Assignment-1

1. $T(1) = 10$; $T(n+1) = 2n + T(n)$ for $n \geq 1$

$$T(n+1) \begin{cases} 2n + T(n) & n \geq 1 \\ 10 & n = 0 \end{cases}$$

$$T(n+1) = 2n + T(n)$$

$$\downarrow$$

$$2(n-1) + T(n-1)$$

$$\downarrow$$

$$2(n-2) + T(n-2)$$

$$\downarrow$$

$$\ldots\ldots \quad 2(\overset{n-n+1}{1}) + \overset{10}{T(1)}$$

$$= 2\left(n + (n-1) + (n-2) + \cdots (n-n+1)\right) + 10$$

$$= \cancel{2(\ldots)} \quad 2\,\frac{n(n-1)}{2} + 10$$

$$T(n+1) \approx n^2 - n$$

$$T(n) = O(n^2)$$

2. $$T(n) = \begin{cases} T(n/4) + T(n/2) + cn^2 & n > 1 \\ c & n = 1 \\ 0 & n = 0 \end{cases}$$

⇒ ~~T(n)~~ Here we can see that $T(n/2) > T(n/4)$

∴ We can replace ($T(n/4)$ by $T(n/2)$)

$\Rightarrow T(n) = 2(T(n/2)) + cn^2$

$a = 2$; $b = 2$; $f(n) = cn^2 = n^{\overset{k=2}{}} \log^{\overset{p=0}{}} n \Rightarrow k = 2, p = 0$

Case $\log_b a < k$ & $p = 0$

$\Rightarrow T(n) = \Theta(n^{k=2} \log^{p=0} n)$

$= \Theta(n^2)$

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3. $T(n) = 5T(n/5) + \sqrt{n} \quad n > 1$

$= 1 \quad n = 1$

$= 0 \quad n = 0$

→ Master theorem for dividing function,

$a = 5 \;;\; b = 5 \;;\; f(n) = \sqrt{n} = n^{k=1/2} \log^{p=0} n$

$\log_b a = \underline{1} \qquad \underline{k = \frac{1}{2}} \qquad \underline{p = 0}$

→ Case, $\log_b a > k$

$\Rightarrow \underline{\Theta(n^1)}$

5. $T(n) = 2T(n/2) + n \;;\; T(0) = T(1) = 1$

⇒ Master Theorem for dividing functions,

$a = 2 \;;\; b = 2 \;;\; f(n) = n = n^{k=1} \log^{p=0} n$

$\underline{\log_b a = 1} \;;\; \underline{k = 1} \;;\; \underline{p = 0}$

Case $\log_b a = k \quad p > -1$

$\Rightarrow T(n) = \Theta(n^1 \log^1 n)$

$= \Theta(n \log n)$

$\cancel{T(n) = \Theta(n\log n)}$ $\because n^2 > n\log n$

$T(n) = O(n^2)$

$T(n) = n^2$ is false

6. A(n) {
if $n \le 2$ $\rightarrow 1$
return 1
else
return $A(\sqrt{n})$ $\rightarrow T(\sqrt{n})$
}

$$T(n) \begin{cases} T(\sqrt{n}) + 1 & n > 2 \\ 1 & n \le 2 \end{cases}$$

$\Rightarrow$ Let $n = 2^m$, $m = \log_2 n$

$$T(2^m) = T(\sqrt{2^m}) + 1$$
$$= T(2^{m/2}) + 1$$

Let, $S(m) = T(2^m)$
$S(m/2) = T(2^{m/2})$

$$\Rightarrow S(m) = S(m/2) + 1$$
$$\approx T(n) = a\,T(n/b) + 1$$

master theorem for dividing function,
$a = 1$; $b = 2$; $f(n) = 1 = m^{k=0} \log^{p=0} m$
$\log_b a = \underline{0}$; $\underline{k = 0}$; $\underline{P = 0}$

Case, $\log_b a = k$; $p > -1$
$\Rightarrow S(m) = (m^0 \log^1 m) = \underline{(\log m)}$

$\Rightarrow \underline{(\log \log n)}$

7. $T(n) = \begin{cases} T(\sqrt{n}) & n > 2 \\ 1 & n \leq 2 \end{cases}$

~~$\Rightarrow n = 2^m$~~

~~$\Rightarrow T(2^m) = T(2^{m/2}) + 2^m$~~

~~$S(m) = S(2^m)$~~ ~~Let, $S(m) = T(2^m)$~~

~~$S(m/2) = S(2^{m/2})$~~ ~~$S(m/2) = T(2^{m/2})$~~

~~$\Rightarrow S(m) = S(m/2) + 2^m$~~

~~$\approx T(n) = aT(n/b) + f(n)$~~

$a = 1$; $b = 2$;

Let $n = 2^m$

$\Rightarrow T(2^m) = T(2^{m/2}) + c_1 \quad -①$

Let,

$S(m) = T(2^m)$

$\therefore S(m/2) = T(2^{m/2})$

equation becomes,

$S(m) = S(m/2) + 1$

By Master Theorem, $F(n) = \Theta(n^k \log^p n)$

$\therefore a = 1$; $b = 2$; $f(n) = 1 \Rightarrow k = 0, p = 0$

<u>Case</u> $\log_b a = k$ & $p > -1$

$\therefore S(m) = \Theta(m^0 \log^{p+1} m)$

$= \Theta(\log m)$

$\therefore T(2^m) = \Theta(\log m)$

$m = \log_2 n$

$\therefore \underline{T(n) = \Theta(\log \log n)}$

8) $T(n) = 1 + T(n-1) + T(n-1)$ $\quad n \neq 1$

$\quad\quad 1$ $\quad n = 1$

$T(n) = 2T(n-1) + 1$

By, Recurrence Equation.

