MATH 437: Homework 5

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Based on Problems 1, 2, 3 and 4 we make the following conjecture.

Conjecture 1. There exists 4 distinct cases that we may encounter between the primal and the dual LP.

1. If a primal LP has a feasible region but an unbounded objective function, then there exists no feasible solution for the dual problem.
2. If there is no feasible region for the primal problem, then the dual problem is unbounded.
3. If both problems are feasible and bounded, then an optimal solution exists for both.
4. If both problem are neither feasible nor bounded, no optimal solution exists for either.

Problem 1

Construct and graph a primal problem with two decision variables and two functional constraints that has feasible solutions and an unbounded objective function. Then construct the dual problem and demonstrate graphically that it has no feasible solutions.

Solution 1. In class, we had an example of an LP with an unbounded feasible region.

$$\begin{array}{lrcrcl} \text{Minimize} & x_1 & + & 2x_2 & & \\ \text{subject to} & |x_1 & - & x_2| & \leq & 1 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

which was equivalent to:

$$\begin{array}{lrrcl} \text{Minimize} & x_1 & +\,2x_2 & & \\ \text{subject to} & x_1 & -\,x_2 & \leq & 1 \\ & -x_1 & +\,x_2 & \leq & 1 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

Therefore, we just need to revise our objective function into one that is unbounded; say $z = x_1 + x_2$, since its level curves would run parallel to our the constraints defining our feasible region. Then, our tableau looks like:

$$\left[\begin{array}{c|cc|c} z & x_1 & x_2 & b \\ \hline -1 & 1 & 1 & 0 \\ \hline 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

and a graph with our feasible region can be seen in Figure 1.

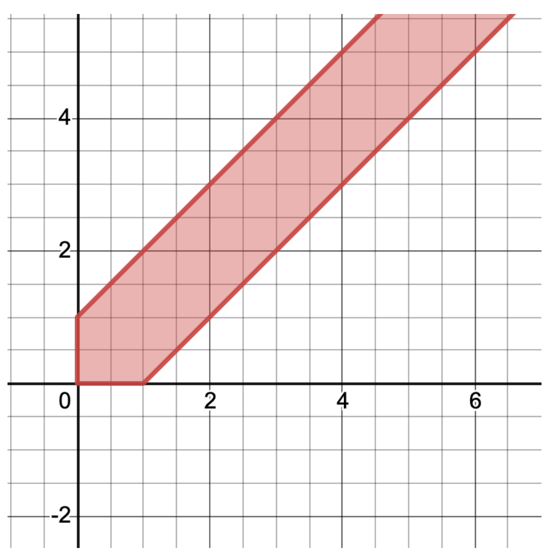

Figure 1: A graphical representation of our feasible region of our primal LP for Problem 1.

Then to construct our dual problem, we take A^T, b^T, c^T, and switch b and c arriving at:

$$\left[\begin{array}{c|cc|c} z & x_1 & x_2 & b \\ \hline -1 & 1 & 1 & 0 \\ \hline 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

which happens to be exactly the same tableau. However, the key difference is our inequalities in our constraints are now 'flipped'. So our dual LP resembles:

$$\begin{array}{lrrcl} \text{Maximize} & x_1 & +\,x_2 & & \\ \text{subject to} & x_1 & -\,x_2 & \geq & 1 \\ & -x_1 & +\,x_2 & \geq & 1 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

with a graphical representation given by Figure 2.

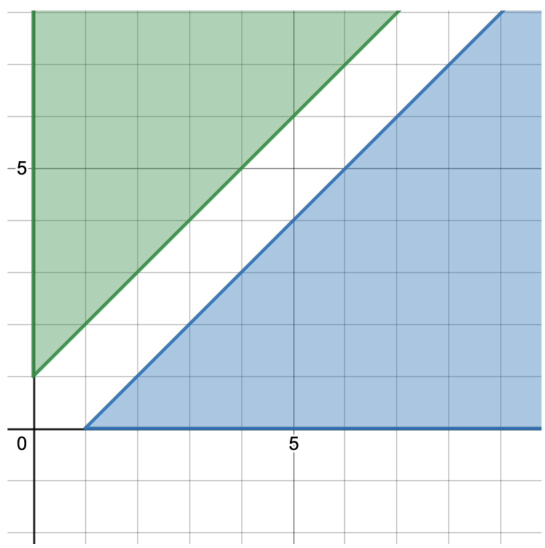

Figure 2: A graphical representation for the dual LP for Problem 1.

