4.12

Problem (22)**.** Solve the Blasius problem of an unsteady boundary layer flow in a semiinfinite body of viscous fluid enclosed by an infinite horizontal disk at $z=0$. The governing equation and the boundary and initial conditions are

$$\begin{aligned}
&\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial z^2}, \quad z>0, \quad t>0,\\
&u(z,t)=Ut \quad \text{on } z=0, \quad t>0,\\
&u(z,t)\to 0 \quad \text{as } z\to\infty, \quad t>0,\\
&u(z,t)=0 \quad \text{at } t\leq 0, \quad z>0.
\end{aligned}$$

Solution. *Apply $\mathscr{L}$ to the system yields:*

$$\begin{aligned}
&s\bar{u}-u(0)=k\bar{u}''\\
&\bar{u}=U\cdot 1/s^2 \ at \ z=0\\
&\bar{u}(z,s)\to 0 \ as \ z\to\infty\\
&\bar{u}=0 \ for \ \forall t\leqslant 0 \quad 1z>0
\end{aligned}$$

Then our ODE to solve is

$$\begin{aligned}
&s\bar{u}=k\bar{u}''\Rightarrow u''-\frac{s}{k}u=0\\
&\text{Auxiliary equation: } m^2-\frac{s}{k}=0, m=\pm\sqrt{\frac{s}{k}}\\
&\Rightarrow \bar{u}=C_1e^{z\sqrt{\frac{s}{k}}}+C_2e^{-z\sqrt{\frac{s}{k}}}
\end{aligned}$$

Since $\bar{u}\to 0$ as $z\to\infty$, then $C_1=0$. At $z=0, \bar{u}(0,s)=\frac{U}{s^2}\Rightarrow C_2=\frac{U}{s^2}$ Therefore, $\bar{u}=\frac{U}{s^2}e^{-z\sqrt{\frac{s}{k}}}$. Applying $\mathscr{L}^{-1}$ gives us

$$u(z,t)=Ut\left[\left(1+2\zeta^2\right)\operatorname{erfc}(\zeta)-\frac{2\zeta}{\sqrt{\pi}}e^{-\zeta^2}\right], \quad where \ \zeta=\frac{z}{2\sqrt{kt}}$$

Problem (25)**.** Solve the following integral and integro-differential equation:

$$f(t)=\sin 2t+\int_0^t f(t-\tau)\sin\tau d\tau$$

Solution.

Theorem (3.5.1)**.** *If $\mathscr{L}\{f(t)\}=\bar{f}(s)$, and $\mathscr{L}\{g(t)\}=\bar{g}(s)$, then $\mathscr{L}\{f(t)*g(t)\}=\bar{f}(s)g(s)$*

Applying $\mathscr{L}$ gives us $\bar{f}(s)=\frac{2}{s^2+2^2}+\bar{f}(s)\cdot\frac{1}{s^2+1}$ by Theorem 3.5.1. Then $\bar{f}(s)\left(s^2+1\right)-\bar{f}(s)=\frac{2\left(s^2+1\right)}{s^2+z^2}$;

$$\bar{f}(s)=\frac{2\left(s^2+1\right)}{s^2\left(s^2+2^2\right)}; \quad \bar{f}(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}$$

$$\begin{aligned}
&\Rightarrow As^3+4As+Bs^2+4B+Cs^3+Ds^2\\
&\Rightarrow s^3(A+C)+s^2(B+D)+4As+4B=s^2+1
\end{aligned}$$

$$\begin{pmatrix}1&0&1&0\\0&1&0&1\\4&0&0&0\\0&4&0&0\end{pmatrix}\begin{pmatrix}s^3\\s^2\\s\\c\end{pmatrix}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\Rightarrow \begin{matrix}A=0\\B=\frac{1}{2}\\C=0\\D=\frac{3}{2}\end{matrix}$$

$$\therefore \bar{f}(s) = \frac{1}{2s^2} + \frac{3}{4} \cdot \frac{2}{s^2+4}$$
$$\textit{Applying } \mathscr{L}^{-1} \Rightarrow f(t) = \frac{t}{2} + \frac{3}{4}\sin 2t$$

