# MATH 437: Homework 6

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## Problem 1

Problems 6.1-8, 6.1-9, and 6.1-10 suggest a general statement about dual and primal LP problems. We discussed drafts of these statements in class. Please state and prove the general statement in your solution to this problem.

**Solution 1.** Based on Problems 6.1-8, 6.1-9, and 6.1-10, we made the following conjecture.

**Conjecture 1.** There exists 4 distinct cases that we may encounter between the primal and the dual LP.

1. If a primal LP has a feasible region but an unbounded objective function, then there exists no feasible solution for the dual problem.

2. If there is no feasible region for the primal problem, then the dual problem is unbounded or infeasible.

3. If both problems are either infeasible or unbounded, no optimal solution exists for either.

*Proof:*

By the weak duality theorem, if a primal (dual) LP problem has a feasible solution $x$, then the dual (primal) LP problem has an objective function $b^T\pi$ bounded below by $c^Tx$. Therefore, by the contrapositive of the previous statement, if the primal (dual) problem is unbounded then the dual (primal) problem has no feasible solution. This shows claim (1).

By the contrapositive of the strong duality theorem, if a primal (dual) LP has no optimal solution, then the dual (primal) LP has no optimal solution. Therefore, if a primal (dual) LP is infeasible, then the dual (primal) LP is either infeasible or unbounded. This shows claim (2).

Claim (3) holds by the definitions of infeasibility and unboundedness.

Problem 6.1-8 illustrated that a feasible and unbounded primal problem yielded an infeasible dual problem. Problem 6.1-9 showed that both problems were infeasible. And problem 6.1-10 showed that an infeasible primal problem gave a dual problem that was unbounded. 6.1-8 and 6.1-10 are consequences of Weak Duality. That is, if the primal (dual) is unbounded, then the dual (primal) is infeasible.

□

## Problem 2

We proved half of the Strong Duality theorem in class. Please prove the remaining half. That is, please prove that if the dual problem has a finite optimal solution, then so does the primal problem and the optimal value of the primal problem is equal to the optimal value of the dual problem.

**Solution 2.** Consider the dual problem;

$$\begin{aligned} \text{Minimize} \quad & W = b^T\pi \\ \text{subject to} \quad & A^T\pi \geq c \\ & \pi \geq 0 \end{aligned}$$

We then add the slack variables to obtain strict equality:

$$\begin{aligned} \text{Minimize} \quad & W = b^T\pi + 0\pi_s \\ \text{subject to} \quad & A^T\pi - \mathbb{I}\ \pi_s = c \\ & \pi, \pi_s \geq 0 \end{aligned}$$

After applying the revised simplex algorithm, we get an optimal solution $(\pi^*, \pi_s^*)$ with row multipliers $x^*$. Then, the cost coefficients are:

$$\left(\bar{b}^T, \bar{b}_s^T\right) = \left(b^T, 0^T\right) - \left((x^*)^T A^T, -\mathbb{I}(x^*)^T\right) = \left(b^T - (x^*)^T A^T, (x^*)^T\right).$$

Since $(\pi^*, \pi_s^*)$ is optimal, and the dual is a minimizing objective function, the costs $(\bar{b}^T, \bar{b}_s^T) \geq 0$. So,

$$b^T - (x^*)^T A^T \geq 0 \text{ and } (x^*)^T \geq 0,$$

and after rewriting and transposing, we have

$$Ax^* \leq b \text{ and } x^* \geq 0.$$

Thus, the row multipliers $x^*$ are feasible for the primal problem.
We now show that $c^T x^* = b^T\pi^*$ by looking at the revised simplex algorithm again. In the optimal solution to the dual problem, $W^* = c^T x^*$, and we also have that $W^* = b^T\pi^*$, so we get

$$c^T x^* = b^T\pi^*.$$

Therefore by optimality conditions, $x^*$ is optimal.

□

## Problem 3

Let $A$ be an $m \times n$ matrix ($m$ rows), $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Please produce a dual problem to the problem

$$\begin{aligned} \text{Minimize} \quad & W = b^T \pi \\ \text{subject to} \quad & A^T \pi \geq c \\ & \pi \geq 0 \end{aligned}$$

using the process that was described in class. Use your work to show that the dual of the dual problem is the primal problem.

**Solution 3.** The dual to the problem above will be a maximization problem. We 'flip' the inequality for the functional constraints because $\pi \geq 0$ in the original problem and we switch from minimization to maximization. Since $(A^T)^T = A$ and $(b^T)^T = b$, the new dual problem is

$$\begin{aligned} \text{Maximize} \quad & V = c^T y \\ \text{subject to} \quad & Ay \leq b \\ & y \geq 0 \end{aligned}$$

.

To show that the dual of the dual is the primal, consider our initial primal problem:

$$\begin{aligned} \text{Minimize} \quad & W = b^T \pi \\ \text{subject to} \quad & A^T \pi \geq c \\ & \pi \geq 0 \end{aligned}$$

Where the dual is:

$$\begin{aligned} \text{Maximize} \quad & V = c^T y \\ \text{subject to} \quad & Ay \leq b \\ & y \geq 0 \end{aligned}$$

If we rewrite our dual problem, then we have:

$$\begin{aligned} \text{Minimize} \quad & -V = (-c)^T y \\ \text{subject to} \quad & (-A)^T y \geq -b \\ & y \geq 0 \end{aligned}$$

The dual of our (rewritten) dual is then:

$$\begin{aligned} \text{Maximize} \quad & \bar{V} = (-b)^T \pi \\ \text{subject to} \quad & ((-A)^T)^T \pi \leq -c \\ & \pi \geq 0 \end{aligned}$$

which is exactly our initial primal problem since $((-A)^T)^T \pi \leq -c \equiv A^T \pi \geq c$. Therefore, the dual of the dual is the primal.

$\square$

## Problem 4

Please solve problem 6.1-11. Use the weak duality property to prove that if both the primal and the dual problem have feasible solutions, then both must have an optimal solution.

**Solution 4.**

**Theorem 1** (Weak Duality)**.** *If $x$ is a feasible solution for the primal problem, and $\pi$ is a feasible solution to the dual problem, then:*

$$c^T x \leq b^T \pi$$

First, define our primal and dual problems to start:

$$\begin{array}{llcrr} \text{Max} & Z = c^T x & & \text{Minimize} & W = b^T \pi \\ \text{subject to} & Ax \leq b & \text{and} & \text{subject to} & A^T \pi \geq c \\ & x \geq 0 & & & \pi \geq 0 \end{array}$$

Let $x^*$ be a feasible solution for the primal problem and $\pi^*$ be a feasible solution for the dual problem. Using weak duality, we have that $c^T x^* \leq b^T \pi^*$. Then we have that $c^T x \leq b^T \pi^*$ and $c^T x^* \leq b^T \pi$ for any feasible solutions $x$ and $\pi$ for the primal and dual problems respectively. This means that because both the primal and dual problem are feasible (since they are bounded in the previous two inequalities) and the objective function is bounded, it must have an optimal solution. Indeed, if strong duality holds then we have that optimal solutions are

$$c^T x = b^T \pi$$

□

# References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.