Infinitely primitively renormalizable polynomials with bounded combinatorics

A priori bounds, local connectivity, and rigidity

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Polynomial-like maps

Definition

A *polynomial-like* map $f : U \to V$ is a holomorphic branched covering map of *degree* $d \geq 2$, where

- U and V are topological disks properly contained in $\mathbb{C}$, and
- $\overline{U}$ is a compact subset of V.

The (filled) *Julia* set of f is

$$K(f) = \bigcap_{n=0}^{\infty} f^{-n}(\overline{U}).$$

Example

The restriction of a polynomial to a sufficiently big disk is polynomial-like.

Polynomial-like maps 2

Theorem

$K(f)$ is connected if and only if $\mathrm{Crit}(f) \subset K(f)$.

Theorem (Douady-Hubbard)

Any polynomial-like map is hybrid conjugate to a polynomial.

A *hybrid* conjugacy between f and g is a quasiconformal map ϕ, taking a neighborhood of $K(f)$ to a neighborhood of $K(g)$, such that

- $\phi \circ f = g \circ \phi$, and
- $\mathrm{Dil}(\phi) = 0$ almost everywhere on $K(f)$.

Primitive renormalization

Let $f : U \to V$ be a polynomial-like map, and let c belong to $\mathrm{Crit}(f)$.

Definition

Given an integer $p \geq 2$, we say that f is *primitively renormalizable around* c with *period* p if there are topological disks U' and V' such that

- c belongs to U',
- $U', f(U'), \ldots, f^{p-1}(U')$ are pairwise disjoint, and
- $f^p : U' \to V'$ is a polynomial-like map, with $K' = K(f^p|U')$ connected.

We let $\mathcal{K} = \mathcal{K}_p = \bigcup_{j=0}^{p-1} f^j(K')$ denote the *little Julia sets* corresponding to the p-renormalization around c.

Infinite primitive renormalization

Definition

We say that f is *infinitely primitively renormalizable around c* if there are infinitely many integers

$$p_1 < p_2 < \cdots$$

such that for each j, f is primitively renormalizable around c with period p_j.

Definition

We say that the renormalizations have *bounded combinatorics* if there exists $B \geq 2$ such that for each j,

$$\frac{p_{j+1}}{p_j} \leq B.$$

My results

Theorem (A.)

Let f be a polynomial, with $K(f)$ connected, admitting infinitely many primitive renormalizations, with bounded combinatorics, around each of its critical points. Then

- *f has a priori bounds: there exists $\mu > 0$ such that for each renormalization $f^p : U' \to V'$,*

$$\operatorname{mod}(V' \setminus \overline{U'}) \geq \mu;$$

- *$K(f)$ is locally connected;*
- *f is rigid: for another polynomial g,*

$$f \sim_{\text{top}} g \Rightarrow f \sim_{\text{conf}} g.$$

History
A priori bounds

- real quadratic polynomials of bounded type (Sullivan '88)
- quadratic polynomials of high type (Lyubich '97)
- essentially bounded but unbounded type (Lyubich-Yampolsky '97, Graczyk-Świątek '96, Levin-van Strien '98)
- unicritical polynomials of bounded type (Kahn '06)
- quadratic polynomials satisfying the decoration condition or the molecule condition (Kahn-Lyubich '07)

History
Local connectivity

- the Feigenbaum quadratic polynomial (Hu-Jiang 98)
- quadratic polynomials with unbranched *a priori* bounds (Jiang '00)
- quadratic polynomials satisfying the secondary limbs condition and *a priori* bounds (Lyubich '97)
- real polynomials $z \mapsto z^d + c$ with connected Julia sets (Levin-van Strien '98, Lyubich-Yampolsky '97)

History

Rigidity

- no invariant line fields for robust infinitely renormalizable quadratic polynomials (McMullen '94)
- quadratic polynomials satisfying the secondary limbs condition and *a priori* bounds (Lyubich '97)
- no invariant line fields for (certain) robust infinitely renormalizable polynomials (Inou '02)
- real polynomials of degree ≥ 2 with connected Julia set, real and nondegenerate critical points, and no neutral periodic points: topological conjugacy implies quasiconformal conjugacy (Kozlovski-Shen-van Strien '07)
- unicritical polynomials satisfying the secondary limbs condition and *a priori* bounds (Cheraghi '09)

Improvement of life

Jeremy Kahn's "improvement of life" philosophy:

bad today $\Rightarrow$ even worse yesterday.

