Chapter 7

Real Projective 3-Space

Contents

7.1 Introduction

The goal of this chapter is to generalize $\mathbb{RP}^2$ to $\mathbb{RP}^3$ so that we see how things work in the higher-dimensional case.

We will proceed without as much rigor as $\mathbb{RP}^2$ because many of the calculations are similar and produce fairly intuitive results.

7.2 Real Projective 3-Space

7.2.1 Definition

Definition 7.2.1.1. Define *real projective 3-space* denoted $\mathbb{RP}^3$ as the set of nonzero vectors $[X;Y;Z;W]$ in $\mathbb{R}^4$ with the condition that two vectors are considered to be equivalent if they are nonzero multiples of one another.

Definition 7.2.1.2. We then say that a *projective point* (or just a point when the context is clear) $\mathbf{P}$ in $\mathbb{RP}^3$ is an equivalence class of vectors. Typically we will give just one vector but don't forget that any other equivalent vector is the same point.

Example 7.1. The point $\mathbf{P} = [1;2;3;4]$ represents the set

$$\{\lambda[1;2;3;4] \mid \lambda \neq 0\}$$

Thus $\mathbf{P} = [6;12;18;24]$ and $\mathbf{P} = [-1;-2;-3;-4]$.

Exercise 7.1. List some vectors which are in the equivalence class of $\mathbf{P} = [4;2;1;7]$ and some which are not.

7.2.2 Euclidean Space is Inside

Definition 7.2.2.1. Consider the subset of $\mathbb{RP}^3$ defined by:

$$E^3 = \{[X;Y;Z;W] \mid W \neq 0\}$$

We call this the *Euclidean patch*.

Notice that considering equivalence we have:

$$E^3 = \{[X;Y;Z;1]\}$$

Since all such projective points are distinct when written this way we see that E^3 is essentially a copy of $\mathbb{R}^3$ existing inside $\mathbb{RP}^3$.

Thus we have a mapping from $\mathbb{R}^3$ to $E^3 \subset \mathbb{RP}^3$:

$$\begin{array}{rcl} \mathbb{R}^3 & \to & E^3 \subset \mathbb{RP}^3 \\ [x; y; z] & \mapsto & [x; y; z; 1] = \{[xW; yW; zW; W] \mid W \neq 0\} \end{array}$$

And we have a mapping from $E^3 \subset \mathbb{RP}^3$ to $\mathbb{R}^3$:

$$\begin{array}{rcl} E^3 \subset \mathbb{RP}^3 & \to & \mathbb{R}^3 \\ [X; Y; Z; W] \equiv [X/W; Y/W; Z/W; 1] & \mapsto & \left[\frac{X}{W}; \frac{Y}{W}; \frac{Z}{W}\right] \end{array}$$

Example 7.2. The Euclidean point $[5; 7; 3] \in \mathbb{R}^3$ correponds to the projective point $[5; 7; 3; 1] \in \mathbb{RP}^3$ which consists of the set of vectors $\{[5W; 7W; 3W; W] \mid W \neq 0\}$

Example 7.3. The projective point $[3; 5; 4; 2] \equiv [3/2; 5/2; 2; 1] \in E^3 \subset \mathbb{RP}^3$ corresponds to the Euclidean point $\left[\frac{3}{2}; \frac{5}{2}; 2\right] \in \mathbb{R}^3$.

Example 7.4. The projective point $[-1; 7; -5; 0] \in \mathbb{RP}^3$ doesn't correspond to any point in $\mathbb{R}^3$ because $[-1; 7; -5; 0] \notin E^3$.

Exercise 7.2. Which projective points correspond to each of the following points in $\mathbb{R}^3$.

(a) $[2; 1; 6]$

(b) $[1; 6; 2]$

(c) $[0.1; 0.7; 1.2]$

Exercise 7.3. Which Euclidean points correspond to each of the following points in $\mathbb{RP}^3$. One is a trick.

(a) $[0; 2; 5; -2]$

(b) $[1; 10; 5; 7]$

(c) $[7; 7; 2; 0]$

(d) $[8.1; 6; 0.2; 0.15]$

It's worth noting that there are other ways we could have selected a copy of $\mathbb{R}^3$ inside $\mathbb{RP}^3$, including fixing $X \neq 0$, $Y \neq 0$ or $Z \neq 0$.

7.2.3 More Than Euclidean Space

Notice that nothing in $\mathbb{R}^3 \equiv E^3$ matches up with vectors in $\mathbb{RP}^3$ which have the form $[X; Y; Z; 0]$. This means that we've enlarged $E^3 \equiv \mathbb{R}^3$ by adding on these vectors, even given the equivalences.

These vectors themselves form a copy of $\mathbb{RP}^2$ and so it follows that the points at infinity for $\mathbb{RP}^3$ look like $\mathbb{RP}^2$, meaning they form a sphere with opposite points identified.

Another intuitive way to see this is to recall that one way to visualize $\mathbb{RP}^2$ is as $\mathbb{R}^2$ with a point at infinity for each direction in the plane (opposite directions are considered the same) and one way to visualize $\mathbb{RP}^3$ is as $\mathbb{R}^3$ with a point at infinity for each direction in space (opposite directions are considered the same). This is precisely the same, a sphere with opposite points identified.

7.2.4 Compatibility with Euclidean Space

It's important to note that in $\mathbb{RP}^3$ objects must behave as espected when we focus on the Euclidean patch. For example a plane in $\mathbb{RP}^3$ must look like a plane when we restrict our view to the Euclidean patch, otherwise we've just built something completely new and useless. We'll see how this works as we move forward.

7.3 Standard Transformations of Points

7.3.1 Rotations

Consider rotation. In our earlier case we initially took our concept of rotation in $\mathbb{R}^2$ and extended it to $\mathbb{RP}^2$. The result transformed $\mathbb{RP}^2$ in a way which behaved like a rotation on the Euclidean patch.

Consider first the rotation:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since this matrix is acting on vectors of the form $[X; Y; Z; W]$ it's a bit unclear exactly what's going on. However we can observe that this matrix will fix projective points of the form $[0; 0; Z; W]$, so in in some sense we could argue that it's a rotation around (meaning it fixes) the ZW-plane, hence the notation.

However projective points of this form actually form a circle so the action is more of a rotation "about" a circle as far as $\mathbb{RP}^3$ is concerned.

However this matrix, when applied to $\mathbb{R}^3 \equiv E^3 \subset \mathbb{RP}^3$, does in fact fix the z-axis, and rotates around it according to the right-hand rule where the thumb points in the direction of the positive z-axis.

The notation R_Z is used for this reason, even though as far as $\mathbb{RP}^3$ is concerned it's a bit dishonest.

Likewise the following matrix fixes points of the form $[X; 0; 0; W]$ and effectively rotates about the x-axis in the Euclidean patch.

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And the following matrix fixes points of the form $[0; Y; 0; W]$ and effectively rotates about the Y-axis in the Euclidean patch.

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice the order of the signs here. It might be tempting to have the $-\sin\theta$ in the upper right nonzero entry as with the other two. We can see why this one is a little different by considering: Rotation about the z-axis takes the positive x-axis to the positive y-axis via the right-hand rule. Rotation about the x-axis takes the positive y-axis to the positive z-axis via the right-hand rule. However rotation about the y-axis takes the positive z-axis to the positive x-axis. If we had casually placed the negative sign in the tempting position we would have ended up with a rotation taking the positive x-axis to the positive z-axis, not the desired rotation. The sign change results from replacing θ by $-\theta$ and simplifying.

Example 7.5. To rotate the point $[51; 106; 21]$ by 3.02 radians about the y-axis we calculate:

$$R_Y(3.02)[51; 106; 21; 1] \approx [-52.14; 106; -17; 99; 1]$$

Exercise 7.4. Calculate the result when the two points $[30; 10; 5]$ and $[100; -20; 50]$ are rotated by 8.3 radians about the x-axis.

7.3.2 Translations

We cannot translate all four coordinates in $\mathbb{RP}^3$ simultaneously because such a transformation is not linear. However we can translate in any three of them.

The following matrix is a shear which translates points in the Euclidean patch by $[x; y; z; 1] \mapsto [x + a; y + b; z + c; 1$.

$$T(a; b; c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7.3.3 Reflections

There are three standard reflections, one in each of the coordinate planes

This flips in the xy-plane, negating z:

$$F_{XY} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This flips in the xz-plane, negating y:

$$F_{XZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This flips in the yz-plane, negating x:

$$F_{YZ} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7.3.4 Combinations

As in $\mathbb{RP}^3$ and E^3 the fact that these are linear and represented by matrices implies that we can multiply these together to get combinations of these.

