101 年微積分期末小考詳解

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再怎麼忙，也要記得......

Theorem

$$
\begin{aligned}
y' = \frac{dy}{dx} \quad &\Longrightarrow \quad \int y'\,dx = \int dy. \\
(uv)' = u'v + uv' \quad &\Longrightarrow \quad uv = \int v\,du + \int u\,dv \\
(\ln|y|)' = \frac{y'}{y} \quad &\Longrightarrow \quad \ln|y| = \int \frac{dy}{y}.
\end{aligned}
$$

$\int x^2 \sin(x)\,dx$

Solution

設 $u = x^2$ 與 $v = \cos(x)$，則 $u' = 2x$ 且 $v' = -\sin(x)$。

$$\int x^2 \sin(x)\,dx = -\int u\,dv = \int 2x\cos(x)\,dx - x^2\cos(x).$$

設 $u = x$ 與 $v = \sin(x)$，則 $u' = 1$ 且 $v' = \cos(x)$。

$$\int x\cos(x)\,dx = \int u\,dv = x\cos(x) - \int \cos(x)\,dx.$$

所以

$$\int x^2 \sin(x)\,dx = 2x\sin(x) + \left(2 - x^2\right)\cos(x).$$

神奇的導函數與判別式

Theorem

$$4a\left(ax^2 + bx + c\right) = (2ax + b)^2 + 4ac - b^2.$$

Proof.

$$\begin{aligned}
(2ax + b)^2 &= 4a^2x^2 + 4abx + b^2 \\
(2ax + b)^2 + 4ac - b^2 &= 4a^2x^2 + 4abx + 4ac \\
4a\left(ax^2 + bx + c\right) &= 4a^2x^2 + 4abx + 4ac \\
4a\left(ax^2 + bx + c\right) &= (2ax + b)^2 + 4ac - b^2.
\end{aligned}$$

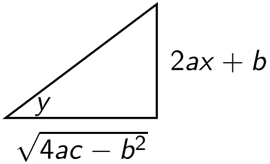
□

Solution

設 $y = \arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$。

$$\frac{dy}{dx} = \frac{2a}{\sqrt{4ac-b^2}\left(\frac{(2ax+b)^2}{4ac-b^2}+1\right)} = \frac{\sqrt{4ac-b^2}}{2ax^2+2bx+2c}.$$

$$\int \frac{1}{ax^2+bx+c}\,dx = \int \frac{2}{\sqrt{4ac-b^2}}\,dy = \frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}.$$



$\displaystyle\int \frac{1}{ax^2+bx+c}\,dx$，其中 $4ac < b^2$

Solution

ax^2+bx+c 的兩根為 $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$。

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{1}{\beta-\alpha}\left(\frac{1}{x-\beta} - \frac{1}{x-\alpha}\right)$$

$$\int \frac{1}{(x-\alpha)(x-\beta)}\,dx = \frac{\ln\left|\frac{x-\beta}{x-\alpha}\right|}{\beta-\alpha} = \frac{\ln\left|\frac{2ax-2a\beta}{2ax-2a\alpha}\right|}{\beta-\alpha}$$

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|}{\sqrt{b^2-4ac}}.$$

Solution

$$\begin{aligned}
&\int \frac{px+q}{ax^2+bx+c}\,dx \\
&= \frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx + \int \frac{q-\frac{bp}{2a}}{ax^2+bx+c}\,dx \\
&= \frac{p\ln\left|ax^2+bx+c\right|}{2a} + \int \frac{2aq-bp}{2a\left(ax^2+bx+c\right)}\,dx.
\end{aligned}$$

$\int \frac{x}{x^2 - 2x + 2}\,dx$

Solution

$$
\begin{aligned}
&\int \frac{x}{x^2 - 2x + 2}\,dx \\
&= \int \frac{2x - 2}{2\left(x^2 - 2x + 2\right)}\,dx + \int \frac{1}{x^2 - 2x + 2}\,dx \\
&= \frac{\ln\left|x^2 - 2x + 2\right|}{2} + \arctan\left(\frac{2x - 2}{2}\right) \\
&= \frac{\ln\left|x^2 - 2x + 2\right|}{2} + \arctan(x - 1).
\end{aligned}
$$

多項式除法

對於任意兩個多項式 A, B，其中 $B \neq 0$，我們都可以找到商 Q 和餘式 R 使得

$$A = BQ + R \tag{1}$$

其中 $\deg(R) < \deg(B)$。

Definition

為了方便，對於 (1) 式，我們定義以下新符號：

$$\lceil B \rceil A = R.$$

擴展的餘式算子

Definition

設 B, D, N, R 皆為多項式，其中 B 與 D 互質，我們定義

$$\lceil B \rceil \frac{N}{D} = R$$

等價於

$$\lceil B \rceil N = \lceil B \rceil DR$$

其中 $\deg(R) < \deg(B)$。

餘式算子的性質

Theorem

設 B, D, N 為多項式，F 與 G 為有理函數且 B 與 D 互質。

- $\lceil B \rceil (F + G) = \lceil B \rceil F + \lceil B \rceil G$
- $\lceil B \rceil FG = \lceil B \rceil (\lceil B \rceil F \lceil B \rceil G)$
- $\lceil B \rceil \dfrac{N}{D} = \lceil B \rceil \dfrac{\lceil B \rceil N}{\lceil B \rceil D}$

