Math 315 - Fall 2021
Homework 7
James Della-Giustina

Problem 1. (Chapter 7, Exercise 20)
How many permutations of length n contain at least one of the 2-cycles (12) and (34)?

Solution 1. Let's examine different values of n to see if we can deduce some relation for $n \geq 2$.

$$\begin{aligned} n = 2;\ 1 \text{ permutation } (12); \\ n = 3;\ 1 \text{ permutation } (12); \\ n = 4;\ 3 \text{ permutations } (12), (34), (12)(34); \end{aligned}$$

So we have that if a permutation contains (12) then all other elements are free to permute and we have a total of $(n-2)!$ permutations. The same argument can be made for the 2-cycle (34), and putting these two together we can see that there are $(n-4)!$ available. Now by the inclusion-exclusion we let $A = (12)$ and $B = (34)$ and so:

$$\begin{aligned} |A| + |B| - |A \cap B| &= (n-2)! + (n-2)! - (n-4)! \\ &= 2(n-2)! - (n-4)! \end{aligned}$$

□

Problem 2. (Chapter 7, Exercise 21)
How many n-permutations $p = p_1 p_2 \cdots p_n$ are there in which at least one of p_1 and p_n is even?

Solution 2. Let's try our usual method of breaking this down into more manageable cases. First consider the case when the first permutation p_1 is even. Though there is a restriction on p_1, if we are to fix any of the even permutations p_i and assign it to the first entry then we have a total of $(n-1)!$ permutations and $\lfloor \frac{n}{2} \rfloor$ choices for p_1. So in total there are:

$$\left\lfloor \frac{n}{2} \right\rfloor \cdot (n-1)!$$

The exact same argument can be used to find the number of permutations where p_n is even, and we denote these sets as A_1 and A_2 respectively. We know from the principle of inclusion/exclusion that:

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= \left| \left\lfloor \frac{n}{2} \right\rfloor \cdot (n-1)! \right| + \left| \left\lfloor \frac{n}{2} \right\rfloor \cdot (n-1)! \right| - |A_1 \cap A_2| \end{aligned}$$

Then all that needs to be found is the intersection $A_1 \cap A_2$ that contains all the elements where p_1 and p_n are even. This can easily be expressed as:

$$\left\lfloor \frac{n}{2} \right\rfloor \cdot (n-2)! \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right)$$

So in total we have:

$$\begin{aligned}|A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= \left|\left\lfloor \frac{n}{2} \right\rfloor \cdot (n-1)!\right| + \left|\left\lfloor \frac{n}{2} \right\rfloor \cdot (n-1)!\right| - \left\lfloor \frac{n}{2} \right\rfloor \cdot (n-2)! \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \\ &= 2\left|\left\lfloor \frac{n}{2} \right\rfloor \cdot (n-1)!\right| - \left\lfloor \frac{n}{2} \right\rfloor \cdot (n-2)! \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\end{aligned}$$

□

Problem 3. (Chapter 7, Exercise 28)
How many three-digit positive integers are divisible by at least one of six and seven?

Solution 3. Let's denote A and B to be the sets of three-digit positive integers that are divisible by six and seven respectively. Now we must first deduce the cardinalities of these sets, and then employ the inclusion-exclusion principle to answer the question. First:

$$\begin{aligned}|A| = \left\lfloor \frac{999}{6} \right\rfloor - \left\lfloor \frac{99}{6} \right\rfloor &= 166 - 16 \\ &= 150\end{aligned}$$

Notice that we are finding all integers below 1000 that are divisible by 6 and subtracting off the number of 2-digit numbers divisible by 6. Similarly:

$$\begin{aligned}|B| = \left\lfloor \frac{999}{7} \right\rfloor - \left\lfloor \frac{99}{7} \right\rfloor &= 142 - 14 \\ &= 128\end{aligned}$$

Now to find the intersection $|A \cap B|$ we simply note that the least common multiple of 6 and 7 is 42, so:

$$\begin{aligned}|A \cap B| = \left\lfloor \frac{999}{42} \right\rfloor - \left\lfloor \frac{99}{42} \right\rfloor &= 23 - 2 \\ &= 21\end{aligned}$$

Therefore:

$$\begin{gathered}|A| + |B| + |A \cap B| = \\ 150 + 128 - 21 = 257\end{gathered}$$

Therefore we have 257 3-digit positive integers that are divisible by either 7 or 6. □

Problem 4. (Chapter 7, Exercise 35)
How many 2×2 matrices are there with entries from the set $\{0, 1, \ldots, k\}$ in which there are no zero rows and no zero columns?

Solution 4. Lets denote the set of all matrices $A \in \mathbb{M}_{2\times 2}(k)$ as K. From our previous work with matrices, we know that $|K| = (k+1)^{2^2} = (k+1)^4$. We just need to compute how many of these matrices have columns or rows consisting of all 0's. We can immediately note that the zero matrix is one such matrix. Now let B be the set of all matrices with zeros in the top row and C be the set of all matrices with zeros in the bottom row. $|B| = |C| = (k+1)^2$ and the only element that is in both sets is the zero matrix. The same exact argument can be made for matrices with either the first or second columns consisting of all zeros and we denote these sets as D and E respectively. It should be clear that the intersection of any of these sets is just the zero matrix. So we have:

$$|A| + |B| + |C| + |D| - 1 = 4(k+1)^2 - 1$$

Then there are:

$$\begin{aligned} &(k+1)^4 - (4(k+1)^2 - 1) \\ &= k^4 + 4k^3 + 6k^2 + 4k + 1 - 4k^2 - 8k - 4 + 1 \\ &= k^4 + 4k^3 + 2k^2 - 4k - 2 \end{aligned}$$

matrices of size 2×2 with integer entries from $0, 1, \dots, k$ that have no rows or columns of zeros. □

Problem 5. (Chapter 7, Exercise 22)
Give a combinatorial proof of the identity

$$D(n+1) = n(D(n) + D(n-1))$$

for $n \geq 1$. Do not use the formula for the numbers $D(n)$ proved in the text. Set $D(0) = 1$ and $D(1) = 0$.

Solution 5. Let's try to break this down into cases:

- Consider when we have exactly one fixed element $k \in [1, n]$ such that $p_1 = k$ and where every other element is permuted but $p_k \neq 1$. The number of such derangements is simply $(n)D(n)$.
- Now consider the derangement where $p_1 = k$ and $p_k = 1$ where any other element is mapped to any of the other elements $[2, n]/\{1, k\}$. Then for any k we have $nD(n-1)$ many derangements.

It should be clear that the multiplicative term n outside of the derangement notation indicates that we have n choices for our fixed element k, thus:

$$D(n+1) = n(D(n) + D(n-1))$$

□

References

[1] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.