Math 315 - Fall 2021
Homework 1
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Problem 1. Before class, a TU student walked up to the chalk board and wrote six distinct positive integers. She pointed out that none of them had a prime factor larger than 10. Prove that there are two integers on the board that have a common prime divisor. *If instead of "six" integers, she wrote "five" on the board, could we arrive at the same conclusion?*

Solution 1. There exists four prime integers less than ten: 2, 3, 5, & 7. Denote the six distinct positive integers written by the student as $a_1, a_2, a_3, a_4, a_5,$ & a_6 and consider their prime factor decomposition which by the hypothesis cannot contain any prime greater than 7. In a worst case scenario it follows that:

$$2|a_1 \quad 3|a_2 \quad 5|a_3 \quad 7|a_4$$

Then by the Pigeonhole Principle, we have four 'holes' (prime integers less than 10) and six 'pigeons' (number of positive integers written by the student with the given condition). Therefore it easily follows that a_5 and a_6 must be divisible by any one of these four positive integers. If we consider the case when there are only five positive integers written on the board by the student, then the same logic applies and the last positive integer must be divisible by one of the four primes. $\square$

Problem 2. Five classmates race each other every day during the last four months of the year. Surprisingly, there are never any ties. Prove that there are two races which end the same way.

Solution 2. Assume that in the four last months of the year there are $2(31 + 30) = 122$ days and therefore 122 total races occur. Since there are 5 classmates in each race, the total number of all possible outcomes is $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$. Then by the Pigeonhole Principle there must exist at least 2 races that have the same outcome as 2 of the possible 120 total race outcomes. $\square$

Problem 3. Let S be a set of 17 points inside a cube of side length 1. Prove that there exists a sphere of radius 1/2 which encloses at least three of the points.

Solution 3. Let's divide the cube into eight equally sized cubes and evenly distribute the points of S inside each smaller cube. Since we have a unit cube, then to construct these smaller cubes we bisect each side length. By the Pigeonhole Principle there must be one box that contains 3 points and all other boxes contain 2 points. Since $\pi > 3$ then we can approximate the volume of the sphere to equal $\frac{1}{2}$ even though in truth we know it to be slightly larger. Now the volume of one of the smaller cubes is given by $v_{subcube} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, with the largest segment being from one corner to its opposite diagonal. Now the length of the diagonal D is given by:

$$
\begin{aligned}
D &= \sqrt{3}\left(\frac{1}{2}\right) \\
D &= \frac{\sqrt{3}}{2} \\
&\approx 0.86602540378445
\end{aligned}
$$

Since a sphere of radius $\frac{1}{2}$ has diameter equal to $1 > 0.87$, then we can completely encapsulate the single cube that contains 3 points of S and therefore the sphere will also have 3 points inside it as seen in Figure 1

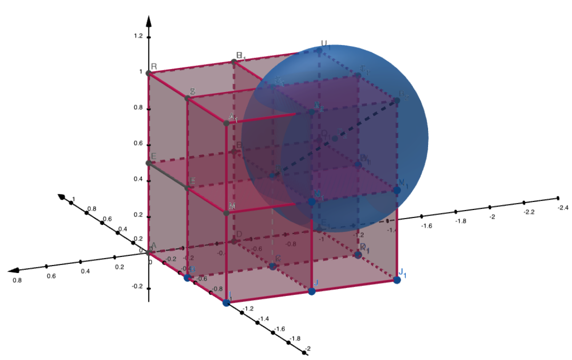

FIGURE 1. The sphere of radius $\frac{1}{2}$ completely encapsulate the cube containing 3 points.

□

Problem 4. (Chapter 1, Exercise 25) Prove that there exists a positive integer n so that $44^n - 1$ is divisible by 7.

