Exam 2

Problem (1a)**.** Find the Laplace Transforms of the function

$$f(t) = \int_0^t \frac{\sin ax}{x} dx$$

Solution.

Theorem (3.64)**.**

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{\bar{f}(s)}{s}$$

By the Theorem 3.6.4, let $x := t$, then

$$\mathscr{L}\{f(t)\} = \frac{1}{s}\int_s^\infty \mathscr{L}\{\sin at\}\ ds = \frac{1}{s}\int_s^\infty \frac{a}{s^2+a^2}\ ds$$

$$= \frac{1}{s}\arctan\left(\frac{s}{a}\right)\Big|_s^\infty = \frac{1}{s}\left(\frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)\right)$$

But since $\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right) = \arctan\left(\frac{y}{x}\right)$, then

$$\therefore \mathscr{L}\{f(t)\} = \frac{1}{s}\arctan\left(\frac{a}{s}\right)$$

□

Problem (1b)**.** Find the Laplace Transforms of the function

$$f(t) = tH(t-a)$$

Solution.

Theorem (3.4.2)**.**

$$\textit{If } \mathscr{L}\{f(t)\} = \bar{f}(s), \textit{ then } \mathscr{L}\{f(t)H(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$$

By Theorem 3.4.2,

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{tH(t-a)\} = e^{-as}\mathscr{L}\{t+a\} \implies e^{-as}\left(\mathscr{L}\{t\} + \mathscr{L}\{a\}\right) = e^{-as}\left(\frac{1}{s^2} + \frac{a}{s}\right)$$

□

Problem (2a)**.** Find the inverse Laplace transform of the function $f(s) = \dfrac{s}{(s-a)(s^2+b^2)}$ $(a > 0, b > 0)$ by using partial fraction decomposition.

Solution.

$$\frac{s}{(s-a)(s^2+b^2)} = \frac{A}{s-a} + \frac{Bs+C}{s^2+b^2}$$

$$s = A\left(s^2+b^2\right) + (Bs+C)(s-a)$$

If $s = a$, then $A\left(a^2+b^2\right) = a \Rightarrow A = \dfrac{a}{a^2+b^2}$. Now, let's isolate $Bs+C$

$$Bs + C = \frac{s - A\left(s^2+b^2\right)}{s-a}$$

$$= \frac{s - \left(\dfrac{a}{a^2+b^2}\right)\left(s^2+b^2\right)}{s-a}$$

$$= \frac{sa^2 + sb^2 - as^2 - ab^2}{\left(a^2+b^2\right)(s-a)}$$

$$Bs + C = \frac{s\left(a^2+b^2\right) - a\left(s^2+b^2\right)}{\left(a^2+b^2\right)(s-a)}$$

If $s = 0$*, then* $C = \dfrac{ab^2}{a} \implies C = b^2$*, and if* $b = 0$*, then*

$$Bs = \frac{a^2 s - as^2}{s - a}$$
$$B = -a$$

Then our partial fraction decomposition gives us

$$\bar{f}(s) = \frac{1}{a^2 + b^2}\left(\frac{b^2}{b^2 + s^2} - \frac{as}{b^2 + s^2} + \frac{a}{s - a}\right)$$

And the inverse Laplace transform $\mathscr{L}^{-1}$ *gives us*

$$\frac{1}{a^2 + b^2}\left(b \sin bt - a \cos at + ae^{at}\right)$$

□

Problem (2b)**.** Find the inverse Laplace transform of the function $f(s) = \dfrac{s}{(s-a)(s^2 + b^2)}$ $(a > 0, b > 0)$ by using the convolution theorem.

Solution.

