Problem 1. Use result (3.4.12) to find (a) $\mathscr{L}\{\cos at\}$ and (b) $\mathscr{L}\{\sin at\}$.

Solution 1. *(a) Since*

$$\frac{d}{dt}\frac{\sin(at)}{a} = \cos(at)$$

then

$$\begin{aligned}\mathscr{L}\{\cos(at)\} &= \frac{s}{a}\mathscr{L}\left\{\sin at\right\} - \sin(0)\\ &= \frac{s}{a}\cdot\frac{a}{s^2+a^2} = \frac{s}{s^2+a^2}\end{aligned}$$

(b) Since

$$\frac{d}{dt}\left(-\frac{1}{a}\cos at\right) = \sin(at)$$

then

$$\begin{aligned}\mathscr{L}\left\{\sin(at)\right\} &= \frac{-s}{a}\mathscr{L}\{\cos(at)\} + \frac{\cos(0)}{a}\\ &= \frac{-s}{a}\cdot\frac{s}{s^2+a^2} + \frac{1}{a} = \frac{-s^2}{as^2+a^3} + \frac{1}{a}\\ &= \frac{-as^2+as^2+a^3}{a^2s^2+a^4} = \frac{a^3}{a^2s^2+a^4} = \frac{a}{s^2+a^2}\end{aligned}$$

Problem 2. Show that $\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}\ du\right\} = \frac{1}{s}\int_s^\infty \overline{f}(x)\ dx$

Solution 2.

Theorem 1 (3.6.3)**.**

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \hat{f}(s)\ ds$$

Theorem 2 (3.4.3)**.** *If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then*

$$\mathscr{L}\left\{f'(t)\right\} = s\mathscr{L}\{f(t)\} - f(0) = s\bar{f}(s) - f(0),$$

Let $F(t) = \int_0^t \frac{f(u)}{u}du$, then by the Fundamental Theorem of Calculus $F'(t) = \frac{f(t)}{t}$. By Theorem 3.4.3

$$\mathscr{L}\{F'(g)\} = \mathscr{L}\left\{\frac{f(t)}{t}\right\} = s\overline{F}(s) - F(0)$$

Assume $F(0) = 0$, then

$$s\overline{F}(s) = \int_{\mathbb{R}+} \frac{f(t)}{t}e^{-st}\ dt$$

But

$$\int_{\mathbb{R}+} \frac{e^{-st}}{t}\ dt = \int_s^\infty e^{-st}\ ds$$

So

$$s\overline{F}(s) = \int_s^\infty \int_{\mathbb{R}+} e^{-st} f(t)\ dt\ ds$$

Since

$$\int_{\mathbb{R}+} e^{-st} f(t)\ dt = \overline{f}(s)$$

then

$$\overline{F}(s) = \frac{1}{s}\int_s^\infty \overline{f}(s)\ ds$$

Problem 3. Obtain the inverse Laplace transforms of the following

$$\frac{1}{s^2\left(s^2+c^2\right)}$$

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Solution 3.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2\left(s^2+c^2\right)\}}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}\mathscr{L}^{-1}\left\{\frac{1}{s^2+c^2}\right\}=t*\frac{\sin(ct)}{c}=\frac{1}{c}\int_0^t(t-\tau)\sin(c\tau)\ d\tau$$

Integration by parts: $u=t-\tau, du=-d\tau \wedge dv=\sin(c\tau), v=\frac{-\cos((\tau)}{c}$

$$\Rightarrow \frac{1}{c}\left[\frac{-t\cos(c\tau)}{c}+\frac{\tau\cos(c\tau)}{c}-\frac{1}{c}\int_0^t\cos(c\tau)d\tau\right]$$

$$=\frac{1}{c}\left[\frac{-t\cos(c\tau)}{c}+\frac{\tau\cos(c\tau)}{c}-\frac{1}{c^2}\sin(c\tau)\Big|_0^t\right.$$

$$=\frac{t}{c^2}-\frac{\sin(ct)}{c^3}=\frac{ct-\sin(ct)}{c^3}$$

Problem 4. Use the Convolution Theorem to find the inverse Laplace transforms of the following

$$\frac{s^2}{\left(s^2+a^2\right)^2}$$

Solution 4. *This is clearly*

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos(at)*\cos(at)=\int_0^t\cos(a\tau)\cos(at-a\tau)\ d\tau$$

