微分

極限、微分、番外篇

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前情提要

- 上回有些關於極限的東西被略過了，這次要來把它補完
  - 上確界、下確界
  - 上極限、下極限
  - 夾擠定理
  - 一些常用的極限值
- 我們現在討論的範圍是實數，因為複數不能比大小

上確界與下確界

Definition

對於一個實數的集合 S，它的上確界 $\sup(S)$ 是大於等於 S 中所有成員的最小實數

- $\sup\{1, 2, 3\} = 3$
- $\sup\{0 < x < 1\} = \sup\{0 \le x \le 1\} = 1$
- $\sup\{x \in \mathbb{Q} : x^2 < 2\} = 1$
  - 這是個有理數的集合，但它的上確界是無理數！
  - 所以有理數是不完備的，也就是 $\mathbb{Q}$ 這個集合有洞
- $\sup \mathbb{N} = \infty$
- $\sup \varnothing = -\infty$

Definition

對於一個實數的集合 S，它的下確界 $\inf(S)$ 是小於等於 S 中所有成員的最大實數

$$\inf(S) = -\sup(-S)$$

上極限與下極限

Definition

函數的上極限與下極限定義為

$$\limsup_{x \to c} f(x) := \lim_{\delta \to 0} \left(\sup\{f(x) : 0 < |x - c| < \delta\} \right)$$
$$\liminf_{x \to c} f(x) := \lim_{\delta \to 0} \left(\inf\{f(x) : 0 < |x - c| < \delta\} \right)$$

當 δ 縮小時，右式的範圍也單調地縮小。所以以上兩式也可以寫作

$$\limsup_{x \to c} f(x) := \inf_{\delta > 0} \left(\sup\{f(x) : 0 < |x - c| < \delta\} \right)$$
$$\liminf_{x \to c} f(x) := \sup_{\delta > 0} \left(\inf\{f(x) : 0 < |x - c| < \delta\} \right)$$

極限的存在

Theorem

$\lim_{x \to c} f(x) = L$ 等價於以下敘述

- $\lim_{x \to c^+} f(x) = L$ 且 $\lim_{x \to c^-} f(x) = L$
- $\limsup_{x \to c} f(x) = L$ 且 $\liminf_{x \to c} f(x) = L$

- 除了用傳統的 ϵ-δ 證明，我們也可以利用極限的方向來驗證極限是否存在
- 反正就是 D&C

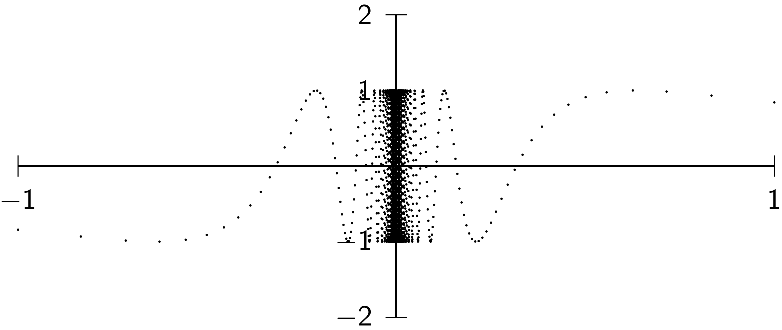
Divide and conquer!

上下極限必存在

- 左右極限可能不存在，但上下極限必存在

$$\limsup_{x\to 0} \sin\left(\frac{1}{x}\right) = 1, \quad \liminf_{x\to 0} \sin\left(\frac{1}{x}\right) = -1$$

- $\lim\limits_{x\to 0^-} \sin\left(\frac{1}{x}\right)$ 與 $\lim\limits_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ 均不存在



夾擠定理

Theorem

- 設一包含 c 點的區間 I，又 f, g, h 為定義在 $I\backslash\{c\}$ 上的函數
- 對於所有 $x \in I\backslash\{c\}$，均有 $g(x) \le f(x) \le h(x)$
- $\lim_{x\to c} g(x) = L$ 且 $\lim_{x\to c} h(x) = L$

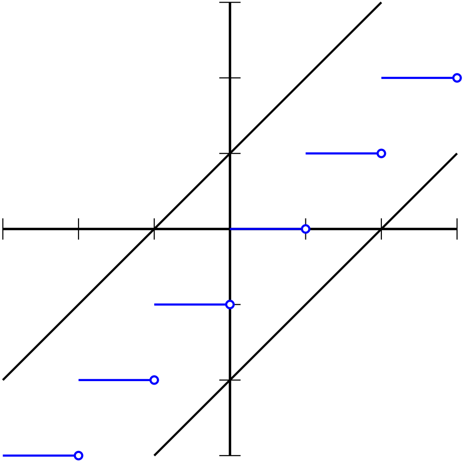
$$\lim_{x\to c} f(x) = L$$

Proof.

$$L = \lim_{x\to c} g(x) \le \liminf_{x\to c} f(x) \le \limsup_{x\to c} f(x) \le \lim_{x\to c} h(x) = L$$

□

假夾擠，真詐財

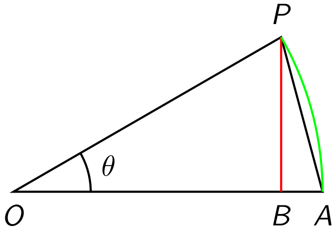


$$f(x) := \lfloor x \rfloor$$
$$g(x) := x + 1$$
$$h(x) := x - 2$$

$$g(x) \le f(x) \le h(x)$$

$\lim_{x \to 0} f(x)$ 不存在！

$$\lim_{\theta\to 0}\sin\theta = 0$$



Proof.