部份分式分解算法

Theorem

對於最簡真分式 A/D，其中

$$D = D_1 D_2 \cdots D_n$$

且 D_k 兩兩互質。則 A/D 可以表達為

$$\frac{A_1}{D_1} + \frac{A_2}{D_2} + \cdots + \frac{A_n}{D_n}$$

其中對於所有 k，若 $1 \le k \le n$ 則

$$A_k = \lceil D_k \rceil \frac{AD_k}{D}.$$

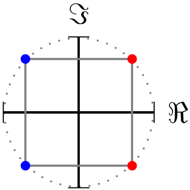
期末小考

何震邦

$$x^4+1=\left(x^2-\sqrt{2}x+1\right)\left(x^2+\sqrt{2}x+1\right).$$

設 $D_1 = x^2-\sqrt{2}x+1$ 與 $D_2 = x^2+\sqrt{2}x+1$。

$$\begin{aligned}\lceil D_1 \rceil \frac{1}{D_2} &= \lceil D_1 \rceil \frac{1}{2^{3/2}x} = \frac{x-\sqrt{2}}{-2^{3/2}} \\ \lceil D_2 \rceil \frac{1}{D_1} &= \lceil D_2 \rceil \frac{1}{-2^{3/2}x} = \frac{x+\sqrt{2}}{2^{3/2}}.\end{aligned}$$



$\int \frac{1}{x^4+1}\,dx$ 之二

Solution

$$\begin{aligned}
&\int \frac{1}{x^4+1}\,dx \\
&= \int \frac{x+\sqrt{2}}{2^{3/2}\left(x^2+\sqrt{2}x+1\right)}\,dx - \int \frac{x-\sqrt{2}}{2^{3/2}\left(x^2-\sqrt{2}x+1\right)}\,dx \\
&= \frac{\ln\left(x^2+\sqrt{2}x+1\right)}{2^{5/2}} - \frac{\ln\left(x^2-\sqrt{2}x+1\right)}{2^{5/2}} + \frac{\arctan\left(\sqrt{2}x+1\right)}{2^{3/2}} \\
&\quad + \frac{\arctan\left(\sqrt{2}x-1\right)}{2^{3/2}}.
\end{aligned}$$

Hermite reduction

假設 $m \geq 2$ 否則 D 已經 squarefree 了。首先設 $V = D_m$ 與 $U = D/V^m$。因為 UV' 與 V 互質，我們能以擴展的輾轉相除法求兩多項式 B, C 使得

$$\frac{A}{1-m} = BUV' + CV$$

其中 $\deg(B) < \deg(V)$。兩端同乘 $(1-m)/(UV^m)$ 得

$$\begin{aligned}
\frac{A}{UV^m} &= \frac{(1-m)\,BV'}{V^m} + \frac{(1-m)\,C}{UV^{m-1}} \\
&= \frac{B'}{V^{m-1}} - \frac{(m-1)\,BV'}{V^m} - \frac{B'U + (m-1)\,C}{UV^{m-1}}
\end{aligned}$$

$$\int \frac{A}{UV^m} = \frac{B}{V^{m-1}} - \int \frac{B'U + (m-1)\,C}{UV^{m-1}}.$$

$\int \frac{6x^2-15x+22}{(x+3)\left(x^2+2\right)^2}\,dx$，頁 I

設 $A = 6x^2 - 15x + 22$，又設 $U = x + 3$ 及 $V = x^2 + 2$。

$$\begin{aligned}
UV' &= 2V + 6x - 4 \\
18V &= (3x+2)(6x-4) + 44 \\
&= (3x+2)\left(UV' - 2V\right) + 44 \\
44 &= (3x+2)\,UV' + (14x-6)\,V \\
-A &= -6V + 15x - 10 \\
&= -6V + \frac{5(6x-4)}{2} \\
&= \frac{5UV'}{2} - 11V.
\end{aligned}$$

$$\int \frac{A}{UV^2} = \frac{5}{2V} + \int \frac{11}{UV}.$$

分解部份分式：

$$\begin{aligned}
\lceil x + 3 \rceil \, \frac{11}{x^2 + 2} &= \frac{11}{11} \\
\lceil x^2 + 2 \rceil \, \frac{11}{x + 3} &= \frac{11\,(x - 3)}{-11}.
\end{aligned}$$

$$\frac{11}{(x+3)\left(x^2+2\right)} = \frac{1}{x+3} + \frac{3-x}{x^2+2}.$$

$$\int \frac{6x^2 - 15x + 22}{(x+3)\left(x^2+2\right)^2}\,dx$$

$$= \frac{5}{2x^2+4} + \ln|x-3| - \frac{\ln\left(x^2+2\right)}{2} + \frac{3\arctan\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2}}.$$