Example 7.6. For example to rotate $E^e \equiv \mathbb{R}^3$ by 1.2 radians about the vertical axis consisting of the line $x = 2$, $y = 3$ we use the matrix:

$$T(2; 3; 0)R_Z(1.2)T(-2; -3; 0)$$

And so for example to rotate the point $[4; -2; 5] \in \mathbb{R}^3$ this way we would do:

$$T(2; 3; 0)R_Z(1.2)T(-2; -3; 0)[4; -2; 5; 1] \approx [7.39; 3.05; 5; 1]$$

Exercise 7.5. Find the result when the point $[102; -54; 82]$ is rotated by 31 radians about the the line $x = -5$, $z = 1$.

Of course if we wish to rotate about an axis which is not so simple then this becomes quite difficult using projective methods. Instead this is what quaternions were for.

However in simple cases we can shift the axis of rotation to a familiar axis.

Example 7.7. To rotate by 0.7 radians about the axis $y = x$, $z = 0$ we rotate by $\pi/4$ about the z-axis to move this axis to the y-axis, then rotate, then rotate back. In other words we use the matrix:

$$R_Z(-\pi/4)R_Y(0.7)R_Z(\pi/4)$$

So if we wished to rotate the point $[20; 30; 40]$ we would do:

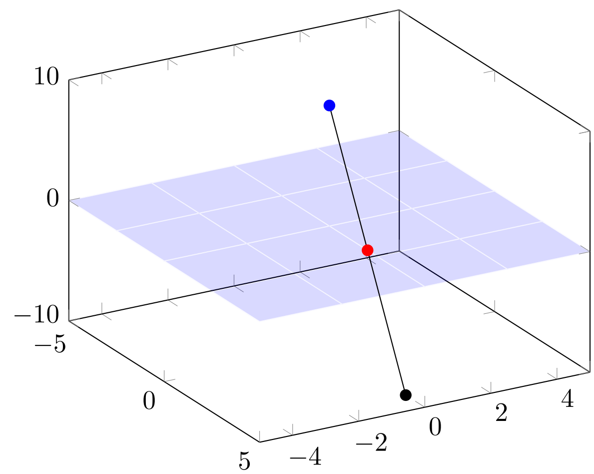
$$R_Z(-\pi/4)R_Y(\theta)R_Z(\pi/4)[20; 30; 40; 1] \approx [39.40; 10.60; 35.15; 1]$$

Exercise 7.6. Find the result when the two points $[5; 8; 10]$ and $[-5; -8; 10]$ are rotated by 3.3 radians about the axis $x = z$, $y = 0$.

7.3.5 Projective Projection

Projecting an object in $\mathbb{R}^3$ means imagining the object below the xy-plane, imagining your eye at the point $[0; 0; d]$ with $d > 0$, and projecting the object to the xy-plane with perspective.

In the following picture the eye is above the xy-plane and a single point is projected with perspective to the xy-plane:



As with $\mathbb{R}^3$ using similar triangles it's straighforward to show that this mapping must act as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} dx/(d-z) \\ dy/(d-z) \\ 0 \end{bmatrix}$$

As with the $\mathbb{R}^2$ case this is not linear but can be represented by a matrix acting on $\mathbb{RP}^3$ with $\mathbb{R}^3$ as a subset.

That matrix is:

$$P(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix}$$

Observe that it works as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 0 \\ 1 - Z/d \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 0 \\ (d-Z)/d \end{bmatrix} \equiv \begin{bmatrix} dX/(d-Z) \\ dY/(d-Z) \\ 0 \\ 1 \end{bmatrix}$$

Thus treating the Euclidean patch as $\mathbb{R}^3$ we have the desired:

$$[x, y; z] \to \left[\frac{dx}{d-z}; \frac{dy}{d-z}; 0\right]$$

Example 7.8. To project the point $[20; 50; -40]$ with $z = 25$ we calculate:

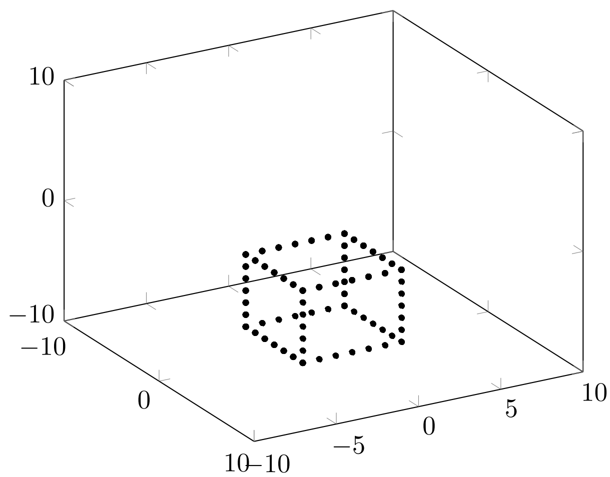
$$P(25)[20; 50; -40; 1] = [20; 50; 0; 13/5] \equiv [7.69; 19.23; 0; 1]$$

yielding a result of $[7.69; 19.23; 0]$ in the xy-plane.

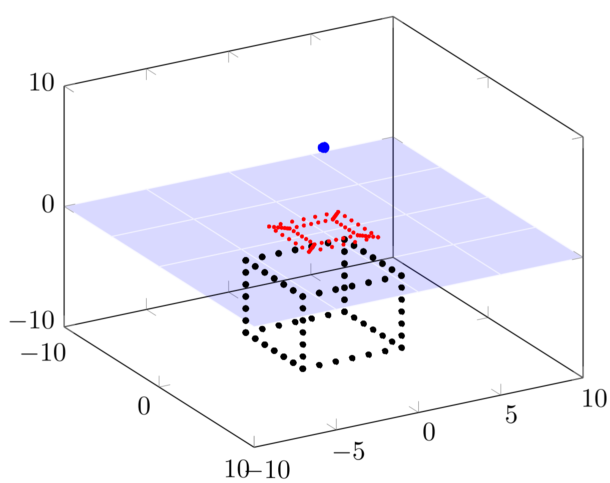
Exercise 7.7. Find the result when the points $[34; 45; -100]$ and $[-46; 50; -50]$ are projected with $z = 50$.

For a visual example with many more points:

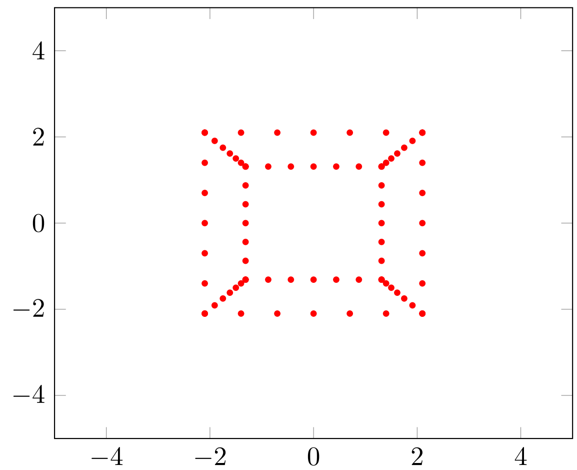
Example 7.9. Consider the box of side length 6 aligned with the axes and with center at $[0; 0; -6]$. Here is this box drawn with seven points per side (points at integer positions).



If we place our eye at $[0; 0; 7]$; treat these points as vectors $[x; y; z; 1] \in \mathbb{RP}^3$ and project these points using the above matrix calculation with $d = 7$ we get the following result with the original points in black, the resulting points in red, and the eye point in blue:



By itself in the xy-plane we see the result, a cube projected with perspective:



7.3.6 Other Perspective Viewpoints

It is restrictive that we can only view our objects from the positive z-axis. In order to view objects from other positions the standard approach would be to rotate space so that the viewpoint is on the positive z-axis and then view. This has the bonus of yielding a result in the xy-plane (rather than some other plane) which makes for easy plotting.

7.4 Points, Planes, Duality and Standard Transformations

7.4.1 Points

As we've seen, a point in $\mathbb{RP}^3$ is defined by the equivalence class of a triple $\mathbf{P} = [X; Y; Z; W]$. For simplicity we'll just say "the point" $\mathbf{P} = [X; Y; Z; W]$ when we mean the entire equivalence class.

