

## Chapter 2

# 2D Graphics Basics



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## 2.1 Intro: Geometric Transformations of 2D

Suppose have an object in  $\mathbb{R}^2$  that we wish to manipulate. For the sake of simplicity this object will be composed of points. Some standard manipulation might be things like:

- (a) Translations: Shifting the object horizontally and vertically.
- (b) Rotations: Rotating the object about an arbitrary point.
- (c) Reflections: Reflecting the object in an arbitrary line.
- (d) Perspective: Although this makes more sense in  $\mathbb{R}^3$  it's worth taking a look at this in  $\mathbb{R}^2$  because it makes the problems clear in a simpler context.

Moreover ideally we would like to use linear algebra do do these things because it is fast and cheap, computationally speaking.

More specifically given a point represented by a vector  $\mathbf{v}$  we would like to represent each of the transformations above by a matrix  $M$  so that applying the transformation can be done by doing the product  $M\mathbf{v}$ .

Note: This is an ideal situation. In reality much of the gruntwork calculation is baked into the software or hardware and transparent to the user. However it's still certainly true that simpler calculations are faster, all other things being equal. Consequently our ongoing goal is simplicity wherever possible.

## 2.2 Translations

Immediately we have a problem. Given a point represented by a vector  $\mathbf{v} = [x, y]^T$  a translation by  $a$  units horizontally and  $b$  units vertically would have to be a transformation  $T([x, y]^T) = [x + a, y + b]^T$ . But this cannot be represented by a matrix because it is not linear and it is not linear because  $T([0, 0]^T) = [a, b]^T \neq [0, 0]^T$  unless  $a = b = 0$  which is no translation at all.

As we've mentioned above this is not a huge issue in the sense that addition is a fairly easy operation but it is a separate operation from matrix multiplication, so it's worth keeping this in mind.

**Exercise 2.1.** Consider the mapping  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$  given by the matrix product:

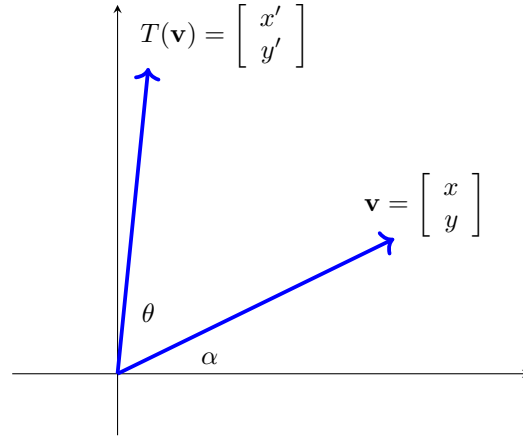
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- (a) Complete the calculation on the right.
- (b) What does this mapping do to the subspace consisting of vectors of the form  $[x, y, 1]^T$ ?
- (c) Does the set of such points form a subspace of  $\mathbb{R}^3$ ? Explain.

## 2.3 Rotations

Rotations about points other than the origin are not linear for the same reason argued above. If you rotate the origin about a point other than the origin it will end up off the origin, unless the rotation is by  $2\pi$ , which is no rotation at all.

So how about rotation about the origin? Let's construct it, so consider the following picture:



First note that:

$$\begin{aligned} x &= \sqrt{x^2 + y^2} \cos \alpha \\ y &= \sqrt{x^2 + y^2} \sin \alpha \end{aligned}$$

It then follows that from the addition formula for cosine:

$$\begin{aligned} x' &= \sqrt{x^2 + y^2} \cos(\alpha + \theta) \\ &= \sqrt{x^2 + y^2} (\cos \alpha \cos \theta - \sin \alpha \sin \theta) \\ &= \sqrt{x^2 + y^2} \cos \alpha \cos \theta - \sqrt{x^2 + y^2} \sin \alpha \sin \theta \\ &= x \cos \theta - y \sin \theta \end{aligned}$$

And that from the addition formula for sine:

$$\begin{aligned} y' &= \sqrt{x^2 + y^2} \sin(\alpha + \theta) \\ &= \sqrt{x^2 + y^2} (\sin \alpha \cos \theta + \sin \theta \cos \alpha) \\ &= \sqrt{x^2 + y^2} \sin \alpha \cos \theta + \sqrt{x^2 + y^2} \sin \theta \cos \alpha \\ &= y \cos \theta + x \sin \theta \\ &= x \sin \theta + y \cos \theta \end{aligned}$$

This can be written as a matrix product:

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

It follows that rotation about the origin counterclockwise by  $\theta$  radians is linear with the matrix representation:

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Because we've explicitly constructed a matrix which does exactly what we wanted we know for a fact that rotation about the origin counterclockwise by  $\theta$  radians is linear and that we have our matrix.

