MATH 437: Homework 2

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Problem 1

This is your lucky day. You have just won a \$20,000 prize. You are setting aside \$8,000 for taxes and partying expenses, but you have decided to invest the other \$12,000. Upon hearing this news, two different friends have offered you an opportunity to become a partner in two different entrepreneurial ventures, one planned by each friend. In both cases, this investment would involve expending some of your time next summer and putting up cash. Becoming a full partner in the first friend's venture would require an investment of \$10,000 and 400 hours, and your estimated profit (ignoring the value of your time) would be \$9,000. The corresponding figures for the second friend's venture are \$8,000 and 500 hours, with an estimated profit to you of \$9,000. However, both friends are flexible and would allow you to come in at any fraction of a full partnership you would like. If you choose a fraction of a full partnership, all the above figures given for a full partnership (money investment, time investment, and your profit) would be multiplied by this same fraction. Because you were looking for an interesting summer job anyway (maximum of 600 hours), you have decided to participate in one or both friends' ventures in whichever combination would maximize your total estimated profit. You now need to solve the problem of finding the best combination.

- Describe the analogy between this problem and the Wyndor Glass Co. problem discussed in Sec. 3.1. Then construct and fill in a table like Table 3.1 for this problem, identifying both the activities and the resources.
- Formulate a linear programming model for this problem.
- Use the graphical method to solve this model. What is your total estimated profit?

(a) If you chose to reduce your partying budget by an additional \$900 (and so have \$12,900 to invest), what effect would this have on the maximal profit?

(b) If you chose to spend \$10,000 on taxes and partying expenses and had only \$10,000 for investing, what effect would it have on the maximal profit?

	x_1	x_2
Profit	\$9,000	\$9,000
Investment	\$10,000	\$8,000
Hours	400	500

Table 1.1: Relevant quantities for defining a linear program corresponding to Problem 1.

Solution 1. It is not hard to see how this problem is analogous to the Wyndor Glass Co. problem. Instead of two products, there are two investments and the constraints pertaining to plants producing x amount of windows or doors is instead split into money and hours invested in which respective venture. While the auxiliary information to each problem is distinct, we can nonetheless formulate a solution in exactly the same manner.

We need to begin by identifying the relevant variables for this problem, as seen in Table 1.1. Let x_1 and x_2 denote the amount of investment in ventures 1 and 2 respectively (note that because these are percentages of our involvement in each venture, so $x_1, x_2 \in [0, 1]$). Since we are guaranteed to gain 9,000\$ in each investment, then we want to maximize the objective function:

$$9000x_1 + 9000x_2$$

Venture 1 required 10,000$ and 400 hours, and venture 2 required 8,000$ and 500 hours. Now we wanted to spend a total of 600 hours working this Summer, so we can instead split this time in between the two ventures. So our constraints would be:

$$\begin{aligned} 400x_1 + 500x_2 &\leq 600 \\ 10000x_1 + 8000x_2 &\leq 12000 \\ 0 \leq x_1, x_2 &\leq 1 \end{aligned}$$

We can manipulate these constraints in terms of x_2 and plot these lines to identify the feasible region and therefore obtain our boundary points.

$$\begin{aligned} x_2 &\leq \frac{6}{5} - \frac{4x_1}{5} \\ x_2 &\leq \frac{3}{2} - \frac{5x_1}{4} \end{aligned}$$

Testing all boundary points shown in Figure 1, we find that our maximum $(x_1, x_2) = \frac{1}{3}(2, 2)$. which gives us a maximum profit of $12,000$$.

Interpretation: Investing $\frac{2}{3}$ in venture 1 equates to \$6,666.67 in cash and 266.67 hours, whereas investing $\frac{2}{3}$ into venture 2 means \$5,333.33 dollars and 333.33 hours. It is not hard to see that the sum of these quantities is exactly the equality in our constraints, and therefore signify binding constraints.