We would like to prove something like this:

If $f^p : U' \to V'$ is a renormalization of $f : U \to V$, then

$$\mathrm{mod}(V' \setminus \overline{U'}) < \mu \Rightarrow \mathrm{mod}(V \setminus \overline{U}) < \mu/100.$$

(Here, $100 = 100(B, \text{degree of } f)$ is just some constant > 1.)

This is false!

There's too much flexibility in choosing domains:

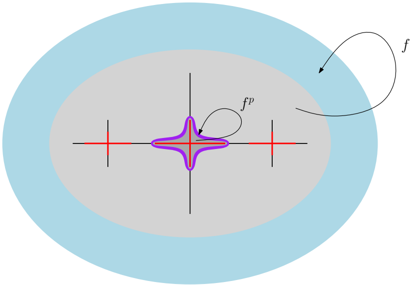

Figure: Life got worse...

When a renormalization is good

Definition

Let $f : U \to V$ be a polynomial-like map admitting a primitive p-renormalization around $c \in \mathrm{Crit}(f)$. Let $\mathcal{K} = \mathcal{K}_p$ denote the corresponding little Julia sets. We call this renormalization *good* if

$$f(\mathrm{Crit}(f)) \subset \mathcal{K}.$$

Remark

This condition is trivially satisfied when there is only one critical point.

Then $f : U \setminus f^{-1}(\mathcal{K}) \to V \setminus \mathcal{K}$ is a covering map.

Canonical renormalization

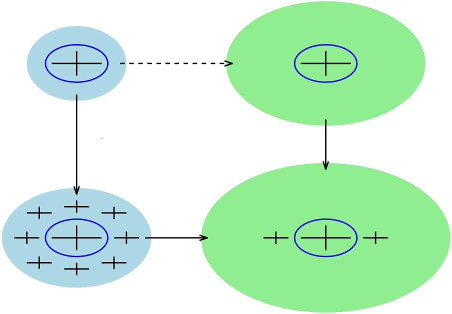

Figure: The vertical arrows are the covering maps, the bottom arrow corresponds to f^p and the inclusion, and the dashed arrow corresponds to the lifted maps.

Pseudo-polynomial-like maps

Definition

A *pseudo-polynomial-like map* $\mathbf{f}$ consists of $i, f : \mathbf{U} \to \mathbf{V}$, where

- $\mathbf{U}$ and $\mathbf{V}$ are toplogical disks properly contained in $\mathbb{C}$,
- $f : \mathbf{U} \to \mathbf{V}$ is a holomorphic branched covering map of degree ≥ 2,
- $i : \mathbf{U} \to \mathbf{V}$ is a holomorphic immersion, and
- there nondegenerate continua $\mathbf{K} \subset \mathbf{V}$ and $\mathbf{K}' \subset \mathbf{U}$ such that $\mathbf{K}' = i^{-1}(\mathbf{K}) = f^{-1}(\mathbf{K})$.

The *modulus* of $\mathbf{f}$ is

$$\mathrm{mod}(\mathbf{f}) = \mathrm{mod}(\mathbf{V} \setminus \mathbf{K}).$$

Example

Any polynomial-like map is a pseudo-polynomial-like map.

Pseudo-polynomial-like maps 2

Theorem (Pseudo-polynomial-like is like polynomial-like)

Let $\mathbf{f}$ *be a pseudo-polynomial-like map* $i, f : (\mathbf{U}, \mathbf{K}') \to (\mathbf{V}, \mathbf{K})$ *of degree* d*. Then*

- i *is injective near* $\mathbf{K}'$,
- *there exist* U *and* V *such that* $f \circ i^{-1} : U \to V$ *is a polynomial-like map with* $K(f \circ i^{-1}) = \mathbf{K}$*, and*
- $\operatorname{mod}(\mathbf{f}) \geq m > 0$ *implies* $\operatorname{mod}(V \setminus \overline{U}) \geq \mu(d, m) > 0$.

Theorem (Compactness)

Fix $\mu > 0$*. The space of pseudo-polynomial-like maps* $\mathbf{f}$ *of degree* d*, with* $\operatorname{mod}(\mathbf{f}) \geq \mu$*, is compact, up to normalization.*

The improvement of life theorem

Theorem (A.)

