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Inter-sample avoidance in trajectory optimizers using mixed-integer linear programming

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SUMMARY

This paper proposes an extension to trajectory optimization using mixed-integer linear programming. The purpose of the extension is to ensure that avoidance constraints are respected at all times between discrete samples, not just at the sampling times themselves. The method is very simple and involves applying the same switched constraints at adjacent time steps. This requires fewer additional constraints than the existing approach and is shown to reduce computation time. A key benefit of efficient inter-sample avoidance is the facility to reduce the number of time steps without having to compensate by enlarging the obstacles. A further extension to the principle is presented to account for curved paths between samples, proving useful in cases where narrow passageways are traversed. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: trajectory optimization; mixed integer linear programming; collision avoidance

1. INTRODUCTION

Mixed-integer linear programming (MILP) is a class of optimization extending linear programming by constraining some of the decision variables to take integer values only. This framework enables discrete decision making and logical constraints to be combined with continuous optimization [1–3]. Global solutions can be found by branch and bound algorithms [4, 5] implemented in efficient commercial solvers such as Gurobi [6] or CPLEX [7]. MILP is widely employed across operations research and was proposed for trajectory optimization by Schouwenaars [8], who used binary decision variables to select which 'side' of an avoidance box separated each pair of vehicles. This has been shown to be analagous to a disjunction, allowing the optimizer to choose from a set of constraints where only one must be enforced [9]. Since then, the MILP approach has been proposed for spacecraft formations [10] including plume impingement avoidance [11], air traffic control [12–14], mobile sensor agents [15, 16] and UAV path planning [17–19]. The advantage of MILP is its provision of a globally-optimized trajectory for non-convex avoidance problems, improving performance over locally-optimizing nonlinear methods [9] and avoiding the loss of repeatability and determinism associated with randomized methods [20].

The problem addressed by this paper is inter-sample avoidance. MILP trajectory optimizers typically employ a linearized discrete-time dynamics model, requiring a set of binary decision variables to enforce avoidance at each time step, allowing the trajectory between time steps to encroach upon the obstacles. This incursion can be bounded as a function of the inter-sample distance, and hence an appropriate margin added to the obstacles [21], ensuring that the trajectory remains outside the real obstacles between samples. However, this leads to a dilemma regarding the choice of discretization step: too short, and computation time becomes excessive, typically growing exponentially with the

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number of binary variables; too long, and the extra margin becomes excessive, blocking potentially useful passages between obstacles and giving conservative results. In 2009, Maia and Galvão published a way to prevent incursions [22] by restricting the way in which the binary variables could change from sample to sample. This modification ensured that the line joining any two samples cannot cross a constraint. The method is effective and has been further adopted by the authors to permit a user to specify the sense of turn in a conflict resolution [23]. However, the modification was also shown to add a large number of constraints to the optimization. The contribution of this paper is an alternative modification for inter-sample avoidance requiring fewer constraints. An extension is also provided to prevent constraint violation while accounting for the fact that the vehicle does not follow a straight line between samples.

The paper begins with a problem definition, including a review of the fundamentals of MILP avoidance constraints. Then, the modifications for inter-sample avoidance are presented in Section 3, along with examples and computational evaluations in comparison with Reference [22]. The extension to account for curved motion is presented in Section 4 followed by conclusions.

2. PROBLEM STATEMENT

Without loss of generality, this section defines the problem in terms of just one vehicle and one obstacle. Inclusion of multiple vehicles and obstacles is possible simply by extending the indices appropriately. Let $\mathbf{x}(k)$ be decision variables representing the dynamic state of the vehicle at time step k. Standard methods can be employed to construct linear constraints that link variables $\mathbf{x}(\cdot)$ to corresponding controls, limits of authority and boundary conditions: see, for example, Reference [8] or [18] for details.

Define a polyhedral obstacle $\mathcal{R}$ in state space as the intersection of a set of N half spaces

$$\mathcal{R} := \left\{\mathbf{x} \,:\, \mathbf{p}_i^T \mathbf{x} \leqslant q_i \;\forall i \in 1, \ldots, N\right\}. \tag{1}$$

Then, avoidance of the obstacle, $\mathbf{x}(k) \notin \mathcal{R}$, is ensured if $\mathbf{p}_i^T \mathbf{x}(k) \geqslant q_i$ for *any* i in $1, \ldots, N$. This can be ensured by the following MILP constraints employing 'big-M' relaxation [3]:

$$\mathbf{p}_i^T \mathbf{x}(k) \geqslant q_i - M b_i(k) \;\forall i \in \{1, \ldots, N\} \;\forall k \in \{1, \ldots, T\} \tag{2a}$$

$$\sum_i^N b_i(k) \leqslant N - 1 \;\forall k \in \{1, \ldots, T\} \tag{2b}$$

where $b_i(k)$ represents a binary decision variable and M is a large positive constant, such that setting $b_i(k) = 1$ effectively relaxes the corresponding constraint. The constraints (2) ensure that each sampled state $\mathbf{x}(k)$ lies outside the obstacle $\mathcal{R}$ because (2b) ensures that $b_i(k) = 0$ for at least one i. However, avoidance at the discrete samples k does not imply avoidance in between. Figure 1(a) shows an example result where the trajectory between samples cuts across obstacle corners.

