# A rolling horizon framework for the inventory routing problem

# Delivery Cost Approximations for Inventory Routing Problems in a Rolling Horizon Framework

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The inventory routing problem considered in this paper is concerned with the repeated distribution of a commodity, such as heating oil, over a long period of time to a large number of customers. The problem involves a central depot as well as various satellite facilities which the drivers can visit during their shift to refill their vehicles. The customers maintain a local inventory of the commodity. Their consumption varies daily and cannot be predicted deterministically. In case of a stockout, a direct delivery is made and a penalty cost is incurred. In this paper, we present incremental cost approximations to be used in a rolling horizon framework for the problem of minimizing the total expected annual delivery costs.

## Introduction

Inventory routing problems (IRPs) usually arise when consumers rely on a central supplier to provide a given commodity on a repeated basis. For the central supplier, these problems usually involve the daily specification of a sequence of locations that must be visited by a fleet of delivery vehicles for restocking purposes. Whenever a customer unexpectedly stocks out, a special delivery must be made immediately, with a high incurred cost.

Although IRPs appeared in the literature as early as the 1970s (e.g., Beltrami and Bodin 1974, Russell and Igo 1979), much of the research is more recent (e.g., Anily and Federgruen 1990, 1993; Bell et al. 1983; Bard et al. 1998 a and b; Berman and Larson 2001; Chan et al. 1998; Chien et al. 1989; Christofides and Beasley 1984; Dror and Ball 1987; Dror et al. 1985; Federgruen and Simchi-Levi 1995; Federgruen and Zipkin 1984; Fisher et al. 1983; Gaudioso and Paletta 1992; Golden et al. 1984; Kleywegt et al. 1998, 2000; Kumar et al. 1995; Reiman et al. 1999; Rubio 1995; Trudeau and Dror 1992; Webb and Larson 1995).

For typical IRPs, a customer's consumption rate is difficult to predict with certainty and can only be represented at best by a random variable with known probability distribution. If one is willing to assume such probability distributions, a natural objective is to minimize the total expected annual delivery cost, including the delivery cost of stock-outs. Planning the entire annual distribution scheme in advance would, however, be unreliable and prone to many needed adjustments. Moreover, for computational feasibility, actual routing plans can only be computed for a much shorter time period than an annual horizon.

In Bard et al. 1998a, we proposed to deal with these issues by introducing a comprehensive decomposition scheme (based on a rolling horizon framework), in which vehicle routing problems are repeatedly solved

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over a two-week moving period. More specifically, for a given two-week planning horizon, the method first identifies customers who are to be visited during this time. After an adjustment is made to balance daily demand and accommodate weekend days, customers scheduled for the first week only are routed. The two-week planning horizon is then shifted by a week and the process is repeated. The overall scheme was tested on data sets generated from field experience with a national liquid propane distributor. One of the unique aspects of the week-long routing subproblems that we considered is the presence of satellite facilities where vehicles can be reloaded and customer deliveries continued until the closing time is reached. In Bard et al. 1998a, we proposed and tested three heuristics for solving these problems. In another paper, Bard et al. 1998b, we proposed and tested a branch and cut algorithm for solving similar routing problems called VRP with satellite facilities.

This paper is a close companion to Bard et al. 1998a and contains the main justifications behind the identification processes (selection of the customers to be routed on a specific day of a given week) as previously described. Interested readers should refer to Bard et al. 1998a for any details and clarifications on the overall approach and on the use of the material developed in this paper. The next section contains mathematical derivations and approximations. Section 2 is concerned with some numerical testing based on Monte Carlo simulation.

# 1. Expected Total Delivery Cost and Optimal Deliveries for a Customer Over a Given Time Period

## 1.1. Problem Statement

Our approach attempts to incorporate long-term delivery costs into shorter planning horizons. In doing so, we develop "optimal" a priori strategies based on expected consumption of customers, as opposed to reactive strategies based on accurate current knowledge of all local inventories. We are assuming that the central supplier has no reliable monitoring of local inventories. Strategies, based on accurate real-time monitoring of local inventories, are feasible with the adoption of advanced technologies; however, the quantification of the (long-term) benefits associated with the use of such real-time information is far from obvious and is an important research topic on its own, but beyond the scope of the current paper.

