MATH 437: Homework 1

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Problem 11.3-5

A county chairwoman of a certain political party is making plans for an upcoming presidential election. She has received the services of six volunteer workers for Precinct work, and she wants to assign them to four Precincts in such a way as to maximize their effectiveness. She feels that it would be inefficient to assign a worker to more than one Precinct, but she is willing to assign no workers to any one of the Precincts if they can accomplish more in other Precincts. The following table gives the estimated increase in the number of votes for the party's candidate in each Precinct if it were allocated various numbers of workers:

Workers	Precinct 1	Precinct 2	Precinct 3	Precinct 4
0	0	0	0	0
1	4	7	5	6
2	9	11	10	11
3	15	16	15	14
4	18	18	18	16
5	22	20	21	17
6	24	21	22	18

Solution 1. Let's break down the quantities that we are given and classify them according to a dynamic program setup;

- Stages: Stage n is the n-th Precinct.
- States: The stage s_n is the number of volunteers that are not allocated to Precincts 1 through n.
- Decision Variables: the decision variable x_n is the number of volunteers allocated to Precinct n.
- We seek to maximize the function $f(s, x_n)$ which is the total number of voters we gain by carefully choosing how we allocate workers to each Precinct.

Let $p_n(x_n)$ denote the expected increase in the number of votes if x_n volunteers are assigned to the n-th Precinct. Let

$$f_n(s_n, x_n) = p_n(x_n) + \max_{\sum_{j=n+1}^{4} x_j = s_n} \left\{ \sum_{i=n+1}^{4} p_i(x_i) \right\}.$$

Let $f_n^*(s_n) = \max_{x_n \in [s_n]}\{f_n(s_n, x_n)\}$. Then we have the equation:

$$f_n(s_n, x_n) = p_n(x_n) + f_{n+1}^*(s_{n+1}).$$

We can then solve for $f_n^*(s_n)$ recursively starting from the last stage. There are three optimal paths that attain an expected increase of 33 votes:

$$6 \to 6 \to 5 \to 2 \to 0$$
$$6 \to 3 \to 2 \to 1 \to 0$$
$$6 \to 3 \to 2 \to 2 \to 0$$

The three corresponding assignments of volunteers are:

1. 0 workers to Precinct 1, 1 worker to Precinct 2, 3 workers to Precinct 3, and 2 workers to Precinct 4
2. 3 workers to Precinct 1, 1 worker to Precinct 2, 1 worker to Precinct 3, and 1 worker to Precinct 4,
3. 3 workers to Precinct 1, 1 to Precinct 2, 0 workers to Precinct 3, and 2 workers to Precinct 4.

Workers	Precinct 1	f_2^*	Precinct 2	f_3^*	Precinct 3	Precinct 4
0	0	0	0	0	0	0
1	4	7	7	6	5	6
2	9	13	11	11	10	11
3	15	18	16	16	15	14
4	18	23	18	21	18	16
5	22	28	20	25	21	17
6	24	33	21	29	22	18

Table 1.1: Finding the f_n* for $n = 2, 3$ replicates the graphical technique we use in Figure 1 for finding an optimal policy.

Non-technical Summary:
Our goal is to maximize the number of votes that a county chairwoman can sway in the upcoming presidential election. We can accomplish this by carefully choosing where to send six available workers to four different Precincts in which we already know the measurable outcome of votes we gain by sending x amount of workers to Precinct n. We structure the problem in terms of a graph consisting of vertices (nodes or dots) and edges (lines) and begin at the left most node, working to the right through the stages. Each stage represents the different Precincts, and the nodes indicate how many workers remain to allocate in the subsequent stages. And though every stage has 7 nodes labeled 0-6, it is not possible to move between certain nodes because of the constraint we have on the total workers available. We begin at the left blue node has a label of 6, indicating our initial amount of workers we can begin with. Say we wish to allocate 6 workers to Precinct 1, then we would gain 24 total votes and move to the node labeled 0 in the next stage. However, we could then only move along the bottom most path, allocating 0 workers to Precincts 2, 3, & 4.

