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SHORT COMMUNICATION

On the use of mixed-integer linear programming for predictive control with avoidance constraints

Marcelo H. Maia and Roberto K. H. Galvão*,†

Instituto Tecnológico de Aeronáutica, Divisão de Engenharia Eletrônica, São José dos Campos, SP 12228-900, Brazil

SUMMARY

This technical note concerns the predictive control of discrete-time linear models subject to state, input and avoidance polyhedral constraints. Owing to the presence of avoidance constraints, the optimization associated with the predictive control law is non-convex, even though the constraints themselves are convex. The inclusion of the avoidance constraints in the predictive control law is achieved by the use of a modified version of a mixed-integer programming approach previously derived in the literature. The proposed modification consists of adding constraints to ensure that linear segments of the system trajectories between consecutive sampling times do not cross existing obstacles. This avoids the significant extra computation that would be incurred if the sampling time was reduced to prevent these crossings. Simulation results show that the inclusion of these additional constraints successfully prevents obstacle collisions that would otherwise occur. Copyright © 2008 John Wiley & Sons, Ltd.

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KEY WORDS: constrained predictive control; vehicle maneuvering; mixed-integer programming

1. INTRODUCTION

In many applications, such as vehicle maneuvering in obstacle fields, the control problem involves non-convex avoidance constraint sets. Such constraints can be included in a predictive control law by using an elegant mixed-integer programming formulation [1, 2]. However, as shown in this technical note, the discrete-time nature of such an approach may lead to physical violation of the

*Correspondence to: Roberto K. H. Galvão, CTA-ITA, Divisão de Engenharia Eletrônica, São José dos Campos, SP 12228-900, Brazil.
†E-mail: kawakami@ita.br

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avoidance constraints in some cases, as the system trajectory between sampling instants is not taken into account.

A possible solution to this shortcoming consists of reducing the sampling time. However, such an approach may lead to significant excess computation and does not always solve the problem, as the gaps between sampling instants are reduced but not eliminated. The present paper proposes an alternative approach that involves the inclusion of appropriate additional constraints in the optimization problem.

Simulations are provided to illustrate possible problems associated with the formulation presented in [1, 2] and to show that the inclusion of the proposed constraints effectively prevents these problems from happening. For the sake of simplicity, all simulations were carried out in the nominal case, but a constraint tightening approach [2–4] could be employed to compensate the effects of persistent external disturbances.

2. BACKGROUND

2.1. Problem statement

This paper is concerned with the control of discrete-time linear models of the form $\mathbf{x}[k+1]=A\mathbf{x}[k]+B\mathbf{u}[k]$ subject to state and input constraints

$$\mathbf{x}[k]\in\mathscr{X}\subset\mathbb{R}^n \quad \forall k\geqslant 0 \tag{1}$$

$$\mathbf{u}[k]\in\mathscr{U}\subset\mathbb{R}^p \quad \forall k\geqslant 0 \tag{2}$$

and avoidance constraints

$$\mathbf{x}[k]\notin\mathscr{Z}\subset\mathbb{R}^n \quad \forall k\geqslant 0 \tag{3}$$

It is assumed that the constraint sets $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{Z}$ are polyhedral, that $\mathscr{U}$ contains the origin, that the pair (A,B) is reachable and that full-state information is available.

The control aim is to steer the system in finite time to a bounded polyhedral target set $\mathscr{Q}$, while guaranteeing satisfaction of the constraints defined in Equations (1)–(3).

2.2. Model predictive control formulation

The model predictive control (MPC) formulation presented below is a nominal version of the algorithm described in [2], which uses a variable prediction horizon $N[k]$ to ensure finite entry time in the terminal set $\mathscr{Q}$.