當 $0 < \theta < \dfrac{\pi}{2}$

$$0 < \triangle OAP < \text{扇形 } OAP$$

$$0 < \sin\theta < \theta$$

$$\lim_{\theta\to 0^+} 0 = \lim_{\theta\to 0^+} \theta = 0$$

$$\lim_{\theta\to 0^+} \sin\theta = 0$$

$$\lim_{\theta\to 0^-} \sin\theta = -\lim_{\theta\to 0^+} \sin\theta = 0$$

$\square$

三角函數是連續函數

Proof.

- 若 $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$，則 $\cos\theta = \sqrt{1 - \sin^2\theta}$

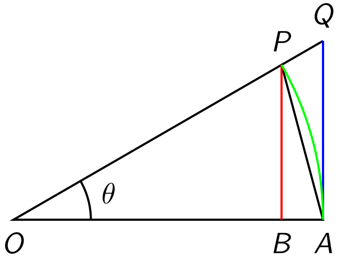
$$\lim_{\theta\to 0} \cos\theta = 1$$

- sin 是連續函數

$$\begin{aligned}
\lim_{x\to c} \sin x &= \lim_{h\to 0} \sin(c+h) \\
&= \lim_{h\to 0} (\sin c \cos h + \cos c \sin h) \\
&= 1 \sin c + 0 \cos c \\
&= \sin c
\end{aligned}$$

- 其他三角函數均可以自變數與 sin 的有理式表達

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



Proof.

當 $0 < \theta < \frac{\pi}{2}$

$$\triangle OAQ > \text{扇形 } OAP > \triangle OAP$$

$$\tan \theta > \theta > \sin \theta$$

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

$$\lim_{\theta \to 0^+} \cos \theta = \lim_{\theta \to 0^+} 1 = 1$$

$$\lim_{\theta \to 0^-} \frac{\sin \theta}{\theta} = \lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1$$

$\square$

$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$，天啓解法

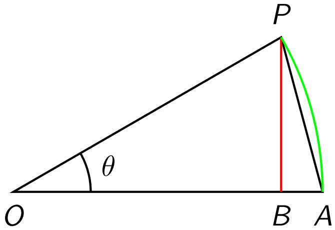
Proof.

$$
\begin{aligned}
\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0} \left(\frac{1-\cos\theta}{\theta}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right) \\
&= \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} \\
&= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \lim_{\theta\to 0} \frac{\sin\theta}{1+\cos\theta} \\
&= 0
\end{aligned}
$$

$\square$

$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$，幾何解法

Proof.



$$
\begin{aligned}
&\lim_{\theta \to 0^+} \frac{1 - \cos\theta}{\theta} \\
&= \lim_{\theta \to 0^+} \frac{\overline{AB}}{\text{弧 } AP} \\
&= \lim_{\theta \to 0^+} \frac{\overline{BP} \tan \angle APB}{\text{弧 } AP} \\
&= \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \lim_{\theta \to 0^+} \tan\frac{\theta}{2} \\
&= 0
\end{aligned}
$$

$$
\lim_{\theta \to 0^-} \frac{1 - \cos\theta}{\theta} = -0 = 0
$$

$$\lim_{x\to 0} x\left\lfloor \frac{1}{x} \right\rfloor = 1$$

Proof.

$$\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}$$

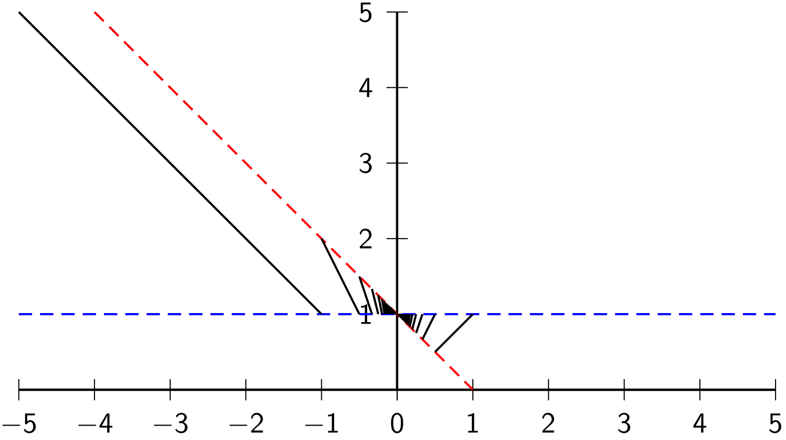
- 當 $x > 0$，$1 - x < x\left\lfloor \frac{1}{x} \right\rfloor \leq 1$
- 當 $x < 0$，$1 - x > x\left\lfloor \frac{1}{x} \right\rfloor \geq 1$

$$\lim_{x\to 0}(1 - x) = \lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0} x\left\lfloor \frac{1}{x} \right\rfloor = 1$$

