Math 315 - Fall 2021
Homework 6
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Problem 1. (Chapter 6, Exercise 27)
Find the number of permutations of length six whose square is the identity permutation.

Solution 1. We can find one solution immediately if we consider the permutation of six singletons i.e. the identity permutation. Next let's think of the permutation of length two and four singletons since the length two permutations have order two. Then from Theorem 6.9([2]) there are $\frac{6!}{1^4 \cdot 4! \cdot 2^1} = 3 \cdot 5 = 15$ total permutations of this cycle type. Building on this we can consider 2 length two cycles and two singletons, of which there are:

$$\frac{6!}{2^2 \cdot 2! \cdot 1^2 \cdot 2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2^4} = 3^2 \cdot 5 = 45$$

The last case we need to consider is if we were to have 3 cycles each of length 2:

$$\frac{6!}{2^3 \cdot 3!} = \frac{5!}{2^3} = 5 \cdot 3 = 15.$$

Then the sum of all of these together gives $1+15+45+15 = 76$ total number of permutations of length six whose order is 2 in S_6. $\square$

Problem 2. (Chapter 6, Exercise 28)
What is the number of permutations of length 20 whose longest cycle is of length 11?

Solution 2. Let's first examine the longest cycle of length 11, which is given by $\binom{20}{11}$. Now remembering back to our circular table seating arrangement problem, there are 10! different cycles of this form. Now we have 9 more elements that we must account for, but we are given no restriction on what possible cycles must be included, and so we simply have 9! different possibilities. Then the product of these quantities is:

$$\binom{20}{11} \cdot 10! \cdot 9! = \frac{20! \cdot 10! \cdot 9!}{11! \cdot 9!} = \frac{20!}{11}$$

$\square$

Problem 3. (Chapter 6, Exercise 30)
A group of ten children want to play cards. They split into three groups, one of these groups has four children in it, the other two have three each. Then each group sits around a table. Two seatings are considered the same if everyone's left neighbor is the same.
(a) In how many ways can this be done if the three tables are identical?
(b) In how many ways can this be done if the three tables are distinct?

Solution 3. (a) We have ten children total and they are split into three groups, one with 4 players and two with 3 players. Then in total we have $\binom{10}{4} \cdot \binom{6}{3} \cdot \binom{3}{3} = 210 \cdot 20 \cdot 1 = 4200$ different groups. But we need to divide this quantity in half, since 2 groups are of size 3 and therefore we are counting duplicate groups, so in total there are 2100 different groups possible of the 10 children. Now thinking back to our circular arrangement problem we have three identical tables for each group. And so to find the number of unique arrangements for each we simply 'fix' the position of one member and find the number of permutations of the other three. Therefore it should be clear that there are $3! \cdot 2! \cdot 2! \equiv 4! = 24$ arrangements at each table. Then our number of unique arrangements for identical circular tables is $2100 \cdot 24 = 50,400$.
(b) Since we are told that each circular table is now distinct, then we can simply build off of the work that we did for part (a). Each table is unique, and so for each of the 50,400 arrangements we have found, each arrangement can be placed at one of the three tables, another at two of the tables with the last arrangement at the remaining table. Therefore we simply have $50400 \cdot 3! = 302,400$ possible groups and arrangements of the 10 children at three distinct circular tables.

Problem 4. (Chapter 6, Exercise 32)
Let $p = p_1p_2 \cdots p_n$ be a permutation. An *inversion* of p is a pair of entries (p_i, p_j) so that $i < j$ but $p_i > p_j$. Let us call a permutation *even* (resp. *odd*) if it has an even (resp. odd) number of inversions.
Prove that the permutation consisting of the one cycle $(a_1a_2 \cdots a_k)$ is even if k is odd, and is odd if k is even.

Solution 4. Let's consider the case when k is odd and work towards a contradiction. Assume that $p = (a_1a_2 \cdots a_k)$ is odd iff p has an odd number of inversions. Let's consider the maximum number of inversions we can find for each a_i if we have a totally ordered permutation that such that $a_1 > a_2 > a_3 > \ldots > a_k$. A concrete example of such a permutation could be given by (10 9 8 7 6 5 4 3 2 1). Now:

- For a_1, there are $k-1$ inversions.
- For a_2, there are $k-2$ inversions.
- For a_3, there are $k-3$ inversions.
- $\vdots$
- For a_{k-1}, there is 1 inversion.

Then the sum of all inversions is given by:

$$(k-1) + (k-2) + (k-3) + \ldots + 1$$

Since k is odd, then there are an even number of sums. We can group these sums together into even and odd terms

$$((k-1) + (k-3) + \cdots + 2) + ((k-2) + (k-4) + \cdots + 1)$$

And since the number of sums of odd numbers is even, then the total number of inversions is then even.

$$\rightarrow\leftarrow$$

Therefore it should be clear that if k is odd, then the maximum number of inversions is even. The same argument can be shown for k even. □

[**Alternate Solution**] An algebraic approach to this problem is recalling the sign function of a cycle $\tau \in S_n$;

$$sgn(\tau) = \begin{cases} 1 & \text{iff } \tau \text{ is even} \\ -1 & \text{iff } \tau \text{ is odd} \end{cases}$$

where τ is decomposed into an either an even or odd product of transpositions, which are cycles of length 2 [1]. So for the cycle $(a_1 a_2 \cdots a_k)$ then for k even we have:

$$(a_1 a_2)(a_1 a_3)(a_1 a_4) \cdots (a_1 a_k)$$

where the number of transpositions is $k-1$ and therefore τ is odd. The same type of argument can be used to show that for k odd then the parity of τ will be even since the decomposition of τ into transpositions will be even.

Problem 5. (Chapter 6, Exercises 38, 39)
(a) How many permutations $p \in S_6$ satisfy $p^3 = \text{id}$, where id $= 123456$ (the identity permutation)?
(b) How many even permutations $p \in S_6$ satisfy $p^2 = \text{id}$?

Solution 5. (a) Just as in our first problem we can automatically include the identity element in our list of permutations such that $p^3 = e$. Now we simply need to count the number of cycles of length 3 in S_6 which can be found by $\frac{6 \cdot 5 \cdot 4}{3} = 40$. We also need to count all permutations that are two cycles of length 3 since their order is also 3. This is:

$$\frac{6!}{3^2 \cdot 2!} = 5 \cdot 4 \cdot 2 = 40$$

Then summing these together we have $1 + 40 + 40 = 81$ total number of permutations of order 3 in S_6.
(b) This question is almost identical to Problem 1 except that we are seeking the number of **even** permutations of order 2 in S_6. But this is simpler than it seems, and though the book provides quite a few theorems to guide us we can simply just examine what the question is really seeking. We want all even permutations of order 2 in S_6 so we inherently need 2-cycles, but we cannot consider all elements of order 2. If we were to look at one 2-cycles then we would still have 4 singletons, which violate the imposed even condition. Two 2-cycles gives 2 singletons and so 4 elements in total and thus even. Three 2-cycles gives 3 elements and so it is odd. Therefore we just need to calculate all permutations consisting of two 2-cycles, given by:

$$\frac{6!}{2^2 \cdot 2! \cdot 1^2 \cdot 2!} = 5 \cdot 3^2 = 45$$

Using our knowledge of group theory, the identity element is in A_n and therefore an even transposition, and we know $e^k = e$ for $\forall k in \mathbb{Z}$. Therefore we have $45 + 1 = 46$ even permutations of order 2 in S_6. $\square$

References

[1] John A. Beachy and William D. Blair. *Abstract Algebra*. 3rd ed. Waveland Press, 2019.
[2] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.