Exam 3

Problem 1. Solve the non-homogeneous diffusion problem by the Hankel transform.

$$\begin{aligned} u_t &= a\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \quad 0 < r < \infty, t > 0 \\ u(r,0) &= f(r) \text{ for } 0 < r < \infty. \end{aligned}$$

Solution 1. *Applying the Hankel transform $\mathscr{H}$ to the system gives us*

$$\tilde{u}' = a\left(-k^2\widetilde{u}\right) + \widetilde{Q}(k,t) \quad \Rightarrow \quad \tilde{u}' + ak^2\widetilde{u} = \widetilde{Q}(k,t)$$

This is a linear non-homogeneous ODE with constant coefficients; therefore we use an integrating factor:

$$\mu(t) = e^{ak^2 \int dt} = e^{ak^2t} \text{ is our integrating factor}$$

$$\frac{d}{dt}\left(\widetilde{u}e^{ak^2t}\right) = \widetilde{Q}(k,t)e^{ak^2t}$$

Integrating both sides gives us

$$\widetilde{u}e^{ak^2t} = \int_0^t \widetilde{Q}(k,\tau)e^{ak^2\tau}d\tau + C_1$$

Since $\widetilde{u}(k,0) = \widetilde{f}(k)$, then $C_1 = \widetilde{f}(k)$. Now, divide both sides by e^{ak^2t} gives us

$$\widetilde{u} = \int_0^t \widetilde{Q}(k,\tau)e^{-ak^2(t-\tau)}d\tau + \widetilde{f}(k)e^{-ak^2t}$$

Then applying the inverse Hankel transform $\mathscr{H}^{-1}$ gives us

$$u(r,t) = \int_{\mathbb{R}^+} kJ(kr)\left[\int_0^t \widetilde{Q}(k,\tau)e^{-ak^2(t-\tau)}dt + \widetilde{f}(k)e^{-ak^2t}\right]dk$$

Problem 2. Find the solution of the wave equation by the Hankel transform

$$\begin{aligned} u_{tt} =& c^2\left(u_{rr} + \frac{1}{r}u_r + u_{zz}\right) \quad 0 < r < \infty, z > 0t > 0 \\ u_z\big|_{z=0} &= g(r,t) \end{aligned}$$

where c is a constant and $g(r,t)$ is a given function.

Solution 2. *Let $u = e^{i\omega t}\varphi(r,z)$ and $g(r,t) = e^{i\omega t}f(r)$, then our system becomes*

$$c^2\left(\varphi_{rr} + \frac{1}{r}\varphi_r + \varphi_{zz}\right) + \omega^2\varphi = 0$$

$$\varphi_{rr} + \frac{1}{r}\varphi_r + \varphi_{zz} + \left(\frac{\omega}{c}\right)^2\varphi = 0$$

Applying the Hankel transform $\mathscr{H}$ gives us:

$$\tilde{\varphi}'' - \left(k^2 - \frac{\omega^2}{c^2}\right)\varphi = 0$$

With auxiliary equation

$$m^2 = \left(k^2 - \frac{\omega^2}{c^2}\right); \quad m = \pm\left(k^2 - \frac{\omega^2}{c^2}\right)^{1/2} := \ell$$

Therefore, we have

$$\begin{aligned} \widetilde{\varphi} &= C_1e^{\ell z} + C_2e^{-\ell z} \\ \widetilde{\varphi}' &= C_1\ell e^{\ell z} - C_2\ell e^{-\ell z} \end{aligned}$$

We are given that our solution is bounded, and so $C_1 = 0$. *Now, because* $\tilde{\varphi}' = \widetilde{f}(k)$ *for* $z = 0$, *then* $-C_2\ell = \widetilde{f}(k)$ *which implies* $C_2 = -\dfrac{\widetilde{f}(k)}{\ell}$ *and so*

$$\widetilde{\varphi} = \frac{-\widetilde{f}(k)}{\ell}e^{-\ell z}$$

Applying the inverse Hankel transform $\mathscr{H}^{-1}$ *gives us the final answer as:*

$$\varphi = \int_{\mathbb{R}+} \frac{-\widetilde{f}(k)}{\ell}e^{-\ell z} \cdot kJ(kr)\ dk$$

Problem 3. Solve the following integral equation by the Mellin transform

$$f(x) = \sin ax + \int_0^\infty \frac{f(xt)}{1+t^2}\ dt$$

Solution 3.