$\square$

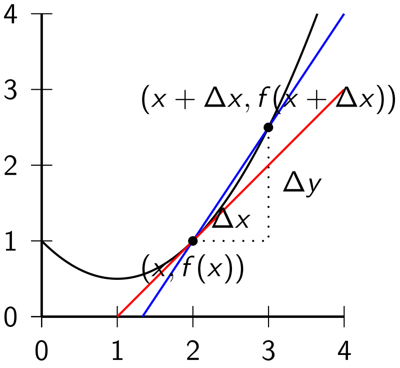
$y = x\left\lfloor \frac{1}{x} \right\rfloor$ 的圖形



微分的用途

- 最佳化
  - 手繪函數圖形
- 微分方程
  - 微分就是變化，所以常用來描述物理/化學變化
- 級數逼近
  - 用多項式或有理函數來逼近無理函數
  - 數值上，我們只會算四則運算！
- 處理隱函數
  - 圓錐曲線

割線與切線



Lemma

- 切線是割線的極限
- 割線的斜率為

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- 切線的斜率為

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

導數（derivative）

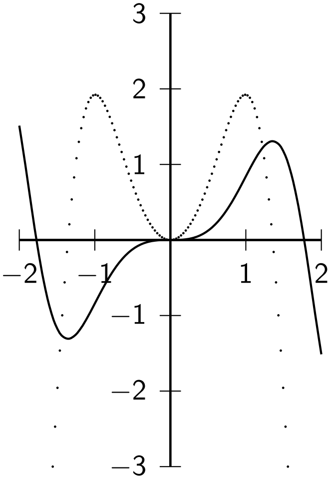
Theorem

對於函數 $y = f(x)$，它的在 $x = c$ 處的導數定義為

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

- 導數也是函數在該點的切線斜率
- 導數是微分（differential）的商，故又稱微商
- 求導的過程叫微分（differentiation）

導函數



對於定義域中找得到 $f'(c)$ 的 c，可以把這些導數們又當成一個函數來看，就是導函數

Definition

對於函數 $y = f(x)$，它的導函數 $y' = f'(x)$如下

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

導函數的記法

- 拉格朗日（Lagarange）記法

$$y' = f'(x)$$

- 萊布尼茲（Leibniz）記法

$$\frac{dy}{dx} = \frac{d}{dx}f(x)$$

- 牛頓記法

$$\mathbf{F} = \dot{\mathbf{P}} = m\dot{\mathbf{v}} = m\ddot{\mathbf{x}} = m\mathbf{a}$$

- 算子記法，由黑維塞（Heaviside）發明

$$Dy = D_x y = Df(x)$$

從定義計算導函數

Example

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= (-\sin x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)+(\cos x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\
&= \cos x
\end{aligned}
$$

可微必連續

Theorem

若函數 f 在 c 上可微，則它必在此連續

Proof.

$$
\begin{aligned}
f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\
\lim_{x \to c} f(x) &= \lim_{x \to c} \left(\left(\frac{f(x) - f(c)}{x - c} \right) (x - c) + f(c) \right) \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \lim_{x \to c} (x - c) + \lim_{x \to c} f(c) \\
&= f(c)
\end{aligned}
$$

$\square$

高階導函數

Definition

把導函數拿來微分，結果就是二階導函數

$$f''(x) = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x)$$

$n+1$ 階導函數，是 n 階導函數的導函數

$$f^{(n+1)}(x) = \frac{d}{dx}f^{(n)}(x)$$

高階導函數的記法

一階導數	$y' = f'(x)$	$Dy = Df(x)$	$\dfrac{dy}{dx} = \dfrac{d}{dx} f(x)$
二階導數	$y'' = f''(x)$	$D^2 y = D^2 f(x)$	$\dfrac{d^2 y}{dx^2} = \dfrac{d^2}{dx^2} f(x)$
三階導數	$y''' = f'''(x)$	$D^3 y = D^3 f(x)$	$\dfrac{d^3 y}{dx^3} = \dfrac{d^3}{dx^3} f(x)$
四階導數	$y^{(4)} = f^{(4)}(x)$	$D^4 y = D^4 f(x)$	$\dfrac{d^4 y}{dx^4} = \dfrac{d^4}{dx^4} f(x)$
n 階導數	$y^{(n)} = f^{(n)}(x)$	$D^n y = D^n f(x)$	$\dfrac{d^n y}{dx^n} = \dfrac{d^n}{dx^n} f(x)$

微分法則

Theorem

- 線性法則

$$
\begin{aligned}
(u + v)' &= u' + v' \\
(cy)' &= cy'
\end{aligned}
$$

- 乘法法則

$$
(uv)' = u'v + uv'
$$

- 連鎖法則

$$
\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}
$$

線性法則

Proof.

$$
\begin{aligned}
(f(x)+g(x))' &= \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x) \\
(cf(x))' &= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\
&= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= cf'(x)
\end{aligned}
$$

$\square$

乘法法則

Proof.

