MATH 437: Homework 7

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Problem 1

$$\begin{array}{lrrrrrrl}
\text{Maximize} & -5x_1 & +5x_2 & +13x_3 & & & = & Z \\
\text{Subject to:} & -x_1 & +x_2 & +3x_3 & +s_1 & & = & 20 \\
 & 12x_1 & +4x_2 & +10x_3 & & +s_2 & = & 90 \\
 & & & & & x_1, x_2, x_3, s_1, s_2 & \geq & 0
\end{array}$$

If we let x4 and x5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{lrrrrrrl}
Z & & & +2x_3 & +5s_1 & & = & 100 \\
 & -x_1 & +x_2 & +3x_3 & +s_1 & & = & 20 \\
 & 16x_1 & & -2x_3 & +4s_1 & +s_2 & = & 10
\end{array}$$

(a) Change to $b_1 = 30$.

(b) Change to $b_2 = 70$.

(c) Change the vector b to:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}$$

(d) Change the coefficient of x_3 in the objective function to $c_3 = 8$.

(e) Change the coefficients of x_1:

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

(f) Change the coefficients of x_2:

$$\begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$$

(g) Introduce a new variable x_4:

$$\begin{bmatrix} c_4 \\ a_{14} \\ a_{24} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}$$

(h) Introduce a third constraint $2x_1 + 3x_2 + 5x_3 \leq 50$.

(i) Change the second constraint to $10x_1 + 5x_2 + 10x_3 \leq 100$.

Solution 1. We are given the final form of our simplex yields an optimal solution, and therefore we know that $c_B = [x_2, s_2] = [20, 10]$. Then the corresponding basis matrix and its inverse are:

$$B = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

Our optimal vector $\pi^* = c_B^T B^{-1}$;

$$\pi^* = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix}$$

(a) We simply need to replace $b_1 = 30$ and compute $B^{-1}b$ and check for optimality;

$$B^{-1}b = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 90 \end{bmatrix} = \begin{bmatrix} 30 \\ -30 \end{bmatrix} \ngeq 0$$

So the optimal solution does not hold anymore.

(b) Similar to part (a):

$$B^{-1}b = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 70 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \end{bmatrix} \ngeq 0$$

Again, the optimal solution no longer holds.

(c) Similar to part (a) & (b):

$$B^{-1}b = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 100 \end{bmatrix} = \begin{bmatrix} 10 \\ 60 \end{bmatrix} \geq 0$$

Our optimal solution still holds true.

(d) We have c_3 change from 13 to 8; i.e. a change of -5. Then we simply take the value of $c_3 = 2$ in our optimal tableau and subtract this change; $c_3 = 2 - (-5) = 7$.

(e) We need to change the whole column that corresponds to a non-basic variable x_1. The coefficients change from:

$$\begin{bmatrix} -5 \\ -1 \\ 12 \end{bmatrix} \rightarrow \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then we need to update our costs to ensure we are still in an optimal solution. We accomplish this by performing $x^T A_j - c_j$ for updated coefficients A_j and c_j. Then:

$$\begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} - (-2) = 2 \geq 0$$

And so it follows that our optimal solution still holds.

(f) We must check whether $\pi^* = [5, 0]^T$ is still optimal.

$$(\pi^*)^T A_2 - c_2 = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} - 6 = 4 \geq 0$$

Our optimal solution is still optimal.

(g) We need to introduce a new variable x_4 with coefficients:

$$\begin{bmatrix} c_4 \\ a_{14} \\ a_{24} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}$$

We check if our current solution holds under this new additional variable by calculating $x^T A_j - c_j$:

$$\begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} - (-10) = 25 \geq 0$$

And so our optimal solution still holds.

(h) We need to introduce a third constraint $2x_1 + 3x_2 + 5x_3 \leq 50$. But all we need to see is if our current solution satisfies this constraint, because our feasible region can only grow smaller. But since $x_2 = 20$ in our optimal solution, then clearly we cannot satisfy this third constraint with our current solution.

(i) We need to change the second constraint to $10x_1 + 5x_2 + 10x_3 \leq 100$. Similar to part h), we just need to ensure that our current solution does not violate this new constraint. Since $x_2 = 20$, then we can see that our optimal solution still holds and in fact this newly added constraint is binding.