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s}{(s-a)(s^2+b^2)}\right\} &= e^{at} * \cos bt \\ &= \int_0^t e^{a\tau} \cos b(t-\tau) d\tau\end{aligned}$$

Converting $\cos b(t - \tau)$ *to exponential form gives*

$$\begin{aligned}&\frac{1}{2}\int_0^t e^{ib(t-\tau)+a\tau} d\tau + \frac{1}{2}\int_0^t e^{-ib(t-\tau)+a\tau} d\tau \\ &= \frac{1}{2}\left(\frac{e^{ib(t-\tau)+a\tau}}{a - ib} + \frac{e^{-ib(t-\tau)+a\tau}}{a + ib}\bigg|_0^t\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left(\frac{e^{at} - e^{ibt}}{a - ib} + \frac{e^{at} - e^{-ibt}}{a + ib}\right) &= \frac{1}{2(a^2 + b^2)}\left(ae^{at} - a\left(e^{ibt} + e^{-ibt}\right) + b\left(e^{ibt} - e^{-ibt}\right)\right) \\ &= \frac{a(e^{at} - \cos bt) + b \sin bt}{a^2 + b^2}\end{aligned}$$

□

Problem (3a)**.** Evaluate the improper definite integral

$$\int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx \quad (\text{a}, \text{t} > 0)$$

Solution. *Let* $f(t) = \int_{\mathbb{R}} \frac{\cos tx}{x^2+a^2}\, dx$*. Since the numerator and denominator are both even functions, then we have that* $2f(t) = \int_{\mathbb{R}^+} \frac{\cos tx}{x^2+a^2}\, dx$*. Applying* $\mathscr{L}$ *we get:*

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \int_{\mathbb{R}^+} e^{-st} \cos tx dt \int_{\mathbb{R}^+} \frac{dx}{x^2 + a^2} \\ &= \int_{\mathbb{R}^+} \frac{s}{(x^2 + a^2)(s^2 + a^2)} dx\end{aligned}$$

By partial fraction decomposition, we have

$$\frac{s}{s^2 - a^2}\int_{\mathbb{R}^+}\left(\frac{1}{x^2 + a^2} - \frac{1}{s^2 + x^2}\right) dx;$$
$$\frac{s}{s^2 - a^2}\left(\frac{1}{a}\arctan\left(\frac{x}{a}\right)\bigg|_0^\infty - \frac{1}{s}\arctan\left(\frac{x}{s}\right)\bigg|_0^\infty\right)$$

$$= \frac{s}{s^2-a^2}\left(\frac{\pi}{2a}-\frac{\pi}{2s}\right) = \frac{\pi}{2}\left(\frac{s}{s^2-a^2}\left(\frac{s-a}{as}\right)\right)$$

$$\frac{\pi}{2a}\cdot\frac{1}{s+a}$$

Applying $\mathscr{L}^{-1}$ gives us $f(t)=\frac{\pi}{2a}e^{-at}$, but since $f(t)=\frac{\cos tx}{x^2+a^2}$ is an even function, we restricted the bounds of the integral from $\mathbb{R}$ to $\mathbb{R}^+$, and therefore we have $2f(t)=\frac{\pi}{a}e^{-at}$. □

Problem (3b)**.**

$$\text{Show that } \int_0^\infty \frac{\sin(\pi t x)}{x\left(1+x^2\right)}dx = \frac{\pi}{2}\left(1-e^{-\pi t}\right), \quad t>0.$$

Solution. *Applying $\mathscr{L}$ to the integral gives us*

$$\int_{\mathbb{R}^+}\frac{dx}{x\left(1+x^2\right)}\int_{\mathbb{R}^+} e^{-st}\sin(\pi t x)dt$$

$$\int_{\mathbb{R}^+}\frac{\pi x}{x\left(1+x^2\right)\left(s^2+(\pi x)^2\right)}\,dx = \int_{\mathbb{R}^+}\frac{\pi\ dx}{\left(1+x^2\right)\left(s^2+(\pi x)^2\right)}$$