Using product to sum

$$\Rightarrow \frac{1}{2}\int_0^t\cos(a\tau-(at-a\tau))+\cos\left(a\tau+(at-a^{\tau}\tau)\right)\ d\tau$$

$$=\frac{1}{2}\int_0^t\cos(2a\tau-at)\ d\tau+\frac{1}{2}\int_0^t\cos(at)\ d\tau$$

For the first integral in our sum, let $u=2a\tau-at, du=2ad\tau$

$$=\frac{1}{4a}\int\cos(u)\ du=\frac{\sin(2a\tau-at)}{4a}\Big|_0^t=\frac{\sin(at)-\sin(-at)}{4a}$$

$$=\frac{\sin(at)}{2a}$$

For the second integral $\int_0^t\cos(at)\ d\tau=t\cos(at)$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\frac{\sin(at)-at\cos(at)}{2a}$$

Problem 5. Show that

$$\text{(a)}\quad \mathscr{L}\left\{\frac{1}{t}(\sin at-at\cos at)\right\}=\tan^{-1}\left(\frac{a}{s}\right)-\frac{as}{s^2+a^2}$$

$$\text{(b)}\quad \mathscr{L}\left\{\int_0^t\frac{1}{\tau}(\sin a\tau-a\tau\cos a\tau)d\tau\right\}=\frac{1}{s}\left[\tan^{-1}\left(\frac{a}{s}\right)-\frac{as}{s^2+a^2}\right]$$

Solution 5. *(a) Breaking up the sum inside our Laplace transform, we have:*

$$\mathscr{L}\left\{\frac{\sin(at)}{t}\right\}-a\mathscr{L}\{\cos(at)\}$$

Theorem 3 (3.6.3)**.**

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\}=\int_s^\infty\hat{f}(s)\ ds$$

By Theorem 3.6.3,

$$\mathscr{L}\left\{\frac{\sin(at)}{t}\right\} = \int_s^\infty \frac{a}{a^2+s^2}\ ds = a\int_s^\infty \frac{ds}{a^2+s^2} = \frac{1}{a}\int_s^\infty \frac{ds}{\frac{s^2}{a^2}+1}$$

let $u = \frac{s}{a}, du = \frac{ds}{a}$, *then*

$$\int_s^\infty \frac{du}{u^2+1} = \arctan\left(\frac{s}{a}\right)\Big|_s^\infty = \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)$$

$$= \arctan\left(\frac{a}{s}\right)$$

For the second term in the difference, just apply the definition of the Laplace transform $\mathscr{L}$ *and the exponential form of* $\cos(at)$

$$= \frac{a}{2}\left(\int_{\mathbb{R}^+} e^{-t(s-a)}\ dt + \int_{\mathbb{R}^+} e^{-t(s+a)}\ dt\right)$$

$$= \frac{a}{2}\left(-\frac{e^{-t(s-a)}}{s-a} - \frac{e^{-t(s+a)}}{s+4}\Big|_0^\infty\right) = \frac{a}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)$$

$$= \frac{as}{s^2-a^2}$$

$$\therefore \mathscr{L}\left\{\frac{\sin(at)}{t}\right\} - a\mathscr{L}\left\{\cos(at)\right\} = \arctan\left(\frac{a}{s}\right) - \frac{as}{s^2-a^2}$$

(b)

Theorem 4 (3.6.4)**.**

$$\mathscr{L}\left\{\int_0^t f(t)dt\right\} = \frac{\hat{f}(s)}{s}$$

This implies

$$\Rightarrow \mathscr{L}\left\{\int_0^t \left(\frac{1}{\tau}\sin(a\tau) - a\cos(a\tau)\right)\ d\tau\right\} = \frac{1}{s}\int_s^\infty \widehat{\sin(at)}\ ds - \frac{1}{s}\int_s^\infty \widehat{\cos(at)}\ ds$$

Since we found these save expressions in part (a), then we have

$$\therefore \mathscr{L}\left\{\int_0^t \left(\frac{1}{t}\sin(a\tau) - a\cos(a\tau)\right)\ d\tau\right\} = \frac{1}{s}\left(\arctan\left(\frac{a}{s}\right) - \frac{as}{s^2-a^2}\right)$$

Problem 6. Show that $\mathscr{L}\left\{t^n \exp(at)\right\} = n!\,(s-a)^{-(n+1)}$

Solution 6.

