathbf{P}_1 \times \mathbf{P}_2$$

□

Example 6.16. The line joining the projective points $\mathbf{P_1} = [2; 2; -1]$ and $\mathbf{P_2} = [3; 1; 5]$ is represented by the vector:

$$\mathbf{P_1} \times \mathbf{P_2} = [11; -13; -4]$$

In $\mathbb{R}^2 \equiv E^2$ this line has equation $11x - 13y - 4 = 0$.

Exercise 6.18. Use the above method to calculate the vector representing the line joining the projective points $[1; 2; 0]$ and $[-5; 2; 2]$. What is the equation of the line in the Euclidean patch?

Exercise 6.19. Use the above method to calculate the Euclidean equation of the line joining the points $[2; 3]$ and $[-4; 9]$.

6.5.5 Duality

We thus have a duality. To find where two lines meet, take their cross product. To find which line joins two points, take their cross product. Essentially points and lines behave equivalently in $\mathbb{RP}^2$.

This allows for some convenient calculations. For example suppose we wish to know if the three points $\mathbf{P_1}$, $\mathbf{P_2}$ and $\mathbf{P_3}$ are colinear. Well the line joining $\mathbf{P_1}$ and $\mathbf{P_2}$ is represented by the vector $\mathbf{P_1} \times \mathbf{P_2}$ and the point $\mathbf{P_3}$ is on this line if:

$$(\mathbf{P_1} \times \mathbf{P_2}) \cdot \mathbf{P_3} = 0$$

Example 6.17. To check if the points $\mathbf{P_1} = [1;2;1]$, $\mathbf{P_2} = [2;0;4]$, and $\mathbf{P_3} = [0;-1;5]$ are colinear we observe:

$$\begin{aligned}(\mathbf{P_1} \times \mathbf{P_2}) \cdot \mathbf{P_3} &= ([1;2;1] \times [2;0;4]) \cdot [0;-1;5] \\ &= [8;-2;-2] \cdot [0;-1;5] \\ &= -8 \neq 0\end{aligned}$$

Therefore the points are not colinear.

Exercise 6.20. Check if the following triples are colinear.

(a) $[2;2;2]$, $[5;13;1]$, and $[-6;-24;-3]$

(b) $[5;-1;1]$, $[16;-8;2]$ and $[2;2;0]$

Interestingly if we were to consider $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$ as lines instead then this calculation still has merit. The cross product $\mathbf{P_1} \times \mathbf{P_2}$ represents the point where they meet and the dot product $(\mathbf{P_1} \times \mathbf{P_2}) \cdot \mathbf{P_3}$ asks if this point is on the line determined by $\mathbf{P_3}$. Thus this same exact calculation checks if the intersection of two lines lies on another line.

Exercise 6.21. Does the intersection of the lines $y = x$ and $y = 2x - 3$ line on the line $y = -5x + 8$?

A fun duality summary:

Point $\times$ Point	Line Connecting the Points
Line $\times$ Line	Intersection Point of the Lines
Point $\cdot$ Line	Determines if the Point Lies on the Line

Exercise 6.22. Suppose $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4 \in \mathbb{RP}^2$. Consider the calculation:

$$(\mathbf{V}_1 \times \mathbf{V}_2) \times (\mathbf{V}_3 \times \mathbf{V}_4)$$

What does this calculation do in the following cases:

(a) If the $\mathbf{V}_i$ represent lines?

(b) If the $\mathbf{V}_i$ represent points?

Exercise 6.23. Consider the four lines $y = 2x + 1$, $y = 3x - 4$, $y = -5x + 2$, and $y = x - 1$. Find the equation of the line containing the point arising from the intersection of the first two and the point arising from the intersection of the second two.

Exercise 6.24. Consider the four points $(1, 3)$, $(-8, 2)$, $(0, 1)$, and $(6, 10)$. Find the point where the line joining the first two points meets the line joining the second two points.

6.5.6 Standard Transformations of Lines

Given that a line is represented by a vector $\mathbf{L}$ we might ask what happens if a projective transformation is applied to the line.

