Conditional Probabilities

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Formulas, Vocabulary, and Notation

Sample Space(Ω) - Set of all possibilities for an outcome.

Event($A \subseteq \Omega$) - A is a subset of Ω such that both Ω and A are finite.

Probability of an event $P(A)$ - $|A|$ divided by $|\Omega|$ where A is our subset and Ω is the set of all possibilities:

$$P(A) = \frac{|A|}{|\Omega|}$$

[1]Counting -

- n is the number of distinct items or events to choose from and we choose r of them.
    - Combinations (order **does not** matter) without repetition: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Permutations (order **does** matter) without repetition: $\frac{n!}{(n-r)!}$

Conditional Probability

Now suppose that we are interested in determining the probability of an event occurring from a sample space Ω. However, we are also aware that there is a history of another event from Ω that has previously occurred and may potentially impact our probability. So then how can we begin to determine the probability of our event while simultaneously accounting for this past occurrence?

Conditional Probability

Definition: If $P(B) > 0$, the **conditional probability** of A given B, denoted by $P(A \mid B)$ is:

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

This definition had been used implicitly even before it was eventually introduced by De Moivre. [3]

Recall that $P(AB)$ is $P(A \cap B)$, the intersection of both events occurring.

Simple Example

Suppose that we have a set of families each with **exactly two** children. From a certain family it is found that one child is a girl. What is the probability that the other child is also a girl? What is the probability that the other child is a boy?
So we know that there are four possibilities of the children in the family.

$$\Omega = \{[girl, girl], [girl, boy], [boy, boy], [boy, girl]\}$$

Notice that it looks like there is a redundancy with [girl, boy] and [boy, girl], but **order matters** so we need to include both cases.

Simple Example Cont'd.

Now denote B as the child selected being a girl. Then, let A_1 be the event that the other child is **also** a girl and A_2 be the event that the other child is **not** a girl, i.e. boy. It follows that $P(A_1B) = \frac{1}{4}$, $P(A_2B) = \frac{1}{2}$, and $P(B) = \frac{3}{4}$.
First, we will solve $P(A_1 \mid B)$ the probability of A_1 given B:

$$P(A_1 \mid B) = \frac{1/4}{3/4} = \frac{1}{3}$$

Second, we will solve $P(A_2 \mid B)$ the probability of A_2 given B:

$$P(A_2 \mid B) = \frac{1/2}{3/4} = \frac{2}{3}$$

Definition of Dependence and Independence

Events A and B are said to be **independent** if:

$$P(A \cap B) = P(A) \cdot P(B)$$

or equivalently, P(A|B) = P(A)

This means that if A is independent of B, the occurrence of B **does not change** the probability of the occurrence of A.

If two events are not independent they are called **dependent**.

Example of Independence

Think of flipping a fair coin twice, where A and B are the events of getting heads on the first and second tosses, respectively. This would mean that A and B are **independent** of each other as the first flip does not necessarily alter the second flip.

Proof:

Note as they're fair coins $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$

We then have a sample space of four equal events :

$$\Omega = \{HH, HT, TH, TT\}$$

This means we have:

$$P(AB) = P(HH) = P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Example of Dependence

Being **dependent** really just means your chances **are affected** by other events. We begin with 5 green marbles (A), 3 blue marbles (B), and 2 red marbles(C) to make a total of $n = 10$ marbles. Let's say we draw a marble and know that it is **not red**, which would create the event $\neg C$. Let's take a look at our updated subsets: $|\neg C| = 8$, $|B| = 3$, $|B \cap \neg C| = 3$ and the conditional probability $P(B|\neg C) = 3/8$ but the probability $P(B)$ is 3/10 and thus not equivalent. So this makes B and $\neg C$ dependent and the same would follow for other combinations.

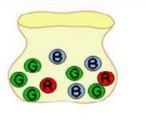
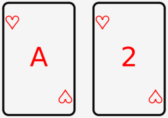


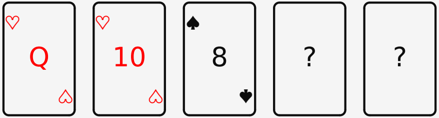
Figure 1: Probabilities of choosing from a total of 10 marbles. [2]

Conditional Probability in Texas Hold'em

This is your hand:



These are the cards on the table:



So, what is the probability that you will hit a flush in this hand? (5 total hearts between your hand and the cards on the table)

Solution

Note: You can get a flush on the fourth card (turn) or the fifth card (river).
Note: There are 52 cards in the deck with 13 hearts.

Let A = Heart on Turn, B = Heart on River

$$P(A) = \frac{13-4}{52-5} = \frac{9}{47} \approx 0.1915$$

Assume you miss the Flush on the Turn, then the P(B) is:

$$P(B \cap \neg A) = P(B \mid \neg A) * P(\neg A) = \left(\frac{9}{46}\right) * \left(1 - \frac{9}{47}\right) \approx 0.1581$$

In conclusion, you have a **19**% chance to hit a flush on the Turn and a **16**% chance to hit a flush on the River if you miss on the Turn. So, in total, you have a $(19 + 16) =$ **35%** chance to hit a Flush after the Flop.

References

[1] Rod Pierce DipCE BEng. *Combinations and Permutations*. 2017. URL: `https://www.mathsisfun.com/combinatorics/combinations-permutations.html`.

[2] Rod Pierce DipCE BEng. *Conditional Probability*. 2017. URL: `https://www.mathsisfun.com/data/probability-events-conditional.html`.

[3] Saeed Ghahramani. *Fundamentals of Probability, with Stochastic Processes, 3rd Edition*. 3rd Edition. Prentice Hall, 2004. ISBN: 0131453408,9780131453401.

Questions?