CMSC427
Computer Graphics

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Fall 2018

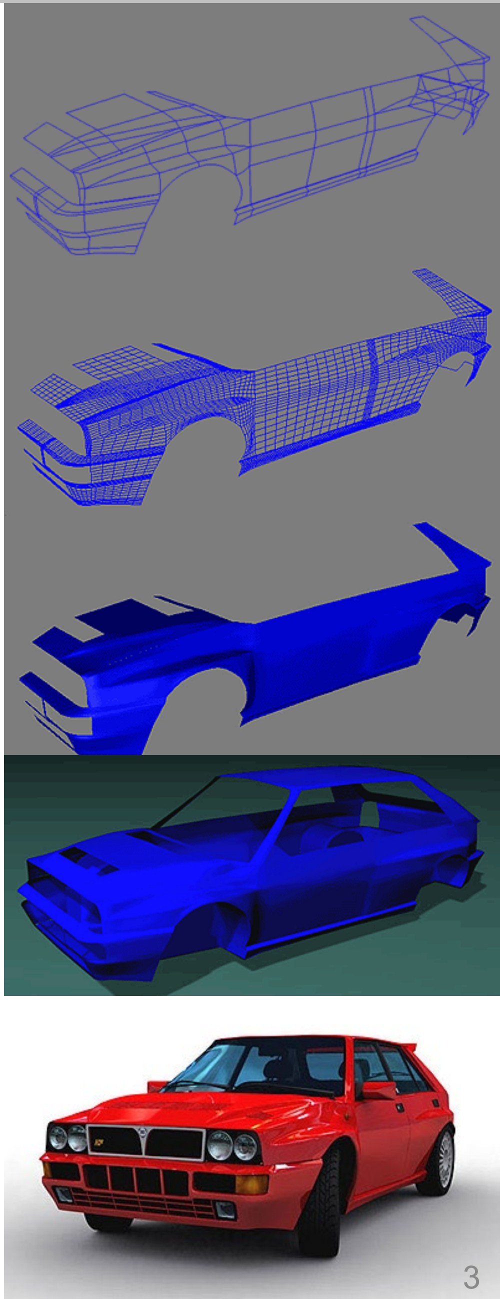
Today

Curves

- Introduction
- Polynomial curves
- Bézier curves
- Drawing Bézier curves
- Piecewise curves

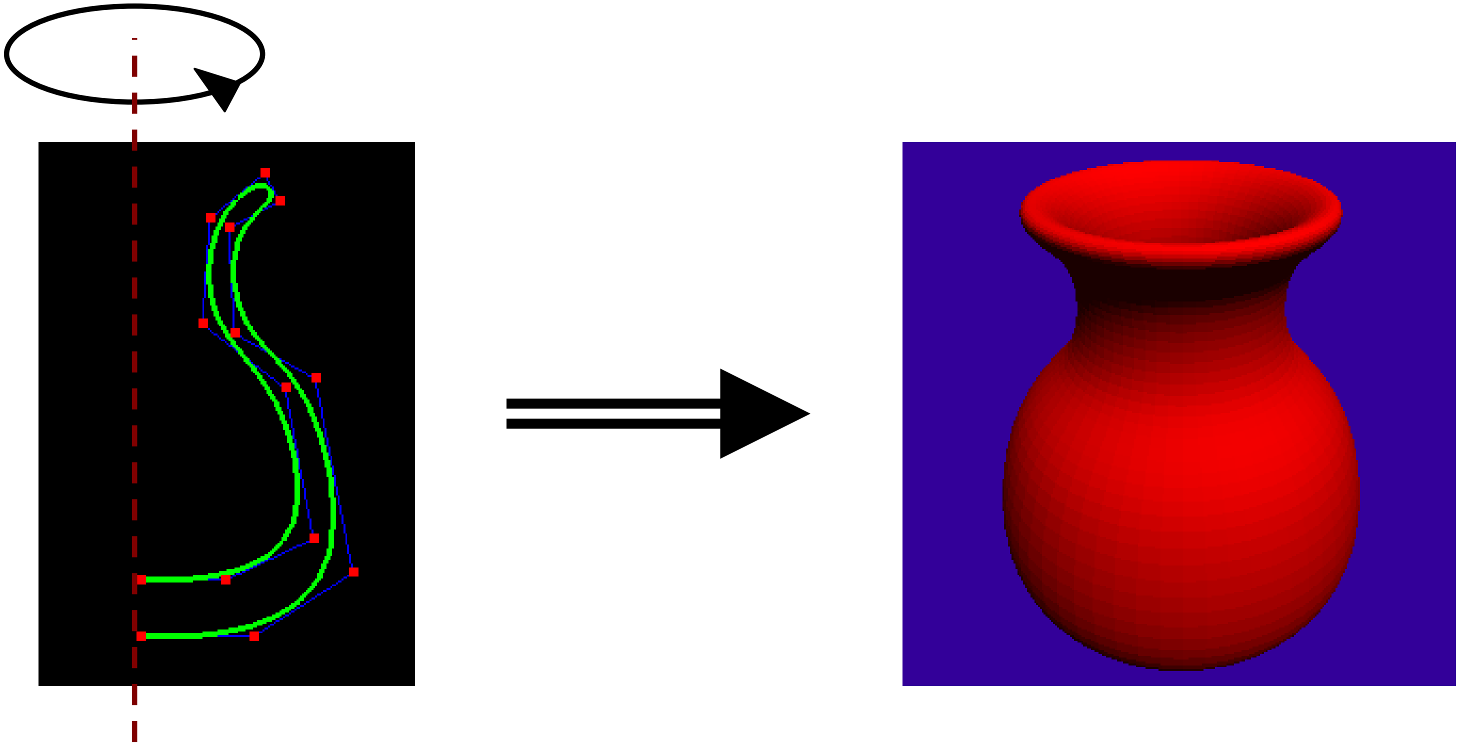
Modeling

- Creating 3D objects
- How to construct complicated surfaces?
- Goal
  - Specify objects with few control points
  - Resulting object should be visually pleasing (smooth)
- Start with curves, then generalize to surfaces



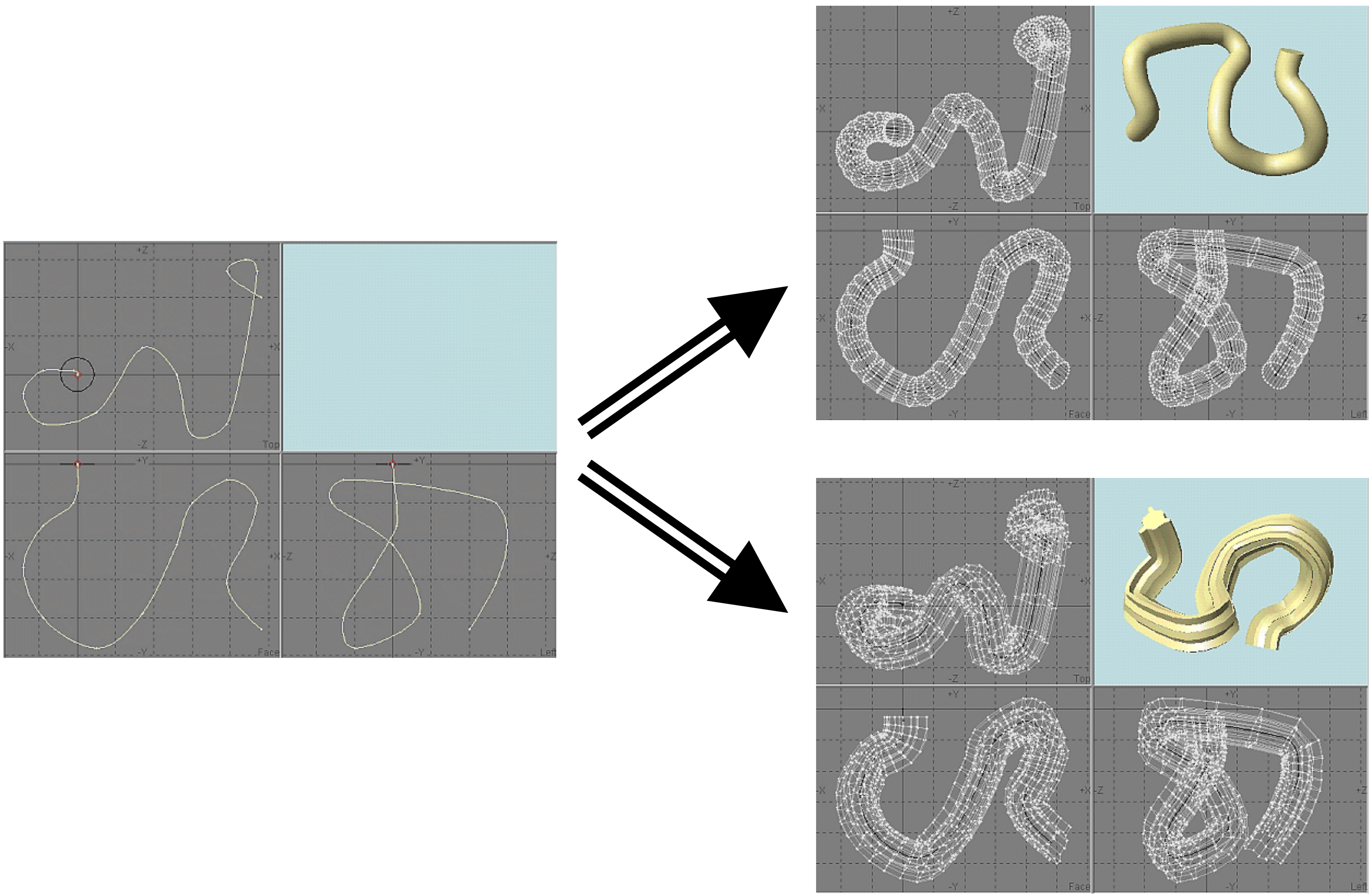
Usefulness of curves

- Surface of revolution



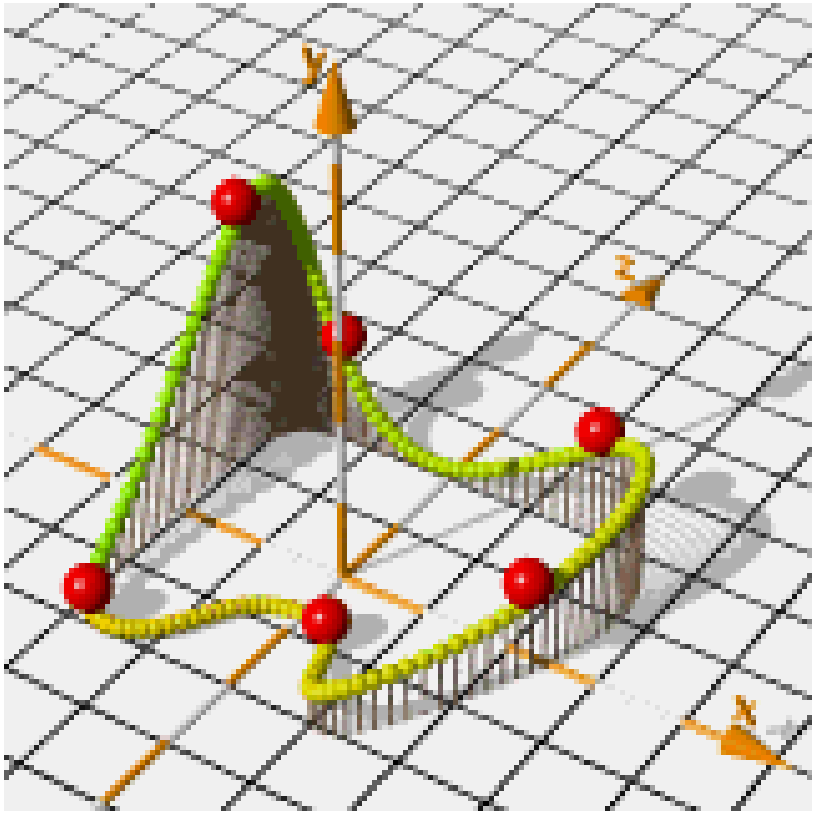
Usefulness of curves

- Extruded/swept surfaces



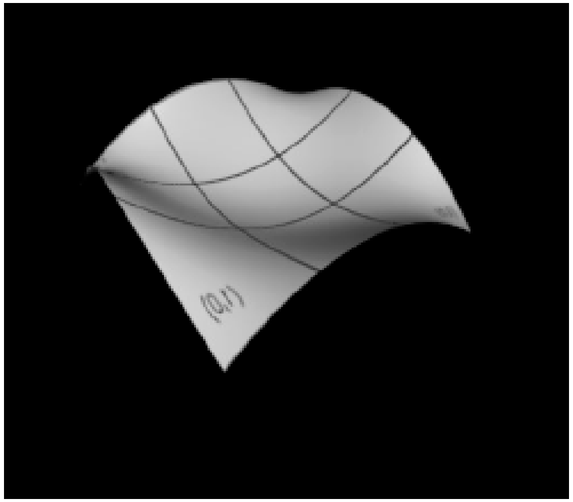
Usefulness of curves

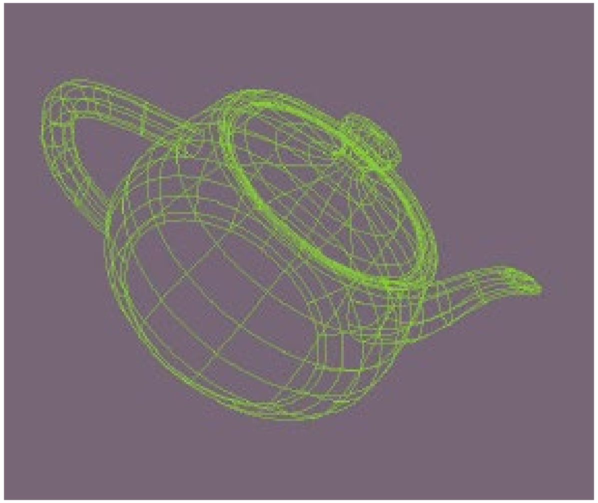
- Animation
  - Provide a “track” for objects
  - Use as camera path



