Topics in Linear Algebra

Basis Change & Linear Transformations

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Standard Basis and Basis Change

Standard basis for $\mathbb{R}^n$

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_1, ..., \mathbf{v}_n\}$ in a vector space V is a basis for V when the conditions the conditions below are true [1]:

1. S spans V
2. S is linearly independent.

A standard basis for $\mathbb{R}^n$ is:

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, 0, ..., 0) \\ \mathbf{e}_2 &= (0, 1, 0, ..., 0) \\ &\vdots \\ \mathbf{e}_n &= (0, 0, 0, ..., 1) \end{aligned}$$

Basis Change

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that [1]:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n$$

The scalars $c_1, c_2, .., c_n$ are the **coordinates of x relative to the basis** B. The **coordinate matrix of x relative to** B is the column matrix in $\mathbb{R}^n$ whose components are the coordinates of $\mathbf{x}$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Example (from Linear Refresher 1)

Determine the components of:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

relative to the basis B where:

$$B = \begin{bmatrix} 7 & 3 \\ -1 & 1 \end{bmatrix}$$

We are looking for each c_i where

$$\mathbf{e}_1 = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2$$

$$\mathbf{e}_2 = c_3 \mathbf{b}_1 + c_4 \mathbf{b}_2$$

where each $\mathbf{b}_i$ denotes the column vectors of B.

Solution for $\mathbf{e}_1$

Construct the augmented matrix:

$$\begin{bmatrix} 7c_1 & 3c_2 & 1 \\ -c_1 & c_2 & 0 \end{bmatrix}$$

After performing Gauss-Jordan elimination to put matrix into reduced row echelon form we obtain:

$$\begin{bmatrix} c_1 & 0 & \frac{1}{10} \\ 0 & c_2 & \frac{1}{10} \end{bmatrix}$$

And so

$$\mathbf{e}_1 = \frac{1}{10}\mathbf{b}_1 + \frac{1}{10}\mathbf{b}_2$$

Then the coordinate matrix of $\mathbf{e}_1$ relative to the basis B is:

$$[\mathbf{e}_1]_B = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{10} \end{bmatrix}$$

Alternative Solution for $\mathbf{e}_1$

Construct a system of equations:

$$7c_1 + 3c_2 = 1$$

$$-c_1 + c_2 = 0$$

Thus,

$$c_1 = c_2$$

and

$$7c_1 + 3c_1 = 1$$

Resulting in:

$$c_1 = c_2 = \frac{1}{10}$$

Solution for $\mathbf{e}_2$

Construct the augmented matrix:

$$\begin{bmatrix} 7c_3 & 3c_4 & 0 \\ -c_3 & c_4 & 1 \end{bmatrix}$$

After performing Gauss-Jordan elimination to put matrix into reduced row echelon form we obtain:

$$\begin{bmatrix} c_3 & 0 & \frac{-3}{10} \\ 0 & c_4 & \frac{7}{10} \end{bmatrix}$$

And so

$$\mathbf{e}_2 = \frac{-3}{10}\mathbf{b}_1 + \frac{7}{10}\mathbf{b}_2$$

Then the coordinate matrix of $\mathbf{e}_2$ relative to the basis B is:

$$[\mathbf{e}_2]_B = \begin{bmatrix} \frac{-3}{10} \\ \frac{7}{10} \end{bmatrix}$$

Alternative Solution for $\mathbf{e}_2$

Construct a system of equations:

$$7c_3 + 3c_4 = 0$$

$$-c_3 + c_4 = 1$$

Thus,

$$c_4 = c_3 + 1$$

and

$$7c_3 + 3c_3 + 3 = 10c_3 + 3 = 0$$

Resulting in:

$$c_3 = \frac{-3}{10}$$

and

$$c_4 = 1 - \frac{3}{10} = \frac{7}{10}$$

Relation to Inverse of a Matrix

Since we are finding the components of the standard basis relative to the basis B, we can simply find the inverse of basis B:

$$B^{-1} = \begin{bmatrix} 7 & 3 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{-3}{10} \\ \frac{1}{10} & \frac{7}{10} \end{bmatrix} = \begin{bmatrix} [\mathbf{e}_1]_B & [\mathbf{e}_2]_B \end{bmatrix}$$

Example 2

Let's examine another example. Determine the components of:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

relative to the basis B where:

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

We are looking for each scalar c_i where

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$$

and each $\mathbf{b}_i$ denotes the column vectors of B.

Solution

Construct the augmented matrix:

$$\begin{bmatrix} c_1 & 3c_2 & 1 \\ 2c_1 & c_2 & 7 \end{bmatrix}$$

After performing Gauss-Jordan elimination to put matrix into reduced row echelon form we obtain:

$$\begin{bmatrix} c_1 & 0 & 4 \\ 0 & c_2 & -1 \end{bmatrix}$$

And so

$$\mathbf{v} = 4\mathbf{b}_1 - \mathbf{b}_2$$

Then the coordinate matrix of $\mathbf{v}$ relative to the basis B is:

$$[\mathbf{v}]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Linear Transformations

Definition

Let V and W be vector spaces. The function

$$T : V \to W$$

is a **linear transformation** of V into W when two properties below are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and for any scalar c [1].