$$
\begin{aligned}
&(f(x)\,g(x))' \\
=&\lim_{h\to 0}\frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h} \\
=&\lim_{h\to 0}\tfrac{f(x+h)\,g(x+h)-f(x)\,g(x+h)}{h}+\lim_{h\to 0}\tfrac{f(x)\,g(x+h)-f(x)\,g(x)}{h} \\
=&\lim_{h\to 0}\tfrac{f(x+h)-f(x)}{h}\lim_{h\to 0}g(x+h)+\lim_{h\to 0}f(x)\lim_{h\to 0}\tfrac{g(x+h)-g(x)}{h} \\
=&f'(x)\,g(x)+f(x)\,g'(x)
\end{aligned}
$$

$\square$

連鎖法則 I

$$
\begin{aligned}
(f \circ g)'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \left(\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right) \left(\frac{g(x) - g(c)}{x - c} \right)
\end{aligned}
$$

設一分段定義函數 $Q(y) := \begin{cases} \frac{f(y) - f(g(c))}{y - g(c)}, & y \neq g(c) \\ f'(g(c)), & y = g(c) \end{cases}$

$$
\lim_{y \to g(c)} Q(y) = \lim_{y \to g(c)} \frac{f(y) - f(g(c))}{y - g(c)} = f'(g(c))
$$

連鎖法則 II

$$
\begin{aligned}
(f \circ g)'(c) &= \lim_{x \to c} Q(g(x)) \left(\frac{g(x) - g(c)}{x - c} \right) \\
&= \lim_{x \to c} Q(g(x)) \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= f'(g(x))\, g'(x)
\end{aligned}
$$

連續使用乘法法則和連鎖法則

Theorem

$$g(x) := \prod_{i=1}^{n} f_i(x) = f_1(x)\, f_2(x) \cdots f_n(x)$$

$$\begin{aligned} g'(x) &= \sum_i \left(f_i'(x) \prod_{j \neq i} f_j(x) \right) \\ &= f_1'(x)\, f_2(x) \cdots f_n(x) + f_1(x)\, f_2'(x) \cdots f_n(x) + \cdots \\ &\quad + f_1(x)\, f_2(x) \cdots f_n'(x) \end{aligned}$$

$$y := f_1(f_2(\cdots f_n(x) \cdots))$$

$$\frac{dy}{dx} = \prod_{i=1}^{n-1} f_i'(f_{i+1}(f_{i+2}(\cdots f_n(x) \cdots))) = \frac{df_1}{df_2} \frac{df_2}{df_3} \cdots \frac{df_{n-1}}{df_n}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Proof.

$$\begin{aligned}
\frac{d}{dx}\ln x &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h} \\
&= \lim_{h\to 0}\ln\left(\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right) = \lim_{h\to 0}\frac{\ln\left(\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right)}{x} \\
&= \frac{\ln\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}}}{x} = \frac{\ln e}{x} \\
&= \frac{1}{x}
\end{aligned}$$

$\square$

$$\frac{d}{dx}\ln|x| = \frac{1}{x},\ x \in \mathbb{R}\backslash\{0\}$$

Proof.

- 當 $x > 0$，$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$
- 當 $x < 0$

$$\begin{aligned}
\frac{d}{dx}\ln|x| &= \frac{d}{dx}\ln(-x) \\
&= \left(\frac{d}{du}\ln u\right)\frac{du}{dx} && u = -x \\
&= \left(\frac{1}{u}\right)(-1) \\
&= \frac{1}{x}
\end{aligned}$$

$\square$

對數微分法（logarithmic differentiation）

當實函數 $y := f(x)$ 難於微分時，先取其絶對值的對數 $\ln|f(x)|$

$$\frac{d}{dx}\ln|f(x)| = \left(\frac{d}{dy}\ln|y|\right)\frac{dy}{dx} = \left(\frac{1}{y}\right)f'(x) = \frac{f'(x)}{f(x)}$$

$$f'(x) = f(x)\,\frac{d}{dx}\ln|f(x)|$$

廣義冪法則

$$
\begin{aligned}
\ln\left|f(x)^{g(x)}\right| &= g(x)\ln|f(x)| \\
(g(x)\ln|f(x)|)' &= g'(x)\ln|f(x)| + \frac{f'(x)\,g(x)}{f(x)} \\
\left(f(x)^{g(x)}\right)' &= f(x)^{g(x)}\left(\frac{f'(x)\,g(x)}{f(x)} + g'(x)\ln|f(x)|\right)
\end{aligned}
$$

除法法則

$$
\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{d}{dx} f(x)\, g(x)^{-1} \\
&= f'(x)\, g(x)^{-1} - f(x)\, g(x)^{-2} g'(x) \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)\, g'(x)}{g(x)^2} \\
&= \frac{f'(x)\, g(x) - f(x)\, g'(x)}{g(x)^2}
\end{aligned}
$$

有理函數的導函數

Example

Given $f(x) = \dfrac{x^2(1-x)^3}{1+x}$, find $f'(2)$.

