Math 315 - Fall 2021
Homework 4
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Problem 1. (Chapter 4, Exercise 37)
How many northeastern lattice paths are there from (0, 0) to (10, 10) that do not touch the point (5, 5), but do touch the point (3, 3)?

Solution 1. First we need to consider the total number of paths from $(0,0)$ to $(3,3)$. Since we need to take 6 steps to get to $(3,3)$ then it can be thought of a combination of $\binom{3+3}{3} = \binom{6}{3}$. Next we need to consider all possible paths from $(3,3)$ to $(10,10)$. There are 14 steps we need to take and so in total we have $\binom{7+7}{7} = \binom{14}{7}$. But wait, we need to take into account the restriction that we cannot touch the lattice point $(5,5)$, so let's count the total number of paths from $(3,3)$ to $(5,5)$ which equals $\binom{2+2}{2} = \binom{4}{2}$ and the total number of paths from $(5,5)$ to $(10,10)$ equals $\binom{5+5}{5} = \binom{10}{5}$. So with the restriction we have $\binom{4}{2}\binom{10}{5}$. This leads us to a total of:

$$\begin{aligned}\binom{6}{3}\left(\binom{14}{7} - \binom{4}{2}\binom{10}{5}\right) &= 20(3432 - 6 \cdot 252)\\ &= 20 \cdot 1920\\ &= 38400\end{aligned}$$

$\square$

Problem 2. (Chapter 4, Exercise 39)
Prove that for all positive integers n,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Solution 2. Let's use an identity for the right hand side of the equation:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

Now for the left hand side, we can clearly see that this is a binomial expansion of:

$$(1+x)^{2n} = ((1+x)^n)^2 = \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right)^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

Clearly the right and left hand sides can be expressed in the exact same way, and therefore are equal. $\square$

Problem 3. (Chapter 4, Exercise 40)
Prove that for all positive integers n,

$$n\binom{2n-1}{n-1} = \sum_{k=1}^{n} k\binom{n}{k}^2.$$

Solution 3. This is most easily explained through the committee example that is used for the proof of Theorem 4.6. On the right hand side let's setup a committee of n members from $2n$ people where one of the n people is chosen to head the committee president. Once a president is chosen, we then have $2n-1$ choices for the committee, out of $n-1$, and so we arrive at:

$$n\binom{2n-1}{n}$$

Now for the left hand side, we can choose k people that are eligible for president, and so for the rest of the committee we have:

$$\binom{n}{k}\binom{n}{n-k}$$

But wait, we still have to consider the k combinations of people that can be chosen as president, and so we arrive at:

$$k\binom{n}{k}\binom{n}{n-k}$$

Problem 4. (Chapter 4, Exercise 46)
Prove that for all positive integers n the equality,

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}.$$

Solution 4. Let's use the binomial expansion of $(1+x)^n$ for $x = \pm 2$:

$$\begin{aligned} x := 2; \quad (1+2)^n &= \sum_{k=0}^{n} \binom{n}{k} 2^k \\ (3)^n &= \sum_{k=0}^{n} \binom{n}{k} 2^k \end{aligned}$$

and;

$$\begin{aligned} x := -2; \quad (1+(-2))^n &= \sum_{k=0}^{n} \binom{n}{k} - 2^k \\ (-1)^n &= \sum_{k=0}^{n} \binom{n}{k} - 2^k \end{aligned}$$

Putting the two together we have:

$$3^n + (-1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k + \sum_{k=0}^{n} \binom{n}{k} - 2^k$$

$$= \left[\binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + ... + \binom{n}{n}2^n\right] + \left[\binom{n}{0} - \binom{n}{1}2 + \binom{n}{2}2^2 - ... + (-1)^n\binom{n}{n}2^n\right]$$

$$3^n + (-1)^n = 2\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} 2^k$$

$\square$

Problem 5. (Chapter 4, Exercise 50)
What is the coefficient of x^n in the power series form of $\sqrt[3]{1-2x}$?

Solution 5. Using the generalized binomial theorem:

$$\sum_{k\geq 0} \binom{1/3}{k} (-2x)^k (1)^{1/3-k}$$

Thus the coefficient of x^n is when $k = n$ which now gives us:

$$\begin{aligned}
\binom{1/3}{k}(-2)^n &= (-2)^n \frac{1/3(-2/3)(-5/3)...(1/3-n+1)}{n!} \\
&= (-1)^n 2^n \frac{1/3(-2/3)(-5/3)...(1/3-n+1)}{n!} \\
&= (-1)^n (2)^n \frac{-2(-5)...(-3n+4)}{3^n n!} \\
&= (-1)^{2n-1} 2^n \frac{(3n-4)!!!}{3^n n!} \\
&= -2^n \frac{(3n-4)!!!}{3^n n!} \\
&= -2^n \frac{\prod_{k=1}^{n-1}(3k-1)}{3^n n!}
\end{aligned}$$

$\square$

References

[1] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.