Definition 7.4.1.1. We say that points of the form $\mathbf{P} = [X; Y; Z; 0]$ are *points at infinity*. These are points which are not in $E^3 \equiv \mathbb{R}^3$.

7.4.2 Planes

In order to define a plane in $\mathbb{RP}^3$ we need to establish exactly what a plane is. There are many ways to approach this but one common-sense one (which is perfect for us) would be to define a plane in $\mathbb{RP}^3$ to be a set of points $\mathcal{P}$ such that the restriction of $\mathcal{P}$ to $E^3 \equiv \mathbb{R}^3$ is a plane in the normal sense.

Definition 7.4.2.1. A *plane in projective space* or a *projective plane* $\mathcal{P}$ in $\mathbb{RP}^3$ is defined by giving a nonzero vector $\mathbf{P} = [a; b; c; d] \in \mathbb{RP}^3$ and looking at all those $\mathbf{x} = [X; Y; Z; W]$ satisfying $\mathbf{P} \cdot \mathbf{x} = \mathbf{P}^T \mathbf{x} = 0$. That is:

$$\mathcal{L} = \{[X; Y; Z; W] \,|\, aX + bY + cZ + dW = 0\}$$

Equivalently $[a, b, c, d][X; Y; Z; W] = 0$ or $[a; b; c; d] \cdot [X; Y; Z; W] = 0$.

Notice that any nonzero multiple of $[a; b; c; d]$ defines the same plane. In this way planes behave a bit like points. This is more true than we might realize right now.

Theorem 7.4.2.1. When at least one of a, b, and c is nonzero this definition of a plane matches our intuition in the sense that a plane in projective space is a Euclidean plane when restricted to $E^3 \equiv \mathbb{R}^3$ and every Euclidean plane in $\mathbb{R}^3 \equiv E^3$ arises from this definition.

Proof. Given $[a; b; c; d]$ consider the set

$$\mathcal{L} = \{[X; Y; Z; W] \,|\, aX + bY + cZ + dW = 0\}$$

If $a = b = c = 0$ then $d \neq 0$ and

$$\begin{aligned} \mathcal{P} &= \{[X; Y; Z; W] \,|\, (0)X + (0)Y + (0)Z + dW = 0\} \\ &= \{[X; Y; Z; 0] \,|\, X, Y, Z \in \mathbb{R}\} - \{[0; 0; 0; 0]\} \end{aligned}$$

which is precisely the set of points at infinity.

If one of a, b, and c is nonzero then if we isolate our view to the Euclidean patch. The point $[x; y; z] \in \mathbb{R}^3$ corresponds to $[x; y; z; 1] \in E^3 \subset \mathbb{RP}^3$ so for this point to be on the line we must have

$$a(x) + b(y) + c(z) + d(1) = 0$$

This is a plane in $\mathbb{R}^2$.

Likewise every plane in $\mathbb{R}^3 \equiv E^3$ may be written in the form $ax + by + cz + d = 0$ and hence in $\mathbb{RP}^3$ may be represented by $[a; b; c; d]$. $\square$

Theorem 7.4.2.2. A plane in $\mathbb{RP}^3$ represented by $[a; b; c; d]$ with at least one of a, b, and c nonzero contains infinitely many points at infinity.

Proof. To identify these points, note that they must have the form $[X; Y; Z; 0]$ not all zero and must satisfy $aX + bY + cZ + d(0) = 0$.

However this is precisely a projective line in $\mathbb{RP}^2 \subset \mathbb{RP}^3$. It follows that a plane in $\mathbb{RP}^3$ consists of a plane in E^3 as well as a projective line of points at infinity. $\square$

Example 7.10. For example the vector $\mathbf{P} = [1; 2; 3; 4]$ defines the plane consisting of the set of points $[X; Y; Z; W]$ satisfying:

$$\begin{aligned} [1, 2, 3, 4][X; Y; Z; W] &= 0 \\ X + 2Y + 3Z + 4Z &= 0 \end{aligned}$$

In $\mathbb{R}^3 \equiv E^3$ the points which lie on this projective plane satisfy the equation:

$$x + 2y + 3z + 4 = 0$$

The points at infinity on this plane are those nonzero $[X; Y; Z; 0]$ with $X + 2Y + 3Z = 0$. This is a projective line.

Exercise 7.8. Consider the plane represented by the vector $\mathbf{P} = [5; 0; -2; 1]$. Find the points in E^3 and the points at infinity on this plane.

If we start with a Euclidean plane and move it into projective space we pick up a projective line at infinity. Intuitively it extends to pick up those additional points.

Example 7.11. Consider the Euclidean plane with equation $2x + y - 5z = 10$ Suppose the projective point $[X; Y; Z; W]$ satisfied this equation. If $Z \neq 0$ then since $[X; Y; Z; W] \equiv [X/W; Y/W; Z/W; 1]$ we know that $2(X/W) + (Y/W) - 5(Z/W) = 10$ so $2X + Y - 5Z - 10W = 0$ and so in projective space this plane is represented by the vector $[2; 1; -5; -10]$.

We also pick up the projective line at infinity consisting of points $[X; Y; Z; 0]$ satisfying $2X + Y - 5Z = 0$.

Exercise 7.9. If the Euclidean plane with equation $x + y = 5$ is moved into $\mathbb{RP}^3$, which vector represents it and which points at infinity are picked up?

Exercise 7.10. Find the vector which represented the plane containing the points $[1; 1; 2]$, $[0; 2; 2]$ and $[5; 3; -2]$. Which points at infinity does it contain?

We close with a small theorem which will be useful later:

Theorem 7.4.2.3. The plane with normal $\mathbf{n}$ and passing through $\mathbf{p}$ is represented by the vector:

$$[\mathbf{n}; -\mathbf{n} \cdot \mathbf{p}]$$

Proof. Suppose $\mathbf{n} = [a; b; c]$ and $\mathbf{p} = [x_0; y_0; z_0]$. Then the plane has equation:

$$\begin{aligned} a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz - (ax_0 + by_0 + cz_0) &= 0 \\ \mathbf{n} \cdot \mathbf{x} - \mathbf{n} \cdot \mathbf{p} &= 0 \end{aligned}$$

□

7.4.3 Duality

What happens here is that a duality appears but it is a duality between points and planes rather than points and lines. Ignoring special cases the intuition emerges from the fact that three points determine a plane and three planes intersect at a point. From here a duality emerges.

Lines have no such duality. Ignoring special cases again we see that two points make a line but it's generally not the case that two lines must intersect at a point and two planes intersect in a line but it's generally not the case that two lines form a plane.

To see how this point-plane duality manifests in intersections from a computational standpoint we need to understand what to do with cross products in $\mathbb{R}^4$. It turns out that the cross product between two vectors in $\mathbb{R}^3$ is actually the determinant of a matrix. If we represent the vector $[X; Y; Z]$ by $X\hat{\boldsymbol{\imath}} + Y\hat{\boldsymbol{\jmath}} + Z\hat{\boldsymbol{k}}$ then we have:

$$[X_1; Y_1; Z_1] \times [X_2; Y_2; Z_2] = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

When we move to $\mathbb{R}^4$ we need three vectors in order to take a cross product. If we represent the vector $[X; Y; Z; W]$ by $X\hat{\boldsymbol{\imath}} + Y\hat{\boldsymbol{\jmath}} + Z\hat{\boldsymbol{k}} + W\hat{\boldsymbol{l}}$ Then if we have three such vectors: $\mathbf{V}_1 = [X_1; Y_1; Z_1; W_1]$, $\mathbf{V}_2 = [X_2; Y_2; Z_2; W_2]$, and $\mathbf{V}_3 = [X_3; Y_3; Z_3; W_3]$ then we can define $\mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$ by:

$$[X_1; Y_1; Z_1; W_1] \times [X_2; Y_2; Z_2; W_2] \times [X_3; Y_3; Z_3; W_3] = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} & \hat{\boldsymbol{l}} \\ X_1 & Y_1 & Z_1 & W_1 \\ X_2 & Y_2 & Z_2 & W_2 \\ X_3 & Y_3 & Z_3 & W_3 \end{vmatrix}$$

We can evaluate this using a cofactor expansion across the top row.