Because the total number of paths through the graph is $7^3 = 343$, it is inefficient to calculate all paths possible and find an optimal policy on where to send our 6 workers. Therefore, we work backwards reducing our total number of calculations that we need to make to just $(7 \cdot 7) + (7 \cdot 7) = 98$. We accomplish this by first examining the outcomes we get for sending x amount of workers to Precinct 4. From here, we examine the path from each individual node in the second stage to all other possible nodes in stage three that will result in the greatest number of votes won. Once we have determined this, we draw an edge to the corresponding node (or nodes) in the third stage and write the subsequent optimal value above the node in stage 2. We continue this method until we are able to reach the left most node, giving us an optimal policy and therefore the maximal value of our objective function. Both Table 1.1 and Figure 1 display numerical and graphical representations of how we calculated our optimal policies.

There are three possible ways to allocate the six volunteers to achieve the maximum expected increase in the number of votes for the candidate. To best allocate her six volunteers, the chairwoman can (1) assign zero volunteers to Precinct 1, one volunteer to Precinct 2, three volunteers to Precinct 3, and the remaining two volunteers to Precinct 4; she can (2) assign three volunteers to Precinct 1 and one volunteer to each of Precinct 2, 3, and 4; or she can (3) assign three volunteers to Precinct 1, one volunteer to Precinct 2, zero volunteers to Precinct 3, and the remaining two volunteers to Precinct 4. It may be worth considering that the first and third allocations each assign zero volunteers to one of the four Precincts. This may not have the best optics in terms of political liability; perhaps these Precincts represent communities with large minority demographics. Ignoring these communities may have an equal effect in the short term in terms of votes, but may prove costly in the future when another re-election campaign starts.

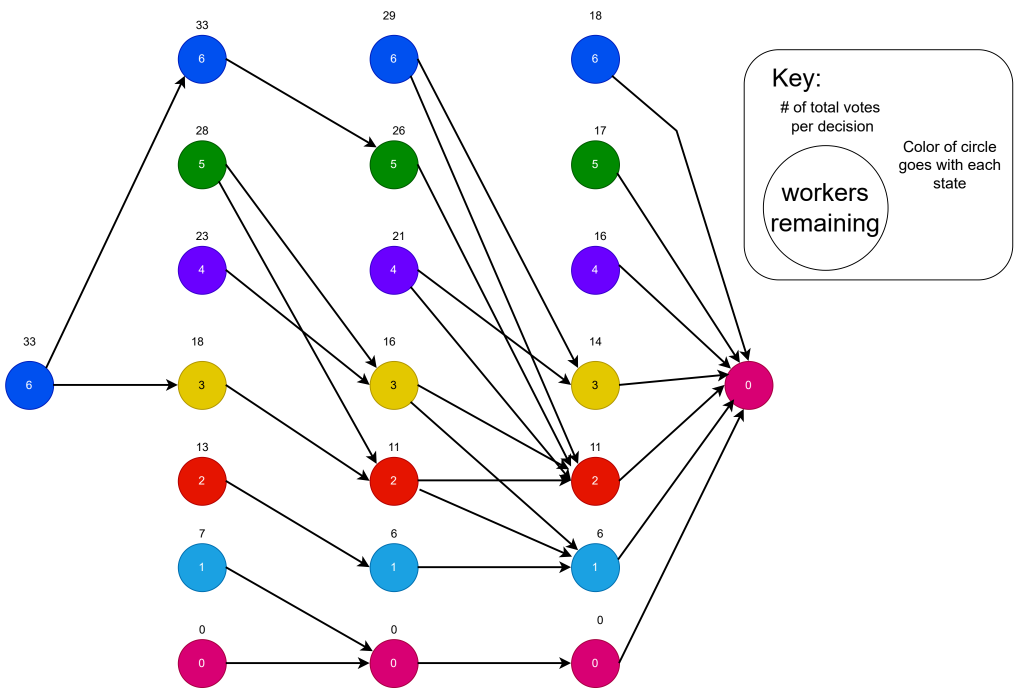


Figure 1: Recursively finding the optimal paths through the graph to determine an optimal policy decision.