We are interested in evaluating the long-term strategies of repeatedly serving a given customer. To do this, we assume that the cost for each scheduled delivery at a given customer is a fixed (but customer dependent) given constant $b$. Note that this is an approximation because the cost due to the service of a customer is, in fact, dependent on the set of other customers to be served that day (and thus can vary depending on who else is served). In Bard et al. 1998a, we provide examples of how to estimate $b$ in practice for a given customer. We assume that the customer's tank capacity is $T$. We also assume that the daily demand of a customer never exceeds $T$ (i.e., a customer never requires more than one delivery per day).

We define $S$ as the cost of stock-out, where $S > b$. Note that we assume $S$ to be fixed (not dependent on the volume of stock-out). This is acceptable in case the demand of the product is best approximated by a continuous rate of consumption (as opposed to a discrete lumpy demand). Also, this assumes that as soon as the tank becomes empty, an immediate, and thus costly, special delivery is made. Additionally, we assume that if a delivery is scheduled on a specific day and no stock-out happened before that day, then no stock-out penalty can occur on the delivery day (in other words, deliveries are always made "first thing in the morning").

## 1.2. Preliminaries

First, consider the simple situation of a customer whose daily demand over a given time period is constant and known, say $\mu$ ($\mu \leq T$). Assume also that the tank is initially full. Based on the assumption of a constant demand, it is clear that the optimal policy (minimizing the total delivery cost) is to fill up the tank every $\lfloor T/\mu \rfloor$ days (i.e., just before the customer runs out of stock). For an $n$-day period, the total delivery cost (not including the initial delivery cost associated with the initial full tank) would then be for $n \geq 1$:

$$TC_n = \left\lfloor \frac{n-1}{\lfloor T/\mu \rfloor} \right\rfloor b.$$

If over the given time period, the daily demand is not constant, but takes different rates for different subperiods (e.g., due to seasonality), one can divide the overall period into shorter periods for which a constant consumption rate is valid, and then patch all solutions together. In the general case for which the daily demand differs for each day of the $n$-day period in a known fashion, say $\mu_1, \mu_2, \ldots, \mu_n$, an optimal policy is still to serve the customer just before he or she runs out of stock. For the $n$-day period, let $1 < m_1 < m_2 \cdots < m_{k(n)} \leq n$ be the optimal days for a visit. These are defined as follows for $1 \leq j \leq k(n)$:

$$\sum_{i=1}^{m_j-1} \mu_i \leq jT \quad \text{and} \quad \sum_{i=1}^{m_j} \mu_i > jT.$$

The overall total delivery cost is then

$$TC_n = k(n)b.$$

### 1.3. Random Demand and Infinite Horizon

Now concentrate on the effect of true random fluctuations. Assume that the daily demand corresponds to a stochastic process $(U_i)_{1\leq i\leq n}$, where $U_i$ is the demand on day $i$. Assuming that seasonality and known variations have first been "removed" (see above), it is then natural to assume that the $U_i$s are i.i.d. random variables with a finite expected value $\mu$. Again, we assume that $\text{Prob}(U_i > T) = 0$ and that the tank is initially full. As mentioned in the Introduction, we assume that the central supplier cannot monitor what is left in the tank at any time, except at the time of a delivery. This precludes any strategies that would schedule deliveries based on the current remaining level of inventories. Under these conditions, we have the following proposition.

Proposition 1. *In case of an infinite horizon, there exists an optimal policy that (i) fills up the tank during each delivery, and (ii) following any prior (scheduled or stock-out) delivery, plans the next delivery $d^*$ days after. The optimal replenishment interval $d^*$ is a constant chosen to minimize the expected daily cost.*

Proof. Because the cost of serving the customer is $b$ for each scheduled visit and $S$ for each stock-out visit, irrespective of the quantity served, serving the maximum amount during every delivery cannot be suboptimal.