Theorem 6.5.6.1. If a line represented by the vector $\mathbf{L}$ is transformed by the matrix M then the resulting line is represented by the matrix:

$$\left(M^{-1}\right)^T \mathbf{L}$$

Proof. Suppose $\mathbf{x}$ is on the transformed line. Then $M^{-1}\mathbf{x}$ is on the original line and so:

$$\begin{aligned}
\mathbf{L} \cdot \left(M^{-1}\mathbf{x}\right) &= 0 \\
\mathbf{L}^T M^{-1}\mathbf{x} &= 0 \\
\left(\left(M^{-1}\right)^T \mathbf{L}\right)^T \mathbf{x} &= 0 \\
\left(\left(M^{-1}\right)^T \mathbf{L}\right) \cdot \mathbf{x} &= 0
\end{aligned}$$

□

Example 6.18. If we translate the line represented by $\mathbf{L} = [2; 3; -1]$ using $T_Z(a; b)$ the resulting line is represented by

$$\begin{aligned}
\left(T_Z(a,b)^{-1}\right)^T \mathbf{v} &= T_Z(-a,-b)[2;3;-1] \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ -2a - 3b - 1 \end{bmatrix} \\
&= [2; 3; -2a - 3b - 1]
\end{aligned}$$

This becomes potentially much clearer if we examine the behavior in the Euclidean patch:

Example 6.19. In the above example the vector $\mathbf{L} = [2; 3; -1]$ represents the line $2x + 3y - z = 0$. The resulting vector represents the line:

$$2x + 3y + (-2a - 3b - 1) = 0$$
$$2(x - a) + 3(y - b) - 1 = 0$$

This is clearly the result of translating this line by $[a; b]$.

Example 6.20. To rotate the line $y = 2x$ by 2.1 radians about the point $[2; -3]$ we first note that the line is represented by the vector $\mathbf{L} = [2; -1; 0]$. So we calculate the new vector:

$$((T_Z(2, -3)R_Z(2.1)T_Z(-2, 3))^{-1})^T\mathbf{L} \approx [-0.1465; 2.2313; 13.9868]$$

This new line therefore has approximate equation in $\mathbb{R}^2$:

$$-0.1465x + 2.2313y + 13.9868 = 0$$

Exercise 6.25. Suppose $\mathbf{L} = [4; 0; 2]$ represents a line in $\mathbb{RP}^2$. If we translate this line by $T_Z(-1, 5)$ which vector represents the resulting line?

Exercise 6.26. Suppose $\mathbf{L} = [5; 3; 1]$ represents a line in $\mathbb{RP}^2$. If we rotate this line by $R_Z(\pi/4)$ which vector represents the resulting line?

Exercise 6.27. Find the resulting line in $\mathbb{R}^2$ when the line $y = 3x+1$ is rotated by $\pi/3$ about the point $[5; 2]$.

Exercise 6.28. Find the resulting line in $\mathbb{R}^2$ when the line $x = 3$ is reflected in the line $y = 3x - 1$.

6.6 (Non-Degenerate) Conics

6.6.1 Definition

Definition 6.6.1.1. Given $a, b, c, d, e, f \in \mathbb{R}$ we can define a *conic* as the set of points $[X; Y; Z]$ satisfying the equation:

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0$$

For now don't worry about where these points lie. Eventually it will be in $\mathbb{RP}^2$.

Classic examples of conics include ellipses (including circles), parabolas, and hyperbolas.

All of these are examples of conics but not every conic is precisely one of these. For example if $a = c = 1$ and $b = d = f = 0$ then the result is a point. Arguably this is a circle of radius 0 but formally this is a *degenerate conic*, specifically a degenerate ellipse. Other examples of degenerate conics are intersecting lines (a degenerate hyperbola) and parallel lines or a single line (a degenerate parabola).

We will focus only on non-degenerate conics and we will use the term "conic" to refer to just these.

Notice that in Euclidean space these are very different. In $\mathbb{RP}^2$ these are exactly the same thing as one another, as we'll see.

First note that for the vector:

$$\mathbf{x} = [X; Y; Z]$$

the equation of a conic can be rewritten as:

$$\mathbf{x}^T \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \mathbf{x} = 0$$

This is simply because the matrix product above yields the conic equation.

So in other words a conic is defined by a matrix C via $\mathbf{x}^T C\mathbf{x} = 0$, meaning $\mathbf{x}$ lies on the conic iff $\mathbf{x}^T C\mathbf{x} = 0$.

If the construction of this matrix is a bit confusing think of the columns as corresponding to coefficients:

$$C = \begin{array}{c} \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \end{array}$$

In this sense each entry contains a contribution to the coefficient of the corresponding row/column border product.

Example 6.21. Consider the conic:

$$2X^2 + 3XY + 1Y^2 + 10XZ - 7Z^2 = 0$$

We put 2, 1, -7 in the main diagonal and then $3/2$ in the XY and YX positions and $10/2 = 5$ in the XZ and ZX positions:

$$C = \begin{array}{c} \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 2 & 3/2 & 5 \\ 3/2 & 1 & 0 \\ 5 & 0 & -7 \end{bmatrix} \end{array}$$

As with lines we will see if we take a conic in $\mathbb{RP}^2$ and restrict it to $E^2 \equiv \mathbb{R}^2$ we get a conic as well. However there may be one or more points at infinity which lie on the conic in $\mathbb{RP}^2$ which are not in $E^2 \equiv \mathbb{R}^2$.

