MATH 437: Homework 3

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Problem 1

Consider the following linear program:

$$\begin{array}{lcccccc} \text{Maximize} & 2x_1 & + & 3x_2 & = & z \\ \text{subject to} & x_1 & + & 3x_2 & \leq & 6 \\ & 3x_1 & + & 2x_2 & \leq & 6 \\ & x_1, x_2 \geq 0 \end{array}$$

(a) State the problem in a canonical feasible form.

(b) Determine all the basic solutions of the problem (you will have 4 variables, any 2 of them can be basic, so there will be total of $\binom{4}{2} = 6$ basic solutions). Classify each of the basic solutions as feasible or infeasible.

(c) Solve graphically the original linear program.

(d) Illustrate how all the basic solutions (both feasible and infeasible) are represented on the 2-dimensional graphical solution space.

(e) Use the simplex algorithm to solve the program in the canonical form.

Solution 1. (a) Our program in canonical feasible form is:

$$\begin{array}{lccccc} \text{Minimize} & -2x_1 & -3x_2 & & & = & -z \\ \text{subject to} & x_1 & +3x_2 & +x_3 & & = & 6 \\ & 3x_1 & +2x_2 & & +x_4 & = & 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

(b) A table of all basic solutions and their feasibility is shown below in Table 1.1:

x_B	x_N	$x =$	Feasibility	z
x_3, x_4	x_1, x_2	(0,0,6,6)	CPF	0
x_1, x_2	x_3, x_4	$\left(\frac{6}{7}, \frac{12}{7}, 0, 0\right)$	CPF	$-\frac{48}{7}$
x_1, x_3	x_2, x_4	(2,0,4,0)	CPF	-4
x_1, x_4	x_2, x_3	(6,0,0,-12)	Infeasible	-12
x_2, x_3	x_1, x_4	(0,3,-3,0)	Infeasible	-9
x_2, x_4	x_1, x_3	(0,2,0,6)	CPF	-6

Table 1.1: Considering all $\binom{4}{2}$ basic solutions, their feasibility, and the value of the objective function at that point.

(c) Figure 1 is the graphical representation of our problem: Clearly, we can see that the point $\left(\frac{6}{7}, \frac{12}{7}\right)$ gives us an optimal solution with $z = \frac{48}{7}$.

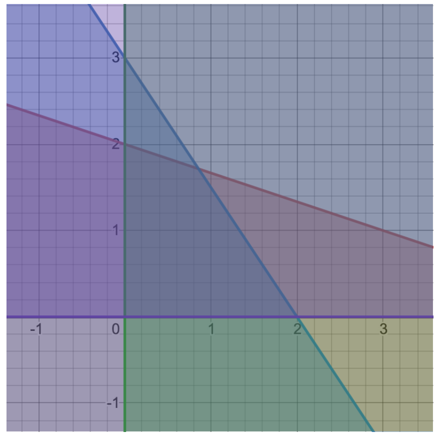

Figure 1: A graphical representation of the feasible region for Problem 1 with CPF at (0,0), (2,0), (0,2), and $\left(\frac{6}{7}, \frac{12}{7}\right)$.

(d) Since we are in $\mathbb{R}^n$ with $n = 2$, then by definition each basic solution is the intersection of $n = 2$ constraints. Therefore, it is clear that each solution is the intersection of two lines defined by our constraints.

(e) We formulate our problem in tableau form:

$$\left[\begin{array}{c|cccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline -1 & -2 & -3 & 0 & 0 & 0 \\ \hline 0 & 1 & 3 & 1 & 0 & 6 \\ 0 & 3 & 2 & 0 & 1 & 6 \end{array}\right]$$

Since $x_2 < x_1$, we choose x_2 to be our incoming basic variable. Our ratios are $\frac{6}{3} = 2$ for row R_2 and $\frac{6}{2} = 3$ for row R_3 and so we choose to pivot on x_2 in R_3 meaning x_3 is our outgoing basic variable. We perform the following row operations:

$$\begin{aligned} R_2 + R_1 &\to R_1 \\ \frac{1}{3}R_2 &\to R_2 \\ -2R_2 + R_3 &\to R_3 \end{aligned}$$

