MATH 437: Homework 4

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Problem 1

Consider the following problem:

$$\begin{array}{lrrrrrrcr}
\text{Maximize} & 8x_1 & +\,4x_2 & +\,6x_3 & +\,3x_4 & +\,9x_5 & & = & z \\
\text{subject to} & x_1 & +\,2x_2 & +\,3x_3 & +\,2x_4 & +\,2x_5 & +\,x_6 & \leq & 180 \\
 & 4x_1 & +\,3x_2 & +\,2x_3 & +\,x_4 & +\,x_5 & & \leq & 270 \\
 & x_1 & +\,3x_2 & & +\,x_4 & +\,3x_5 & & \leq & 180 \\
 & x_i \geq 0 & & & & & & &
\end{array}$$

Solution 1. (a) We are given that in our optimal solution $x_N = (x_2, x_4)$ and the basic variables are $x_B = (x_3, x_1, x_5)$ and that their corresponding basis matrix B is:

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad B^{-1} = \frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}$$

Since we know that these variables are non-basic in the solution, then we are really looking to solve $Bx = b$ which is the system:

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_3 \\ x_1 \\ x_5 \end{bmatrix} = \begin{bmatrix} 180 \\ 270 \\ 180 \end{bmatrix}$$

Then we simply need to multiply our vector $\bar{b}$ on the left by B^{-1} to recover our vector x:

$$\begin{aligned}
\frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}\begin{bmatrix} 180 \\ 270 \\ 180 \end{bmatrix} &= \begin{bmatrix} x_3 \\ x_1 \\ x_5 \end{bmatrix} \\
&= \begin{bmatrix} 50 \\ 30 \\ 50 \end{bmatrix}
\end{aligned}$$

Which gives us a maximal value of $z = 990$.

(b) In order to answer this question, we first need to know exactly what is the meaning of a shadow price.

Definition 1. [1] The **shadow price** for a resource i (denoted y_i^*) measure the marginal value of this resource, i.e., the rate at which Z could be increased by *slightly* increasing the amount of this resource b_i being made available. The simplex method identifies this shadow price by y_i^* = coefficient of the i^{th} slack variable in row 0 of the final simplex tableau.

Since we are told $x_B = (x_3, x_1, x_5)$ in the optimal solution, then we know that the columns corresponding to these variables are just basis vectors e_1, e_2, and e_3. Then for x_3:

$$\frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{27} \\ -\frac{2}{9} \\ \frac{2}{27} \end{bmatrix}$$

The shadow price of x_3 is:

$$\begin{aligned}\begin{bmatrix} 8 & 4 & 6 & 3 & 9 \end{bmatrix}\begin{bmatrix} -\frac{2}{9} \\ \frac{11}{27} \\ \frac{2}{27} \end{bmatrix} &= 8 \cdot -\frac{2}{9} + 6 \cdot \frac{11}{27} + 9 \cdot \frac{2}{27} \\ &= \frac{4}{3}\end{aligned}$$

For x_1:

$$\frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{1}{3} \\ -\frac{1}{9} \end{bmatrix}$$

The shadow price of x_1 is:

$$\begin{aligned}\begin{bmatrix} 8 & 4 & 6 & 3 & 9 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{9} \\ -\frac{1}{9} \end{bmatrix} &= 8 \cdot \frac{1}{3} + 6 \cdot -\frac{1}{9} + 9 \cdot -\frac{1}{9} \\ &= 1\end{aligned}$$

For x_5:

$$\frac{1}{27}\begin{bmatrix} 11 & -3 & 1 \\ -6 & 9 & -3 \\ 2 & -3 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{27} \\ -\frac{1}{9} \\ \frac{10}{27} \end{bmatrix}$$

The shadow price of x_5 is:

$$\begin{aligned}\begin{bmatrix} 8 & 4 & 6 & 3 & 9 \end{bmatrix}\begin{bmatrix} -\frac{1}{9} \\ \frac{1}{27} \\ \frac{10}{27} \end{bmatrix} &= 8 \cdot -\frac{1}{9} + 6 \cdot \frac{1}{27} + 9 \cdot \frac{10}{27} \\ &= \frac{8}{3}\end{aligned}$$

Shadow prices for $(x_1, x_3, x_5) = \left(1, \frac{4}{3}, \frac{8}{3}\right)$

$\square$

Problem 2

Please solve Problem 5.2-2 from the text. Bonus points if you figure out how to do it without having to write the entire 3×9 table multiple times, using the insight of Problem 1.