Let $\mathbf{U}[i|k]\triangleq[\mathbf{u}[i|k]^{\mathrm{T}},\ldots,\mathbf{u}[i+N[k]-1|k]^{\mathrm{T}}]^{\mathrm{T}}$, where the index notation $[k+i|k]$ indicates the prediction, made at time k, of a value at time $k+i$. Define the optimization problem $P(\mathbf{x}[k])$ as

$$J^*(\mathbf{x}[k])=\min_{\mathbf{U}[k|k],N[k]} J \tag{4}$$

subject to

$$J=N[k]+\rho\sum_{i=0}^{N[k]-1}\sum_{j=1}^{p}|u_j[k+i|k]| \tag{5}$$

$$\mathbf{x}[k|k]=\mathbf{x}[k] \tag{6}$$

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$$\mathbf{x}[k+i+1|k] = A\mathbf{x}[k+i|k] + B\mathbf{u}[k+i|k] \tag{7}$$

$$\mathbf{x}[k+i|k] \in \mathscr{X}, \quad 1 \leqslant i < N[k] \tag{8}$$

$$\mathbf{x}[k+i|k] \notin \mathscr{Z}, \quad 1 \leqslant i < N[k] \tag{9}$$

$$\mathbf{u}[k+i|k] \in \mathscr{U}, \quad 0 \leqslant i < N[k] \tag{10}$$

$$\mathbf{x}[N[k]+k|k] \in \mathscr{Q} \tag{11}$$

where u_j is the jth component of $\mathbf{u}$ and ρ is a positive scalar. The MPC algorithm can be described by the following steps:

Step 1: Solve optimization problem $P(\mathbf{x}[k])$.

Step 2: Apply $\mathbf{u}[k] = \mathbf{u}^*[k|k]$ to the plant, where $\mathbf{u}^*[k|k]$ is the first element of the optimal control sequence resulting from step 1.

Step 3: Let $k = k+1$ and return to step 1.

If the target set $\mathscr{Q}$ is control invariant and $P(\mathbf{x}[0])$ is feasible, then it can be shown that the closed-loop system obtained using the above algorithm is stable.

2.3. MILP implementation

$P(\mathbf{x}[k])$ can be expressed as a mixed-integer linear programming (MILP) problem by an appropriate modification of Equations (5), (8), (9) and (11), as shown in [2]. For the sake of simplicity, only the MILP representation of the avoidance constraint will be presented, as it is important for the following discussion.

The variable horizon feature is achieved by solving the optimization problem over a maximum allowed horizon $N_{\max}$ with a set of additional $N_{\max}$ binary decision variables, $b_i \in \{0, 1\}$, which enable the MILP solver to select the optimal horizon and to relax constraints beyond the selected horizon. The b variables are such that $b_i = 1$ if $N[k] = i$ and $b_i = 0$ otherwise, with the following additional constraint to ensure that the value 1 is assigned to only one of the b decision variables

$$\sum_{i=1}^{N_{\max}} b_i = 1 \tag{12}$$

Assuming that the polyhedral set $\mathscr{Z}$ is defined by N_z inequalities,

$$\mathscr{Z} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{s}_z[j]^{\mathrm{T}} \cdot \mathbf{x} \leqslant r_z[j], 1 \leqslant j \leqslant N_z\} \tag{13}$$

the exclusion condition $\mathbf{x} \notin \mathscr{Z}$ can be expressed as

$$\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^n | \mathbf{s}_z[1]^{\mathrm{T}} \cdot \mathbf{x} \geqslant r_z[1] \text{ or } \ldots \text{ or } \mathbf{s}_z[N_z]^{\mathrm{T}} \cdot \mathbf{x} \geqslant r_z[N_z]\} \tag{14}$$