Fix $\lambda > 1$ and a degree $D \geq 2$. There exists $\underline{p} = \underline{p}(\lambda, D) \geq 2$ such that for any $\overline{p} \geq \underline{p}$, there exists $\mu = \mu(D, \overline{p}) > 0$ satisfying the following property:

> *Let $\mathbf{f}$ be a pseudo-polynomial-like map of degree D admitting a canonical renormalization $\mathbf{f}'$ of period p, with $\underline{p} \leq p \leq \overline{p}$. Then*
>
> $$\mathrm{mod}(\mathbf{f}') < \mu \Rightarrow \mathrm{mod}(\mathbf{f}) < \mu/\lambda.$$

Jeremy Kahn proved this theorem for degree 2 maps, but with a few trivial changes, his proof is still valid for higher degree, unicritical maps.

Non-associativity of canonical renormalization

Theorem (A.)

Let $\mathbf{f}$ *be a pseudo-polynomial-like map admitting canonical renormalizations of periods* p *and* pq*. Then* $\mathbf{f}_{pq}$ *and* $(\mathbf{f}_p)_q$ *are different. In fact,*

$$\mathrm{mod}(\mathbf{f}_{pq}) > \mathrm{mod}((\mathbf{f}_p)_q).$$

Beau bounds

Theorem (A.)

Fix a degree d and a combinatorial bound B. There exist $\mu = \mu(B, d) > 0$ and $N = N_{B,d} : (0, +\infty) \to \mathbb{Z}_{\geq 0}$ satisfying the following property:

> *Let $\mathbf{f}$ be a pseudo-polynomial-like map, of degree d, admitting some number of canonical renormalizations around $c \in \mathrm{Crit}(\mathbf{f})$ with combinatorics bounded by B; then*
>
> $$\mathrm{mod}(\mathbf{f}') \geq \mu$$
>
> *whenever $\mathbf{f}'$ is one of these renormalizations with depth at least $N(\mathrm{mod}\, \mathbf{f})$.*

Decomposition

How can we apply the improvement of life theorem to a polynomial admitting infinitely many primitive renormalizations around a critical point?

How can we guarantee that $f(\mathrm{Crit}(f)) \subset \mathcal{K}$?

Around any c in $\mathrm{Crit}(f)$, we have infinitely many polynomial-like renormalizations

$$f^{p_n(c)} : U^n(c) \to V^n(c)$$

and corresponding little Julia sets

$$K^n(c) = K(f^{p_n(c)}|U^n(c)).$$

Define

$$\begin{aligned}
\mathcal{A}^n(c) &= \bigcup_{j=0}^{p_n(c)-1} f^j(K^n(c)) \\
&= \text{the orbit of the little Julia set } K^n(c), \\
\mathcal{C}^n(c) &= \mathrm{Crit}(f) \cap \mathcal{A}^n(c) \\
&= \text{the critical points of } f \text{ appearing in the renormalization}, \\
\mathcal{K}^n(c) &= U^n(c) \cap \mathcal{A}^{n+1}(c) \\
&= \text{the little Julia sets in } U^n(c).
\end{aligned}$$

Draw a picture...

Fact

- $\mathcal{C}^n(c) \supset \mathcal{C}^{n+1}(c) \neq \emptyset$.
- *There exists an integer* $N(c) > 0$ *such that*

$$\mathcal{C}^n(c) = \mathcal{C}^N(c) \text{ whenever } n \geq N.$$

- *The critical values of* $f^{p_n(c)}|U^n(c)$ *are contained in* $\mathcal{K}^n(c)$ *whenever* $n \geq N(c)$.

In other words, we can apply the improvement of life theorem to

$$f^{p_N}(c) : U^N(c) \to V^N(c),$$

its canonical renormalizations, the canonical renormalizations of its canonical renormalizations, and so on.

A priori bounds

Let $\{\mathbf{f}_\alpha = \mathcal{R}_\alpha f\}_\alpha$ be set of all canonical renormalizations of all deeper levels of the maps

$$\{f^{p_N(c)} : U^N(c) \to V^N(c)\}_{c \in \mathrm{Crit}(f)}.$$

Theorem (A.)