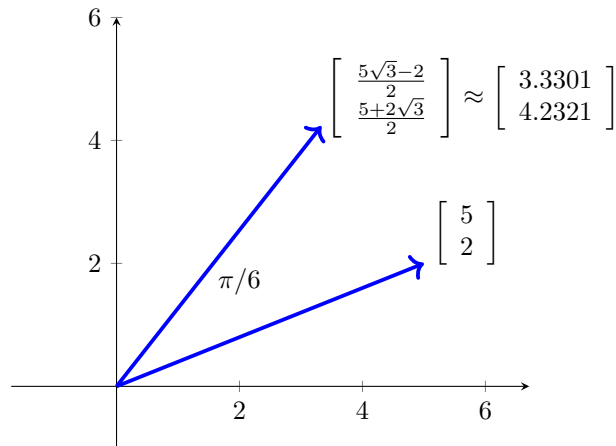
We can now use it to rotate.

**Example 2.1.** The result of rotating the point  $(5, 2)$  counterclockwise by  $\pi/6$  radians about the origin is:

$$R_{\pi/6} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5(\sqrt{3}/2) - 2(1/2) \\ 5(1/2) + 2(\sqrt{3}/2) \end{bmatrix}$$

Or the point

$$(5(\sqrt{3}/2) - 2(1/2), 5(1/2) + 2(\sqrt{3}/2)) = \left( \frac{5\sqrt{3} - 2}{2}, \frac{5 + 2\sqrt{3}}{2} \right)$$



**Exercise 2.2.** Find the result of rotating the point  $(-3, 7)$  counterclockwise by  $\pi/4$  radians.

**Exercise 2.3.** Find the result of rotating the point  $(-2, -7)$  clockwise by  $7\pi/6$  radians.

**Exercise 2.4.** Show computationally that a rotation by  $\theta_1$  radians followed by a rotation by  $\theta_2$  radians results in a rotation by  $\theta_1 + \theta_2$  radians.

**Exercise 2.5.** A non-vertical line in space can be given by the slope-intercept form  $y = mx + b$  for  $m, b \in \mathbb{R}$ .

- Find the image of the point  $(0, b)$  under rotation by  $\theta$ .
- Find the image of the point  $(1, m + b)$  under rotation by  $\theta$ .
- Find the slope of the image of the line. This should not contain  $b$ . Why does this make sense? For which values of  $\theta$  will this be undefined? Explain geometrically what is happening for such values.
- Write the equation of the new line in the form  $y = m'x + b'$  where  $m'$  and  $b'$  may depend on  $m, b, \theta$ .
- Where does the line  $y = 2x + 6$  get mapped to when  $\theta = \pi/6$ ?
- In the case where the slope is undefined what will the equation of the new line be?

**Exercise 2.6.** Any line in 2D space can be given by an equation of the form  $ax + by = c$  for  $a, b, c \in \mathbb{R}$ .

- If a rotation by  $\theta$  radians is applied to this line, find the equation of the resulting line. Your final answer should be in the form

$$a'x + b'y = c'$$

where  $a', b'$ , and  $c'$  may depend on  $a, b, c, \theta$ . You may need more than one case.

- Where does the line  $y = 5$  get mapped to when  $\theta = \pi/3$ ?
- Where does the line  $x = 5$  get mapped to when  $\theta = \pi/4$ ?
- Where does the line  $2x + 4y = 7$  get mapped to when  $\theta = \pi/6$ ?

