Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

Tabular Method:

(+)	u	dv
(-)	du	v
(+)	du^2	∫v
(-)	du^3	∫(∫v)

❖The table can be continued using the same pattern
❖Circled entities are multiplied in front of the integral
❖Last two entities are multiplied inside of the integral
❖Signs at left indicate whether entities are added or subtracted from those around them

Techniques:

➢Choose *u* and *dv* so that *u* becomes "simpler" when differentiated and *dv* is easily integrated to become *v*

➢If the integrand is a product of two functions from different categories in the list below, let *u* be the function whose category occurs earlier in the list and let *dv* be the rest of the integrand.

- Logarithmic
- Inverse trig
- Algebraic
- Trig
- Exponential

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left|\frac{u}{a}\right| + C$$

ex: $\int x \sin x \, dx$

ex: $\int x^2 e^x dx$

ex: $\int e^{2x} \sin x \, dx$

ex: $\int_1^2 \ln x \, dx$

Basic Integration Rules

Procedures for fitting Integrands to Basic Rules

Technique	Example
Expand (numerator)	$(1+e^x)^2 = 1+2e^x+e^{2x}$
Separate numerator	$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$
Complete the square	$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$
Divide improper rational function	$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$
Add and subtract terms in numerator	$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$
Use trigonometric identities	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate	$\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x}$ $= \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$

ex: $\int \frac{4x^2}{x^2+9}\,dx$

ex: $\int \frac{x+3}{\sqrt{4-x^2}}\,dx$

ex: $\int \frac{1}{1+e^x}\,dx$

ex: $\int (\cot x)\left[\ln(\sin x)\right]dx$

Trig Integrals

Today we will look at integrals of the form $\int \sin^m x \cos^n x \, dx$

We will use the following identities:

$$\sin^2 x + \cos^2 x = 1 \text{ so } \begin{cases} \sin^2 x = 1 - \cos^2 x \\ \cos^2 x = 1 - \sin^2 x \end{cases}$$

$$\cos 2x = 1 - 2\sin^2 x \text{ so } \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos 2x = 2\cos^2 x - 1 \text{ so } \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin A \cos B = \frac{1}{2}\left(\sin(A - B) + \sin(A + B)\right)$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A - B) - \cos(A + B)\right)$$

$$\cos A \cos B = \frac{1}{2}\left(\cos(A - B) + \cos(A + B)\right)$$

Some helpful guidelines:

1) If the power of cosine is odd and positive, "save" one cosine factor and convert the remaining factors to sine.
2) If the power of sine is odd and positive, "save" one sine factor and convert the remaining factors to cosine.

If the powers of both sine and cosine are even and nonnegative, use the identities:

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

ex: $\int \cos^3(5x)\,dx =$

ex: $\int \cos^3 x \sin^2 x\,dx =$

ex: $\int \sin^2(3x)\cos^2(3x)\,dx =$

ex: $\int \sin(5x)\cos(4x)\,dx =$

Trig Substitution

This lesson involves substituting trigonometric expressions. This technique is especially helpful when a radical is in the integrand.

General Guidelines for Trig Substitution

1. $\cos^2 x + \sin^2 x = 1 \rightarrow \cos^2 x = 1 - \sin^2 x$

   $a^2 - x^2$

2. $1 + \tan^2 x = \sec^2 x$

   $a^2 + x^2$

3. $\tan^2 x = \sec^2 x - 1$

   $x^2 - a^2$

ex: $\int \frac{dx}{x^2\sqrt{4-x^2}}$

ex: $\int \frac{dx}{\left(x^2+1\right)^{3/2}} =$

Partial Fractions

Integrating Rational Functions by Partial Fractions

Rational functions can be challenging to integrate. One technique that can be very helpful is to break the rational function down to smaller pieces which are usually easier to integrate. Once decomposed, the rational function is integrated piece by piece.