Theorem (Convolution Type)**.**

$$\textit{If}\ \ \mathscr{M}\{f \circ g\} = \mathscr{M}\left\{\int_{\mathbb{R}^+} f(x\tau)g(\tau)d\tau\right\} = \widetilde{f}(p)\widetilde{g}(1-p)$$

Applying the Mellin transform $\mathscr{M}$ *and applying the convolution type theorem gives us*

$$\widetilde{f}(p) = a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right) + \tilde{f}(p)\frac{\Gamma\left(\frac{1-p}{2}\right)\Gamma\left(1-\frac{1-p}{2}\right)}{2\Gamma(1)}$$

$$\widetilde{f}(p) = a^{-P}\Gamma(p)\sin\left(\frac{\pi p}{2}\right) + \tilde{f}(p)\frac{\Gamma\left(\frac{1}{2}-\frac{p}{2}\right)\Gamma\left(\frac{1}{2}+\frac{p}{2}\right)}{2}$$

$$\tilde{f}(p) = a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right) + \tilde{f}(p)\frac{\pi}{2}\sec\left(\frac{\pi p}{2}\right)\ \ \textit{by identity}$$

$$\tilde{f}(p)\left(\frac{2-\pi\sec\left(\frac{\pi p}{2}\right)}{2}\right) = a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)$$

$$\tilde{f}(p) = \frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2-\pi\sec\left(\frac{\pi p}{2}\right)}$$

$$\tilde{f}(p) = \frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)\cos\left(\frac{\pi p}{2}\right)}{2\cos\left(\frac{\pi p}{2}\right)-\pi}$$

$$\tilde{f}(p) = \frac{a^{-p}(p)\sin(\pi p)}{\cos\left(\frac{\pi p}{2}\right)-\frac{\pi}{2}}$$

Applying the inverse Mellin transform $\mathscr{M}^{-1}$ *gives us*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\left(\frac{a^{-p}(p)\sin(\pi p)}{\cos\left(\frac{\pi p}{2}\right)-\frac{\pi}{2}}\right)\ dp$$

Problem 4. Solve the following partial differential equation by the Mellin transform

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0, \quad 0 < r < \infty, 0 < \theta < \pi$$

subject to

$$\phi(r,0) = \begin{cases} (1-r)^2, & 0 < r < 1 \\ 0, & r \geq 1 \end{cases} \quad \text{and} \quad \phi(r,\pi) = \begin{cases} 1, & 0 \leq r \leq 1 \\ 0, & r > 1 \end{cases}$$

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Solution 4. *Applying the Mellin transform $\mathscr{M}$ to the system gives us*

$$\widetilde{\varphi}'' + p^2\widetilde{\varphi} = 0$$

With auxiliary equation,

$$m^2 + p^2 = 0, m = \pm ip \Rightarrow \widetilde{\varphi} = C_1 \cos(\theta p) + C_2 \sin(\theta p)$$

and the boundary conditions become:

$$\widetilde{\varphi}(p, 0 = \begin{cases} \frac{\Gamma(3)\Gamma(p)}{\Gamma(3+p)}, & p \in (0,1) \\ 0, & otherwise \end{cases} \quad and \quad \widetilde{\varphi}(p, \pi) = \begin{cases} \frac{1}{p}, & p \in [0,1] \\ 0, & otherwise \end{cases}$$

At $\theta = 0$ we have $C_1 = \frac{\Gamma(3)\Gamma(p)}{\Gamma(3+p)}$ for $p \in (0,1) \Rightarrow C_2 = 0$. And at $\theta = \pi$, we have $C_1 \cos(p\pi) = \frac{1}{p}$; $C_1 = \dfrac{1}{p\cos(p\pi)}$ for $p \in [0,1]$. Then altogether, we have

$$\widetilde{\varphi} = \frac{\Gamma(3)\Gamma(p)}{p\cos(p\pi)\ \Gamma(3+p)} \cos(\theta p)$$

Applying the inverse Mellin transform $\mathscr{M}^{-1}$ yields

$$\varphi(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \left(\frac{\Gamma(3)\Gamma(p)}{p\cos(p\pi)\ \Gamma(3+p)} \cos(\theta p) \right)\ dp$$

Problem 5. Use initial Value Theorem to find $f(0)$, and use Final Value Theorem to find $\lim_{n\to\infty} f(n)$ for $F(z) = \dfrac{z^2 - z\cos x}{z^2 - 2z\cos x + 1}$.