Example 7.12. If we define: $\mathbf{P_1} = [1;2;3;1]$, $\mathbf{P_2} = [0;1;2;1]$, and $\mathbf{P_3} = [4;6;-1;1]$. Then we calculate:

$$\mathbf{P_1} \times \mathbf{P_2} \times \mathbf{P_3} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} & \hat{\boldsymbol{l}} \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 4 & 6 & -1 & 1 \end{vmatrix}$$

$$= \hat{\boldsymbol{\imath}} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 5 & 6 & -1 \end{bmatrix} 1 - \hat{\boldsymbol{\jmath}} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 4 & -1 & 1 \end{bmatrix} + \hat{\boldsymbol{k}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 4 & 6 & 1 \end{bmatrix} - \hat{\boldsymbol{l}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 6 & -1 \end{bmatrix}$$

$$= \hat{\boldsymbol{\imath}}(8) - \hat{\boldsymbol{\jmath}}(7) + \hat{\boldsymbol{k}}(-1) + \hat{\boldsymbol{l}}(9)$$
$$= 8\hat{\boldsymbol{\imath}} - 7\hat{\boldsymbol{\jmath}} - 1\hat{\boldsymbol{k}} + 9\hat{\boldsymbol{l}}$$
$$= [8;-7;-1;9]$$

The result of this sustains duality as follows:

Theorem 7.4.3.1. We have the following:

- The cross product of three vectors representing points results in the vector representing the plane containing these points. If the points are all colinear then the result will be $\mathbf{0}$ and does not apply. This statement holds if one or more of the points are at infinity.
- The cross product of three vectors representing planes results in the vector representing the point in all three planes. If all three planes meet in a line, rather than in a point, then the result will be $\mathbf{0}$ and does not apply.

Proof. Omitted. The proof is similar to the duality proofs for points and lines in the previous chapter. □

As before these lead to quick calculations in the Euclidean patch.

Example 7.13. Consider the three points in $\mathbb{R}^3$ given by $[1;2;3]$; $[0;1;2]$; and $[4;6;-1]$. In $\mathbb{RP}^3$ these are represented by the vectors $\mathbf{P_1} = [1;2;3;1]$, $\mathbf{P_2} = [0;1;2;1]$, and $\mathbf{P_3} = [4;6;-1;1]$. So we calculate:

$$\mathbf{P_1} \times \mathbf{P_2} \times \mathbf{P_3} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} & \hat{\boldsymbol{l}} \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 4 & 6 & -1 & 1 \end{vmatrix} = 8\mathbf{i} - 7\mathbf{j} - 1\mathbf{k} + 9\mathbf{l} = [8;-7;-1;9]$$

Thus in the Euclidean patch the equation of the plane is:

$$8x - 7y - z + 9 = 0$$

Example 7.14. To illustrate the dual nature of the calculation we could also say that if we had three planes in the Euclidean patch with equations $x+2y+3z+1=0$; $y+2z+1=0$, and $4x+6y-z+1=0$ then the same calculation shows that they meet at the point

$$[8;-7;-1;9] \equiv [8/9;-7/9;-1/9;1]$$

which in the Euclidean patch is the point $\left[\frac{8}{9};-\frac{7}{9};-\frac{1}{9}\right]$.

To note some special cases:

Example 7.15. Consider the three points in $\mathbb{R}^3$ given by $[1;2;3]$; $[2;4;6]$; and $[3;6;9]$. In $\mathbb{RP}^3$ these are represented by the vectors $\mathbf{P_1}=[1;2;3;1]$, $\mathbf{P_2}=[2;4;6;1]$, and $\mathbf{P_3}=[3;6;9;1]$. We calculate:

$$\mathbf{P_1} \times \mathbf{P_2} \times \mathbf{P_3} = 0$$

This is expected since these three points are colinear.

Example 7.16. In this previous example if these three vectors had represented planes instead of points then they would represent the planes $x+2y+3z+1=0$, $2x+3y+6z+1=0$, and $3x+6y+9z+1=0$. These three planes are parallel and in fact meet at a line, that line being the line consisting of the points at infinity which line on each plane.

Example 7.17. Consider the three planes in R^3 with equations $x+y=2$, $x=0$ and $y=0$. In R^3 these meet pairwise in vertical lines. In $\mathbb{RP}^3$ these are represented by the vectors $\mathbf{P_1}=[1;1;0;-2]$, $\mathbf{P_2}=[1;0;0;0]$, and $\mathbf{P_3}=[0;1;0;0]$. We calculate:

$$\mathbf{P_1} \times \mathbf{P_2} \times \mathbf{P_3} = [0;0;-2;0] \equiv [0;0;1;0]$$

This is the point at infinity which is encountered if we allow $Z \to \pm\infty$ for any $[X_0;Y_0;Z;1]$.

In the following example one of the points is a point at infinity. This is not a problem, there is still a plane.

Example 7.18. Consider the two points in $\mathbb{R}^3$ given by $[1;4;3]$ and $[0;5;2]$ and the single point at infinity $[6;2;1;0]$. In $\mathbb{RP}^3$ these are represented by the vectors $\mathbf{P_1}=[1;4;3;1]$, $\mathbf{P_2}=[0;5;2;1]$, and $\mathbf{P_3}=[6;2;1;0]$. We calculate:

$$\mathbf{P_1} \times \mathbf{P_2} \times \mathbf{P_3} = [3;-5;8;41]$$

Exercise 7.11. Find the intersection point of the three planes with Euclidean equations $x+2y-3z=3$, $4x-y+2z=5$ and $x+6y-z=10$.

Exercise 7.12. Find the equation of the plane containing the three points $[1; 0; 0]$, $[5; 10; 3]$ and $[6; -3; 4]$.

Exercise 7.13. Suppose $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4 \in \mathbb{R}^3$. Consider the calculation:

$$(\mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3) \cdot \mathbf{V}_4$$

What does this calculation tell you...

1. If the $\mathbf{V}_i$ represent points?
2. If the $\mathbf{V}_i$ represent planes?

7.4.4 Standard Transformations of Planes

Given that a plane is represented by a vector $\mathbf{P}$ we might ask what happens if a projective transformation is applied to the plane. This theorem should remind you of a similar theorem about lines in the previous chapter.

Theorem 7.4.4.1. If a plane represented by the vector $\mathbf{P}$ is transformed by the matrix M then the resulting plane is represented by the matrix:

$$\left(M^{-1}\right)^T \mathbf{P}$$

Proof. Suppose $\mathbf{x}$ is on the transformed plane. Then $M^{-1}\mathbf{x}$ is on the original plane and so:

$$\begin{aligned} \mathbf{P} \cdot \left(M^{-1}\mathbf{x}\right) &= 0 \\ \mathbf{P}^T M^{-1}\mathbf{x} &= 0 \\ \left(\left(M^{-1}\right)^T \mathbf{P}\right)^T \mathbf{x} &= 0 \\ \left(\left(M^{-1}\right)^T \mathbf{P}\right) \cdot \mathbf{x} &= 0 \end{aligned}$$

□

Example 7.19. To calculate the result when the plane $2x+y-z=3$ is rotated by 5 radians about the y-axis we assign $\mathbf{P} = [2; 1; -1; -3]$ and then calculate:

$$(R_Y(5)^{-1})^T P \approx [1.53; 1; 1.63; -3]$$

so the Euclidean equation for the result is approximately $1.53x+y+1.63z-3 = 0$.

Exercise 7.14. Find the result when the plane $x + 5y - 2z = 10$ is rotated by 4.2 radians about the line $x = 2$, $y = -3$.

7.5 Plücker Coordinates for Lines

7.5.1 Definition of Plücker Coordinates

Lines are not easy to describe in $\mathbb{RP}^3$ just as they are not easy to describe in $\mathbb{R}^3$.

It's certainly true that two points make a line but these points are not unique and even so, knowing two points on a line is not particularly helpful. Similarly a point and a vector make a line but these don't necessarily lead to easy calculations of intersections, coplanarity and so on.

Instead we introduce Plücker coordinates which are a computationally convenient way to store the information about line.

Definition 7.5.1.1. Suppose a line $\mathcal{L}$ in $\mathbb{R}^3$ is determined by a point $\mathbf{v}$ and a direction vector $\mathbf{d}$. If we define the *moment vector* of the line:

$$\mathbf{m} = \mathbf{v} \times \mathbf{d}$$

then the pair of vectors in $\mathbb{R}^3$

$$[\mathbf{d}; \mathbf{m}]$$

make up the *Plücker Coordinates* for $\mathcal{L}$.

Given that any multiple $\beta\mathbf{d}$ and any other point on the line $\mathbf{v} + \alpha\mathbf{d}$ can be used to construct the same line we have to ask what happens to the Plücker coordinates if we use a variation.