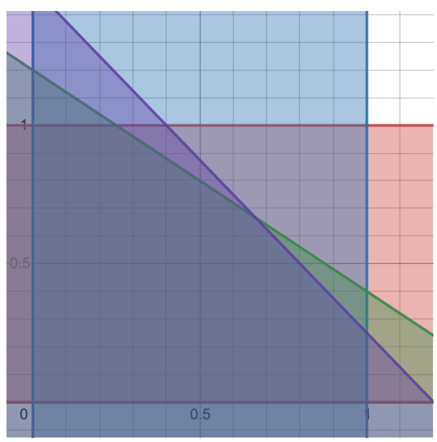

Figure 1: A graphical representation of the feasible region for the original conditions of Problem 1 gives boundary points at the origin, $(0, 1), (1, 0) \left(\frac{2}{3}, \frac{2}{3}\right), \left(\frac{1}{4}, 1\right)$ and $\left(1, \frac{1}{4}\right)$.

(a) Now we need to consider the case when reduce the partying budget and therefore increasing our amount of money we can invest. This only affects our one constraint into $10000x_1 + 8000x_2 \leq 12900$. So we have our new line is $x_2 \leq \frac{129}{80} - \frac{5x_1}{4}$. Our new graph feasible region can be seen in Figure 2. By testing all boundary points, we see that our objective function is maximized at $(x_1, x_2) \sim (.92, .47)$ and gives a maximum profit of $12,456$. Therefore, we could argue that spending the extra 900$ on the investment only returns around 450$ more, and therefore may not be the best recommendation. **Interpretation:** In this scenario we are

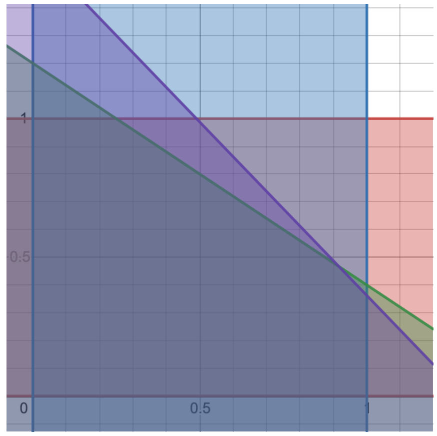

Figure 2: A graphical representation of the feasible region for the increased amount of investment for Problem 1 (a) gives boundary points at the origin, $(0,1),(1,0),\left(\frac{1}{4},1\right),(0.92,.47)$, and $(1,.36)$.

investing $.92 \times 10000 = \$9,200$ dollars and $.92 \times 400 = 368$ hours into venture 1. For venture 2, we are $.47 \times 8000 = \$3,760$ dollars and $\dot{4}7 \times 500$ hours. Noting that the values $(.92,.47)$ have been simplified to two significant digits, we can see that we reach equality for both our constraints and therefore these are binding constraints.

(b) Similarly to part (a), we only need to change the constraint $10000x_1 + 8000x_2 \leq 10000$. Simplifying x_2 once more, we obtain $x_2 \leq \frac{5}{4}(1-x_1)$. The new feasible region can be seen in Figure 3. Testing our new boundary points, we get a new maximum of \$10,800 at $(x_1,x_2) = (\frac{1}{5},1)$. **Interpretation:** In this scenario we are giving \$2,000 in cash and 80 hours to venture 1 and \$6,400 and 400 hours into venture 2. This signifies that we are not using the full amount of either money or cash that we set aside for investing this Summer, and therefore there are no binding constraints.

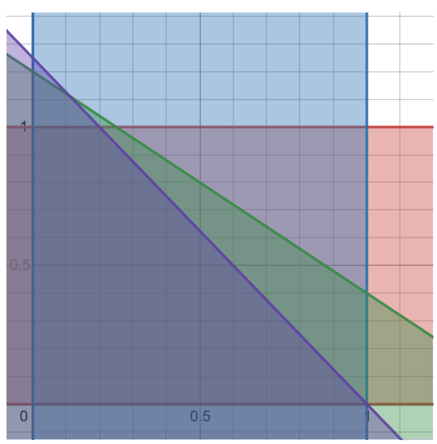

Figure 3: A graphical representation of the feasible region for the decreased amount of investment for Problem 1 (b) gives boundary points at the origin, $(0,1),(1,0)$ and $\left(\frac{1}{5},1\right)$.