Problem 2

A company will soon be introducing a new product into a very competitive market and is currently planning its marketing strategy. The decision has been made to introduce the product in three phases. Phase 1 will feature making a special introductory offer of the product to the public at a greatly reduced price to attract first-time buyers. Phase 2 will involve an intensive advertising campaign to persuade these first-time buyers to continue purchasing the product at a regular price. It is known that another company will be introducing a new competitive product at about the time that Phase 2 will end. Therefore, Phase 3 will involve a follow-up advertising and promotion campaign to try to keep the regular purchasers from switching to the competitive product.

A total of \$40 million has been budgeted for this marketing campaign. The problem now is to determine how to allocate this money most effectively to the three phases. Let m denote the initial share of the market (expressed as a percentage) attained in Phase 1, f_2 the fraction of this market share that is retained in Phase 2, and f_3 the fraction of the remaining market share that is retained in Phase 3. Use dynamic programming to determine how to allocate the \$40 million to maximize the final share of the market for the new product, i.e., to maximize m, f_2 and f_3.

1. Assume that the money must be spent in integer multiples of \$10 million in each Phase, where the minimum permissible multiple is 1 for Phase 1 and 0 for Phases 2 and 3. The following table gives the estimated effect of expenditures in each Phase:

Millions of dollars	m	f_2	f_3
0	-	.2	.3
10	20%	.4	.5
20	30%	.5	.6
30	40%	.6	.7
40	50%	-	-

Table 2.1: m denotes the initial market share we gain by allocating $n \in [1, 4]$ millions of dollars in Phase 1, while f_2 and f_3 indicate the percentage of market share we retain by investing n million dollars in Phases 2 & 3.

2. Now assume that any amount within the total budget can be spent in each Phase, where the estimated effect of spending an amount x_i (in units of tens of millions of dollars) in Phase i for $(i = 1, 2, 3)$ is:

$$m = 10x_1 - x_1^2 \tag{2.1}$$
$$f_2 = .4 + .1x_2 \tag{2.2}$$
$$f_3 = .6 + .07x_3 \tag{2.3}$$

[Hint: After solving for the $f_2^*(s)$ and $f_3^*(s)$ functions analytically, solve for x_1^* graphically.]"

Solution 2. (a)

Let's begin by listing out all relevant quantities to our dynamic programming problem:

- Stages: Phases n in the products roll out strategy $(n \in [1, 3])$.

- Decision variables: The amount of money x_n we spend in stage n.
- States: The state s_n is the amount of money in millions not allocated in stages 1 through n. Symbolically, $s_n = 40 - \sum_{i\in[n-1]} x_i$.

We can solve this problem easily if we are careful about how we define our operation. In Phase 1, we are determining what percent of the market share we will initially capture with our 'introductory offer', and the subsequent phases will be advertising and marketing campaigns that will determine how much of that market share we retain as time progresses. Therefore, we can not simply sum the values and must therefore take the product at each stage; a simple example will help. Say that in our first phase m we invest 10 million dollars and therefore capture 20% of the market. If we were to invest no money in phase two, then we would only retain $20\% \cdot .02 = 4\%$ of our customers. Investing the remaining 30 million dollars in Phase 3 would allow us to retain $.04 \cdot .7 = 2.8\%$ of the overall market.

Therefore, we can use our usual recursive operation to find the maximum amount of the market we can persuade using our product through the special introductory offer in Phase 1 and the number of customers that will remain loyal to us after making advertising investments in Phases 2 and 3. We are constrained that we must invest at least 10 million in Phase. Although we are working with \$40 total, we omit the 0 to simplify our calculations.

Then our objective function to maximize is simply the product of the market shares at stage n:

$$f_n = \prod_{n=1}^{3} f(s_i, x_i)$$

Let's begin at the end with $s_3 = x_3 \in [0, 3]$. Then, $x_3^* = \{0, 1, 2, 3\}$ which corresponds exactly to $f_3^* = \{.3, .5, .6, .7\}$. Now we move to Phase 2; our state variables for s_2 are exactly those of s_3 i.e. $s_2 \in [0, 3]$. Figure 2 gives a representation of our problem as a full graph, while Table 2.2 and Figure 3 gives an optimal path to maximize our market share retention and all relevant quantities.

s_1	$f_1^*(s_1)$	x_1^*	s_2	$f_2^*(s_2)$	x_2^*	s_3	$f_3^*(s_3)$	x_3^*
4	.06	2	3	.25	1	3	.7	3
			2	.2	1	2	.6	2
			1	.12	1	1	.5	1
			0	.06	0	0	.3	0

Table 2.2: Values for our recursive procedure to determine an optimal investment of the \$40 million dollars.