Similarly if we take a conic in $\mathbb{R}^2 \equiv E^2$ and apply its equation to $\mathbb{RP}^2$ we may pick up one or more points at infinity.

Some examples will clarify. Here are three starting with a conic equation in $\mathbb{RP}^2$.

Example 6.22. Consider the conic defined by the equation:

$$X^2 + YZ = 0$$

If we look for $[X;Y;1] \in E^2 \equiv \mathbb{R}^2$ we find:

$$\begin{aligned} X^2 - (Y)(1) &= 0 \\ Y &= X^2 \end{aligned}$$

If we look for points at infinity $[X;Y;0] \notin E^2$ we find:

$$\begin{aligned} X^2 - (Y)(0) &= 0 \\ X^2 &= 0 \\ X &= 0 \end{aligned}$$

This is the single point at infinity $[0;Y;0] \equiv [0;1;0] \in \mathbb{RP}^2$.

Therefore this conic contains a parabola in E^2 and contains the single point at infinity $[0;1;0]$.

Note in $\mathbb{R}^2 \equiv E^2$ this parabola is $y = x^2$.

Moreover observe that in E^2 on the parabola with $y \neq 0$ we have $[x;y;1] = [x;x^2;1] \equiv [1/x;1;1/x^2]$ and so as $x \to \pm\infty$ we have $[x;y;1] \to [0;1;0]$ so the ends of the parabola actually meet at the point at infinity and the parabola forms a loop. It actually forms a ellipse as we'll see later.

Also note that the matrix representing this conic is:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}$$

Example 6.23. Consider the conic defined by the equation:

$$X^2 + Y^2 - Z^2 = 0$$

If we look for $[X;Y;1] \in E^2 \equiv \mathbb{R}^2$ we find:

$$\begin{aligned} X^2 + Y^2 - 1^2 &= 0 \\ X^2 + Y^2 &= 0 \end{aligned}$$

If we look for points at infinity $[X;Y;0] \notin E^2$ we find:

$$X^2 + Y^2 = 0$$

which has no solutions.

Therefore this conic contains a circle in E^2 and contains no points at infinty.

Note in $\mathbb{R}^2 \equiv E^2$ this circle is $x^2 + y^2 = 1$.

Also note that the matrix representing this conic is:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Example 6.24. Consider the conic defined by the equation:

$$XY - Z^2 = 0$$

If we look for $[X;Y;1] \in E^2 \equiv \mathbb{R}^2$ we find:

$$\begin{aligned} XY - 1^2 &= 0 \\ XY &= 1 \end{aligned}$$

If we look for points at infinity $[X;Y;0] \notin E^2$ we find:

$$\begin{aligned} XY - 0^2 &= 0 \\ XY &= 0 \end{aligned}$$

This consists of two points at infinity. If $X = 0$ then $Y \neq 0$ and we get $[0;Y;0] \equiv [0;1;0] \in \mathbb{RP}^2$ and if $Y = 0$ then $X \neq 0$ and we get $[X;0;0] \equiv [1;0;0] \in \mathbb{RP}^2$.

Therefore this conic contains a hyperbola in E^2 and contains the two points at infinity $[0;1;0]$ and $[1;0;0]$.

Note in $\mathbb{R}^2 \equiv E^2$ this hyperbola is $xy = 1$.

Moreover observe that in E^2 on the hyperbola we have $[x;y;1] = [x;1/x;1] \equiv [1;1/x^2;1/x]$ and so as $x \to \pm\infty$ we have $[x;y;1] \to [1;0;0]$. But also we have $[x;y;1] = [1/y;y;1] \equiv [1/y^2;1;1/y]$ and so as $y \to \pm\infty$ we have $[x;y;1] \to [0;1;0]$.

Thus the right and left ends of the hyperbola actually meet at the point at infinity $[1;0;0]$ and the top and bottom ends of the hyperbola actually meet at the point at infinity $[0;1;0]$. As with the parabola the hyperbola then forms a loop. It actually also forms an ellipse as we'll see later.

Also note that the matrix representing this conic is:

$$C = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise 6.29. For each of the following conics: Find the corresponding equation in $\mathbb{R}^2 \equiv E^2$, categorize, find all points at infinity, and write down the corresponding matrix.