By partial fraction decomposition, we get

$$\frac{1}{\pi^2-s^2}\int_{\mathbb{R}^+}\left(\frac{-1}{1+x^2}+\frac{\pi^2}{(\pi x)^2+s^2}\right)dx = \frac{1}{\pi^2-s^2}\left(\frac{\pi}{s}\arctan\left(\frac{\pi x}{s}\right)\Big|_0^\infty - \arctan(x)|_0^\infty\right)$$

$$= \frac{1}{\pi^2-s^2}\left(\frac{\pi^2}{2s}-\frac{\pi}{2}\right) = \frac{\pi}{2\left(\pi^2-s^2\right)}\left(\frac{\pi-s}{s}\right) = \frac{\pi}{2(\pi+s)(\pi-s)}\left(\frac{\pi-s}{s}\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{s(\pi+s)}\right)$$

Applying the inverse Laplace transform and by Example 3.7.1, we get

$$\frac{\pi}{2}\mathscr{L}^{-1}\left\{\frac{1}{s(\pi+s)}\right\} = \frac{\pi}{2}\left(1-e^{-\pi t}\right)$$

□

Problem (4a)**.** Apply the Laplace transform to solve the following IVP problem

$$y''+2ay'+\left(a^2+4\right)y=f(t), \quad y(0)=1, y'(0)=-a$$

Solution. *Applying $\mathscr{L}$ to the system gives us*

$$s^2\bar{y}-sy(0)-y'(0)+2a(s\bar{y}-y(0))+\left(a^2+4\right)\bar{y}=\bar{f}(s)$$

Since $y(0)=1, y'(0)=-a$, then

$$s^2\bar{y}-s+a+2as\bar{y}-2a+\bar{y}\left(a^2+4\right)=\bar{f}(s)$$

$$\bar{y}\left(s^2+2as+a^2+4\right)=\bar{f}(s)+a+s$$

$$\bar{y}=\frac{\bar{f}(s)+a+s}{s^2+2as+a^2+4}$$

$$\bar{y}=\frac{\bar{f}(s)+a+s}{(s+a)^2+2^2}$$

$$\bar{y}=\frac{\bar{f}(s)}{(s+a)^2+2^2}+\frac{s+a}{(s+a)^2+2^2}$$

Applying the inverse Laplace transform gives us

$$y=\int_0^t f(t-\tau)\frac{\sin(2\tau)}{2}e^{-a\tau}d\tau+e^{-at}\cos(2t)$$

□

Problem (4b)**.** Apply the Laplace transform to solve the following IVP problem

$$u_{tt} = c^2 u_{xx} + \sin x, \quad 0 < \pi < x, \quad t > 0$$
$$u(0,t) = u(\pi,t) = 1, \quad u(x,0) = u_t(x,0) = 0.$$

Solution. *Applying the Laplace transform to the system gives us*

$$s^2\bar{u} - su(x,0) - u'(x,0) = c^2\bar{u}'' + \frac{\sin x}{s}$$

where $\bar{u}(0,s) = \bar{u}(\pi,s) = \frac{1}{s}, \bar{u}(x,0) = u'(x,0) = 0$*, and therefore* $s^2\bar{u} = c^2u'' + \frac{\sin x}{s}$*. First, solve the homogeneous case*

$$s^2\bar{u} = c^2\bar{u}''; c^2\bar{u}'' - s^2\bar{u} = 0$$

The auxiliary equation is $m^2 = \frac{s^2}{c^2}$*;* $m = \pm\frac{s}{c}$ *and therefore* $\bar{u} = C_1e^{xs/c} + C2e^{-xs/c}$*.*

$$\bar{u}(0,s) = \frac{1}{s} \Rightarrow C_1 + C_2 = \frac{1}{s}, \quad C_1 = \frac{1}{s} - C_2$$
$$\bar{u} = \left(\frac{1}{s} - C_2\right)e^{xs/c} + C_2e^{-xs/c}$$

And $\bar{u}'(x,0) = 0 \Rightarrow \left(\frac{1}{s} - C_2\right)\frac{x}{c} - C_2\frac{x}{c} = 0$

$$x\left(\frac{1}{s} - C_2 - C_2\right) = 0 \Rightarrow C_2 = \frac{1}{2s} \text{ and } C_1 = -\frac{1}{2s}$$