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

Claim: The following Linear Mapping is a Linear Transformation

$$f(x_1, x_2) = (x_2, -x_1)$$

How to Solve: We need to confirm that this Linear Mapping satisfies the two conditions of a Linear Transformation

Visualizing This Transformation

$$f(x_1, x_2) = (x_2, -x_1)$$

This function takes any vector **u** and rotates it clockwise 90 degrees or from one quadrant to it's following quadrant.

Transformation

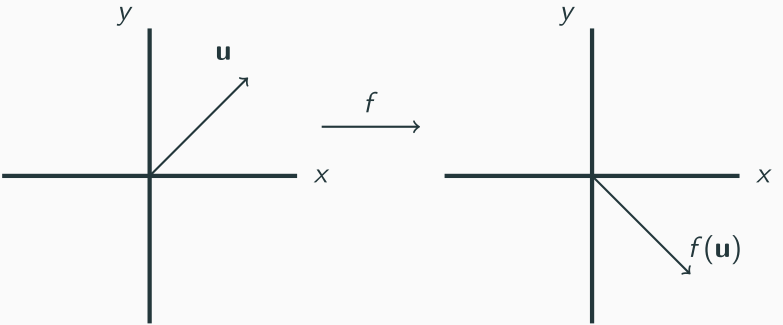


Figure 1: Visual interpretation of the linear transformation f acting on the vector $\mathbf{u}$.

Ex1 Solution

Claim: The following Linear Mapping is a Linear Transformation

$$f(x_1, x_2) = (x_2, -x_1)$$

Part 1: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$**.** Suppose $\mathbf{u}$, $\mathbf{v} \in V$. Let $\mathbf{u} =$ (a,b) and $\mathbf{v} =$ (c,d). Then $\mathbf{u} + \mathbf{v} =$ (a,b) + (c,d) = (a+c,b+d). Then $T(\mathbf{u} + \mathbf{v}) =$ (b+d, -a-c). Secondly, let $T(\mathbf{u}) =$ (b, -a) and $T(\mathbf{v}) =$ (d, -c). Then $T(\mathbf{u}) + T(\mathbf{v}) =$ (b+d,-a-c).
Thus

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

.

Solution (Continued)

Part 2: $T(c\mathbf{u}) = cT(\mathbf{u})$**.** Suppose $\mathbf{u} \in V$. Let $\mathbf{u}$ = (a,b). c$\mathbf{u}$ = c(a,b) = (ca,cb). $T(c\mathbf{u}) = T(ca, cb)$ = (cb,-ca). Secondly, c$T(\mathbf{u})$ = c$T(a, b)$ = c(b, -a) = (cb,-ca).

Conclusion: Thus both requirements are satisfied for a linear transformation. Thus

$$f(x_1, x_2) = (x_2, -x_1)$$

is a linear transformation.

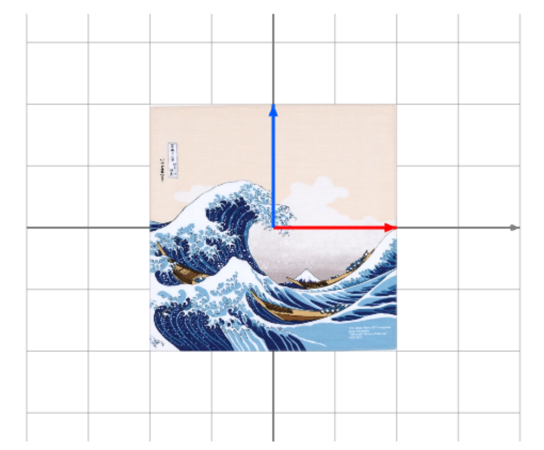
Real World Application

Visualizing a Matrix

One way to visualize a matrix is to think of a digital image. If you were to zoom in and see every pixel, then you could think of a vertical strip of pixels as representing a vector and each pixel as an entry consisting of an ordered triple representing a RGB value.

Visualizing a Matrix

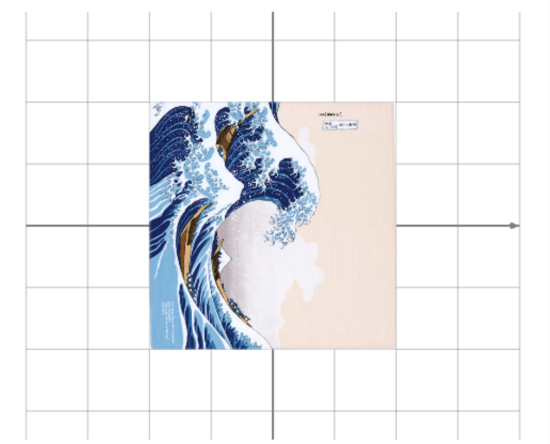
Now imagine this image in a plane [2]:



Visualizing a Linear Transformation

Then applying to our image the same linear transformation specified above, i.e.

$$f(x_1, x_2) = (x_2, -x_1)$$



References

Ron Larson. *Elementary Linear Algebra*. 8th ed. Brooks Cole, 2016. ISBN: 1305658000,9781305658004.

Yuri Sulyma. URL: https://web.ma.utexas.edu/users/ysulyma/matrix/.

Questions?