Solution

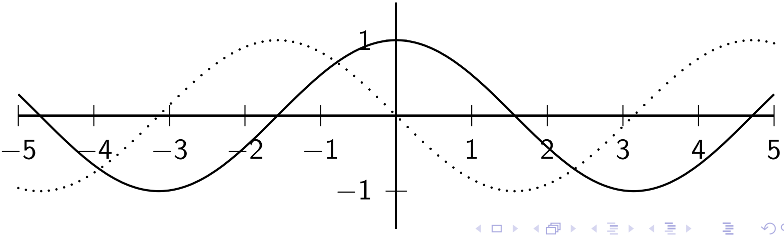
$$\begin{aligned}
f'(x) &= \frac{\left(x^2(1-x)^3\right)'}{1+x} - \frac{x^2(1-x)^3}{(1+x)^2} \\
&= \frac{2x(1-x)^3 - 3x^2(1-x)^2}{1+x} - \frac{x^2(1-x)^3}{(1+x)^2} \\
f'(2) &= \frac{-4-12}{3} + \frac{4}{9} = \frac{-44}{9}
\end{aligned}$$

$$\frac{d}{dx}\cos x = -\sin x$$

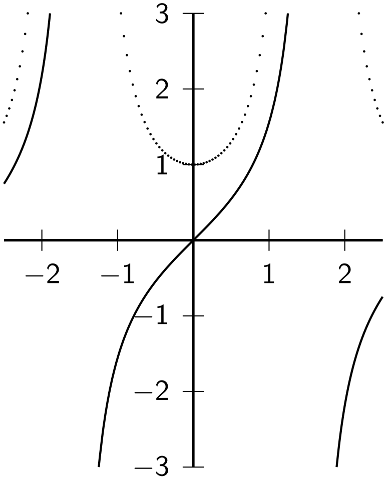
Proof.

$$\begin{aligned}\frac{d}{dx}\cos x &= \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right)\\ &= -\cos\left(\frac{\pi}{2} - x\right)\\ &= -\sin x\end{aligned}$$

$\square$



$$\frac{d}{dx}\tan x = \sec^2 x$$

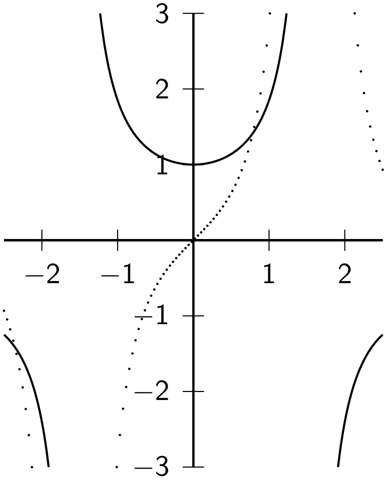


Proof.

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} \\ &= \frac{\cos x}{\cos x} - \frac{-\sin^2 x}{\cos^2 x} \\ &= 1 + \tan^2 x \\ &= \sec^2 x\end{aligned}$$

$\square$

$$\frac{d}{dx}\sec x = \sec x \tan x$$



Proof.

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$$
$$= -\frac{-\sin x}{\cos^2 x}$$
$$= \sec x \tan x$$

□

初等函數的導函數 I

Example

Given $f(x) = \ln\left(\sec^4 x + \tan^2 x\right)$, find $f'\left(\frac{\pi}{4}\right)$.

Solution

$$
\begin{aligned}
\frac{d}{dx}\left(\sec^4 x + \tan^2 x\right) &= 4\left(\sec^3 x\right)\left(\sec x \tan x\right) + 2\sec^2 x \tan x \\
&= 4\sec^4 x \tan x + 2\sec^2 x \tan x \\
f'(x) &= \frac{4\sec^4 x \tan x + 2\sec^2 x \tan x}{\sec^4 x + \tan^2 x} \\
f'\left(\frac{\pi}{4}\right) &= \frac{16+4}{4+1} = 4
\end{aligned}
$$

初等函數的導函數 II

Example

給定 $f(x) = \left(x^2+1\right)^{\sin x}$，求 $f'\left(\frac{\pi}{2}\right)$

Solution

$$
\begin{aligned}
\ln|f(x)| &= \sin x \ln\left(x^2+1\right) \\
\frac{f'(x)}{f(x)} &= \cos x \ln\left(x^2+1\right) + \frac{2x\sin x}{x^2+1} \\
f'(x) &= \left(x^2+1\right)^{\sin x}\left(\cos x \ln\left(x^2+1\right) + \frac{2x\sin x}{x^2+1}\right) \\
f'\left(\frac{\pi}{2}\right) &= \left(\frac{\pi^2}{4}+1\right)\left(\frac{\pi}{\frac{\pi^2}{4}+1}\right) = \pi
\end{aligned}
$$

隱函數

- 有時候兩個變數的關係並非函數關係，而是
  - 隱函數 $\phi(x, y) = 0$，如$x^2 + y^2 = 1$
  - 參數式 $x = t - \sin t;\ y = 1 - \cos t$
- 當我們要做 $d\phi/dx$ 時，因為 y 會隨 x 變動，所以**不能視為常數**，要用**連鎖法則**把他做掉
- 詳細原理、證明等期中考後我們有空再來解決，它跟多變數的微分有關