We find that the optimal policy is to invest 20 million dollars during Phase 1 with the special introductory offer, capturing 30% of the market, with 10 million dollar investments in both Phases 2 and 3, resulting in retaining 6% of the total market share.

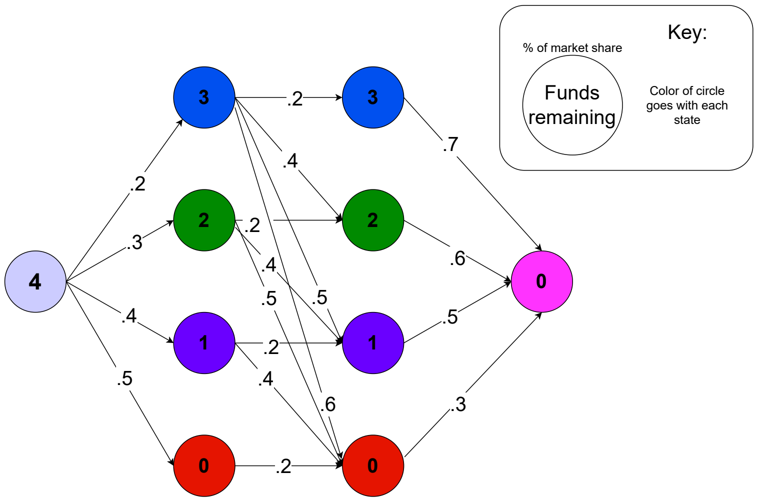


Figure 2: Full formulation of Problem 2 as a graph.

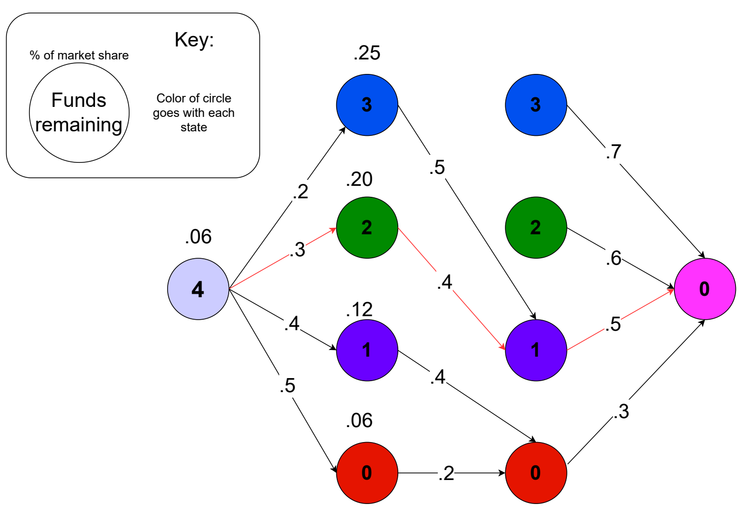


Figure 3: Optimal policy of Problem 2 and relevant quantities.

(b) For this part, we can simply emulate many of the same procedures that we used from the example we did in class and Problem 3. We are given f_3 and so we know that $f_3^* = .6 + .07s$ where we have constraints of $s_i \in [0, 4]$. However, since s is dependent on the amount of money we spent in the previous stage, then we substitute a difference i.e. $s = s - x_2$. Since this problem uses a product of decision variables instead of sums, then we can simply multiply f_2 by f_3^*; then:

$$f_2(s_3, x_2) = f_2 \cdot f_3^* \tag{2.4}$$

$$\begin{aligned} &= (.4 + .1x_2)(.6 + .7(s - x_2)) \\ &= 0.007s\ x_2 + 0.028s - 0.007x_2^2 + 0.032x_2 + 0.24 \end{aligned}$$