It is quite clear to see that our feasible regions defined by the constraints have an empty intersection, and so there exists no feasible solution in the dual.

$\square$

Problem 2

Construct a pair of primal and dual problems, each with two decision variables and two functional constraints, such that both problems have no feasible solutions. Demonstrate this property graphically.

Solution 2. Consider the following primal problem:

$$\begin{array}{lrrcl} \text{Maximize} & x_1 & & & \\ \text{subject to} & x_1 & -\, x_2 & \leq & 0 \\ & -x_1 & +\, x_2 & \leq & 1 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

We can see a graphical representation of our feasible region for the primal LP in Figure 3.

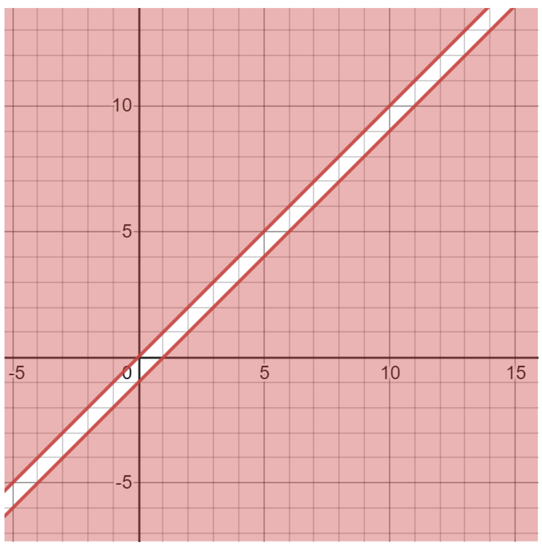

Figure 3: A graphical representation of our feasible region of our primal LP for Problem 2.

Therefore, the dual of our primal is then:

$$\begin{array}{lrrcl} \text{Minimize} & -x_2 & & & \\ \text{subject to} & x_1 & -\, x_2 & \geq & 1 \\ & -x_1 & +\, x_2 & \geq & 0 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

With a graphical representation of the feasible region given by Figure 4.

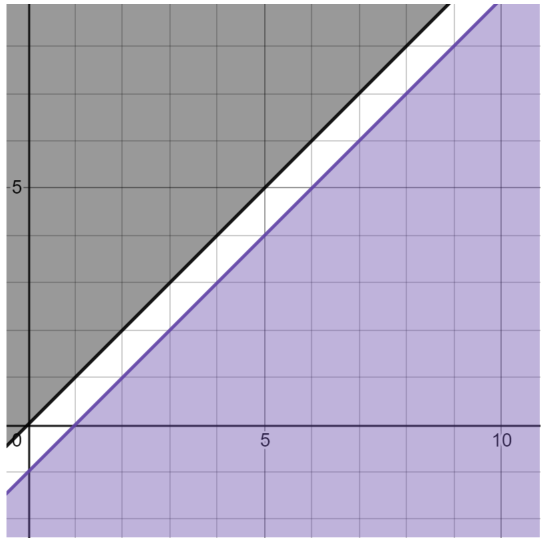

Figure 4: A graphical representation of our feasible region of our dual LP for Problem 2.

It is clear graphically that neither the primal nor the dual problem have a feasibility region.

□

Problem 3

Construct a pair of primal and dual problems, each with two decision variables and two functional constraints, such that the primal problem has no feasible solutions and the dual problem has an unbounded objective function.