回顧錐線切線公式 I

求錐線 $\phi(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$ 上一點 (x_0, y_0) 上的切線

$$\frac{d\phi}{dx} = 2ax + b\left(y + xy'\right) + 2cyy' + d + ey' = 0$$

$$\frac{d\phi}{dx}(x_0, y_0) = \left(bx_0 + 2cy_0 + e\right)y' + 2ax_0 + by_0 + d = 0$$

$$y' = -\frac{2ax_0 + by_0 + d}{bx_0 + 2cy_0 + e}$$

代入點斜式得切線為

$$y - y_0 = \left(-\frac{2ax_0 + by_0 + d}{bx_0 + 2cy_0 + e}\right)(x - x_0)$$

咦，怎麼沒有 f？這不是高中背的樣子啊！

回顧錐線切線公式 II

$$\left(2ax_0 + by_0 + d\right)\left(x - x_0\right) + \left(bx_0 + 2cy_0 + e\right)\left(y - y_0\right) = 0$$

$$\begin{aligned} & 2ax_0x + b\left(y_0x + x_0y\right) + 2cy_0y + d\left(x + x_0\right) + e\left(y + y_0\right) \\ =& 2\left(ax_0^2 + bx_0y_0 + cy_0^2 + dx_0 + ey_0\right) \\ =& -2f \end{aligned}$$

$$ax_0x + b\left(\frac{y_0x + x_0y}{2}\right) + cy_0y + d\left(\frac{x + x_0}{2}\right) + e\left(\frac{y + y_0}{2}\right) + f = 0$$

求曲線上的切線

Example

Find the equation of the tangent line of the curve $y^3 - xy^2 + xy = -14$ at $(1, -2)$

Solution

$$3y^2y' - \left(2xyy' + y^2\right) + \left(xy' + y\right) = 0$$

$$\left(3y^2 - 2xy + x\right) y' = y^2 - y$$

$$y'(1, -2) = \frac{(-2)^2 - (-2)}{3(-2)^2 - 2(1)(-2) + 1} = \frac{6}{17}$$

故切線方程式為

$$y + 2 = \frac{6(x - 1)}{17}$$

求曲線上的法線

Example

Find the equation of the normal line of the curve $y^3 - xy^2 + xy = -14$ at $(1, -2)$

Solution

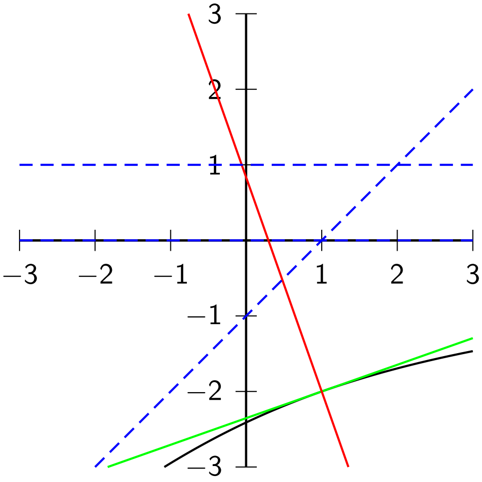
$$y'(1,-2) = \frac{6}{17}$$

$$\frac{-1}{y'(1,-2)} = \frac{-17}{6}$$

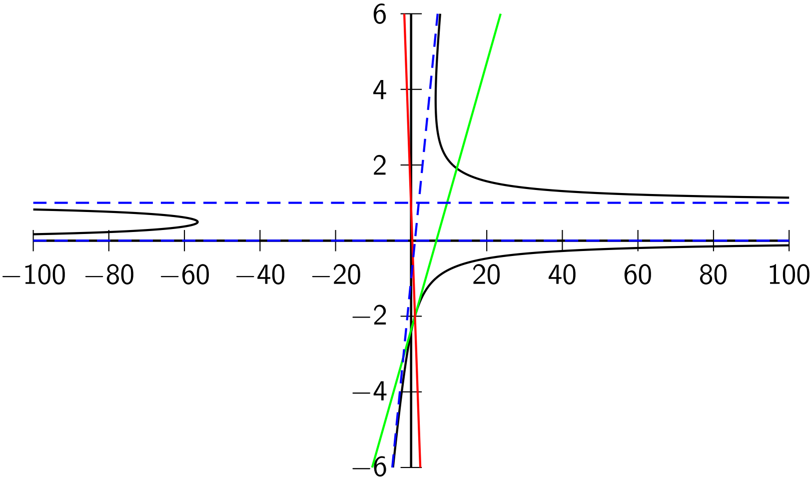
故法線方程式為

$$y + 2 = \frac{-17(x-1)}{6}$$

$y^3 - xy^2 + xy = -14$ 的特寫



$y^3 - xy^2 + xy = -14$ 的全貌



反函數的導數

Theorem

$$\left(f^{-1}\right)'(c) = \frac{1}{f'(f^{-1}(c))}$$

Proof.

$$\begin{aligned} f\left(f^{-1}(x)\right) &= x \\ f'\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x) &= 1 \\ \left(f^{-1}\right)'(c) &= \frac{1}{f'(f^{-1}(c))} \end{aligned}$$

□

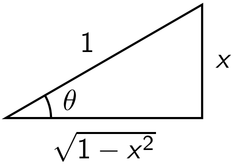
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

Proof.

設 $\theta := \arcsin x$，則 $x = \sin\theta$ 且 $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\begin{aligned}\frac{dx}{d\theta} &= \cos\theta\\ \frac{d\theta}{dx} &= \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

$\square$



$$\frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

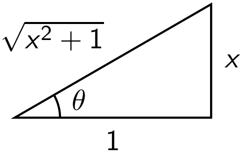
Proof.