Theorem 7.5.1.1. For any line the Plücker coordinates are unique up to a constant multiple.

Proof. Suppose we replace $\mathbf{d}$ by $\mathbf{d}' = \beta\mathbf{d}$ and $\mathbf{v}$ by $\mathbf{v}' = \mathbf{v} + \alpha\mathbf{d}$. Then observe that:

$$\begin{aligned}
\mathbf{m}' &= \mathbf{v}' \times \mathbf{d}' \\
&= (\mathbf{v} + \alpha\mathbf{d}) \times (\beta\mathbf{d}) \\
&= \beta\mathbf{v} \times \mathbf{d} + \alpha\beta\mathbf{d} \times \mathbf{d} \\
&= \beta\mathbf{m} + \alpha\beta(0) \\
&= \beta\mathbf{m}
\end{aligned}$$

So the new Plücker coordinates are:

$$[\mathbf{d}'; \mathbf{m}'] = \beta[\mathbf{d}; \mathbf{m}]$$

□

Next we reverse the question and ask if any sextuplet (meaning any $\mathbf{d}$ and $\mathbf{m}$) can arise from a line.

Observe that we have the following:

Theorem 7.5.1.2. If $[\mathbf{d}; \mathbf{m}]$ arises from a line then we have $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{d} \cdot \mathbf{m} = 0$.

Proof. For the first part of course $\mathbf{d} \neq \mathbf{0}$ or else the line would not have a direction.

For the second part suppose $\mathbf{v}$ is on the line and $\mathbf{d}$ is the direction. Then we have $\mathbf{d} \cdot \mathbf{m} = \mathbf{d} \cdot (\mathbf{v} \times \mathbf{d}) = 0$ because $(\mathbf{v} \times \mathbf{d}) \perp \mathbf{d}$. □

Theorem 7.5.1.3. If $\mathbf{d} \neq 0$ and $\mathbf{d} \cdot \mathbf{m} = 0$ then $[\mathbf{d}; \mathbf{m}]$ represents a line.

Proof. Given a sextuplet $[\mathbf{d}; \mathbf{m}]$ with $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{d} \cdot \mathbf{m} = 0$ we claim that this sextuplet arises from the line with direction $\mathbf{d}$ and point $\mathbf{v} = (\mathbf{d} \times \mathbf{m})/(\mathbf{d} \cdot \mathbf{d})$. To see this observe that we only need to show that this $\mathbf{v}$ gives rise to our $\mathbf{m}$. In other words that $\mathbf{m} = \mathbf{v} \times \mathbf{d}$.

$$
\begin{aligned}
\mathbf{v} \times \mathbf{d} &= -\mathbf{d} \times \mathbf{v} \\
&= -\mathbf{d} \times \frac{\mathbf{d} \times \mathbf{m}}{\mathbf{d} \cdot \mathbf{d}} \\
&= -\frac{1}{\mathbf{d} \cdot \mathbf{d}} \left[\mathbf{d} \times (\mathbf{d} \times \mathbf{m})\right] \\
&= -\frac{1}{\mathbf{d} \cdot \mathbf{d}} \left[(\mathbf{d} \cdot \mathbf{m})\mathbf{d} - (\mathbf{d} \cdot \mathbf{d})\mathbf{m}\right] \\
&= -\frac{1}{\mathbf{d} \cdot \mathbf{d}} \left[0 - (\mathbf{d} \cdot \mathbf{d})\mathbf{m}\right] \\
&= \mathbf{m}
\end{aligned}
$$

□

It follows from the previous theorems that any given line will yield $[\mathbf{d}; \mathbf{m}]$ with $\mathbf{d} = \mathbf{0}$ and $\mathbf{d} \cdot \mathbf{m} = 0$ unique up to scalar multiples and any $[\mathbf{d}; \mathbf{m}]$ represents a line provided $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{d} \cdot \mathbf{m} = 0$, with scalar multiples representing the same line.

We'll continue to use the phrase "the Plücker coordinates" even though they're not unique with the understanding that nonzero multiples are equivalent.

Example 7.20. To find the Plücker coordinates for the line containing the points $[4; 5; 0]$ and $[10; 13; 3]$ we calculate:

$$
\begin{aligned}
\mathbf{d} &= [6; 8; 3] \\
\mathbf{m} &= [4; 5; 0] \times [6; 8; 3] = [15; -12; -2]
\end{aligned}
$$

Thus the result is $[\mathbf{d};\mathbf{m}] = [6;8;3;15;-12;-2]$.

Exercise 7.15. Find the Plücker coordinates for the line containing the points $[5;0;2]$ and $[8;1;2]$.

Exercise 7.16. Show that for Plücker coordinatex $[\mathbf{d};\mathbf{m}]$ we have $\mathbf{m} = \mathbf{0}$ if and only if the lines goes through the origin.

Intuitively this is a bit like the closest-point representation of a line in $2D$ where the moment vector gives information about where the line is in relation to the origin.

7.5.2 Pre-Computation Note and Facts

The reason that Plücker coordinates are useful is that they allow for convenient computation. While most of the formulas here apply fully in $\mathbb{RP}^3$ we are essentially thinking of them as tools to manage objects in $\mathbb{R}^3$ and so all proofs, calculations, etc., will use $\mathbb{R}^3$ as their setting.

First off here are some lemmas:

Lemma 7.5.2.1. For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ we have:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

Proof. Brute force. □

Lemma 7.5.2.2. For vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ we have:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

Proof. Brute force. □

Lemma 7.5.2.3. For vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ we have:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

Proof. Brute force. Notice this is easy to remember because the letters are always in cyclical order. □

The following arose earlier but we'll give it again as a lemma:

Lemma 7.5.2.4. The line $[\mathbf{d};\mathbf{m}]$ contains the point

$$\frac{\mathbf{d}\times\mathbf{m}}{\mathbf{d}\cdot\mathbf{d}}$$

Proof. See earlier. □

7.5.3 Point on Line

Theorem 7.5.3.1. The point $\mathbf{v} \in \mathbb{R}^3$ is on the line $[\mathbf{d};\mathbf{m}]$ iff $\mathbf{v} \times \mathbf{d} = \mathbf{m}$.

Proof. Suppose $\mathbf{v}$ is on the line. We also know that $(\mathbf{d} \times \mathbf{m})/\mathbf{d} \cdot \mathbf{d}$ is on the line. It follows that the the difference is parallel to $\mathbf{d}$ and so:

$$\begin{aligned}
\left[\mathbf{v} - \frac{\mathbf{d} \times \mathbf{m}}{\mathbf{d} \cdot \mathbf{d}}\right] \times \mathbf{d} &= 0 \\
\mathbf{v} \times \mathbf{d} + \frac{1}{\mathbf{d} \cdot \mathbf{d}}\mathbf{d} \times (\mathbf{d} \times \mathbf{m}) & \\
\mathbf{v} \times \mathbf{d} + \frac{1}{\mathbf{d} \cdot \mathbf{d}}\left[(\mathbf{d} \cdot \mathbf{m})\mathbf{d} - (\mathbf{d} \cdot \mathbf{d})\mathbf{m}\right] &= 0 \\
\mathbf{v} \times \mathbf{d} + \frac{1}{\mathbf{d} \cdot \mathbf{d}}\left[(0)\mathbf{d} - (\mathbf{d} \cdot \mathbf{d})\mathbf{m}\right] &= 0 \\
\mathbf{v} \times \mathbf{d} - \mathbf{m} &= 0 \\
\mathbf{v} \times \mathbf{d} &= \mathbf{m}
\end{aligned}$$

On the other hand suppose $\mathbf{v}$ satisfies $\mathbf{v} \times \mathbf{d} = \mathbf{m}$. Let $\mathbf{v}_0$ be any point on the line. Then we have:

$$(\mathbf{v} - \mathbf{v}_0) \times \mathbf{d} = \mathbf{v} \times \mathbf{d} - \mathbf{v}_0 \times \mathbf{d} = \mathbf{m} - \mathbf{m} = \mathbf{0}$$

Thus $\mathbf{v} - \mathbf{v}_0$ is parallel to $\mathbf{d}$ and since $\mathbf{v}_0$ is on the line, so is $\mathbf{v}$. □

Example 7.21. Consider the line $[\mathbf{d};\mathbf{m}] = [8;2;-1;-3;1;22]$. We can check some points:

- $[1;3;0]$: We find $[1;3;0] \times [8;2;-1] = [-3;1;22] = \mathbf{m}$ so it's on the line.
- $[9;5;-1]$: We find $[9;5;-1] \times [8;2;-1] = [-3;1;22] = \mathbf{m}$ so it's on the line.
- $[10;3;10]$: We find $[10;3;10] \times [8;2;-1] = [-23;90;-4] \neq \mathbf{m}$ so it's not on the line.

