Differential Equations, Day 1

•A ***differential equation*** is an equation that contains an unknown function and one or more of its derivatives.

•The ***order*** of a differential equation is the order of the highest derivative that occurs in the equation. Ex: $y' = xy$ is a first-order differential equation.

•A function f is classed a ***solution*** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. Thus, f is a solution of the example above if: $f'(x) = xf(x)$

•An ***equilibrium solution*** is a constant solution. Ex: $y = 5$ or $P = 0$

Separable equations are a type of differential equation that can be solved explicitly. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written like:

$$\frac{dy}{dx} = g(x)\,f(y)$$

<u>To solve this type of equation:</u>

1. Rewrite it by moving all y's to one side and all x's to the other side.
2. Integrate both sides of the equation.
3. If the equation can be solved for y, do so.

ex: Is $y = e^{-t}$ a solution to the differential equation

$$y'' + 2y' + y = 0?$$

ex: Given the differential equation, solve for y as a function of x.

$$\frac{dy}{dx} = \frac{2x}{y},\ and\ y(1) = -3$$

ex: Solve for y as a function of x.

$$y' = x^2\sqrt{y} \text{ and } y(3) = 4$$

Differential Equations, Day 2

ex: $\frac{dy}{dx} = 9x^2 y$ and $y(0) = -2$

ex: $y' = \frac{y+3}{x^2}$, $x \neq 0$, and $y(2) = 1$

ex: $\frac{dy}{dx} = 4x^3 y + 20x^3$ and $y(1) = -3$

ex: Consider the differential equation $\frac{dy}{dx} = \frac{x}{y}$. Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-3) = -1$ and state its domain

Slope Fields

Unfortunately, its impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution, but we can still learn a lot about the solution through a graphical approach called direction or slope fields.

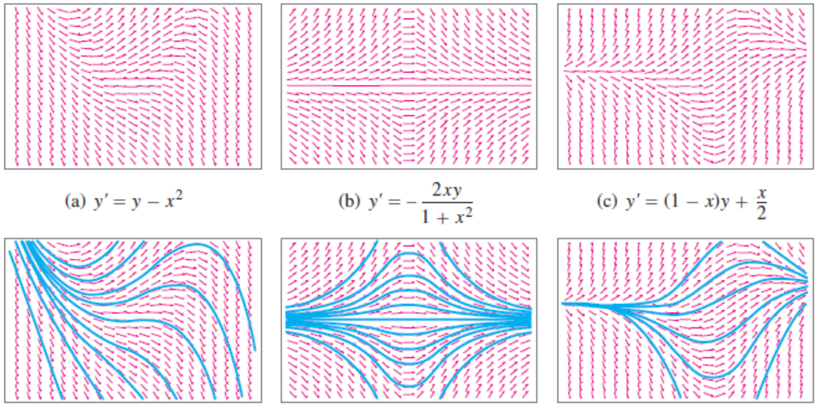
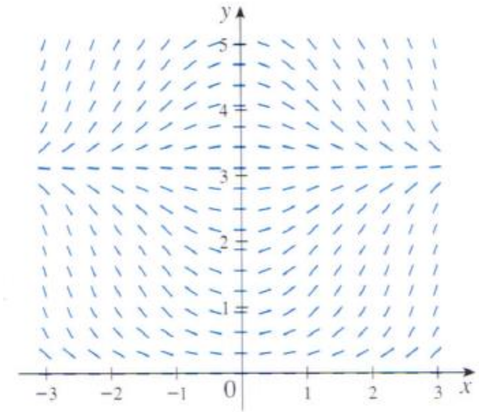


FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.

ex: A direction field for the differential equation is shown $y' = x \sin y$

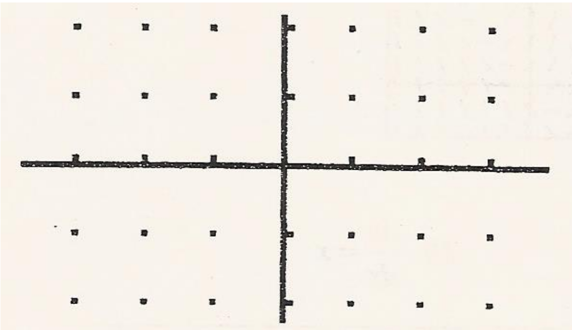


a) Sketch the graphs of the solutions that satisfy the given initial conditions.
   i) y(0) = 1 ii) y(0) = π iii) y(0) = 5