Because the underlying stochastic process is stationary, and because we have an infinite horizon, delivery days correspond to renewal events associated with a simple stationary renewal process, and an optimal a priori planning strategy (i.e., minimizing the expected daily delivery cost) will always schedule the next delivery a constant number of days after any given delivery. □

### 1.4. Optimal Policies Over a Finite Horizon

**1.4.1. Constant Replenishment Interval Strategies.** Because (regular or stock-out) delivery costs are fixed, irrespective of the amount delivered, we can safely assume that, even for a finite horizon, any valid strategies will fill up the tank at each delivery.

For an $n$-day period, an intelligent strategy should, however, adapt itself and choose the next scheduled visit based not only on the tank size and probability distribution of the daily demand of the customer, but also on the number of days remaining until the end of the period. In other words, an optimal general policy may lead to nonidentical replenishment intervals over a given $n$-day period.

However, because of the conditions of our problem (e.g., each refill costs the same, irrespective of the amount delivered), we believe that such an adaptive strategy would not bring a significant difference (in the total expected delivery cost) over a simpler (and analytically tractable) nonadaptive (i.e., with constant replenishment intervals) strategy. This is why we will restrict ourselves to the following set of acceptable policies.

*Policy Restriction.* For a given $n$-day period, feasible policies will be chosen among the restricted set of constant replenishment interval strategies, i.e., strategies whose scheduled replenishment intervals are constant over the time period.

A consequence of this restriction is that any optimal such policies over a $n$-day period should lead to a constant replenishment interval $d^*(n)$ that should, in fact, be independent of $n$, when $n$ is large enough. We will test this approximation in §2.

**1.4.2. Optimal Constant Replenishment Interval Strategies.** Define $\mathrm{ETC}_n(d)$ to be the expected total cost over an $n$-day period of a constant replenishment interval strategy of $d$ days.

As before, we assume an initially full tank. Let us derive a detailed analytical formula for $\mathrm{ETC}_n(d)$. Because we assume that any scheduled deliveries are always done "first thing in the morning" and that the daily demand can never exceed the tank size $T$, there will never be any stock-out if $d = 1$.

Assume now that $d > 1$ and that we are at the beginning of day 1 with a full tank (the next scheduled visit is then on day $d+1$). If on one day $i$ during the next $d-1$ days there is a stock-out, then one makes a delivery on that day and the next scheduled visit is updated to $d+i$ and so on. We then have a *renewal* stochastic process where the renewal points are the times of refill (regular or due to stock-out).

Let $p_j$ be the probability that a stock-out first appears on the $j$th day following day $1, 1 \le j \le d-1$. This probability is given as

$$\begin{aligned} p_j &= \mathrm{Prob}(D_j \le T \text{ and } D_{j+1} > T) \\ &= \int_0^T P(U_{j+1} > T - x) f_{D_j}(x)\,dx, \end{aligned}$$

where $D_k = \sum_{i=1}^k U_i$ is the total consumption from day 1 to $k$ and $f_{D_k}(x)$ is the density function of the distribution of $D_k$.

Let $p = p_1 + p_2 + \cdots + p_{d-1}$ be the probability that there is (at least) a stock-out in the next $d-1$ days and $p_d \triangleq 1 - p$ be the probability of no stock-out during that period. We then have the following relationships:

For $d \ge n \ge 2$,

$$\mathrm{ETC}_n(d) = \sum_{j=1}^{n-1} p_j(\mathrm{ETC}_{n-j}(d) + S),$$

where $\mathrm{ETC}_1(d) = 0$.

For $n \ge d+1$,

$$\begin{aligned} \mathrm{ETC}_n(d) = &\sum_{j=1}^{d-1} p_j(\mathrm{ETC}_{n-j}(d) + S) \\ &+ (1-p)(\mathrm{ETC}_{n-d}(d) + b). \end{aligned}$$