Exercise 7.17. Check which of the following points are on the line $[\mathbf{d};\mathbf{m}] = [2;3;3;-9;-10;16]$.

1. $[4;-2;1]$
2. $[6;1;4]$
3. $[10;8;10]$
4. $[14;13;16]$

7.5.4 Intersection Line Between Planes

Theorem 7.5.4.1. Suppose $\mathcal{P}_1$ is represented by $[\mathbf{n}_1; n_1]$ and $\mathcal{P}_2$ is represented by $[\mathbf{n}_2; n_2]$. If these planes are not parallel then they meet in the line

$$[\mathbf{d}; \mathbf{m}] = [\mathbf{n}_1 \times \mathbf{n}_2; n_1\mathbf{n}_2 - n_2\mathbf{n}_1]$$

Proof. Clearly $\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2$. Suppose $\mathbf{v} \in \mathbb{R}^3$ is on both planes so that $\mathbf{v} \cdot \mathbf{n}_1 + n_1 = 0$ and $\mathbf{v} \cdot \mathbf{n}_2 + n_2 = 0$. Then observe that:

$$\begin{aligned}
\mathbf{m} &= \mathbf{v} \times \mathbf{d} \\
&= \mathbf{v} \times (\mathbf{n}_1 \times \mathbf{n}_2) \\
&= (\mathbf{v} \cdot \mathbf{n}_2)\mathbf{n}_1 - (\mathbf{v} \cdot \mathbf{n}_1)\mathbf{n}_2 \\
&= (-n_2)\mathbf{n}_1 - (-n_1)\mathbf{n}_2 \\
&= n_1\mathbf{n}_2 - n_2\mathbf{n}_1
\end{aligned}$$

□

As a side note I've found some sources (including Ken Shoemake's original notes on Plücker coordinates, widely regarded as a good computational source), which give the formula as $[\mathbf{n}_1 \times \mathbf{n}_2; n_2\mathbf{n}_1 - n_1\mathbf{n}_2]$. As far as I can tell this formula is incorrect as a simple example suggests:

Example 7.22. Consider the planes $x + y + z - 1 = 0$ and $x + y + 2z - 1 = 0$.

For reference clearly the points $[1; 0; 0]$ and $[0; 1; 0]$ lie on both planes and so the line can be constructed with $\mathbf{d} = [-1; 1; 0]$ and $\mathbf{m} = [1; 0; 0] \times [-1; 1; 0] = [0; 0; 1]$ yielding Plücker coordinates $[\mathbf{d}; \mathbf{m}] = [-1; 1; 0; 0; 0; 1]$.

Assigning $[\mathbf{n}_1; n_1] = [1; 1; 1; -1]$ and $[\mathbf{n}_2; n2] = [1; 1; 2; -1]$ the formula above finds:

$$\begin{aligned}
\mathbf{n}_1 \times \mathbf{n}_2 &= [1; -1; 0] \\
n_1\mathbf{n}_2 - n_2\mathbf{n}_1 &= [0; 0; -1]
\end{aligned}$$

Yielding Plücker coordinates $[1; -1; 0; 0; 0; -1] \equiv [-1; 1; 0; 0; 0; 1]$.

Shoemake's notes in this case would use $n_2\mathbf{n}_1 - n_1\mathbf{n}_2 = [0; 0; 1]$ and would yield $[1; -1; 0; 0; 0; 1] \not\equiv [-1; 1; 0; 0; 0; 1]$.

Perhaps these other sources are not wrong, and I'm simply misunderstanding something in the way they're framing their calculations.

Example 7.23. Consider the planes $x + 2y - 3z + 10 = 0$ and $2x + y - 5 = 0$. These are represented by $[1; 2; -3; 10]$ and $[2; 1; 0; -5]$. These meet in the line represented by:

$$[\mathbf{d}; \mathbf{m}] = [[1; 2; -3] \times [2; 1; 0]; 10[2; 1; 0] - (-5)[1; 2; -3]] = [3; -6; -3; 25; 20; -15]$$

Note that if we took a more traditional Calculus approach to this we would find the direction vector for the line, taking $[1; 2; -3] \times [2; 1; 0] = [3; -6; -3]$, so that part is similar. Then we would find a point on both planes, which involves finding a point satisfying both equations. This can be algebraically awkward in general. In this case it's not so bad and $[5; -5; 5/3]$ works. Then to get the Plücker coordinates we find:

$$\mathbf{m} = [5; -5; 5/3] \times [3; -6; -3] = [25; 20; -15]$$

Exercise 7.18. Find the Plücker coordinates $[\mathbf{d}; \mathbf{m}]$ of the line of intersection of the planes $x + y - z = 3$ and $2x + y + 3z = 10$.

7.5.5 Detecting Coplanarity of Lines

Theorem 7.5.5.1. Suppose $\mathcal{L}_1$ is represented by $[\mathbf{d}_1; \mathbf{m}_1]$ and $\mathcal{L}_2$ is represented by $[\mathbf{d}_2; \mathbf{m}_2]$. Then the lines are coplanar iff $\mathbf{d}_1 \cdot \mathbf{m}_2 + \mathbf{d}_2 \cdot \mathbf{m}_1 = 0$.

Proof. We can check if two lines cross by taking a point on each line, taking the vector connecting them and checking if this vector is perpendicular to the a vector which is perpendicular to both lines. If it is then the lines cross. Pick a point $\mathbf{v}_1$ on $\mathcal{L}_1$ and a point $\mathbf{v}_2$ on $\mathcal{L}_2$. Then observe that we know $\mathbf{d}_1 \times \mathbf{d}_2$ is perpendicular to both and so:

$$\begin{aligned}
(\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{d}_1 \times \mathbf{d}_2) &= \mathbf{v}_1 \cdot (\mathbf{d}_1 \times \mathbf{d}_2) - \mathbf{v}_2 \cdot (\mathbf{d}_1 \times \mathbf{d}_2) \\
&= \mathbf{d}_2 \cdot (\mathbf{v}_1 \times \mathbf{d}_1) - \mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{v}_2) \\
&= \mathbf{d}_2 \cdot \mathbf{m}_1 - \mathbf{d}_1 \cdot (-\mathbf{m}_2) \\
&= \mathbf{d}_2 \cdot \mathbf{m}_1 + \mathbf{d}_1 \cdot \mathbf{m}_2
\end{aligned}$$

The result follows immediately. □

Example 7.24. Consider the line $\mathcal{L}_1$ joining $[1; 0; 0]$ and $[8; 2; -3]$ and the line $\mathcal{L}_2$ joining $[5; 5; -1]$ and $[0; 3; 3]$. These have Plücker coordinates $[\mathbf{d}_1; \mathbf{m}_1] = [7; 2; -3; 0; 3; 2]$ and $[\mathbf{d}_2; \mathbf{m}_2] = [-5; -2; 4; 18; -15; 15]$ respectively. To see if they meet we calculate:

$$\mathbf{d}_1 \cdot \mathbf{m}_2 + \mathbf{d}_2 \cdot \mathbf{m}_1 = 53 \neq 0$$

So they do not meet.

Example 7.25. The lines with Plücker coordinates $[\mathbf{d}_1, \mathbf{m}_1] = [4; -8; 7; 38; 5; -16]$ and $[\mathbf{d}_2, \mathbf{m}_2] = [9; -5; -3; 9; 30; -23]$ do meet because:

$$\mathbf{d}_1 \cdot \mathbf{m}_2 + \mathbf{d}_2 \cdot \mathbf{m}_1 = 0$$

Exercise 7.19. Determine whether each of the pairs of lines meet:

(a) $[\mathbf{d}_1; \mathbf{m}_1] = [0; 3; 10; 10; -50; 15]$ and $[\mathbf{d}_2; \mathbf{m}_2] = [5; 1; -1; -1; 5; 0]$.

(b) $[\mathbf{d}_1; \mathbf{m}_1] = [0; -3; -10; -10; 50; -15]$ and $[\mathbf{d}_2; \mathbf{m}_2] = [4; -8; -5; -18; 46; -88]$.