Below is the `Sage` code that helped verify our results:

```
p=MixedIntegerLinearProgram(maximization=True)
v = p.new_variable(real=True, nonnegative=True)
x1, x2 = v['x1'], v['x2']
p.set_objective(9000*x1+9000*x2)
p.add_constraint(400*x1+500*x2<=600)
p.add_constraint(10000*x1+8000*x2<=10000)
p.add_constraint(x1>=0)
p.add_constraint(x2>=0)
p.add_constraint(x1<=1)
p.add_constraint(x2<=1)
p.show()
p.solve()
d=p.get_values(v)
for k in d:
    print(k,d[k])
print('Max=',p.get_objective_value())
```

Problem 2

A company has a contract to deliver 100, 200, 300, 500, and 400 units of some commodity over the next five weeks. Your goal is to determine the optimal hiring, firing, producing, and storing schedule to minimize the cost of the contract under the following conditions:

- Each unit not delivered on schedule incurs an \$80 penalty;
- Any production ahead of schedule requires storage at \$30/unit/week;
- All deliveries must be met by the end of the fifth week;
- Initially, there are 20 workers and 10 units on hand;
- Each worker used in production can make 8 units per week;
- Each worker who is training the recruits can train 5 new workers;
- Wages of every worker and recruit are \$300/week, when used in production, training, or when idle;
- The cost to fire a worker is \$300.

Please do the following:

(a) provide a careful description of all the variables and summarize information in a table form;

(b) formulate a suitable linear program;

(c) obtain a solution using `lp_solve` (or some other method);

(d) write a paragraph with an executive summary (your recommendation in non-mathematical terms).

Solution 2. (a) For simplicity, we assume that workers can only be fired at the beginning of each work week. To model the company's operations, we define the following non-negative decision variables:

1. $x_{i,1}$:= The number of units delivered on week i,
2. $x_{i,2}$:= The number of units on hand at the start of week i,
3. $x_{i,3}$:= The number of recruits trained during week i,
4. $x_{i,4}$:= The number of workers fired at the start of week i.
5. $\forall x_{i,j} \geq 0$

With our decision variables formulated, we have the table below.

(b) To formulate our objective function and constraints, we assume that we may hire, recruit, or fire fractional numbers of workers. We also assume that fractional recruits may be trained by fractional workers. Fractional workers may be explained by part-time positions. Since we have not yet started integer programming in this course, we can justify our formulation in a part-time sense.

Week n	Delivery Quota Q_n	Units Delivered	Units on hand
1	100	$x_{1,1}$	10
2	200	$x_{2,1}$	$x_{2,2}$
3	300	$x_{3,1}$	$x_{3,2}$
4	500	$x_{4,1}$	$x_{4,2}$
5	400	$x_{5,1}$	$x_{5,2}$
Week n	Workers	Recruits	Workers fired
1	$20 - x_{1,4}$	$x_{1,3}$	$x_{1,4}$
2	$20 + x_{1,3} - \sum_{i=1}^{2} x_{i,4}$	$x_{2,3}$	$x_{2,4}$
3	$20 + \sum_{i=1}^{2} x_{i,3} - \sum_{i=1}^{3} x_{i,4}$	$x_{3,3}$	$x_{3,4}$
4	$20 + \sum_{i=1}^{3} x_{i,3} - \sum_{i=1}^{4} x_{i,4}$	$x_{4,3}$	$x_{4,4}$
5	$20 + \sum_{i=1}^{4} x_{i,3} - \sum_{i=1}^{5} x_{i,4}$	$x_{5,3}$	$x_{5,4}$

We note that one worker producing $n < 8$ units or training $m < 5$ recruits incurs the same cost as $1 - n/8$ idle workers and $n/8$ workers each producing 8 units per week or $1 - m/5$ idle workers and $m/5$ workers each training 5 recruits respectively. So without loss of generality, we assume each worker produces 8 units per week or trains 5 recruits per week.

We now need to formulate an objective function and constraints. We start by considering the costs incurred on Week 1. Since the company must deliver $Q_1 = 100$ units and must pay a \$80 penalty for each unit not delivered, there is a $\$80(100 - x_{1,1})$ fee. The company must also pay a $\$30x_{2,2}$ storage fee. Since it costs a \$300 to pay, recruit, or fire a worker and since a fired worker does not receive a \$300 wage, the total cost of workers, recruits, and firing for week 1 is

$$\$300\left[(20 - x_{1,4}) + x_{1,3} + x_{1,4}\right] = \$300(20 + x_{1,3}).$$

Therefore, the total cost incurred on Week 1 is

$$Z_1 := 80(100 - x_{1,1}) + 30x_{2,2} + 300(20 + x_{1,3}).$$

Next, we consider constraints for the variables involved in the costs incurred on Week 1. Our first constraint regarding production is as follows:

$$\text{Workers training recruits} + \text{Workers producing units} + \text{Idle workers} = \text{Total workers}.$$