**Exercise 2.7.** It's fairly clear intuitively that rotation preserves distances (formally it's an example of an *isometry*) but show computationally that this is true. In other words show that if  $P$  and  $Q$  are points then we have:

$$\text{dist}(R_\theta(P), R_\theta(Q)) = \text{dist}(P, Q)$$

## 2.4 Rotation Addendum in 3D

It's possible to rewrite our rotation formula in a way that extends its use. Consider that the mapping works as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix} = \cos \theta \begin{bmatrix} x \\ y \end{bmatrix} + \sin \theta \begin{bmatrix} -y \\ x \end{bmatrix}$$

The vector  $[-y, x]^T$  is the result of rotating  $[x, y]^T$  counterclockwise by  $\pi/2$  about the origin. Consequently what this reformulation is saying is that if  $\mathbf{v}$  is a vector and if  $\mathbf{w}$  is the result of rotating  $\mathbf{v}$  counterclockwise by  $\pi/2$  about the origin then the rotation map is:

$$\mathbf{v} \mapsto (\cos \theta)\mathbf{v} + (\sin \theta)\mathbf{w}$$

Since this mapping is a simple linear combination it is actually valid even in 3D if we understand the context. Imagine a plane through the origin and a perpendicular axis  $\hat{\mathbf{u}}$  (in this case  $\hat{\mathbf{u}}$  is a unit normal vector).

Suppose  $\mathbf{v}$  is a vector in this plane and  $\mathbf{w}$  is the result of rotating  $\mathbf{v}$  by  $\pi/2$  about the  $\hat{\mathbf{u}}$  axis obeying the right-hand rule (thumb follows  $\hat{\mathbf{u}}$  and fingers curl from  $\mathbf{v}$  to  $\mathbf{w}$ ). Then the result of rotating  $\mathbf{v}$  by  $\theta$  radians about  $\hat{\mathbf{u}}$  (again following the right-hand rule) will obey the same rotation map above.

We'll use the following notation when it's clear what the axis and angle are or when the specifics are irrelevant:

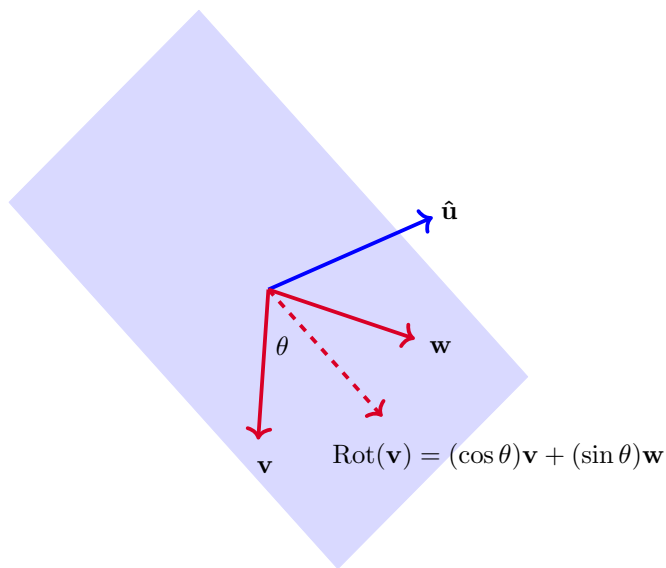
$$\text{Rot}(\mathbf{v}) = (\cos \theta)\mathbf{v} + (\sin \theta)\mathbf{w}$$

Note that the perpendicular axis  $\hat{\mathbf{u}}$  is useful only for understanding the direction of rotation (how  $\mathbf{v}$  and  $\mathbf{w}$  need to relate), it doesn't initially show in the formula. Really the formula just indicates that  $\mathbf{v}$  rotates toward  $\mathbf{w}$  within the plane they span and  $\hat{\mathbf{u}}$  doesn't come into play using that interpretation.

It is of course critical that  $|\mathbf{v}| = |\mathbf{w}|$  in order to use the formula.

This picture might clarify.