設 $\theta := \arctan x$，則 $x = \tan\theta$ 且 $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

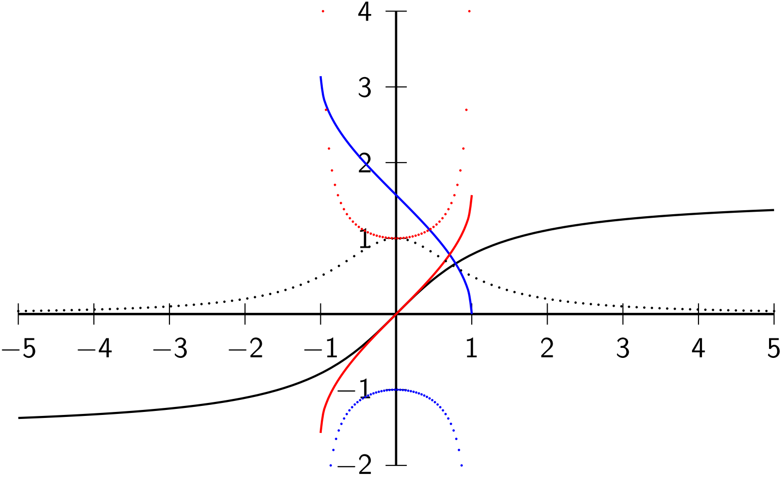
$$\frac{dx}{d\theta} = \sec^2\theta$$

$$\frac{d\theta}{dx} = \cos^2\theta = \frac{1}{x^2+1}$$

□



反三角函數的圖形



微分（differential）

Definition

對於函數 $y = f'(x)$ 而言，它的微分 dy 為

$$dy := f'(x)dx$$

Theorem

- 它具有線性性

$$d(u+v) = du + dv$$
$$d(cy) = c\,dy$$

- 乘法定則

$$d(uv) = u\,dv + v\,du$$

例題

Example

In the late 1830s, French physiologist Jean Poiseuille discovered the formula we use today to predict how much the radius of a particular clogged artery decreases the normal volume of flow. His formula,

$$V = kr^4$$

say that volume of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How dose a 10% decrease in r affect V?

解題

Solution

- 標準答案：

$$dV = 4kr^3 dr$$

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = \frac{4dr}{r}$$

  - V 的相對變化率為 r 的四倍，故減少 *40%*
- 嘴炮答案：

$$(1 - 0.1)^4 = 0.6561$$

  - 故變為原本的 *65.61%*

線性近似

線性近似就是用函數的切線來對該函數進行近似。當 $x \approx c$

$$f(x) \approx f(c) + f'(c)(x - c)$$

Example

Use the differentials to approximate the quantity $\sqrt{4.6}$ to four decimal places.

Solution

$$\sqrt{x} \approx \sqrt{c} + \frac{x - c}{2\sqrt{c}}$$

$$\sqrt{4.6} \approx 2 + \frac{4.6 - 4}{4} = 2.15$$

$$\sqrt{4.6} \approx 2.15 + \frac{4.6 - (2.15)^2}{4.3} = \frac{3689}{1720} \approx 2.145$$

無關微積分的題目

- 期中考的範圍其實蠻簡單的
- 為了湊滿 10 題，考卷中會有一些醬油題
- 去年醬油題的題型是
  - 比較係數法
  - 線性回歸

比較係數法

Example

Water boils at 212°F at sea level and 200°F at an elevation of 6000 ft. Assume that the boiling point B varies linearly with altitude α. Find the function $B = f(\alpha)$ that describes the dependence. Comment on whether a linear function gives a realistic model.

Solution

設 $f(\alpha) := m\alpha + k$

$$\begin{aligned} f(0) &= k = 212 \\ f(6000) &= 6000m + 212 = 200 \\ m &= \frac{200 - 212}{6000} = -0.002 \\ B &= f(\alpha) = -0.002\alpha + 212 \end{aligned}$$

線性回歸 I

- 假設我們想要觀察 p 個自變數 $x_1, \ldots, x_p$ 如何影響 y
- 我們收集了 n 組數據 $\{y_i, x_{i1}, \ldots, x_{ip}\}_{i=1}^n$
- 我們試圖用線性關係來表達它，其中仍有誤差 ϵ_i

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \epsilon_i = \mathbf{x}_i^\mathsf{T} \boldsymbol{\beta} + \epsilon_i$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ x_{i1} \\ \vdots \\ x_{ip} \end{pmatrix}$$

線性回歸 II

- 包裝成矩陣的樣子就是

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} \mathbf{x}_1^{\mathsf{T}} \\ \mathbf{x}_2^{\mathsf{T}} \\ \vdots \\ \mathbf{x}_n^{\mathsf{T}} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

什麼是內積

Definition

在幾何向量空間 $\mathbb{R}^n$ 中，向量 $\mathbf{x}$ 和 $\mathbf{y}$ 的內積為

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^{\mathsf{T}} \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