Equivalently, since all numbers of workers are non-negative,

$$\text{Workers training recruits} + \text{Workers producing units} \leq \text{Total workers}.$$

We initially have 20 workers and we fire $x_{1,4}$, so we have $20 - x_{1,4}$ total workers during week 1. Each of the $20 - x_{1,4}$ workers can produce 8 units or train 5 recruits. The total number of units produced equals the $x_{1,1}$ units delivered plus the units sent to storage. The number of units sent to storage equals the number of units on hand at the start of Week 2 minus the number of units on hand at the start of Week 1. So we have the following inequality:

$$\frac{x_{1,3}}{5} + \frac{x_{1,1} + x_{2,2} - 10}{8} \leq 20 - x_{1,4}.$$

We cannot deliver more than 100 units the first week. Each variable and the number of workers must be non-negative, so we have the following constraints for Week 1:

$$\begin{aligned} 5x_{1,1} + 8x_{1,3} + 40x_{1,4} + 5x_{2,2} &\leq 850 \\ x_{1,1} &\leq 100 \\ x_{1,4} &\leq 20 \\ \text{and } x_{1,1},\ x_{1,3},\ x_{1,4},\ x_{2,2} &\geq 0. \end{aligned}$$

In general, the costs incurred on Week n is 80 times the number of units undelivered plus 30 times the number of units stored plus 300 times the number of people working, recruited, or fired. Symbolically,

$$Z_n := 80(Q_n - x_{n,1}) + 30x_{n+1,2} + 300\left(20 + \sum_{i=1}^{n-1}(x_{i,3} - x_{i,4}) + x_{n,3}\right).$$

For each week, we must provide a constraint equivalent to the following:

$$\text{Workers training recruits} + \text{Workers producing units} + \text{Idle workers} = \text{Total workers}.$$

We note here that the number of idle workers acts as a non-negative slack variable. However, it does not seem necessary to introduce a new quantity if we do not absolutely need to, especially given how it will not influence the final optimal value of the program. Therefore, we do not include a term for idle workers and require the inequality:

$$\text{Workers training recruits} + \text{Workers producing units} \leq \text{Total workers}.$$

Since each worker can only train 5 recruits or produce 8 units, we may take values from our table to obtain the following inequality:

$$\frac{x_{n,3}}{5} + \frac{x_{n,1} + x_{n+1,2} - x_{n,2}}{8} \leq 20 + \sum_{k=1}^{n-1} x_{k,3} - \sum_{k=1}^{n} x_{k,4}$$

where $x_{6,i} = 0$ for all $i = 1, \ldots n$.

To formulate a suitable linear system, we will now define the objective function to be the total cost over Weeks 1 through 5 and collect our constraints.

Objective function:

$$\begin{aligned} Z := \sum_{n=1}^{5} Z_n = 150000 - 80x_{1,1} + 1500x_{1,3} - 1200x_{1,4} \\ -80x_{2,1} + 30x_{2,2} + 1200x_{2,3} - 900x_{2,4} \\ -80x_{3,1} + 30x_{3,2} + 900x_{3,3} - 600x_{3,4} \\ -80x_{4,1} + 30x_{4,2} + 600x_{4,3} - 300x_{4,4} \\ -80x_{5,1} + 30x_{5,2} + 300x_{5,3} \end{aligned}$$

We must consider that all our variables must be non-negative and that all deliveries must be completed by the end of the 5 weeks to find the constraints. We must also consider that deliveries can not be made early.

$$
\begin{aligned}
5x_{1,1} + 8x_{1,3} + 40x_{1,4} + 5x_{2,2} &\leq 850 \\
-40x_{1,3} + 40x_{1,4} + 5x_{2,1} - 5x_{2,2} + 8x_{2,3} + 40x_{2,4} + 5x_{3,2} &\leq 800 \\
-40x_{1,3} + 40x_{1,4} - 40x_{2,3} + 40x_{2,4} + 5x_{3,1} - 5x_{3,2} + 8x_{3,3} + 40x_{3,4} + 5x_{4,2} &\leq 800 \\
-40x_{1,3} + 40x_{1,4} - 40x_{2,3} + 40x_{2,4} & \\
-40x_{3,3} + 40x_{3,4} + 5x_{4,1} - 5x_{4,2} + 8x_{4,3} + 40x_{4,4} + 5x_{5,2} &\leq 800 \\
-40x_{1,3} + 40x_{1,4} - 40x_{2,3} + 40x_{2,4} - 40x_{3,3} & \\
+40x_{3,4} - 40x_{4,3} + 40x_{4,4} + 5x_{5,1} - 5x_{5,2} + 8x_{5,3} + 40x_{5,4} &\leq 800
\end{aligned}
$$

