4.12

Problem (28e). Using the Laplace transform or otherwise, evaluate the following integral:

$$\int_0^\infty \exp\left(-tx^2\right) dx, \quad t > 0$$

Solution. *Let*

$$f(t) = \int_{\mathbb{R}^+} e^{-tx^2}\ dx, \text{ then } \mathscr{L}\{f(t)\} = \int_{\mathbb{R}^+} e^{-tx^2} e^{-st}\ dt; \int_{\mathbb{R}^+} e^{-t\left(x^2+s\right)}\ dt$$

$$= \frac{-e^{-t\left(x^2+s\right)}}{(x^2+s)}\Bigg|_0^\infty = \frac{1}{x^2+s}; \quad \int_{\mathbb{R}^+} \frac{dx}{x^2+s} \Rightarrow \frac{1}{s}\int_{\mathbb{R}^+} \frac{1}{\frac{x^2}{s}+1}\ dx$$

$$\text{Let } u = \frac{x}{\sqrt{s}}, du = \frac{1}{\sqrt{s}}dx \Rightarrow \frac{1}{\sqrt{s}}\int_{\mathbb{R}^+} \frac{du}{u^2+1} = \frac{1}{\sqrt{s}}\arctan\left(\frac{x}{\sqrt{s}}\right)\Bigg|_0^\infty = \frac{\pi}{2\sqrt{s}}$$

Now, apply the inverse Laplace transform

$$\frac{\pi}{2}\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{\pi}{2}\int_{\mathbb{R}^+} \frac{e^{-st}}{\sqrt{s}}\ ds = \frac{\pi}{2}\left(\frac{1}{\sqrt{\pi t}}\right) \text{ by Example 3.7.10}$$

$$\therefore \int_{\mathbb{R}^+} e^{-tx^2} dx = \sqrt{\frac{\pi}{4t}}$$

Problem. Show that

$$\int_0^\infty e^{-ax}\left(\frac{\sin qx - \sin px}{x}\right) dx = \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right), \quad a > 0$$

Solution.

Theorem (3.6.7).

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\ ds$$

Let $a = s$ and $x = t$, then the integral becomes:

$$\int_{\mathbb{R}^+} e^{-st}\left(\frac{\sin qt \cdot \sin pt}{t}\right) dt; \text{ By Therem 3.6.7} \Rightarrow \int_s^\infty \mathscr{L}\{\sin qt \cdot \sin pt\}\ ds$$

$$\int_s^\infty \left(\frac{q}{s^2+q^2} - \frac{p}{s^2+p^2}\right) ds = \arctan\left(\frac{s}{q}\right)\Bigg|_s^\infty - \arctan\left(\frac{s}{p}\right)\Bigg|_0^\infty$$

$$= \left(\frac{\pi}{2} - \arctan\left(\frac{s}{q}\right)\right) - \left(\frac{\pi}{2} - \arctan\left(\frac{s}{p}\right)\right)$$

Since $\pi/2 - \arctan\left(\frac{x}{y}\right) = \arctan\left(\frac{y}{x}\right)$ and $a = s$, then

$$\therefore \left(\frac{\pi}{2} - \arctan\left(\frac{s}{q}\right)\right) - \left(\frac{\pi}{2} - \arctan\left(\frac{s}{p}\right)\right) = \arctan\left(\frac{q}{a}\right) - \arctan\left(\frac{p}{a}\right)$$

Problem (35a). Using the Laplace transform, solve the following difference equation

$$\Delta u_n - 2u_n = 0, \quad u_0 = 1,$$

Solution.

$$\Delta u_n = u_{n+1} - u_n \Rightarrow u_{n+1} - 3u_n = 0$$

Applying $\mathscr{L}$ to the difference equation yields:

$$e^s\left[\bar{u} - u_0\bar{S}_0\right] - 3\bar{u} = 0$$

$$e^s\bar{u} - e^s\bar{S}_0 - 3\bar{u} = 0$$

$$\bar{u}\left(e^s - 3\right) = e^s\bar{S}_0$$

$$\bar{u} = \frac{e^s\bar{S}_0}{e^s - 3}$$

Applying the inverse Laplace transform

$$u = \mathscr{L}^{-1}\left\{\bar{S}_0 \frac{e^s}{e^s - 3}\right\} \Rightarrow u = 3^n$$

Problem (35b)**.** Using the Laplace transform, solve the following difference equation

$$\Delta^2 u_n - 2u_{n+1} + 3u_n = 0, \quad u_0 = 0 \quad \text{and} \quad u_1 = 1$$

