MATH 437: Homework 8

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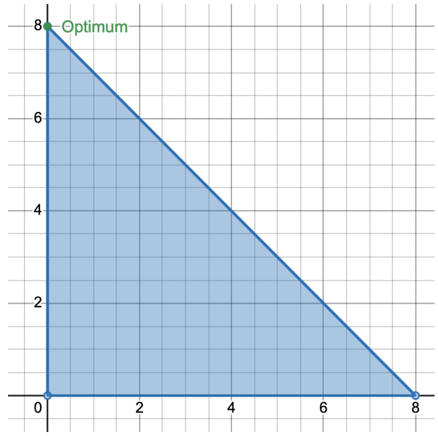

Figure 1: Feasible region of Problem 1 and optimal CPF solution at $(0, 8)$.

Problem 1

Consider the following problem:

$$\begin{array}{lcccccc} \text{Maximize} & x_1 & + & 2x_2 & = & z \\ \text{subject to} & x_1 & + & x_2 & = & 8 \\ & x_1 & & x_2 & \geq & 0 \end{array}$$

Solve the problem graphically (so that you know what to expect).

(a) Notice that this is a maximization problem. Please modify the interior point process that was described in class to make it work for the maximization problem. You can check that your modification is correct by reading Section 7.4. List the steps in the modified algorithm.

(b) Take $x^0 = [4, 4]^T$. Perform two iterations of the interior point method with $\alpha = 0.9$ to obtain an approximation to the optimal solution.

Solution 1. A graphical representation of our feasible region with an optimal solution (0,8) can be seen in Figure 1.

(a) The appropriate change in our interior point method is to change the sign in step (7), meaning instead of choosing a direction in the steepest descent, we look instead for the direction of steepest *ascent*. Therefore, we choose $\hat{p}^t := \hat{P}\hat{c}$.

(b) We are given $x^0 = [4, 4]^T$ as our initial point and so we calculate $\hat{A} = Ac$ and $\hat{c} = Dc^T$:

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}; \quad \hat{A} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \end{bmatrix}; \quad \hat{c} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Then our projection matrix:

$$\hat{P} = \mathbb{I} - \hat{A}^T(\hat{A}\hat{A}^T)^{-1}\hat{A}; \quad (\hat{A}\hat{A}^T)^{-1} = \frac{1}{32};$$

$$\hat{P} = \frac{1}{32}(\mathbb{I} - \hat{A}^T\hat{A}) = \frac{1}{32}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix}\begin{bmatrix} 4 & 4 \end{bmatrix};$$

$$\hat{P} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Then $\hat{p}^t = \hat{P}\hat{c}$:

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Clearly $\theta = 2$, translating our solution back:

$$\hat{x}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{.9}{2}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} .1 \\ 1.9 \end{bmatrix}$$

So:

$$x^1 = D\hat{x}^1 \longrightarrow \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} .1 \\ 1.9 \end{bmatrix} = \begin{bmatrix} .4 \\ 7.6 \end{bmatrix}$$

Iteration (2):

$$D = \begin{bmatrix} .4 & 0 \\ 0 & 7.6 \end{bmatrix}; \quad \hat{A} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} .4 & 0 \\ 0 & 7.6 \end{bmatrix} = \begin{bmatrix} .4 \\ 7.6 \end{bmatrix}; \quad \hat{c} = \begin{bmatrix} .4 & 0 \\ 0 & 7.6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} .4 \\ 15.2 \end{bmatrix}$$

$$\hat{P} = \mathbb{I} - \hat{A}^T(\hat{A}\hat{A}^T)^{-1}\hat{A}; \quad (\hat{A}\hat{A}^T)^{-1} = \frac{1}{57.92};$$

$$\hat{P} = \frac{1}{32}(\mathbb{I} - \hat{A}^T\hat{A}) = \frac{1}{57.92}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .4 \\ 7.6 \end{bmatrix}\begin{bmatrix} .4 & 7.6 \end{bmatrix};$$

$$\hat{p} = \frac{1}{362}\begin{bmatrix} 361 & -19 \\ -19 & 1 \end{bmatrix}$$

So

$$\hat{p} = \frac{1}{362}\begin{bmatrix} 361 & -19 \\ -19 & 1 \end{bmatrix}\begin{bmatrix} .4 \\ 15.2 \end{bmatrix} = \frac{1}{905}\begin{bmatrix} -361 \\ 19 \end{bmatrix}$$

Clearly $\theta = \frac{361}{905}$; then:

$$\hat{x}^2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{.9}{361/905}\begin{bmatrix} -\frac{361}{905} \\ \frac{19}{905} \end{bmatrix} = \begin{bmatrix} .1 \\ \frac{199}{190} \end{bmatrix}$$

So:

$$x^2 = \begin{bmatrix} .4 & 0 \\ 0 & 7.6 \end{bmatrix} \begin{bmatrix} .1 \\ \frac{199}{190} \end{bmatrix} = \begin{bmatrix} .04 \\ 7.96 \end{bmatrix}$$