3. INTER-SAMPLE AVOIDANCE

The simple observation of this paper is that inter-sample avoidance can be achieved by directly applying the same avoidance constraints from time step k at the preceding time step $k-1$, ensuring that whichever constraint is chosen at time k is also enforced at time $k-1$. This is carried out by introducing the following additional constraints:

$$\mathbf{p}_i^T \mathbf{x}(k-1) \;\geqslant\; q_i - M b_i(k) \;\forall i \in \{1, \ldots, N\} \;\forall k \in \{2, \ldots, T\}. \tag{3}$$

It should be noted that (3) is not equivalent to (2a): in going from (2a) to (3), the index on position $\mathbf{x}$ has changed from k to $k-1$ but the index on the binaries b has not. Hence, (2a) applies the constraints selected at time k, according to $b_i(k)$, to the position $\mathbf{x}(k-1)$. Together, (2) and (3) ensure that for each time step k there is a 'common' constraint i_k such that $\mathbf{p}_{i_k}^T \mathbf{x}(k-1) \geqslant q_{i_k}$

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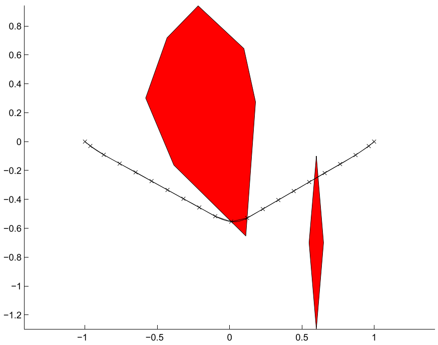
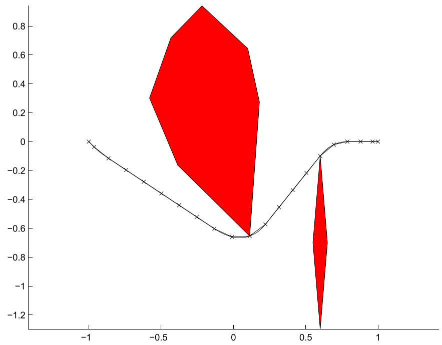
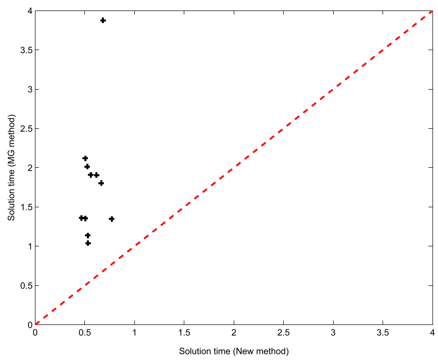


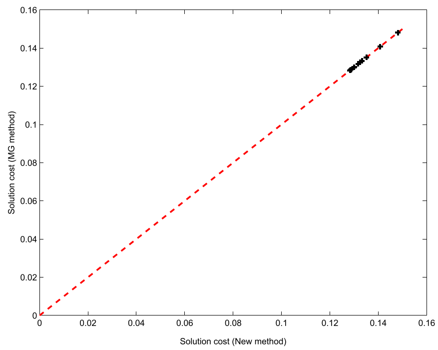
(a) Example result without inter-sample avoidance



(b) Example result with inter-sample avoidance



(c) Comparison of solution times as number of time steps changed from 10 to 20



(d) Comparison of optimized costs as number of time steps changed from 10 to 20

Figure 1. Obstacle avoidance examples.

and $\mathbf{p}_{i_k}^T\mathbf{x}(k) \geqslant q_{i_k}$. Therefore, no point on the line between $\mathbf{x}(k-1)$ and $\mathbf{x}(k)$ can be within the obstacle $\mathcal{R}$. The method adds $N(T-1)$ constraints to the problem, but no more decision variables.