There exists $\mu > 0$ such that for all α,

$$\mathrm{mod}(\mathbf{f}_\alpha) \geq \mu.$$

Corollary

Let $K^1 \supset K^2 \supset \cdots$ be a decreasing sequence of little Julia sets. As $n \to \infty$, $\mathrm{diam}(K^n) \to 0$.

Local connectivity

Theorem (A.)

K is locally connected.

Proof.

1. If $K^1 \supset K^2 \supset \cdots$, then $\bigcap K^n = \{\text{point}\}$.
2. For each n, we can find Yoccoz puzzle pieces shrinking to K^n.
3. K is locally connected at each point of the postcritical set.
4. By Koebe distortion, K is locally connected at each point of $K \setminus (\text{postcritical set})$.

$\square$

Preparing to prove rigidity

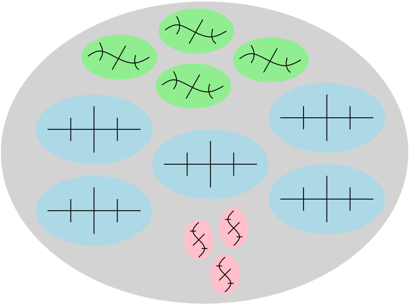

Figure: Everything is disjoint for deep enough levels.

This is what we've done so far:

1. decompose
2. improvement of life $\rightsquigarrow$ A.P.B.
3. little K_j^n are disjoint when $n \gg 0$
4. shrink domains
5. improvement of life $\rightsquigarrow$ A.P.B.

The strategy for rigidity

We want to prove

$$\begin{aligned} \text{topological equivalence} &\Rightarrow \text{quasiconformal equivalence} && (1)\\ &\Rightarrow \text{hybrid equivalence} && (2)\\ &\Rightarrow \text{conformal equivalence.} && (3) \end{aligned}$$

- (2) is a consequence of Inou's no invariant line fields theorem.
- (3) follows from gluing external and hybrid conjugacies.
- (1) is our job.

The combinatorial model

1. Assume that f admits a good renormalization of period p. Then the corresponding little Julia sets $\mathcal{K} = \mathcal{K}_p = \bigcup_{j=0}^{p-1} K_j$ of f contain the critical values and at least one critical point.
2. Collapse each little Julia set to a point. The induced topological "branched covering map" is Thurston equivalent to a superattracting polynomial P.

Definition

The *combinatorial model* of f, denoted $\mathrm{comb}_p(f)$, is the polynomial P (considered up to affine conjugacy).

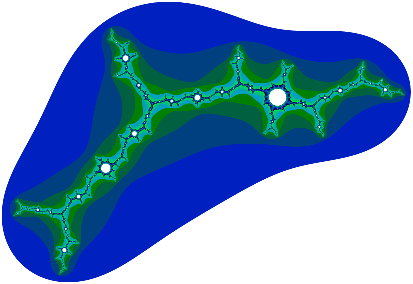

Figure: This dolphin is the Julia set of the superattracting cubic polynomial $P : z \mapsto z^3 + az + b$, where $a \approx -1.09847 - 1.09321i$ and $b \approx -0.903066 + 0.0891784i$.

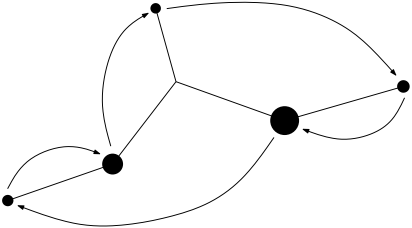

Figure: The arrows indicate the action of P. Near the two biggest disks, P acts as a degree two map. The renormalization has degree 4.

One level of renormalization

Given a superattracting model F (of period p), an integer $d \geq 2$, and positive numbers B and $m < M$, let

$$\mathcal{X} = \mathcal{X}(B, d, p, F, m, M)$$

be the set of pseudo-polynomial-like maps $\mathbf{f}$ of degree d such that

1. $\mathbf{f}$ admits a good renormalization of period p,
2. $\mathrm{comb}_p(\mathbf{f}) = F$,
3. $\mathbf{f}$ admits infinitely good renormalizations,
4. the relative renormalization periods are $\leq B$, and
5. $m \leq \mathrm{mod}(\mathbf{f}) \leq M$.