Problem (25)**.** Solve the following integral and integro-differential equations:

$$f(t) = \frac{t}{2}\sin t + \int_0^t f(\tau)\sin(t-\tau)d\tau$$

Solution.

$$\textit{Apply } \mathscr{L} : \bar{f}(s) = \frac{s}{(s^2+1)^2} + \bar{f}(s)\frac{1}{s^2+1};$$
$$\bar{f}(s)\left(s^2+1\right) - \bar{f}(s) = \frac{s}{s^2+1}; \quad \therefore \bar{f}(s) = \frac{s}{s^2(s^2+1)}$$

Partial fraction decomposition gives us:

$$\frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s\left(s^2+1\right)}; \quad A\left(s^2+1\right) + Bs^2 + Cs = 1$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} A = 1 \\ B = -1 \\ C = 0 \end{matrix}$$

$$\therefore \bar{f}(s) = \frac{1}{s} - \frac{1}{s^2+1}$$

Applying the inverse Laplace transform $\mathscr{L}^{-1}$ *gives the final answer as* $f(t) = 1 - \cos t$

Problem (25)**.** Solve the following integral and integro-differential equation:

$$f(t) = \sin t + \int_0^t f(\tau)\sin\{2(t-\tau)\}d\tau$$

Solution. *Applying* $\mathscr{L}$ *yields*

$$\bar{f}(s) = \frac{1}{s^2+1} + \bar{f}(s)\frac{2}{s^2+4}$$
$$\bar{f}(s)\left(s^2+4\right) - 2\bar{f}(s) = \frac{s^2+4}{s^2+1}$$
$$\bar{f}(s) = \frac{s^2}{\left(s^2+1\right)\left(s^2+2\right)} + \frac{4}{\left(s^2+1\right)\left(s^2+2\right)}$$

Partial fractions gives us $\bar{f}(s) = \frac{3}{s^2+1} - \frac{2}{s^2+2}$. *Applying the inverse Laplace transform* $\mathscr{L}^{-1}$ *gives us a final answer of* $f(t) = 3\sin t - \sqrt{2}\sin\sqrt{2}t$.

Problem. A uniform horizontal beam of length 2ℓ is clamped at the end $x = 0$ and freely supported at $x = 2\ell$. It carries a distributed load of constant value W in $\frac{\ell}{2} < x < \frac{3\ell}{2}$ and zero elsewhere. Obtain the deflection of the beam which satisfies the boundary value problem

$$EI\frac{d^4y}{dx^4} = W\left[H\left(x - \frac{\ell}{2}\right) - H\left(x - \frac{3\ell}{2}\right)\right], \quad 0 < x < 2\ell$$
$$y(0) = 0 = y'(0), \quad y''(2\ell) = 0 = y'''(2\ell)$$

Solution. *Apply* $\mathscr{L}$ *to the system with respect to* x *(where* $\bar{y}(s) = \int_{R^+} e^{-sx}y(x)dx$*) yields:*

$$EI\left(s^4y(s) - s^3y(0) - s^2y(0) - sy''(0) - y'''(0)\right) = \frac{W}{s}\left(e^{-\ell s/2} - e^{-3\ell s/2}\right)$$

Applying the initial conditions gives us:

$$EI\left(s^4\bar{y} - 0 - 0 - sy''(0) - y'''(0)\right) = \frac{W}{s}\left(e^{-\ell s/2} - e^{-3\ell s/2}\right)$$

$$\therefore \bar{y} = \frac{W}{EIs^5}\left(e^{-\ell s/2} - e^{-e\ell s/2}\right)$$

Now applying the inverse Laplace transform $\mathscr{L}^{-1}$ *gives us a final solution of:*

$$y = \frac{W}{EI} \cdot \frac{1}{24}\big((x-\ell/2)^4 \cdot H(x-\ell/2) - (x-3\ell/2)^4 H(x-3\ell/2)\big) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y''(0)$$