According to Richards and How [2], one way to represent the avoidance constraint in the MILP form is to add a set of $N_{\max} \cdot N_z$ binary decision variables, $d_j[i]$, with $1 \leqslant i < N_{\max}$ and $1 \leqslant j \leqslant N_z$, that indicates which of the inequalities in Equation (14) hold. Assuming that $d_j[i] = 0$ if the associated inequality holds and $d_j[i] = 1$ otherwise, the MILP form of the avoidance constraint becomes

$$\mathbf{s}_z[j]^{\mathrm{T}} \cdot \mathbf{x}[k+i|k] \geqslant r_z[j] - M_z d_j[i], \quad 1 \leqslant j \leqslant N_z, \quad 1 \leqslant i < N_{\max} \tag{15}$$

where the scalar M_z is positive and sufficiently large to make the inequalities in Equation (15) hold for all states in the feasibility region [2]. To ensure that at each time step i at least one of the inequalities in Equation (14) holds, the following constraints must be added:

$$\sum_{j=1}^{N_z} d_j[i] \leqslant N_z - 1 + \sum_{h=1}^{i} b_h, \quad 1 \leqslant i < N_{\max} \tag{16}$$

3. PROPOSED MODIFICATION OF THE AVOIDANCE CONSTRAINT REPRESENTATION

The MILP avoidance constraint representation described above ensures satisfaction of the avoidance constraint at every sampling instant. However, there is no guarantee that the system trajectory between sampling instants does not violate such constraints.

Consider, for example, the movement of an object in a plane with a triangular-shaped obstacle, as depicted in Figure 1. In order to satisfy the avoidance constraint, the object must always be above line 1 (constraint 1) or above line 2 (constraint 2). The points A and B may represent valid positions at two consecutive sampling instants. However, if the trajectory between these two sampling instants was the straight line connecting A and B, the object would have actually crashed into the obstacle.

If the sampling period is small enough so that the trajectory segments between sampling instants can be approximated by straight lines, the problem described above can be circumvented by ensuring that these linear segments also satisfy the avoidance constraint. This can be accomplished by imposing that at least one of the satisfied inequalities ($d_j = 0$) on the previous sampling instant remains satisfied at the next sampling instant. Therefore, in Figure 1, if the object is to move from A (where constraint 1 is satisfied) to B (where constraint 2 is satisfied), then this approach would make the object go, for instance, to C (where both constraints are satisfied) before moving to B.

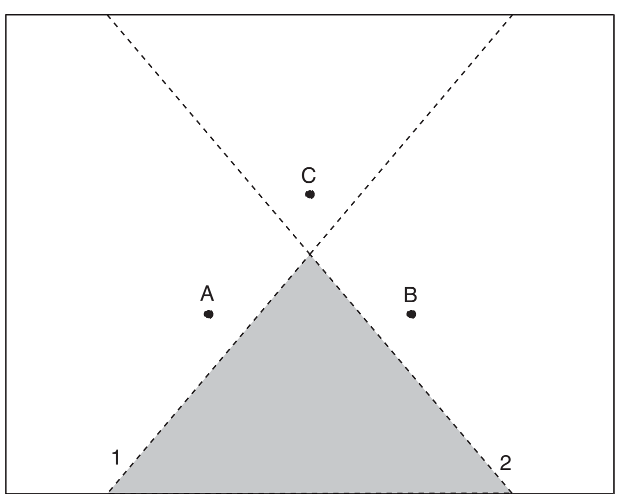

Figure 1. Illustration of a problem involving the movement of an object around a triangular-shaped obstacle (shaded area).

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In the general case, this approach can be implemented by adding the following set of constraints to the MILP formulation presented in Section 2.3:

$$\mathbf{q}^{\mathrm{T}} \cdot \mathbf{d}[i] \leqslant N_z - 1 + \mathbf{v}^{\mathrm{T}} \cdot \mathbf{d}[i-1] + \sum_{h=1}^{i} b_h, \quad 1 \leqslant i < N_{\max} \ \ \forall \mathbf{q} \in \{0,1\}^{N_z} \tag{17}$$

$$v_j = \begin{cases} -1 & \text{if } q_j = 0 \\ +1 & \text{if } q_j = 1 \end{cases}, \quad 1 \leqslant j \leqslant N_z \tag{18}$$

where $\mathbf{d}[0] \in \{0,1\}^{N_z}$ represents an input to the optimization problem and is defined as

$$d_j[0] = \begin{cases} 0 & \text{if } \mathbf{s}_z[j]^{\mathrm{T}} \cdot \mathbf{x}[k] \geqslant r_z[j] \\ 1 & \text{otherwise} \end{cases}, \quad 1 \leqslant j \leqslant N_z \tag{19}$$