The following are instructions for creating partial fractions:

Decomposition of N(x)/D(x) into Partial Fractions

1. *Divide if improper:* If N(x)/D(x) is improper (degree of N(x) ≥ degree of D(x)), use long division to prepare rational expression for decomposition into partial fractions.
2. *Factor the denominator:* completely factor the denominator into linear or irreducible quadratic factors with real coefficients.
3. *Linear factors:* For each repeated linear factor, the partial fraction decomposition must be written like:

$$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}+\ldots$$

4. *Quadratic factors:* For each repeated quadratic factor, the partial fraction decomposition must be written:

$$\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(ax^2+bx+c)^2}+\frac{Ex+F}{(ax^2+bx+c)^3}+\ldots$$

ex: $\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2}\,dx$

ACCUMULATION AND POPULATION DENSITY

<u>**Density:**</u> A material's density is its mass per unit volume. In practice, however, we tend to use units we can conveniently measure.

(For wires, rods, and narrow strips we use mass per unit length):

<u>Population on segment</u> = density $\cdot$ length of interval = p(x) $\cdot$ Δx = people/mile $\cdot$ mile

(For flat sheets and plates we use mass per unit area):

<u>Population on a ring</u> = density $\cdot$ area = p(r) $\cdot$ 2πr $\cdot$ Δr = people/mile^2 $\cdot$ mile $\cdot$ mile

<u>Population on a strip</u> = density $\cdot$ area = p(x) $\cdot$ length $\cdot$ Δx

<u>**Examples of problems that involve density:**</u>

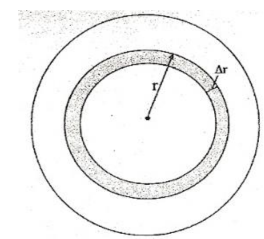
-> A population density measured in units like people per mile (along the edge of a road), or people per unit area (in a city), or bacteria per cubic cm (in a test tube).

-> The density of a substance (e.g. air, wood, or metal), is the mass of a unit volume of the substance and can be measured in grams per cubic cm, etc.

ex: A city located along a straight highway has a population whose density can be approximated by the function $p(x) = 1500(4 - x)$ where *x* is the distance from the town square, measured in miles, where $-4 \le x \le 4$, and $p(x)$ is measured in people per mile. The town extends for four miles on either side of the town square. Find the population of the city.

ex: Water is pumped out of a holding tank at a rate of $4-4e^{-0.12t}$ liters/minute, where *t* is in minutes since the pump is started. If the holding tank contains 300 liters of water when the pump is started, how much water does it hold 20 minutes later?

ex: A city can be approximated by a circle of radius 8 miles. Moving away from the center of the city along a radius, the population density can be approximated by the function $p(r)=2000e^{-0.15r}$ people per square mile. Find the population of the city.



ex: A city is located by a river. The city runs for 8 miles along the river and is 3 miles wide. If p(x) represents the population density, write an integral expression used to calculate the population of the city.

Improper Integrals-Day 1

Integrals such as $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{b} f(x)\,dx$, and $\int_{-\infty}^{\infty} f(x)\,dx$ are called **improper integrals**.

These improper integrals can be evaluated through their definitions listed below. **If the limit exists**, the improper integral is said to converge to the value of the limit, **if the limit does not exist**, the improper integral is said to diverge and is not assigned a value.

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{k\to a^+}\int_k^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{k\to b^-}\int_a^k f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

ex: $\int_1^{\infty}\frac{1}{x}\,dx =$

$$\ln^d a < x^c < x^c \ln^d a < x^{c+d} < a^x < x! < x^x$$

rates of growth

ex: $\int_{1}^{\infty} e^{-x} dx$

ex: $\int_{-\infty}^{0} e^{\frac{x}{4}} dx$

ex: Let $f(x) = e^{-x}$, and let R be the region in the first quadrant between the graph of f and the x-axis. Find the volume of the solid formed generated when R is revolved about the x-axis.

Improper Integrals-Day 2

If f has an infinite discontinuity at a or at b or at some c in (a, b) then the integral $\int_a^b f(x)\,dx$ is an improper integral. These improper integrals are evaluated by rewriting the integral as a proper integral and then using limits.

ex: $\int_0^1 \frac{1}{x^{1/3}}\,dx =$

ex: $\int_0^1 \frac{1}{x^3}\,dx =$

ex: $\int_0^{27} \frac{dx}{\sqrt[3]{27-x}} =$

ex: $\int_0^{3} \frac{dx}{(x-1)^{2/3}} =$

ex: Let $f(x) = 2\ln x$ and let R be the unbounded region in the fourth quadrant between the graph of f and the x-axis. Find the area of R.