Usefulness of curves

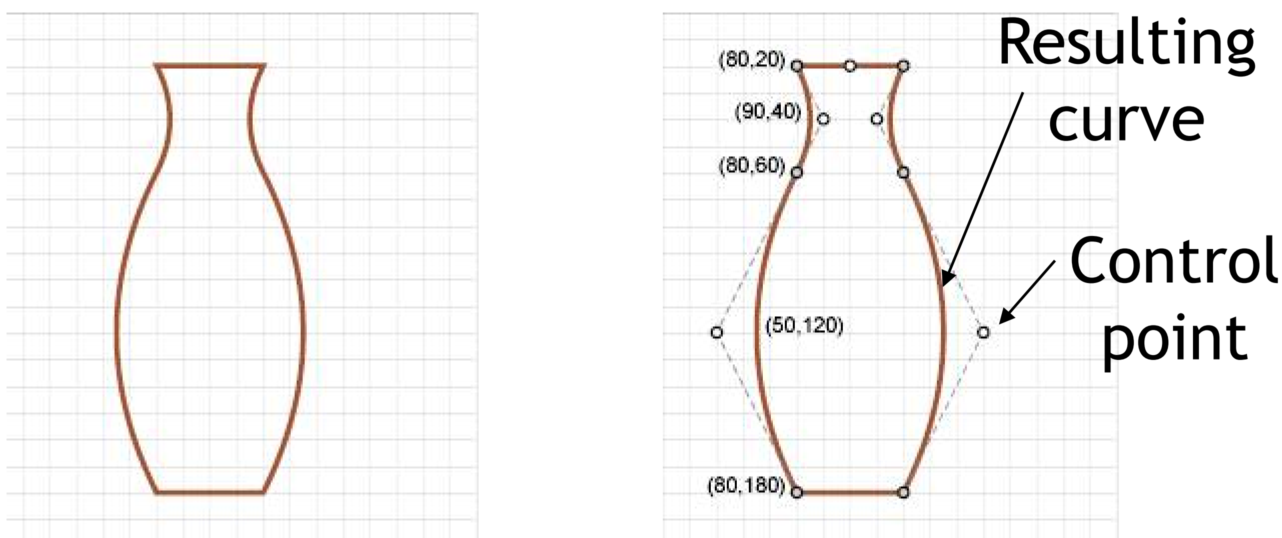
- Generalize to surface patches using “grids of curves”, next class





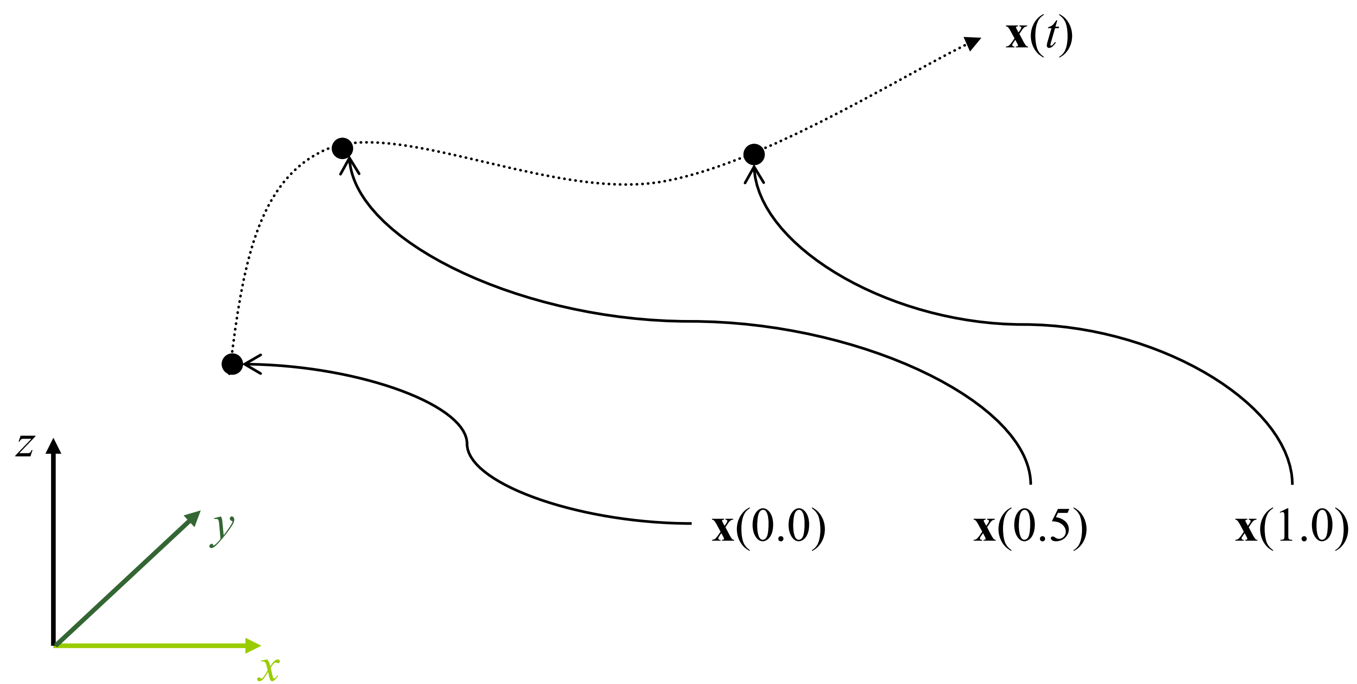
How to represent curves

- Specify every point along curve?
  - Hard to get precise, smooth results
  - Too much data, too hard to work with
- Idea: specify curves using small numbers of control points
- Mathematics: use polynomials to represent curves



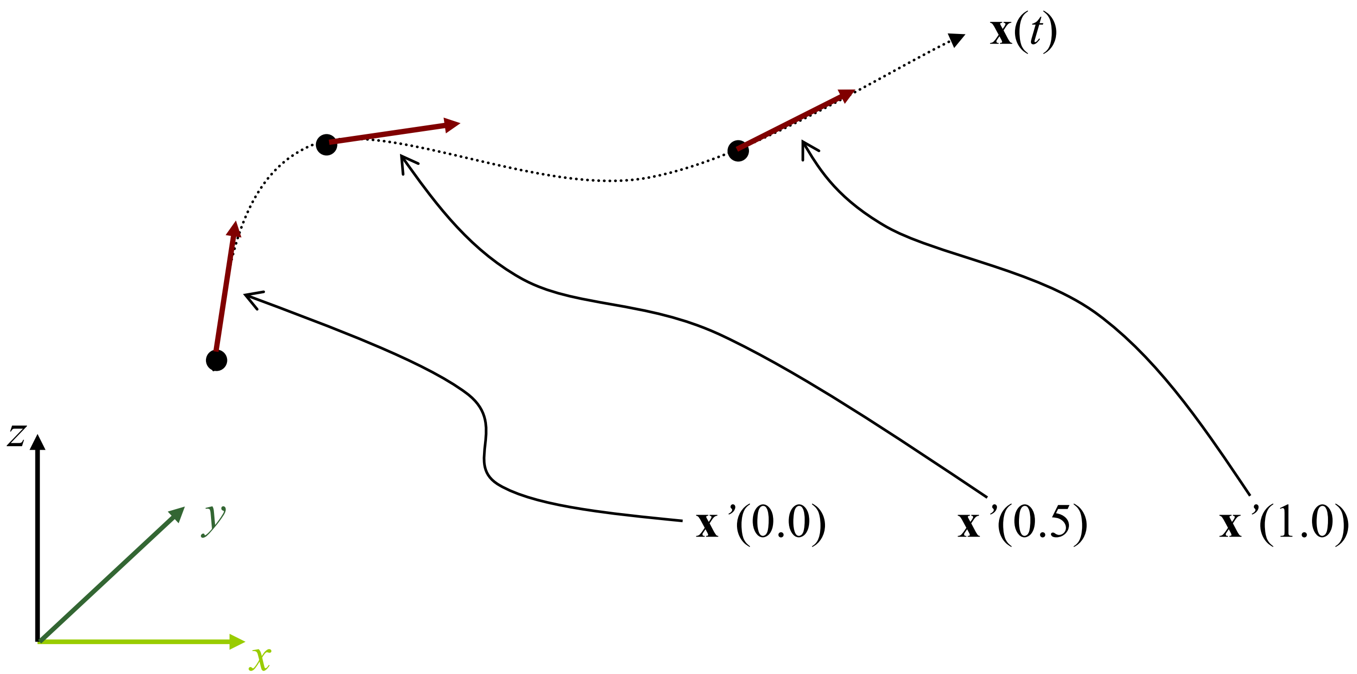
Mathematical definition

- A vector valued function of one variable $\mathbf{x}(t)$
  - Given t, compute a 3D point $\mathbf{x}=(x,y,z)$
  - May interpret as three functions $x(t)$, $y(t)$, $z(t)$
  - "Moving a point along the curve"



Tangent vector

- Derivative $\mathbf{x}'(t) = \frac{d\mathbf{x}}{dt} = (x'(t), y'(t), z'(t))$
- A vector that points in the direction of movement
- Length of $\mathbf{x}'(t)$ corresponds to speed



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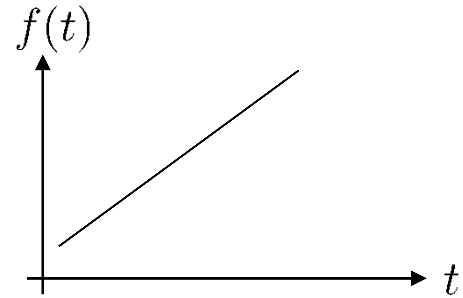
Polynomial functions

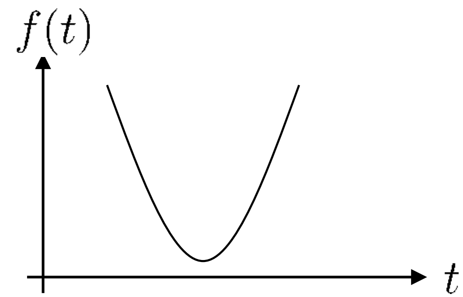
- Linear: $f(t) = at + b$

  (1st order)



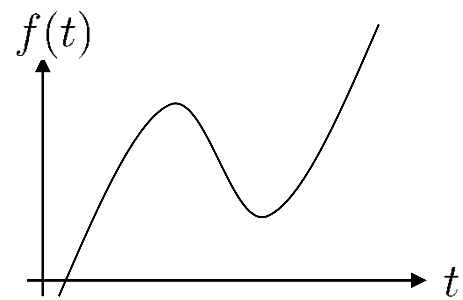
- Quadratic: $f(t) = at^2 + bt + c$

  (2nd order)



- Cubic: $f(t) = at^3 + bt^2 + ct + d$

  (3rd order)



Polynomial curves

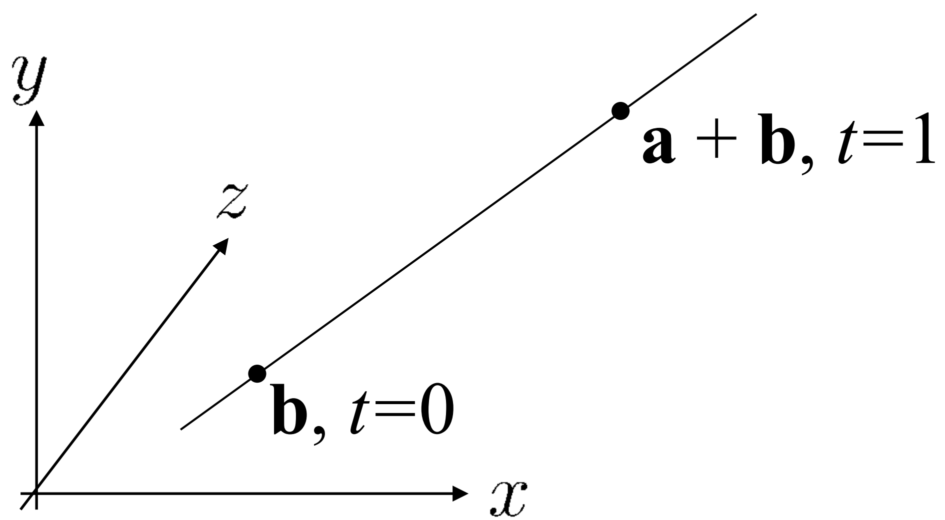
- Linear $$\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$$

$$\mathbf{x} = (x, y, z), \mathbf{a} = (a_x, a_y, a_z), \mathbf{b} = (b_x, b_y, b_z)$$

