Renormalization and rigidity

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Overview

1. Background
2. Dynamics
3. Little copies
4. Renormalization
5. *A priori* bounds and rigidity

The big picture

- Introduce the fundamental objects in complex dynamical systems
- Define renormalization of quadratic dynamical systems
- Give sufficient conditions for the existence of *a priori* bounds
- Prove that these conditions imply combinatorial rigidity
- To non-mathematicians: Sorry... This is a math talk.
- To mathematicians: Sorry... I'll skip all the details.

Complex numbers

Definition

A *complex number* has the form $a + bi$, where a and b are real numbers, and $i = \sqrt{-1}$. $\mathbb{C}$ denotes the set of all complex numbers.

- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Multiplication: $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$
- We can visualize the complex numbers as a 2-dimensional plane.

Complex functions

- Elementary calculus considers functions $f : \mathbb{R} \to \mathbb{R}$.
- Elementary calculus generalizes to functions $f : \mathbb{C} \to \mathbb{C}$.
- A differentiable complex function f has a derivative f'. Zeros of the derivative are *critical points*
- Complex functions can be iterated: $f^2(z) = f(f(z))$. The *n-th iterate* of f is $f^n(z) = \underbrace{f(...(f}_{n \text{ times}}(z))$.

Example

A *complex quadratic* has the form $f(z) = z^2 + c$, where c is a complex number.

- $f'(z) = 2z$, so 0 is the critical point.
- $f^2(z) = (z^2 + c)^2 + c$.

The Fatou set

Consider a complex quadratic $f_c(z) = z^2 + c$.

Definition

The *Fatou set* K_c of f_c is the set of z such that $f_c^n(z) \not\to \infty$ as $n \to \infty$.

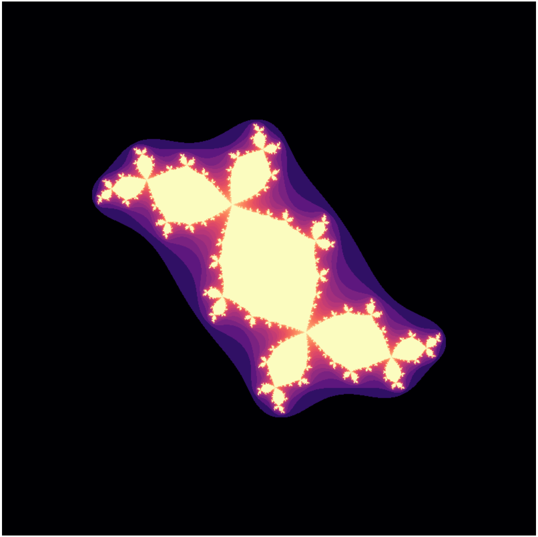
Definition

The *Julia set* is the boundary of K_c.

For historical reasons, we'll sometimes call K_c the Julia set.

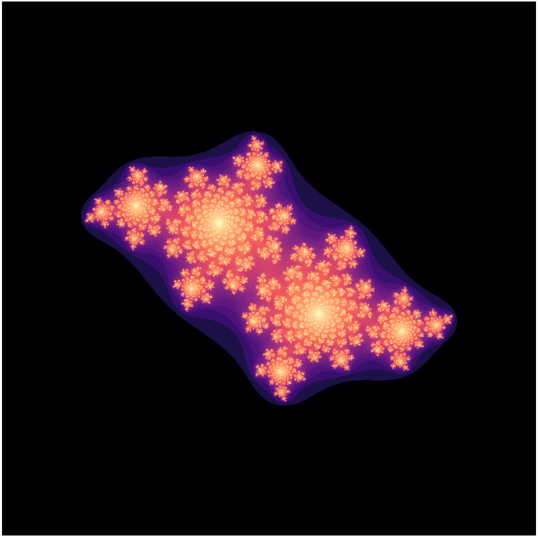
The Julia set can be connected

$$c = -0.1226 + 0.7449i$$



The Julia set can be disconnected

$c = -0.781 + 0.234j$



Parameter space for quadratic dynamics

Theorem (The fundamental dichotomy)

K_c is either connected or totally disconnected.