$\int e^{3x} \sin(2x)\, dx$

Solution

$$\int e^{3x} \sin(2x)\, dx = \frac{e^{3x} \sin(2x)}{3} - \int \frac{2e^{3x} \cos(2x)}{3}\, dx$$

$$\int e^{3x} \cos(2x)\, dx = \frac{e^{3x} \cos(2x)}{3} + \int \frac{2e^{3x} \sin(2x)}{3}\, dx.$$

$$\begin{aligned} &\int e^{3x} \sin(2x)\, dx \\ &= \frac{e^{3x} \left(3 \sin(2x) - 2 \cos(2x)\right)}{9} - \int \frac{4e^{3x} \sin(2x)}{9}\, dx \\ &= \frac{e^{3x} \left(3 \sin(2x) - 2 \cos(2x)\right)}{13}. \end{aligned}$$

$\int_1^5 \frac{1}{x}\,dx$，分別取左、右黎曼和，$n = 4$

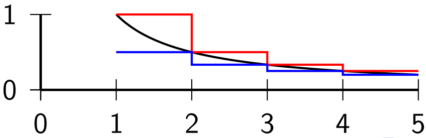
Solution

左黎曼和為

$$\int_1^5 \frac{1}{x}\,dx \approx 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

右黎曼和為

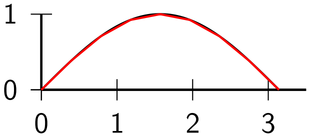
$$\int_1^5 \frac{1}{x}\,dx \approx \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{7}{60}.$$



$\int_0^{\pi} \sin(x)\,dx$，梯形法則，$n = 8$

Solution

$$\begin{aligned}\int_0^{\pi} \sin(x)\,dx &\approx \frac{\pi}{16}\left(\sin(0) + 2\sum_{k=1}^{7} \sin\left(\frac{k\pi}{8}\right) + \sin(\pi)\right)\\ &= \frac{\pi}{8}\left(2\sin\left(\frac{\pi}{8}\right) + 2\sin\left(\frac{3\pi}{8}\right) + \sqrt{2} + 1\right)\\ &\approx 1.974231601945551.\end{aligned}$$

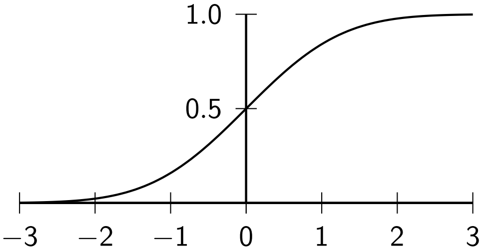


累積分布函數（CDF）

Definition

實值隨機變數 X 的**累積分布函數**為

$$F_X(x) = P(X \leq x).$$

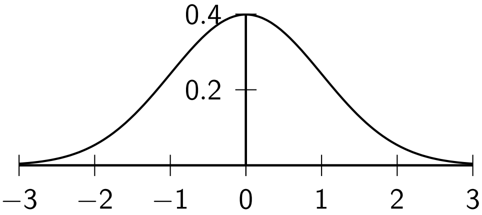


機率密度函數（PDF）

Definition

隨機變數 X 的**機率密度函數**是 f，其中 f 是非負可積函數，若

$$P(a \le X \le b) = \int_a^b f(x)\,dx.$$



CDF 與 PDF 的關係

Theorem

若 F 是 X 的累積分布函數，則

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

Theorem

若 $F'(c)$ 存在，則

$$F'(c) = f(c).$$

期望值

Definition

若隨機變數 X 的機率密度函數為 f，則 X 的期望值為

$$E(X) = \int X\,dP = \int_{-\infty}^{\infty} xf(x)\,dx.$$

Theorem

若隨機變數 X 的機率密度函數為 f，則函數 $g(X)$ 的期望值為

$$E(g(X)) = \int g(X)\,dP = \int_{-\infty}^{\infty} g(x)\,f(x)\,dx.$$

應用題

In reliability theory, the random variable is often the lifetime of some item, such as a laptop computer battery. The PDF can be used to find probabilities and expectations about the lifetime. Suppose then that the lifetime in hours of a battery is a continuous random variable x with PDF

$$f(x) = \begin{cases} \frac{12x^2(5-x)}{625} & \text{if } 0 \leq x \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

1. Find the probability that the battery lasts at least three hours.
2. Find the expected value of the lifetime.

詳解

Solution

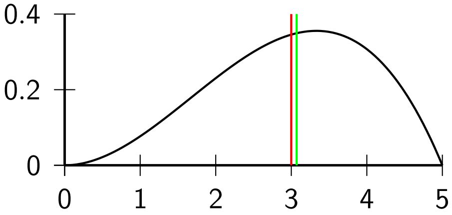
$$\int_3^5 \frac{12x^2(5-x)}{625}\,dx = \left[\frac{20x^3-3x^4}{625}\right]_3^5 = \frac{328}{625}.$$

Solution

$$\int_0^5 \frac{12x^3(5-x)}{625}\,dx = \left[\frac{3\left(25x^4-4x^5\right)}{3125}\right]_0^5 = 3.$$

機率密度函數的圖

- 期望值：3
- 中位數：3.071362159338052



Thanks for your attention!