- Evaluated as $$x(t) = a_x t + b_x$$

$$y(t) = a_y t + b_y$$

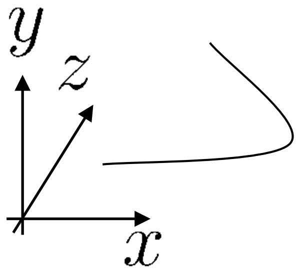
$$z(t) = a_z t + b_z$$



Polynomial curves

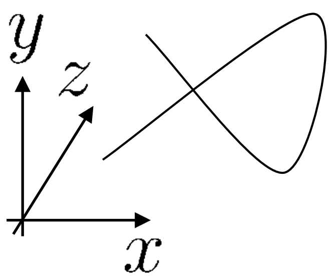
- Quadratic: $\mathbf{x}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$

  (2nd order)



- Cubic: $\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$

  (3rd order)



- We usually define the curve for $0 \le t \le 1$

Control points

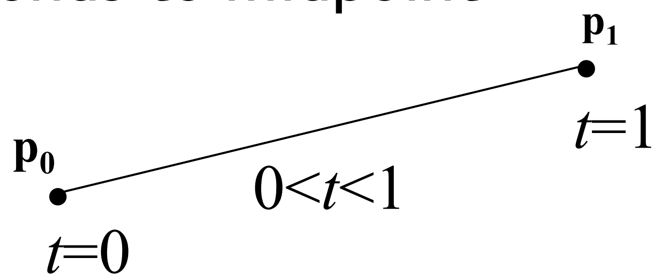
- Polynomial coefficients **a, b, c, d** etc. can be interpreted as 3D control points
  - Remember **a, b, c, d** have x,y,z components each
- Unfortunately, polynomial coefficients don't intuitively describe shape of curve
- Main objective of curve representation is to come up with intuitive control points
  - Position of control points predicts shape of curve

Control points

- How many control points?
  - Two points define a line (1st order)
  - Three points define a quadratic curve (2nd order)
  - Four points define a cubic curve (3rd order)
  - $k+1$ points define a k-order curve
- Let's start with a line…

First order curve

- Based on linear interpolation (LERP)
  http://en.wikipedia.org/wiki/Linear_interpolation
  - Weighted average between two values
  - "Value" could be a number, vector, color, …
- Interpolate between points $\mathbf{p}_0$ and $\mathbf{p}_1$ with parameter t
  - Defines a "curve" that is straight (first-order curve)
  - $t=0$ corresponds to $\mathbf{p}_0$
  - $t=1$ corresponds to $\mathbf{p}_1$
  - $t=0.5$ corresponds to midpoint



$$\mathbf{x}(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = (1-t)\mathbf{p}_0 + t\ \mathbf{p}_1$$

Linear interpolation

- Three different ways to write it
  - Equivalent, but different properties become apparent
  - Advantages for different operations, see later

1. Weighted sum of control points

$$\mathbf{x}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t$$

2. Polynomial in t

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0 t^0$$

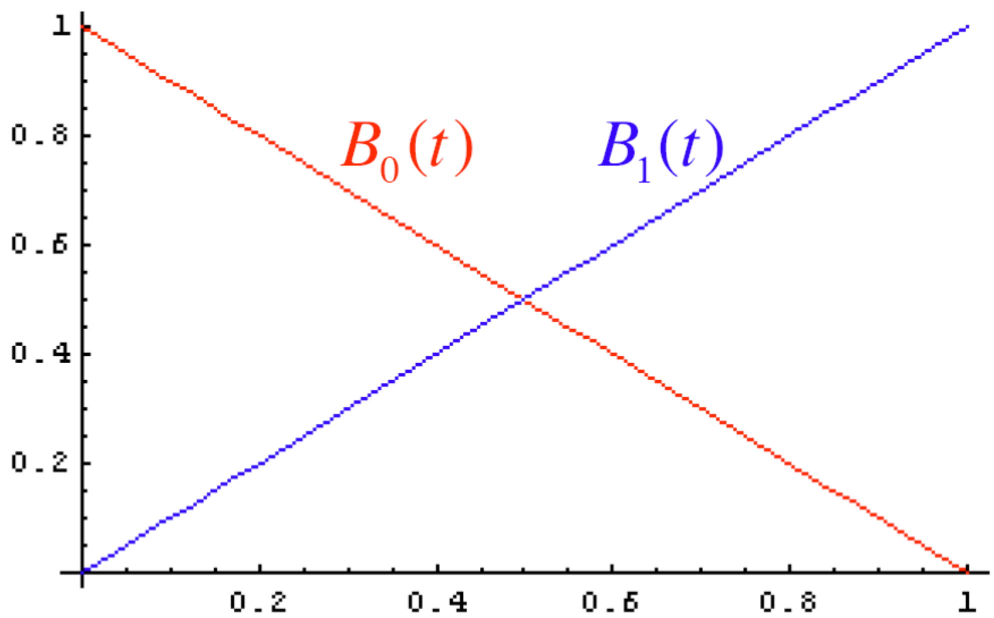
3. Matrix form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Weighted sum of control points

$$\mathbf{x}(t) = (1-t)\mathbf{p}_0 + \quad (t)\mathbf{p}_1$$
$$= B_0(t)\,\mathbf{p}_0 + B_1(t)\mathbf{p}_1, \text{ where } B_0(t) = 1-t \text{ and } B_1(t) = t$$

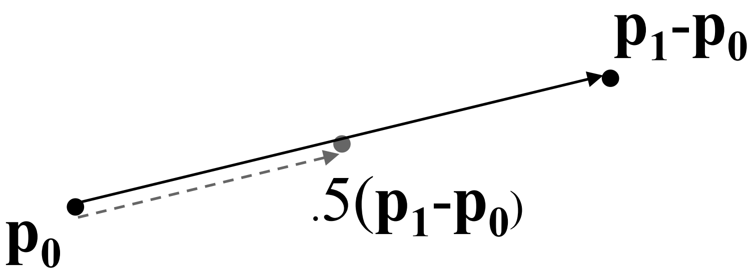
- Weights $\mathrm{B}_0(t)$, $\mathrm{B}_1(t)$ are functions of t
  - Sum is always 1, for any value of t
  - Also known as basis or blending functions



Linear polynomial

$$\mathbf{x}(t) = \underbrace{(\mathbf{p}_1 - \mathbf{p}_0)}_{\substack{\text{vector} \\ \mathbf{a}}}\, t + \underbrace{\mathbf{p}_0}_{\substack{\text{point} \\ \mathbf{b}}}$$

- Curve is based at point $\mathbf{p_0}$
- Add the vector, scaled by t



Matrix form

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \mathbf{GBT}$$

- Geometry matrix $\mathbf{G} = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix}$
- Geometric basis $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$
- Polynomial basis $T = \begin{bmatrix} t \\ 1 \end{bmatrix}$
- In components

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} \\ p_{0y} & p_{1y} \\ p_{0z} & p_{1z} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Tangent

- For a straight line, the tangent is constant

$$\mathbf{x}'(t) = \mathbf{p}_1 - \mathbf{p}_0$$

- Weighted average

$$\mathbf{x}(t) = \mathbf{p}_0(1-t) + \mathbf{p}_1 t \;\longrightarrow\; \mathbf{x}'(t) = (-1)\mathbf{p}_0 + (+1)\mathbf{p}_1$$

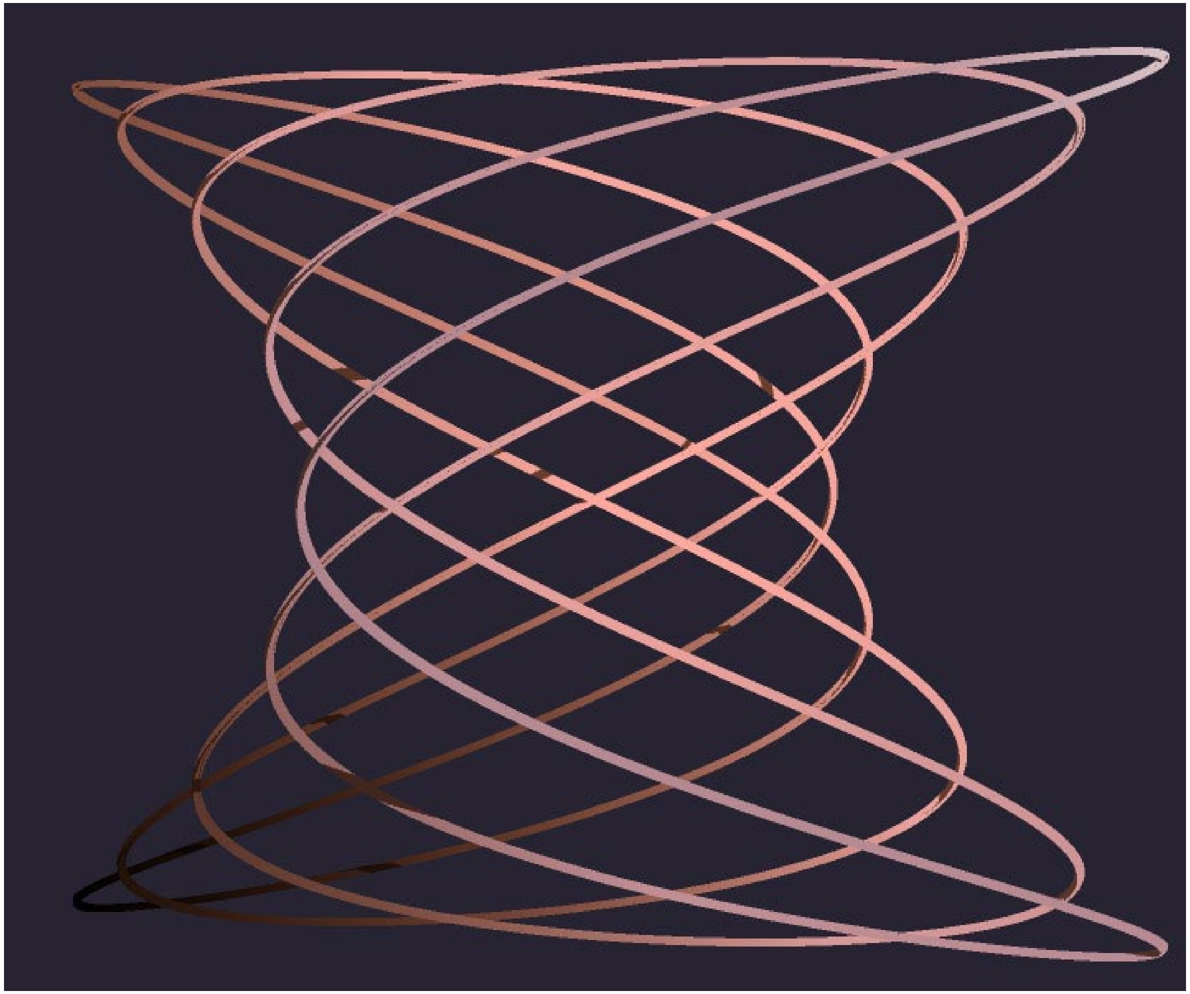
- Polynomial

$$\mathbf{x}(t) = (\mathbf{p}_1 - \mathbf{p}_0)t + \mathbf{p}_0 \;\longrightarrow\; \mathbf{x}'(t) = 0t + (\mathbf{p}_1 - \mathbf{p}_0)$$

- Matrix form

$$\mathbf{x}'(t) = \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Side note: Lissajous curves



http://en.wikipedia.org/wiki/Lissajous_curve

What type of mathematical function is used here?