And from our graphical solution, we can see that we have converged to our optimal solution $x = (0, 8)$

$\square$

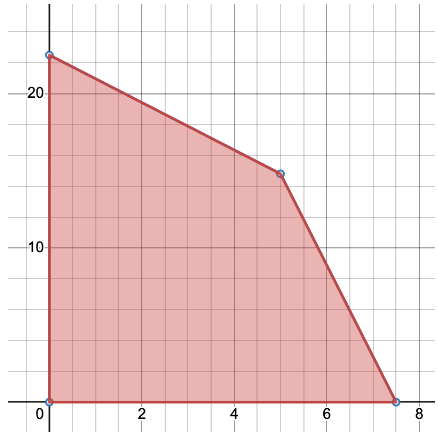

Figure 2: The feasible region for Problem 2.

Problem 2

Consider the following program:

$$
\begin{array}{lrcrcr}
\text{Maximize} & 3x_1 & + & x_2 & = & z \\
\text{subject to} & 3x_1 & + & 2x_2 & \leq & 45 \\
 & 6x_1 & + & x_2 & \leq & 45 \\
 & x_1 & & x_2 & \geq & 0
\end{array}
$$

(a) Solve the problem graphically.

(b) Take $x^0 = (1, 2)^T$. Note that this problem is not in a canonical form, so we need to introduce two slack variables (and then you need to determine what are the values of the slack variables when $x_1^0 = 1$ and $x_2^0 = 2$). Perform two iterations of the interior point method with $\alpha = 0.9$ to obtain an approximation to the optimal solution. Plot the approximations on the (x_1, x_2)-plane.

Solution 2. (a) A feasible region can be shown Figure 2 with 4 CPF solutions. Table 2.1 lists all objective function values when evaluated at these 4 possible solutions with $(5, 15)$ giving us our optimum.

CPF	$3x_1 + x_2 = z$
(0,0)	0
(0,22.5)	22.5
(7.5, 0)	22.5
(5,15)	30

Table 2.1: All possible CPF solutions and the respective objective function values.

(b) After introducing slack variables, our LP is:

$$\begin{array}{lrcrcrcrcl} \text{Maximize} & 3x_1 & + & x_2 & & & & & = & z \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & = & 45 \\ & 6x_1 & + & x_2 & & & + & x_4 & = & 45 \\ & & & & & & & x_i & \geq & 0 \end{array}$$

We use `Mathematica` to help us attain a solution:

```
In[63]:= A = {{3, 2, 1, 0}, {6, 1, 0, 1}}
c = {{3, 1, 0, 0}}
b = {{45}, {45}}

Out[63]= {{3, 2, 1, 0}, {6, 1, 0, 1}}

Out[64]= {{3, 1, 0, 0}}

Out[65]= {{45}, {45}}



In[66]:= d = DiagonalMatrix[{1, 2, 38, 37}]

Out[66]= {{1, 0, 0, 0}, {0, 2, 0, 0}, {0, 0, 38, 0}, {0, 0, 0, 37}}

In[67]:= Ahat = A . d

Out[67]= {{3, 4, 38, 0}, {6, 2, 0, 37}}

In[68]:= chat = d . Transpose[c]

Out[68]= {{3}, {2}, {0}, {0}}

In[18]:= Inverse[Ahat . Transpose[Ahat]]

Out[18]= {{1409/2069145, -(2/159165)}, {-(2/159165), 113/159165}}

In[69]:= phat =
 IdentityMatrix[
   4] - (Transpose[Ahat] . Inverse[Ahat . Transpose[Ahat]] . Ahat)

Out[69]= {{222724/
  229905, -(11252/689715), -(51566/689715), -(8288/53055)}, {-(11252/
   689715), 2041141/
  2069145, -(212192/2069145), -(8066/159165)}, {-(51566/689715), -(
   212192/2069145), 34549/2069145, 2812/
```

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 159165}, {-(8288/53055), -(8066/159165), 2812/159165, 4468/159165}}

In[70]:= littlephat = phat . chat

Out[70]= {{1982012/689715}, {3981014/
 2069145}, {-(888478/2069145)}, {-(90724/159165)}}

In[71]:= theta = -Min[Transpose[littlephat]]

Out[71]= 90724/159165

In[72]:= xhat = {{1, 1, 1, 1}} + (.9/theta) Transpose[littlephat]

Out[72]= {{5.53737, 4.03788, 0.322009, 0.1}}

In[73]:= xtrans = d . Transpose[xhat]

Out[73]= {{5.53737}, {8.07576}, {12.2364}, {3.7}}

In[74]:= d2 =
 DiagonalMatrix[{5.537373199526543, 8.075760802840737,
   12.236358795738893, 3.6999999999999993}]

Out[74]= {{5.53737, 0., 0., 0.}, {0., 8.07576, 0., 0.}, {0., 0.,
 12.2364, 0.}, {0., 0., 0., 3.7}}