Define $y_n = \mathrm{ETC}_n(d)$ ($y_n$ is, of course, dependent on $d$, but for notational simplicity we will not make this dependence explicit). One can rewrite the previous relationships as follows:

$$y_n = \begin{cases} \displaystyle\sum_{j=1}^{n-1} p_j y_{n-j} + S \sum_{j=1}^{n-1} p_j & \text{if } 2 \le n \le d, \\ \displaystyle\sum_{j=1}^{d} p_j y_{n-j} + pS + (1-p)b & \text{if } n \ge d+1, \end{cases} \tag{1}$$

with the initial condition $y_1 = 0$. From this initial condition and recursions (1), one can numerically compute $y_j$ for all $j$. However, we would like to obtain an analytical expression for $y_n$, especially when $n$ is large compared to $d$. To do so, let us consider the recursions when $n \ge d+1$, i.e.,

$$y_n - \sum_{j=1}^{d} p_j y_{n-j} = pS + (1-p)b \tag{2}$$

and the associated homogeneous equation

$$y_n - \sum_{j=1}^{d} p_j y_{n-j} = 0. \tag{3}$$

The corresponding characteristic function (see e.g., Kelley and Peterson 1991) for the definition of a characteristic function and its use for solving homogeneous linear recursions) is given by

$$x^d - \sum_{j=1}^{d} p_j x^{d-j} = 0. \tag{4}$$

Let us characterize the roots of Equation (4).

LEMMA 1. *A is a root of* (4) *with multiplicity one. Any other root $r$ is such that $|r| \le 1$, where $|r|$ is the absolute value of $r$ if $r$ is real, and the modulus of $r$ if $r$ is complex.*

PROOF. Because $\sum_{i=1}^d p_i = 1$, 1 is root of (4). One can thus rewrite (4) as

$$\begin{aligned} (x-1)[x^{d-1} + (1-p_1)x^{d-2} \\ + \cdots + (1 - p_1 - p_2 - \cdots - p_{d-1})] = 0, \end{aligned}$$

which shows that 1 is a root with multiplicity 1 only. (The second factor on the left-hand side of the above equation is a polynomial of degree $d-1$ of which 1 is not a root. This can be seen by simple substitution.)

Any other root $r$ will have $|r| \le 1$. Indeed, if $|r| > 1$, then $|r^d| > |r^i|$ for all $1 \le i \le d$. But this is clearly impossible because (4) implies that $|r^d| \le \sum_{i=1}^{d} p_i |r^{d-i}| \le \max_{1\le i\le d} |r^{d-i}|$. □

Because the general solutions $y_n$ of (3) are given by weighted sums of the roots of the characteristic equations raised to the $n$th power, the implication of Lemma 1 is that $y_n = f(n, d) + \alpha(d)$, where $f(n, d)$ goes to zero exponentially fast as $n$ goes to infinity ($f(n, d)$ is the part corresponding to the weighted sum of roots $r$ such that $r < 1$, each raised to the $n$th power).

Now look at a particular solution to (2). Because 1 is a root of order 1 in the characteristic function, a particular solution will be of the form $y_n = \beta(d)n$, where $\beta(d)$ is a constant. Replacing $y_n$ by $\beta(d)n$ in (2), we get $\beta(d) = (pS + (1-p)b)/\sum_{j=1}^{d} jp_j$.

We just proved the following result.

Theorem 1. *The expected total cost of filling up a customer's tank using a constant replenishment period strategy of d days over an n-day period ($n \ge d+1$) is given by*

$$\mathrm{ETC}_n(d) = f(n, d) + \alpha(d) + \beta(d)n, \tag{5}$$

*where*

$$\beta(d) = \frac{pS + (1-p)b}{\sum_{j=1}^{b} jp_j}, \tag{6}$$

*and where $f(n, d)$ goes to zero exponentially fast as n goes to infinity.*

This leads to our final approximation result.

Corollary 1. *An optimal constant replenishment period strategy of serving a customer over a large n-day period will correspond to choosing $d^*$ to minimize $\beta(d)$.*

Proof. From Theorem 1, an optimal policy will correspond to finding $d$ to minimize $\mathrm{ETC}_n(d) = f(n, d) + \alpha(d) + \beta(d)n$. When $n$ is large enough, $\beta(d)n$ is the main component of the total cost. □

**Notes:**

(1) From (5), we have $\lim_{n\to\infty} \mathrm{ETC}_n(d)/n = \beta(d)$. From an application of a discrete version of the renewal reward theorem (see, for example, Ross 1997 for an elementary treatment of renewal reward theory), we know that the expression for $\beta(d)$ should have a simple interpretation. Indeed, the numerator of $\beta(d)$ is the expected cost of a delivery (i.e., the expected "cost of a cycle") and the denominator is the expected number of days between deliveries (i.e., the expected "duration of a cycle").