**Example 2.2.** Observe that the vectors

$$\mathbf{v} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}} \text{ and } \mathbf{w} = 0\hat{\mathbf{i}} + \frac{3}{\sqrt{5}}\hat{\mathbf{j}} - \frac{6}{\sqrt{5}}\hat{\mathbf{k}}$$

are perpendicular and have the same magnitude. Consequently if we wish to rotate  $\mathbf{v}$  toward  $\mathbf{w}$  by  $3\pi/4$  radians we calculate:

$$\begin{aligned} \text{Rot}(\mathbf{v}) &= (\cos(3\pi/4))\mathbf{v} + (\sin(3\pi/4))\mathbf{w} \\ &= -\frac{\sqrt{2}}{2}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 1\hat{\mathbf{k}}) + \frac{\sqrt{2}}{2}\left(0\hat{\mathbf{i}} + \frac{3}{\sqrt{5}}\hat{\mathbf{j}} - \frac{6}{\sqrt{5}}\hat{\mathbf{k}}\right) \end{aligned}$$

It's worth noting that since  $\frac{3\pi}{4} > \frac{\pi}{2}$  this actually rotates  $\mathbf{v}$  past  $\mathbf{w}$  in the plane they span.

**Exercise 2.8.** Observe that the vectors  $\mathbf{v} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$  and  $\mathbf{w} = 0\hat{\mathbf{i}} + \sqrt{6}\hat{\mathbf{j}} + \sqrt{6}\hat{\mathbf{k}}$  are perpendicular and have the same magnitude. Find the vector which results when  $\mathbf{v}$  is rotated  $\frac{\pi}{3}$  radians towards  $\mathbf{w}$  in the plane they span and the vector which results when  $\mathbf{w}$  is rotated  $\frac{\pi}{3}$  radians towards  $\mathbf{v}$  in the plane they span.

**Exercise 2.9.** The vectors  $\mathbf{v} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$  and  $\mathbf{w} = 1\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$  are perpendicular but do not have the same magnitude. Find the vector which results when  $\mathbf{v}$  is rotated  $\frac{5\pi}{6}$  radians towards  $\mathbf{w}$  in the plane they span.

**Exercise 2.10.** The vectors  $\mathbf{v} = 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}$  and  $\mathbf{w} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$  are neither perpendicular nor have the same magnitude. They do of course span a plane.

Find the vector which results when  $\mathbf{v}$  is rotated  $\frac{5\pi}{6}$  radians towards  $\mathbf{w}$  in the plane they span.

In a more likely scenario it's  $\hat{\mathbf{u}}$  and  $\mathbf{v}$  which are given. In this context  $\hat{\mathbf{u}}$  defines both the plane and a sense of counterclockwise within that plane again with the right-hand rule. The thumb follows  $\hat{\mathbf{u}}$  and the curl of the fingers are designated as counterclockwise.

In this scenario  $\hat{\mathbf{u}}$  becomes essential in finding  $\mathbf{w}$  because  $\mathbf{w} = \hat{\mathbf{u}} \times \mathbf{v}$ .

Why is this? Well the right-hand rule shows that  $\mathbf{w}$  points in the direction we need it to and it's the right length because we know that  $\hat{\mathbf{u}}$  and  $\mathbf{v}$  are perpendicular and  $\hat{\mathbf{u}}$  is a unit vector (important!) and so we know that

$$|\mathbf{w}| = |\hat{\mathbf{u}} \times \mathbf{v}| = |\hat{\mathbf{u}}||\mathbf{v}| \sin \theta = (1)|\mathbf{v}| \sin(\pi/2) = |\mathbf{v}|$$

Thus in reality we can write:

$$\text{Rot}(\mathbf{v}) = (\cos \theta)\mathbf{v} + (\sin \theta)(\hat{\mathbf{u}} \times \mathbf{v})$$

**Example 2.3.** Observe that the vectors  $\hat{\mathbf{u}} = \frac{1}{\sqrt{14}}(1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$  and  $\mathbf{v} = 3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 1\hat{\mathbf{k}}$  are perpendicular. Consequently to rotate  $\mathbf{v}$  about the axis  $\mathbf{u}$  by  $\pi/4$  radians we calculate:

$$\text{Rot}(\mathbf{v}) = (\cos(\pi/4))\mathbf{v} + (\sin(\pi/4))(\hat{\mathbf{u}} \times \mathbf{v})$$

