Math 315 - Fall 2021
Homework 8
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Submit all of the following; three (3) will be graded:

Problem 1. (Chapter 8, Exercise 31)
Find an explicit formula for the numbers a_n if $a_{n+1} = (n+1)a_n + 2(n+1)!$ for $n \geq 0$, and $a_0 = 0$.

Solution 1. Let's first divide everything by $(n+1)!$:

$$\frac{a_{n+1}}{(n+1)!} = \frac{a_n(n+1)}{(n+1)!} + 2$$
$$\frac{a_{n+1}}{(n+1)!} - \frac{a_n}{(n)!} = 2$$

Now multiply by x^{n+1} and take the sum over all n:

$$\sum_{n\geq 0} x^{n+1}\left(\frac{a_{n+1}}{(n+1)!} - \frac{a_n}{(n)!}\right) = \sum_{n\geq 0} 2x^{n+1}$$

Let $G(x) = \sum_{n\geq 1} \frac{a_n x^{n+1}}{n+1!}$, then we have:

$$G(x) - xG(x) = 2x\left(\frac{1}{1-x}\right)$$
$$G(x)(1-x) = \frac{2x}{1-x}; \quad G(x) = \frac{2x}{(1-x)^2}$$

□

Problem 2. (Chapter 8, Exercise 27)
A certain kind of insect population multiplies so that at the end of each year, its size is the double of its size a year before, plus 1000 more insects. Assuming that originally we released 50 insects, how many of them will we have at the end of the nth year?

Solution 2. Let's list out a few terms before we try to express this in terms of a generating function. We have:

$$a_1 = 2a_0 + 1000 = 2^1 a_0 + (2^1 - 1)1000$$
$$a_2 = 2a_1 + 1000 = 2^2 a_0 + (2^2 - 1)1000$$
$$a_3 = 2a_2 + 1000 = 2^3 a_0 + (2^3 - 1)1000$$
$$\vdots$$

Therefore we can clearly see the relation:

$$a_{n+1} = 2a_n + 1000$$
$$\text{or } a_{n+1} = 2^n a_0 + (2^n - 1)1000$$

Now to express this relation with a generating function $G(x) = \sum_{n\geq 0} a_n x^n$ we can multiply the whole equation by x^{n+1} and sum over all n:

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = \sum_{n\geq 0} x^{n+1}(2a_n + 1000)$$
$$\sum_{n\geq 0} a_{n+1}x^{n+1} = 2\sum_{n\geq 0} x^{n+1}a_n + 1000\sum_{n\geq 0} x^{n+1}$$
$$G(x) - a_0 = 2xG(x) + 1000x\left(\frac{1}{1-x}\right)$$
$$G(x) = \frac{a_0 + \frac{1000x}{1-x}}{1-2x}$$
$$G(x) = \frac{a(0)}{1-2x} + \frac{1000x}{(1-2x)(1-x)}$$

Now let's examine the two terms on the right hand side of the equation:

$$\frac{a(0)}{1-2x} = \frac{50}{1-2x}; \quad 50\sum_{n\geq 0}(2x)^n$$
$$= 50 \cdot 2^n$$

By partial fraction decomposition, the other term on the right hand side:

$$\frac{1000x}{(1-2x)(1-x)} = \frac{1000}{1-2x} - \frac{1000}{1-x}$$
$$= 1000\left(\sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0} x^n\right)$$
$$= 1000(2^n - 1)$$

Then putting everything together we have:

$$G(x) = 50(2^n) + 1000(2^n - 1)$$

To check we see that at $n = 0$:

$$50(2^0) + 1000(2^0 - 1) = 50$$

Which is exactly what we expect, for $n = 1$ we have:

$$50(2) + 1000(2 - 1) = 1050$$

$\square$

Problem 3. (Chapter 8, Exercise 32)
Let $a_0 = a_1 = 1$, and let $a_n = na_{n-1} + n(n-1)a_{n-2}$ for $n \geq 2$. Find the exponential generating function of the numbers a_n. Compare your result to the result of Exercise 5.

Solution 3. Instead of multiplying by x^n, we are asked for the exponential generating function and so we multiply the whole expression by $\frac{x^n}{n!}$:

$$a_n \frac{x^n}{n!} = \frac{x^n}{n!} \cdot na_{n-1} + \frac{x^n}{n!} n(n-1)a_{n-2}$$

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^n}{n!} \cdot na_{n-1} + \sum_{n\geq 0} \frac{x^n}{n!} n(n-1)a_{n-2}$$

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^n}{(n-1)!} \cdot a_{n-1} + \sum_{n\geq 0} \frac{x^n}{(n-2)!} a_{n-2}$$

Let $G(x) = \sum_{n\geq 2} a_n \frac{x^n}{n!}$, since we need to start at $n-2$ we can 'peel off' the first two terms and re-index. Then:

$$G(x) - a_0 - xa_1 = xa_0 + xG(x) + x^2 G(x)$$

$$G(x) - xG(x) - x^2 G(x) = a_0(1-x) + xa_1$$

Since $a_0 = a_1 = 1$, substitute these values in:

$$G(x) = \frac{1}{1 - x - x^2}$$

□

Problem 4. (Chapter 8, Exercise 37)
Find a closed form (no summation signs) for the generating function

$$G(x) = \sum_{n\geq 0} S(n,k) \frac{x^n}{n!}.$$

Solution 4. We can immediately recognize that the term $\frac{x^n}{n!}$ is the power series form of the exponential e^x, and we are looking for the ways to partitions n objects into k blocks (nonempty), so we have $e^x - 1$. There are $(e^x - 1)^k$ ways we can partition n, but since order does not matter we divide by $k!$. Therefore we have:

$$G(x) = \frac{(e^x - 1)^k}{k!}$$

□

Problem 5. (Chapter 8, Exercise 43)
We have two bookshelves. Let t_n be the number of ways to first partition a set of n distinct books into two non-empty blocks, and then to line up one block on the top bookshelf and to line up the other block on the bottom bookshelf. Find a closed formula for t_n.

Solution 5. We've got n books and we need to split these into two non-empty groups ('blocks'). Therefore if we choose r books for the first group, we have $\binom{n}{r}$ choices, of which we can arrange $r!$ ways. Similarly for the second group we have $(n-r)!$ ways to arrange the $n-r$ books. Altogether we have $2 \cdot r!\binom{n}{r} \cdot (n-r)!$ ways of arranging either groups on the two bookshelves.

□

References

[1] Miklós Bóna. *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory,* 4th. World Scientific, 2016. ISBN: 9813148845.