(2) The expression given in (6) could naturally be obtained directly by the renewal reward theorem for the infinite horizon case. Our derivation can precisely measure the approximation made from a finite to an infinite horizon period.

(3) As an illustration, when $d = 2$, Equation (4) has two roots, 1 and $-(1-p)$, and the exact expression for $y_n$, $n \ge 3$ is given by

$$y_n = \frac{pS-b}{(2-p)^2}[-(1-p)]^n - \frac{pS + b(1-p)(3-p)}{(2-p)^2} + \frac{pS+(1-p)b}{2-p}n.$$

When $n$ is large, approximating this expression by the linear term $\{[pS + (1-p)^b]/(2-p)\}n$ overestimates the true total cost by a constant $[pS + b(1-p)(3-p)]/(2-p)^2$.

## 1.5. Incremental Costs

In the previous section, we have shown that the optimal (constant replenishment interval) strategy to serve a given customer corresponds to planning on replenishing the tank every $d^*(n)$ days, where $d^*(n)$ is the value of $d$ that minimizes $\mathrm{ETC}_n(d) = f(n, d) + \alpha(d) + \beta(d)n$.

We would now like to estimate the impact on the overall total cost (over an $n$-day period) of switching a given scheduled delivery of a customer to another day. This incremental cost is an essential input in Bard et al. 1998a used in the second phase of the customer selection (balancing the preselected customers among the days of planning the current week).

As before, we assume that we are at the beginning of day 1 with a full tank. Now suppose that on the first visit, the optimal a priori strategy is not followed, and the customer is scheduled to be replenished on day $j+1$, $j \ne d^*$. Also, suppose that after the first visit, the optimal replenishment period of $d^*$ is resumed. The difference between the expected total cost incurred by following this strategy and the optimal expected total cost will hereafter be called the incremental cost, denoted by $c_j$. By definition $c_{d^*} = 0$. Let us derive the general incremental cost $c_j$. Let $X(j)$ be the random variable representing the number of days until the next actual delivery, assuming a scheduled replenishment interval of $j$ days. (In case of a stock-out, $X(j) < j$, otherwise $X(j) = j$.) We then have

Theorem 2. *The incremental cost $c_j$ of serving a customer after $j$ days instead of $d^*$ days is given by*

$$c_j = g(n, j, d^*) + E[X(j)](\beta(j) - \beta(d^*)), \tag{7}$$

*where, for any $k \geq 2$,*

$$E[X(k)] = k\left(1 - \sum_{i=1}^{k-1} p_i\right) + \sum_{i=1}^{k-1} ip_i, \tag{8}$$

*and where $g(n, d^*, d^*) = 0$ and $g(n, j, d^*)$ go to zero exponentially fast as $n$ goes to infinity.*

Proof. Let $\mathrm{SUB}_n(j, d^*)$ be the expected total cost corresponding to the suboptimal strategy. We have

$$\begin{aligned}\mathrm{SUB}_n(j, d^*) &= \sum_{i=1}^{j-1} p_i(\mathrm{ETC}_{n-i}(d^*) + S) \\ &\quad + \left(1 - \sum_{i=1}^{j-1} p_i\right)(\mathrm{ETC}_{n-j}(d^*) + b) \\ &= \alpha(d^*) + \beta(d^*)n \\ &\quad + E[X(j)](\beta(j) - \beta(d^*)) + h(n, j, d^*),\end{aligned}$$

where $h(n, j, d^*)$ (sum of the terms $f(n-i, d^*)$ as defined in (5)) goes to zero exponentially fast as $n$ goes to infinity. By writing $c_j = \mathrm{SUB}_n(j, d^*) - \mathrm{ETC}_n(d^*)$ and defining $g(n, j, d^*) = h(n, j, d^*) - f(n, d^*)$, we get the result. □

**Notes:**

(1) When $n$ is large enough, the incremental cost given in (7) can be approximated as follows:

$$c_j \approx E[X(j)](\beta(j) - \beta(d^*)). \tag{9}$$

(2) Note that if the expected total cost $\mathrm{ETC}_n(d)$ is approximated by its linear component $\beta(d)n$ to start with, then $\beta(d)$ can be interpreted as the daily cost rate associated with the strategy of filling the customer every $d$ days. Under this interpretation, one can then approximate $\mathrm{ETC}_n(j, d^*)$ as follows:

$$\mathrm{SUB}_n(j, d^*) \approx \beta(j)E[X(j)] + \beta(d^*)(n - E[X(j)]).$$

The first component is the expected number of days with a suboptimal cost of $\beta(j)$ per day, and the second term corresponds to the remaining total expected cost under the optimal policy. This leads to the same approximation of $c_j$ as given in (9).