Since it does not make sense to fire more workers than we currently have, we need to place a constraint on the workers we can fire each week by taking into account the workers hired, trained, and fired over the previous weeks.

$$
\begin{aligned}
x_{1,4} &\leq 20 \\
x_{1,4} + x_{2,4} - x_{1,3} &\leq 20 \\
x_{1,4} + x_{2,4} + x_{3,4} - x_{1,3} - x_{2,3} &\leq 20 \\
x_{1,4} + x_{2,4} + x_{3,4} + x_{4,4} - x_{1,3} - x_{2,3} - x_{3,3} &\leq 20 \\
x_{1,4} + x_{2,4} + x_{3,4} + x_{4,4} + x_{5,4} - x_{1,3} - x_{2,3} - x_{3,3} - x_{4,3} &\leq 20
\end{aligned}
$$

We cannot 'over deliver' units for the weeks that have elapsed, so $\sum_{i=1}^{n} x_{i,1} \leq \sum_{i=1}^{n} Q_n$ for $n = 1, 2, \ldots, 4$. And all deliveries must be complete by the end of week 5.

$$
\begin{aligned}
x_{1,1} &\leq 100 \\
x_{1,1} + x_{2,1} &\leq 300 \\
x_{1,1} + x_{2,1} + x_{3,1} &\leq 600 \\
x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} &\leq 1100 \\
x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} + x_{5,1} &= 1500 \\
\text{and } x_{i,j} &\geq 0 \, \forall i, j = 1, \ldots, 5.
\end{aligned}
$$

(c) From `lp_solve`, we find the following tables and a minimal cost of \$71,007 for the objective function.

$x_{i,j}$	$x_{i,1}$	$x_{i,2}$	$x_{i,3}$	$x_{i,4}$
$x_{1,j}$	100	10	0.375	0
$x_{2,j}$	200	69.4	20.25	0
$x_{3,j}$	300	0	15.625	0
$x_{4,j}$	450	0	0	0
$x_{5,j}$	450	0	0	0

(d) For the company to deliver the total 1,500 units required over weeks 1 through 5 at minimal cost, we recommend the company hires 36 workers during the weeks 2 and 3 according to the

Week n	Delivery Quota Q_n	Units Delivered	Units on hand
1	100	100	10
2	200	200	69.4
3	300	300	0
4	500	450	0
5	400	450	0
Week n	Workers	Recruits	Workers fired
1	20	0.375	0
2	20.375	20.25	0
3	40.625	15.625	0
4	56.25	0	0
5	56.25	0	0

following plan. Allowing for fractional employment and units of product in storage, the optimal plan is to produce 159.4 units, deliver the required 100 units, send the 59.4 units to storage, and train 0.375 recruits on Week 1. On Week 2, the optimal strategy is to train 20.25 workers and deliver all 200 required units on time, delivering the 69.4 units still in storage. The company should have 40.625 workers at the beginning of Week 3. Of the 40.625 workers, 37.5 workers should produce 300 units to be delivered and the remaining 3.125 should train 15.625 workers. By Week 4, there are sufficiently many workers to produce 450 units a week. To minimize costs, the company should not hire any more workers beyond this point. The company will have to pay $\$4,000$ in penalties for late deliveries during Week 4. The overall total cost of hiring, firing, storage, and late penalties will be $\$71,007$. If we may only hire full-time workers and produce whole units, then the optimal strategy may be very different.