Solution 2. Our formulated problem is:

$$\begin{array}{llllllllll} \text{Minimize} & 5x_1 & +\,8x_2 & +\,7x_3 & +\,4x_4 & +\,6x_5 & & & = & -z \\ \text{subject to} & 2x_1 & +\,3x_2 & +\,3x_3 & +\,2x_4 & +\,2x_5 & +\,x_6 & & = & 20 \\ & 3x_1 & +\,5x_2 & +\,4x_3 & +\,2x_4 & +\,4x_5 & & +\,x_7 = & & 30 \\ & x_i \geq 0 \end{array}$$

Therefore, our tableau looks like:

$$\left[\begin{array}{c|ccccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & b \\ \hline -1 & 5 & 8 & 7 & 4 & 5 & 0 & 0 & 0 \\ \hline 0 & 2 & 3 & 3 & 2 & 2 & 1 & 0 & 20 \\ 0 & 3 & 5 & 4 & 2 & 4 & 0 & 1 & 30 \end{array}\right]$$

So the basis B corresponds to our slack variables and therefore $B = B^{-1} = \mathbb{I}_{2,2}$. Then:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$$

Then $c_B = \begin{bmatrix} 0 & 0 \end{bmatrix}$; $c_N = \begin{bmatrix} 5 & 8 & 7 & 4 & 5 \end{bmatrix}$ and since x_2 is the largest value in c_N, then x_2 is entering i.e $k = 2$. Ratios $R_1 = \frac{20}{3}$ and $R_2 = \frac{30}{6} = 5$ so x_7 is leaving x_B. Then our $\eta = \begin{bmatrix} -\frac{3}{5} \\ \frac{1}{5} \end{bmatrix}$ and so then $B^{-1} = \begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \mathbb{I}_{2,2}$. Then:

$$B^{-1}b = \begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Then $c_B = \begin{bmatrix} 0 & 8 \end{bmatrix}$ so $\pi^T = \begin{bmatrix} 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0 & \frac{8}{5} \end{bmatrix}$. Let's update our cost coefficients $c^T - \pi^T A$:

$$\begin{aligned} &\begin{bmatrix} 5 & 8 & 7 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 0 & \frac{8}{5} \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & 0 & \frac{3}{5} & \frac{4}{5} & -\frac{7}{5} & 0 & -\frac{8}{5} \end{bmatrix} \end{aligned}$$

Since we still have positive values in our cost coefficients, then we are yet to reach an optimal solution for this LP. Recalculating A:

$$\begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{bmatrix}$$

We choose x_4 to be our incoming basic variable, but we need to find the current values of x_4 in order to choose which variable is outgoing:

$$B^{-1}x_4 = \begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix}$$

So ratios $R_1 = \frac{2}{4/5} = \frac{5}{2}$ and $R_2 = \frac{6}{2/5} = 15$ and so we pivot on R_1 which means x_6 is outgoing. Let's update our basis matrix inverse B^{-1} by observing $k = 4, r = 1$, and $\eta =] \, [$

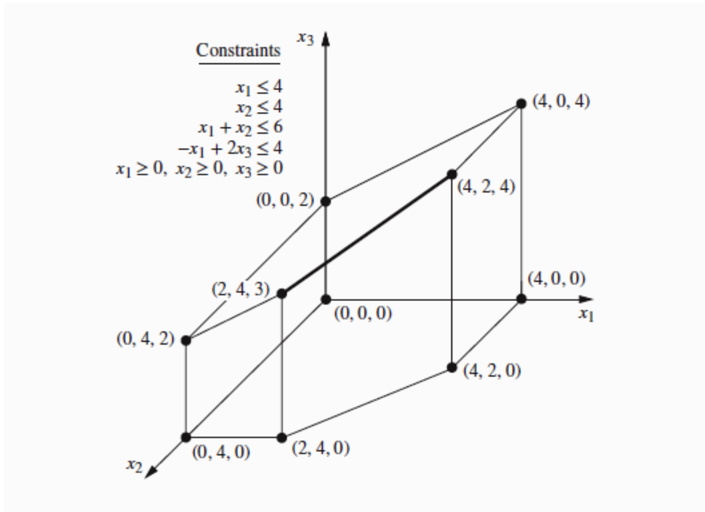

Figure 1: The feasibility region for Problem 3 and associated constraints.