□

Problem 2

Consider the LP:

$$\begin{array}{lrrrrrcl}
\text{Maximize} & 2x_1 & +\,7x_2 & -\,3x_3 & & & = & Z \\
\text{subject to} & x_1 & +\,3x_2 & +\,4x_3 & +\,s_1 & & = & 30 \\
 & x_1 & +\,4x_2 & -\,x_3 & & +\,s_2 & = & 10 \\
 & & & & & x_1, x_2, x_3, s_1, s_2 & \geq & 0
\end{array}$$

Using the simplex method yields the following final set of equations:

$$\begin{array}{rrrrrcl}
Z & +\,x_2 & +\,x_3 & & +2s_2 & = & 20 \\
 & -x_2 & +\,5x_3 & +\,s_1 & -\,s_2 & = & 20 \\
x_1 & +\,4x_2 & +\,-\,x_3 & & +\,s_2 & = & 10
\end{array} \tag{2.1}$$

Conduct sensitivity analysis by independently investigating each of the following seven changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations and convert it to proper form from the Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the vector b to:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

(b) Change the coefficients of x_3:

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

(c) Change the coefficients of x_1:

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

(d) Introduce a new variable x_4 with coefficients of:

$$\begin{bmatrix} c_4 \\ a_{14} \\ a_{24} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

(e) Change the objective function to $x_1 + 5x_2 - 2x_3$.

(f) Introduce a new constraint $3x_1 + 2x_2 + 3x_3 \leq 25$.

(g) Change constraint 2 to $x_1 + 2x_2 + 2x_3 \leq 35$.

Solution 2. Since we are given that $c_B = [s_1, x_1]$, then our corresponding basis matrix and its inverse are:

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Our optimal vector $\pi^* = c_B^T B^{-1}$;

$$\pi^* = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}$$

(a) We simply need to replace the values in b, compute $B^{-1}b$, and check for optimality;

$$B^{-1}b = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \end{bmatrix} \not\geq 0$$

So our optimal solution no longer holds. Unlike Problem (1), we need to go ahead and re-optimize our changed problem. After multiplying by the tableau by -1, our new tableau is:

$$\left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & & b \\ \hline -1 & 0 & -1 & -1 & 0 & -2 & & -60 \\ \hline 0 & 0 & 1 & -5 & -1 & 1 & & 10 \\ 0 & -1 & -4 & 1 & 0 & -1 & & -30 \end{array}\right]$$

Since s_2 has the smallest value, we choose s_2 to be incoming. Rather than compute the basis matrix and its associated inverse to update the matrix, we simply perform Gaussian elimination to make s_2 an incoming basic variable in with s_1 as an outgoing (since R_2 is the only positive ratio). Performing row operations $2R_2 + R_1 \to R_1$; $\quad R_2 + R_3 \to R_3$ we get:

$$\left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & & b \\ \hline -1 & 0 & 1 & -11 & -2 & 0 & & -40 \\ \hline 0 & 0 & 1 & -5 & -1 & 1 & & 10 \\ 0 & -1 & -3 & -4 & -1 & 0 & & -20 \end{array}\right]$$

Now choosing x_2 to be an incoming basic variable, with ratios $R_2 = 10$ and $R_3 = \frac{-20}{-3}$ and so we choose to pivot on row 2. Performing row operations $\frac{1}{3}R_3 \to R_3, R_3 + R_2 \to R_2$, and $R_3 + R_1 \to R_1$, we get:

$$\left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & & b \\ \hline 1 & \frac{1}{3} & 0 & \frac{37}{3} & \frac{7}{3} & 0 & & \frac{140}{3} \\ \hline 0 & -\frac{1}{3} & 0 & -\frac{19}{3} & -\frac{4}{3} & 1 & & \frac{10}{3} \\ 0 & \frac{1}{3} & 1 & \frac{4}{3} & \frac{1}{3} & 0 & & \frac{20}{3} \end{array}\right]$$

(b) We need to change the whole column that corresponds to a non-basic variable x_3. The coefficients change from:

$$\begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} \to \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

Then we need to update our costs to ensure we are still in an optimal solution. We accomplish this by performing $x^T A_3 - c_3$ for updated coefficients A_3 and c_3. Then:

$$\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} - (-2) = -2 \not\geq 0$$