Theorem

K_c is connected if and only if $f_c^n(0) \not\to \infty$.

Definition

The *Mandelbrot set M* is the set of c such that K_c is connected.

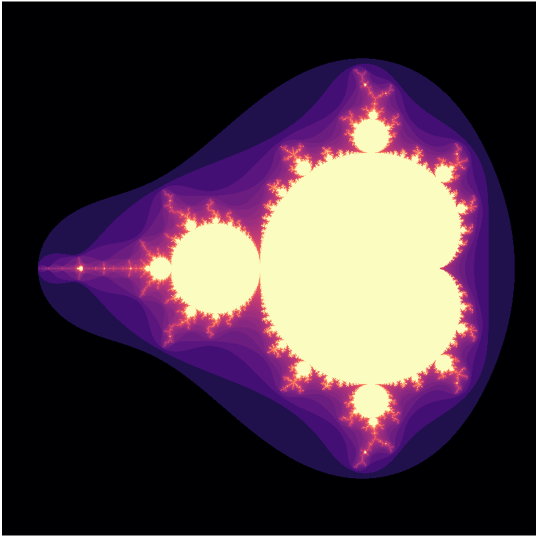
The Mandelbrot set

Why study the Mandelbrot set?

- Dynamical systems are things that change over time.
- Complex dynamical systems are fundamental to mathematics.
- If we understand the Mandelbrot set, then we understand the dynamics of all quadratics.
- The Mandelbrot set is universal: M is dense in the bifurcation locus of any holomorphic family of rational maps.

In this field, the most famous conjecture is called MLC: The Mandelbrot set is locally connected.

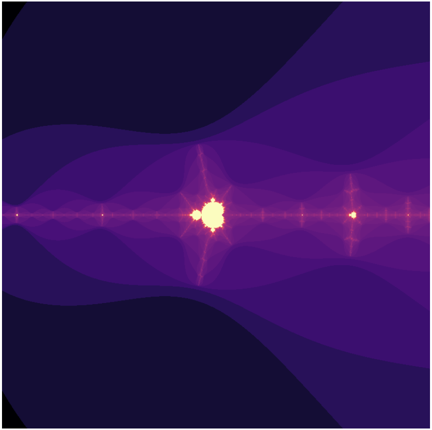
The Mandelbrot set

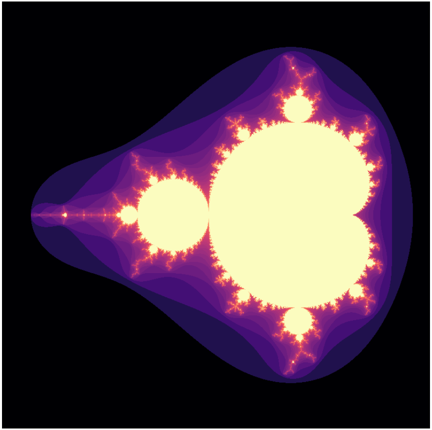


Can you see the little copies of the Mandelbrot set?