(a) $X^2 + YZ = 0$

(b) $4X^2 + Y^2 - 16Z^2 = 0$

(c) $XY - YZ - Z^2 = 0$

(d) $X^2 + Y^2 + 4YZ - 2Z^2 = 0$

(e) $X^2 - 4XY - YZ + Z^2 = 0$

(f) $X^2 + Y^2 - 2YZ = 0$

Here are three starting with a familar equation in $\mathbb{R}^2$.

Example 6.25. Consider the parabola defined in Euclidean geometry by:

$$y = x^2 + 2$$

First if we take a point $[X;Y;Z] \equiv [X/Z;Y/Z,1] \in E^2 \subset \mathbb{RP}^2$ this point is equivalent to $[X/Z;Y/Z] \in \mathbb{R}^2$. To satisfy our equation we must have:

$$\begin{aligned} (Y/Z) &= (X/Z)^2 + 2 \\ YZ &= X^2 + 2Z^2 \\ X^2 - YZ + 2Z^2 &= 0 \end{aligned}$$

Therefore this must be the conic which defines this parabola in E^2.

To check if we pick up any points at infinity with this equation we look for $[X;Y;0] \in\subset \mathbb{RP}^2 - E^2$ satisfying:

$$\begin{aligned} X^2 - Y(0) + 2(0)^2 &= 0 \\ X^2 &= 0 \\ X &= 0 \end{aligned}$$

We get the single point $[0;Y;0] \equiv [0;1;0]$.

Also note that the matrix representing this conic is:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 2 \end{bmatrix}$$

Example 6.26. Consider the circle defined in Euclidean geometry by:

$$(x-1)^2 + y^2 = 4$$

First if we take a point $[X;Y;Z] \equiv [X/Z;Y/Z,1] \in E^2 \subset \mathbb{RP}^2$ this point is equivalent to $[X/Z;Y/Z] \in \mathbb{R}^2$. To satisfy our equation we must have:

$$\begin{aligned}(X/Z-1)^2+(Y/Z)^2&=4\\ \frac{X^2}{Z^2}-\frac{2X}{Z}+1+\frac{Y^2}{Z^2}-4&=0\\ X^2-2XZ+Y^2-3Z^2&=0\end{aligned}$$

Therefore this must be the conic which defines this circle in E^2.

To check if we pick up any points at infinity with this equation we look for $[X;Y;0] \in\subset \mathbb{RP}^2 - E^2$ satisfying:

$$\begin{aligned}X^2-2X(0)+Y^2-3(0)^2&=0\\ X^2+Y^2&=0\end{aligned}$$

There are no such points.

Also note that the matrix representing this conic is:

$$C=\begin{bmatrix}1&0&-1\\0&1&0\\-1&0&-3\end{bmatrix}$$

Example 6.27. Consider the hyperbola defined in Euclidean geometry by:

$$x^2-y^2=1$$

First if we take a point $[X;Y;Z] \equiv [X/Z;Y/Z,1] \in E^2 \subset \mathbb{RP}^2$ this point is equivalent to $[X/Z;Y/Z] \in \mathbb{R}^2$. To satisfy our equation we must have:

$$\begin{aligned}(X/Z)^2-(Y/Z)^2&=1\\ X^2-Y^2&=Z^2\\ X^2-Y^2-Z^2&=0\end{aligned}$$

Therefore this must be the conic which defines this hyperbola in E^2.

To check if we pick up any points at infinity with this equation we look for $[X;Y;0] \in\subset \mathbb{RP}^2 - E^2$ satisfying:

$$\begin{aligned}X^2-Y^2-(0)^2&=0\\ Y^2&=X^2\\ Y&=\pm X\end{aligned}$$

This consists of two points at infinity. If $Y = X$ then we get $[X;X;0] \equiv [1;1;0] \in \mathbb{RP}^2$ and if $Y = -X$ then we get $[X;-X;0] \equiv [1;-1;0] \in \mathbb{RP}^2$

Also note that the matrix representing this conic is:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise 6.30. For each of the following familiar equations in $\mathbb{R}^2$: Categorize, find the corresponding conic equation, find any points at infinity which are picked up by this equation, and write down the corresponding matrix.

(a) $x^2 + y^2 = 9$

(b) $x^2 - y^2 = 9$

(c) $(x-3)^2 + (y+1)^2 = 4$

(d) $2x^2 + 10y^2 = 20$

(e) $y + x^2 = 0$

(f) $y = x^2 + 2x + 8$

6.6.2 Standard Transformations of Conics

Given that a conic is represented by a matrix C we might ask what happens if a projective transformation is applied to the conic.