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3. $\mathbf{f}$ admits infinitely good renormalizations,
   This implies that $\mathrm{Int}(\mathbf{K}) = \emptyset$, so $\mathbf{f} \mapsto \mathbf{K}(\mathbf{f})$ is continuous.
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4. the relative renormalization periods are $\leq B$, and
   This implies $\mathcal{X}$ is closed.
5. $m \leq \text{mod}(\mathbf{f}) \leq M$.
   The lower bound implies that $\overline{\mathcal{X}}$ is compact.

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1. $\mathbf{f}$ admits a good renormalization of period p,
2. $\text{comb}_p(\mathbf{f}) = F$,
   This condition says that the maps are topologically equivalent.
3. $\mathbf{f}$ admits infinitely good renormalizations,
   This implies that $\text{Int}(\mathbf{K}) = \emptyset$, so $\mathbf{f} \mapsto \mathbf{K}(\mathbf{f})$ is continuous.
4. the relative renormalization periods are $\leq B$, and
   This implies $\mathcal{X}$ is closed.
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The S domain

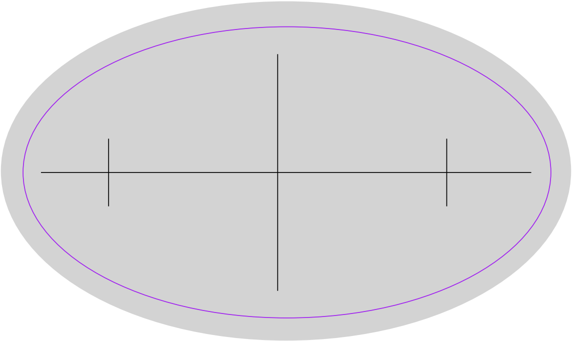

Figure: The geodesic $\Gamma \subset \mathbf{V} \setminus \mathbf{K}$ around $\mathbf{K}$ is purple.

The S domain 2

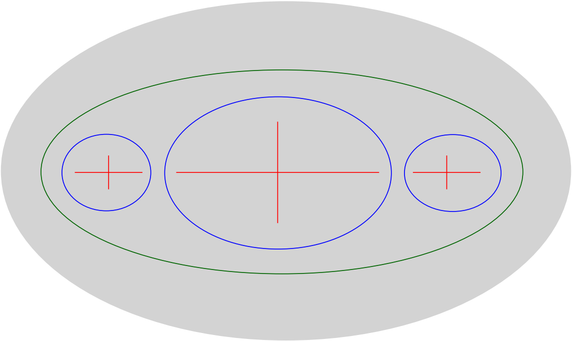

Figure: The equidistant curves $\bigcup_{j=0}^{p-1} \gamma_j^{1/2} \subset \mathbf{V} \setminus \mathcal{K}$ around each little Julia set are blue. The geodesic in the homotopy class of Γ is green.

The S domain 3

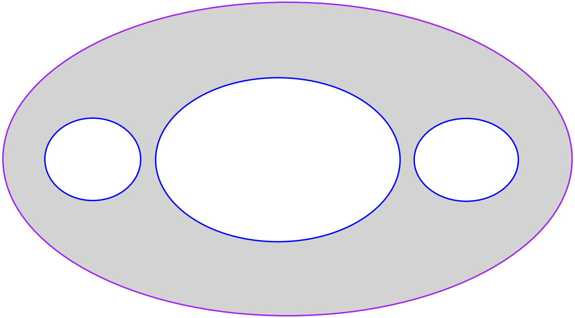

Figure: We define $S(\mathbf{f}) = S_p(\mathbf{f})$ as the domain bounded by $\Gamma \cup \bigcup_{j=0}^{p-1} \gamma_j^{1/2}$.

Choose a basepoint $\mathbf{f}_* \in \mathcal{X}$. Suppose that $\mathbf{f}_n$ and $\mathbf{f}$ are close in $\mathcal{X}$.

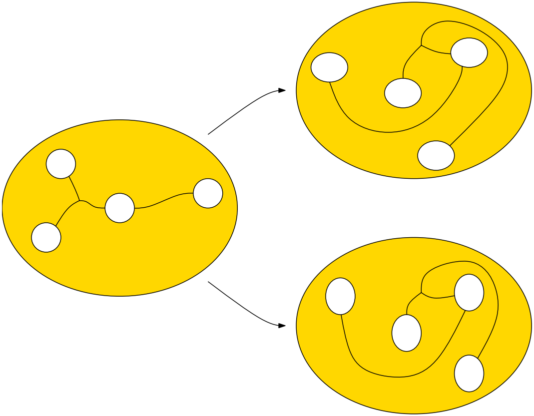

Figure: The domain $S(\mathbf{f}_*)$ is on the left. The domains $S(\mathbf{f})$ and $S(\mathbf{f}_n)$ are on the right. The black curves are the markings by the combinatorial Hubbard trees.