3.1. Comparison with the method of Maia and Galvão

In Reference [22], Maia and Galvão showed that inter-sample avoidance could also be enforced by the following requirement:

$$\sum_{i=1}^{N} |b_i(k) - b_i(k-1)| \leqslant 1. \tag{4}$$

Condition (4) means that only one binary can be changed from step to step. Hence, the current active constraint cannot be relaxed until another has first been made active, again ensuring at least one common active constraint from one step to the next. However, Maia and Galvão go on to show that it requires $2^N(T-1)$ linear constraints to enforce the condition (4), in contrast to just $N(T-1)$ extra constraints for the method proposed herein. However, (4) are ‘tight’ constraints, without the ‘big-M’ form that is known to challenge solvers because of poor relaxation [24], so the effect on computation is not immediately clear. The two methods are therefore compared in the following section.

3.2. Example

All examples in this paper were modeled in AMPL [25] and solved using Gurobi [6] version 5.5 on an Intel Core-i7 3.1 GHz Desktop PC with 8 GB RAM.

Figure 1(a) shows an example without any inter-sample avoidance constraints in which the trajectories can be seen to cut into the obstacles, especially at sharper corners. Figure 1(b) shows the result from the modified optimizer with inter-sample avoidance using the extra constraints (3). Now, the

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trajectories are collision-free throughout. Figure 1(c) and 1(d) compare results for the same problems solved using Maia and Galvão's method [22] and using the new method proposed in this paper. Figure 1(c) plots the solution times from the two methods against each other as the number of time steps is increased from $T = 10$ to $T = 20$, reducing the step length to keep the finishing time the same. The new method is faster in all cases. Figure 1(d) does the same comparison for the optimized costs, showing that the results of the two methods are identical (as are the trajectories, which are not shown).

4. ACCOUNTING FOR DYNAMICS

Both the new method proposed in this paper and the method of Maia and Galvão ensure that the straight line joining any two adjacent samples does not intersect with the obstacle. However, this does not ensure actual avoidance at all times because the vehicle typically follows a curve, not that straight line. This section proposes a simple further extension to the constraints to ensure avoidance at all times for a particular type of dynamics. It depends on the following simple lemma.

Lemma 1
A point mass moving with constant acceleration from position $\mathbf{p}(0)$ and velocity $\mathbf{v}(0)$ must remain in the triangle formed by points $\{\mathbf{p}(0), \mathbf{p}(0) + h\mathbf{v}(0), \mathbf{p}(h)\}$ at all times $t \in [0, h]$.

Proof
The position at time t is given by the following:

$$\mathbf{p}(t) = \mathbf{p}(0) + t\mathbf{v}(0) + \frac{1}{2}t^2\mathbf{a} \tag{5}$$

where $\mathbf{a}$ is the constant acceleration. Similarly, the position at time h is given by the following:

$$\mathbf{p}(h) = \mathbf{p}(0) + h\mathbf{v}(0) + \frac{1}{2}h^2\mathbf{a}. \tag{6}$$

Combining (5) and (6) to eliminate acceleration $\mathbf{a}$ yields the following:

$$\mathbf{p}(t) = \underbrace{\left(1 - \frac{t}{h}\right)}_{\alpha}\mathbf{p}(0) + \underbrace{\left(\frac{t}{h} - \frac{t^2}{h^2}\right)}_{\beta}(\mathbf{p}(0) + h\mathbf{v}(0)) + \underbrace{\left(\frac{t^2}{h^2}\right)}_{\gamma}\mathbf{p}(h). \tag{7}$$

Noting that the quantities labeled α, β, γ sum to one and all lie in interval $[0, 1]$ for $t \in [0, h]$, the point $\mathbf{p}(t)$ must lie in the convex hull of $\{\mathbf{p}(0), \mathbf{p}(0) + h\mathbf{v}(0), \mathbf{p}(h)\}$, proving the result. □

Thus, if the dynamic model is that of a point mass subject to constant accelerations between time steps, inter-sample avoidance for the real trajectory—not just the straight line between samples—can be ensured by adding the following extra constraints:

$$\mathbf{p}_i^T\mathbf{A}\mathbf{x}(k-1) \geqslant q_i - Mb_i(k) \ \forall i \in \{1, \dots, N\} \ \forall k \in \{2, \dots, T\} \tag{8}$$

where the matrix $\mathbf{A}$ is the matrix propagating the state for a single time step without acceleration. For example, if the state is defined to be $\mathbf{x} = \left(\mathbf{p}^T, \mathbf{v}^T\right)^T$, i.e. position stacked above velocity, the matrix is given by $\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{I}h \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$. Taken together, constraints (2), (3), and (8) ensure that there is a common hyperplane separating the obstacle from the triangle of points $\{\mathbf{x}(k-1), \mathbf{A}\mathbf{x}(k-1), \mathbf{x}(k)\}$ and hence no overlap between the obstacle and the trajectory.