## 2. Monte Carlo Simulation and Testing

The purpose of this section is to use Monte Carlo simulation to (i) validate one consequence of the Policy Restriction of §1.4.1 and (ii) validate the limiting approximation formula (9) for the computation of the incremental costs.

For the first testing, the input of the simulation is the total number of days $n$ for the horizon and the frequency of visit $d$ (in days). The output is the total delivery cost. For a fixed $n$, we vary $d$ to find the optimal a priori frequency $d^*(n)$. The Policy Restriction implies that $d^*(n)$ should be independent of $n$, when $n$ is large. The code is a fixed-time-increment simulation in which time is advanced day by day. For each day, one first checks whether there is a scheduled delivery. If yes, the level in the tank is set back to $T$ and the total cost is incremented by $b$. Then, the consumption of the customer is randomly generated from a truncated normal distribution and is deducted from the current level of the tank. If the new tank level is negative, then an emergency visit is scheduled, the level of the tank is set back to $T$, and the total cost is incremented by $S$.

The other testing (incremental cost) is done via a similar simulation, except that the input is $n, d^*(n)$ (as computed above), and $j$ (which represents the initial number of days before the first scheduled visit). In both cases, for a given input, we run 10,000 independent simulations and average the output to obtain an approximation of the expected value under consideration.

We assume that the customer's consumption rate follows a truncated normal distribution $N(\mu, \sigma)$, where negative values of the normal distribution are ignored. We initially assume that $T = 1{,}000$, $\mu = 40$, $\sigma = 10$, $b = 16.27$, and $S = 68$. Note that $T/\mu = 25$ would represent the number of days between each refill, i.e., the optimal a priori frequency if the consumption rate were a constant equal to $\mu$.

### 2.1. Results for Test 1

By varying the time period $n$ from 0.5, 1, 2, and 5–10 years, we found that the optimal a priori policy was always to refill the customer every $d^* = 23$ days. We repeated the experiments with different $\sigma$'s, from $\sigma = 5$ to $\sigma = 30$, and reached the same conclusion, i.e., that $d^*$ was independent of the time period $n$.

For $n = 10$ years, Figure 1 presents the simulated expected total cost for $d$ varying from 15 to 30, as well as the calculated expected total cost based on Theorem 1 and approximated by $\beta(d)n$ (i.e., with only the linear part). The fit between the two curves is almost perfect. To apply Theorem 1, the probability $p_i$s must be evaluated. The easiest way to do this is via Monte Carlo simulation. To achieve a satisfactory level of precision, about 100,000 independent simulations were conducted for each $p_j$.