Fact

There is a quasiconformal map $S(\mathbf{f}) \to S(\mathbf{f}_n)$ that respects the markings, and its dilatation is small when $\mathbf{f}$ and $\mathbf{f}_n$ are close in $\mathcal{X}$.

Consequently, we obtain a continuous map

$$\Phi : \mathcal{X} \to \mathrm{Teich}^{\#}(S(\mathbf{f}_*)).$$

Fact

$\Phi(\mathcal{X})$ is bounded in Teichmüller space.

Proof of rigidity

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial admitting infinitely many primitive renormalizations, with combinatorics $\leq B$, around each of its critical points.

1. Decompose f into finitely many "good" polynomial-like restrictions $\{f_1, \dots, f_n\}$ of iterates of f.

Let $\{f_\alpha\}_\alpha$ be all of the canonical renormalizations around all of the little Julia sets of the maps $\{f_j\}_j$.

2. Apply the improvement of life theorem to the maps $\{f_j\}_j$.
3. For all α,

$$\mathrm{mod}(f_\alpha) \geq m.$$

Proof of rigidity 2

Let $g : \mathbb{C} \to \mathbb{C}$ be another polynomial topologically conjugate to f:

$$\mathcal{B} \circ f = g \circ \mathcal{B}.$$

4. Decompose g into "good" pieces $\{g_j = \mathcal{B} \circ f_j \circ \mathcal{B}^{-1}\}_j$.
5. Enumerate the canonical renormalizations of the maps $\{g_j\}_j$ according to the conjugacy $\mathcal{B}$.
6. Apply the improvement of life theorem to the maps $\{g_j\}_j$.
7. For all α,

$$\operatorname{mod}(g_\alpha) \geq m.$$

Proof of rigidity 3

8 For all α,

$$\operatorname{mod}(f_\alpha), \operatorname{mod}(g_\alpha) \leq M.$$

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Fact

For any $D, B \geq 2$, there are finitely many superattracting models (up to affine conjugacy) of degree $\leq D$ and period $\leq B$.

Let $\{\mathcal{X}_i\}_i$ be the spaces of pseudo-polynomial-like maps based on the finitely many superattracting models, with moduli in $[m, M]$.

9 For each α, f_α and g_α have the same combinatorial model.

10 There exists a K-quasiconformal map

$$h_\alpha : S(f_\alpha) \to S(g_\alpha)$$

in the homotopy class determined by $\mathcal{B}$.

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in the homotopy class determined by $\mathcal{B}$.
K is independent of α!

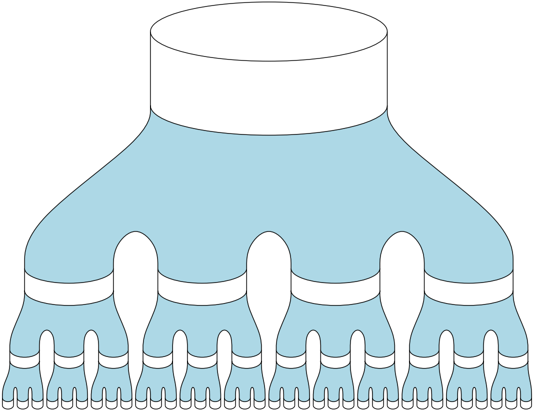

Figure: The blue pieces are the domains $S(f_\alpha)$. Interpolate on the separating annuli between the maps h_α. Don't spoil dilatation too much!

Proof of rigidity 4

Now, we have built a quasiconformal map

$$h : (\mathbb{C}, P_f) \to (\mathbb{C}, P_g)$$

such that $h \circ f = g \circ h$ on P_f.

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Proof of rigidity 4

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such that $h \circ f = g \circ h$ on P_f.
Unfortunately, h is probably not homotopic to a conjugacy...
But the homotopy class is only boundedly wrong, so we can fix it!