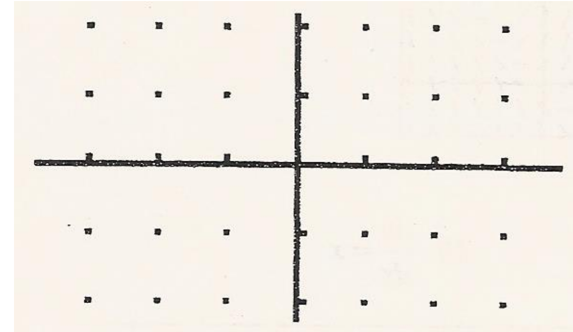
b) Find all equilibrium solutions.

Draw a slope field for each of the following.

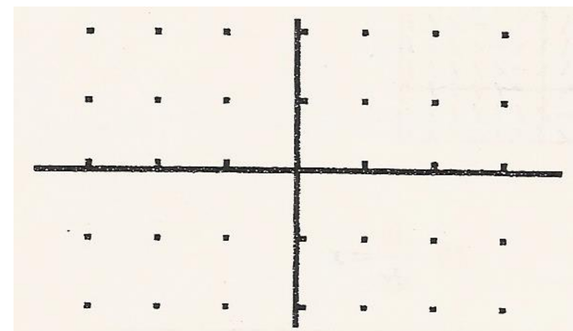
ex: $\frac{dy}{dx} = x - 1$



ex: $\frac{dy}{dx} = \frac{1}{2}y$

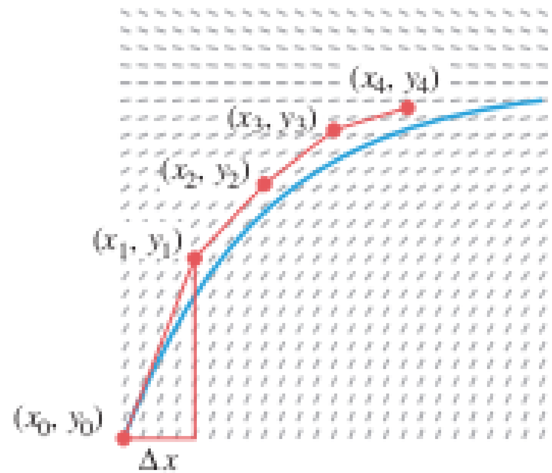


ex: $\frac{dy}{dx} = x - y$



Euler's Method

To approximate the solution of the initial-value problem
$y' = f(x, y)$ and $y(x_0) = y_0$ proceed as follows:



<u>Step 1</u>. Choose a nonzero number Δx to serve as an ***increment*** or ***step size*** along the x-axis, and let

$$x_1 = x_0 + \Delta x, \qquad x_2 = x_1 + \Delta x, \qquad x_3 = x_2 + \Delta x, \ldots$$

<u>Step 2</u>. Compute successively

$$y_1 = y_0 + f(x_0, y_0)\,\Delta x$$
$$y_2 = y_1 + f(x_1, y_1)\,\Delta x$$
$$y_3 = y_2 + f(x_2, y_2)\,\Delta x$$

$$y_{n+1} = y_n + f(x_n, y_n)\,\Delta x$$

The numbers $y_1, y_2, y_3, \ldots$ in these equations are the approximations of $y(x_1), y(x_2), y(x_3), \ldots$

Accuracy of Euler's Method

Absolute Error $\quad$ |exact value – approximation|

Percentage Error $\quad \dfrac{|\text{exact value} - \text{approximation}|}{|\text{exact value}|} \times 100\%$

Ex: Given the differential equation $\frac{dy}{dx} = x + y$ Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(4) = -1$. Use Euler's method, starting at x = 4 with two steps of equal size, to approximate $f(4.2)$. Show the work that leads to your answer.

Ex: Use Euler's method with a step size of 0.2 to estimate the value of y(0.4) where y is the solution of the initial-value problem y' = y, y(0) = 1. If the exact solution of the initial value problem is $y = e^x$ find the error made in using Euler's to estimate.

Applications of D.E.