Now let's take the derivative to find a critical point of x_2:

$$\begin{aligned} \frac{\partial f_2}{\partial x_2} &= .007s - .014x_2 + .032 \\ x_2 &= \frac{.007s + .032}{.014} \\ &= \frac{7s + 32}{14} \end{aligned}$$

Set $x_2 = s$ and solve giving $s = \frac{32}{7}$ which falls outside our feasible region. Therefore, it is clear that we need to spend whatever remaining money into s_2 i.e. $x_2^* = s_2$ and $x_3 = 0$. Plug s into Equation 2.4:

$$\begin{aligned} &= (.4 + .1s)(.6 + .7(s - s)) \\ &= (.4 + .1s)(.6) \\ f_2^*(s) &= .06s + .024 \end{aligned}$$

Now we find $f_1(s, x_1)$ and assume the highest value, i.e. $s = 4$. As in previous steps, we multiply the proceeding f_2^*:

$$\begin{aligned} f_1(4, x_1) &= (10x_1 - x_1^2)(.06(4 - x_1) + .24) \\ &= 0.06x_1^3 - 1.08x_1^2 + 4.8x_1 \end{aligned}$$

Again take the partial with respect to x_1 to find the critical point:

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= .18x_1^2 - 2.16x_1 + 4.8 := 0 \\ 0.18(x_1 - 6)^2 &= 1.68 \\ (x_1 - 6)^2 &= 9.3 \\ x_1 &= \pm\sqrt{9.3} + 6 \\ x_1 &= 9.06 \text{ or } 2.95 \end{aligned}$$

A graphical representation of $\frac{\partial f_1}{\partial x_1}$ can be seen in Figure 4 which shows that 2.95 is our critical point on a concave parabola for $x_1 \in [0, 4]$, and therefore $x_1^* = 2.95$. Then working backwards and subbing in our optimal values, we obtain:

$$\begin{aligned} m &= 10(2.95) - 2.95^2 \\ f_2 &= .4 + .1(4 - 2.95) \\ f_3 &= .6 + .07(0) \\ m \cdot f_2 \cdot f_3 &= 6.3\% \end{aligned}$$

Which shows a market share capture of 6.3%. It should be noted that these additional calculations and relaxing the amount we can spend from tens of millions into any number resulted in only a 0.3% increase as opposed to the answer we found in part (a).

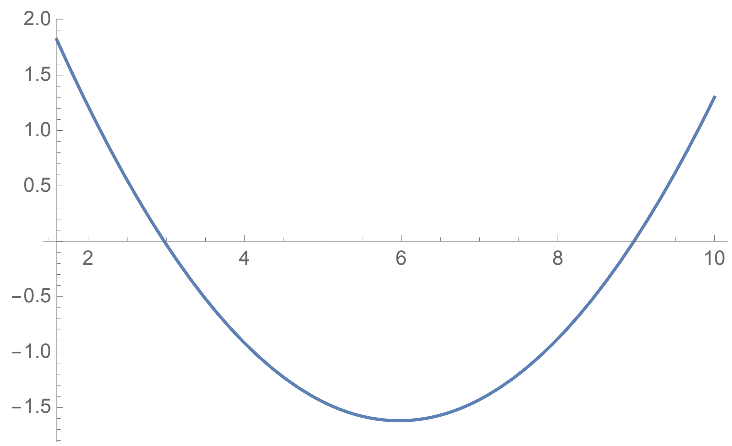


Figure 4: Graphical verification of x_1^* in the feasible region $x_1 \in [0, 4]$.

Problem 3

Referring to the seasonal employment problem from class, recall that the set-up is the same as in Example 4 of Section 11.3, but we change the numbers.

For the homework, please consider the following seasonal requirements:

Season	Spring	Summer	Autumn	Winter	Spring
Requirement	26	22	23	19	26

The cost of an idle worker is \$8,000 per season and the cost to hire or fire a worker is \$2,000 times the square of the change in the staffing level.

Please determine the optimal staffing strategy for the company.