4.1. Example results

Figure 2(a) and 2(b) show an example of the problem addressed here, extending the example from Section 3.2 with a third obstacle. The inter-sample avoidance method from (3) ensures that the

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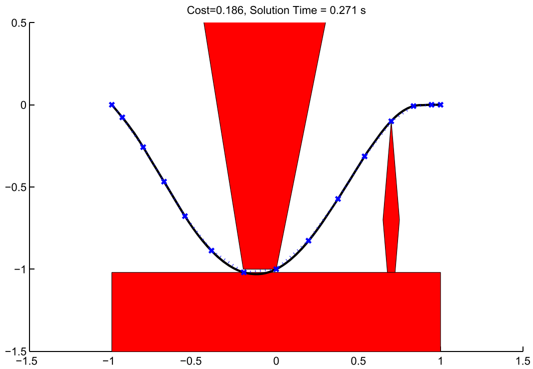
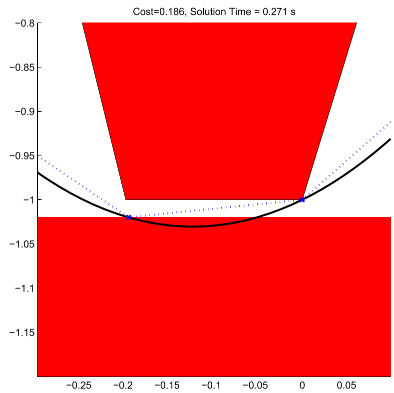
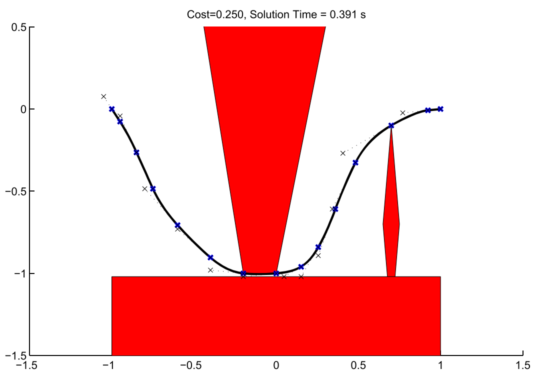


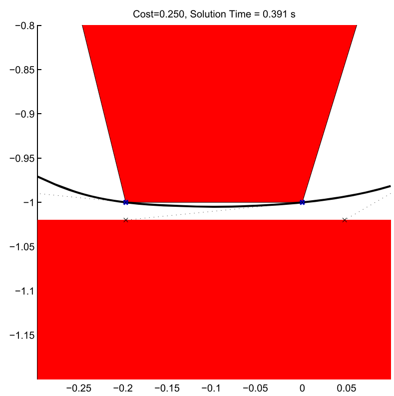
(a) Example with only inter-sample line avoidance



(b) Closer view of channel with only inter-sample line avoidance. Dashed lines join between samples.



(c) Example with continuous time avoidance



(d) Closer view with continuos time avoidance. Dashed lines indicate propagations.

Figure 2. Effect of continuous-time avoidance. The vehicle moves from right to left.

straight lines joining two samples (shown dashed) are collision free. However, the continuous time trajectory (shown solid) cuts into the lower obstacle. Figure 2(c) and 2(d) show the effect of adding the continuous time consideration using (8). The trajectory takes a greater diversion to line up better with the channel, and the close up view shows that the entire trajectory is now collision free. As expected, because the problem has been made more constrained, the cost has increased. With extra constraints, the problem takes longer to solve, up to 0.57 s from 0.44 s, but that remains a relatively fast solution for a problem of this sort.

5. CONCLUSIONS

This paper has introduced a simple but effective method to enforce inter-sample avoidance constraints in a trajectory optimization using MILP. This avoids the effect of cutting obstacle corners or straight through narrow obstacles. The method performs favorably in terms of computation to a prior solution to the problem with no loss of trajectory quality. A key consequence of inter-sample avoidance is the ability to reduce significantly the number of time steps employed. This was particularly exploited in a multi-vehicle problem whose separation requirement was far smaller than the inter-sample spacing.

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A further modification has been presented, exploiting the same principle, to ensure that the true curved trajectory joining two samples does not encroach on the obstacles. This method is particularly helpful in problems with narrow passageways to traverse. The core feature of both of these developments was the enforcement of identical relaxation options, that is, binary variables, at adjacent time steps. This ensures a common plane separating adjacent samples and the obstacles, preventing incursion between.

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