Concepts of Mechanics for Background Validation

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1 Mechanics of Continuous Media-Elasticity

1.1 Young's modulus

The Young's modulus E is also known as the elastic modulus. It is used to predict how much a material extends under tension or shortens under compression.

$$E = \frac{\sigma}{\epsilon} \qquad [\mathrm{Pa} = \mathrm{N\,m^{-2}}]$$

The experiment might be done using a rod as conceptually illustrated in Figure 1.

We need a rod attached to the ceiling, a length-measuring tool (eg : a Vernier scale) and some slotted masses of known mass. First we measure the length L of the rod and its cross-sectional area A when no mass is attached to it. We then attach the different masses and measure for each one the length of the rod and deduce the extension ΔL_i. From the weight of each mass m_i, we compute the associated force $F_i = m_i\, g$. We now have all needed information to plot the stress-strain curve of the material. We observe in Figure 2, which shows the stress-strain curve of steel, that the slope varies. The tangent of the initial, linear portion of the curve (during which is material is referred as elastic) is the Young's modulus.

Its order of magnitude is in GPa for steel materials.

Stress and strain

Note that *stress* σ is a physical quantity that expresses the internal forces that neighboring particles of a continuous material exert on each other, while *strain* ϵ is the measure of the deformation in terms of relative displacement of particles in the material.

Figure 1 – Conceptual experiment to determine Young's modulus $E = \sigma/\epsilon$ from the stress $\sigma = F/A$ and strain $\epsilon = \Delta l/L$.

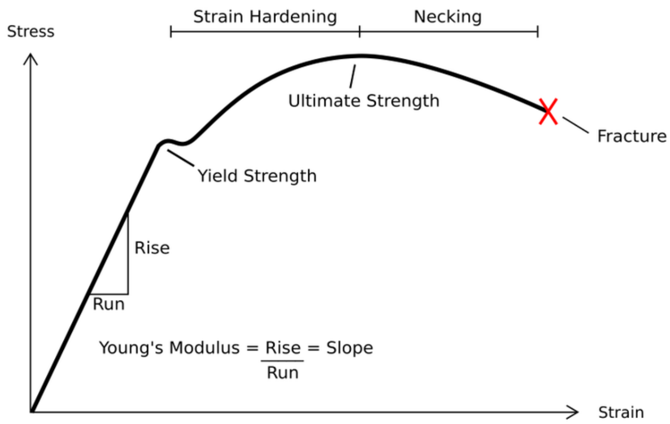

FIGURE 2 – Stress-strain curve of steel.

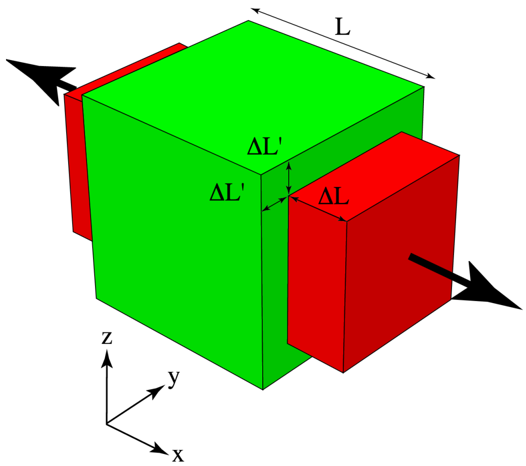

FIGURE 3 – Conceptual experiment to determine the Poisson ratio, defined in this picture as $\mu = -|\Delta L'/\Delta L|$.

1.2 Poisson ratio

The Poisson ratio ν is also known as the coefficient of expansion on the tranverse axial. It is the negative ratio of tranverse to axial strain. If the material is stretched along the axial direction

$$\nu = -\frac{\mathrm{d}\varepsilon_{\mathrm{trans}}}{\mathrm{d}\varepsilon_{\mathrm{axial}}} = -\frac{\mathrm{d}\varepsilon_{\mathrm{y}}}{\mathrm{d}\varepsilon_{\mathrm{x}}} = -\frac{\mathrm{d}\varepsilon_{\mathrm{z}}}{\mathrm{d}\varepsilon_{\mathrm{x}}}$$

Note that $\varepsilon_{\mathrm{axial}}$ is defined positive for axial tension (thus negative for compression).

An experiment to determine Poisson ratio can easily be done using an iron rod and a system which measures the displacements of the rod in the 3 directions. We extend the rod following one direction, then measure the axial strain and the tranverse one.

The Poisson Ratio is dimensionless, it has no units. Its value for most materials varies between -1.0 and 0.5 and has an average value of 0.3.

1.3 Strain tensor

The components of the strain matrix ε are obtained using this relation

$$\varepsilon_{\mathrm{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right)$$

where $u = (u_x, u_y, u_z)$ is the displacement vector.