The term $\sum_{h=1}^{i} b_h$ in Equation (17) is added to force relaxation of these constraints beyond the selected prediction horizon. Note that, by the inclusion of this set of constraints, at least one of the active inequalities at a given time step will be forced to remain active at the next time step.

In order to clarify the use of Equation (17), consider an example in which the avoidance constraint is defined by two inequalities ($N_z = 2$). In this situation, assuming the constraints are not relaxed due to the choice of the prediction horizon ($\sum_{h=1}^{i} b_h = 0$), Equation (17) reduces to

$$q_1 \cdot d_1[i] + q_2 \cdot d_2[i] \leqslant 1 + v_1 \cdot d_1[i-1] + v_2 \cdot d_2[i-1] \tag{20}$$

Table I presents all the inequalities defined by Equation (20), one for each possible value of $\mathbf{q}$, and Table II indicates which of these inequalities represents the only restrictive condition for a given value of $\mathbf{d}[i-1]$.

It can be seen that the inequality in the first row of Table I always holds, due to Equation (16), and that the proposed constraints force at least one of the previously satisfied inequalities to remain satisfied at the next sampling instant, as expected.

Table I. Proposed constraints for $N_z = 2$.

q_1	q_2	Associated inequality
0	0	$0 \leqslant 1 - d_1[i-1] - d_2[i-1]$
1	0	$d_1[i] \leqslant 1 + d_1[i-1] - d_2[i-1]$
0	1	$d_2[i] \leqslant 1 - d_1[i-1] + d_2[i-1]$
1	1	$d_1[i] + d_2[i] \leqslant 1 + d_1[i-1] + d_2[i-1]$

Table II. Proposed constraints effective restriction.

$d_1[i-1]$	$d_2[i-1]$	Associated inequalities	Restrictive inequality
0	0	$d_1[i] \leqslant 1$, $d_2[i] \leqslant 1$, $d_1[i] + d_2[i] \leqslant 1$	$d_1[i] + d_2[i] \leqslant 1$
1	0	$d_1[i] \leqslant 2$, $d_2[i] \leqslant 0$, $d_1[i] + d_2[i] \leqslant 2$	$d_2[i] \leqslant 0$
0	1	$d_1[i] \leqslant 0$, $d_2[i] \leqslant 2$, $d_1[i] + d_2[i] \leqslant 2$	$d_1[i] \leqslant 0$

Before presenting the simulation results, it is important to consider the limitations associated with the use of the proposed additional constraints. The first limitation is that some admissible straight paths, besides the ones that cross the obstacles, are not allowed by the modified formulation. For example, in Figure 1, the straight paths that begin in the triangular region that contains point A and end in the triangular region that contains point B are not allowed, whether they cross the obstacle or not. Therefore, the modified formulation is more conservative (in the sense that the predicted trajectories may be longer and require more time steps to be completed than the ones predicted by the original formulation), but, on the other hand, it benefits from a stronger guarantee of satisfaction of the avoidance constraints.

The second one is that the modified MILP formulation has an increased computational burden compared with its unmodified counterpart, due to the $(2^{Nz})\cdot N_{\max}$ constraints that are added for each obstacle considered. It is worth noting that the number of additional constraints grows exponentially with the number of inequalities that define the obstacles, which could represent a significant increase in computation time for situations in which the obstacles have many facets. In these cases, however, conservative approximations of the obstacles, with fewer facets, could be used to reduce the number of additional constraints.