Period 3: Little copy

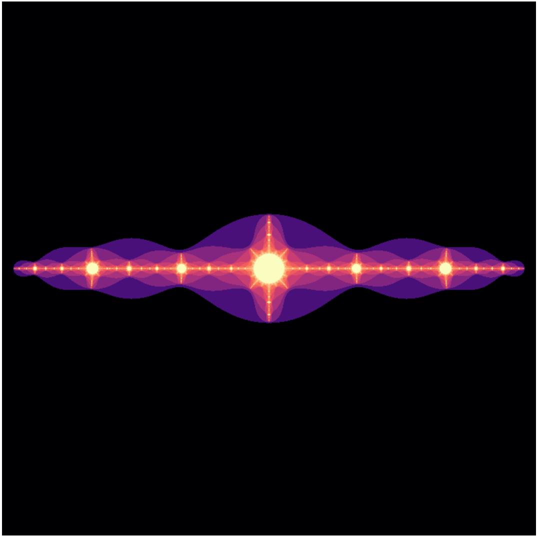
$c = -1.754877666246693$





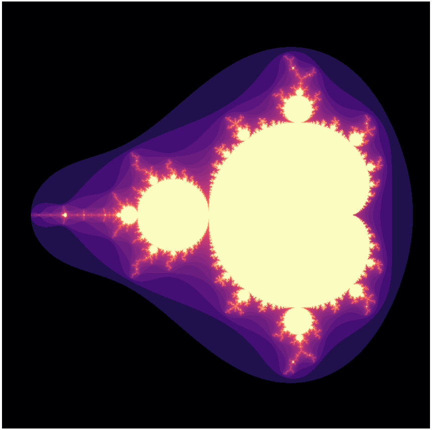
Period 3: Julia set

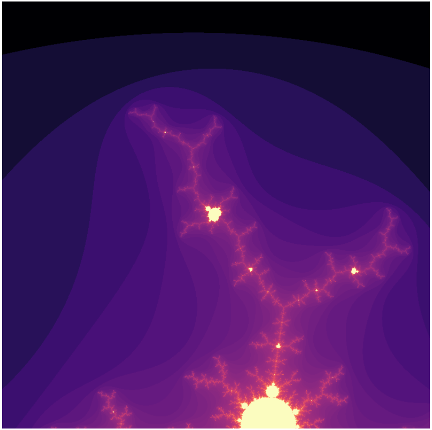
$c = -1.754877666246693$



Period 4: Little copy

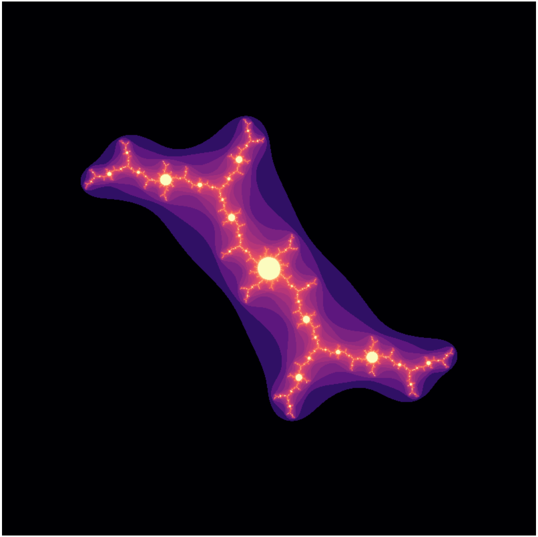
$$c = -0.156520166833755 + 1.032247108922832i$$





Period 4: Julia set

$c = -0.156520166833755 + 1.032247108922832i$



How do we explain these little copies?

Observations

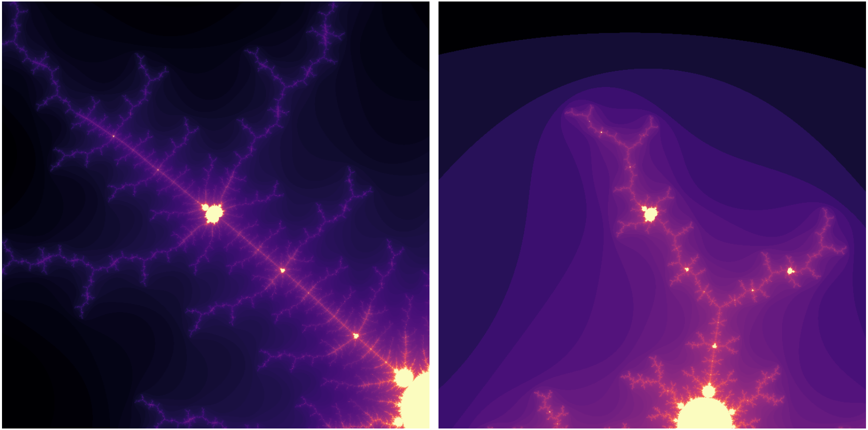
- Little copies look like the Mandelbrot set.
- We've seen little copies for periods 3 and 4.

Question

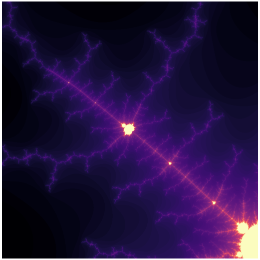
Is there a little-little copy in the period 4 little copy corresponding to the period 3 little copy in the Mandelbrot set?