Solution 4. By the hypothesis we can say that for some $n \in \mathbb{Z}^+$:

$$44^n - 1 \equiv 0 \ mod \ 7; \quad 44^n \equiv 1 \ mod \ 7$$

Let's break the quantity on the left hand side of the equivalence relation into a factor that is divisible by 7:

$$(42 + 2)^n \equiv 1 \ mod \ 7$$

We can expand this using the Binomial Theorem:

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \to \sum_{k=0}^{n} \binom{n}{k} 42^k \ 2^{n-k}$$

It should be obvious that all but one term in this series has a factor that is some power of 42, so each of these terms will have 7 in their prime factor decomposition and therefore are divisible by 7. Now we need to handle the single term not divisible by 7, which is 2^n for the initial term for $k = 0$ and coefficient $\binom{n}{0} = 1$. Since 2 and 7 are relatively prime, then:

$$44^n \equiv 1 \ mod \ 7; \quad 44^n \equiv 2^n \ mod \ 7; \quad 44^n - 1 \equiv 2^n - 1 \ mod \ 7$$

Which is easily satisfied for $n = 3$. It follows that $44^3 - 1 = 85183$ and $7|85183 = 12169$. $\square$

Additional Solution: A result of Euler states that if $(m, n) = 1$, then $m^{\phi(n)} \equiv 1 \ mod \ n$ where $\phi(n)$ is Euler's totient function: $\phi(n) = n \times \prod_{i=1}^{k}(1 - \frac{1}{p_i})$ for all k-factors in the prime decomposition of n. Since $(44, 7) = 1$ and $\phi(7) = 7(1 - \frac{1}{7}) = 6$, then it follows that $44^6 \equiv 1 \ mod \ 7 \ \to 44^6 - 1 \equiv 0 \ mod \ 7$. Explicitly this quantity is: $7256313856 \equiv 1 \ mod \ 7; 7256313856 - 1 \equiv 0 \ mod \ 7; 7|7256313855 = 1036616265$ $\square$

Problem 5. (Chapter 1, Exercise 27) Find all 4-tuples (a, b, c, d) of distinct positive integers so that $a < b < c < d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1.$$

Solution 5. Since we are given the equivalence relations that $a < b < c < d$, then it must follow that $\frac{1}{a} > \frac{1}{b} > \frac{1}{c} > \frac{1}{d}$ for $a, b, c,$ & $d \in \mathbb{Z}^+$. It should be obvious that $a \neq 1$ or else there would be no solution for the given equation. Let's try $a = 2$; and so we end up with the equation:

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{2}$$

Clearly $b > 2$ so let's try $b = 3$:

$$\frac{1}{c} + \frac{1}{d} = \frac{1}{6}$$

Since $c > 6$ and following from $c < d$ we let $c = 8$ and $d = 24$ then $\frac{1}{8} + \frac{1}{24} = \frac{1}{6}$ and therefore one solution of the form $(a, b, c, d) = (2, 3, 8, 24)$. Let's determine the rest of the solutions for $b = 3$:

(1) $c := 7; \frac{1}{d} = \frac{1}{6} - \frac{1}{7} = \frac{1}{42}; \to d = 42.$
(2) $c := 9; \frac{1}{d} = \frac{1}{6} - \frac{1}{9} = \frac{3}{54} = \frac{1}{18} \to d = 18.$
(3) $c := 10; \frac{1}{d} = \frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15} \to d = 15.$
(4) $c := 11; \frac{1}{d} = \frac{1}{6} - \frac{1}{11} = \frac{5}{66} \to\leftarrow d \notin \mathbb{Z}^+.$

Therefore we have four quadruples of $(2,3,7,42),(2,3,8,24),(2,3,9,18),(2,3,10,15)$. We can now set $b := 4$ and find solutions from $\frac{1}{c}+\frac{1}{d}=\frac{1}{4}$ where $c > 4$:

(1) $c := 5; \frac{1}{d}=\frac{1}{4}-\frac{1}{5}=\frac{1}{20} \to d = 20$.
(2) $c := 6; \frac{1}{d}=\frac{1}{4}-\frac{1}{6}=\frac{1}{12} \to d = 12$.
(3) $c := 7; \frac{1}{d}=\frac{1}{4}-\frac{1}{7}=\frac{3}{28} \to\leftarrow d \notin \mathbb{Z}^+$.

We have two more solutions of $(2,4,5,20)$ and $(2,4,6,12)$. Let $b := 5$ then $c > 5$ and so $\frac{1}{c}+\frac{1}{d}=\frac{3}{10}$. But wait, there exists no positive integers c and d such that $\frac{1}{c}+\frac{1}{d}=\frac{3}{10}$. Now we must consider if we choose $a > 2$. If the next smallest integer is chosen, that is $a := 3$, then by the relation that $a < b < c < d$ the largest possible sum we can obtain is:

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}<1$$

Therefore we have identified all 6 quadruples that satisfy both the equivalences and equation given. $\square$

References

[1] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.