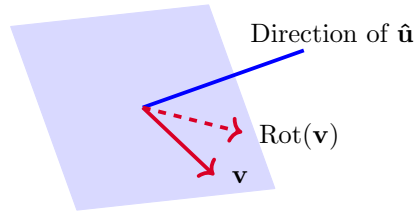
Since we have:

$$\hat{\mathbf{u}} \times \mathbf{v} = \frac{1}{\sqrt{14}}(-2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

we get the result:

$$\text{Rot}(\mathbf{v}) = \frac{\sqrt{2}}{2}(3\hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 1\hat{\mathbf{k}}) + \frac{\sqrt{2}}{2} \frac{1}{\sqrt{14}}(-2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

The following picture is the result. We've put the direction of  $\hat{\mathbf{u}}$  rather than  $\hat{\mathbf{u}}$  itself since  $\hat{\mathbf{u}}$  is a unit vector and consequently is so short as to be tricky to have in the picture.



**Exercise 2.11.** Let  $\mathbf{u} = 5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$  (not a unit vector) and let  $\mathbf{v} = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ . Find  $\hat{\mathbf{u}}$  (normalize  $\mathbf{u}$ ), prove that  $\hat{\mathbf{u}}$  and  $\mathbf{v}$  are perpendicular and then find the result when  $\mathbf{v}$  is rotated by  $\pi/6$  around  $\hat{\mathbf{u}}$ .

**Exercise 2.12.** Consider the plane defined by  $x + 2y + 4z = 0$  and assumed normal vector arising from the normalization of  $\mathbf{N} = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ . Find a generic formula for rotation of this plane about this vector and give this formula as a mapping of a point  $(x, y, z) \mapsto (?, ?, ?)$ .

**Exercise 2.13.** Given a plane  $\mathcal{P}$ , a point  $\mathbf{p} \in \mathcal{P}$ , and a unit vector  $\hat{\mathbf{u}}$  anchored at  $\mathbf{p}$ , we can still rotate  $\mathcal{P}$  about  $\mathbf{p}$  (and about the axis  $\hat{\mathbf{u}}$ ) in a counterclockwise direction as defined by  $\hat{\mathbf{u}}$  even when  $\mathbf{p} \neq \mathbf{0}$ , although this transformation is not linear. We do so by translating  $\mathbb{R}^3$  so that  $\mathbf{p}$  is at the origin, then rotate, then translate back.

Let  $\mathcal{P}$  be the plane  $x + 2y + 3z = 6$ . Let the center and axis be defined by  $\mathbf{p} = [2, 2, 0]^T$  and  $\hat{\mathbf{u}}$  be the normalization of  $\mathbf{N} = 1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ . Find the resulting point when  $(0, 3, 0)$  is rotated by  $\pi/6$  radians in this way.

## 2.5 Reflections

As with rotations it's clear that a transformation which reflects in a line not through the origin will not be linear since it will take the origin off itself. But how about reflection in a line through the origin?

There are various approaches to this but the most elementary one is to approach the problem as follows. A line through the origin can be represented by a vector  $\mathbf{v}$ . What we can do is rotate about the origin to move  $\mathbf{v}$  to the positive  $x$ -axis, then reflect in the  $x$ -axis using the transformation:

$$F_{\mathbf{e}_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and then rotate back. Notice that the above transformation simply negates the  $y$ -coordinate. The notation  $F_{\mathbf{e}_1}$  is used because we can think of it as reflection in (flipping over) the vector  $\mathbf{e}_1$ .

So then in other words if the vector  $\mathbf{v}$  makes an angle of  $\theta$  with the positive  $x$ -axis then we would simply reflect by:

$$F_{\mathbf{v}} = R_{\theta} F_{\mathbf{e}_1} R_{-\theta}$$

Since the product is a matrix the resulting transformation is linear.

Interestingly if we work the details of this out we find:

$$\begin{aligned}
F_{\mathbf{v}} &= R_{\theta} F_{\mathbf{e}_1} R_{-\theta} \\
&= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \\
&= \dots \\
&= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= R_{2\theta} F_{\mathbf{e}_1}
\end{aligned}$$

So reflecting in the line (vector) with angle  $\theta$  is equivalent to first reflecting in the  $x$ -axis and then rotating counterclockwise by  $2\theta$ .

**Exercise 2.14.** Fill in the ... part of the calculation above.

**Exercise 2.15.** Use the above formula to calculate the reflection of the point  $(5, -3)$  in the vector  $\mathbf{v} = 2\hat{\mathbf{i}} + 1\hat{\mathbf{j}}$ .