Before the general statement we can get inspiration from the following example: Suppose $\mathbf{x} = [X; Y; Z]$ lies on the circle of radius 2 centered at $[1; 2]$. This means that if we shift $\mathbf{x}$ using $T_Z(1,2)^{-1}$ then the result lies on the circle of radius 2 centered at $(0; 0)$. In other words the vector $T_Z(1,2)^{-1}\mathbf{x}$ satisfies:

$$\left(T_Z(1,2)^{-1}\mathbf{x}\right)^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \left(T_Z(1,2)^{-1}\mathbf{x}\right) = 0$$

which shows that the matrix

$$\left(T_Z(1,2)^{-1}\right)^T C T_Z(1,2)^{-1}$$

represents the circle of radius 2 centered at $[1; 2]$. In fact we can see this by actually multiplying to get:

$$\left(T_Z(1,2)^{-1}\right)^T C T_Z(1,2)^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

which represents the equation

$$X^2 + Y^2 - 2XZ - 4YZ + Z^2 = 0$$

Notice that this makes sense, since in Euclidean geometry the circle would have equation

$$(x-1)^2 + (y-2)^2 - 4 = 0$$

and for a projective point $[X;Y;Z] \equiv [X/Z;Y/Z;1]$ to satisfy this we would require

$$\begin{aligned}(X/Z-1)^2 + (Y/Z-2)^2 - 4 &= 0\\ X^2/Z^2 - 2X/Z + 1 + Y^2/Z^2 - 2Y/Z + 4 - 4 &= 0\\ X^2 - 2XZ + Y^2 - 2YZ + Z^2 &= 0\end{aligned}$$

This is precisely the same, of course.

Theorem 6.6.2.1. If a conic represented by the matrix C is transformed by the matrix M then the resulting conic is represented by the matrix:

$$(M^{-1})^T C M^{-1}$$

Proof. Suppose $\mathbf{x}$ is on the transformed conic. Then $M^{-1}\mathbf{x}$ is on the original conic and so:

$$\begin{aligned}\left(M^{-1}\mathbf{x}\right)^T C \left(M^{-1}\mathbf{x}\right) &= 0\\ \mathbf{x}^T \left(M^{-1}\right)^T C M^{-1}\mathbf{x} &= 0\end{aligned}$$

□

Example 6.28. To rotate the ellipse $4x^2 + y^2 = 16$ by $\pi/4$ radians about the origin we first find the conic equation so we can calculate the matrix:

$$\begin{aligned}4(X/Z)^2 + (Y/Z)^2 &= 16\\ 4X^2 + Y^2 &= 16Z^2\\ 4X^2 + Y^2 - 16Z^2 &= 0\end{aligned}$$

Thus we have:

$$C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -16 \end{bmatrix}$$

The matrix representing the rotated ellipse is then:

$$(R_Z(\pi/4)^{-1})^T C R_Z(\pi/4)^{-1} = \begin{bmatrix} 5/2 & 3/2 & 0 \\ 3/2 & 5/2 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

The conic equation is then:

$$\frac{5}{2}x^2 + 3XY + \frac{5}{2}Y^2 + 16Z^2 = 0$$

The equation in $\mathbb{R}^2$ is then:

$$\frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2 + 16 = 0$$

Example 6.29. To rotate the parabola $y = x^2$ by $\pi/2$ radians about the point $[2; 1]$ we first find the conic equation so we can calculate the matrix:

$$\begin{aligned} Y/Z &= (X/Z)^2 \\ YZ &= X^2 \\ X^2 - YZ &= 0 \end{aligned}$$

Thus we have:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix}$$

The matrix representing the rotated parabola is then:

$$\begin{aligned} &\left((T_Z(2,1)R_Z(\pi/2)T_Z(-2,-1))^{-1}\right)^T C\,(T_Z(2,1)R_Z(\pi/2)T_Z(-2,-1))^{-1} \\ &= \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 1 \\ 1/2 & 1 & -2 \end{bmatrix} \end{aligned}$$

The conic equation is then:

$$XZ + Y^2 + 2YZ - 2Z^2 = 0$$

The equation in $\mathbb{R}^2$ is then:

$$\begin{aligned} x + y^2 + 2y - 2 &= 0 \\ x + (y+1)^2 - 3 &= 0 \\ x &= 3 - (y+1)^2 \end{aligned}$$

Note that this makes sense. If you're not convinced, draw a picture!

Exercise 6.31. Draw a picture to convince yourself that the previous example makes sense.

Exercise 6.32. Given the equation $x^2 + y^2 = 1$ in $\mathbb{R}^2$, show that translation $T_z(1, 2)$ does as expected.