7.5.6 Plane Formed by Intersecting Lines

Theorem 7.5.6.1. Suppose $\mathcal{L}_1$ is represented by $[\mathbf{d}_1; \mathbf{m}_1]$ and $\mathcal{L}_2$ is represented by $[\mathbf{d}_2; \mathbf{m}_2]$. If the two lines are coplanar then their common plane is represented by:

$$[\mathbf{d}_1 \times \mathbf{d}_2; \mathbf{d}_1 \cdot \mathbf{m}_2]$$

Proof. The plane has normal vector $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$. We know that the point $\mathbf{p} = \mathbf{d}_2 \times \mathbf{m}_2 / \mathbf{d}_2 \cdot \mathbf{d}_2$ is on the line and hence on the plane and so by an earlier theorem the plane is represented by $[\mathbf{n}, -\mathbf{n} \cdot \mathbf{p}]$. Observe that:

$$\begin{aligned}
-\mathbf{n} \cdot \mathbf{p} &= -(\mathbf{d}_1 \times \mathbf{d}_2) \cdot \left(\frac{\mathbf{d}_2 \times \mathbf{m}_2}{\mathbf{d}_2 \cdot \mathbf{d}_2} \right) \\
&= -\frac{1}{\mathbf{d}_2 \cdot \mathbf{d}_2} (\mathbf{d}_1 \times \mathbf{d}_2) \cdot (\mathbf{d}_2 \times \mathbf{m}_2) \\
&= -\frac{1}{\mathbf{d}_2 \cdot \mathbf{d}_2} \left[(\mathbf{d}_1 \cdot \mathbf{d}_2)(\mathbf{d}_2 \cdot \mathbf{m}_2) - (\mathbf{d}_1 \cdot \mathbf{m}_2)(\mathbf{d}_2 \cdot \mathbf{d}_2) \right] \\
&= -\frac{1}{\mathbf{d}_2 \cdot \mathbf{d}_2} \left[(\mathbf{d}_1 \cdot \mathbf{d}_2)(0) - (\mathbf{d}_1 \cdot \mathbf{m}_2)(\mathbf{d}_2 \cdot \mathbf{d}_2) \right] \\
&= \mathbf{d}_1 \cdot \mathbf{m}_2
\end{aligned}$$

□

Example 7.26. The lines with Plücker coordinates $[\mathbf{d}_1, \mathbf{m}_1] = [4; -8; 7; 38; 5; -16]$ and $[\mathbf{d}_2, \mathbf{m}_2] = [9; -5; -3; 9; 30; -23]$ meet as we saw earlier. Their common plane is represented by the vector:

$$[\mathbf{d}_1 \times \mathbf{d}_2; \mathbf{d}_1 \cdot \mathbf{m}_2] = [59; 75; 52; -365]$$

In Euclidean coordinates this is:

$$59x + 75y + 52z - 365 = 0$$

Exercise 7.20. Determine the plane formed by the intersecting lines with Plücker coordinates $[\mathbf{d}_1; \mathbf{m}_1] = [0; 3; 10; 10; -50; 15]$ and $[\mathbf{d}_2; \mathbf{m}_2] = [5; 1; -1; -1; 5; 0]$.

7.5.7 Intersection Point Between Coplanar Lines

Theorem 7.5.7.1. Suppose $\mathcal{L}_1$ is represented by $[\mathbf{d}_1; \mathbf{m}_1]$ and $\mathcal{L}_2$ is represented by $[\mathbf{d}_2; \mathbf{m}_2]$ and suppose that neither contains the origin. If the two lines are coplanar but not parallel then they meet at the point:

$$[\mathbf{m}_1 \times \mathbf{m}_2; \mathbf{d}_2 \cdot \mathbf{m}_1] \equiv \left[\frac{\mathbf{m}_1 \times \mathbf{m}_2}{\mathbf{d}_2 \cdot \mathbf{m}_1}; 1\right]$$

Proof. We show that this point is on both lines. We have:

$$\begin{aligned}
\frac{\mathbf{m}_1 \times \mathbf{m}_2}{\mathbf{d}_2 \cdot \mathbf{m}_1} \times \mathbf{d}_2 &= -\frac{1}{\mathbf{d}_2 \cdot \mathbf{m}_1}\mathbf{d}_2 \times (\mathbf{m}_1 \times \mathbf{m}_2) \\
&= -\frac{1}{\mathbf{d}_2 \cdot \mathbf{m}_1}\left[(\mathbf{d}_2 \cdot \mathbf{m}_2)\mathbf{m}_1 - (\mathbf{d}_2 \cdot \mathbf{m}_1)\mathbf{m}_2\right] \\
&= -\frac{1}{\mathbf{d}_2 \cdot \mathbf{m}_1}\left[(0)\mathbf{m}_1 - (\mathbf{d}_2 \cdot \mathbf{m}_1)\mathbf{m}_2\right] \\
&= \mathbf{m}_2
\end{aligned}$$

Keeping in mind that coplanarity implies $\mathbf{d}_2 \cdot \mathbf{m}_1 = -\mathbf{d}_1 \cdot \mathbf{m}_2$ we have:

$$\begin{aligned}
\frac{\mathbf{m}_1 \times \mathbf{m}_2}{\mathbf{d}_2 \cdot \mathbf{m}_1} \times \mathbf{d}_1 &= \frac{1}{\mathbf{d}_2 \cdot \mathbf{m}_1}\mathbf{d}_1 \times (\mathbf{m}_1 \times \mathbf{m}_2) \\
&= \frac{1}{\mathbf{d}_2 \cdot \mathbf{m}_1}\left[(\mathbf{d}_1 \cdot \mathbf{m}_2)\mathbf{m}_1 - (\mathbf{d}_1 \cdot \mathbf{m}_1)\mathbf{m}_2\right] \\
&= \frac{1}{\mathbf{d}_2 \cdot \mathbf{m}_1}\left[(\mathbf{d}_1 \cdot \mathbf{m}_2)\mathbf{m}_1 - (0)\mathbf{m}_2\right] \\
&= \mathbf{m}_1
\end{aligned}$$

Note: If either lines passes through the origin then we have one of the $\mathbf{m}_i = \mathbf{0}$ and then $\mathbf{d}_2 \cdot \mathbf{m}_1 = \mathbf{d}_1 \cdot \mathbf{m}_2 = 0$ and the calculation returns an invalid result. If this is the case then the formula is far more complicated. □

As a side note I've found some sources (including Ken Shoemake's original notes on Plücker coordinates, widely regarded as a good computational source), which give the formula as $[\mathbf{m}_1 \times \mathbf{m}_2; \mathbf{d}_1 \cdot \mathbf{m}_2]$ As far as I can tell this formula is incorrect as a simple example suggests:

Example 7.27. Consider $\mathcal{L}_1$ passing through $[1; 1; 1]$ and $[3; 4; 5]$ and $\mathcal{L}_2$ passing through $[1; 1; 1]$ and $[7; 10; 4]$. For reference clearly they meet at $[1; 1; 1]$.

We find $\mathbf{d}_1 = [2; 3; 4]$ and so $\mathbf{m}_1 = [1; 1; 1] \times [2; 3; 4] = [1; -2; 3]$ and we find $\mathbf{d}_2 = [6; 9; 3]$ and so $\mathbf{m}_2 = [1; 1; 1] \times [6; 9; 3] = [-6; 3; 3]$ Then the formula from our theorem yields an intersection point via:

$$\begin{aligned}
\mathbf{m}_1 \times \mathbf{m}_2 &= [-9; -9; -9] \\
\mathbf{d}_2 \cdot \mathbf{m}_1 &= -9
\end{aligned}$$

Yielding $[-9; -9; -9; -9] \equiv [1; 1; 1; 1]$ for the point $[1; 1; 1]$.

Shoemake's notes in this case would use $\mathbf{d}_1 \cdot \mathbf{m}_2 = 9$ and would yield $[-1; -1; -1]$.

Perhaps these other sources are not wrong, and I'm simply misunderstanding something in the way they're framing their calculations.

Example 7.28. The lines with Plücker coordinates $[\mathbf{d}_1, \mathbf{m}_1] = [4; -8; 7; 38; 5; -16]$ and $[\mathbf{d}_2, \mathbf{m}_2] = [9; -5; -3; 9; 30; -23]$ meet as we saw earlier. The point they meet at is:

$$[\mathbf{m}_1 \times \mathbf{m}_2; \mathbf{d}_2 \cdot \mathbf{m}_1] = [365; 730; 1095; 365] \equiv [1; 2; 3; 1]$$

Exercise 7.21. Determine the intersection point of the lines with Plücker coordinates $[\mathbf{d}_1; \mathbf{m}_1] = [0; 3; 10; 10; -50; 15]$ and $[\mathbf{d}_2; \mathbf{m}_2] = [5; 1; -1; -1; 5; 0]$.