Which gives us the current tableau:

$$\left[\begin{array}{c|cccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline -1 & -1 & 0 & 1 & 0 & 6 \\ \hline 0 & \frac{1}{3} & 1 & \frac{1}{3} & 0 & 2 \\ 0 & \frac{7}{3} & 0 & -\frac{2}{3} & 1 & 2 \end{array}\right]$$

All cost coefficients are not non-negative and so we choose x_1 as an incoming basic variable. Since $\frac{2}{1/3} = 6$ for row R_2 and $\frac{2}{7/3} = \frac{6}{7}$ for row R_3, we choose the last row to pivot on. The following row operations are performed:

$$\frac{3}{7}R_3 \to R_3$$
$$-\frac{1}{3}R_3 + R_2 \to R_2$$
$$R_3 + R_1 \to R_1$$

Which gives us a final tableau form:

$$\left[\begin{array}{c|cccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline -1 & 0 & 0 & \frac{5}{7} & \frac{3}{7} & \frac{48}{7} \\ \hline 0 & 0 & 1 & \frac{3}{7} & -\frac{1}{7} & \frac{12}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{3}{7} & \frac{6}{7} \end{array}\right]$$

Since all cost coefficients $c_j > 0$, then we have reached our optimality stopping condition. Letting $x_N = (x_3, x_4)$ equal to 0 and $x_B = (x_1, x_2)$ equal to $\bar{b}$, we see that $(x_1, x_2) = \left(\frac{12}{7}, \frac{6}{7}\right)$ gives us a minimal value of $z = -\frac{48}{7}$; which is exactly the solution we found in parts (b) and (c).

$\square$

Problem 2

Suppose that a linear program is given in a canonical feasible form:

$$\begin{array}{rcrcrcr} -z & + & 0x_B & + & \bar{c}^T x_N & = & -\bar{z}_0, \\ & & Ix_B & + & \bar{A}x_N & = & \bar{b}. \end{array}$$

Carefully prove that the basic feasible solution corresponding to the above form is the unique minimal feasible solution if $\bar{c}_j > 0$ for all non-basic variables.

Solution 2. First we will show that the basic feasible solution x^* corresponding to the above form is a minimizer. Then we will show that the minimal feasible solution is unique. To see that x^* is a minimizer, note that

$$z = \bar{z}_0 + 0x_B + \bar{c}^T x_N.$$

If we change values of our basic variables, the value of z does not change. Since $x_N = 0$ and since we have non-negativity constraints, we can only increase the values of our non-basic variables. Since $\bar{c}^T > 0$, increasing the value of any non-basic variable increases the objective value z. So x^* is a minimizer.

To see that x^* is a unique minimizer, first note that if we change any non-basic variable then the value of the objective function increases. Since we must satisfy

$$Ix_B + \bar{A}x_N = \bar{b},$$

if we change the values of the basic variables then we must change the values of the non-basic variables. Then x^* is the unique minimizer. $\square$

Problem 3

Suppose that a linear program is given in a canonical feasible form:

$$\begin{array}{rcrcrcr} -z & + & 0x_B & + & \bar{c}^T x_N & = & -\bar{z}_0, \\ & & Ix_B & + & \bar{A}x_N & = & \bar{b}. \end{array}$$

As usual, we will let $\bar{a}_{ij}$ denote the elements of the matrix $\bar{A}$ and let $\bar{A}_s$ denote the column of the matrix $\bar{A}$ that corresponds to the variable x_s.

Suppose that the column of the matrix $\bar{A}$ that corresponds to a non-basic variable x_s contains a positive element (using mathematical notation, the set $\{i \mid \bar{a}_{is} > 0\}$ is not empty). Let

$$\bar{x}_s = \frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\{i \mid \bar{a}_{is} > 0\}} \frac{\bar{b}_i}{\bar{a}_{is}}.$$

Let $\bar{x}$ be a vector obtained by the following process:

- For each non-basic variable $j \neq s$, keep $\bar{x}_j = 0$.
- Let $\bar{x}_B = \bar{b} - \bar{A}_s \bar{x}_s$.