In[75]:= A2hat = A . d2

Out[75]= {{16.6121, 16.1515, 12.2364, 0.}, {33.2242, 8.07576, 0., 3.7}}

In[77]:= c2hat = d2 . Transpose[c]

Out[77]= {{16.6121}, {8.07576}, {0.}, {0.}}

In[78]:= P2hat =
 IdentityMatrix[
   4] - (Transpose[A2hat] . Inverse[A2hat . Transpose[A2hat]] . A2hat)

Out[78]= {{0.0444144, -0.126571, 0.106772, -0.122561}, {-0.126571,
 0.493923, -0.480126, 0.0584941}, {0.106772, -0.480126, 0.488794,
 0.0891792}, {-0.122561, 0.0584941, 0.0891792, 0.972868}}

In[79]:= littlep2hat = P2hat . c2hat

Out[79]= {{-0.284342}, {1.88619}, {-2.10367}, {-1.56361}}
```

```
In[80]:= theta2 = -Min[Transpose[littlep2hat]]

Out[80]= 2.10367

In[83]:= x2hat = {{1, 1, 1, 1}} + (.9/theta2) Transpose[littlep2hat]

Out[83]= {{0.878352, 1.80696, 0.1, 0.33105}}

In[84]:= x2trans = d2 . Transpose[x2hat]

Out[84]= {{4.86376}, {14.5925}, {1.22364}, {1.22489}}
```

We can see that after just two iterations, we obtain approximately our optimal solution at $(5, 15)$ as shown in Figure 3.

Figure 3: Two iterations of the interior point method quickly approach the optimum at $(5, 15)$.

□

Problem 3

Chapter 9 of the textbook describes several network optimization models. Each of them can be formulated as a linear program (but this is not done in the text).

(a) Formulate a shortest-path problem described in Section 9.1 (Seervada Park, first of the three problems) as a linear program.

(b) Formulate the minimum spanning tree problem (the second problem mentioned in Section 9.1) as a linear program.

Solution 3.
(a)
We can formulate the shortest-path problem as the following linear program.

$$\text{Minimize } z = \sum_{a \in \mathcal{A}} w_a x_a$$

where $\mathcal{A} = \{OA, OB, \ldots, ET\}$ is the set of arcs, x_a is a decision variable that equals 1 if a is in the path and 0 otherwise, and w_a is the weight corresponding to the arc a. We minimize z subject to the constraints

$$\begin{aligned} x_{OA} + x_{OB} + x_{OC} &= 1 \\ x_{DT} + x_{ET} &= 1 \\ \sum_{ik \in \mathcal{A}} x_{ik} - \sum_{kj \in \mathcal{A}} x_{kj} &= 0 \quad \text{for } k = A, B, C, D, E \\ x_a &\geq 0 \quad \forall a \in \mathcal{A}. \end{aligned}$$

The first constraint notes that there should be one arc from O. The second constraint notes that there should be one arc to T. The remaining functional constraints demand that the number of incoming arcs equals the number of outgoing arcs for any node besides O and T. Taken together, the above constraints are equivalent to requiring that the arcs a such that $x_a = 1$ form a path from O to T.

(b)
We can formulate the minimum spanning tree problem as the following linear program.

$$\text{Minimize } z = \sum_{a \in \mathcal{A}} w_a x_a$$

where $\mathcal{A} = \{OA, OB, \ldots, ET\}$ is the set of arcs, x_a is a decision variable that equals 1 if a is in the path and 0 otherwise, and w_a is the weight corresponding to the arc a. Define arcs $x_{OA} = x_1$, and so on where OA=AO=1, OB=2, OC=3, AB=4, BC=5, AD=6, BD=7, BE=8, CE=9, DE=10, DT=11, ET=12. We minimize z subject to the constraints:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_{11} + x_{12} = 1 \\ \sum x_i = 6 \text{ for all } i = 1, 2, \ldots, 12 \\ x_1 + x_4 + x_6 \geq 1 \\ x_2 + x_4 + x_5 + x_7 + x_8 \geq 1 \\ x_3 + x_5 + x_9 \geq 1 \\ x_6 + x_7 + x_{10} + x_{11} \geq 1 \\ x_8 + x_9 + x_{10} + x_{12} \geq 1 \\ x_i \leq 1 \text{ for all } i = 1, 2, \ldots, 12. \end{cases}$$

In general,to guarantee that we do not have any cycles in our spanning tree, we impose that any subset of k vertices $S \subseteq N$, there exists at most $k-1$ arcs between them. This is obviously not an optimal formulation, since you need to introduce $2^{|N|}$ constraints. However, for this problem we did not need to impose such constraints.

The first constraint guarantees that the set of arcs contains the correct number of arcs for a spanning tree. The remaining functional constraints give us that each node is adjacent to an arc. Since each node is adjacent to at least one of six (one less than the number of nodes) arcs, we have a spanning tree.

□

References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.