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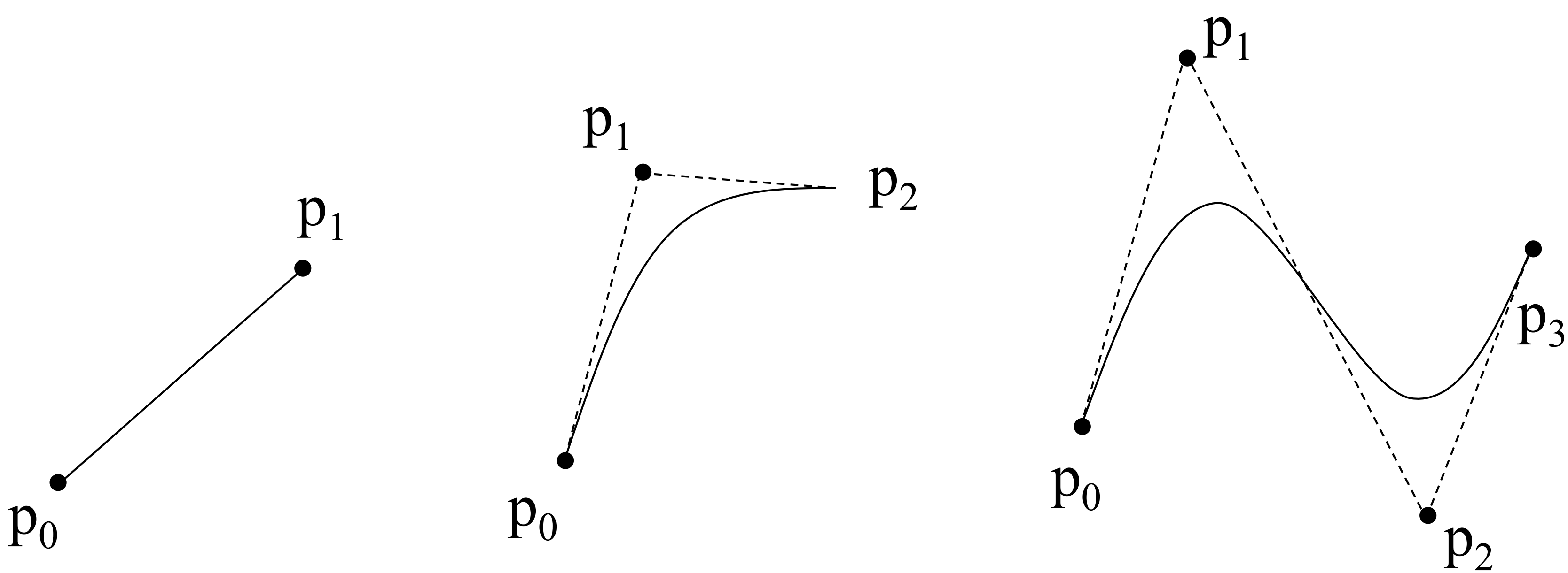
Bézier curves

http://en.wikipedia.org/wiki/B%C3%A9zier_curve

- A particularly intuitive way to define control points for polynomial curves
- Developed for CAD (computer aided design) and manufacturing
  - Before games, before movies, CAD was the big application for CG
- Pierre Bézier (1962), design of auto bodies for Peugeot, http://en.wikipedia.org/wiki/Pierre_B%C3%A9zier
- Paul de Casteljau (1959), for Citroen

Bézier curves

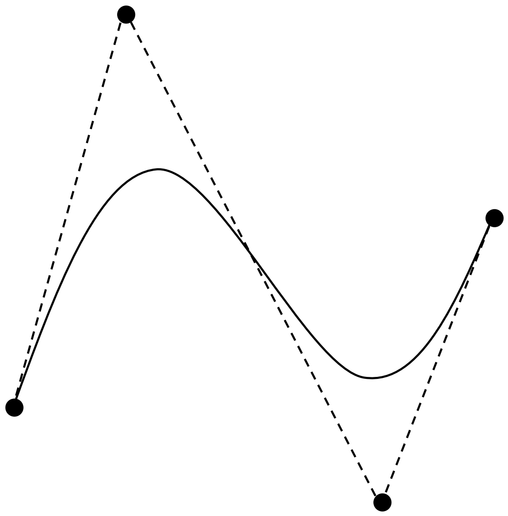
- Can be considered higher order extension of linear interpolation
- Control points $p_0, p_1, \ldots$



Linear Quadratic Cubic

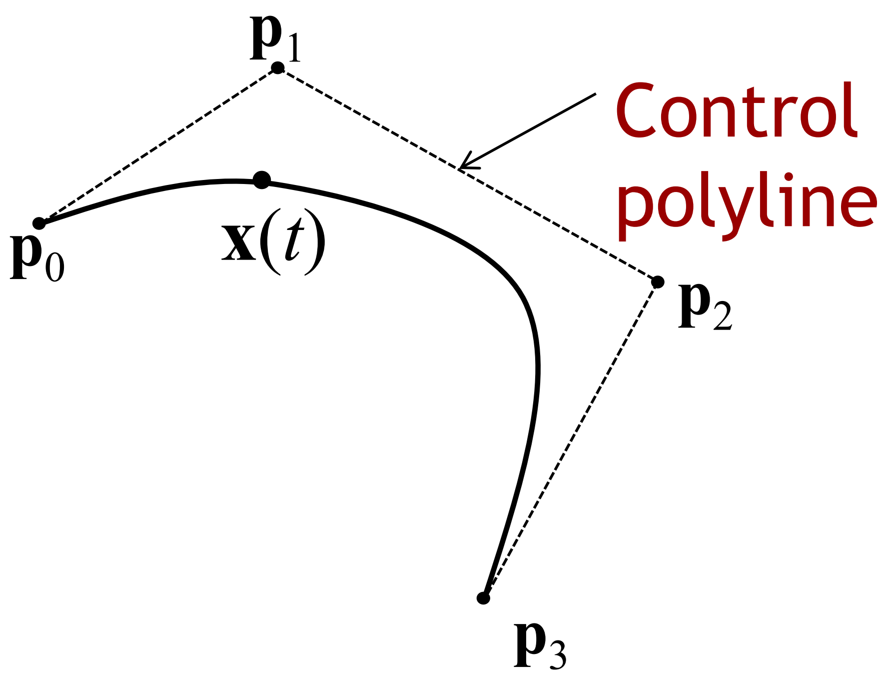
Bézier curves

- Intuitive control over curve given control points
  - Endpoints are interpolated, intermediate points are approximated
- Many demo applets online
  - http://ibiblio.org/e-notes/Splines/Intro.htm



Cubic Bézier curve

- Cubic polynomials, most common case
- Defined by 4 control points
- Two interpolated endpoints
- Two midpoints control the tangent at the endpoints



Bézier Curve formulation

- Three alternative formulations, analogous to linear case

1. Weighted average of control points
2. Cubic polynomial function of t
3. Matrix form

- Algorithmic construction
  - de Casteljau algorithm

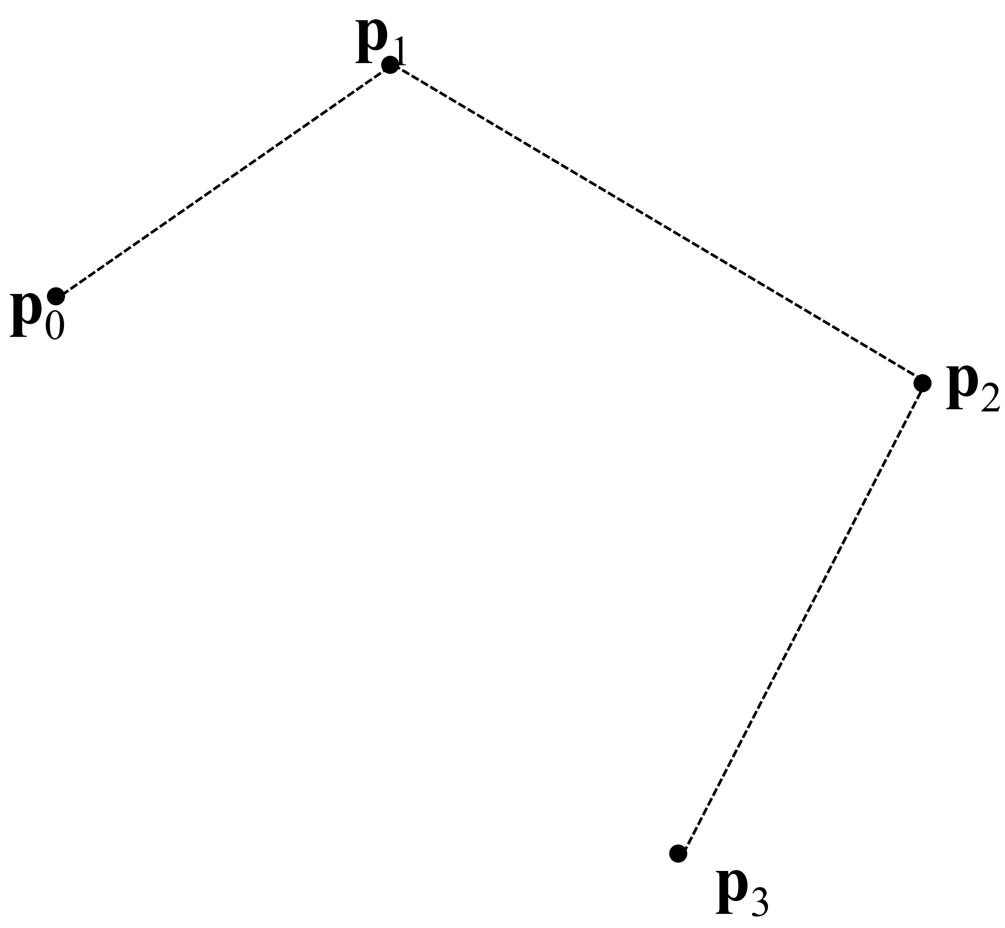
de Casteljau Algorithm

http://en.wikipedia.org/wiki/De_Casteljau's_algorithm

- A recursive series of linear interpolations
  - Works for any order, not only cubic
- Not terribly efficient to evaluate
  - Other forms more commonly used
- Why study it?
  - Intuition about the geometry
  - Useful for subdivision (later today)

de Casteljau Algorithm

- Given the control points
- A value of t
- Here $t \approx 0.25$

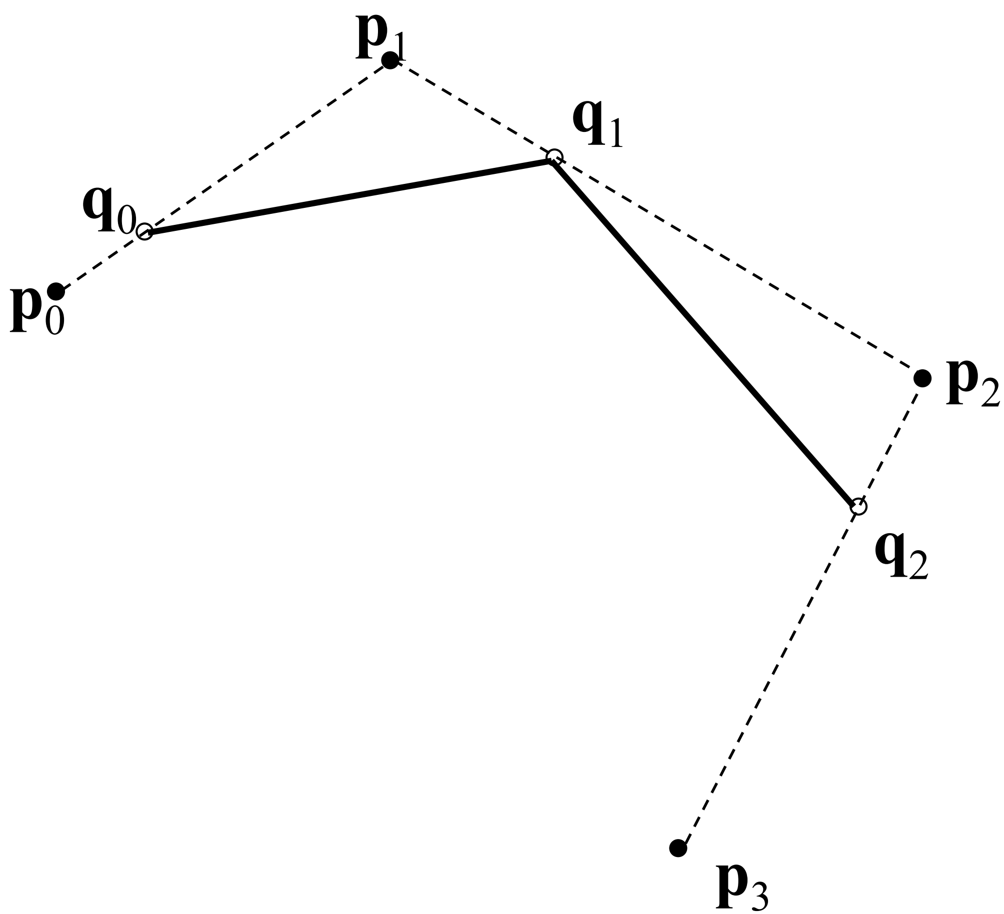


de Casteljau Algorithm

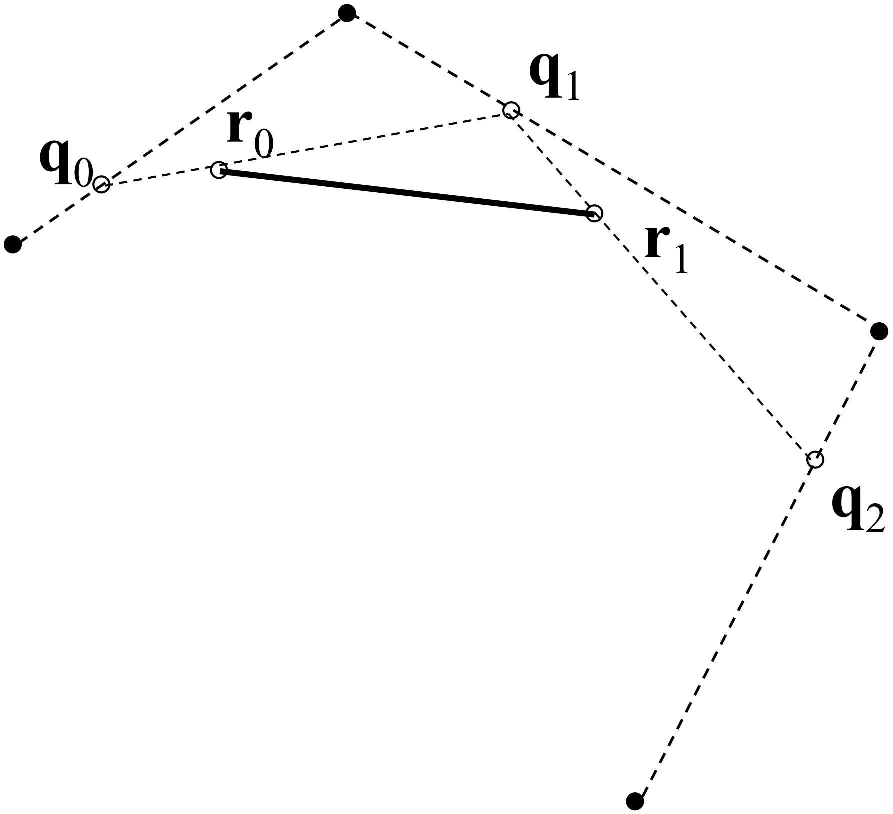
$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right)$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right)$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right)$$