Problem 3

Please solve Problem 5.1-18 in the text. No calculations are needed for this one; but you need to provide ample explanations. You have three original variables x_1, x_2, and x_3 and four slack variables $x_4, \ldots, x_7$. Assume that you are starting out at the origin; for each step in the simplex algorithm, specify which of the variables will be basic.

Solution 3. We are told to use purely geometric reasoning and Figure 1 to find an optimal solution to the LP:

$$\begin{array}{lrrrrrrrcl}
\text{Maximize} & 3x_1 & +\,4x_2 & +\,3x_3 & & & & & = & z \\
\text{subject to} & x_1 & +\,x_2 & & +\,x_4 & & & & = & 6 \\
& -x_1 & & +\,2x_3 & & +\,x_5 & & & = & 4 \\
& x_1 & & & & & +\,x_6 & & = & 4 \\
& & x_2 & & & & & +\,x_7 & = & 4 \\
& x_i \geq 0 & & & & & & & &
\end{array}$$

Now, our initial solution is always the trivial one, i.e. the origin at $(x_1, x_2, x_3) = (0, 0, 0)$. To find the path to our optimal solution, we examine the 3 points where $x_i \neq 0$ for exactly one x_i. So we have:

(x_1, x_2, x_3)	z
(4,0,0)	12
(0,4,0)	16
(0,0,2)	6

Initially, we have $x_B = (x_4, x_5, x_6, x_7)$, and we travel along the line connecting the origin and $(0, 4, 0)$ making x_2 our incoming basic variable and x_7 is our leaving basic variable. From here we have two choice of adjacent CFP solutions: (2,4,0) and (0,4,2). Checking the values they yield in the objective function, we obtain $z = 22$ for both. Since both of these solutions are optimal, then we arbitrarily choose (2,4,0); so our basic variables change from $x_B = (x_2, x_4, x_5, x_6)$ to $x_B = (x_2, x_3, x_4, x_6)$. It is interesting to note that if we were performing the simplex algorithm, this would be a case where the cost coefficients for the two potential outgoing variables were the same quantity.

Now, we are currently at the point (2,4,0) with adjacent CFP solutions (2,4,3) and (4,2,0). These yield z values of 31 and 20 respectively, and so we choose the point (2,4,3) to be our next stop on the path. At this juncture, our incoming basic variable is x_1 and our outgoing basic is x_4, changing $x_B = (x_2, x_3, x_4, x_6)$ to $x_B = (x_1, x_2, x_3, x_6)$. We now have choices of (4,2,4) and (0,4,2) of which we have already considered the latter. At (4,2,4) we obtain $z = 32$, but adjacent CFP solutions are (4,2,0) which has already been evaluated and (4,0,4) which cannot be maximal solely from the comparison of the value of x_2 between the two points. It is clear to see that $(4, 2, 4)$ gives us our optimal solution and our final change is x_7 incoming basic with x_6 leaving. Therefore, our final vector $x_B = (x_1, x_2, x_3, x_7)$ corresponding to $x = (4, 2, 4, 0, 0, 0, 2)$.

$\square$

Problem 4

Suppose that A in an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$. Let $a = (a_1, \ldots, a_m)$ be a vector in $\mathbb{R}^m$.

Carefully prove that the following two statements are equivalent:

- There is a vector $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.
- The linear program

$$\begin{aligned} \text{Minimize} \quad & W = a_1 + \cdots + a_m \\ \text{subject to} \quad & Ax + Ia = b \\ & x \geq 0,\ a \geq 0 \end{aligned}$$

  has a feasible solution with $W^* = 0$.

Solution 4. The linear program

$$\begin{aligned} \text{Minimize} \quad & W = a_1 + \cdots + a_m \\ \text{subject to} \quad & Ax + Ia = b \\ & x \geq 0,\ a \geq 0 \end{aligned}$$

has a feasible solution with $W^* = 0$ if and only if there exists some $x \in \mathbb{R}^n$ such that $Ax + I\vec{0} = b$ and $x \geq 0$ because $a_1 + \cdots + a_m = 0$ and $a \geq 0$ is only satisfied if $a_1 = \cdots = a_m = 0$. There exists some $x \in \mathbb{R}^n$ such that $Ax + I\vec{0} = b$ and $x \geq 0$ if and only if there exists some $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.