Now, solving for a particular solution using the method of undetermined coefficients, we have

$$\bar{u} = A\sin x + B\cos x$$
$$\bar{u}' = A\cos x - B\sin x$$
$$\bar{u}'' = -A\sin x - B\cos x$$

Then $c^2\bar{u}'' - s^2\bar{u} = c^2(-A\sin x - B\cos x) - s^2(A\sin x + B\cos x)$ *and*

$$\sin x\left(A\left(-c^2 - s^2\right)\right) + \cos x\left(B\left(c^2 + s^2\right)\right) = \sin x/s$$
$$\Rightarrow B = 0 \text{ and } A\left(-c^2 - s^2\right) = \frac{1}{s}; \quad \therefore A = \frac{-1}{s\left(c^2 + s^2\right)}$$

Then

$$\bar{u} = \frac{-\sin x}{s\left(c^2 + s^2\right)} + \frac{1}{2s}e^{-xs/c} - \frac{1}{2s}e^{xs/c}$$

Then taking the inverse Laplace transform $\mathscr{L}^{-1}$*:*

$$u = \frac{\sin x(\cos ct - 1)}{c^2} - \frac{1}{2}\left(H\left(t + \frac{x}{c}\right) + H\left(t - \frac{x}{c}\right)\right)$$

□

Problem (5)**.** Apply the Laplace transform to solve the following wave problem

$$\frac{\partial u^2(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2} + f(t), \quad x > 0,\ t > 0$$

subject to

$$u(0,t) = 0, t > 0, \text{ and } u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0,\ x > 0.$$

Solution. *Applying the Laplace transform to the system gives*

$$s^2\bar{u} - su(0) - u'(0) = c^2u'' + \bar{f}(s)$$
$$\text{where } \bar{u}(0,s) = 0, \bar{u}(x,0) = 0, u'(0) = 0$$
$$\Rightarrow s^2\bar{u} = c^2u'' + \bar{f}(s)$$
$$u'' - \frac{s^2}{c^2}\bar{u} + \frac{f(s)}{c^2} = 0$$

Let's first solve for the homogeneous case

$$c^2u'' - s^2\bar{u} = 0$$

$$\textit{Auxiliary equation} \quad c^2m^2 - s^2 = 0;\ m^2 = \frac{s^2}{c^2};\ m = \pm\frac{s}{c}$$

$$\Rightarrow \bar{u} = C_1e^{xs/c} + C_2e^{-xs/c}$$

If $\bar{u}(0, s) = 0$, *then* $0 = C_1 + C_2 \Rightarrow C_1 = -C_2$. *Now, we must deal with the non-homogeneous term* $f(t)$ *to find a particular solution. Let* $y_1 = e^{xs/c}$ *and* $y_2 = e^{-xs/c}$, *then the Wronskian is*

$$W = \begin{vmatrix} e^{-xs/c} & e^{xs/c} \\ -\frac{s}{c}e^{-xs/c} & \frac{s}{c}e^{xs/c} \end{vmatrix} = \frac{2s}{c}$$

and

$$W_1 = \begin{vmatrix} 0 & e^{-xs/c} \\ \frac{f(s)}{c^2} & -\frac{s}{c}e^{-xs/c} \end{vmatrix} = -\frac{\bar{f}(s)e^{-xs/c}}{c^2}; \quad W_2 = \begin{vmatrix} e^{xs/c} & 0 \\ \frac{x}{s}e^{xs/c} & \frac{\bar{f}(s)}{c^2} \end{vmatrix} = \frac{\bar{f}(s)e^{xs/c}}{c^2}$$