So our optimal solution no longer holds, and therefore we must reoptimize. We switch to the dual, in which we have a tableau:

$$\left[\begin{array}{c|ccccc|c} z & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & b \\ \hline -1 & 30 & 10 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & -1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 0 & -1 & 0 & 7 \\ 0 & 3 & -2 & 0 & 0 & -1 & -2 \end{array}\right]$$

with $c_B = (\pi_2, \pi_4, \pi_5) = (10, 0, 0)$ and the associated inverse basis matrix:

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

Then:

$$\bar{c} = c - c_B^T B^{-1} A =$$
$$\begin{bmatrix} 30 & 10 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 3 & 4 & 0 & -1 & 0 \\ 3 & -2 & 0 & 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 20 & 0 & 10 & 0 & 0 \end{bmatrix} \geq 0$$

So by the complementary slackness, we know that $c_B = (x_1, x_3)$ in our primal solution. Therefore, we switch *back* to our primal problem and find the new basis matrix and its inverse:

$$\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

Now checking the updated cost coefficients for optimality:

$$\bar{c} = c - c_B^T B^{-1} A =$$

$$\begin{bmatrix} 2 & 7 & -2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{3}{5} & 0 & -\frac{2}{5} & -\frac{8}{5} \end{bmatrix} \leq 0$$

And so we have found our new optimal solution for the change in all coefficients of x_3. Since we already have our new cost coefficients, we can calculate $B^{-1}[A|b]$ to reach our final tableau:

$$\begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 1 & 0 & 30 \\ 1 & 4 & -2 & 0 & 1 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{18}{5} & 0 & \frac{2}{5} & \frac{3}{5} & 18 \\ 0 & -\frac{1}{5} & 1 & \frac{1}{5} & -\frac{1}{5} & 4 \end{bmatrix}$$

And so our final tableau is:

$$\left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & & b \\ \hline -1 & 0 & -\frac{3}{5} & 0 & -\frac{2}{5} & -\frac{8}{5} & & 0 \\ \hline 0 & 1 & \frac{18}{5} & 0 & \frac{2}{5} & \frac{3}{5} & & 18 \\ 0 & 0 & -\frac{1}{5} & 1 & \frac{1}{5} & -\frac{1}{5} & & 4 \end{array}\right]$$

with $x^* = [18, 0, 4, 0, 0]$.

(c) We need to change the coefficients of a basic variable x_1:

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

First, we update our basis matrix B and its inverse:

$$B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Then our updated cost coefficients:

$$\bar{c} = c - c_B^T B^{-1} A =$$

$$\begin{bmatrix} 4 & 7 & -3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 4 & 1 & 0 \\ 2 & 4 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 & 0 & -2 \end{bmatrix} \leq 0$$

And so our updated tableau is:

$$\left[\begin{array}{c|ccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & b \\ \hline -1 & 0 & -1 & -1 & 0 & -2 & 0 \\ \hline 0 & 0 & -3 & \frac{11}{2} & 1 & -\frac{3}{2} & 15 \\ 0 & 1 & 2 & -\frac{1}{2} & 0 & \frac{1}{2} & 5 \end{array}\right]$$

(d) We introduce a new variable x_4 with coefficients of:

$$\begin{bmatrix} c_4 \\ a_{14} \\ a_{24} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

To check if our current solution still holds, we perform $x^T A_4 - c_4$ for updated coefficients A_4 and c_4.

$$\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} - (-3) = 13$$

And so the introduction of this new variable and its associated coefficients in the constraints does not affect our optimal solution, i.e. $x_4 \in x_N$.

(e) We must change the objective function to $x_1 + 5x_2 - 2x_3$. First, let's check if our 'old' basis yields an optimal solution:

$$\bar{c} = c - c_B^T B^{-1} A =$$

$$\begin{bmatrix} 1 & 5 & -2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 & 1 & 0 \\ 1 & 4 & -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 & 0 & -1 \end{bmatrix} \not\leq 0$$

So our optimal basis no longer holds, and we must reoptimize. Since $c_2 > 0$, we compute $B^{-1}A_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $B^{-1}b = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$. Since R_3 is the only positive ratio, we choose to pivot on the third row with outgoing basic variable x_1. Our new basis matrix for $x_B = [s_1, x_2]$ and row multipliers are:

$$B = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$\pi = c_B B^{-1} = \begin{bmatrix} 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix}$$