Problem 5

Formulate a linear program to find a non-negative solution to the system of equations $Ax = b$, where

$$A = \begin{bmatrix} 0 & 1 & 3 & -6 & 1 \\ -2 & 4 & 0 & 3 & -1 \\ 1 & -6 & -3 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 9 \\ -14 \end{bmatrix}$$

or determine that no such solutions exist.

Use `lp_solve` or some other software to run the program you have formulated.

Solution 5. In order to discover whether a feasible solution exists, we use the infeasibility form as the objective function:

$$\left[\begin{array}{c|cccccccc|c} W & x_1 & x_2 & x_3 & x_4 & x_5 & a_1 & a_2 & a_3 & b \\ \hline -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 3 & -6 & 1 & 1 & 0 & 0 & -1 \\ 0 & -2 & 4 & 0 & 3 & -1 & 0 & 1 & 0 & 9 \\ 0 & 1 & -6 & -3 & 0 & 1 & 0 & 0 & 1 & -14 \end{array}\right]$$

Formulation of our LP in `Sage`:

```
p = MixedIntegerLinearProgram(maximization=False)
v = p.new_variable(real=True, nonnegative=True)
x1, x2, x3, x4, x5, a1, a2, a3 = v['x1'], v['x2'], v['x3'], v['x4'], v['x5'],
v['a1'], v['a2'], v['a3']
p.set_objective(a1+a2+a3)
p.add_constraint(x2+3*x3-6*x4+x5+a1==-1)
p.add_constraint(-2*x1+4*x2+3*x4-x5+a2==9)
p.add_constraint(x1-6*x2-3*x3+x5+a3==-14)
p.show()
p.solve()
d = p.get_values(v)
for k in d:
    print(k,d[k])
print('Min=',p.get_objective_value())
```

Which yields output:

```
Minimization:
  x_5 + x_6 + x_7

Constraints:
  -1.0 <= x_1 + 3.0 x_2 - 6.0 x_3 + x_4 + x_5 <= -1.0
  9.0 <= -2.0 x_0 + 4.0 x_1 + 3.0 x_3 - x_4 + x_6 <= 9.0
  -14.0 <= x_0 - 6.0 x_1 - 3.0 x_2 + x_4 + x_7 <= -14.0
```

```
Variables:
  x_0 is a continuous variable (min=0.0, max=+oo)
  x_1 is a continuous variable (min=0.0, max=+oo)
  x_2 is a continuous variable (min=0.0, max=+oo)
  x_3 is a continuous variable (min=0.0, max=+oo)
  x_4 is a continuous variable (min=0.0, max=+oo)
  x_5 is a continuous variable (min=0.0, max=+oo)
  x_6 is a continuous variable (min=0.0, max=+oo)
  x_7 is a continuous variable (min=0.0, max=+oo)
x1 1.6000000000000008
x2 2.6
x3 0.0
x4 0.6
x5 0.0
a1 -4.440892098500626e-16
a2 8.881784197001252e-16
a3 0.0
Min= 4.440892098500626e-16
```

Then since the minimum value is essentially zero, a feasible solution exists with $x_B = (x_1, x_2, x_4)$. So we can construct a basis matrix just as in Problem 1 by looking at the columns of x_B:

$$B = \begin{bmatrix} 0 & 1 & -6 \\ -2 & 4 & 3 \\ 1 & -6 & 0 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} -\frac{2}{5} & -\frac{4}{3} & -\frac{3}{5} \\ -\frac{1}{15} & -\frac{2}{15} & -\frac{4}{15} \\ -\frac{8}{45} & -\frac{1}{45} & -\frac{2}{45} \end{bmatrix}$$

Then finding our vector x by multiplying on the left by B^{-1} we get:

$$x = \begin{bmatrix} -\frac{2}{5} & -\frac{4}{3} & -\frac{3}{5} \\ -\frac{1}{15} & -\frac{2}{15} & -\frac{4}{15} \\ -\frac{8}{45} & -\frac{1}{45} & -\frac{2}{45} \end{bmatrix} \begin{bmatrix} -1 \\ 9 \\ -14 \end{bmatrix}; \qquad x = \begin{bmatrix} \frac{8}{5} \\ \frac{13}{5} \\ \frac{3}{5} \end{bmatrix}$$

And so our non-negative solution to the equation $Ax = b$ is $x = \left(\frac{8}{5}, \frac{13}{5}, 0, \frac{3}{5}, 0\right)$.

□

References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.