Two levels of renormalization

Let B, d, p, and r be integers ≥ 2. Let F and G be superattracting models models with periods p and rp, respectively. Fix positive numbers $m < M$. Let

$$\mathcal{Y} = \mathcal{Y}(B, d, p, r, F, G, m, M)$$

be the set of pseudo-polynomial-like maps $\mathbf{f}$ of degree d such that

1. $\mathbf{f}$ admits good renormalizations of periods p and rp,
2. $\mathrm{comb}_p(\mathbf{f}) = F$, and $\mathrm{comb}_{rp}(\mathbf{f}) = G$,
3. $\mathbf{f}$ admits infinitely many good renormalizations,
4. the relative renormalization periods are $\leq B$, and
5. $m \leq \mathrm{mod}(\mathbf{f}) \leq M$.

Given $\mathbf{f} \in \mathcal{Y}$, define

$$T(\mathbf{f}) = S_{rp}(\mathbf{f}).$$

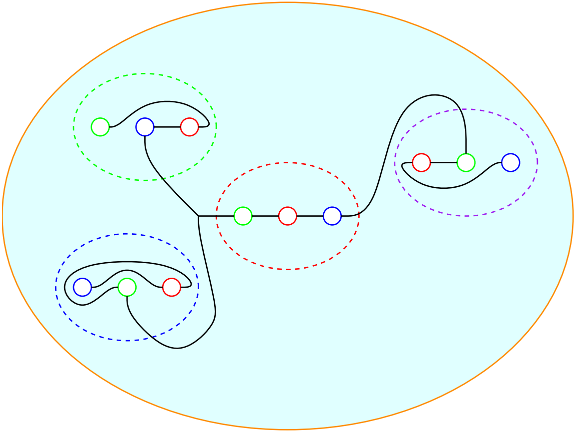

Figure: This is a domain $T(\mathbf{f})$ marked by the combinatorial Hubbard tree.

Two levels of renormalization 2

Choosing a basepoint $\mathbf{f}_*$ in $\mathcal{Y}$, we obtain a continuous map

$$\Psi : \mathcal{Y} \to \text{Teich}^{\#}(T(\mathbf{f}_*)).$$

Theorem

$\Psi(\mathcal{Y})$ *is bounded in Teichmüller space.*

Proof of rigidity 5

11 The domains $\{T(f_\alpha), T(g_\alpha)\}_\alpha$ have bounded geometry.

12 For each α, there exists a K'-quasiconformal map

$$t_\alpha : T(f_\alpha) \to T(g_\alpha)$$

in the homotopy class determined by $\mathcal{B}$.

13 The composition

$$h^{-1} \circ t_\alpha : T(f_\alpha) \to \mathbb{C} \setminus P_f$$

is a quasiconformal embedding with dilatation independent of α.

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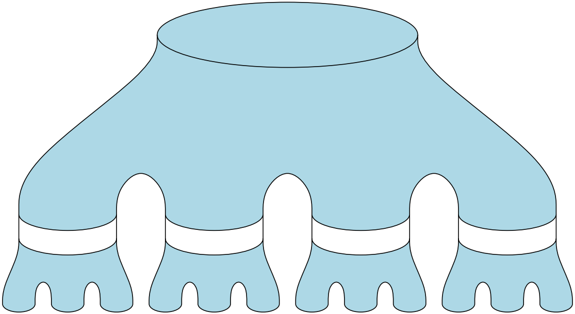

Figure: This is $T(f_\alpha)$. The quasiconformal map $h^{-1} \circ t_\alpha$ is almost homotopic to the identity map: It only differs by some number of Dehn twists around the white annuli. Bounded geometry for these domains implies that the order of the twist around each white annulus is bounded independent of α.

Proof of rigidity 6

The annuli separating the domains $S(f_\alpha)$ have modulus bounded below, and the twists around these annuli are bounded above.

- Adjust h by untwisting the appropriate amount on each annulus.
- Forget the old h, and let h denote the adjusted map.

We obtain a quasiconformal map $h : (\mathbb{C}, P_f) \to (\mathbb{C}, P_g)$ such that

- $h \circ f = g \circ h$ on P_f, and
- h is homotopic rel P_f to a conjugacy from f to g.

Proof of rigidity 7

Now, standard arguments show that f and g are hybrid conjugate.

14. Promote h to a quasiconformal conjugacy using Sullivan's pullback argument.
15. Apply Inou's no invariant line fields theorem.

This completes the proof of rigidity.