Our particular solution is of the form

$$\bar{u} = v_1y_1 + v_2y_2$$

For

$$v_1 = \int \frac{W_1}{W} = -\int \frac{\bar{f}(s)e^{-xs/c}}{c^2 2s/c}\ ds = \frac{-1}{2c}\int \frac{\bar{f}(s)e^{-xs/c}}{s}\ ds$$

and similarly, $v_2 = \int \frac{W_2}{W} = \frac{1}{2c}\int \frac{f(s)e^{xs/c}}{s}\ ds$ *So*

$$\begin{aligned} \bar{u} &= v_1y_1 + v_2y_2 \\ &= \frac{1}{2c}\left(e^{xs/c}\int \frac{e^{-xs/c}f(s)}{s}\ ds + e^{-xs/c}\int \frac{e^{xs/c}f(s)}{s}\ ds\right) \\ &= \frac{1}{2c}\left(\int \frac{f(s)}{s}\ ds + \int \frac{f(s)}{s}\ ds\right) \\ &= \frac{1}{c}\int \frac{f(s)}{s}\ ds \end{aligned}$$

Altogether, we have

$$\bar{u} = C_1e^{xs/c} - C_1e^{-xs/c} + \frac{1}{c}\int \frac{f(s)}{s}ds$$

If $\bar{u}(x, 0) = 0$, *then* $C_1 = 0$ *and therefore*

$$\bar{u} = \int \frac{f(s)}{s}\ ds \implies u = \int\int \frac{f(s)}{s}\ ds\ dt$$

□

Problem (6)**.** Solve the following integral equation by the Laplace transform

$$f(t) = t\cos at + a\int_0^t f(\tau)\sin a(t-\tau)d\tau$$

Solution. *Applying the Laplace transform to the system gives us*

$$\begin{aligned}
\bar{f}(s) &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2} + a\mathscr{L}\{f(t) * \sin at\} \\
\bar{f}(s) &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2} + a\bar{f}(s)\cdot\frac{a}{s^2+a^2} \\
\bar{f}(s) &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2} + \frac{a^2}{s^2+a^2}\bar{f}(s) \\
\bar{f}(s) - \frac{a^2}{s^2+a^2}\bar{f}(s) &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2} \\
\bar{f}(s)\left(1-\frac{a^2}{s^2+a^2}\right) &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2} \\
\bar{f}(s)\left(\frac{s^2+a^2-a^2}{s^2+a^2}\right) &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2} \\
\bar{f}(s)\left(\frac{s^2}{s^2+a^2}\right) &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2} \\
\bar{f}(s) &= \frac{s^2-a^2}{s^2\left(s^2+a^2\right)}
\end{aligned}$$

Partial fractions gives us

$$\bar{f}(s) = \frac{2}{a^2+s^2} - \frac{1}{s^2}$$

Finally, the inverse Laplace transform gives us $f(t) = \frac{2\sin(at)}{a} - t$ □

Problem (7)**.** Apply the Laplace transform to solve the following diffusion problem

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= K\frac{\partial u^2(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \\
u(x,0) &= 0, \quad 0 < x < 1 \\
u(0,t) &= f(t), \quad \frac{\partial u(1,t)}{\partial x} = 0, \quad t > 0
\end{aligned}$$

Solution. *Applying the Laplace transform to the system gives us*

$$\begin{aligned}
s\bar{u} &= k\bar{u}'' \\
\bar{u}'' - \frac{s}{k}\bar{u} &= 0
\end{aligned}$$

Then the auxiliary equation is give as

$$m^2 = \frac{s}{k};\ m = \pm\sqrt{\frac{s}{k}}$$

Then

$$\bar{u} = C_1\cosh(x\sqrt{s/k}) + C_2\sinh(x\sqrt{s/k})$$

and

$$\bar{u}' = C_1\sqrt{\frac{s}{k}}\sinh\left(x\sqrt{\frac{s}{k}}\right) + C_2\sqrt{\frac{s}{k}}\cosh\left(x\sqrt{\frac{s}{k}}\right)$$