Solution.

$$\Delta^2 u_n = \Delta\left(u_{n+1} - u_n\right) = u_{n+2} - u_{n+1} - \left(u_{n+1} - u_n\right) = u_{n+2} - 2u_{n+1} + u_n$$

Then our difference equation is

$$u_{n+2} - 4u_{n+1} + 4u_n = 0$$

Apply $\mathscr{L}$*:*

$$e^{2s}\left[\bar{u} - \bar{S}_0\left(u_0 + u_1 e^{-s}\right)\right] - 4e^s\left[\bar{u} - u_0\bar{S}_0\right] + 4\bar{u} = 0$$

$$\textit{Since } \bar{u}_0 = 0 \wedge \bar{u}_1 = 1 \Rightarrow \bar{u}e^{2s} - e^s\bar{S}_0 - 4e^s\bar{u} + 4\bar{u} = 0$$

$$\bar{u}\left(e^{2s} - 4e^s + 4\right) = e^s\bar{S}_0$$

$$\Rightarrow \bar{u} = \bar{S}_0\tfrac{e^s}{e^{2s} - 4e^s + 4} = \bar{S}_0\tfrac{e^s}{(e^s - 2)^2}$$

$$\therefore \bar{u} = \tfrac{n}{2} \cdot 2^n \textit{ by Example } 4.7.3$$

Problem (36)**.** Show that the solution of the difference equation

$$u_{n+2} + 4u_{n+1} + u_n = 0, \text{ with } u_0 = 0 \text{ and } u_1 = 1,$$

is

$$u_n = \frac{1}{2\sqrt{3}}\left[(\sqrt{3} - 2)^n + (-1)^{n+1}(2 + \sqrt{3})^n\right].$$

Solution. *Apply the Laplace transform:*

$$e^{2s}\left[\bar{u} - \bar{S}_0\left(u_0 + u_1 e^{-s}\right)\right] + 4e^s\left[\bar{u} - \bar{S}_0 u_0\right] + \bar{u} = 0$$

$$\bar{u}\left(e^{2s} + 4e^s + 1\right) = \bar{S}_0 e^s$$

$$\bar{u} = \bar{S}_0\frac{e^s}{e^{2s} + 4e^s + 1}; \quad \bar{u} = \bar{S}_0\frac{e^s}{\left(e^s + 2 - \sqrt{s}\right)\left(e^s + 2 + \sqrt{s}\right)}$$

The inverse transform gives us

$$u = \frac{1}{2\sqrt{3}}\left[(\sqrt{3} - 2)^n + (-1)^{n+1}(\sqrt{3} + 2)^n\right].$$

Problem (37)**.** Show that the solution of the differential-difference equation

$$\dot{u}(t) - u(t-1) = 2, \quad u(0) = 0$$

is

$$u(t) = 2\left[t - \frac{(t-1)^2}{2!} + \frac{(t-2)^3}{3!} + \cdots + \frac{(t-n)^{n+1}}{(n+1)!}\right], \quad t > n$$

Solution. *Applying* $\mathscr{L}$ *to the differential difference equation gives us:*

$$s\bar{u} - u(0) - e^{-s}\left(\bar{u} - \bar{S}_0 u(0)\right) = \frac{2}{s}$$

$$\bar{u}\left(s - e^{-s}\right) = \frac{2}{s}; \quad \bar{u} = \frac{2}{s\left(s - e^{-s}\right)}$$

$$\Rightarrow \bar{u} = \frac{2}{s^2}\left(1 - \frac{e^{-s}}{s}\right); \quad \bar{u} = 2\left[\frac{1}{s^2} + \frac{e^{-s}}{s^3} + \frac{e^{-2s}}{s^4} + \cdots\right] \textit{ By Example 4.7.5}$$

By Example 4.7.4, $\mathscr{L}^{-1}\left\{\dfrac{e^{-as}}{s^n}\right\} = \dfrac{(t-a)^{n-1}}{(n-1)!}\ H(t-a)$, *then applying* $\mathscr{L}^{-1}$ *gives us:*

$$u = 2\left(t - \frac{(t-1)^2}{2} + \frac{(t-2)^3}{6} + \cdots + \frac{(t-n)^{n+1}}{(n+1)!}\right)$$

$$\therefore u = 2\left(t - \sum_{i=1}^{n+1} \frac{(t-i)^{i+1}}{(i+1)!}\right)$$