Solution 3. Just as in our example, it is not efficient to consider employment levels that surpass our busiest season (Spring at 26 workers), and therefore we consider this as an upper bound for the decision variables. Our stages are the different seasons and our decision variables are the employment levels at each stage. Define r_n as the minimum employment level at stage n; so we have $r_1 = 22, r_2 = 23, r_3 = 19, r_4 = 26$. The cost at the current stage depends on only by the current decision x_n and the employment that we inherit from the past season x_{n-1}, and so the state at stage n is given by $s_n = x_{n-1}$; so $s_1 = x_0 = x_4 = 26$. We simplify our problem and calculations by changing 8000 to 8 and remembering this decision. Our cost function at stage n is given by:

$$\begin{aligned} c_n &= 8(x_n - r_n) + 2(x_n - x_{n-1})^2 \\ &= 8(x_n - r_n) + 2(x_n - s_n)^2 \end{aligned}$$

We want to minimize $\sum_{i=1}^{4} c_i$, and so our function we wish to minimize at each stage is:

$$f_n(s_n, x_n) = 8(x_n - r_n) + 2(x_n - s_n)^2 + f_{n+1}^*(x_n)$$

where $f_n^*(s_n) = \sum_{i=n}^{4} \min_{r_i \leq x_i \leq 26} \{8(x_i - s_i) + 2(x_i - r_i)^2\}$

A table of relevant information is presented in Table 3.1:

n	r_n	Feasible x_n	Possible $s_n = x_{n-1}$	Cost
1	22	$22 \leq x_1 \leq 26$	$s_1 = 26$	$8(x_1 - 22) + 2(x_1 - 22)^2$
2	23	$23 \leq x_2 \leq 26$	$22 \leq s_2 \leq 26$	$8(x_2 - 23) + 2(x_2 - x_1)^2$
3	19	$19 \leq x_3 \leq 26$	$23 \leq s_3 \leq 26$	$8(x_3 - 19) + 2(x_3 - x_2)^2$
4	26	$x_4 = 26$	$19 \leq s_4 \leq 26$	$2(26 - x_3)^2$

Table 3.1: Formulation of Decision Variables, Feasible and Possible Regions, and Costs

Since $x_4 = r_4 = 26$, we have that

$$\begin{aligned} f_4(s_4, x_4) &= 8(x_4 - r_4) + 2(x_4 - s_4)^2 + f_5^*(x_4) \\ &= 2(x_4 - s_4)^2 + \sum_{i=5}^{4} \min_{r_i \leq x_i \leq 26} \{8(x_i - s_i) + 2(x_i - r_i)^2\} \\ &= 2(26 - s_4)^2. \end{aligned}$$

Therefore $f_4^*(s_4) = f_4(s_4, x_4^*) = 2(26 - s_4)^2$.

To find x_3^*, first note that $r_3 = 19$. Then $f_3(s_3, x_3) = 2(x_3 - s_3)^2 + 8(x_3 - 19) + f_4^*(x_3)$. By setting the partial derivative of f_3 with respect to x_3 equal to 0, we find $x_3^* = \frac{24+s_3}{2}$. Since it is possible to inherit any $x_3 = s_4 \in [19, 26]$. Then:

$$19 \leq x_3 \leq 26 \implies 19 \leq \frac{24 + s_3}{2} \leq 26 \implies 14 \leq s_3 \leq 28.$$

So our assignment of x_3^* is feasible for all possible values of s_3.

To find x_2^*, first note that $r_2 = 23$. Then $f_2(s_2, x_2) = 2(x_2 - s_2)^2 + 8(x_2 - 23) + f_3^*(x_2)$. By setting the partial derivative of f_2 with respect to x_2 equal to 0, we find $x_2^* = 2(10 + s_2)/3$. Since it is possible to inherit any $x_2 = s_3 \in [23, 26]$. Then

$$23 \leq x_2 \leq 26 \implies 23 \leq 2(10 + s_2)/3 \leq 26 \implies 24.5 \leq s_2 \leq 29.$$

Our assignment of $x_2^* = 2(10 + s_2)/3$ is not feasible for all possible values of s_2. If $s_2 \in [22, 24.5)$, then the minimizing value of x_2 is 23. So let

$$x_2^* = \begin{cases} 23 & \text{if } 22 \leq s_2 < 24.5 \\ 2(10 + s_2)/3 & \text{if } 24.5 \leq s_2 \leq 26 \end{cases}.$$