Period 12: Little copy

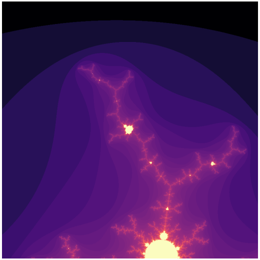
$$c = -0.167349208205021 + 1.041178661132973i$$



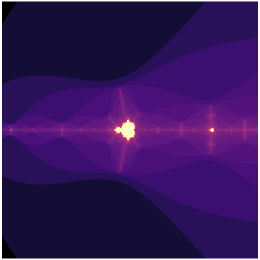
The period 12 LC corresponds to the period 3 LC



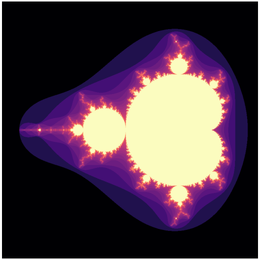
is to



as

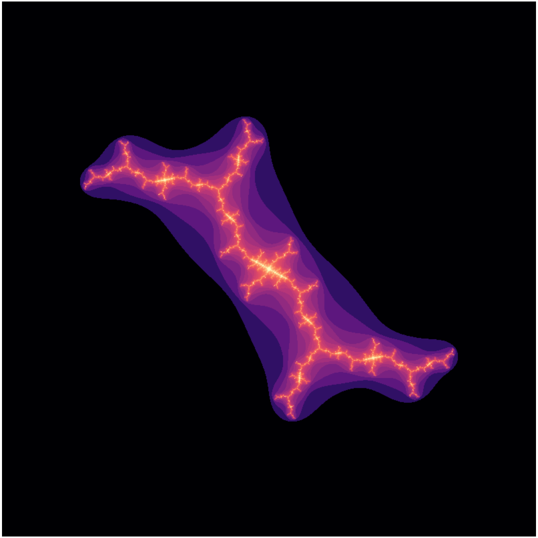


is to



Period 12: Julia set

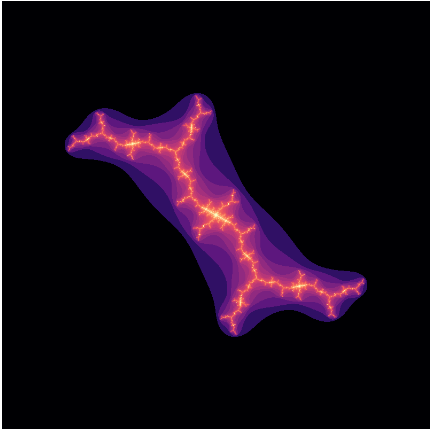
$$c = -0.167349208205021 + 1.041178661132973i$$

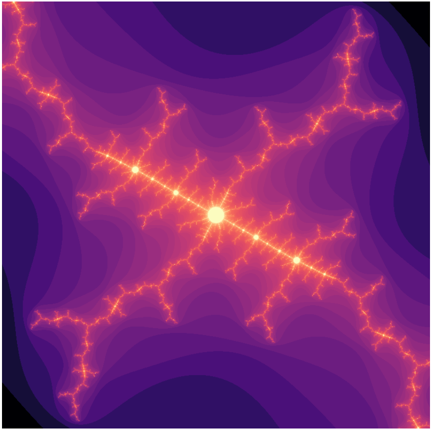


It looks like a little period 3 Julia set is embedded in the period 4 Julia set.