Exercise 6.33. Find the equation in $\mathbb{R}^2$ that results when the parabola $y = x^2$ is rotated by $\pi/4$ about the origin.

Exercise 6.34. Find the equation in $\mathbb{R}^2$ that results when the ellipse $4x^2+9y^2 = 36$ is rotated by $\pi/3$ about the origin.

Exercise 6.35. Find the equation in $\mathbb{R}^2$ that results when the hyperbola $xy = 1$ is reflected in the line $y = 3x$.

6.6.3 Equivalency of Conics

An ellipse is a parabola is a hyperbola.

It turns out that the two ends of a parabola meet, and the parabola forms an ellipse.

It turns out that the four ends of a hyperbola meet in pairs and the hyperbola forms an ellipse.

Basically projectively they're all the same shape, the only difference is that an ellipse doesn't touch the line at infinity, a parabola touches it once and a hyperbola touches it twice.

But let's be more formal. What does "the same" mean? In other words if we say that a parabola is "the same as" an ellipse what exactly do we mean by this?

Here's an example from Euclidean geometry:

Example 6.30. Consider that in Euclidean space the equations $y = x^2$ and $x = y^2$ are both parabolas because all we've done is switch variables.

So now as a semi-formal example from $\mathbb{RP}^2$:

Example 6.31. Consider the two equations:

$$X^2 + Y^2 - Z^2 = 0 \text{ and } X^2 + Z^2 - Y^2 = 0$$

In any reasonable sense these ought to define the same shape because all we've done is switch the variables around.

However if we look at the Euclidean points $[x; y] \equiv [x; y; 1]$ which satisfy each of these we see that in the first one we get:

$$\begin{aligned} x^2 + y^2 - 1^2 &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$$

and in the second one we get:

$$\begin{aligned} x^2 + 1^2 - y^2 &= 0 \\ x^2 - y^2 &= 1 \end{aligned}$$

So that the first is a circle and the second is a hyperbola.

What's actually happening here is that these are the same shape in $\mathbb{RP}^2$ but when we look only at the Euclidean patch of $\mathbb{RP}^2$ we get a circle for the first and a hyperbola for the second.

This shouldn't surprise you at this point. Earlier on we observed that this hyperbola $x^2 - y^2 = 1$ picks up two points at infinity, $[1; 1; 0]$ and $[1; -1; 0]$.

Moreover in E^2 if we follow the hyperbola along the branch $y = +\sqrt{x^2 - 1}$ as $x \to \infty$ we see:

$$[x; y; 1] = [x; \sqrt{x^2 - 1}; 1] \equiv [1; \sqrt{1 - 1/x^2}; 1/x] \to [1; 1; 0]$$

and along the branch $y = -\sqrt{x^2 - 1}$ as $x \to -\infty$ we see:

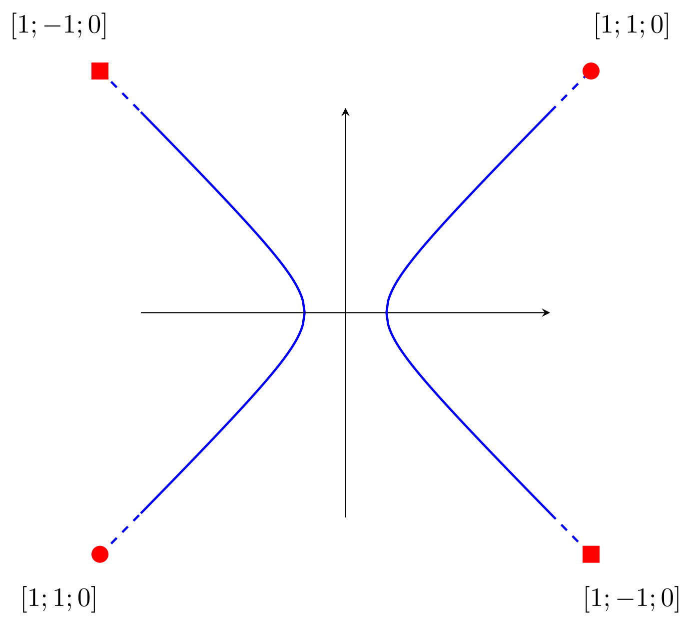
$$[x; y; 1] = [x; -\sqrt{x^2 - 1}; 1] \equiv [1; \sqrt{1 + 1/x^2}; 1/x] \to [1; 1; 0]$$

So those two branches meet at $[1; 1; 0]$.