- 對稱性（交換律）

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$$

- 雙線性

$$\langle \mathbf{x}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{x}, \mathbf{u} \rangle + \langle \mathbf{x}, \mathbf{v} \rangle$$
$$\langle \mathbf{x}, c\mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle$$

- 向量與自身的內積不為負，且等號僅發生於零向量

$$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$$

推廣到複向量空間

- 實向量空間所定義的內積，在 $\mathbb{C}^n$ 仍然有效嗎？

$$\|\mathbf{ix}\|^2 = \langle \mathbf{ix}, \mathbf{ix} \rangle = -\|\mathbf{x}\|^2 \ ?$$

- 這違反了我們對距離不為負值的期待！

Definition

在複向量空間 $\mathbb{C}^n$ 中，向量 $\mathbf{x}$ 和 $\mathbf{y}$ 的內積為

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \mathbf{y} = \sum_{i=1}^{n} \bar{x}_i y_i$$

- 其中 $\overline{x}$ 代表 x 的共軛複數，而 $\mathbf{x}^*$ 為 $\mathbf{x}$ 的共軛轉置

內積空間

Definition

符合以下四個條件，即廣義內積條件，的向量空間，稱為內積空間

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
- $\langle \mathbf{x}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{x}, \mathbf{u} \rangle + \langle \mathbf{x}, \mathbf{v} \rangle$
- $\langle \mathbf{x}, c\mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ 且 $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$

常見的例子有

- 實向量空間和複向量空間的標準內積
- 矩陣標準內積

$$\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{tr}(\mathbf{A}^* \mathbf{B})$$

正射影是最佳的近似

Theorem

在內積空間 V 中有一向量 $\mathbf{v}$，而空間 W 為該內積空間的子空間，則 W 中對 $\mathbf{v}$ 的最佳近似值為 $\operatorname{proj}_W \mathbf{v}$

Proof.

- 對於所有 W 中的向量 $\mathbf{w}$，我們有

$$\mathbf{v} - \mathbf{w} = (\mathbf{v} - \operatorname{proj}_W \mathbf{v}) + (\operatorname{proj}_W \mathbf{v} - \mathbf{w})$$

- 因為 $\operatorname{proj}_W \mathbf{v}$ 和 $\mathbf{w}$ 都在 W 中，所以 $\operatorname{proj}_W \mathbf{v} - \mathbf{w}$ 也是
- $\mathbf{v} - \operatorname{proj}_W \mathbf{v} = \operatorname{proj}_{W^\perp} \mathbf{v}$，故與 W 中任一向量垂直

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v} - \operatorname{proj}_W \mathbf{v}\|^2 + \|\operatorname{proj}_W \mathbf{v} - \mathbf{w}\|^2$$

$$\|\mathbf{v} - \mathbf{w}\| \geq \|\mathbf{v} - \operatorname{proj}_W \mathbf{v}\|$$

$\square$

$\mathbf{A}^*$ 的零空間與 $\mathbf{A}$ 的行空間垂直

Theorem

對矩陣 $\mathbf{A}$ 而言，若有一向量 $\mathbf{x}$ 滿足 $\mathbf{A}^*\mathbf{x} = \mathbf{0}$，則 $\mathbf{x}$ 與 $\mathbf{A}$ 的所有行向量垂直

Proof.

$$\mathbf{A} := \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{pmatrix}$$

$$\mathbf{A}^*\mathbf{x} = \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{x} \rangle \\ \langle \mathbf{a}_2, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{a}_n, \mathbf{x} \rangle \end{pmatrix} = \mathbf{0}$$

□

- 行是 column，列是 row

最小平方法

- 最小平方法的目標，在於讓 $\|\epsilon\|$ 最小
- $\epsilon = \mathbf{y} - \mathbf{X}\beta$

Solution

$$\begin{aligned}
\mathbf{X}\hat{\beta} &= \operatorname{proj} \mathbf{y} \\
\mathbf{y} - \mathbf{X}\hat{\beta} &= \operatorname{proj}_{\perp} \mathbf{y} \\
\mathbf{X}^*(\mathbf{y} - \mathbf{X}\hat{\beta}) &= \mathbf{0} \\
\mathbf{X}^*\mathbf{X}\hat{\beta} &= \mathbf{X}^*\mathbf{y}
\end{aligned}$$

單變數線性回歸

當 $p = 1$ 且 $\mathbf{X}$ 是實矩陣

Solution

$$\begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\beta_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$\beta_0 = \frac{\sum y_i - \beta_1 \sum x}{n}$$

例題

Example

In a study of five industrial areas, a researcher obtained these data relating the average number of units of a certain pollutant in the air and the number of incidences (per 100 000 people) of a certain disease:

Units of pollutant	3	4	5	8	10
Incidences of the disease	48	52	58	70	96

Find the equation of the least-square line $y = Ax + B$ (to two decimal places.)

Given that $A = \dfrac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$ and $B = \dfrac{\sum y - A\sum x}{n}$.

解題

Solution

	x	y	x^2	xy
	3	48	9	144
	4	52	16	208
	5	58	25	290
	8	70	64	560
	10	96	100	980
$\sum$	30	324	214	2162

$$A = \frac{109}{17}, \quad B = \frac{2238}{85}$$

$$y = 6.41x + 26.33$$

謝謝聆聽！

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