Solution 3. For this problem, we can quite simply 'take the converse' of what we did in Solution 1. Therefore, our primal problem can be formulated as:

$$\begin{array}{rrrcl} \text{Maximize} & x_1 & +\,x_2 & & \\ \text{subject to} & x_1 & -\,x_2 & \geq & 1 \\ & -x_1 & +\,x_2 & \geq & 1 \\ & & x_1, x_2 & \geq & 0 \end{array}$$

and in tableau form:

$$\left[\begin{array}{c|cc|c} z & x_1 & x_2 & b \\ \hline -1 & 1 & 1 & 0 \\ \hline 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

which can be represented graphically by Figure 5.

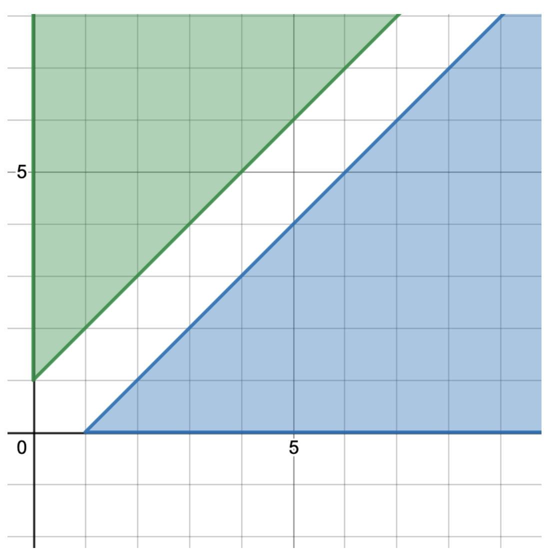

Figure 5: A graphical representation for the primal LP for Problem 3.

Our feasible regions are disjoint, and so there exists no feasible solution. Now, to construct our dual problem, we take A^T, b^T, c^T, and switch b and c arriving at the same exact tableau. However, the inequalities in the constraints are now reversed, and so we have a graphical representation seen in Figure 6.

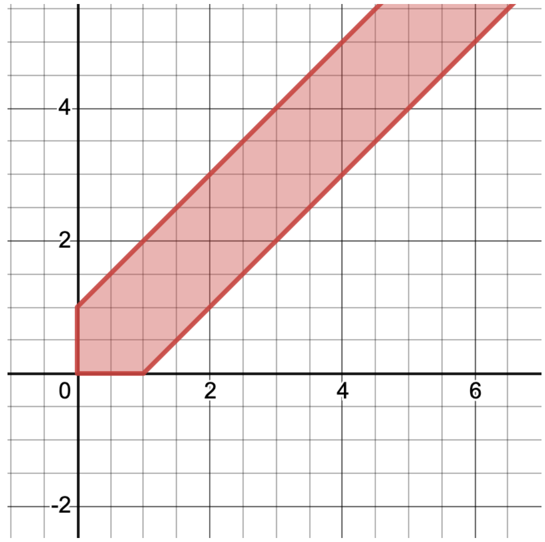

Figure 6: A graphical representation of our feasible region of our dual LP for Problem 3.

Therefore, it should be clear that we now have a feasible region but since our objective function is to minimize $x_1 + x_2$ then our objective function is unbounded.

$\square$

Problem 4

Use the weak duality property to prove that if both the primal and the dual problem have feasible solutions, then both must have an optimal solution.

Theorem 1 (Weak Duality). *If x is a feasible solution for the primal problem, and π is a feasible solution to the dual problem, then:*

$$c^T x \leq b^T \pi$$

Solution 4. Let x^* be a feasible solution for the primal problem and π^* be a feasible solution for the dual problem. Using weak duality, we have that $c^T x^* \leq b^T \pi^*$. Then we have that $c^T x \leq b^T \pi^*$ and $c^T x^* \leq b^T \pi$ for any feasible solutions x and π for the primal and dual problems respectively. This means that because both the primal and dual problem are feasible (since they are bounded in the previous two inequalities) and the objective function is bounded, it must have an optimal solution. Indeed, if strong duality holds then we have that optimal solutions are

$$c^T x = b^T \pi$$

$\square$

References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.