From $\bar{u}(0,s) = \bar{f}(s)$ *implies* $C_1 = \bar{f}(s)$ *and* $\bar{u}(x,0) = 0$ *shows us* $C_2 = 0$ *and* $\bar{f}(0) = 0$*, then*

$$\bar{u} = \bar{f}(s)\cosh\left(x\sqrt{\frac{s}{k}}\right)$$

Taking the inverse Laplace transform gives us our final answer as

$$u = \int_0^t f(t-\tau)\cosh\left(x\sqrt{\frac{\tau}{k}}\right)\ d\tau$$

□

Problem (8a)**.** Solve the following difference equation,

$$u_{n+2} - 7u_{n+1} + 10u_n = 0,\ u_0 = 1, u_1 = 2$$

Solution. *Applying the Laplace transform to the difference equation yields*

$$e^{2s}\left[\bar{u} - \bar{S}_0\left(u_0 + u_1 e^{-s}\right)\right] - 7e^s\left[\bar{u} - \bar{S}_0 u_0\right] + 10\bar{u} = 0$$

where $u_0 = 1$ *and* $u_1 = 2$*. Then*

$$e^{2s}\left[\bar{u} - \bar{S}_0\left(1 + 2e^{-s}\right)\right] - 7e^s\bar{u} + 7e^s\bar{S}_0 + 10\bar{u} = 0$$
$$\bar{u}\left[e^{2s} - 7e^s + 10\right] = e^{2s}\left(1 + 2e^{-s}\right)\bar{S}_0 - 7e^s\bar{S}_0$$
$$\bar{u} = \frac{e^{2s} - 5e^s}{e^{2s} - 7e^s + 10}\bar{S}_0$$

Let $x = e^s$*, then*

$$\bar{u} = \frac{x^2 - 5x}{x^2 - 7x + 10}\bar{S}_0; \quad \bar{u} = \frac{x(x-5)}{(x-5)(x-2)}\bar{S}_0; \quad \bar{u} = \frac{x}{x-2}\bar{S}_0 \implies \bar{u} = \frac{e^s}{e^s - 2}\bar{S}_0$$

Applying the inverse Laplace transform $\mathscr{L}^{-1}$

$$u = \mathscr{L}^{-1}\left\{\frac{e^s}{e^s - 2}\bar{S}_0\right\}$$
$$\therefore u = 2^n$$

□

Problem (8b)**.** Solve the following differential difference equation

$$\frac{du}{dt} - 2u(t-1) = 0,\ u(0) = 1$$

Solution. *Applying* $\mathscr{L}$ *gives us*

$$s\bar{u} - u(0) - 2e^{-s}\left(\bar{u} - \bar{S}_0 u(0)\right) = 0 \qquad \textit{where } u(0) = 1$$
$$s\bar{u} - 1 - 2e^{-s}\bar{u} + 2e^{-s}\bar{S}_0 = 0; \quad \bar{u}\left(s - 2e^{-s}\right) = 1 - 2e^{-s}\bar{S}_0$$

Since $\bar{S}_0 = \dfrac{1}{s}\left(1 - \dfrac{e^s}{s}\right)$*, then*

$$\bar{u}\left(s - 2e^{-s}\right) = 1 - \frac{2e^{-s}}{s}\left(1 - \frac{e^{-s}}{s}\right)$$
$$\bar{u}\left(s - 2e^{-s}\right) = 1 - \frac{2e^{-s}}{s} + \frac{2e^{-2s}}{s}$$
$$\bar{u} = \left(\frac{1}{s - 2e^{-s}} - \frac{2e^{-s}}{s\left(s - 2e^{-s}\right)}\right) + \frac{2e^{-2s}}{s}$$
$$\bar{u} = \frac{1}{s} + \frac{2e^{-2s}}{s}$$

By applying the inverse Laplace transform $\mathscr{L}^{-1}$ *and Example 4.7.4*

$$u = 1 + \sum_{k=1}^{n}\frac{2^k(t-2)^k}{(k-2)!} \textit{ for } t > n$$

□