Problem 3

Consider the following program:

$$
\begin{array}{lrcrcr}
\text{Minimize} & -2x_1 & + & x_2 & = & z \\
\text{subject to} & -x_1 & + & x_2 & \leq & 2 \\
 & 2x_1 & + & 3x_2 & \leq & 21 \\
 & 2x_1 & - & x_2 & \leq & 9 \\
 & x_1 & & & \geq & 2 \\
 & & & x_2 & \geq & 1
\end{array}
$$

Please solve the problem graphically and determine:

(a) What are the binding constraints;

(b) For the binding constraints, determine the ranges for the right-hand side coefficients such that the constraints remain binding if the coefficients are changed one at a time;

(c) Determine the ranges for the cost coefficients (the coefficients in the objective functions) such that the optimal solution does not change if we vary the coefficients one at a time;

(d) Obtain a solution using `lp_solve` and write a short explanation where the program reports the values you found above.

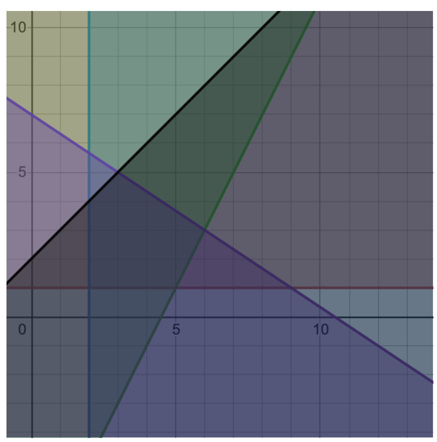

Figure 4: Graphical representation of Problem 3 showing the feasible region with boundary points at $(2,1), (2,4), (3,5), (5,1)$ and $(6,3)$.

Solution 3. (a) A binding constraint is one such that the constraint holds with equality at the optimal solution [1]. Therefore, looking at our constraints and knowing our optimal solution $(x_1, x_2) = (5,1)$, then we can see that $2x_1 - x_2 = 9$ and $x_2 = 1$ are binding constraints. But we also see that (6,3) is an optimal solution just by checking the values given by the objective function at each of the 5 boundary points. It is clear that any point on the line $2x_1 - x_2 = 9$ gives an optimal solution; and so we can disregard $x_2 = 1$ as a binding constraint.

(b) The right coefficient in the constraint $2x_1 - x_2 = c$ remains binding if we vary $c \in [3, 17]$. Any point past 17 is not within the feasible region, and $c = 3$ (i.e. the point (2,1)) gives the last point such that this constraint holds with equality.

(c) The most intuitive way to explain this concept is to remember that our objective function $-2x_1+x_2 = z$ is indeed a plane, and the coefficients of both x_1 and x_2 determine the gradient of the plane. Therefore, our minimal solution within the feasible region will be where the level curves point *away* from and vice versa for a maximal solution. The cost coefficient of x_1 in the objective function may vary within the range $[-2, 0)$ without the optimal solution changing. The cost coefficient of x_2 in the objective function may vary within the range $[1, \infty)$ without the optimal solution changing.

(d) The following is the linear program written and solved in `Sage`:

```
p = MixedIntegerLinearProgram(maximization=False)
v = p.new_variable(real=True, nonnegative=True)
x1, x2 = v['x1'], v['x2']
p.set_objective(-2*x1+x2)
p.add_constraint(-x1+x2<=2)
p.add_constraint(2*x1+3*x2<=21)
p.add_constraint(2*x1-x2<=9)
p.add_constraint(x1>=2)
p.add_constraint(x2>=1)
p.show()
p.solve()
d = p.get_values(v)
for k in d:
    print(k,d[k])
print('Min=',p.get_objective_value())
```

Which gives output:

```
Minimization:
  -2.0 x_0 + x_1

Constraints:
  - x_0 + x_1 <= 2.0
  2.0 x_0 + 3.0 x_1 <= 21.0
  2.0 x_0 - x_1 <= 9.0
  - x_0 <= -2.0
  - x_1 <= -1.0
Variables:
  x_0 is a continuous variable (min=0.0, max=+oo)
  x_1 is a continuous variable (min=0.0, max=+oo)
x0 5.0
x1 1.0
```

Problem 4

Using `lp_solve` (or some other method), please solve Problem 3.4-7. You solution should include a careful formulation of the linear programming model; the answer given by software; and practical, non-technical interpretation of the answer.