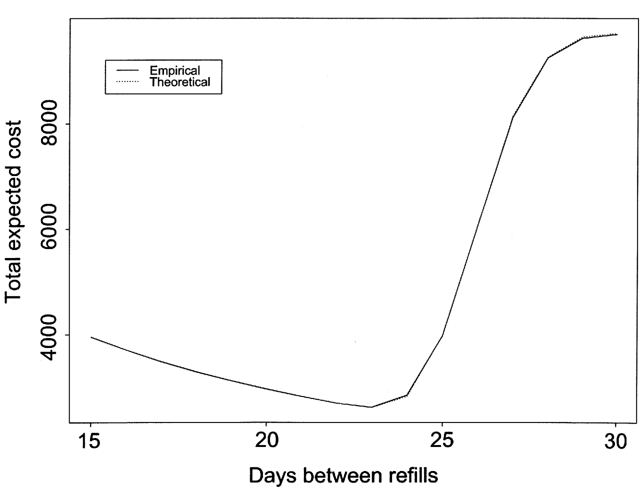


**Figure 1** **Annual Expected Total Cost**

**Table 1** **Comparison of Numerical Results and Analytic Results**

| Cases | Analytical Calculation, $x$ s | | Numerical Simulation, $z$ s | | Fitness Ratio |
|---|---|---|---|---|---|
| $\sigma$ | $d^*$ | $\mathrm{ETC}_{\min}$ | $d^*$ | $\mathrm{ETC}_{\min}$ | $r$ |
| 5 | 24 | 2,477.43 | 24 | 2,477.67 | 0.14 |
| 10 | 23 | 2,618.62 | 23 | 2,624.04 | 0.16 |
| 15 | 22 | 2,790.16 | 22 | 2,791.82 | 0.23 |
| 20 | 21 | 2,969.13 | 21 | 2,996.05 | 0.28 |
| 25 | 19 | 3,210.13 | 19 | 3,222.46 | 0.57 |
| 30 | 18 | 3,459.64 | 18 | 3,493.97 | 1.07 |

Again, we repeated the experiments with $n = 10$ and different $\sigma$'s, ranging from $\sigma = 5$ to $\sigma = 30$. To compare the simulated results with the analytic results, we defined a fitness ratio $r$ as follows. Suppose $x_i (i = 1, \ldots, n)$ is a set of numbers and $z_i (i = 1, \ldots, n)$ another set. Then

$$r = \frac{\sum_{i=1}^{n} |x_i - z_i|}{\sum_{i=1}^{n} x_i / n}$$

is our measure of fitness. Note that the numerator is simply the $L_1$ measure of discrepancy between the two sets of numbers and the denominator allows for a measure independent of units. The smaller $r$ is, the better the fitness between the data sets $x$ and $z$. In Table 1, we present six cases with $\sigma$ varying from 5 to 30. We give the minimal expected total cost for each case, using $d^*$ to represent the best frequency and $\mathrm{ETC}_{\min}$ to indicate the minimal expected total cost. The data show that for all six cases, we get the same $d^*$ with both the simulation and analytic methods. The fitness ratio $r$ does not exceed 1.07, which indicates a good fit between the two approaches.

### 2.2. Results for Test 2

By assuming that $T = 1{,}000$, $\mu = 40$, $\sigma = 10$, $b = 16.27$, and $S = 68$, the optimal a priori frequency is to serve the customer every $d^* = 23$ days (as indicated in Figure 1).

In Figure 2, we present, for $n = 10$, the simulated expected incremental costs corresponding to $j$ varying from 15 to 30 (the "empirical" curve). Also, we have used (9) to calculate the analytical (theoretical) approximated values for comparison purposes. The

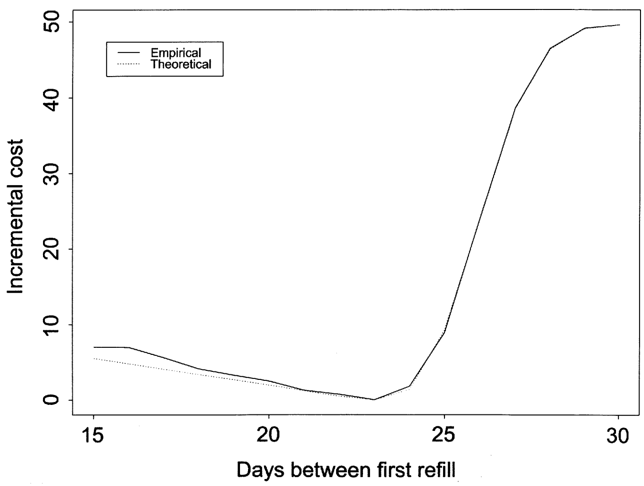


**Figure 2 Incremental Cost**

fit between simulated and calculated is almost perfect. Additional testings for varying $\sigma$ yielded similar results.

## 3. Concluding Remarks

In this paper, we have presented the justifications behind the derivations of the incremental costs as used in the rolling horizon framework of Bard et al. 1998a. The essence of our contribution lies in our ability to reduce the problem from an annual time base to a biweekly rolling planning period via the approximations of delivery costs.

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