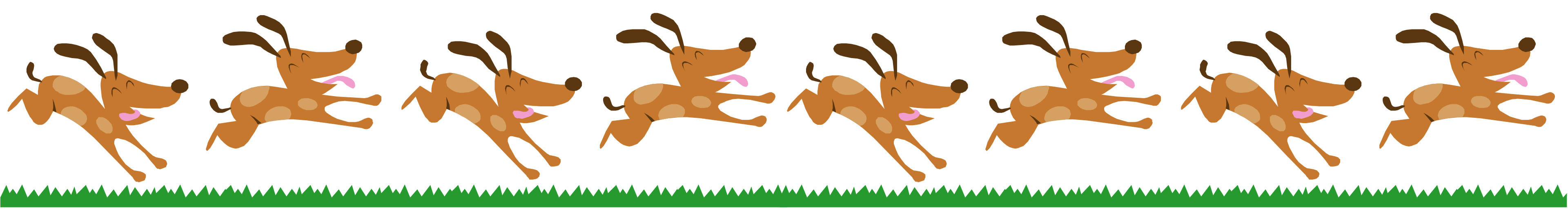
In **exponential growth** (or decay), we assume that the rate of increase (or decrease) of a population at any time *t* is directly proportional to the population *P*, $\frac{dP}{dt} = kP$ where *k* is the relative growth rate. The general solution is $P = P_0 e^{kt}$

ex: The rate of change of y is proportional to y. When t = 0, y = 2. When t = 2, y = 4. What is the value of y when t = 3?

ex: Suppose that 10 grams of the plutonium isotope Pu-239 with a half-life of 24,100 years was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

Newton's Law of Cooling: The rate of change in the temperature of an object is proportional to the difference between the temperature of the object and the temperature of the surrounding medium. $\frac{dT}{dt} = k\left(T_O - T_S\right)$

ex: Suppose a room is kept at a constant temperature of 60° and an object cooled from 100° to 90° in ten minutes. How much longer will it take for its temperature to decrease to 80°?



Logistic Growth-Day 1

In many situations population growth levels off and approaches a limiting number *L* (the carrying capacity) because of limited resources. In this situation the rate of increase (or decrease) is directly proportional to both P and $L-P$. This type of growth is called **logistic growth**. **Logistic growth** is modeled by the differential equation $\frac{dP}{dt} = kP(L-P)$

If we find $\frac{d^2P}{dt^2}$ we can find out an important fact about the time when *P* is growing the fastest.

Ex. 1 The population *P*(*t*) of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where *t* is measured in years, and *P*(0) = 4000

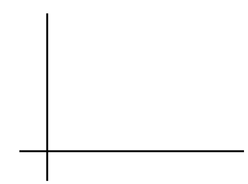
(a) $\lim_{t\to\infty} P(t) =$ __________ (b) What is the range of the solution curve?_______________

(c) For what values of *P* is the solution curve increasing? Decreasing? Justify your answer.

(d) For what values of *P* is the solution curve concave up? Concave down? Justify your answer.

(e) Does the solution curve have an inflection point? Justify your answer.

(f) Use the information you found to sketch the graph of $P(t)$



Ex. 2 The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years, and $P(0) = 10,000$

(a) $\lim_{t \to \infty} P(t) =$ ________ (b) What is the range of the solution curve? ______________

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

(e) Does the solution curve have an inflection point? Justify your answer.

(f) Use the information you found to sketch the graph of $P(t)$

Ex. 3 The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$ where t is measured in years, and $P(0) = 20,000$

(a) $\lim_{t \to \infty} P(t) =$ ________ (b) What is the range of the solution curve? ______________

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

(e) Does the solution curve have an inflection point? Justify your answer.

(f) Use the information you found to sketch the graph of $P(t)$

Logistic Growth-Day 2

Ex. The rate at which the flu spreads through a community is modeled by the logistic differential equation $\frac{dP}{dt} = 0.001P(3000 - P)$ where t is measured in days.

(a) If $P(0) = 50,$, solve for P as a function of t. All work must be shown.

(b) Use your solution to (a) and your graphing calculator to find the size of the population when *t* = 2 days.

(c) Use your solution to (a) and your graphing calculator to find the number of days that have passed when 2400 people have contracted the flu.

Logistic Growth Formulas

$$\frac{dP}{dt} = kP\left(L-P\right) \qquad P = \frac{L}{1+Ae^{-k\,L\,t}} \qquad A = \frac{L-y_0}{y_0}$$

Ex: Given the following differential equation and P(0) = 50, solve for P as a function of t. $\frac{dP}{dt} = 0.001P\left(3000-P\right)$