We derive that, following notation of Figure [TO ADD],

$$\begin{aligned}
\varepsilon_1 &= \varepsilon_{\mathrm{x}} \cos^2\theta_1 + \varepsilon_{\mathrm{y}} \sin^2\theta_1 + 2\varepsilon_{\mathrm{xy}} \sin\theta_1 \cos\theta_1 \\
\varepsilon_2 &= \varepsilon_{\mathrm{x}} \cos^2\theta_2 + \varepsilon_{\mathrm{y}} \sin^2\theta_2 + 2\varepsilon_{\mathrm{xy}} \sin\theta_2 \cos\theta_2 \\
\varepsilon_3 &= \varepsilon_{\mathrm{x}} \cos^2\theta_3 + \varepsilon_{\mathrm{y}} \sin^2\theta_3 + 2\varepsilon_{\mathrm{xy}} \sin\theta_3 \cos\theta_3
\end{aligned}$$

1.4 Stress on a rigid body

If we have

$$\sigma_{zz} = \rho g(z - H) + p$$

the associated body forces and surface forces on each face are

1. at $z = 0$

   $$\sigma_{zz}^{z=0} = \frac{F_{z=0}}{L^2}$$

   which gives

   $$F_{z=0} = -V\rho g + pL^2$$

   We identify two component, the first $-V\rho g$ is the body force (corresponding to its weight) and the second p is the surface force.

2. at $z = H$

   $$F_{z=H} = pL^2$$

   which is only the surface force.

Note that the forces on the other are all equal to 0. We conclude that p is the traction's stress.

The force exerced by the body on its support is the opposite of $F_{z=0}$.

2 Fluid mechanics

2.1 Mass conservation

The *equation of mass conservation* for an incompressible fluid in the stationary case is the following

$$\frac{\partial p}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

It relates its density ρ and its flow velocity $\vec{v}$.

2.2 Viscosity

The viscosity η of a fluid is a measure of its resistance to gradual deformation by shear stress or tensile stress. Its unit is the Pa·s.

To find the viscosity of a fluid, we can do the following experiment. We get a sphere of known radius r_1 and mass m_1, let $\rho_1 = m_1/V_1$ be its density where V_1 is its volume deduced from r_1. A cylinder of height h and cross-section radius r_2 containing a fluid of known mass m_2, let $\rho_2 = m_2/V_2$ be the fluid density where V_2 is deduced from r_2 and h.

Mark with tape a starting point about 2cm below the surface of the liquid (in this way the sphere can reach terminal velocity before we begin taking measurements). The ending point should be marked about 5cm from the bottom. Measure the distance d between the two tapes. Drop the sphere in the cylinder and measure t as the time it takes to travel between the two tapes. We now have the average velocity $v = d/t$ of the sphere. The viscosity η is found using the following formula

$$\eta = \frac{2\Delta\rho\, g\, r_1^2}{9\, v}$$

where $\Delta\rho = \rho_1 - \rho_2$ and g is the acceleration of gravity.

3 Rigid body dynamics

3.1 Velocity of a point on a rigid body

The velocity v of a point with position vector x belonging to the rigid body is given by

$$v = v_C + \omega \times (x - x_C)$$

where v_C and ω be respectively the velocity vector of the center of inertia and the angular velocity vector and x_C the position vector of the center of inertia.

3.2 Moment of inertia

The matrix of inertia is the following

$$J = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix}$$

The general formula for finding its components is given by

$$J_\Delta = \iiint d(u, \Delta)^2 \rho \, \mathrm{d}V \tag{1}$$

where $d(x, \Delta)$ is the distance between a point u in the solid and the axis Δ. The inertia matrices are given here below for the solid sphere and cylinder but no calculations are made. To understand how to apply equation (1) see Appendix A.

The physical unit of the components are kg·m^2. It is a symetric and positive semi-definite matrix. The diagonal components J_{ii} of the matrix are always non-zero since it would mean an infinite angular acceleration around the x-axis. The diagonal components J_{ij} indicates us how the object is going to be accelerated around the j-axis when a torque is applied around the i-axis. This has no sense for symetric objets, therefore non-diagonal components are zero.

Cylinder

We will first derive the components of the inertia matrix of a cylindrical rigid body with mass M, height H, circular cross-section with radius R, with its axis along the z-axis.

$$J = M \begin{bmatrix} \frac{1}{12}H^2 + \frac{1}{4}R^2 & 0 & 0 \\ 0 & \frac{1}{12}H^2 + \frac{1}{4}R^2 & 0 \\ 0 & 0 & \frac{1}{2}MR^2 \end{bmatrix}$$

Sphere

Now with spherical rigid body of mass M and radius R.

$$J = M \begin{bmatrix} \frac{2}{5}MR^2 & 0 & 0 \\ 0 & \frac{2}{5}MR^2 & 0 \\ 0 & 0 & \frac{2}{5}MR^2 \end{bmatrix}$$

4 Strength of materials - Beams

A straigth horizontal beam with Young modulus E, length L, square cross-section S, bending inertia I, is free at its extremity, loaded only by a vertical force F at its mid-length and clamped at the origin. A drawing of the problem is shown in Figure 4. Let $u(x)$ the vertical displacement.