Web Mercantile sells many household products through an online catalog. The company needs substantial warehouse space for storing its goods. Plans now are being made for leasing warehouse storage space over the next 5 months. Just how much space will be required in each of these months is known. However, since these space requirements are quite different, it may be most economical to lease only the amount needed each month on a month-by-month basis. On the other hand, the additional cost for leasing space for additional months is much less than for the first month, so it may be less expensive to lease the maximum amount needed for the entire 5 months. Another option is the intermediate approach of changing the total amount of space leased (by adding a new lease and/or having an old lease expire) at least once but not every month. The space requirement and the leasing costs for the various leasing periods are as follows:

Month	Required Space (Sq.Ft.)	Leasing Period (Months)	Cost per Sq.Ft. Leased
1	30,000	1	65
2	20,000	2	100
3	40,000	3	135
4	10,000	4	160
5	50,000	5	190

The objective is to minimize the total leasing cost for meeting the space requirements. In addition, please answer the following questions:

(I) During what months will the leased warehouse space be completely full? During the months when the leased space is not completely full, what is the excess capacity?

(II) Explain the connection between the questions above and the concept of a "binding constraint".

(III) Suppose that the minimal space requirements are negotiable. You can lower the minimal space requirement for any of the 5 months at the cost of \$67 per square foot. So for example, you could pay \$67,000 and have the minimal space requirement for month 1 be 29,000 square feet instead of 30,000. For which of the months does it make sense to do it? How much can you save?

Let's formulate the relevant quantities that we need for our formulation. Decision variables $x_{i,j}$ is the square footage leased on month i leased for j months, with $i \in [1,5]$ and $j \in [1, 6-i]$. So our objective function to minimize is:

$$65\left(\sum_{i=1}^{5} x_{i,1}\right) + 100\left(\sum_{i=1}^{4} x_{i,2}\right) + 135\left(\sum_{i=1}^{3} x_{i,3}\right) + 160\left(\sum_{i=1}^{2} x_{i,4}\right) + 190x_{1,5}$$

Let's think about the space constraints that we must adhere to.

$$\sum_{j=1}^{5} x_{1,j} \geq 30000$$
$$\sum_{j=2}^{5} x_{1,j} + \sum_{j=1}^{4} x_{2,j} \geq 20000$$
$$\sum_{j=3}^{5} x_{1,j} + \sum_{j=2}^{4} x_{2,j} + \sum_{j=1}^{3} x_{3,j} \geq 40000$$
$$\sum_{j=4}^{5} x_{1,j} + \sum_{j=3}^{4} x_{2,j} + \sum_{j=2}^{3} x_{3,j} + \sum_{j=1}^{2} x_{4,j} \geq 10000$$
$$x_{1,5} + x_{2,4} + x_{3,3} + x_{4,2} + x_{5,1} \geq 50000$$

Our linear program written in Sage:

```
p = MixedIntegerLinearProgram(maximization=False)
v = p.new_variable(real=True, nonnegative=True)
x11,x12,x13,x14,x15,x21,x22,x23,x24,x31,x32,x33,x41,x42,x51 = v['x11'], v['x12'],
    v['x13'], v['x14'],v['x15'], v['x21'],v['x22'], v['x23'],v['x24'], v['x31'],v[
    'x32'], v['x33'],v['x41'], v['x42'],v['x51']
p.set_objective(65*(x11+x21+x31+x41+x51)+100*(x12+x22+x32+x42)+135*(x13+x23+x33)
    +160*(x14+x24)+190*(x15))
p.add_constraint(x11+x12+x13+x14+x15>=30000)
p.add_constraint(x12+x13+x14+x15+x21+x22+x23+x24>=20000)
p.add_constraint(x13+x14+x15+x22+x23+x24+x31+x32+x33>=40000)
p.add_constraint(x14+x15+x23+x24+x32+x33+x41+x42>=10000)
p.add_constraint(x15+x24+x33+x42+x51>=50000)
p.show()
p.solve()
d = p.get_values(v)
for k in d:
    print(k,d[k])
p.get_values()
print('Min=',p.get_objective_value())
```