$T(n) = a\,T(n-b) + f(n)$

$a = 2$; $b = 1$; $f(n) = 1 = n^{k=0}$

$\Rightarrow T(n) = \Theta(1 \cdot 2^{n/1})$

$= \Theta(2^n)$

9) $T(n) = 3T(n/5) + (\log n)^2$

$a = 3$; $b = 5$, $f(n) = (\log n)^2 \Rightarrow k = 0$; $p = 2$

Case $\log_5 3 > k = 0$

$\Rightarrow \Theta(n^{\log_5 3}) = \Theta(n^{0.68})$

10) $T(n) = T(\sqrt{n}) + \Theta(\log \log n)$

$\Rightarrow n = 2^m \Rightarrow T(2^m) = T(2^{m/2}) + \Theta(\log m)$

Let $T(2^m) = S(m)$ & $T(2^{m/2}) = S(m/2)$

$\Rightarrow S(m) = S(m/2) + \Theta(\log m)$

$a = 1$, $b = 2$, $f(n) = \log m = n^{k=0} (\log m)^{p=0}$

Case $\log_b a = k$ & $p > -1$.

$S(m) = \Theta(\log^2 m)$, But $m = \log n$

$\Rightarrow T(n) = \Theta(\log \log n)^2$

11. $T(n) = 2T(n/2) + \log n$

$\underline{a = 2 ; b = 2}$, $f(n) = \log n \Rightarrow \underline{k = 0 \,\&\, p = 1}$

$\underline{\text{Case}}$ $\underline{\log_b a > k}$

$$\underline{T(n) = \Theta(n)}$$

12. $T(1) = 1$

$T(n+1) = T(n) + \lfloor \sqrt{n+1} \rfloor \qquad n \geq 1$

~~Let,~~

~~$\Rightarrow n+1 = m^2$~~

~~$T(m^2) = T(m^2-1) + m$~~

~~Let $m^2 = a$,~~

~~$T(a) = T(a-1)$~~

Let,

$n+1 = m$,

$T(m) = T(m-1) + \lfloor \sqrt{m} \rfloor$

Master Theorem for decreasing function

$a = 1 ; b = 1 ; k = \frac{1}{2}$

$\Rightarrow \underline{\text{Case}}$ $\underline{a = 1}$,

$\Rightarrow \Theta(m\sqrt{m})$

$\Rightarrow \Theta(m^{3/2})$

$\Rightarrow \Theta((n+1)^{3/2})$

$\approx \underline{\Theta(n^{3/2})}$

$\Rightarrow T(m^2) = \Theta((m^2)^{3/2})$

$= \underline{\Theta(m^3)}$

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13) $T(2^k) = 3 \cdot T(2^{k-1}) + 1$ $\quad n \neq 1$

1 $\quad n = 1$

$\Rightarrow 2^k = n$

$\Rightarrow T(n) = 3 T(n/2) + 1$; $a = 3$; $b = 2$; $f(n) = 1 \Rightarrow k = 0$; $p = 0$

Case $\log_2 3 > k$

$\Rightarrow \Theta(n^{\log_2 3})$

$\Rightarrow \Theta(2^{k \log_2 3})$

$\Rightarrow \Theta(3^k)$

14. $T(x, c) = \Theta(x)$ $\quad c \leq 2$

$T(c, y) = \Theta(y)$ $\quad c \leq 2$

$T(x, y) = \Theta(x+y) + T(x/2, y/2)$

$T(x,y) = \Theta(x+y) + \Theta(\frac{x}{2} + y/2) + T(\frac{x}{4}, y/4)$

$= \Theta(x+y) + \Theta(\frac{x}{2} + y/2) + \Theta(\frac{x}{4} + y/4) + T(x/8, y/8)$

$=$ ⋮

$= \Theta\left((x+y) + \left(\frac{x+y}{2}\right) + \left(\frac{x+y}{4}\right) + \cdots\right)$

$= \Theta\left(\frac{(x+y)}{1-1/2}\right)$

$T(x,y) = 2\Theta(x+y)$

$T(x,y) = \Theta(x+y)$

~~$a_n = 6n^2 + 2n + a_{n-1}$~~

15. $f(n) = n$; $g(n) = n^{(1+\sin n)}$

- For $-1 \le \sin n \le 0$

  i.e n is $\Theta(n^k)$ for $0 \le k \le 1$

  $\therefore f(n) \ne g(n)$

  $\Rightarrow$ Not True

- For $0 \le \sin n \le 1$

  i.e. n is $\Omega(n^k)$ for $1 \le k \le 2$

  $\therefore f(n) \ne \Omega\, g(n)$

  $\Rightarrow$ Not True

$\Rightarrow$ Both are not correct.

16. $a_n = 6n^2 + 2n + a_{n-1}$

$= 6[n^2 + (n-1)^2] + 2[n + n-1] + a_{n-2}$

$= 6(n^2 + (n-1)^2 + (n-2)^2) + 2(n + (n-1) + (n-2)) + a_{n-3}$

$= \vdots$

$= 6(n^2 + (n-1)^2 + (n-2)^2 + \cdots + 1^2) + 2(n + (n-1) + (n-2) + \cdots + 1)$

$= 6\, \frac{n(n+1)(2n+1)}{6} + 2 \times \frac{n(n+1)}{2}$

$= n(n+1)(2n+2)$

$\Rightarrow a_n = 2n(n+1)^2$

$a_{99} = 2 \times 99\,(100)^2$

$= \underline{198} \times 10^4$

$\underline{\underline{\quad \to k}}$