Then our updated cost coefficients:

$$\begin{aligned} \bar{c} = c - \pi^T A = \\ \begin{bmatrix} 1 & 5 & -2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 & 1 & 0 \\ 1 & 4 & -1 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} -\frac{1}{4} & 0 & -\frac{3}{4} & 0 & -\frac{5}{4} \end{bmatrix} \leq 0 \end{aligned}$$

So our optimal solution has $x_B = [s_1, x_2]$, our final tableau is:

$$\left[\begin{array}{c|ccccc|c} z & x_1 & x_2 & x_3 & s_1 & s_2 & b \\ \hline -1 & -\frac{1}{4} & 0 & -\frac{3}{4} & 0 & \frac{5}{4} & 0 \\ \hline 0 & \frac{1}{4} & 0 & \frac{19}{4} & 1 & -\frac{3}{4} & \frac{45}{2} \\ 0 & \frac{1}{4} & 1 & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{5}{2} \end{array}\right]$$

With optimal $x^* = [0, \frac{5}{2}, 0, \frac{45}{2}, 0]$

(f) We must introduce a new constraint $3x_1 + 2x_2 + 3x_3 \leq 25$. Since $x_1 = 10$ in our optimal solution, then we are still within the 'new' feasible region and therefore our solution is still optimal.

(g) We need to change constraint (2) to $x_1 + 2x_2 + 2x_3 \leq 35$. Similar to part (f), since $x_1 = 10$ in the optimal solution, then clearly we are not violating our new constraint and therefore our optimal solution holds.

□

Problem 3

Consider the LP:

$$\begin{array}{lrrrrrrcr}
\text{Maximize} & 2x_1 & -x_2 & +x_3 & & & & = & Z \\
\text{subject to} & 3x_1 & -2x_2 & +2x_3 & +s_1 & & & = & 15 \\
& -x_1 & +x_2 & +x_3 & & +s_2 & & = & 3 \\
& x_1 & -x_2 & +x_3 & & & +s_3 & = & 4 \\
& & & & \multicolumn{3}{r}{x_1, x_2, x_3, s_1, s_2, s_3} & \geq & 0
\end{array}$$

Using the simplex method yields the following final set of equations:

$$\begin{array}{rrrrrrrcr}
Z & & & +2x_3 & +s_1 & +s_2 & & = & 18 \\
& & x_2 & +5x_3 & +s_1 & +3s_2 & & = & 24 \\
& & & 2x_3 & & +s_2 & +s_3 & = & 7 \\
& x_1 & & +4x_3 & +s_1 & +2s_2 & & = & 21
\end{array} \tag{3.1}$$

Now you are to conduct sensitivity analysis by independently investigating each of the following eight changes in the original model. For each change, use the sensitivity analysis procedure to revise this set of equations (in tableau form) and convert it to proper form from Gaussian elimination for identifying and evaluating the current basic solution. Then test this solution for feasibility and for optimality. If either test fails, reoptimize to find a new optimal solution.

(a) Change the vector b to:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix}$$

(b) Change $c_3 = 2$.

(c) Change $c_1 = 3$.

(d) Change the coefficients of x_3:

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

(e) Change the coefficients of x_1 and x_2:

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_2 \\ a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \\ 2 \end{bmatrix}$$

(f) Change the objective function to $5x_1 + x_2 + 3x_3$.

(g) Change constraint 1 to $2x_1 - x_2 + 4x_3 \leq 12$.

(h) Introduce a new constraint $2x_1 + x_2 + 3x_3 \leq 60$.

Solution 3. Since we are given that $x_B = [x_2, x_6, x_1]$, then our corresponding basis matrix and its inverse are:

$$B = \begin{bmatrix} -2 & 0 & 3 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}; \quad B^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \text{Our optimal vector } x^* = c_B^T B^{-1};$$

$$x^* = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & -1 \end{bmatrix}$$

(a) We simply need to replace the values in b, compute $B^{-1}b$, and check for optimality;

$$B^{-1}b = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 22 \\ 6 \\ 18 \end{bmatrix} \geq 0$$

Therefore our solution is feasible and optimal.

(b) Changing the coefficient of x_3 in the objective function to $c_3 = 2$ results in a change of 1. Our optimal cost coefficient for x_3 is 2, therefore our new c_3 is $2 - 1 = 1$ which is still feasible and optimal.