Which gave output:

```
Minimization:
  65.0 x_0 + 100.0 x_1 + 135.0 x_2 + 160.0 x_3 + 190.0 x_4 + 65.0 x_5 +
  100.0 x_6 + 135.0 x_7 + 160.0 x_8 + 65.0 x_9 + 100.0 x_10 + 135.0 x_11 +
  65.0 x_12 + 100.0 x_13 + 65.0 x_14

Constraints:
  - x_0 - x_1 - x_2 - x_3 - x_4 <= -30000.0
  - x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7 - x_8 <= -20000.0
```

```
  - x_2 - x_3 - x_4 - x_6 - x_7 - x_8 - x_9 - x_10 - x_11 <= -40000.0
  - x_3 - x_4 - x_7 - x_8 - x_10 - x_11 - x_12 - x_13 <= -10000.0
  - x_4 - x_8 - x_11 - x_13 - x_14 <= -50000.0
Variables:
  x_0 is a continuous variable (min=0.0, max=+oo)
  x_1 is a continuous variable (min=0.0, max=+oo)
  x_2 is a continuous variable (min=0.0, max=+oo)
  x_3 is a continuous variable (min=0.0, max=+oo)
  x_4 is a continuous variable (min=0.0, max=+oo)
  x_5 is a continuous variable (min=0.0, max=+oo)
  x_6 is a continuous variable (min=0.0, max=+oo)
  x_7 is a continuous variable (min=0.0, max=+oo)
  x_8 is a continuous variable (min=0.0, max=+oo)
  x_9 is a continuous variable (min=0.0, max=+oo)
  x_10 is a continuous variable (min=0.0, max=+oo)
  x_11 is a continuous variable (min=0.0, max=+oo)
  x_12 is a continuous variable (min=0.0, max=+oo)
  x_13 is a continuous variable (min=0.0, max=+oo)
  x_14 is a continuous variable (min=0.0, max=+oo)
x11 0.0
x12 0.0
x13 0.0
x14 0.0
x15 30000.0
x21 0.0
x22 0.0
x23 0.0
x24 0.0
x31 10000.0
x32 0.0
x33 0.0
x41 0.0
x42 0.0
x51 20000.0
Min= 7650000.0
```

So we need a 5 month lease in month 1 for 30,000 sq.ft., a 1 month lease in month 3 for 10,000 sq.ft, and a 1 month lease in month 5 for 20,000 sq.ft. for a total minimized cost of 7,650,000$.

Solution 4. (I) To begin answering this, it is useful to just make a table of our current capacity given by the linear program solutions and the minimum space required for that month as seen in Table 4.1.

Month	Required Space (Sq.Ft.)	Leased Space Available	Unused Space
1	30,000	30,000	0
2	20,000	30,000	10,000
3	40,000	40,000	0
4	10,000	30,000	20,000
5	50,000	50,000	0

Table 4.1: Required space, leased space, and unused space for each month for our optimal solution given by a linear program written in `Sage`.

We see that only in months 2 and 4 do we have any extra space, whereas every other month fills our available space to maximum capacity.

(II) A binding constraint is one such that the constraint holds with equality at the optimal solution [1]. Therefore, our binding constraints are exactly those months when we are utilizing 100% of the space that we leased; i.e. months 1, 3, and 5.

(III) Suppose we could reduce the minimal space requirement for any of the five months at the cost of \$67 per square foot. Then reducing the minimal space requirement is not worth the cost for any of the five months.

We can verify that reducing requirements is not worth the cost by adding five non-negative variables $y_k, k \in [5]$ to the objective function with coefficients of 67 and adding one of the five new variables to the left side of each of the five constraints. So we would have:

$$65\left(\sum_{i=1}^{5} x_{i,1}\right) + 100\left(\sum_{i=1}^{4} x_{i,2}\right) + 135\left(\sum_{i=1}^{3} x_{i,3}\right) + 160\left(\sum_{i=1}^{2} x_{i,4}\right) + 190x_{1,5} + 67\left(\sum_{k=1}^{5} y_k\right)$$

With constraints:

$$\begin{aligned}
\sum_{j=1}^{5} x_{1,j} + y_1 &\geq 30000\\
\sum_{j=2}^{5} x_{1,j} + \sum_{j=1}^{4} x_{2,j} + y_2 &\geq 20000\\
\sum_{j=3}^{5} x_{1,j} + \sum_{j=2}^{4} x_{2,j} + \sum_{j=1}^{3} x_{3,j} + y_3 &\geq 40000\\
\sum_{j=4}^{5} x_{1,j} + \sum_{j=3}^{4} x_{2,j} + \sum_{j=2}^{3} x_{3,j} + \sum_{j=1}^{2} x_{4,j} + y_4 &\geq 10000\\
x_{1,5} + x_{2,4} + x_{3,3} + x_{4,2} + x_{5,1} + y_5 &\geq 50000
\end{aligned}$$

The optimal solution sets each of the newly introduced variables to 0.

References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.