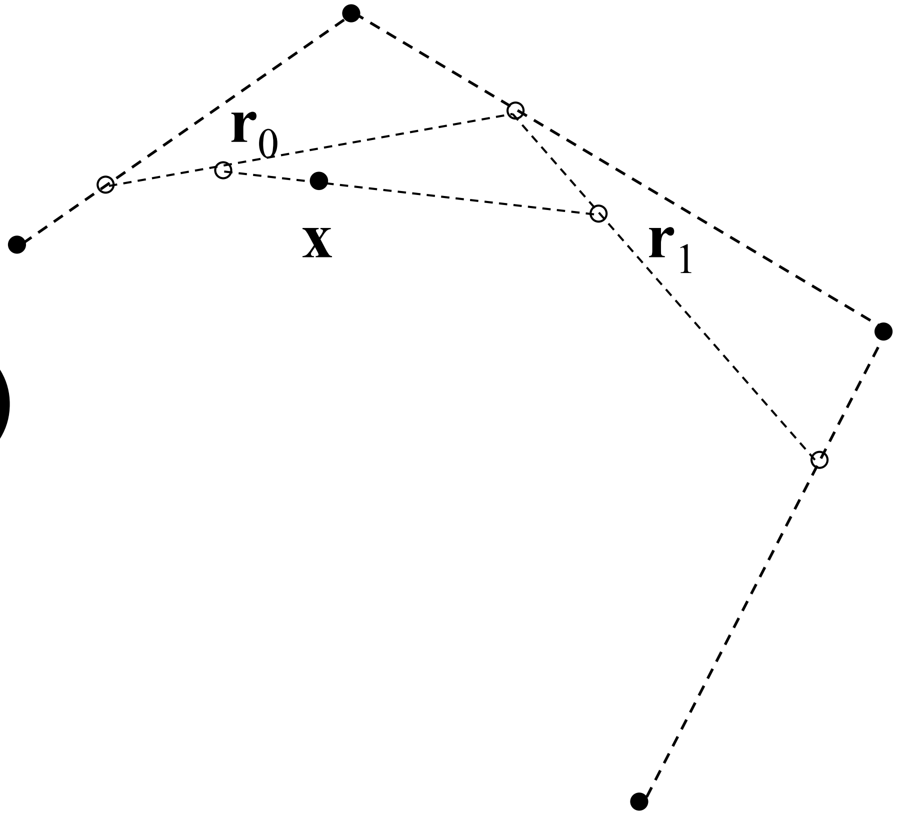
de Casteljau Algorithm



$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right)$$

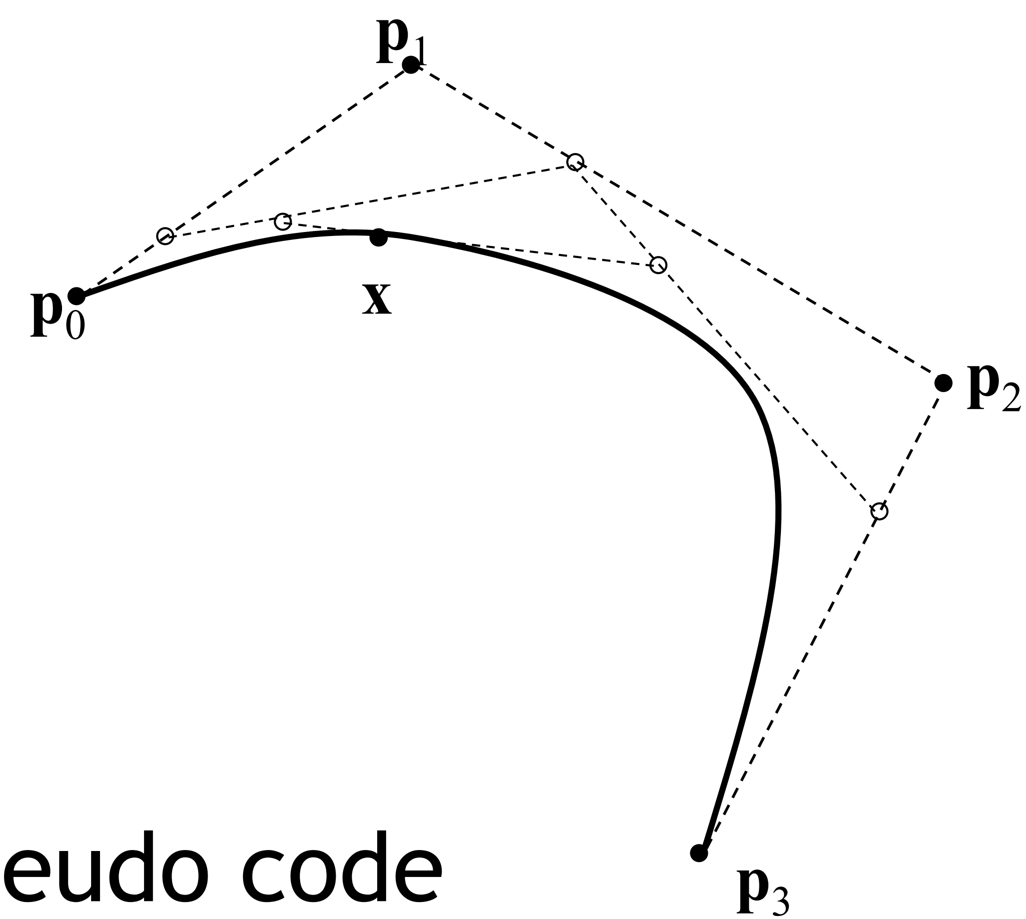
$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right)$$

de Casteljau Algorithm



$$\mathbf{x}(t) = Lerp\big(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\big)$$

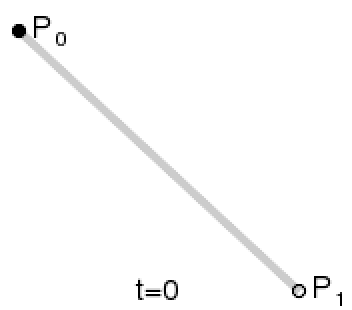
de Casteljau algorithm



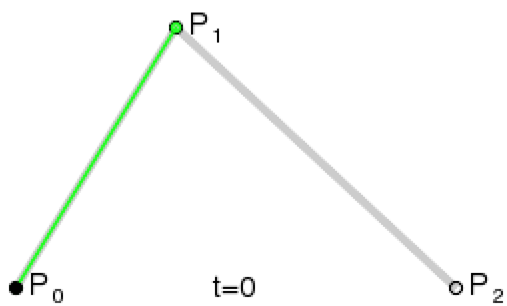
- More details, pseudo code
  - http://ibiblio.org/e-notes/Splines/bezier.html

de Castelljau Algorithm

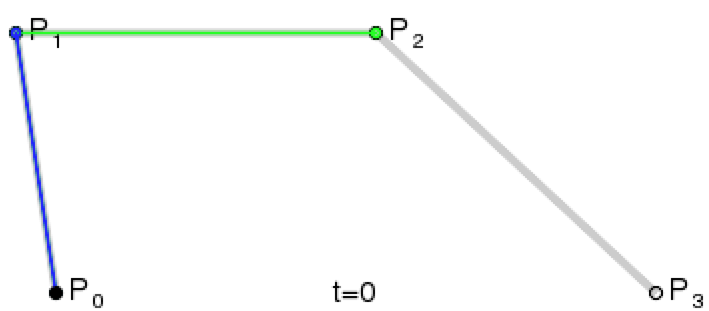
http://en.wikipedia.org/wiki/De_Casteljau's_algorithm



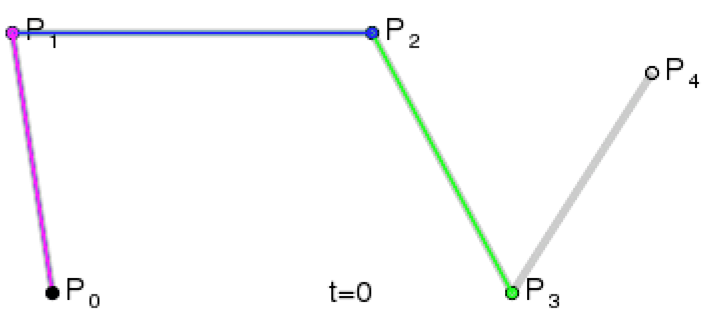
Linear



Quadratic



Cubic



Quartic

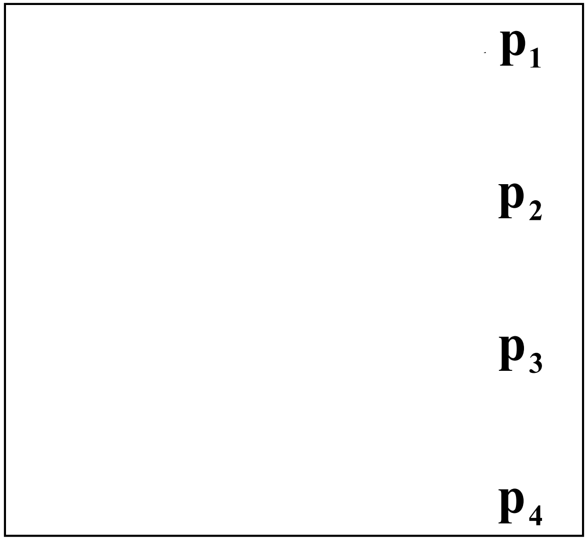
Recursive linear interpolation

$\mathbf{p}_0$

$\mathbf{p}_1$

$\mathbf{p}_2$

$\mathbf{p}_3$



Recursive linear interpolation

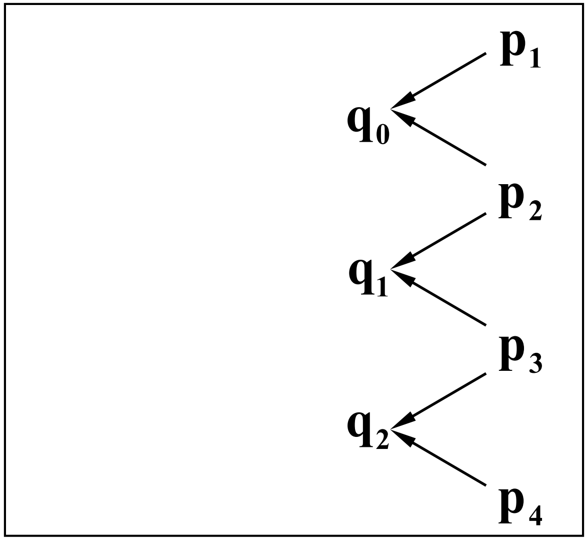
$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1)$$
$$\mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2)$$
$$\mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$

$\mathbf{p}_0$

$\mathbf{p}_1$

$\mathbf{p}_2$

$\mathbf{p}_3$



Recursive linear interpolation

$$\mathbf{r}_0 = Lerp(t, \mathbf{q}_0, \mathbf{q}_1)$$

$$\mathbf{r}_1 = Lerp(t, \mathbf{q}_1, \mathbf{q}_2)$$

$$\mathbf{q}_0 = Lerp(t, \mathbf{p}_0, \mathbf{p}_1)$$

$$\mathbf{q}_1 = Lerp(t, \mathbf{p}_1, \mathbf{p}_2)$$

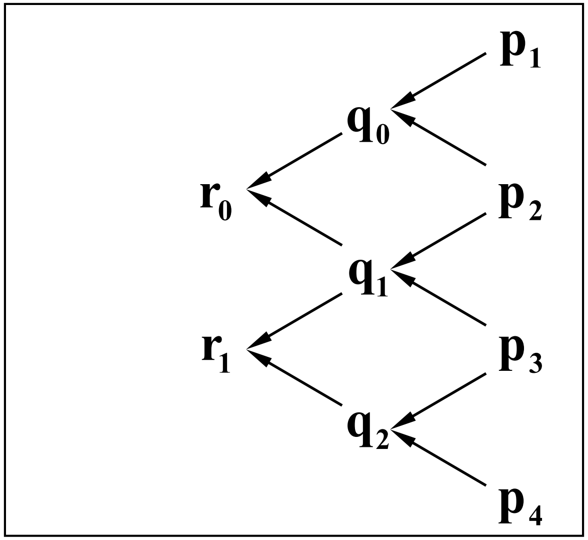
$$\mathbf{q}_2 = Lerp(t, \mathbf{p}_2, \mathbf{p}_3)$$

$$\mathbf{p}_0$$

$$\mathbf{p}_1$$

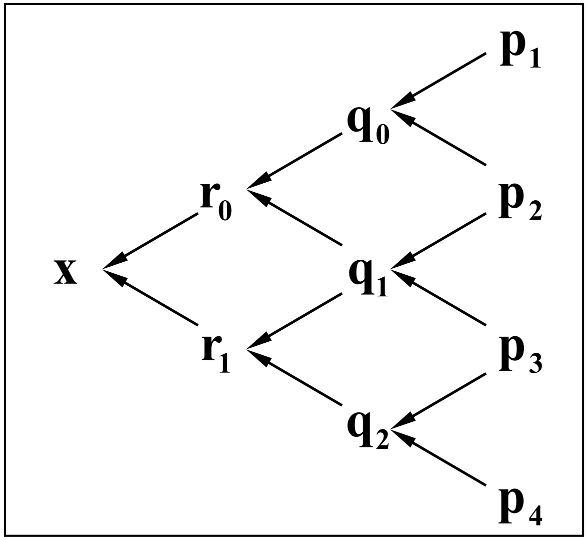
$$\mathbf{p}_2$$

$$\mathbf{p}_3$$



Recursive linear interpolation

$$\mathbf{x} = Lerp\left(t, \mathbf{r}_0, \mathbf{r}_1\right) \quad \begin{matrix} \mathbf{r}_0 = Lerp\left(t, \mathbf{q}_0, \mathbf{q}_1\right) \\ \mathbf{r}_1 = Lerp\left(t, \mathbf{q}_1, \mathbf{q}_2\right) \end{matrix} \quad \begin{matrix} \mathbf{q}_0 = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) \\ \mathbf{q}_1 = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) \\ \mathbf{q}_2 = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) \end{matrix} \quad \begin{matrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{matrix}$$



Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1 - t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1 - t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1 - t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1 - t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1 - t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1 - t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right)$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = \left(1-t\right)\left(\left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = \left(1-t\right)\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left(\left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = \left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = \left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = \left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = \left(1-t\right)\left(\left(1-t\right)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = \left(1-t\right)\left(\left(1-t\right)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left(\left(1-t\right)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

$$\mathbf{x}(t) = Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right)$$

Expand the LERPs

$$\mathbf{q}_0(t) = Lerp\left(t, \mathbf{p}_0, \mathbf{p}_1\right) = (1-t)\mathbf{p}_0 + t\mathbf{p}_1$$

$$\mathbf{q}_1(t) = Lerp\left(t, \mathbf{p}_1, \mathbf{p}_2\right) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$$

$$\mathbf{q}_2(t) = Lerp\left(t, \mathbf{p}_2, \mathbf{p}_3\right) = (1-t)\mathbf{p}_2 + t\mathbf{p}_3$$

$$\mathbf{r}_0(t) = Lerp\left(t, \mathbf{q}_0(t), \mathbf{q}_1(t)\right) = (1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)$$

$$\mathbf{r}_1(t) = Lerp\left(t, \mathbf{q}_1(t), \mathbf{q}_2(t)\right) = (1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)$$

$$\begin{aligned} \mathbf{x}(t) &= Lerp\left(t, \mathbf{r}_0(t), \mathbf{r}_1(t)\right) \\ &= (1-t)\left((1-t)\left((1-t)\mathbf{p}_0 + t\mathbf{p}_1\right) + t\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right)\right) \\ &\qquad + t\left((1-t)\left((1-t)\mathbf{p}_1 + t\mathbf{p}_2\right) + t\left((1-t)\mathbf{p}_2 + t\mathbf{p}_3\right)\right) \end{aligned}$$

Weighted average of control points

- Regroup

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

Weighted average of control points

- Regroup

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\,\mathbf{p}_0 + 3(1-t)^2\,t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

Weighted average of control points

- Regroup

$$\mathbf{x}(t) = (1-t)\Big((1-t)\big((1-t)\mathbf{p}_0 + t\mathbf{p}_1\big) + t\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big)\Big)$$
$$+t\Big((1-t)\big((1-t)\mathbf{p}_1 + t\mathbf{p}_2\big) + t\big((1-t)\mathbf{p}_2 + t\mathbf{p}_3\big)\Big)$$

$$\mathbf{x}(t) = (1-t)^3\,\mathbf{p}_0 + 3(1-t)^2\,t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3$$

$$\mathbf{x}(t) = \overbrace{\left(-t^3 + 3t^2 - 3t + 1\right)}^{B_0(t)}\mathbf{p}_0 + \overbrace{\left(3t^3 - 6t^2 + 3t\right)}^{B_1(t)}\mathbf{p}_1$$
$$+\underbrace{\left(-3t^3 + 3t^2\right)}_{B_2(t)}\mathbf{p}_2 + \underbrace{\left(t^3\right)}_{B_3(t)}\mathbf{p}_3$$

Bernstein polynomials

Cubic Bernstein polynomials

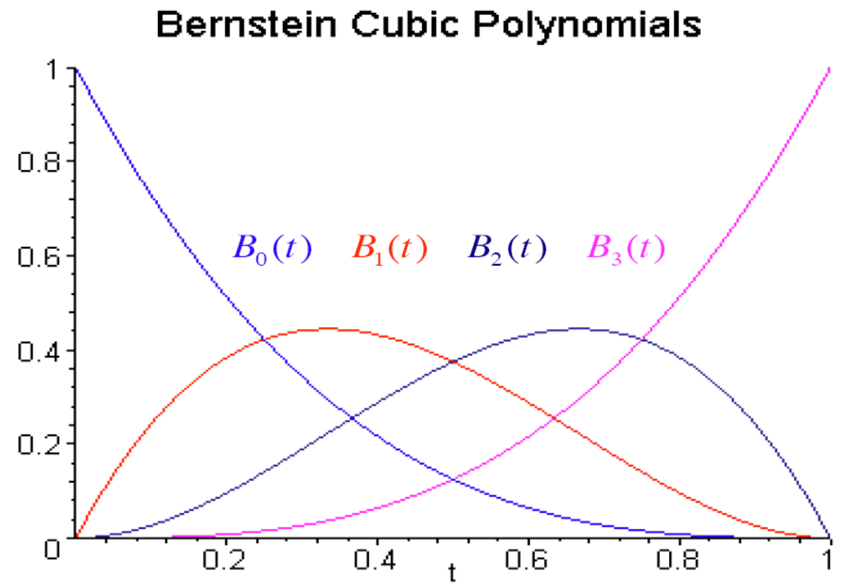
http://en.wikipedia.org/wiki/Bernstein_polynomial

$$\mathbf{x}(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

The cubic *Bernstein polynomials :*

$$B_0(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1(t) = 3t^3 - 6t^2 + 3t$$
$$B_2(t) = -3t^3 + 3t^2$$
$$B_3(t) = t^3$$

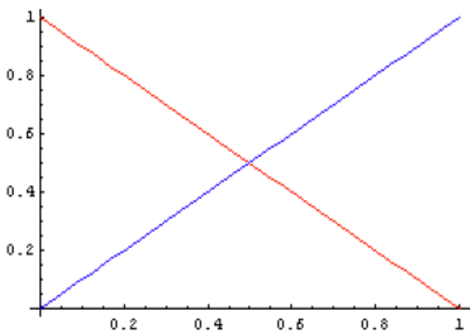
$$\sum B_i(t) = 1$$



- Partition of unity, at each t always add to 1
- Endpoint interpolation, B_0 and B_3 go to 1

General Bernstein polynomials

$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

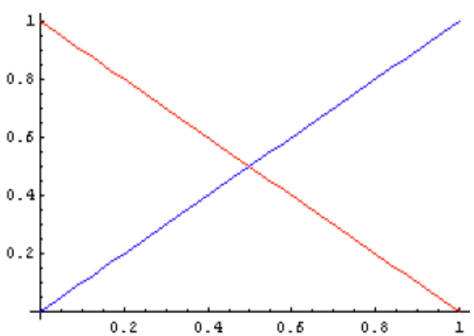


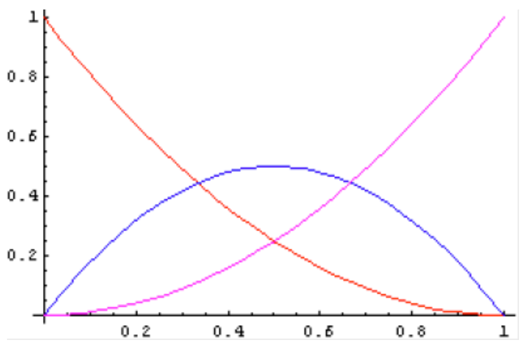
General Bernstein polynomials

$$B_0^1(t) = -t + 1 \qquad B_0^2(t) = t^2 - 2t + 1$$

$$B_1^1(t) = t \qquad B_1^2(t) = -2t^2 + 2t$$

$$B_2^2(t) = t^2$$



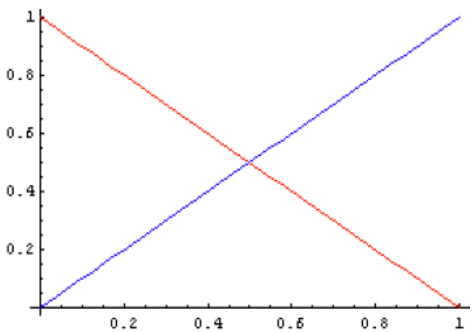


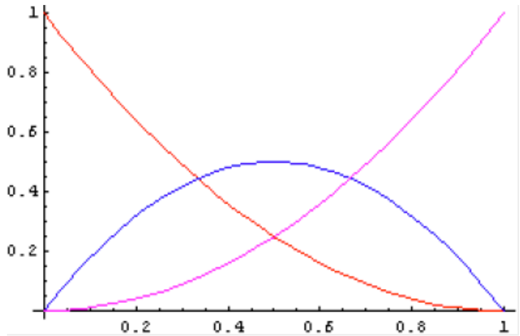
General Bernstein polynomials

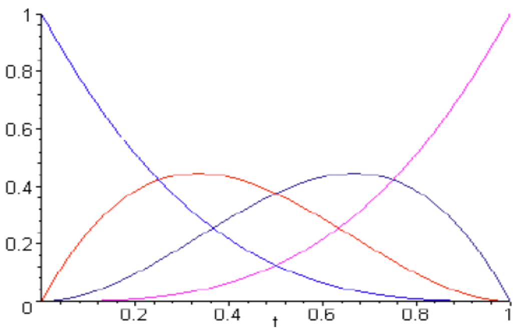
$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$
$$B_2^3(t) = -3t^3 + 3t^2$$
$$B_3^3(t) = t^3$$





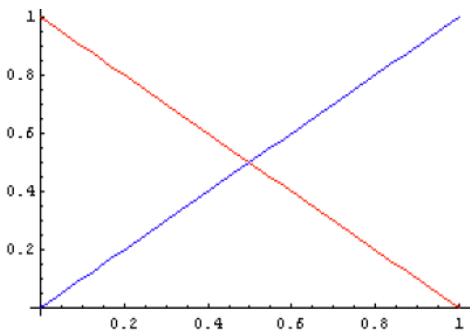


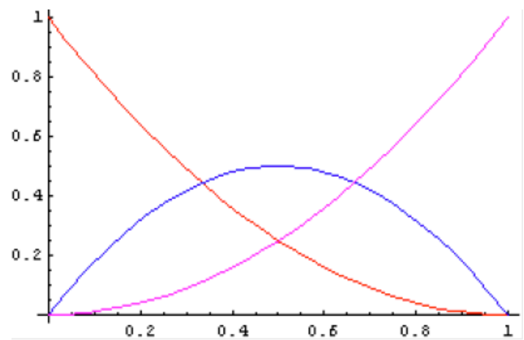
General Bernstein polynomials

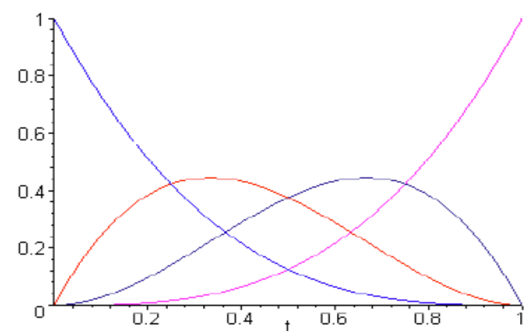
$$B_0^1(t) = -t + 1$$
$$B_1^1(t) = t$$

$$B_0^2(t) = t^2 - 2t + 1$$
$$B_1^2(t) = -2t^2 + 2t$$
$$B_2^2(t) = t^2$$

$$B_0^3(t) = -t^3 + 3t^2 - 3t + 1$$
$$B_1^3(t) = 3t^3 - 6t^2 + 3t$$
$$B_2^3(t) = -3t^3 + 3t^2$$
$$B_3^3(t) = t^3$$







Order n: $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i$ $\qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$

$$\sum B_i^n(t) = 1$$

Partition of unity, endpoint interpolation

General Bézier curves

- nth-order Bernstein polynomials form nth-order Bézier curves
- Bézier curves are weighted sum of control points using nth-order Bernstein polynomials

Bernstein polynomials of order n:

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}(t)^i$$

Bézier curve of order n:

$$\mathbf{x}(t) = \sum_{i=0}^{n} B_i^n(t)\mathbf{p}_i$$

Affine invariance

- Two ways to transform Bézier curves
  1. Transform the control points, then compute resulting point on curve
  2. Compute point on curve, then transform it
- Either way, get the same transform point!
  - Curve is defined via affine combination of points (convex combination is special case of an affine combination)
  - Invariant under affine transformations
  - Convex hull property always remains

For your reference

- Starting from weighted sum of control points using Bernstein polynomials, polynomial and matrix form can be derive easily

Cubic polynomial form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

Cubic polynomial form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

Regroup into coefficients of t :

$$\mathbf{x}(t) = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

Cubic polynomial form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

Regroup into coefficients of t :

$$\mathbf{x}(t) = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

$$\boxed{\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d} \qquad \begin{aligned} \mathbf{a} &= \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right) \\ \mathbf{b} &= \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right) \\ \mathbf{c} &= \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right) \\ \mathbf{d} &= \left(\mathbf{p}_0\right) \end{aligned}}$$

Cubic polynomial form

Start with Bernstein form:

$$\mathbf{x}(t) = \left(-t^3 + 3t^2 - 3t + 1\right)\mathbf{p}_0 + \left(3t^3 - 6t^2 + 3t\right)\mathbf{p}_1 + \left(-3t^3 + 3t^2\right)\mathbf{p}_2 + \left(t^3\right)\mathbf{p}_3$$

Regroup into coefficients of t :

$$\mathbf{x}(t) = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)t^3 + \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)t^2 + \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)t + \left(\mathbf{p}_0\right)1$$

$$\boxed{\mathbf{x}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d} \qquad \begin{aligned} \mathbf{a} &= \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right) \\ \mathbf{b} &= \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right) \\ \mathbf{c} &= \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right) \\ \mathbf{d} &= \left(\mathbf{p}_0\right) \end{aligned}}$$

- Good for fast evaluation, precompute constant coefficients $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$
- Not much geometric intuition

Cubic matrix form

$$\mathbf{x}(t) = \begin{bmatrix} \vec{\mathbf{a}} & \mathbf{b} & \vec{\mathbf{c}} & \mathbf{d} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\vec{\mathbf{a}} = \left(-\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3\right)$$
$$\mathbf{b} = \left(3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2\right)$$
$$\vec{\mathbf{c}} = \left(-3\mathbf{p}_0 + 3\mathbf{p}_1\right)$$
$$\mathbf{d} = \left(\mathbf{p}_0\right)$$

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}}_{\mathbf{G}_{Bez}} \underbrace{\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_{Bez}} \underbrace{\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}}_{\mathbf{T}}$$

- Can construct other cubic curves by just using different basis matrix $\mathbf{B}$
- Hermite, Catmull-Rom, B-Spline, …

Cubic matrix form

- 3 parallel equations, in x, y and z:

$$\mathbf{x}_x(t) = \begin{bmatrix} p_{0x} & p_{1x} & p_{2x} & p_{3x} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{x}_y(t) = \begin{bmatrix} p_{0y} & p_{1y} & p_{2y} & p_{3y} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\mathbf{x}_z(t) = \begin{bmatrix} p_{0z} & p_{1z} & p_{2z} & p_{3z} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Matrix form

- Bundle into a single matrix

$$\mathbf{x}(t) = \begin{bmatrix} p_{0x} & p_{1x} & p_{2x} & p_{3x} \\ p_{0y} & p_{1y} & p_{2y} & p_{3y} \\ p_{0z} & p_{1z} & p_{2z} & p_{3z} \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} \mathbf{x}(t) &= \mathbf{G}_{Bez}\mathbf{B}_{Bez}\mathbf{T} \\ \mathbf{x}(t) &= \mathbf{C}\ \mathbf{T} \end{aligned}}$$

- Efficient evaluation
  - Precompute $\mathbf{C}$
  - Take advantage of existing 4x4 matrix hardware support

Today

Curves

- Introduction
- Polynomial curves
- Bézier curves
- Drawing Bézier curves
- Piecewise curves

Drawing Bézier curves

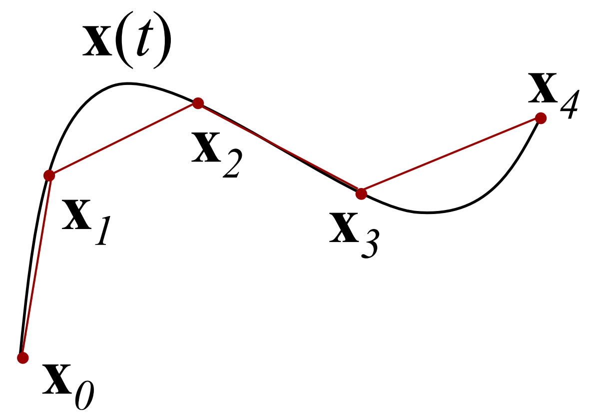
- Generally no low-level support for drawing smooth curves
  - I.e., GPU draws only straight line segments
- Need to break curves into line segments or individual pixels
- Approximating curves as series of line segments called tessellation
- Tessellation algorithms
  - Uniform sampling
  - Adaptive sampling
  - Recursive subdivision

Uniform sampling

- Approximate curve with N straight segments
  - N chosen in advance
  - Evaluate $\mathbf{x}_i = \mathbf{x}(t_i)$ where $t_i = \frac{i}{N}$ for $i = 0, 1, \ldots, N$

$$\mathbf{x}_i = \mathbf{a}\frac{i^3}{N^3} + \mathbf{b}\frac{i^2}{N^2} + \mathbf{c}\frac{i}{N} + \mathbf{d}$$

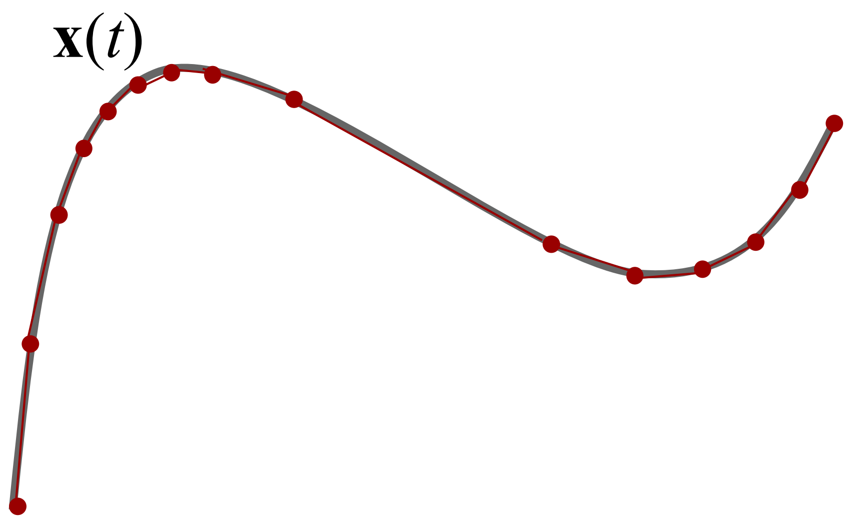
  - Connect the points with lines



- Too few points?
  - Bad approximation
  - "Curve" is faceted
- Too many points?
  - Slow to draw too many line segments
  - Segments may draw on top of each other

Adaptive Sampling

- Use only as many line segments as you need
  - Fewer segments where curve is mostly flat
  - More segments where curve bends
  - Segments never smaller than a pixel
- Various schemes for sampling, checking results, deciding whether to sample more



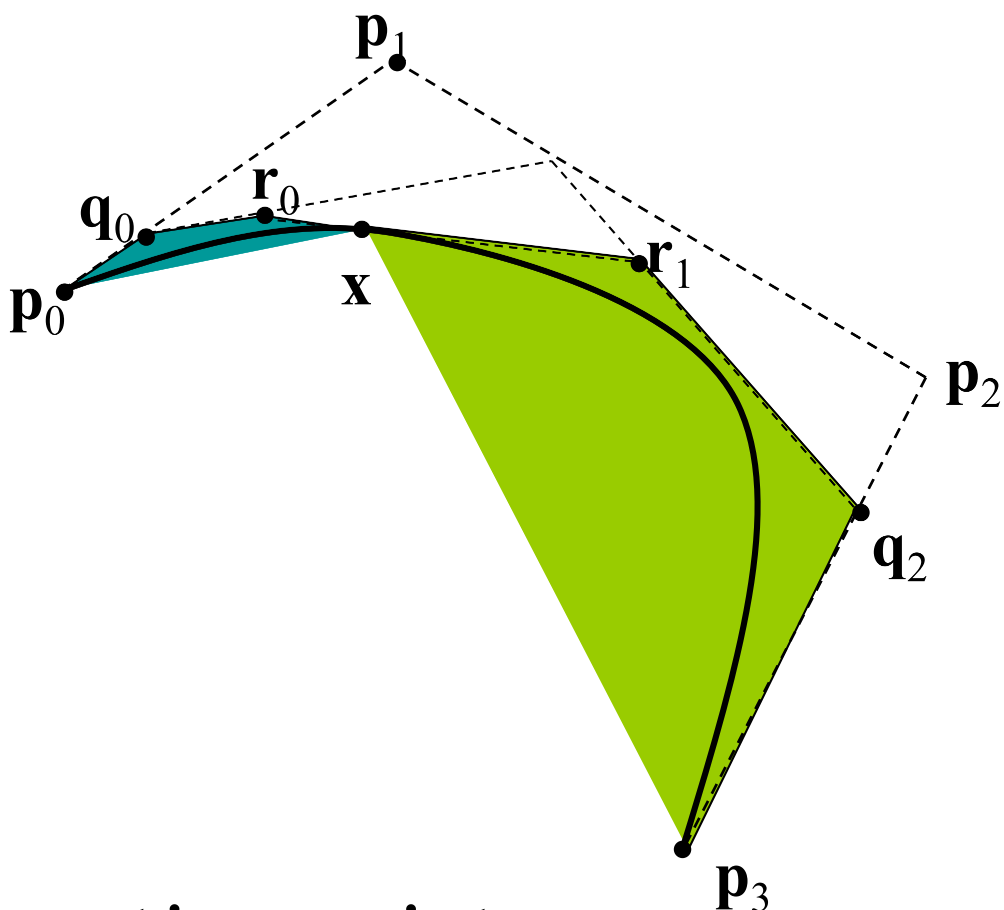
Recursive Subdivision

- Any cubic (or k-th order) curve segment can be expressed as a cubic (or k-th order) Bézier curve

  "Any piece of a cubic (or k-th order) curve is itself a cubic (or k-th order) curve"

- Therefore, any Bézier curve can be subdivided into smaller Bézier curves

de Casteljau subdivision



- de Casteljau construction points are the control points of two Bézier sub-segments $(\mathbf{p}_0,\mathbf{q}_0,\mathbf{r}_0,\mathbf{x})$ and $(\mathbf{x},\mathbf{r}_1,\mathbf{q}_2,\mathbf{p}_3)$

Adaptive subdivision algorithm

1. Use de Casteljau construction to split Bézier segment in middle (t=0.5)

2. For each half

   - If “flat enough”: draw line segment
   - Else: recurse from 1. for each half

- Test how far away midpoints are from straight segment connecting start and end
  - If less than a pixel, flat enough

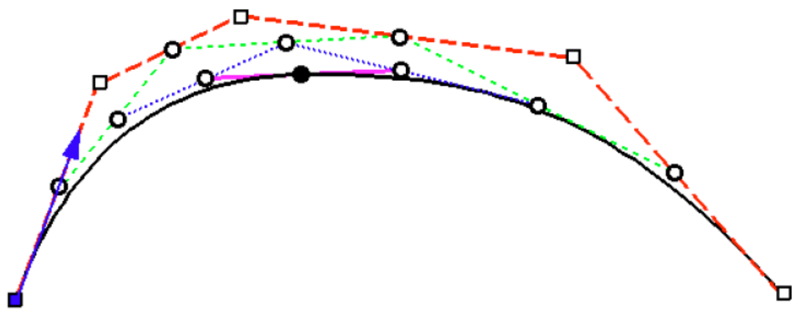
Today

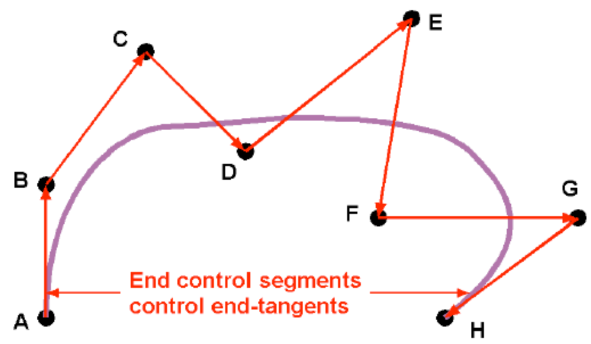
Curves

- Introduction
- Polynomial curves
- Bézier curves
- Drawing Bézier curves
- Piecewise curves

More control points

- Cubic Bézier curve limited to 4 control points
  - Cubic curve can only have one inflection
  - Need more control points for more complex curves
- k-1 order Bézier curve with k control points