(c) Changing the coefficient of x_1 in the objective function to $c_1 = 3$ results in a change of 3, therefore the change in our optimal solution is -3 and $2 - 3 = -1$ is the new c_1 in our new optimal solution. This new solution is feasible and optimal.

(d) We need to change the whole column that corresponds to a non-basic variable x_3. The coefficients change from:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Then we need to update our costs to ensure we are still in an optimal solution. We accomplish this by performing $x^T A_3 - c_3$ for updated coeffecients A_3 and c_3. Then:

$$\begin{bmatrix} 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - (3) = 13 \geq 0$$

And so our optimal solution still holds.

(e) Using `Sage` to solve our updated LP, we have:

```
p = MixedIntegerLinearProgram(maximization=True)
v = p.new_variable(real=True, nonnegative=True)
x1, x2, x3, s1, s2, s3, = v['x1'], v['x2'], v['x3'], v['s1'], v['s2'], v['s3
    '],
p.set_objective(x1-2*x2+x3)
p.add_constraint(x1-2*-2*x2+2*x3+s1==15)
p.add_constraint(-2*x1+3*x2+x3+s2==3)
p.add_constraint(3*x1+2*x2+x3+s3==4)
p.show()
p.solve()
d = p.get_values(v)
for k in d:
    print(k,d[k])
print('Max=',p.get_objective_value())
```

Which gives output:

```
Maximization:
  x_0 -2.0 x_1 + x_2

Constraints:
  15.0 <= x_0 + 4.0 x_1 + 2.0 x_2 + x_3 <= 15.0
  3.0 <= -2.0 x_0 + 3.0 x_1 + x_2 + x_4 <= 3.0
  4.0 <= 3.0 x_0 + 2.0 x_1 + x_2 + x_5 <= 4.0
Variables:
  x_0 is a continuous variable (min=0.0, max=+oo)
  x_1 is a continuous variable (min=0.0, max=+oo)
  x_2 is a continuous variable (min=0.0, max=+oo)
  x_3 is a continuous variable (min=0.0, max=+oo)
  x_4 is a continuous variable (min=0.0, max=+oo)
  x_5 is a continuous variable (min=0.0, max=+oo)
x1 0.2
x2 0.0
x3 3.4
s1 8.0
s2 0.0
s3 0.0
Max= 3.6
```

From above we get $x^* = [.2, 0, 3.4, 8, 0, 0]$ with maximum value $Z = 3.6$

(f) Using `Sage` to solve our updated LP, we have:

```
p = MixedIntegerLinearProgram(maximization=True)
v = p.new_variable(real=True, nonnegative=True)
```

```
x1, x2, x3, s1, s2, s3 = v['x1'], v['x2'], v['x3'], v['s1'], v['s2'], v['s3'
    ]
p.set_objective(5*x1+x2+3*x3)
p.add_constraint(3*x1-2*x2+2*x3+s1==15)
p.add_constraint(-1*x1+x2+x3+s2==3)
p.add_constraint(x1-x2+x3+s3==4)
p.show()
p.solve()
d = p.get_values(v)
for k in d:
    print(k,d[k])
print('Max=',p.get_objective_value())
```

which gives output:

```
Maximization:
  5.0 x_0 + x_1 + 3.0 x_2

Constraints:
  15.0 <= 3.0 x_0 - 2.0 x_1 + 2.0 x_2 + x_3 <= 15.0
  3.0 <= - x_0 + x_1 + x_2 + x_4 <= 3.0
  4.0 <= x_0 - x_1 + x_2 + x_5 <= 4.0
Variables:
  x_0 is a continuous variable (min=0.0, max=+oo)
  x_1 is a continuous variable (min=0.0, max=+oo)
  x_2 is a continuous variable (min=0.0, max=+oo)
  x_3 is a continuous variable (min=0.0, max=+oo)
  x_4 is a continuous variable (min=0.0, max=+oo)
  x_5 is a continuous variable (min=0.0, max=+oo)
x1 21.0
x2 24.0
x3 0.0
s1 0.0
s2 0.0
s3 7.0
Max= 129.0
```

From above we get $x^* = [21, 24, 0, 0, 0, 7]$ with maximum value $Z = 129$.