**Exercise 2.16.** For a general  $\theta$  calculate the explicit formula for  $F_{\mathbf{v}}$ .

**Exercise 2.17.** The line  $y = mx$  is represented by the vector  $\mathbf{v} = [1, m]^T$  which by basic trigonometry satisfies  $1 = \sqrt{1 + m^2} \cos \theta$  and  $m = \sqrt{1 + m^2} \sin \theta$  (a picture can help you see this). Use this fact and the above formula for  $F_{\mathbf{v}}$  to show that reflection in the line  $y = mx$  is represented by the matrix:

$$\frac{1}{m^2 + 1} \begin{bmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{bmatrix}$$

**Exercise 2.18.** Calculate the matrix which reflects in the line  $y = 3x$  and use it to reflect the points  $(1, 2)$  and  $(3, -4)$ .

**Exercise 2.19.** Another approach to reflection in  $y = mx$  is to start with a point  $(x_0, y_0)$ , construct the line perpendicular to  $y = mx$  and through  $(x_0, y_0)$ , and then locate the opposite point on this new line which is the same distance from  $y = mx$  as  $(x_0, y_0)$ . Do so and show that the result is the same as the matrix two exercises above. This is pretty straightforward when done carefully.

**Exercise 2.20.** Work through some of the above calculations by using reflection in the  $y$ -axis instead of reflection in the  $x$ -axis and show that the results are all equivalent.

**Exercise 2.21.** Show that reflecting in the line (vector) with angle  $\theta$  is equivalent to reflecting in the  $x$ -axis and then rotating counterclockwise by  $2\theta$ .

lent to first reflecting in the  $y$ -axis and then rotating counterclockwise by some amount.

## 2.6 Two Reflections Make a Rotation

From the previous calculation:

$$F_{\mathbf{v}} = R_{2\theta} F_{\mathbf{e}_1}$$

We can solve for  $R_{2\theta}$  as follows:

$$\begin{aligned} F_{\mathbf{v}} &= R_{2\theta} F_{\mathbf{e}_1} \\ F_{\mathbf{v}} F_{\mathbf{e}_1}^{-1} &= R_{2\theta} \\ F_{\mathbf{v}} F_{\mathbf{e}_1} &= R_{2\theta} \end{aligned}$$

It follows that any rotation at all can be constructed using reflection in the  $x$ -axis as well as one other reflection.

**Example 2.4.** To rotate by  $\pi/6$  radians we choose  $\mathbf{v}$  making an angle of  $\theta = \pi/12$  radians with the  $x$ -axis and then:

$$R_{\pi/6} = R_{2(\pi/12)} = F_{\mathbf{v}} F_{\mathbf{e}_1}$$

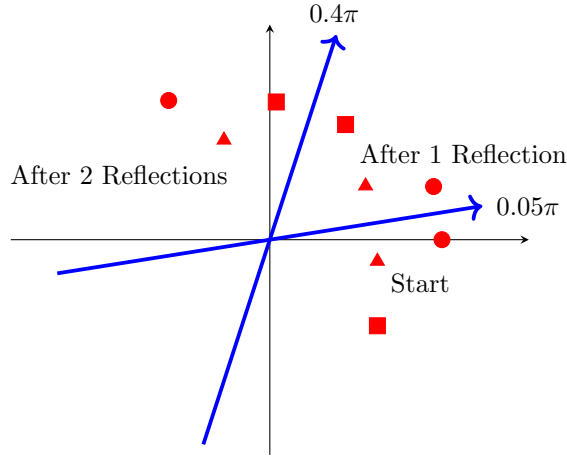
Choosing such a  $\mathbf{v}$  is easy, for example:

$$\mathbf{v} = \begin{bmatrix} \cos(\pi/12) \\ \sin(\pi/12) \end{bmatrix}$$

**Exercise 2.22.** Write  $R_{\pi/3}$  as a product of reflections as in the previous example.

It turns out that the previous statement can be generalized to say that the product of any two reflections equals a rotation.

A picture with a few points can help convince us of this. Here we see a circle, a square and a triangle reflected first in a line with angle of  $0.05\pi$  and then in a line with angle of  $0.4\pi$ . We can see that the successive reflections formed a rotation, the question is how far.