To find x_1^*, first note that $r_1 = 22$. Then $f_1(s_1, x_1) = 2(x_1 - s_1)^2 + 8(x_1 - 22) + f_2^*(x_1)$. By setting the partial derivative of f_1 with respect to x_1 equal to 0 and noting that $s_1 = 26$, we solve for x_1^*. If $x_1^* \in [24.5, 26]$, then by our definition for f_2^* we find that x_1 is a minimizer of f_1 at 23, a contradiction. If $x_1^* \in [22, 24.5)$, then by our definition for f_2^* we find that $x_1 = 24$ is minimizes $f_1|_{x_1 \in [22, 24.5)}$. Thus, $x_1^* = 24$.

n	Possible s_n	$f_n(s_n, x_n)$	x_n^*
4	$19 \leq s_4 \leq 26$	$2(26 - s_4)^2$	26
3	$23 \leq s_3 \leq 26$	$2(x_3 - s_3)^2 + 8(x_3 - 19) + f_4^*(x_3)$	$\frac{24+s_3}{2}$
2	$22 \leq s_2 \leq 26$	$2(x_3 - s_2)^2 + 8(x_2 - 23) + f_3^*(x_2)$	$\begin{cases} 23 & \text{if } 22 \leq s_2 < 24.5 \\ 2(10 + s_2)/3 & \text{if } 24.5 \leq s_2 \leq 26 \end{cases}$
1	26	$2(x_3 - s_1)^2 + 8(x_1 - 22) + f_2^*(x_1)$	24

Table 3.2: Objective Functions and Minimizers

Given our values of x_n^* in terms of s_n for $n = 1, 2, 3, 4$, we find $x_1^* = 24$, $x_2^* = 23$, $x_3^* = 23.5$, and $x_4^* = 26$.

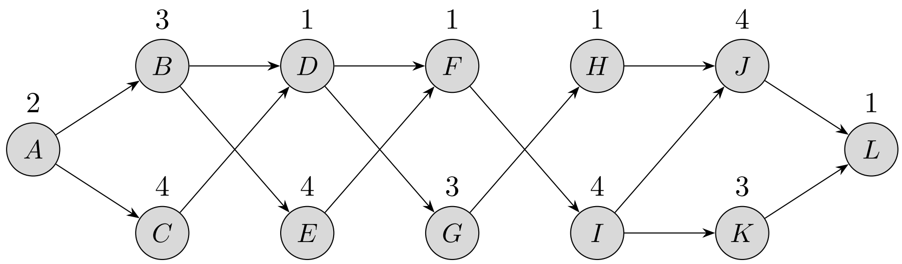
x_1^*	x_2^*	x_3^*	x_4^*
24	23	23.5	26

Table 3.3: Optimal Decision Variables

Problem 4

It is not surprising that OR techniques are used for project management. Imagine a complicated process (for example, construction of a new CHP building or a massive software development project). A project would proceed in a number of stages; some tasks can be done in parallel (for example, foundation work and exterior plumbing), but other tasks can only be started after prerequisite ones have completed (for example, putting in drywall without a roof is a bad idea).

The picture below shows the diagram of sub-tasks in a project. An arrow from node C to node D means that task C is a prerequisite for task D. The numbers above each node say how long does each task take. Your overall goal is to determine how long will this project take.



(a) Structure this problem as a dynamic programming problem. What are the stages, states, decision variables?

(b) Use dynamic programming to solve the problem. Your solution should contain all the needed values of $f^*(s_n)$ and (graphical) indication of the longest path through this "project network".

(c) In the industry, what you did in parts (a) and (b) is called a *critical path method* (CPM). You can read a part of Section 10.8 of the textbook for background. Identify the critical path(s) in this problem and explain their practical significance.

Solution 4. **(a)** Let's identify all relevant quantities to our problem:

- Stages: Stage n is the set of tasks that can be done in parallel after $n-1$ stages of preliminary tasks have been completed.
- States: Tasks that can be completed in parallel after prerequisite tasks are fulfilled. The state s_n is the task being completed at stage n.
- Decision Variables: The decision variable x_n is the next task s_{n+1} to be completed at stage n. $(s_n = x_{n-1})$.