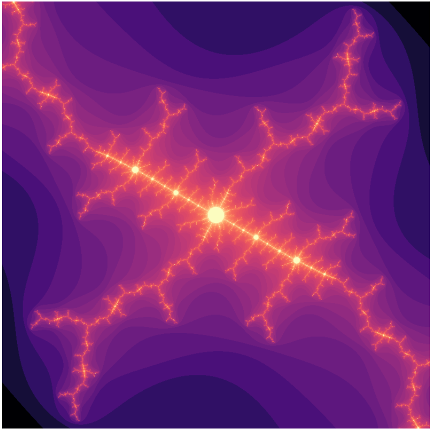
Renormalization 1: Zoom in to the little Julia set

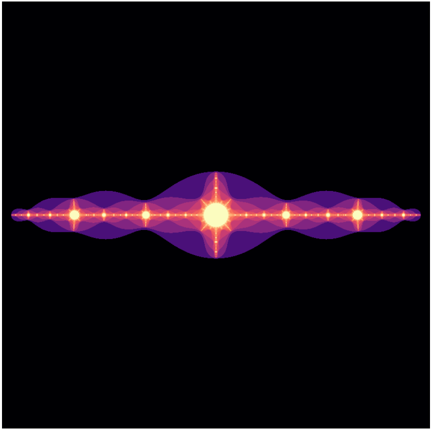
Zoom in to the center of the period 12 Julia set.





Renormalization 2: Straighten out the little Julia set

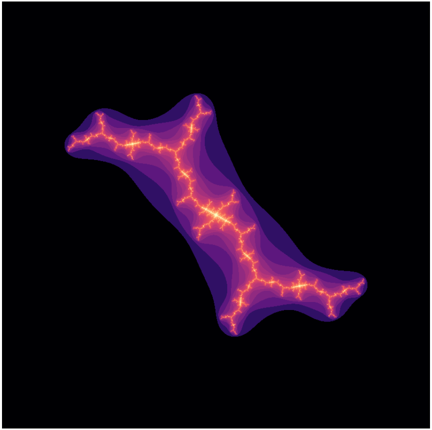


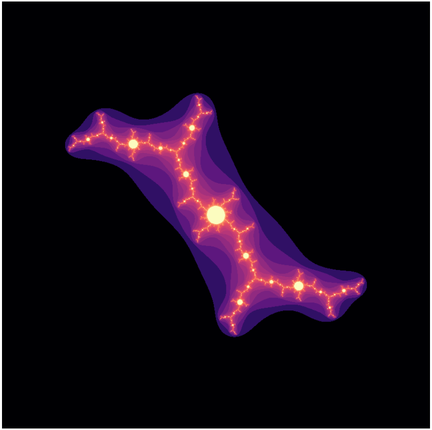


We obtain the period 3 Julia set.

Combinatorial model: Forget about the little Julia set

Blur your eyes so that each little period 3 Julia set just looks like a blob.





We obtain the period 4 Julia set.

Renormalization: A vague definition

Let f be a *renormalizable* quadratic.

Definition

The *renormalization* Rf of f is the quadratic obtained by zooming in to a little embedded Julia set.

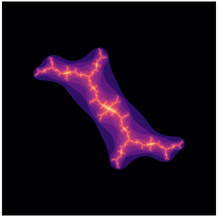
The renormalization scheme we'll discuss is called *primitive* renormalization in mathematical literature.

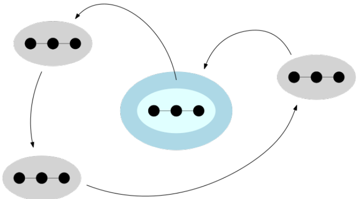
Definition

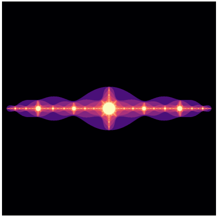
The *combinatorial model* Hf of f is the shape/dynamics of the big Julia set, where we forget about what happens on the little Julia sets.

We can think of Hf as the quadratic obtained by replacing each little Julia set with a blob/disk. In mathematical literature, the combinatorial model coincides with the *Hubbard tree*.

Renormalization: Definition via example







1. Draw a small, topological disk U around the central little Julia set.
2. Iterate U until it returns to the central little Julia set. Let $V = f^4(U)$ denote the topological disk so obtained.
3. The central little Julia set is the set of points that never escape U under iterates of $Rf = f^4$.
4. $Rf = f^4 : U \to V$ is a period 4 renormalization of f.
5. Rf is *quadratic-like* and equivalent to the quadratic associated with the period 3 Julia set.

