Math 315 - Fall 2021
Homework 3
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Problem 1. (a) How many subsets does $[n]$ have that contain exactly one of the elements 1 and 2?
(b) How many subsets does $[n]$ have that contain at least one of the elements 1 and 2?

Solution 1. (a) We know that the total number of subsets of $[n]$ is the power set $\mathcal{P}(n) = 2^n$ and so we must determine how many of these 2^n subsets contain exactly 1 or 2 but not both. If we count the number of subsets that always contain 1 and never contain 2 then we have 2^{n-2} subsets, since 1 is considered to be in every set 2 is not and therefore the cardinality of objects that can be chosen is $n-2$. Similarly, for subsets that always contain 2 and never contain 1 we have 2^{n-2} subsets. The sum of both is the total number of subsets that contain exactly one of the elements:

$$\begin{aligned} &2^{n-2} + 2^{n-2} \\ &= 2(2^{n-2}) \\ &= 2^{n-1} \end{aligned}$$

(b) This is easier to think of in terms of a 'complement'. The total number of sets that don't contain 1 is 2^{n-1} and similarly the total number of sets that don't contain 2 is 2^{n-1}. Therefore we have that the number of subsets that do not contain either as 2^{n-2}. Subtracting this from the total possible number of subsets:

$$\begin{aligned} &2^n - 2^{n-2} \\ &= 2^{n-2}(2^2 - 1) \\ &= 2^{n-2}(3) \end{aligned}$$

□

Problem 2. A student needs to work five days in January. She does not want to work on more than one Sunday. In how many ways can she select her five working days? (Assume that in the year in question, January has five Sundays.)

Solution 2. We have the restriction that there can only be one Sunday that the student may work on and so we must consider two cases and sum them together. Since we are given that there are 5 Sundays in this particular January then for the first case we have:

$$\binom{5}{1}\binom{26}{4}$$

For the second case of no Sundays:

$$\binom{5}{0}\binom{26}{5} = \binom{26}{5}$$

The sum of both is equal to:

$$\begin{aligned}
&\binom{5}{1}\binom{26}{4}+\binom{26}{5}\\
&=5\cdot\frac{26\cdot 25\cdot 24\cdot 23}{4\cdot 3\cdot 2}+\frac{26\cdot 25\cdot 24\cdot 23\cdot 22}{5\cdot 4\cdot 3\cdot 2}\\
&=5\cdot 26\cdot 25\cdot 23+5\cdot 26\cdot 23\cdot 22=140530
\end{aligned}$$

Therefore there are 140,530 different possible ways to select her five working days in January.

□

Problem 3. Andy and Brenda play with dice. They throw four dice at the same time. If at least one of the four dice shows a six, then Andy wins, if not, then Brenda. Who has a greater chance of winning?

Solution 3. We know that Brenda will win if no six appears in any of the four dice, and the probability of this can be given by:

$$\begin{aligned}
&\left(\frac{5}{6}\right)\cdot\left(\frac{5}{6}\right)\cdot\left(\frac{5}{6}\right)\cdot\left(\frac{5}{6}\right)\\
&=\left(\frac{5}{6}\right)^4\\
&=\frac{625}{1296}\\
&=0.4823
\end{aligned}$$

Surprisingly, Andy's chance of winning is $1-0.4823=.5177\equiv 51.77\%$ and therefore Andy has the greater chance of winning.

□

Problem 4. How many $n\times n$ square matrices are there whose entries are 0 or 1 and in which each row and column has an even sum?

Solution 4. Consider eliminating the last row and the last column of every matrix $A\in M_{n,n}(0,1)$ and so we are left with matrices $B\in M_{n-1,n-1}(0,1)$. We claim that for every unique matrix B we can determine the values of the n^{th} row and the n^{th} column such that the sum of each row and each column is even. Now every there are $(n-1)\times(n-1)=(n-1)^2$ possible entries in B and since entries are either 0 or 1, then there are $2^{(n-1)^2}$ unique matrices $B\in M_{n-1,n-1}(0,1)$ where the n^{th} row and column can be determined such that the sum of all columns and rows is even. □

Problem 5. How many ways are there for n people to sit around a circular table if two seating arrangements are considered identical if each person has the same left neighbor in them?

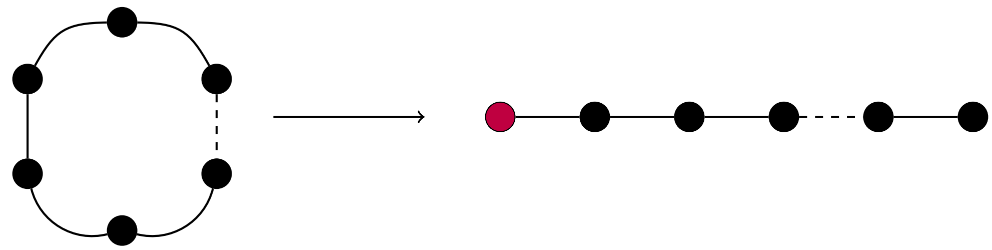

FIGURE 1. A circular seating arrangement of n people can be reduced into a linear problem.

Solution 5. We are familiar with solving counting problems if everyone is sitting in some linear fashion so we can try to frame this problem in a similar manner. Now if every persons left neighbor is the same then we essentially have the same seating arrangement, and for n people at a circular table then there are n seating arrangements that are exactly the same but in different seats. Now consider fixing exactly one person in the seat that they currently occupy, then there are $(n-1)!$ **unique** different seating arrangements now possible. These same $(n-1)!$ arrangements are possible no matter which seat we choose to fix, as long as we fix exactly one. In this way we have done exactly what we set out to do, and transformed a circular seating pattern into one that can be described by a path graph with an end vertex that is fixed as illustrated in Figure 1. $\square$

References

[1] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.