We know that

$$u''(x) = \frac{M(x)}{EI}$$

where $M(x)$ is the bending moment, which we found few lines down from here when computing the shear force and bending moment diagrams.

From equations (3) and (5), we have

1. for $x \in [0, L/2]$

$$\begin{aligned} u''(x) &= \frac{1}{EI} F(x - \frac{L}{2}) \\ u'(x) &= \frac{1}{EI} \left(F\frac{x^2}{2} - \frac{L}{2}x \right) + C_1 \\ u(x) &= \frac{1}{EI} \left(F\frac{x^3}{6} - \frac{L}{4}x^2 \right) + C_1 x + C_2 \end{aligned}$$

2. for $x \in [L/2, L]$

$$\begin{aligned} u''(x) &= 0 \\ u'(x) &= C_3 \\ u(x) &= C_3 x + C_4 \end{aligned}$$

The known boundary conditions are the following

- $u(0) = 0$ as the beam doesn't experience any deflection at the wall,
- $u'(0) = 0$ as the beam is assumed to be horizontal at the wall,
- $u''(L) = 0$ as we assume no bending moment at the free end of the beam,
- $u'''(L) = 0$ as we assume no shearing force acting at the free end of the beam.

Using these conditions we can easily show that

$$C_i = 0 \qquad \forall i \in [1, 4]$$

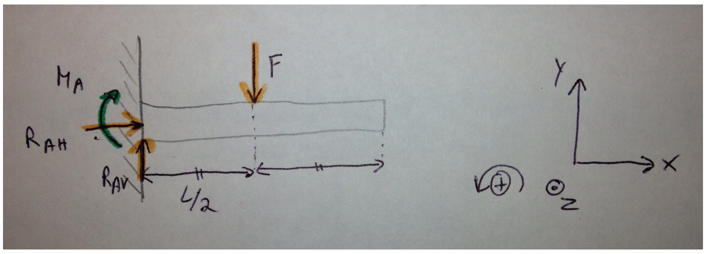

FIGURE 4 – General problem.

We are now going to find the *shear force* $V(x)$ and the *bending moment* $M(x)$ where x varies between $[0, L]$.

First, we set the general conditions.

$$\begin{aligned}\sum F = 0 &\iff R_{AV} - F = 0 \\ &\iff R_{AV} = F \\ \sum \tau = 0 &\iff -M_A - \frac{L}{2}F = 0 \\ &\iff M_A = -\frac{L}{2}F\end{aligned}$$

Note that the torque is defined positive in the counter-clockwise direction, as seen in Figure 4.

Now our goal will be to find V and M along the beam, we will proceed by iterating the above step for each cut. A cut is defined when a new force appears. We define x as being the distance of the cut from the origin.

Thus, we have for $x \in [0, L/2]$, shown in Figure 5.

$$\begin{aligned}\sum F = 0 &\iff R_{AV} - V = 0 \\ &\iff T = F \qquad (2)\\ \sum \tau = 0 &\iff -M_A - xV + M = 0 \\ &\iff M = F(x - L/2) \qquad (3)\end{aligned}$$

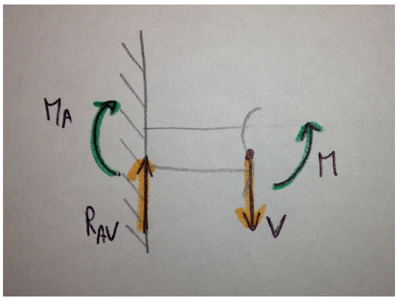

FIGURE 5 – First cut.

Then, for $x \in [L/2, L]$, shown in Figure 6.

$$\begin{aligned}\sum F = 0 &\iff R_{AV} - F - V = 0 \\ &\iff V = 0 \qquad (4)\\ \sum \tau = 0 &\iff -M_A - \frac{L}{2}F - xT + M = 0 \\ &\iff M = 0 \qquad (5)\end{aligned}$$

Using equations (2) to (5), we can draw the diagrams found in Figure 7.

