Math 315 - Fall 2021
Homework 2
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Problem 1. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-2} + a_{n-1}$ for $n \geq 2$. Find and prove an explicit formula for a_n.

Solution 1. We are given $a_0 = 1;\ a_1 = 2$ and $a_n = 2a_{n-2} + a_{n-1}$. If we calculate a few subsequent terms we get: $a_2 = 4,\ a_3 = 8,\ a_4 = 16\ldots$ and so forth. It is clear that $a_n = 2^{n-1}$ is the closed form of the sequence.

(1) Base Case: Let $n := 2$, then $a_2 = 2^{2-1} = 2$ and since base case is satisfied we can proceed to the inductive step.

(2) Following the induction hypothesis $n := k$, we set $n := k+1$. If the given relation holds then:

$$\begin{aligned} a_n = 2a_{n-2} + a_{n-1} \ \longrightarrow\ a_{k+1} &= 2a_{k-1} + a_k \\ a_{k+1} &= 2 \cdot 2^{n-2} + 2^{n-1} \\ a_{k+1} &= 2^{n-1} + 2^{n-1} \\ a_{k+1} &= 2(2^{n-1}) \\ a_{k+1} &= 2^n \end{aligned}$$

Since we have proved that the relation holds for $k+1$, then it must be true that it holds for all $n \in \mathbb{Z}^+$ for $n \geq 2$. $\square$

Problem 2. Prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n = \frac{(n-1)\,n\,(n+1)}{3}.$$

Solution 2. We have a closed form and so we can immediately start the induction procedure.

(1) Base Case: Let $n := 1$, we have:

$$\begin{aligned} (1-1)1 &= \frac{(1-1)(1)(1+1}{3} \\ 0 &= \frac{(0)(1)(2)}{3} \\ 0 &= 0 \end{aligned}$$

(2) Inductive Step: Since the base case is satisfied we can move on to the inductive hypothesis. Let $n := k$ such that:

$$1(2) + 2(3) + \ldots + (k-1)k = \frac{(k-1)k(k+1)}{3}$$

Now we want to show that $n := k+1$ holds true for some $k \in \mathbb{Z}^+$:

$$\begin{aligned} 1(2) + 2(3) + \ldots + (k-1+1)(k+1) &= \frac{(k-1+1)(k+1)(k+1+1)}{3} \\ 1(2) + 2(3) + \ldots + (k)(k+1) &= \frac{(k)(k+1)(k+2)}{3} \end{aligned} \tag{1}$$

All terms (except the very last) in the left hand side of the equation $1(2) + 2(3) + \ldots + (k-1)(k) = \frac{(k-1)\,k\,(k+1)}{3}$. Let's manipulate the left hand side of the equation and leave the right hand side unchanged:

$$\frac{k(k-1)(k+1)}{3} + k(k+1) \;\rightarrow\; \frac{k-1}{3}(k)(k+1) + (k)(k+1)$$

Factor by grouping:

$$\frac{k-1}{3}(k(k+1)) + 1(k(k+1)) \rightarrow \left(\frac{k-1}{3} + 1\right)(k(k+1)) = \frac{(k+2)k(k+1)}{3}$$

This expression is exactly the right hand side in Equation 1 and since we have shown that the relation holds for $k+1$ then by induction the relation holds for all $n \in \mathbb{Z}^+$. $\square$

Problem 3. (Chapter 2, Exercise 32)
Let $a_0 = a_1 = 1$, and let $a_{n+2} = a_{n+1} + 5a_n$ for $n \geq 0$. Prove that $a_n \leq 3^n$ for all $n \geq 0$.

Solution 3. We are given the inequality that $a_n \leq 3^n$ along with the relation that $a_{n+2} = a_{n+1} + 5a_n$.

(1) Base Case: For $n := 0$ we have that $a_0 \leq 3^0 \rightarrow 1 \leq 1$ which holds.
(2) Since the base case is satisfied, then by the induction hypothesis we have that if $n := k+1$ and $a_n \leq 3^n$ then $a_{k+1} \leq 3^{k+1}$. Assume that a_n is the maximum value that it possibly can be then we can set $a_n := 3^n$ and $a_{k+1} := 3^{k+1}$, and so we can obtain from our closed form:

$$\begin{aligned} a_{k+2} &= a_{k+1} + 5a_k \\ a_{k+2} &= 3^{k+1} + 5(3^k) \\ a_{k+2} &= 3^k(3+5) \\ a_{k+2} &= 8 \times 3^k \end{aligned}$$

But we also have that $a_{k+2} := 3^{k+2} = 9(3^k)$ and clearly $3^{k+2} > 3^k \times 8$ where because of our assumption that $a_n := 3^n$ we no longer have that equality between the two is possible. $\square$

Problem 4. (Chapter 2, Exercise 33)
Let H be a ten-element set of two-digit positive integers. Prove that H has two disjoint subsets A and B so that the sum of the elements of A is equal to the sum of the elements of B.

Solution 4. Let $G := \{10, 11, \cdots, 98, 99\}$ be the set of all consecutive integers from 10 to 99. Then the number of ten-element subsets of G is equal to $\mathcal{P}(G) = 2^{10} = 1024$ and we set H to be the collection of all ten-element subsets of G. Now let's find a range of values for all possible sums of entries in H. The smallest sum we can obtain is 0 from $\varnothing$ and the largest sum we can get is $90 + 91 + \ldots + 99 = 945$. Now there exists 1024 total possible subsets, but:

$$0 \leq \sum_{i=1}^{10} |H_i| \leq 945$$

and therefore there are only 945 unique sums of H. Then by the Pigeonhole Principle we have 1024 subsets and only 945 unique sums, and therefore there exists distinct subsets A and B such that their sums are equal. If we suppose that $|A \cap B| \neq \varnothing$, then we can simply remove the elements of $A \cap B$ from both A and B and still have two unique (though of smaller cardinality) subsets whose sums are equal. $\square$

Problem 5. Prove that if $n \geq 2$ is an integer, then n can be written as a product of primes.

Solution 5. The Fundamental Theorem of Arithmetic states that for every $a > 1$ can be factored uniquely as a product of prime numbers in the form:

$$a = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_n^{\alpha_n}$$

where $p_1 < p_2 < \ldots p_n$ and the exponents $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all positive [1]. Now 2 is the only even prime, and so by both induction and this well known property we can split this into two cases. We also note that a prime p has prime factor decomposition $p \times 1 = p$.

(1) Base Case: If $n = 2$, then $2 = 2 \times 1$ and since both 1 and 2 are prime then we have satisfied the base case. Additionally $2 + 1 = 3$ which is also a prime.

(2) By the induction hypothesis we set $n := n + 1$ and so we have two cases based on if n is prime for $n > 2$:
- If n is prime then $n+1$ must be an even integer, and therefore can be decomposed into its prime factor decomposition.
- If n is composite, then $n + 1$ may or may not be prime which again breaks down into a question of parity. For n odd then $n + 1$ will be even and therefore composite, and for n even then $n + 1$ will be odd. If $n + 1$ is odd then either it is prime, or it can be factored into some prime decomposition.

$\square$

References

[1] John A. Beachy and William D. Blair. *Abstract Algebra*. 3rd ed. Waveland Press, 2019.
[2] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.