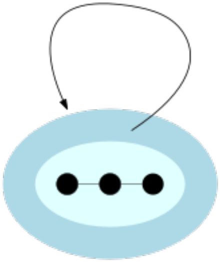
Quadratic-like maps

Definition

A *quadratic-like* map is a degree 2 branched cover $f : U \to V$, where

- U and V are topological disks,
- $\overline{U} \subset V$, and
- the Fatou set, $K = \bigcap_{n=1}^{\infty} f^{-n}(U)$, is connected.

The *modulus* of f, denoted $\operatorname{mod} f$, is the conformal thickness of the annulus $V \setminus \overline{U}$.



U = light blue disk

V = dark blue disk

f = black arrow

K = black continuum within U

Quadratic-like maps

- A quadratic-like map is equivalent, i.e., hybrid conjugate, to a quadratic.
- A renormalization of a quadatic or quadratic-like map is a quadratic-like map.

Example

Let $f(z) = z^2 + c$ have a connected Julia set. The restriction of f to a small disk around its Julia set is a quadratic-like map.

Canonical renormalization leads to improvement of life

Let $f : U \to V$ be a quadratic-like renormalization. There's too much freedom in choosing U and V.

Definition

If U and V are chosen so that they maximize $\operatorname{mod} f$, then we call f a *canonical* renormalization.

Theorem (Improvement of life)

There exists a threshold $\mu > 0$, depending only on certain combinatorial data, such that if Rf is a canonical renormalization of f, then

$$\operatorname{mod} Rf < \mu \Rightarrow \operatorname{mod} f < \mu/2.$$

Infinite renormalization

Definition

Suppose that a quadratic f admits infinitely many renormalizations

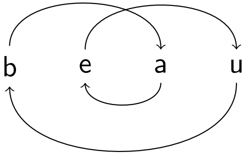
$$f_n = f^{p_n} : (U_n, K_n) \to (V_n, K_n)$$

around its critical point.

- f is *infinitely renormalizable*.
- f has *bounded combinatorics* if $p_{n+1}/p_n \leq B < \infty$ for all n.
- f has *a priori bounds* if $\operatorname{mod} f_n \geq \mu > 0$ for all n.

From now on, we'll assume that the renormalizations f_n are canonical.

Beau bounds



beau = bounded and eventually universally bounded

Beau bounds

Theorem (Beau bounds)

Let $B > 0$. There exists $\mu > 0$, depending only on B, such that if f is an infinitely renormalizable quadratic with combinatorics bounded by B, then

$$\operatorname{mod} f_n \geq \mu > 0$$

for large enough n.

In fact, $\operatorname{mod} f_n \geq \mu$ for all $n \geq N$, where N depends only on B and $\operatorname{mod} f$.

Theorem (*A priori* bounds)

If f is an infinitely renormalizable quadratic with bounded combinatorics, then f has a priori bounds.

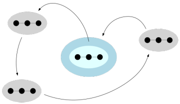
Rigidity

Theorem (Rigidity)

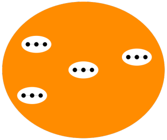
Let f and g be infinitely renormalizable quadratics with bounded combinatorics. Assume that the combinatorial models of f_n and g_n coincide for each n. Then $f = g$.

Proof of rigidity

Consider a renormalization f_n.

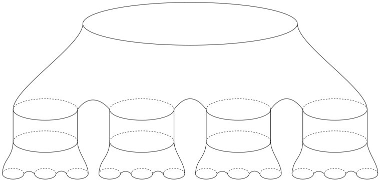


The dynamical picture gives rise to the domain S_n, drawn in orange.



Proof of rigidity

The existence of *a priori* bounds implies that the S_n domains and the annular domains A_n (between S_n and S_{n+1}) have bounded geometry.



Bounded combinatorics and the upper bound on the thickness of the annular domains A_n ensures that the amount of twist across any A_n is bounded.

Proof of rigidity

1. Coincidence of the combinatorial models implies that f and g are topologically equivalent.
2. Bounded geometry and bounded twists, together with Sullivan's pullback argument, allow us to promote the topological equivalence to quasiconformal equivalence.
3. Inou's invariant line theorem allows us to promote the quasiconformal equivalence to hybrid equivalence.
4. Hybrid equivalence here implies conformal equivalence.