Solution 5.

Theorem (Initial Value)**.** *If $\mathcal{Z}\{f(n)\} = F(z)$, then*

$$f(0) = \lim_{z\to\infty} F(z)$$

If $f(0) = 0$, then

$$\lim_{z\to\infty} zF(z) = F(1)$$

Theorem (Final Value)**.** *If $\mathcal{Z}\{f(n)\} = F(z)$, then*

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z)$$

To find $f(0)$, apply the initial value theorem

$$\begin{aligned} f(0) &= \lim_{z\to\infty} \frac{z^2 - z\cos x}{z^2 - 2z\cos x + 1} \\ &= \lim_{z\to\infty} \frac{1 - \frac{1}{z}\cos x}{1 - \frac{2}{z}\cos x + \frac{1}{z^2}} \end{aligned}$$

Therefore $f(0) = 1$, which follows since $F(z) = \mathcal{Z}\{\cos nx\}$ and $\cos 0 = 1$. To find $\lim_{n\to\infty} f(n)$, use the final value theorem:

$$\begin{aligned} \lim_{n\to\infty} f(n) &= \lim_{z\to 1}(z-1)\frac{z^2 - z\cos x}{z^2 - 2z\cos x + 1} \\ &= 0 \quad for\ \forall x \neq 2n\pi \end{aligned}$$

For $x = 2n\pi$, we have

$$\lim_{z\to 1} \frac{(z-1)(z^2 - z)}{(z-1)^2}$$

$$\lim_{z\to 1} \frac{z(z-1)^2}{(z-1)^2} = 1$$

Since the limits do not equal, and clearly $\cos nx$ is an oscillating function, then the limit does not exist.

Problem 6. Solve the following initial value problem by the $\mathcal{Z}$ transform

$$f(n+2) - f(n+1) - 6f(n) = \sin\left(\frac{n\pi}{2}\right) \quad (n \geq 2), f(0) = 0, f(1) = 3$$

Solution 6. *Applying the $\mathcal{Z}$ transform gives us*

$$z^2(F - f(0)) - zF(1) - z(F - f(0)) - 6F = \frac{z\sin\left(\frac{\pi}{2}\right)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1}$$

$$z^2F - 3z - zF - 6F = \frac{z}{z^2+1}$$

$$F\left(z^2 - z - 6\right) = \frac{z}{z^2+1} + 3z$$

$$F = \frac{z}{\left(z^2+1\right)(z-3)(z+2)} + \frac{3z}{(z-3)(z+2)}$$

Applying the inverse transform $\mathcal{Z}^{-1}$ by the formal definition

$$f(n) = \frac{1}{2\pi i}\oint_C F(z)z^{n-1}\,dz$$

$$f(n) = \frac{1}{2\pi i}\oint_C \frac{z^n}{\left(z^2+1\right)(z-3)(z+2)} + \frac{3z^n}{(z-3)(z+2)}\,dz$$

By the residue theorem,

$$g(z) = \frac{z^n}{\left(z^2+1\right)(z-3)(z+2)}$$

has poles at $\pm i, -2,$ & 3 and

$$h(z) = \frac{3z^n}{(z-3)(z+2)}$$

has poles at -2 & 3. Using `Mathematica` *and the residue theorem, we find the final solution as*

$$f(n) = 2\pi i\sum Res(g(z) + h(z))$$

$$f(n) = \frac{\pi}{50}\left((7+i) - (7-i)(-1)^n + 2i\left(31(3i)^n - i^{3n}2^{5+n}\right)\right)$$