A similar argument shows that the two branches $y = +\sqrt{x^2 - 1}$ as $x \to -\infty$ and $y = -\sqrt{x^2 - 1}$ as $x \to \infty$ meet at $[1; -1; 0]$

Graphically the two branches meet at the red circle (single point at infinity) and the other two brances meet at the red square (single point at infinity).

Thus a circle is formed.



Exercise 6.36. Consider the two equations:

$$Y^2 - XZ = 0 \text{ and } Z^2 - XY = 0$$

These ought to be define the same shape because the only difference is the switch of Z and Y. What do they represent in E^2?

Exercise 6.37. Write down two conic equations which are identical other than switching of variables, where one represents a circle and the other a parabola.

Exercise 6.38. The parabola $x = y^2$, when moved into $\mathbb{RP}^2$, picks up a single point at infinity. Find this point and show algebraically that both ends of the parabola meet there.

6.7 Fancy Transformations

All our transformations so far, for obvious reasons, have preserved E^2. However we can certainly define other transformations of $\mathbb{RP}^2$ which don't preserve E^2. These will have interesting effects on the parts of objects which are in E^2.

6.7.1 More Rotations

As a reminder, recall:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can also get fancy and do two other types of rotations. Consider the additional two matrices:

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

This rotation should not be described as rotation around an axis but rather as transformation of $\mathbb{RP}^2$ which fixes X and for each fixed X_0 acts on $[X_0; Y; Z]$ as a rotation with regards to Y and Z.

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

This rotation should not be described as rotation around an axis but rather as transformation of $\mathbb{RP}^2$ which fixes Y and for each fixed Y_0 acts on $[X; Y_0; Z]$ as a rotation with regards to X and Z.

Notice the odd positioning of the signs for the sine functions in R_Y. The reason for this is nuanced. Given that $\mathbb{RP}^2$ is really $\mathbb{R}^3 - \{0\}$ with an equivalence of vectors imposed, these three rotations are actually rotations about axes of $\mathbb{R}^3$ and really, in $\mathbb{R}^3$ it is true that R_Y is a rotation about the Y-axis. Moreover it's natural to want R_Y to obey the right-hand rule and take the positive Z-axis towards the positive X-axis and not the other way around. The repositioning of the signs is what accomplishes this.

Neither of these necessarily preserves $E^2 \subset \mathbb{RP}^2$ and hence can lead to interesting results.

Example 6.32. Observe that:

$$\begin{aligned}
R_Y(\pi/2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} \cos(\pi/2) & 0 & \sin(\pi/2) \\ 0 & 1 & 0 \\ -\sin(\pi/2) & 0 & \cos(\pi/2) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

This shows that rotation $R_Y(\pi/2)$ moves the point $[0;0;1] \in E^2)$ to the point at infinity $[1;0;0] \notin E^2$.

Exercise 6.39. Where does $R_X(\pi/2)$ take $[0;0;1]$?

Exercise 6.40. Find a rotation which takes $[1;0;0]$ to $[0;1;0]$.

6.7.2 More Translations

As a reminder recall:

$$T_Z(a,b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

In light of this we can do translations which fix X and Z. Consider the two additional matrices:

$$T_X(a,b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \text{ with } \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y + aX \\ Z + bX \end{bmatrix}$$

and

$$T_Y(a,b) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \text{ with } \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X + aY \\ Y \\ Z + bY \end{bmatrix}$$

The first of these fixes the set with $X = 1$ while the second fixes the set with $Y = 1$.

Neither of these necessarily preserve the Euclidean patch, however, because both of these can take a point with nonzero third coordinate to a point with zero third coordinate.

Example 6.33. If $Z \neq 0$ but $Z + bX = 0$ then $T_x(a,b)$ takes a point in E^2 to a point at infinity.

Example 6.34. Consider that:

$$T_X(2,3)[X;Y;-3X] = [X;Y+3X;-3X+3X] = [X;Y+3X;0]$$

Thus in fact infinitely many points with nonzero third coordinate are taken to a point with zero third coordinate since this happens to all points with $X \neq 0$.

Exercise 6.41. Find some specific examples whereby T_X and T_Y take points at infinity to points in E^2. Can you create any general rules?

6.7.3 More Reflections

Last but not least recall our reflection which negates Y:

$$F_Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We may also define:

$$F_X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$F_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The second of these negates X, effectively reflecting $\mathbb{R}^2 \equiv E^2$ in the $x \equiv X$-axis.

The third of these actually negates both X and Y because:

$$F_Z \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ -Z \end{bmatrix} \equiv \begin{bmatrix} -X \\ -Y \\ Z \end{bmatrix}$$

In this sense as far as its action on $\mathbb{R}^2 \equiv E^2$ is concerned it is actually a rotation about the origin by π radians.