This is not hard to figure out. Suppose the first reflection makes an angle of  $\theta_1$  while the second makes an angle of  $\theta_2$ . Let's calculate the result of reflecting in the first and then the second. The lengthy matrix calculation here is omitted, just the result is given:

$$\begin{aligned}
 F_{\theta_2} F_{\theta_1} &= R_{\theta_2} F_{\mathbf{e}_1} R_{-\theta_2} R_{\theta_1} F_{\mathbf{e}_1} R_{-\theta_1} \\
 &= \dots \\
 &= \begin{bmatrix} \cos(2(\theta_2 - \theta_1)) & -\sin(2(\theta_2 - \theta_1)) \\ \sin(2(\theta_2 - \theta_1)) & \cos(2(\theta_2 - \theta_1)) \end{bmatrix}
 \end{aligned}$$

This shows that the result rotates counterclockwise about the origin by  $2(\theta_2 - \theta_1)$ . An alternate way of thinking about this is that it rotates the plane from the first reflection toward the second reflection by twice the angle between them.

Another consequence of this is that if we wish to rotate by  $\theta$  radians we simply pick any two vectors  $\mathbf{a}$  and  $\mathbf{b}$  such that the angle from  $\mathbf{a}$  to  $\mathbf{b}$  is  $\theta/2$  and do  $F_{\mathbf{b}} F_{\mathbf{a}}$ . Since one of these can be  $\mathbf{e}_1$  this is rather easy.

**Exercise 2.23.** Explicitly do the product of reflection calculations to show that reflection in  $\mathbf{e}_1$  followed by reflection in  $\sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$  does as expected. Note: What's expected?!

## 2.7 The Orthogonal Group $O(2)$

The set of all rotations about the origin and all reflections in lines through the origin form a *group* called the *orthogonal group*  $O(2)$ . While this is not a course in group theory it's interesting to observe some of the properties of this group as they relate to computation. We'll explore this with some exercises.

**Exercise 2.24.** Show that conjugation of a rotation by a reflection yields the reverse rotation. More specifically show that the following holds true for any  $\mathbf{v}$  and  $\theta$ :

$$F_{\mathbf{v}} R_{\theta} F_{\mathbf{v}} = R_{-\theta}$$

**Exercise 2.25.** Suppose all you have at your disposal is an unknown fixed reflection  $F_{\mathbf{v}}$  and all rotations. Show that you can achieve any reflection by doing a product  $R_{\theta} F_{\mathbf{v}}$  for some  $\theta$ .

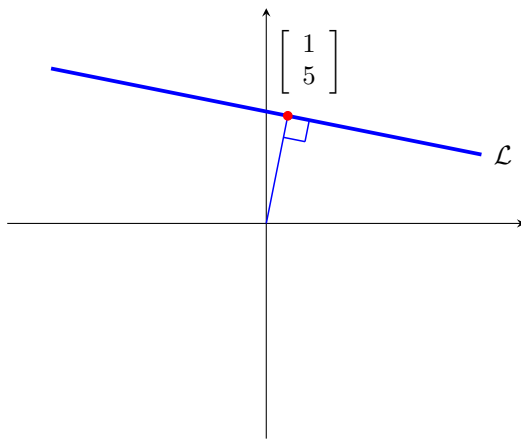
**Exercise 2.26.** Given a rotation  $R_{\theta}$  under what criteria would there exist a positive integer  $n$  such that  $R_{\theta}^n = I$ ? Justify.

## 2.8 Closest Point Parametrization of a Line

Classic ways of representing a line in  $\mathbb{R}^2$  are  $y = mx + b$ ,  $ax + by = c$  and via a parametrization  $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ .

Another interesting way to represent a line  $\mathcal{L}$  which does not pass through the origin is via the *closest point parametrization of a line*. Let  $\mathbf{p} \in \mathbb{R}^2$  with  $\mathbf{p} \neq \mathbf{0}$ . This defines a line as follows - first draw the line from  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  to  $\mathbf{p}$  and then draw the line  $\mathcal{L}$  through  $\mathbf{p}$  perpendicular to this first line.