Finally, the proposed modifications do not completely eliminate the risk of violation of the avoidance constraints, since, in practice, it is not possible to guarantee that the system trajectory between sampling instants is a straight line. However, since the sampling time is typically chosen to be sufficiently small to provide good control performance, it is reasonable to consider that the deviation of the real system trajectory from the assumed straight segment is small.

4. SIMULATION RESULTS

To illustrate the problems that may occur when using the original MILP representation of the avoidance constraints presented in [1, 2] and that the proposed modifications prevent these problems from occurring, a simple situation was considered in simulation. The variable horizon MPC in the MILP formulation was used with $N_{\max}=10$ and $\rho=1$ and the simulation was performed using Matlab® 6.5, Simulink® and the default MILP solver of the Multi-Parametric Toolbox [5].

The discrete-time state-space system was represented by the following matrices:

$$A=\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix},\quad B=\begin{bmatrix}0.5&0\\1&0\\0&0.5\\0&1\end{bmatrix} \tag{21}$$

which were obtained by a zero-order hold discretization, with 1 s sampling time, of a kinematic model for a particle that can move independently in two orthogonal directions. States x_1 and x_3 represent the particle position, states x_2 and x_4 represent the particle velocity and inputs u_1 and u_2 represent the particle acceleration.

The chosen scenario involves a triangular-shaped obstacle hindering movement without entirely blocking the path to the target set. Equations (22)–(25) define, respectively, the state and input constraints, the avoidance constraint (obstacle) and the target set. The initial condition was assumed to be $\mathbf{x}[0]=[-1\ \ 0\ \ 1\ \ 0]^{\mathrm{T}}$.

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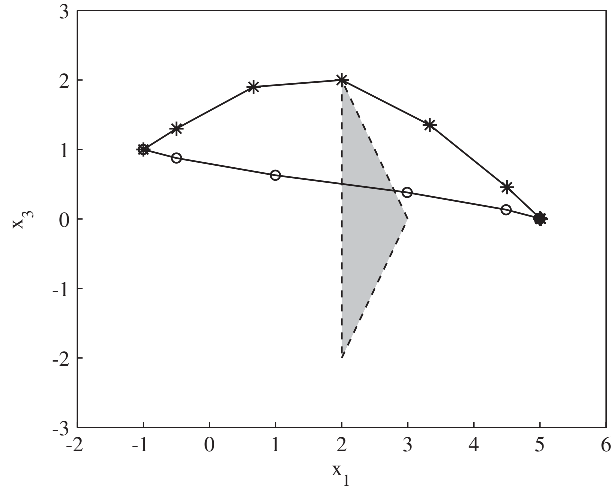

Figure 2. Results of simulation using both the original (marked with circles) and modified (marked with asterisks) avoidance constraint representations. The shaded area represents an obstacle.

$$\mathscr{X}=\{\mathbf{x}\in\mathbb{R}^4|-2\leqslant x_1\leqslant 6,-3\leqslant x_3\leqslant 3\} \tag{22}$$

$$\mathscr{U}=\{\mathbf{u}\in\mathbb{R}^2|-1\leqslant u_1\leqslant 1,-1\leqslant u_2\leqslant 1\} \tag{23}$$

$$\mathscr{Z}=\{\mathbf{x}\in\mathbb{R}^4|x_1\geqslant 2, x_1+0.5x_3\leqslant 3, x_1-0.5x_3\leqslant 3\} \tag{24}$$

$$\mathscr{Q}=\{\mathbf{x}\in\mathbb{R}^4|4.99\leqslant x_1\leqslant 5.01,-0.01\leqslant x_3\leqslant 0.01\} \tag{25}$$

Figure 2 shows the results of the simulations. The trajectory resulting from the use of the avoidance constraint representation presented in [1, 2] is marked with circles, and the one resulting from the use of the modified avoidance constraint representation proposed in this paper is marked with asterisks. It is clear that the unmodified formulation leads the system to pass through part of the obstacle, even though the constraint is enforced at every sampling time. On the other hand, the proposed additional constraints effectively guarantee that the system trajectory contours the obstacle.

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