At each stage n, we want to maximize the value of the function:

$$f_n(s_n, x_n) = p_n(s_n) + p_{n+1}(x_n) + \sum_{i=n+2}^{7} \max\{p_i(s_i)\}$$

where $p_n(s_n)$ is the amount of time it takes to complete task s_n at stage n.

Instead of having values on our edges that correspond to contribution to the objective function, we instead have values that sit above each node that represent the time it takes to complete these tasks. These are the values that we add to our objective function at each stage, and thus the ones that we wish to maximize in order to find the longest path.

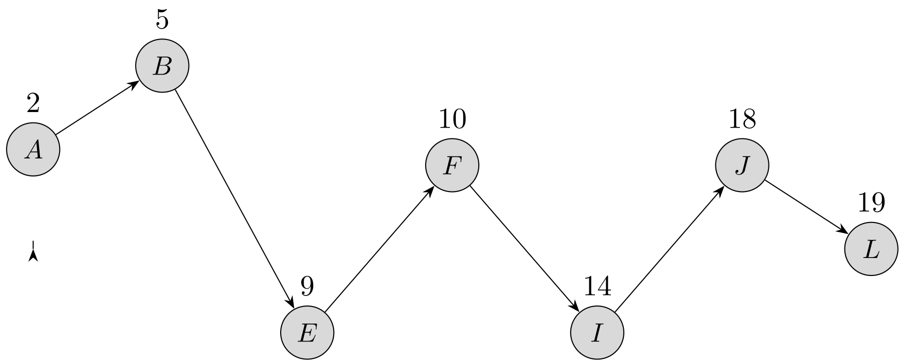


Figure 5: The critical path for the project is the **longest** path possible and gives the project duration.

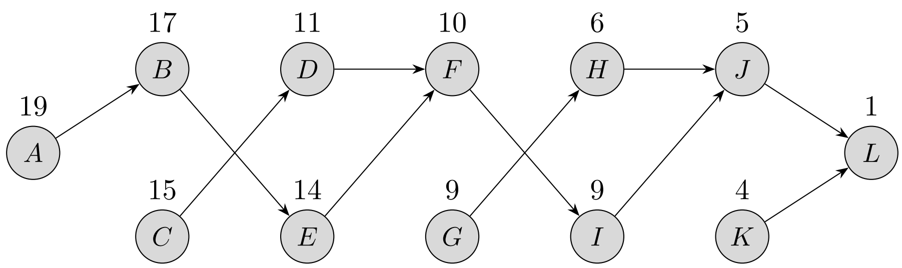


Figure 6: The f_n^* values at each stage.

(b) Let $f_n^*(s_n) = \max_{x_n \in \{s_n\}}\{f_n(s_n, x_n)\}$ be the maximal value attainable by the function f_n. Then:

$$f_n(s_n, x_n) = p_n(s_n) + f_{n+1}^*(s_{n+1}).$$

Since there is only one state s_7 at stage 7 and $p_7(s_7) = 1$, we have $f_7^*(s_7) = 1$. We then find $f_n^*(s_n)$ for each $n = 1, 2, \ldots, 6$ recursively backwards. The result is shown in Figure 6. The longest path through the graph to maximize the total amount of time that this project will take is:

$$A \to B \to E \to F \to I \to J \to L$$

which gives a maximal value of 19 time units.

(c) The critical path is the longest path and therefore the expected time that this project will take to complete (so-called *project duration* which was identified in part (b)). The critical path is significant because it is the sequence of tasks that takes the longest time to complete. Since tasks in a stage may be completed in parallel, the duration of time required to complete the tasks in the critical path determines the duration of time required to complete all the tasks. This can be seen in Table 4.1

Path	Length
$ABDFIJL$	16
$ABDFIKL$	15
$ABDGHJL$	15
$ABEFIJL$	19
$ABEFIKL$	18
$ACDFIJL$	17
$ACDFIKL$	16
$ACDGHJL$	15

Table 4.1: The 8 possible paths through the graph and their corresponding lengths.

References

[1] Frederick Hillier and Gerald Lieberman. *Introduction to Operations Research*. 11th ed. McGraw-Hill Education, 2021.