Please carefully prove that the solution $\bar{x}$ is feasible. On the way to the solution, please illustrate what is happening on an example.

Solution 3. To show that $\bar{x}$ is feasible, we must verify that $A\bar{x} = \bar{b}$ and $\bar{x} \geq 0$.

$$A\bar{x} = I\bar{x}_B + \bar{A}\bar{x}_N = I(\bar{b} - \bar{A}_s\bar{x}_s) + \bar{x}_s\bar{A}_s = \bar{b} + \bar{A}_s\bar{x}_s - \bar{A}_s\bar{x}_s = \bar{b}.$$

Now we will verify that $\bar{x} \geq 0$. Since $\bar{b} \geq 0$, we know that $\bar{x}_s$ is a ratio of positive numbers. So $\bar{x}_s > 0$. Each non-basic variable $\bar{x}_j$ such that $j \neq s$ is zero. By our assignment of $\bar{x}_s$ we have

$$\bar{A}_s\bar{x}_s = (\bar{a}_{1s}, \bar{a}_{2s}, \ldots, \bar{a}_{ms})^T \min_{\{i \mid \bar{a}_{is} > 0\}} \frac{\bar{b}_i}{\bar{a}_{is}}.$$

We may note that for each $j = 1, \ldots, m$ such that $\bar{a}_{js} > 0$, we have

$$\left(\bar{A}_s\bar{x}_s\right)_j = \min_{\{i \mid \bar{a}_{is} > 0\}} \frac{\bar{b}_i}{\bar{a}_{is}} \bar{a}_{js} \leq \frac{\bar{b}_j}{\bar{a}_{js}} \bar{a}_{js} = \bar{b}_j.$$

Then for each $j = 1, \ldots, m$ such that $\bar{a}_{js} > 0$, we have that

$$(\bar{x}_B)_j = \left(\bar{b} - \bar{A}_s\bar{x}_s\right)_j = \bar{b}_j - \left(\bar{A}_s\bar{x}_s\right)_j \geq \bar{b}_j - \bar{b}_j = 0.$$

For each $j = 1, \ldots, m$ such that $\bar{a}_{js} \leq 0$, we have $(\bar{x}_B)_j = (\bar{b} - \bar{A}_s\bar{x}_s)_j = \bar{b}_j - \bar{a}_{js}\bar{x}_s \geq \bar{b}_j \geq 0$. Since $\bar{x}_j \geq 0$ for each $j = 1, \ldots, n$, we have $\bar{x} \geq 0$. $\square$

Example 1. Consider a problem with the following form where $x = (0, 0, 6, 6)^T$, $x_B = (x_3, x_4)^T = (6, 6)^T$, $x_N = (x_1, x_2)^T = (0, 0)^T$, and $\bar{b} = (6, 6)^T$:

$$\left[\begin{array}{c|cccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline -1 & -2 & -3 & 0 & 0 & 0 \\ \hline 0 & 1 & 3 & 1 & 0 & 6 \\ 0 & 3 & 2 & 0 & 1 & 6 \end{array}\right].$$

Suppose $s = 1$. Then $A_s = (1, 3)^T$ contains a positive element. And

$$\bar{x}_1 = \frac{\bar{b}_r}{\bar{a}_{r1}} = \min_{\{i|\bar{a}_{i1}>0\}} \frac{\bar{b}_i}{\bar{a}_{i1}} = \frac{\bar{b}_r}{\bar{a}_{21}} = \frac{6}{3} = 2.$$

Since x_2 is non-basic, we let $\bar{x}_2 = 0$. And we let $\bar{x}_B = (\bar{x}_3, \bar{x}_4)^T = \bar{b} - \bar{A}_s \bar{x}_s = (6, 6)^T - 2(1, 3)^T = (4, 0)^T$. Therefore, $\bar{x} = (2, 0, 4, 0)^T \geq 0$. Then

$$A\bar{x} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \bar{b}.$$

References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.