7.5.8 Intersection Point Between Line and Plane

Theorem 7.5.8.1. Suppose $\mathcal{L}$ is represented by $[\mathbf{d}; \mathbf{m}]$ and the plane $\mathcal{P}$ is represented by $\mathbf{P} = [\mathbf{n}; n_0]$ with $\mathbf{n} \in \mathbb{R}^3$. Then $\mathcal{L}$ meets $\mathcal{P}$ at the point:

$$[\mathbf{n} \times \mathbf{m} - n_0\mathbf{d}; \mathbf{n} \cdot \mathbf{d}] \equiv \left[\frac{\mathbf{n} \times \mathbf{m} - n_0\mathbf{d}}{\mathbf{n} \cdot \mathbf{d}}; 1\right]$$

Proof. We show that the point is on the line and on the plane. For the line observe that:

$$\begin{aligned}
\frac{\mathbf{n} \times \mathbf{m} - n_0\mathbf{d}}{\mathbf{n} \cdot \mathbf{d}} \times \mathbf{d} &= \frac{1}{\mathbf{n} \cdot \mathbf{d}} \left[-\mathbf{d} \times (\mathbf{n} \times \mathbf{m}) - n_0(\mathbf{d} \times \mathbf{d})\right] \\
&= \frac{1}{\mathbf{n} \cdot \mathbf{d}} \left[-\left[(\mathbf{d} \cdot \mathbf{m})\mathbf{n} - (\mathbf{d} \cdot \mathbf{n})\mathbf{m}\right] - n_0(0)\right] \\
&= \frac{1}{\mathbf{n} \cdot \mathbf{d}} \left[-\left[(0)\mathbf{n} - (\mathbf{d} \cdot \mathbf{n})\mathbf{m}\right]\right] \\
&= \mathbf{m}
\end{aligned}$$

For the plane observe that:

$$\begin{aligned}
[\mathbf{n} \times \mathbf{m} - n_0\mathbf{d}; \mathbf{n} \cdot \mathbf{d}] \cdot [\mathbf{n}; n_0] &= (\mathbf{n} \times \mathbf{m} - n_0\mathbf{d}) \cdot \mathbf{n} + (\mathbf{n} \cdot \mathbf{d})n_0 \\
&= (\mathbf{n} \times \mathbf{m}) \cdot \mathbf{n} - n_0\mathbf{d} \cdot \mathbf{n} + (\mathbf{n} \cdot \mathbf{d})n_0 \\
&= 0
\end{aligned}$$

□

Example 7.29. Consider the line through $[0; 0; 1]$ and $[3; 2; -1]$. To see where this line meets the plane $x + 5y + 2z = 10$ we calculate the Plücker coordinates for the line, which turn out to be $[\mathbf{d}; \mathbf{m}] = [3; 2; -2; -2; 3; 0]$ and we assign for the plane $[\mathbf{n}; n_0] = [1; 5; 2; -10]$ and then we calculate:

$$[\mathbf{n} \times \mathbf{m} - n_0\mathbf{d}; \mathbf{n} \cdot \mathbf{d}] = [24; 16; -7; 9] \equiv [24/9; 16/9; -7/9; 1]$$

or the Euclidean point $[24/9; 16/9; -7/9]$.

If we had taken a more traditional Calculus approach we might have found the parametric equations of the line:

$$\begin{aligned} x &= 0 + 3t \\ y &= 0 + 2t \\ z &= 1 - 2t \end{aligned}$$

Then we compute the t value for which it hits the plane:

$$\begin{aligned} 3t + 5(2t) + 2(1 - 2t) &= 10 \\ 9t &= 8 \\ t &= 8/9 \end{aligned}$$

Which then yields the point:

$$\begin{aligned} x &= 24/9 \\ y &= 16/9 \\ z &= -7/9 \end{aligned}$$

Exercise 7.22. Find the intersection of the line through the points $[4; 2; 0]$ and $[-3; 1; 0]$ and the plane with equation $2x - 2y + 5z = 40$.

7.5.9 Plane Containing Point and Line

Theorem 7.5.9.1. Suppose $\mathcal{L}$ is represented by $[\mathbf{d}; \mathbf{m}]$ and the point $\mathbf{Q} = [\mathbf{q}; 1] \in E^3$ with $\mathbf{q} \in \mathbb{R}^3$ is not on $\mathcal{L}$. Then the plane containing both $\mathcal{L}$ and $\mathbf{Q}$ is represented by the vector:

$$[\mathbf{q} \times \mathbf{d} - \mathbf{m}; \mathbf{m} \cdot \mathbf{q}]$$

Proof. Pick some $\mathbf{v}$ on the line (and hence on the plane) so that $\mathbf{v} \times \mathbf{d} = \mathbf{m}$. Then we can obtain a normal for the plane via:

$$\mathbf{n} = (\mathbf{q} - \mathbf{v}) \times \mathbf{d} = \mathbf{q} \times \mathbf{d} - \mathbf{v} \times \mathbf{d} = \mathbf{q} \times \mathbf{d} - \mathbf{m}$$

In addition we have:

$$-\mathbf{n} \cdot \mathbf{q} = -(\mathbf{q} \times \mathbf{d} - \mathbf{m}) \cdot \mathbf{q} = -(\mathbf{q} \times \mathbf{d}) \cdot \mathbf{q} + \mathbf{m} \cdot \mathbf{q} = 0 + \mathbf{m} \cdot \mathbf{q}$$

The result then follows by an earlier theorem. $\square$

Example 7.30. The plane containing the line with Plücker coordinates $[\mathbf{d}; \mathbf{m}] = [3; 2; -2; -2; 3; 0]$ and containing the point $\mathbf{q} = [3; 5; 2]$ is represented by the vector:

$$[\mathbf{q} \times \mathbf{d} - \mathbf{m}; \mathbf{m} \cdot \mathbf{q}] = [-12; 9; -9; 9]$$

The Euclidean equation is then $-12x + 9y - 9z + 9 = 0$.

Exercise 7.23. Find the equation of the plane containing the line $[\mathbf{d}; \mathbf{m}] = [5; 2; -2; -2; 10; 5]$ and the point $[1; 1; 1]$.

7.5.10 Computation Summary

Point $\mathbf{v}$ on Line $[\mathbf{d}; \mathbf{m}]$	If $\mathbf{v} \times \mathbf{d} = \mathbf{m}$
Intersection Line Between Planes $[\mathbf{n}_1; n_1]$ and $[\mathbf{n}_2; n_2]$	$[\mathbf{n}_1 \times \mathbf{n}_2; n_1\mathbf{n}_2 - n_2\mathbf{n}_1]$
Detecting Coplanarity of Lines $[\mathbf{d}_1; \mathbf{m}_1]$ and $[\mathbf{d}_2; \mathbf{m}_2]$	If $\mathbf{d}_1 \cdot \mathbf{m}_2 + \mathbf{d}_2 \cdot \mathbf{m}_1 = 0$
Plane Formed by Intersecting Lines $[\mathbf{d}_1; \mathbf{m}_1]$ and $[\mathbf{d}_2; \mathbf{m}_2]$	$[\mathbf{d}_1 \times \mathbf{d}_2; \mathbf{d}_1 \cdot \mathbf{m}_2]$
Intersection Point Between Coplanar Lines $[\mathbf{d}_1; \mathbf{m}_1]$ and $[\mathbf{d}_2; \mathbf{m}_2]$	$[\mathbf{m}_1 \times \mathbf{m}_2; \mathbf{d}_2 \cdot \mathbf{m}_1]$
Intersection Point between Line $[\mathbf{d}; \mathbf{m}]$ and plane $[\mathbf{n}; n_0]$	$[\mathbf{n} \times \mathbf{m} - n_0\mathbf{d}; \mathbf{n} \cdot \mathbf{d}]$
Plane Containing Point $\mathbf{q}$ and Line $[\mathbf{d}; \mathbf{m}]$	$[\mathbf{q} \times \mathbf{d} - \mathbf{m}; \mathbf{m} \cdot \mathbf{q}]$

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