Math 315 - Fall 2021
Homework 5
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Problem 1. (Chapter 5, Exercise 20)
What is the number of partitions of [8] into two blocks in which the two blocks do not have the same size?

Solution 1. The number of compositions of 8 into 2 parts can be given by the Stirling number of the second kind $S(8, 2)$:

$$\begin{aligned} S(8,2) &= S(7,1) + 2S(7,2) \\ &= 1 + 2(S(6,1) + 2(S(6,2)) \\ &= 1 + 2 + 4(S(5,1) + 2S(5,2)) \\ &= 1 + 2 + 4 + 8(S(4,1) + 2(S(4,2)) \\ = 1 + 2 + 4 + 8 + 16(S(3,1) + 2(S(3,2)) &= 1 + 2 + 4 + 8 + 16 + 32\binom{3}{2} \\ &= 1 + 2 + 4 + 8 + 16 + 96 = 127 \end{aligned}$$

Now we need to exclude all compositions of 8 into 2 equal parts, i.e. all compositions of 8 into two boxes of size 4. This is easier then it looks, since we have 8 integers and we need to choose 4 for one block then it is $\binom{8}{4}$ with the other 4 integers occupying the second block and of these we only need half, since there will be repetition. Therefore we get:

$$\begin{aligned} 127 - \frac{1}{2}\binom{8}{4} \\ = 127 - \frac{70}{2} \\ = 92 \end{aligned}$$

$\square$

Problem 2. (Chapter 5, Exercise 23)
A student has to take twelve hours of classes a week. Due to her extracurricular activities, she must take at least three hours of classes on Monday, at least two on Thursday, and at least one on Friday.
(a) In how many ways can she do this?
(b) In how many ways can she do this if there is only one class on Tuesday that she may take?

Solution 2. (a) Let's make some assumptions to help us solve this problem in a coherent way. We can assume that classes are split into one hour time intervals, and that there are classes offered every day of the week. Now we have 12 hours of classes of which 6 are already accounted for, so we have 6 identical balls and 7 distinct boxes. Therefore the number of weak compositions can be given as:

$$\begin{aligned}\binom{6+7-1}{7-1} &= \binom{12}{6} \\ &= 924\end{aligned}$$

So there are 924 different ways in which we can accomplish this.

(b) Since there is only one class on Tuesday that she may take, then we can take our solution above we have two cases: (1) She takes the class on Tuesday, therefore we have 5 hours left that we need to choose 6 other days to fill.

$$\begin{aligned}\binom{5+6-1}{6-1} &= \binom{10}{5} \\ &= 252\end{aligned}$$

(2) She does not take any class on Tuesdays, therefore we still have 6 hours but also only 6 days to choose from.

$$\begin{aligned}\binom{6+6-1}{6-1} &= \binom{11}{5} \\ &= 462\end{aligned}$$

Therefore under these conditions we sum these possibilities to find the total number possible $462 + 252 = 714$.

$\square$

Problem 3. (Chapter 5, Exercise 24)
Find the number of compositions of n into an even number of parts.

Solution 3. We know from Corollary 5.4 [1] that for all positive integers n, the number of all compositions of n is 2^{n-1}. Additionally, we know that the number of compositions of n into even parts is equal to the number of compositions of n into odd parts. Using these two pieces of information, we can say that $2^{n-1} = c_{even} + c_{odd}$: and therefore let $c = c_{even} + c_{odd}$:

$$\begin{aligned}2c &= 2^{n-1} \\ c &= 2^{n-2}\end{aligned}$$

$\square$

Problem 4. (Chapter 5, Exercise 25)
Find the number of weak compositions of 25 into five odd parts.

Solution 4. First we can fill five 'boxes' with 1 integer each. This reduces our problem down to finding 20 into 5 even parts, but we can reduce it even further. If we consider each *pair* of integers then this is just finding 10 pairs into 5 boxes. Using Theorem 5.2 we have the total number of weak compositions of 10 into 5 parts is:

$$\begin{aligned}\binom{n+k-1}{k-1} &= \binom{14}{4} \\ &= 1001\end{aligned}$$

□

Problem 5. (Chapter 5, Exercise 28)
Find a closed formula for $S(n, n-2)$ if $n \geq 2$.

Solution 5. Let's list out a few terms to get a handle on the behavior of $S(n, n-2)$ for $n \geq 2$.

$$\begin{aligned}n := 2 \quad S(2,0) &= 0 \\ n := 3 \quad S(3,1) &= 1 \\ n := 4 \quad S(4,2) = S(3,1) + 2(S(3,2)) = 1 + 2\binom{3}{2} &= 1 + 6 = 7 \\ n := 5 \quad S(5,3) = S(4,2) + 3(S(4,3)) = 7 + 3&\binom{4}{2} \\ &= 7 + 3 + 6 = 16 \\ n := 6 \quad S(6,4) = S(5,3) + 4(S(5,4)) = 16 + 4\binom{5}{2} &= 56\end{aligned}$$

Clearly we can see a recurrence relation; let's try to generalize.

$$S(n, n-2) = 0 + 1 + 2\binom{3}{2} + 3\binom{4}{2} + 4\binom{5}{2} + 5\binom{6}{2} + \ldots + (n-2)\binom{n-1}{2}$$

This is simply:

$$\sum_{i=3}^{n} (i-2)\binom{i-1}{2}$$

□

References

[1] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.