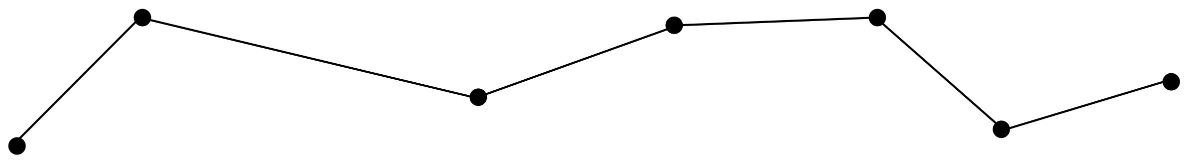




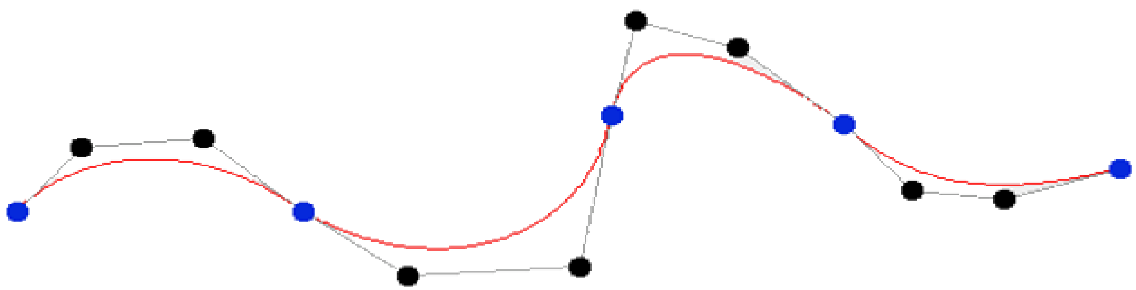
- Hard to control and hard to work with
  - Intermediate points don't have obvious effect on shape
  - Changing any control point changes the whole curve
- Want local support
  - Each control point only influences nearby portion of curve

Piecewise curves (splines)

- Sequence of simple (low-order) curves, end-to-end
  - Piecewise polynomial curve, or splines
    http://en.wikipedia.org/wiki/Spline_(mathematics)
- Sequence of line segments
  - Piecewise linear curve (linear or first-order spline)



- Sequence of cubic curve segments
  - Piecewise cubic curve, here piecewise Bézier (cubic spline)



Piecewise cubic Bézier curve

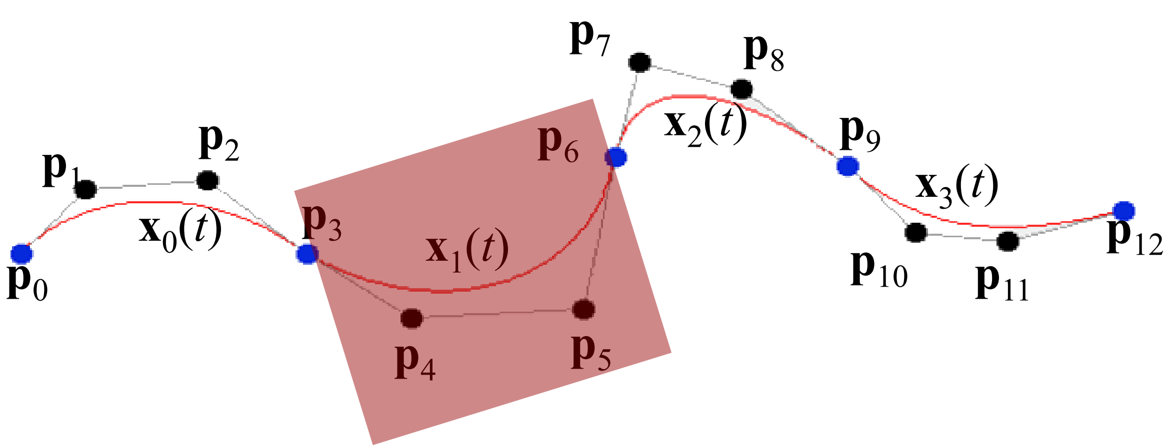
- Given $3N+1$ points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{3N}$
- Define N Bézier segments:

$$\mathbf{x}_0(t) = B_0(t)\mathbf{p}_0 + B_1(t)\mathbf{p}_1 + B_2(t)\mathbf{p}_2 + B_3(t)\mathbf{p}_3$$

$$\mathbf{x}_1(t) = B_0(t)\mathbf{p}_3 + B_1(t)\mathbf{p}_4 + B_2(t)\mathbf{p}_5 + B_3(t)\mathbf{p}_6$$

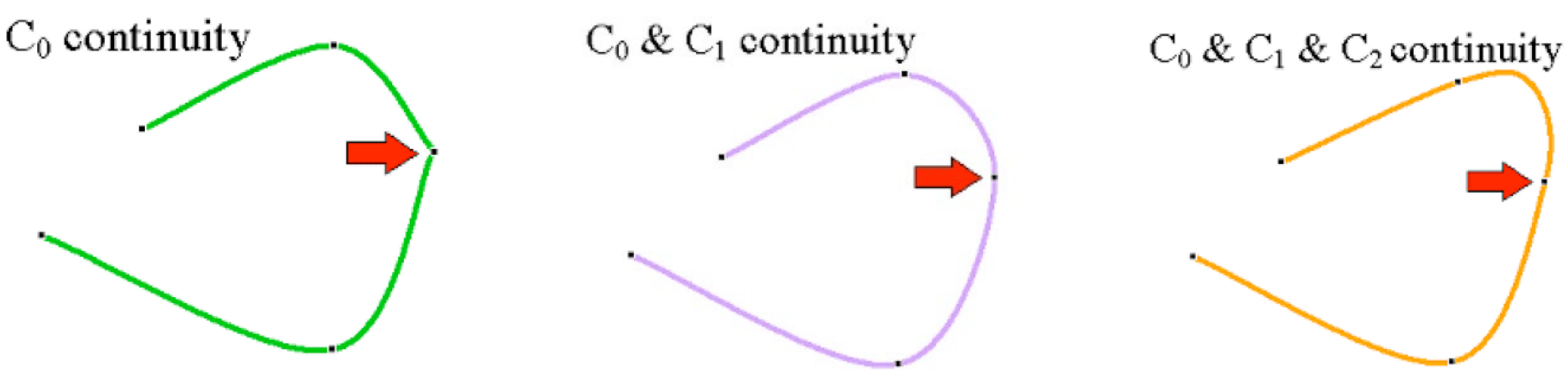
$$\vdots$$

$$\mathbf{x}_{N-1}(t) = B_0(t)\mathbf{p}_{3N-3} + B_1(t)\mathbf{p}_{3N-2} + B_2(t)\mathbf{p}_{3N-1} + B_3(t)\mathbf{p}_{3N}$$



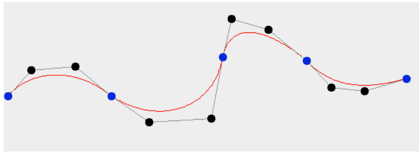
Continuity

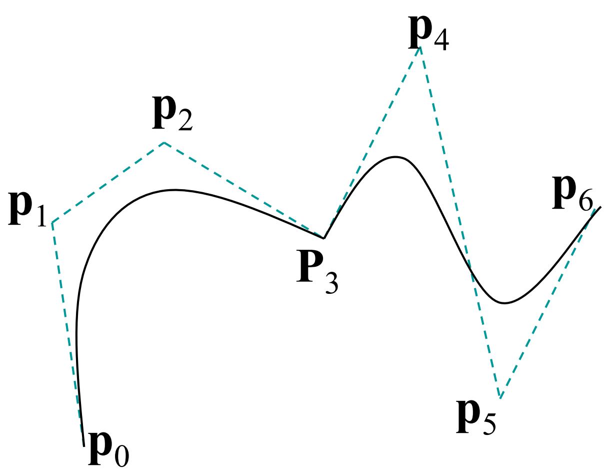
- Want smooth curves
- C^0 continuity
  - No gaps
  - Segments match at the endpoints
- C^1 continuity: first derivative is well defined
  - No corners
  - Tangents/normals are C^0 continuous (no jumps)
- C^2 continuity: second derivative is well defined
  - Tangents/normals are C^1 continuous
  - Important for high quality reflections on surfaces



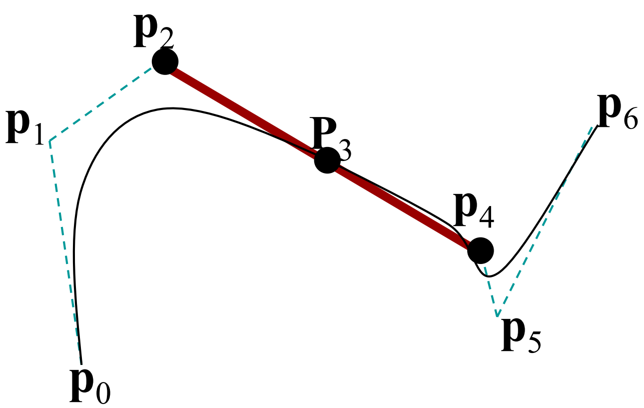
Piecewise cubic Bézier curves

- C^0 continuous if endpoints are shared
- C^1 continuous at segment endpoints $\mathbf{p}_{3i}$ if $\mathbf{p}_{3i} - \mathbf{p}_{3i-1} = \mathbf{p}_{3i+1} - \mathbf{p}_{3i}$
- C^2 is harder to get





C^0 continuous, shared endpoints



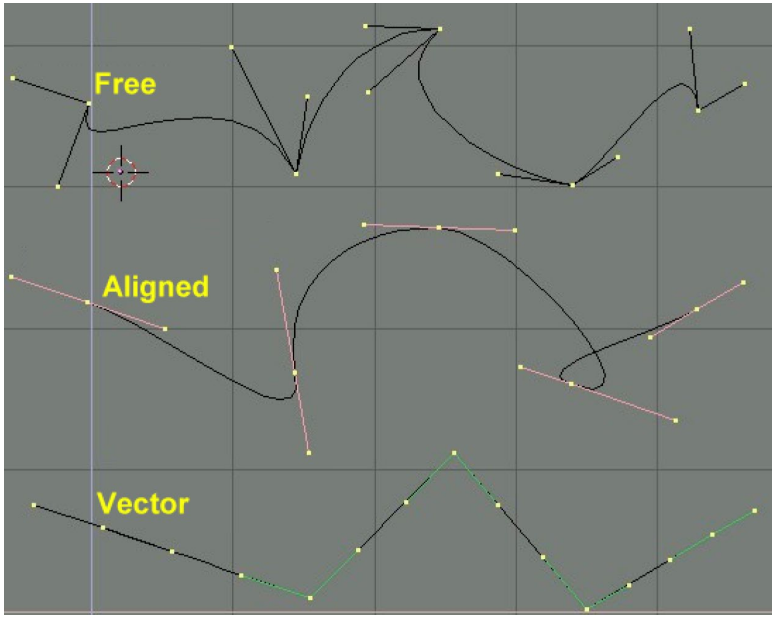
C^1 continuous

Piecewise cubic Bézier curves

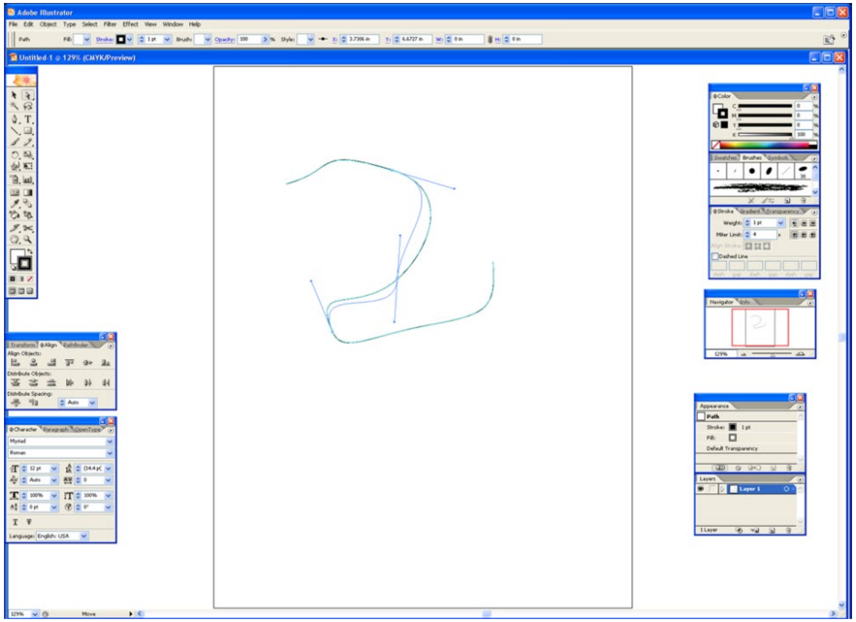
- Used often in 2D drawing programs
- Inconveniences
  - Must have 4 or 7 or 10 or 13 or … (1 plus a multiple of 3) control points
  - Some points interpolate (endpoints), others approximate (handles)
  - Need to impose constraints on control points to obtain C^1 continuity
  - C^2 continuity more difficult
- Solutions
  - User interface using "Bézier handles"
  - Generalization to B-splines, next time

Bézier handles

- Segment end points (interpolating) presented as curve control points
- Midpoints (approximating points) presented as "handles"
- Can have option to enforce C^1 continuity



[www.blender.org]



Adobe Illustrator

Next time

- B-splines and NURBS
- Extending curves to surfaces