(g) We need to change constraint (1) to $2x_1 - x_2 + 4x_3 \leq 12$. Since our optimal solution is $[x_2, s_3, x_1] = [24, 7, 21]$, then clearly $2 \cdot 21 - 24 \not\leq 12$ and so our optimal solution no longer holds. Using `Sage` to solve our updated LP, we have:

```
p = MixedIntegerLinearProgram(maximization=True)
v = p.new_variable(real=True, nonnegative=True)
x1, x2, x3, s1, s2, s3 = v['x1'], v['x2'], v['x3'], v['s1'], v['s2'], v['s3'
    ]
```

```
p.set_objective(2*x1-x2+x3)
p.add_constraint(2*x1-x2+4*x3+s1==12)
p.add_constraint(-1*x1+x2+x3+s2==3)
p.add_constraint(x1-x2+x3+s3==4)
p.show()
p.solve()
d = p.get_values(v)
for k in d:
    print(k,d[k])
print('Max=',p.get_objective_value())
```

Which outputs:

```
  Maximization:
  2.0 x_0 - x_1 + x_2

Constraints:
  12.0 <= 2.0 x_0 - x_1 + 4.0 x_2 + x_3 <= 12.0
  3.0 <= - x_0 + x_1 + x_2 + x_4 <= 3.0
  4.0 <= x_0 - x_1 + x_2 + x_5 <= 4.0
Variables:
  x_0 is a continuous variable (min=0.0, max=+oo)
  x_1 is a continuous variable (min=0.0, max=+oo)
  x_2 is a continuous variable (min=0.0, max=+oo)
  x_3 is a continuous variable (min=0.0, max=+oo)
  x_4 is a continuous variable (min=0.0, max=+oo)
  x_5 is a continuous variable (min=0.0, max=+oo)
x1 8.0
x2 4.0
x3 0.0
s1 0.0
s2 7.0
s3 0.0
Max= 12.0
```

Therefore, our new optimal solution is $x* = [8, 4, 0, 0, 7, 0]$.

(h) We must introduce a new constraint $2x_1 + x_2 + 3x_3 \leq 60$. First let's test if our optimal solution still lies inside our 'new' feasible region. $2 \cdot 21 + 24 \not\leq 60$, and so we must reoptimize to find a new solution. Using `Sage`, our problem is:

```
p = MixedIntegerLinearProgram(maximization=True)
v = p.new_variable(real=True, nonnegative=True)
x1, x2, x3, s1, s2, s3, s4 = v['x1'], v['x2'], v['x3'], v['s1'], v['s2'], v[
    's3'], v['s4']
p.set_objective(2*x1-x2+x3)
```

```
p.add_constraint(3*x1-2*x2+2*x3+s1==15)
p.add_constraint(-1*x1+x2+x3+s2==3)
p.add_constraint(x1-x2+x3+s3==4)
p.add_constraint(2*x1+x2+3*x3+s4==60)
p.show()
p.solve()
d = p.get_values(v)
for k in d:
    print(k,d[k])
print('Max=',p.get_objective_value())
```

Which gives output:

```
  Maximization:
  2.0 x_0 - x_1 + x_2

Constraints:
  15.0 <= 3.0 x_0 - 2.0 x_1 + 2.0 x_2 + x_3 <= 15.0
  3.0 <= - x_0 + x_1 + x_2 + x_4 <= 3.0
  4.0 <= x_0 - x_1 + x_2 + x_5 <= 4.0
  60.0 <= 2.0 x_0 + x_1 + 3.0 x_2 + x_6 <= 60.0
Variables:
  x_0 is a continuous variable (min=0.0, max=+oo)
  x_1 is a continuous variable (min=0.0, max=+oo)
  x_2 is a continuous variable (min=0.0, max=+oo)
  x_3 is a continuous variable (min=0.0, max=+oo)
  x_4 is a continuous variable (min=0.0, max=+oo)
  x_5 is a continuous variable (min=0.0, max=+oo)
  x_6 is a continuous variable (min=0.0, max=+oo)
x1 19.28571428571429
x2 21.42857142857143
x3 0.0
s1 0.0
s2 0.8571428571428577
s3 6.142857142857142
s4 -7.105427357601002e-15
Max= 17.142857142857146
```

And so our new optimal vector is $x^* = [19.29, 21.43, 0, 0, 0.86, 6.14, 0]$.

$\square$

References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.