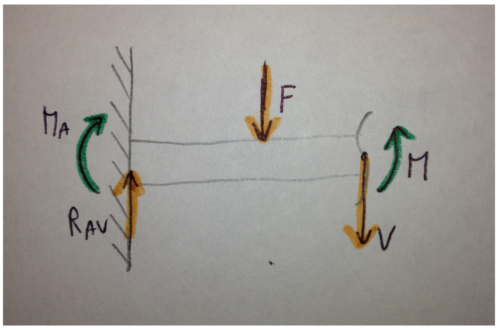

FIGURE 6 – Second cut.

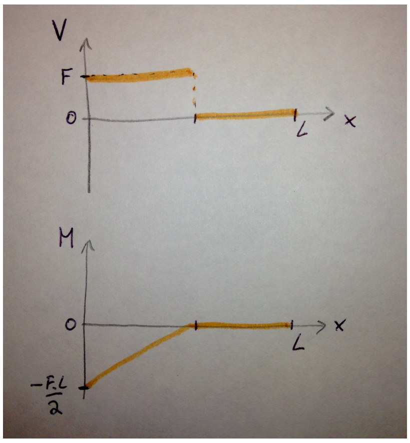

FIGURE 7 – Shear force V diagram and bending moment M diagram at the origin.

Appendix

A Components of inertia matrix

Let's recall equation (1).

$$J_\Delta = \iiint d(u, \Delta)^2 \rho \, \mathrm{d}V$$

Let's compute

$$J_{zz} = \iiint d(u, Oz)^2 \rho \, \mathrm{d}V$$

where u is any point contained in the sphere. In the case of the sphere, we have $J_{xx} = J_{yy} = J_{zz}$.

Using spherical coordinates (r, ϕ, θ) for x as shown in Figure 8, this gives us

$$J_{zz} = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^{R} (r \sin\theta)^2 \rho \, r^2 \sin\theta \, \mathrm{d}r$$

the easy part

$$J_{zz} = \frac{2}{5} \rho \pi R^5 \int_0^{\pi} \sin^3\theta \, \mathrm{d}\theta$$

the need-more-math part, knowing that

$$\int \sin^3\theta \,\mathrm{d}\theta = \frac{1}{3}\cos^3\theta - \cos x$$

gives as expected

$$J_{zz} = \frac{2}{5}MR^2$$

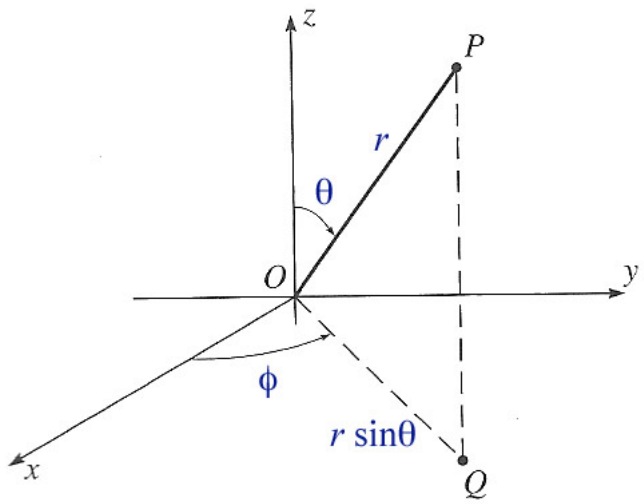

FIGURE 8 – Spherical coordinates used in Appendix A.

Another way to do it, is to slice up the solid sphere into infinitesimally thin solid cylinders, as shown in Figure 9. Knowing that the moment of inertia for a solid cylinder is

$$J_{zz} = \frac{1}{2}MR^2$$

Applied to this problem we get

$$\mathrm{d}J_{zz} = \frac{1}{2}r^2 \,\mathrm{d}m$$

where $\mathrm{d}m = \rho\,\mathrm{d}V$, and

$$\mathrm{d}V = \pi r^2 \,\mathrm{d}x$$

we also notice that $r^2 = R^2 - x^2$, thus

$$\mathrm{d}J_{zz} = \frac{1}{2}\rho\,\pi(R^2 - x^2)^2 \,\mathrm{d}x$$

We integrate over $[-R, R]$, which gives

$$J_{zz} = \frac{8}{15}\rho\,\pi R^5$$

Hence,

$$\rho = \frac{M}{V} = \frac{M}{4/3\pi R^3}$$

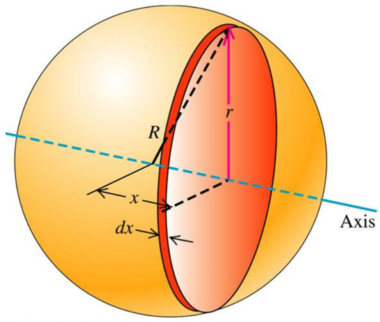

FIGURE 9 – Scheme for computing the moment of inertia of a spherical rigid body of mass M and radius R.

Finally,

$$J_{zz} = \frac{2}{5}MR^2$$