**Example 2.5.** Here is the line parametrized by  $\mathbf{p} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ :



**Exercise 2.27.** Consider now the line  $\mathcal{L}$  parametrized by  $\mathbf{p} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ .

(a) Draw a rough sketch of  $\mathcal{L}$ .

- (b) Find an equation which is satisfied iff  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{L}$  and rewrite it slope-intercept form.

**Exercise 2.28.** Which point parametrizes the line  $y = 3x + 2$ ?

**Exercise 2.29.** Suppose  $\mathbf{p} \neq \mathbf{0}$  represents  $\mathcal{L}$ . Show that  $R_\theta(\mathbf{p})$  represents  $R_\theta(\mathcal{L})$  and  $F_{\mathbf{v}}(\mathbf{p})$  represents  $F_{\mathbf{v}}(\mathcal{L})$ .

**Exercise 2.30.** Suppose two lines are represented by:

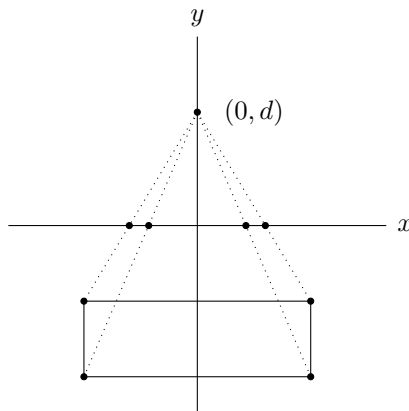
$$\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \mathbf{p}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Find the point at which they meet.

## 2.9 Perspective

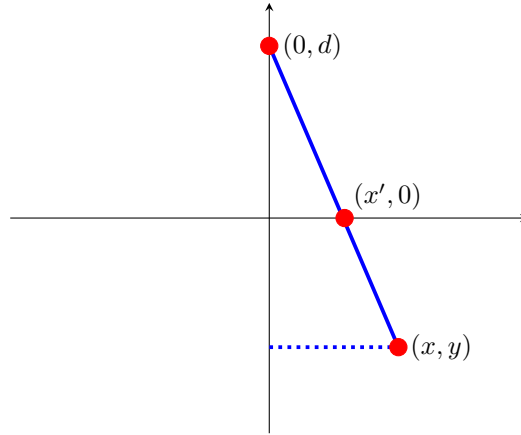
Perspective will make more sense when we move to 3D but we can work up a 2D version that nails down the basic problem.

Imagine an object in the  $xy$ -plane below the  $x$ -axis. If you position your eye at the point  $(0, d)$  and look downwards and imagine that the viewing plane is the  $x$ -axis then you will see the object as if it is projected with perspective to the  $x$ -axis:



Let's look at just one point:





A simple calculation with similar triangles tells us that the point  $(x, y)$  maps to the point  $(x', 0)$  via:

$$\begin{aligned}\frac{x'}{d} &= \frac{x}{d-y} \\ x' &= \frac{dx}{d-y}\end{aligned}$$

This is not a linear transformation, meaning there is no  $2 \times 2$  matrix  $P$  such that:

$$P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

This is easy to see - if such a mapping were a linear transformation then it would take the basis vectors  $[1, 0]^T$  and  $[0, 1]^T$  to  $[d(1)/(d-0), 0]^T = [1, 0]^T$  and  $[d(0)/(d-1), 0]^T = [0, 0]^T$  respectively, meaning it would be the matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

However clearly this matrix does not do what we're asking of it:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \neq \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

**Exercise 2.31.** We showed that this perspective projection was not linear by assuming that it was, constructing the associated matrix, then showing that the matrix failed to do what we want. Instead we could take the mapping

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} dx/(d-y) \\ 0 \end{bmatrix}$$

and show that it failed the definition of linearity. Do so.

**Exercise 2.32.** Explain the difference (in terms of the geometric result) between the object moving away from the viewing plane and the eye moving away from the viewing plane.

## 2.10 Conclusion

In 2D none of translation, rotation, reflection or projection work as we'd like. We get rotation only if it's about the origin and reflection only if it's in a line through the origin. However we've learned a few useful things, such as how two successive reflections yield a rotation as well as how perspective projection ought to work. This knowledge will help us as we move forward.

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