Theorem 5 (Heaviside's)**.**

$$\mathscr{L}\left\{e^{-at}f(t)\right\} = \hat{f}(s+a)$$

Then $\mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a))^{n+1}}$. *To prove this, we use the definition of* $\mathscr{L}$ *:*

$$\mathscr{L}\left\{e^{at}t^n\right\} = \int_{\mathbb{R}^+} e^{-t(s-a)}t^n\ dt$$

Using Kronecker's method with

$$p = t^n, p' = nt^{n-1}, \ldots, p^{(n)} = n! \ \text{ and } F_1 = -\frac{e^{-t(s-a)}}{s-a}, F_2 = \frac{e^{-t(s-a)}}{(s-a)^2}, \ldots, F_{n+i} = (-1)^{n+1}\frac{e^{-t(s-a)}}{(s-a)^{n+1}}$$

gives us:

$$\Rightarrow \frac{-t^n e^{-t(s-a)}}{s-a} - \frac{nt^{n-1} - t(s\cdot a)}{(s-a)^2} \cdots - (-1)^n \frac{n!\,e^{-t(s-a)}}{(s-a)^{n+1}}\Bigg|_0^\infty$$

$$= \frac{n!}{(s-a)^{n+1}}$$

** Note: In Kroneckers method the original function p is used, but the function f is automatically integrated. Therefore if p is n-time differentiable such that $p^{(n+1)} = 0$, then we take the antiderivative of f $n+1$ times.*

Problem 7. Establish the following result

$$\mathscr{L}\left\{\sin^2 at\right\} = \frac{2a^2}{s\left(s^2+4a^2\right)}$$

Solution 7. $\mathscr{L}\left\{\sin^2 at\right\} = \int_{\mathbb{R}^+} e^{-st}\sin^2(at)\,dt$*; convert to exponential form*

$$\sin^2(bt) = \left(\frac{1}{2t}\left(e^{at}-e^{-at}\right)\right)^2 = -\frac{1}{4}\left(e^{2at}-2+e^{-2at}\right)$$

Apply $\mathscr{L}$:

$$-\frac{1}{4}\left[\int_{\mathbb{R}^+} e^{-t(s-2a)}dt - 2\int_{\mathbb{R}^+} e^{-st}dt + \int_{\mathbb{R}^+} e^{-t(s+2c)}dt\right]$$

$$= -\frac{1}{4}\left[-\frac{e^{-t(s-2a)}}{s-2a} + \frac{2e^{-st}}{s} - \frac{e^{-t(s+2a)}}{s+2a}\bigg|_0^\infty\right] = \frac{-1}{4}\left(\frac{-8a^2}{s\left(s^2+4a^2\right)}\right) = \frac{2a^2}{s\left(s^2+4a^2\right)}$$

Problem 8. 23. Show that

$$\mathscr{L}\left\{te^{-bt}\cos at\right\} = \frac{(s+b)^2-a^2}{\left[(s+b)^2+a^2\right]^2}$$

Solution 8.

Theorem 6 (3.4.1)**.**

$$\text{If } \mathscr{L}\{f(t)\} = \hat{F}(s), \text{ then } \mathscr{L}\left\{e^{-at}f(t)\right\} = \hat{f}(s+a)$$

Theorem 7 (3.6.2)**.**

$$\text{If } \mathscr{L}\{f(t)\} = \hat{f}(s), \text{ then } \mathscr{L}\left\{t^n f(t)\right\} = (-1)^n\frac{d^n}{ds^n}\bar{f}(s)$$

By the above Theorems, let's first find $\mathscr{L}\{t\cos(at)\}$ and then use the shifting property.

$$\begin{aligned}\mathscr{L}\{t\cos(at)\} &= -\frac{d}{ds}\frac{s}{s^2+a^2} \\ &= -\frac{s^2+a^2-s(2s)}{\left(s^2+a^2\right)^2} = \frac{s^2-a^2}{\left(s^2+a^2\right)^2}\end{aligned}$$

Then applying Theorem 3.4.1 we obtain

$$\therefore \frac{(s+b)^2-a^2}{\left((s+b)^2+a^2\right)^2}$$