6.7.4 Fancy Transformations of Lines and Conics

In earlier sections we discussed how transformations affect the vectors which represent lines and the matrices which represent conics. Nothing in the theorems changes. As a reminder they were:

Theorem 6.7.4.1. If a line represented by the vector $\mathbf{L}$ is transformed by the matrix M then the resulting line is represented by the matrix:

$$\left(M^{-1}\right)^T \mathbf{L}$$

Theorem 6.7.4.2. If a conic represented by the matrix C is transformed by the matrix M then the resulting conic is represented by the matrix:

$$(M^{-1})^T C M^{-1}$$

Let's look at what can happen if we apply these to lines and conics:

Example 6.35. If we rotate the conic (a parabola) $X^2 - YZ = 0$ using R_X we have:

$$\left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \right)^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The result represents the equation:

$$XY + Z^2 = 0$$

This is another conic (a hyperbola).

Notice in the Euclidean patch the parabola has equation $y = x^2$ and the hyperbola has equation $y = -\frac{1}{x}$ so the rotation has moved the parabola to a hyperbola.

Exercise 6.42. Find the equation in $\mathbb{R}^2$ that results when the circle $x^2+y^2=9$ is translated using $T_X(1,2)$. Is the result a circle? If not, what it it?

Exercise 6.43. Find the equation in $\mathbb{R}^2$ that results when the parabola $y=x^2$ is rotated using $R_X(1.2)$. Is the result a parabola? If not, what it it?

Exercise 6.44. Suppose $\mathbf{L}=[4;0;2]$ represents a line in $\mathbb{RP}^2$. If we translate this line by $T_X(-1;5)$ which vector represents the resulting line? Is this a (Euclidean) translation of the original line?

As an interesting closing example consider the rotation $R_X(\theta)$ applied to the circle $x^2+y^2=1$ for $0\le\theta\le\pi/4$. Note that $\pi/4\approx 0.7854$. The circle has matrix:

$$C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Here are some specific examples of the rotation applied to this circle:

$$(R_X(0.2)^{-1})^T M(R_X(0.2)^{-1} \approx \begin{bmatrix} 1.00 & 0 & 0 \\ 0 & 0.92 & 0.39 \\ 0 & 0.39 & -0.92 \end{bmatrix}$$

$$(R_X(0.2)^{-1})^T M(R_X(0.2)^{-1} \begin{bmatrix} 1.00 & 0 & 0 \\ 0 & 0.70 & 0.72 \\ 0 & 0.72 & -0.70 \end{bmatrix}$$

$$(R_X(0.6)^{-1})^T M(R_X(0.6)^{-1} \approx \begin{bmatrix} 1.00 & 0 & 0 \\ 0 & 0.36 & 0.93 \\ 0 & 0.93 & -0.36 \end{bmatrix}$$

$$(R_X(\pi/4)^{-1})^T M(R_X(\pi/4)^{-1} = \begin{bmatrix} 1.00 & 0 & 0 \\ 0 & 0 & 1.00 \\ 0 & 1.00 & 0 \end{bmatrix}$$

Including the circle and going in turn these represent, most approximately:

The circle $x^2+y^2=1$.

The ellipse $x^2+0.92y^2+0.78y-0.92=0$.

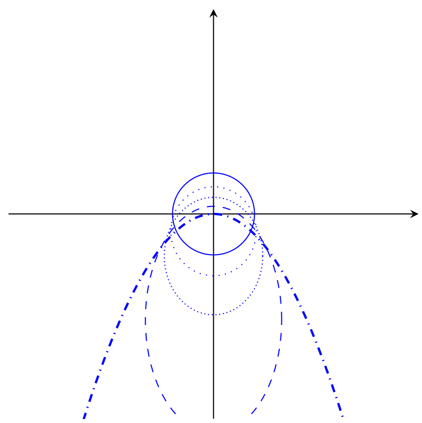
The ellipse $x^2+0.7y^2+1.44y-0.7=0$.

The ellipse $x^2+0.36y^2+1.86y-0.36=0$.

The parabola $x^2+2y=0$.

In the picture below these are in turn solid (the circle), dotted (an ellipse), densely dotted (an ellipse), dashed (an ellipse) and dash-dotted (the parabola).

We see that as θ increases towards $\pi/4$ the circles changes to an ellipse whose center moves down and which, when $\theta = \pi/4$, becomes a parabola as the lowest point moves to a point at infinity.



Exercise 6.45. What happens to the previous example as θ keeps going from $\pi/4$ to $\pi/2$? Experiment and draw some pictures to find out.

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